
**40.012 MANUFACTURING AND SERVICE
OPERATIONS 2.0**

Homework 4

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Question 1 & 2

Let $X(t)$ represent the number of items in inventory at time t . From Homework 1, the Q matrix is given as:

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & \dots & R-1 & R & R+1 & \dots & K-1 & K & K+1 & \dots & R+K-1 & R+K \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ R-1 \\ R \\ R+1 \\ \vdots \\ K-1 \\ K \\ K+1 \\ \vdots \\ R+K-1 \\ R+K \end{matrix} & \begin{bmatrix} -\theta & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta & 0 & \dots & 0 & 0 \\ \lambda & -\lambda-\theta & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \theta & \dots & 0 & 0 \\ 0 & \lambda & -\lambda-\theta & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\theta-\lambda & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \theta & 0 \\ 0 & 0 & 0 & \dots & \lambda & -\lambda-\theta & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda & -\lambda & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda & -\lambda \end{bmatrix} \end{matrix}$$

From question 1, we are given the following values: $K = 4$, $\lambda = 2$ and $\frac{1}{\theta} = 1$. For the 3 different values of R , i.e. $R = 1$, $R = 2$, $R = 3$, we have 3 different matrices for Q , Q_1 , Q_2 , and Q_3 , corresponding to each value of R respectively. Starting with $R = 1$, we have:

$$Q_1 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{matrix} \quad (1)$$

To obtain the long-run probabilities, we need to use the steady-state equations, i.e.

$$\pi Q = \mathbf{0} \quad (2)$$

where π is the steady-state probability matrix with entries p_i , and $\mathbf{0}$ is a vector of zeroes of similar dimension. From Q_1 , we have the following expression:

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the following simultaneous equations:

$$\begin{aligned} -p_0 + 2p_1 &= 0 \\ -3p_1 + 2p_2 &= 0 \\ -2p_2 + 2p_3 &= 0 \\ -2p_3 + 2p_4 &= 0 \\ p_0 - 2p_4 + 2p_5 &= 0 \\ p_1 - p_5 &= 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 &= 1 \end{aligned}$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned}\pi &= [p_0 \ p_1 \ p_2 \ p_3 \ p_4 \ p_5] \\ &= \left[\frac{1}{4} \ \frac{1}{8} \ \frac{3}{16} \ \frac{3}{16} \ \frac{3}{16} \ \frac{1}{16} \right]\end{aligned}$$

Now we repeat the process for Q_2 , with $R = 2$.

$$Q_2 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{matrix} \quad (3)$$

To obtain the long-run probabilities, we have from Q_2 :

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the following simultaneous equations:

$$\begin{aligned}-p_0 + 2p_1 &= 0 \\ -3p_1 + 2p_2 &= 0 \\ -3p_2 + 2p_3 &= 0 \\ -2p_3 + 2p_4 &= 0 \\ -2p_4 + 2p_5 &= 0 \\ p_0 - 2p_4 + 2p_5 &= 0 \\ p_1 - 2p_5 + 2p_6 &= 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1\end{aligned}$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned}\pi &= [p_0 \ p_1 \ p_2 \ p_3 \ p_4 \ p_5 \ p_6] \\ &= \left[\frac{2}{11} \ \frac{1}{11} \ \frac{3}{22} \ \frac{9}{44} \ \frac{9}{44} \ \frac{5}{44} \ \frac{3}{44} \right]\end{aligned}$$

Now we repeat the process for Q_3 , with $R = 3$:

$$Q_3 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{matrix} \quad (4)$$

To obtain the long-run probabilities, we have from Q_3 :

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the following simultaneous equations:

$$\begin{aligned} -p_0 + 2p_1 &= 0 \\ -3p_1 + 2p_2 &= 0 \\ -3p_2 + 2p_3 &= 0 \\ -3p_3 + 2p_4 &= 0 \\ p_0 - 2p_4 + 2p_5 &= 0 \\ p_1 - 2p_5 + 2p_6 &= 0 \\ p_2 - 2p_6 + 2p_7 &= 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1 \end{aligned}$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned} \pi &= [p_0 \ p_1 \ p_2 \ p_3 \ p_4 \ p_5 \ p_6 \ p_7] \\ &= \left[\frac{4}{31} \ \frac{2}{31} \ \frac{3}{31} \ \frac{9}{62} \ \frac{27}{124} \ \frac{19}{124} \ \frac{15}{124} \ \frac{9}{124} \right] \end{aligned}$$

