40.002 OPTIMIZATION

Problem 1

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Question 1

a) Let D and F denote Domestic and Foreign stocks (in millions) respectively. We formulate the Linear Program (LP):

$$
\begin{aligned}
\max_{D,F} \quad & 0.11D + 0.17F \\
\text{s.t.} \quad & D + F \leq 12 \\
& 0 \leq D \leq 10 \\
& 0 \leq F \leq 7 \\
& D \geq \frac{1}{2}F \\
& F \geq \frac{1}{2}D
\end{aligned}
$$

b) Graphically, we have Figure 1 with corner points labelled and the feasible region shaded:

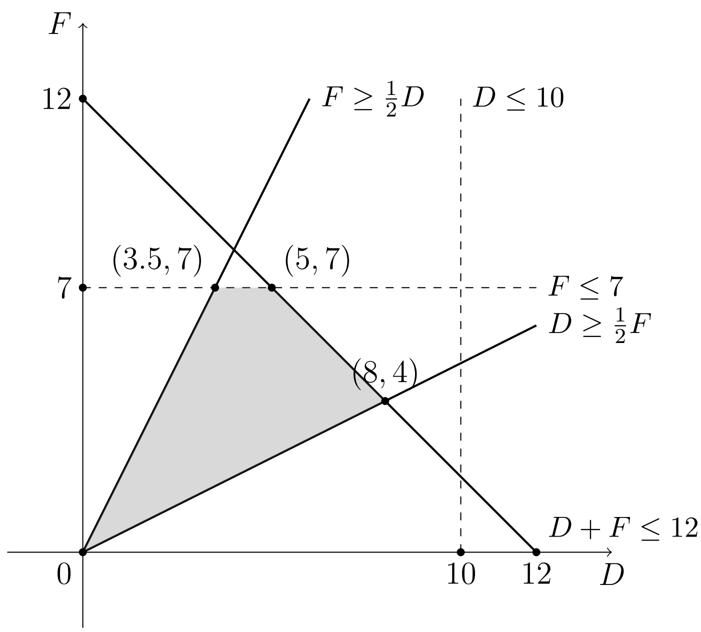

Figure 1: Feasible Region of LP

c) Solving graphically, the optimal value of F and D that maximises returns are $D = 5$, $F = 7$ from Figure 1, for a total (maximum) return of:

$$\boxed{0.11(5) + 0.17(7) = \$1.74 \text{ million}}$$

To prove that this is the global maximum, we brute-force attempt to find the returns of the other corner points:

For $(3.5, 7)$, we have:

$$0.11(3.5) + 0.17(7) = \$1.575 \text{ million} < \$1.74 \text{ million}$$

For $(8, 4)$, we have:

$$0.11(8) + 0.17(4) = \$1.56 \text{ million} < \$1.74 \text{ million}$$

And thus \$1.74 million is indeed the global maximum return for the fund.

Question 2

a) Given the LP constraints

$$\begin{aligned} -x_1 + x_2 &\geq 1 \\ x_1 &\geq 0 \end{aligned}$$

We have the feasible region shaded and corner points labelled:

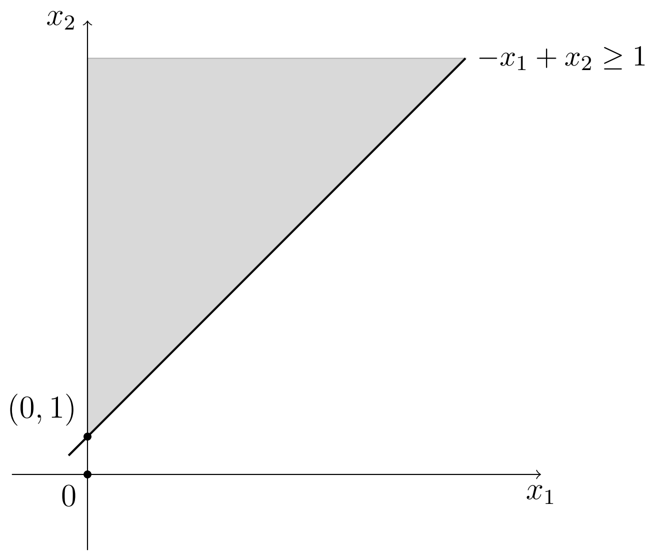

Figure 2: Feasible Region of LP

b) For a minimisation problem, we have the complete LP:

