40.012 MANUFACTURING AND SERVICE OPERATIONS 2.0

Homework 4

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Question 1 & 2

Let $X(t)$ represent the number of items in inventory at time t. From Homework 1, the Q matrix is given as:

$$
Q = \begin{array}{c c}
 & \begin{array}{ccccccccccccccc} 0 & 1 & 2 & \dots & \text{R - 1} & \text{R} & \text{R + 1} & \dots & \text{K - 1} & \text{K} & \text{K + 1} & \dots & \text{R + K - 1} & \text{R + K} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \text{R - 1} \\ \text{R} \\ \text{R + 1} \\ \vdots \\ \text{K - 1} \\ \text{K} \\ \text{K+1} \\ \vdots \\ \text{R + K - 1} \\ \text{R + K} \end{array} &
\begin{bmatrix}
-\theta & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta & 0 & \dots & 0 & 0 \\
\lambda & -\lambda-\theta & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \theta & \dots & 0 & 0 \\
0 & \lambda & -\lambda-\theta & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & -\theta-\lambda & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \theta & 0 \\
0 & 0 & 0 & \dots & \lambda & -\lambda-\theta & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & \lambda & -\lambda & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda & -\lambda & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda & -\lambda & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda & -\lambda
\end{bmatrix}
\end{array}
$$

From question 1, we are given the following values: $K = 4$, $\lambda = 2$ and $\frac{1}{\theta} = 1$. For the 3 different values of R, i.e. $R = 1$, $R = 2$, $R = 3$, we have 3 different matrices for Q, Q_1, Q_2, and Q_3, corresponding to each value of R respectively. Starting with $R = 1$, we have:

$$
Q_1 = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 0 \\
2 & -3 & 0 & 0 & 0 & 1 \\
0 & 2 & -2 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 & -2 & 0 \\
0 & 0 & 0 & 0 & 2 & -2
\end{bmatrix}
\end{array} \tag{1}
$$

To obtain the long-run probabilities, we need to use the steady-state equations, i.e.

$$\pi Q = \mathbf{0} \tag{2}$$

where π is the steady-state probability matrix with entries p_i, and $\mathbf{0}$ is a vector of zeroes of similar dimension. From Q_1, we have the following expression:

$$
\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 0 \\
2 & -3 & 0 & 0 & 0 & 1 \\
0 & 2 & -2 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 & -2 & 0 \\
0 & 0 & 0 & 0 & 2 & -2
\end{bmatrix}
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

which gives us the following simultaneous equations:

$$
\begin{aligned}
-p_0 + 2p_1 &= 0 \\
-3p_1 + 2p_2 &= 0 \\
-2p_2 + 2p_3 &= 0 \\
-2p_3 + 2p_4 &= 0 \\
p_0 - 2p_4 + 2p_5 &= 0 \\
p_1 - p_5 &= 0 \\
p_1 + p_2 + p_3 + p_4 + p_5 &= 1
\end{aligned}
$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned}\pi &= \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}\end{aligned}$$

Now we repeat the process for Q_2, with $R = 2$.

$$Q_2 = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{array} \tag{3}$$

To obtain the long-run probabilities, we have from Q_2:

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the following simultaneous equations:

$$\begin{aligned} -p_0 + 2p_1 &= 0 \\ -3p_1 + 2p_2 &= 0 \\ -3p_2 + 2p_3 &= 0 \\ -2p_3 + 2p_4 &= 0 \\ -2p_4 + 2p_5 &= 0 \\ p_0 - 2p_4 + 2p_5 &= 0 \\ p_1 - 2p_5 + 2p_6 &= 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1 \end{aligned}$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned}\pi &= \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{11} & \frac{1}{11} & \frac{3}{22} & \frac{9}{44} & \frac{9}{44} & \frac{5}{44} & \frac{3}{44} \end{bmatrix}\end{aligned}$$

Now we repeat the process for Q_3, with $R = 3$:

$$Q_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{array} \tag{4}$$

To obtain the long-run probabilities, we have from Q_3:

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the following simultaneous equations:

$$\begin{aligned} -p_0 + 2p_1 &= 0 \\ -3p_1 + 2p_2 &= 0 \\ -3p_2 + 2p_3 &= 0 \\ -3p_3 + 2p_4 &= 0 \\ p_0 - 2p_4 + 2p_5 &= 0 \\ p_1 - 2p_5 + 2p_6 &= 0 \\ p_2 - 2p_6 + 2p_7 &= 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1 \end{aligned}$$

Solving this using an online solver, we get the following values for the steady-state distribution:

$$\begin{aligned} \pi &= \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{31} & \frac{2}{31} & \frac{3}{31} & \frac{9}{62} & \frac{27}{124} & \frac{19}{124} & \frac{15}{124} & \frac{9}{124} \end{bmatrix} \end{aligned}$$

For each of the 3 cases, we need to obtain the average number of items in inventory in steady-state, as well as the proportion of time there is nothing in inventory. This corresponds to the expected value of the number of items in steady-state and p_0 respectively. To calculate the expected value, for π_1:

$$\begin{aligned}
\mathbb{E}(x) &= \sum_{i=0}^{5} i \cdot p_i \\
&= 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{3}{16}\right) + 3\left(\frac{1}{16}\right) + 4\left(\frac{3}{16}\right) + 5\left(\frac{1}{16}\right) \\
&= \frac{17}{8} \\
&= 2.125
\end{aligned}$$

for π_2:

$$\begin{aligned}
\mathbb{E}(x) &= \sum_{i=0}^{6} i \cdot p_i \\
&= 0\left(\frac{2}{11}\right) + 1\left(\frac{1}{11}\right) + 2\left(\frac{3}{22}\right) + 3\left(\frac{9}{44}\right) + 4\left(\frac{9}{44}\right) + 5\left(\frac{5}{44}\right) + 6\left(\frac{3}{44}\right) \\
&= \frac{61}{22} \\
&= 2.7727
\end{aligned}$$

for π_3:

$$\begin{aligned}
\mathbb{E}(x) &= \sum_{i=0}^{7} i \cdot p_i \\
&= 0\left(\frac{4}{31}\right) + 1\left(\frac{2}{31}\right) + 2\left(\frac{3}{31}\right) + 3\left(\frac{9}{62}\right) + 4\left(\frac{27}{124}\right) + 5\left(\frac{19}{124}\right) + 6\left(\frac{15}{124}\right) + 7\left(\frac{9}{124}\right) \\
&= \frac{221}{62} \\
&= 3.5645
\end{aligned}$$

Putting all of these into a concise table, we have:

R	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	$\mathbb{E}(x)$
1	$\mathbf{\frac{1}{4}}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	0	0	**2.125**
2	$\mathbf{\frac{2}{11}}$	$\frac{1}{11}$	$\frac{3}{22}$	$\frac{9}{44}$	$\frac{9}{44}$	$\frac{5}{44}$	$\frac{3}{44}$	0	**2.7727**
3	$\mathbf{\frac{4}{31}}$	$\frac{2}{31}$	$\frac{3}{31}$	$\frac{9}{62}$	$\frac{27}{124}$	$\frac{19}{124}$	$\frac{15}{124}$	$\frac{9}{124}$	**3.5645**

Table 1: Table of Values

Question 3

To compute the limiting distribution $\pi = \begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix}$, we need to use the steady-state balance equations, given by

$$\pi Q = \mathbf{0} \tag{5}$$

Given that we have

$$Q = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix} \tag{6}$$

the balance equation results in

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

which gives us the system of equations:

$$\begin{aligned} -3p_0 + 2p_1 + p_2 &= 0 \\ 2p_0 - 4p_1 + p_2 &= 0 \\ p_0 + 2p_1 - 2p_2 &= 0 \\ p_0 + p_1 + p_2 &= 1 \end{aligned}$$

using an online solver yields the following limiting distribution:

$$\pi = \begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} = \begin{bmatrix} \frac{6}{19} & \frac{5}{19} & \frac{8}{19} \end{bmatrix}$$