For each of the 3 cases, we need to obtain the average number of items in inventory in steady-state, as well as the proportion of time there is nothing in inventory. This corresponds to the expected value of the number of items in steady-state and p_0 respectively. To calculate the expected value,

for π_1 :

$$\begin{aligned}
 \mathbb{E}(x) &= \sum_{i=0}^5 i \cdot p_i \\
 &= 0 \left(\frac{1}{4} \right) + 1 \left(\frac{1}{8} \right) + 2 \left(\frac{3}{16} \right) + 3 \left(\frac{1}{16} \right) + 4 \left(\frac{3}{16} \right) + 5 \left(\frac{1}{16} \right) \\
 &= \frac{17}{8} \\
 &= 2.125
 \end{aligned}$$

for π_2 :

$$\begin{aligned}
 \mathbb{E}(x) &= \sum_{i=0}^6 i \cdot p_i \\
 &= 0 \left(\frac{2}{11} \right) + 1 \left(\frac{1}{11} \right) + 2 \left(\frac{3}{22} \right) + 3 \left(\frac{9}{44} \right) + 4 \left(\frac{9}{44} \right) + 5 \left(\frac{5}{44} \right) + 6 \left(\frac{3}{44} \right) \\
 &= \frac{61}{22} \\
 &= 2.7727
 \end{aligned}$$

for π_3 :

$$\begin{aligned}
 \mathbb{E}(x) &= \sum_{i=0}^7 i \cdot p_i \\
 &= 0 \left(\frac{4}{31} \right) + 1 \left(\frac{2}{31} \right) + 2 \left(\frac{3}{31} \right) + 3 \left(\frac{9}{62} \right) + 4 \left(\frac{27}{124} \right) + 5 \left(\frac{19}{124} \right) + 6 \left(\frac{15}{124} \right) + 7 \left(\frac{9}{124} \right) \\
 &= \frac{221}{62} \\
 &= 3.5645
 \end{aligned}$$

Putting all of these into a concise table, we have:

R	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	$\mathbb{E}(x)$
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	0	0	2.125
2	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{3}{22}$	$\frac{9}{44}$	$\frac{9}{44}$	$\frac{5}{44}$	$\frac{3}{44}$	0	2.7727
3	$\frac{4}{31}$	$\frac{2}{31}$	$\frac{3}{31}$	$\frac{9}{62}$	$\frac{27}{124}$	$\frac{19}{124}$	$\frac{15}{124}$	$\frac{9}{124}$	3.5645

Table 1: Table of Values

Question 3

To compute the limiting distribution $\pi = [p_0 \ p_1 \ p_2]$, we need to use the steady-state balance equations, given by

$$\pi Q = \mathbf{0} \quad (5)$$

Given that we have

$$Q = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix} \quad (6)$$

the balance equation results in

$$[p_0 \ p_1 \ p_2] \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix} = [0 \ 0 \ 0]$$

which gives us the system of equations:

$$\begin{aligned} -3p_0 + 2p_1 + p_2 &= 0 \\ 2p_0 - 4p_1 + p_2 &= 0 \\ p_0 + 2p_1 - 2p_2 &= 0 \\ p_0 + p_1 + p_2 &= 1 \end{aligned}$$

using an online solver yields the following limiting distribution:

$$\pi = [p_0 \ p_1 \ p_2] = \left[\frac{6}{19} \ \frac{5}{19} \ \frac{8}{19} \right]$$

