40.017 PROBABILITY & STATISTICS

Homework 3

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Section 2

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Question 1

Consider two disjoint timelines shown in Figure 1 below.

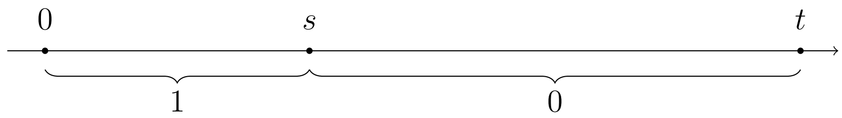

Figure 1: Disjoint Intervals from 0 to t

Using the definition of conditional probability, we have:

$$\begin{aligned}
\mathbb{P}(1 \text{ arrival in } [0,s] | 1 \text{ arrival in } [0,t]) &= \frac{\mathbb{P}(1 \text{ arrival in } [0,s] \cap 1 \text{ arrival in } [0,t])}{\mathbb{P}(1 \text{ arrival in } [0,t])} \\
&= \frac{\mathbb{P}(1 \text{ arrival in } [0,s] \cap 0 \text{ arrival in } [s,t])}{\mathbb{P}(1 \text{ arrival in } [0,t])} \\
&= \frac{\mathbb{P}(N(s)=1) \times \mathbb{P}(N(t-s)=0)}{\mathbb{P}(N(t)=1)} \\
&= \frac{e^{-\lambda s}\frac{(\lambda s)^1}{1!} \times e^{\lambda(t-s)}\frac{[\lambda(t-s)]^0}{0!}}{e^{-\lambda t}\frac{(\lambda t)^1}{1!}} \\
&= \frac{s}{t} e^{-\lambda s} \times e^{-\lambda t} \times e^{\lambda s} \times e^{\lambda t} \\
&= \boxed{\frac{s}{t}}
\end{aligned}$$

Question 2

(a)

The arrival time of a Poisson process is exponentially distributed, with parameter λ for a time period t. If we have a merged Poisson process, the resulting distribution is $\text{Poi}(\lambda_1 + \lambda_2)$. The arrival time X of a Poisson process follows an exponential distribution, so we have

$$X \sim \exp(\lambda_1 + \lambda_2)$$

and they are i.i.d. for both processes. To get the distribution of the 2nd event in the merged process, we add the two arrival times X_1, X_2 which then follows a Gamma distribution (sum of exponentials):

$$X_1 + X_2 \sim \text{Gamma}(2, \lambda_1 + \lambda_2)$$

(b)

It is not necessarily true: The arrival time of the second event in the merged process will follow a gamma distribution from part (a). Consider also the memoryless-ness property, where the second arrival will occur X_2 time after, which is not necessarily $\max(Y_1, Y_2)$.

Question 3

Since the enemy tanks are numbered from 0 to N, then we have a total of $N+1$ tanks. From the hint, we first compute $\mathbb{E}(X_i)$:

$$\begin{aligned}
\mathbb{E}(X_i) &= \sum_{i=0}^{N} x_i \mathbb{P}(X = x_i) \\
&= 0 \times \frac{1}{N+1} + 1 \times \frac{1}{N+1} + \cdots + N \times \frac{1}{N+1} \\
&= \frac{1}{N+1}(0 + 1 + \cdots + N) \\
&= \frac{1}{N+1}\left(\frac{N+1}{2}(N)\right) \\
&= \frac{N(N+1)}{2(N+1)} \\
&= \boxed{\frac{N}{2}}
\end{aligned}$$

For a total of n observations with replacement, we have:

$$\begin{aligned}
\mathbb{E}(\bar{X}_n) &= \mathbb{E}(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n}\left[\mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n)\right] \\
&= \frac{1}{n}\left[n\mathbb{E}(X_i)\right] \\
&= \frac{1}{n}\left[n\left(\frac{N}{2}\right)\right] \\
&= \boxed{\frac{N}{2}}
\end{aligned}$$