Question 4

Similar to the class problems, we consider this to be a Poisson splitting process, where the customer picks either one of the servers with equal probabilities if both are free, i.e. $p = \frac{1}{2}$. We let $X_i(t)$ be the number of customers at server i, and $(X_1(t), X_2(t))$ be a bivariate CTMC, the state of the system at time t, with state space $\mathcal{S} = \{(0,0), (1,0), (0,1), (1,1)\}$. Arrivals to the shop are $PP(\lambda)$ with rate λ, and server i takes $\exp(\mu_i)$ time to serve a customer, with rate μ_i. With this, the rate diagram is given below as

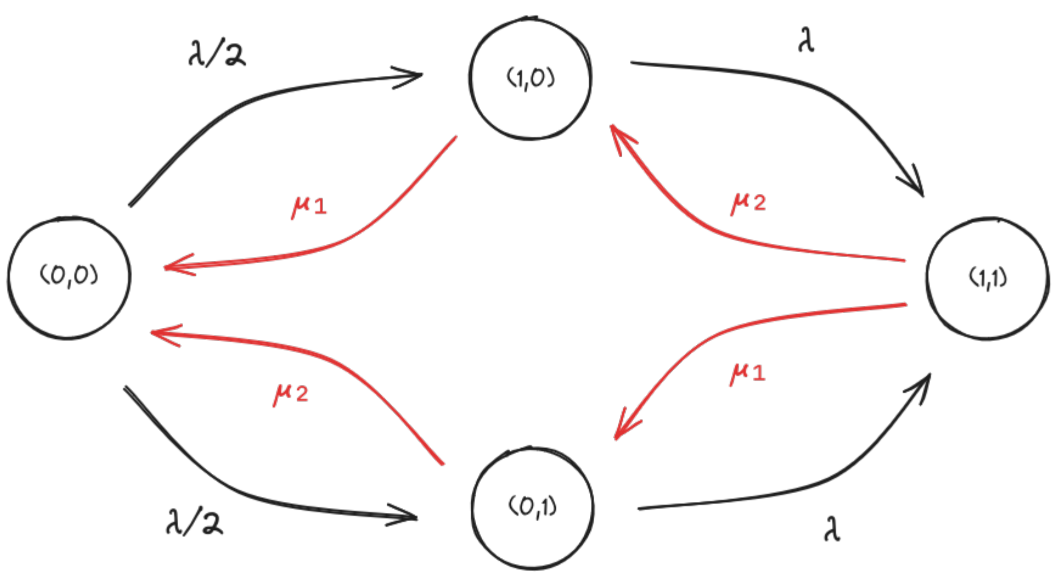

Figure 1: Transition Rate Diagram

The transition rate matrix Q is given by

$$Q = \begin{array}{c} \\ (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{bmatrix} -\lambda & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \mu_1 & -\mu_1 - \lambda & 0 & \lambda \\ \mu_2 & 0 & -\mu_1 - \lambda & \lambda \\ 0 & \mu_2 & \mu_1 & -\mu_2 - \mu_1 \end{bmatrix} \end{array} \tag{7}$$

Question 6

Let $X(t)$ represent the number of customers waiting at the bus station at time t. Customers arrive in a $PP(\lambda)$ fashion with rate λ. The state space here is $\mathcal{S} = \{0, 1, 2, \dots\}$ since we have no limit to the number of customers arriving. Since the bus has no limit to the number of customers boarding for departure, our rate diagram becomes:

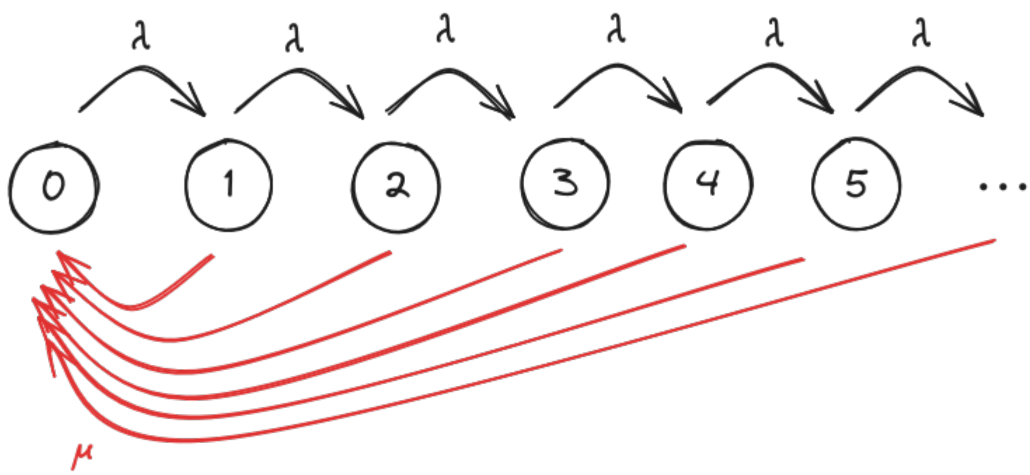

Figure 2: Transition State Diagram

The transition rate matrix is given by:

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \dots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -\mu - \lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & 0 & -\mu - \lambda & \lambda & 0 & 0 & \dots \\ \mu & 0 & 0 & -\mu - \lambda & \lambda & 0 & \dots \\ \mu & 0 & 0 & 0 & -\mu - \lambda & \lambda & \dots \\ \mu & 0 & 0 & 0 & 0 & -\mu - \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array} \tag{8}$$

To find the steady-state distribution, we need to consider the balance equations:

$$\pi Q = \mathbf{0}$$

expressed as

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots \end{bmatrix} \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -\mu-\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & 0 & -\mu-\lambda & \lambda & 0 & 0 & \dots \\ \mu & 0 & 0 & -\mu-\lambda & \lambda & 0 & \dots \\ \mu & 0 & 0 & 0 & -\mu-\lambda & \lambda & \dots \\ \mu & 0 & 0 & 0 & 0 & -\mu-\lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots \end{bmatrix}$$

From this, we get the simultaneous equations:

$$-\lambda p_0 - \mu p_1 - \mu p_2 - \mu p_3 - \dots = 0 \tag{9}$$

$$\lambda p_0 - (\mu+\lambda) p_1 = 0 \qquad \Longrightarrow p_1 = \frac{\lambda}{\mu+\lambda} p_0$$

$$\lambda p_1 - (\mu+\lambda) p_2 = 0 \qquad \Longrightarrow p_2 = \left(\frac{\lambda}{\mu+\lambda}\right)^2 p_0$$

$$\lambda p_2 - (\mu+\lambda) p_3 = 0 \qquad \Longrightarrow p_3 = \left(\frac{\lambda}{\mu+\lambda}\right)^3 p_0$$

$$\vdots \qquad\qquad \vdots$$

$$p_0 + p_1 + p_2 + \dots = 1 \tag{10}$$

From Equation 9, we have

$$\lambda p_0 + \mu\left(p_1 + p_2 + p_3 + \dots\right) = 0 \tag{11}$$

From Equation 10, we have:

$$p_0 + p_1 + p_2 + \dots = 1$$

$$p_1 + p_2 + p_3 + \dots = 1 - p_0 \tag{12}$$

Substituting Equation 12 into Equation 11, we have

$$\lambda p_0 + \mu(1 - p_0) = 0$$

$$\lambda p_0 + \mu - \mu p_0 = 0$$

$$(\lambda - \mu) p_0 = -\mu$$

$$p_0 = -\left(\frac{\mu}{\lambda - \mu}\right)$$

$$p_0 = \boxed{\frac{\mu}{\mu - \lambda}} \tag{13}$$

From the simultaneous equations above, we can see a trend where:

$$p_i = \left(\frac{\lambda}{\mu+\lambda}\right)^i p_0 \tag{14}$$

Here, we substitute Equation 13 into the above to obtain:

$$p_i = \left(\frac{\lambda}{\mu+\lambda}\right)^i \left(\frac{\mu}{\mu-\lambda}\right) \tag{15}$$

Thus, the steady-state distribution of the CTMC is the matrix formed by the values of p_i, i.e. $\pi = \left[p_i\right]$.