Question 4

Similar to the class problems, we consider this to be a Poisson splitting process, where the customer picks either one of the servers with equal probabilities if both are free, i.e. $p = \frac{1}{2}$. We let $X_i(t)$ be the number of customers at server i , and $(X_1(t), X_2(t))$ be a bivariate CTMC, the state of the system at time t , with state space $\mathcal{S} = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Arrivals to the shop are $PP(\lambda)$ with rate λ , and server i takes $\exp(\mu_i)$ time to serve a customer, with rate μ_i . With this, the rate diagram is given below as

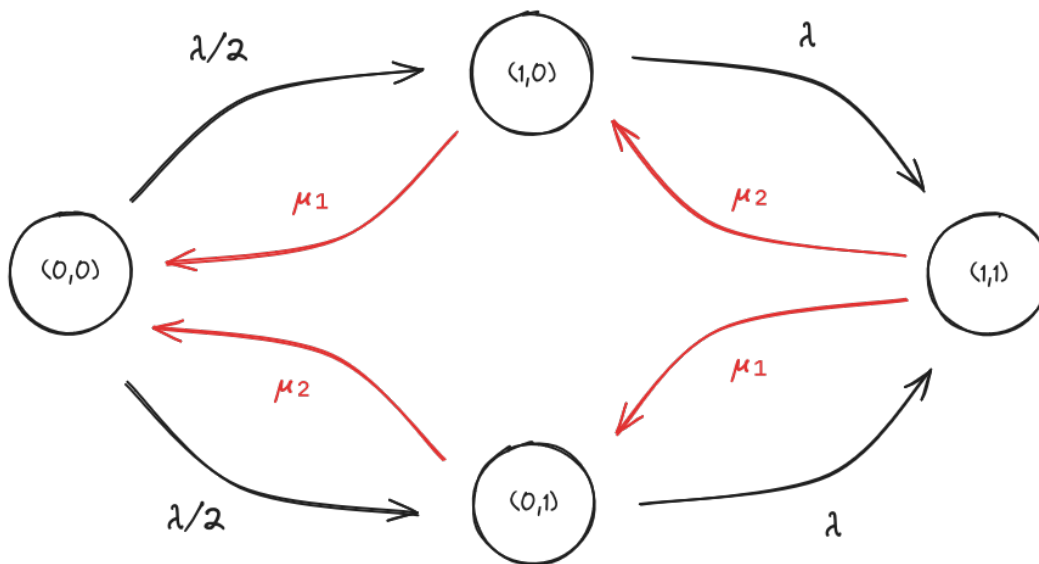


Figure 1: Transition Rate Diagram

The transition rate matrix Q is given by

$$Q = \begin{matrix} & \begin{matrix} (0,0) & (1,0) & (0,1) & (1,1) \end{matrix} \\ \begin{matrix} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{matrix} & \begin{bmatrix} -\lambda & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \mu_1 & -\mu_1 - \lambda & 0 & \lambda \\ \mu_2 & 0 & -\mu_2 - \lambda & \lambda \\ 0 & \mu_2 & \mu_1 & -\mu_2 - \mu_1 \end{bmatrix} \end{matrix} \quad (7)$$

Question 6

Let $X(t)$ represent the number of customers waiting at the bus station at time t . Customers arrive in a $PP(\lambda)$ fashion with rate λ . The state space here is $\mathcal{S} = \{0, 1, 2, \dots\}$ since we have no limit to the number of customers arriving. Since the bus has no limit to the number of customers boarding for departure, our rate diagram becomes:

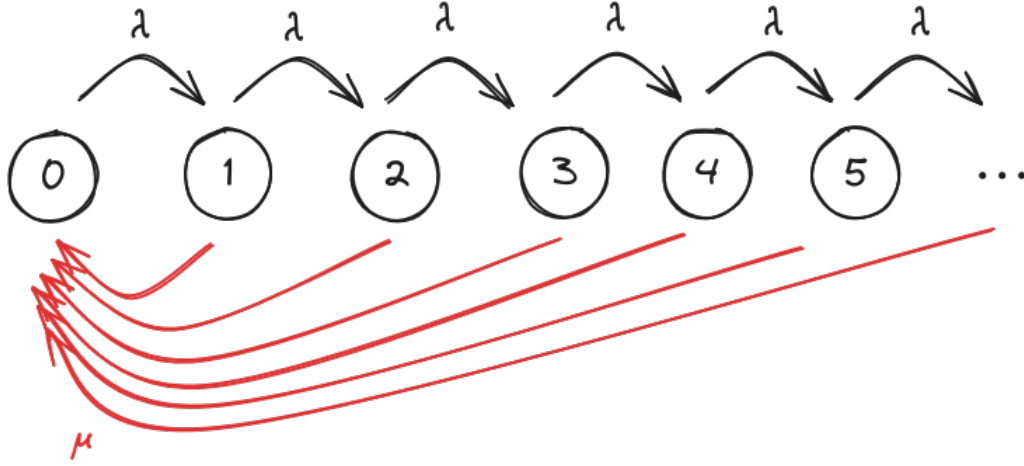


Figure 2: Transition State Diagram

The transition rate matrix is given by:

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & \dots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{matrix} & \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -\mu - \lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & 0 & -\mu - \lambda & \lambda & 0 & 0 & \dots \\ \mu & 0 & 0 & -\mu - \lambda & \lambda & 0 & \dots \\ \mu & 0 & 0 & 0 & -\mu - \lambda & \lambda & \dots \\ \mu & 0 & 0 & 0 & 0 & -\mu - \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{matrix} \quad (8)$$

To find the steady-state distribution, we need to consider the balance equations:

$$\pi Q = \mathbf{0}$$

expressed as

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots \end{bmatrix} \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -\mu - \lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & 0 & -\mu - \lambda & \lambda & 0 & 0 & \dots \\ \mu & 0 & 0 & -\mu - \lambda & \lambda & 0 & \dots \\ \mu & 0 & 0 & 0 & -\mu - \lambda & \lambda & \dots \\ \mu & 0 & 0 & 0 & 0 & -\mu - \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots \end{bmatrix}$$

From this, we get the simultaneous equations:

$$-\lambda p_0 + \mu p_1 + \mu p_2 + \mu p_3 + \dots = 0 \quad (9)$$

$$\lambda p_0 - (\mu + \lambda)p_1 = 0 \quad \implies p_1 = \frac{\lambda}{\mu + \lambda} p_0$$

$$\lambda p_1 - (\mu + \lambda)p_2 = 0 \quad \implies p_2 = \left(\frac{\lambda}{\mu + \lambda}\right)^2 p_0$$

$$\lambda p_2 - (\mu + \lambda)p_3 = 0 \quad \implies p_3 = \left(\frac{\lambda}{\mu + \lambda}\right)^3 p_0$$

$$\vdots \quad \quad \quad \vdots$$

$$p_0 + p_1 + p_2 + \dots = 1 \quad (10)$$

From Equation 9, we have

$$\lambda p_0 = \mu (p_1 + p_2 + p_3 + \dots) \quad (11)$$

From Equation 10, we have:

$$\begin{aligned} p_0 + p_1 + p_2 + \dots &= 1 \\ p_1 + p_2 + p_3 + \dots &= 1 - p_0 \end{aligned} \quad (12)$$

Substituting Equation 12 into Equation 11, we have

$$\begin{aligned} \lambda p_0 &= \mu(1 - p_0) \\ \lambda p_0 &= \mu - \mu p_0 \\ (\lambda + \mu)p_0 &= \mu \\ p_0 &= \left(\frac{\mu}{\lambda + \mu}\right) \end{aligned} \quad (13)$$

From the simultaneous equations above, we can see a trend where:

$$p_i = \left(\frac{\lambda}{\mu + \lambda}\right)^i p_0 \quad (14)$$

Here, we substitute p_0 from Equation 13 into the above to obtain:

$$p_i = \left(\frac{\lambda}{\mu + \lambda}\right)^i \left(\frac{\mu}{\mu + \lambda}\right) \quad (15)$$

Thus, the steady-state distribution of the CTMC is the matrix formed by the values of p_i , i.e. $\pi = [p_i], \forall i \in \mathcal{S}$.