To find an unbiased estimator of the total number of tanks, we need by definition $\mathbb{E}(\hat{N}) = (N+1)$. We note that the total number is $N+1$, and thus:

$$\begin{aligned}
\mathbb{E}(\bar{X}_n) &= \frac{N}{2} \\
2\mathbb{E}(\bar{X}_n) &= N \\
(N+1) &= 2\mathbb{E}(\bar{X}_n) + 1 \\
(N+1) &= \boxed{\mathbb{E}(2\bar{X}_n + 1)} \quad \text{(linearity of expectation)}
\end{aligned}$$

Thus, we conclude that an unbiased estimator for the total number of tanks is $\hat{N} = 2\bar{X}_n + 1$.

Question 4

(a)

We are given that $X_1, X_2, \ldots, X_n$ are i.i.d uniformly distributed with parameter θ. From the hint, we note that if the maximum of X_i's is $< x$, then each of the X_i's is $< x$, we use the property of independence here:

$$\begin{aligned}
\mathbb{P}(X_{\max} < x) &= \mathbb{P}(\max(X_1, X_2, \ldots, X_n) < x) \\
&= \mathbb{P}(X_1 < x) \cap \mathbb{P}(X_2 < x) \cap \cdots \cap \mathbb{P}(X_n < x) \\
&= \prod_{i}^{n} \mathbb{P}(X_i < x) \\
&= \boxed{\left(\frac{x}{\theta}\right)^n} \quad \text{(using cdf of uniform distribution)}
\end{aligned}$$

where x is between 0 and θ.

(b)

From the hint, we differentiate part (a) to get the pdf of $X_{\max}$:

$$\frac{d}{dx}\left[\left(\frac{x}{\theta}\right)^n\right] = n\frac{x^{n-1}}{\theta^n}$$

Now to find $\mathbb{E}(X_{\max})$:

$$\begin{aligned}
\mathbb{E}(X_{\max}) &= \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x \\
&= \int_{0}^{\infty} x \left(n \cdot \frac{x^{n-1}}{n}\right) \mathrm{d}x \\
&= \frac{n}{\theta^n} \int_{0}^{\infty} x^n \, \mathrm{d}x \\
&= \frac{n}{\theta^n} \cdot \frac{x^{n+1}}{n+1}\Bigg|_0^\theta \\
&= \frac{n(\theta^{n+1})}{\theta^n (n+1)} \\
&= \boxed{\frac{n}{n+1} \cdot \theta}
\end{aligned}$$

(c)

To find an unbiased estimator for θ, we need to show that $\mathbb{E}(\hat{\theta}) - \theta = 0$, i.e.

$$\begin{aligned}
\mathbb{E}(X_{\max}) &= \frac{n}{n+1} \cdot \theta \\
\frac{n+1}{n} \cdot \mathbb{E}(X_{\max}) &= \theta \\
\mathbb{E}\left(\frac{n+1}{n} \cdot X_{\max}\right) &= \theta \quad \text{(by linearity of expectation)} \\
\Rightarrow \hat{\theta} &= \boxed{\frac{n+1}{n} \cdot X_{\max}}
\end{aligned}$$

Thus $\frac{n+1}{n} \cdot X_{\max}$ is an unbiased estimator for θ.

Question 5

Let $X \sim \text{uniform}(\theta_1, \theta_2)$. Using the method of moments,

$$\mathbb{E}(X) = \bar{x} = \frac{\theta_1 + \theta_2}{2}, \quad \text{Var}(X) = s_x^2 = \frac{(\theta_2 - \theta_1)^2}{12}$$