$$\begin{aligned} \min_{x_1, x_2} \quad & \alpha x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \geq 1 \\ & x_1 \geq 0 \end{aligned}$$

i) For the optimal solution to be unique, we need $\alpha = 0$ for a minimum at $x_2 = 1$.

ii) For there to be multiple optimal solutions with infinite optimal objective values, we need $\alpha \in \mathbb{R} \backslash \{-1, 0\}$.

iii) For there to be an unbounded optimal objective value, we need $\alpha = -1$.

Question 3

a) Since Z is a discrete random variable that takes values in the set $\{1, 2, \ldots, 10\}$, and $p_i = \text{Prob}(Z = i)\ \forall i \in \{1, 2, \ldots, 10\}$, with sum and non-negative constraints, we can formulate the LP:

$$\begin{aligned}
\max_{p_i} \quad & \sum_{i=5}^{10} p_i \\
\text{s.t.} \quad & \sum_{i=1}^{10} p_i = 1 \\
& \sum_{i=1}^{10} i \cdot p_i = 4 \\
& p_i \geq 0
\end{aligned}$$

We want to optimise the probability distribution of Z such that the likelihood of Z being 5, 6, 7, 8, 9, or 10 is maximised.

Using the following *JuMP* code, we have the final probability distribution of Z being:

$$\boxed{p = \{0.25, 0, 0, 0, 0.75, 0, 0, 0, 0, 0\}}$$

Question 4

For the minimisation problem:

$$
\begin{aligned}
\min \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \leq 0 \\
& x_2,\ x_3 \geq 0
\end{aligned}
$$

We can rearrange the LP into the form:

$$
\begin{aligned}
\max \quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}
$$

For the 3rd constraint, we separate into two cases:

$$
\begin{aligned}
-x_1 - x_2 + 2x_3 + x_4 &\geq 6 \\
x_1 + x_2 - 2x_3 - x_4 &\leq -6
\end{aligned}
$$

where:

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & 4 \\ -3 & -1 & -4 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -3 \\ 6 \\ -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Question 5

The given optimisation problem is non-linear in nature, due to the piecewise absolute value function $f(z)$ in the objective function. We linearise the problem by introducing additional constraints and auxiliary variables. Let y_i be auxiliary variables for each i, and z_i be the absolute value of $\mathbf{a}_i^\top \mathbf{x} - b_i$.

Enforcing additional constraints to the problem in place of the piecewise nonlinear $f(x)$ in the objective function, we now formulate an appropriate LP:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} y_i \\
\text{s.t.} \quad & z_i = \mathbf{a}_i^\top \mathbf{x} - b_i, \quad i = 1, \ldots, m && (1) \\
& y_i \geq z_i - 5, \quad i = 1, \ldots, m && (2) \\
& y_i \geq -(z_i - 5), \quad i = 1, \ldots, m && (3) \\
& y_i \geq 0, \quad i = 1, \ldots, m && (4) \\
& \mathbf{x} \in \mathbb{R}^n && (5)
\end{aligned}
$$

Constraint (2) enforces a lower bound on the LP, in view of the first piece of $f(z)$ where it is 0 when $|z| < 5$. Constraint (3) enforces an upper bound, in view of the second piece of $f(z)$ for $|z| \geq 5$. Lastly, the non-negativity constraint (4) is imposed to ensure the absolute value of z_i.