Combining the two equations, where $\theta_1 = 2\bar{x} - \theta_2$, we start with s_x^2:

$$\begin{aligned}
s_x^2 &= \frac{(\theta_2 - 2\bar{x} + \theta_2)^2}{12} \\
&= \frac{(2\theta_2 - 2\bar{x})^2}{12} \\
&= \frac{4(\theta_2 - \bar{x})^2}{12} \\
3s_x^2 &= (\theta_2 - \bar{x})^2 \\
\sqrt{3}s_x &= \theta_2 - \bar{x} \\
\theta_2 &= \sqrt{3}s_x + \bar{x} \\
\Rightarrow \theta_1 &= 2\bar{x} - \sqrt{3}s_x + \bar{x} \\
&= \bar{x} - \sqrt{3}s_x
\end{aligned}$$

Thus we have the estimators for θ_1 and θ_2:

$$\boxed{\hat{\theta}_1 = \bar{x} - \sqrt{3}s_x, \quad \hat{\theta}_2 = \bar{x} + \sqrt{3}s_x}$$

Question 6

(a)

From Week 6 Class 1 page 11, the Likelihood function of an exponential distribution is given by:

$$\begin{aligned}
L(\theta) &= f(x_1|\theta)f(x_2|\theta)\dots f(x_n|\theta) \\
&= \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) \\
&= \theta^4 \exp\left(-\theta(1.3 + 0.6 + 0.3 + 0.8)\right) \\
&= \theta^4 \exp\left(-3\theta\right)
\end{aligned}$$

where the x_i's are i.i.d, and there are 4 observations of x.

(b)

The prior for θ has a gamma$(5, 2)$ distribution, given by:

$$\begin{aligned}
f(\theta) &= C_0\theta^{5-1}e^{-2\theta} \\
&= C_0\theta^4 e^{-2\theta}
\end{aligned}$$

Using the formula in Week 6 Class 2, posterior = constant × likelihood function × prior, we have:

$$\begin{aligned}
\text{posterior} &= \text{constant} \times \text{likelihood function} \times \text{prior} \\
&= C_1 \times \theta^4 e^{-3\theta} \times C_0\theta^4 e^{-2\theta} \\
&= C_2\theta^{4+4}e^{-3\theta-2\theta} \\
&= \boxed{C_2\theta^8 e^{-5\theta}}
\end{aligned}$$

(c)

The standard gamma function with parameters α, λ is given by:

$$f(\theta) = C\theta^{\alpha-1}e^{-\lambda\theta}$$

for any real constant C. From (b), we see that the posterior is also clearly gamma distributed, where:

$$\text{posterior} = C_2\theta^{9-1}e^{-5\theta}$$

and with parameters $\alpha = 9$, $\lambda = 5$. From Week 3 Class 1, the mean of a $\theta \sim \text{Gamma}(\alpha, \lambda)$ distribution is given by:

$$\begin{aligned}
\mathbb{E}(\theta) &= \frac{\alpha}{\lambda} \\
&= \boxed{\frac{9}{5}}
\end{aligned}$$

Question 7

We start with the general definition of expectation:

$$
\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x \\
&= \int_0^1 x \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot x^{(a+1)-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)} \times \frac{\Gamma(a+1)}{\Gamma((a+1)+b)} \times \frac{\Gamma((a+1)+b)}{\Gamma(a+1)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\Gamma(a+b)}{\cancel{\Gamma(a)}} \times \frac{a\cancel{\Gamma(a)}}{\Gamma((a+1)+b)} \times \frac{\Gamma((a+1)+b)}{\Gamma(a+1)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1}\,\mathrm{d}x \quad (\because \Gamma(\alpha+1) = \alpha\Gamma(\alpha)) \\
&= \int_0^1 \frac{a\cancel{\Gamma(a+b)}}{(a+b)\cancel{\Gamma(a+b)}} \times \frac{\Gamma((a+1)+b)}{\Gamma(a+1)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \frac{a}{a+b} \int_0^1 \frac{\Gamma((a+1)+b)}{\Gamma(a+1)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \boxed{\frac{a}{a+b}} \quad \text{(total area under beta(a + 1, b) r.v. is 1)}
\end{aligned}
$$