Question 6

We are given that $f(\mathbf{x})$ is convex, and we need to show that $\forall t \in \mathbb{R}$, the set of $\mathbf{x}$ (denoted by χ_t) that satisfies $f(\mathbf{x}) \leq t$ is a convex set. Geometrically, for any 2 points $\mathbf{x}_1$ and $\mathbf{x}_2$ in χ_t, the line segment containing them must also be in χ_t. i.e., if $\mathbf{x}_1, \mathbf{x}_2 \in \chi_t$, then $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \chi_t$, $\forall \lambda \in [0, 1]$.

Proof.
$\because$ $\mathbf{x}_1, \mathbf{x}_2 \in \chi_t$, then $f(\mathbf{x}_1) \leq t$ and $f(\mathbf{x}_2) \leq t$. Since $f(\mathbf{x})$ is convex, we use the property:

$$
\begin{aligned}
f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &\leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \\
&\leq \lambda t + (1-\lambda) t \\
&= \lambda t + t - \lambda t \\
&= t
\end{aligned}
$$

$\therefore$ $\forall \lambda \in [0, 1]$, $f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq t$, and thus $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \chi_t$, which is convex. $\square$

An appropriate illustration of this is Figure 3 below.

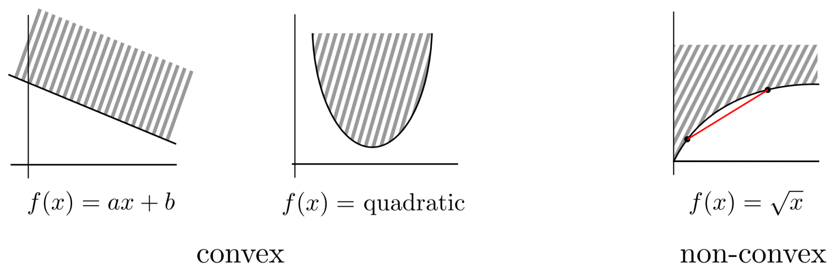

Figure 3: Illustration of Convexity

Question 7

(1)

$$
\begin{aligned}
\max \quad & 5x_1x_2 + 12x_3 \\
\text{s.t.} \quad & x_3 \leq 2 \\
& x_2 \leq 5 \\
& 3.2x_2 + 5.1x_1 = 5.1 \\
& x_1 + \log x_2 - x_3 = 0 \\
& x_1, x_2, x_3 \in \{0, 1\}
\end{aligned}
$$

For this LP, Proportionality, Additivity, and Certainty assumptions hold, while Divisibility does not. The final constraint suggests that x_1, x_2, x_3 are constrained to values either 0 or 1 in the set, which are only binary whole numbers, and not fractional values.

(2)

$$
\begin{aligned}
\max \quad & 5x_1 - x_2 - 2x_3 \\
\text{s.t.} \quad & x_3 \leq 2.9 \\
& x_2 \geq \theta \\
& x_3 + x_2 - x_1^2 = 0 \\
& 0.4\frac{x_1}{x_2} + x_3 \leq 0 \\
& x_1, x_2, x_3 \geq 0 \\
& \theta \in U(0, 1)
\end{aligned}
$$

For this LP, Proportionality, Additivity, hold, while Divisibility and Certainty do not. For Divisibility, the constraint involving the random variable $\theta \in U(0, 1)$ does not explicitly imply that the decision variable x_2 can be assigned to fractional values. For Certainty, since random variables are stochastic in nature, there is no way to know for sure the value of x_2.

(3)

$$\begin{aligned}
\max \quad & x_1 - 15x_2 - 10x_3 \\
\text{s.t.} \quad & \frac{x_3}{x_2} \leq 20 \\
& x_2 - 4x_1 \leq 0 \\
& x_3 + x_2 - x_1 = 0 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

For this LP, all 4 assumptions of Proportionality, Additivity, Divisibility, and Certainty hold.