40.002 OPTIMIZATION

Problem 1

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Question 1

a) Let D and F denote Domestic and Foreign stocks (in millions) respectively. We formulate the Linear Program (LP):

$$\begin{aligned}
\max_{D,F} \quad & 0.11D + 0.17F \\
\text{s.t.} \quad & D + F \leq 12 \\
& 0 \leq D \leq 10 \\
& 0 \leq F \leq 7 \\
& D \geq \frac{1}{2}F \\
& F \geq \frac{1}{2}D
\end{aligned}$$

b) Graphically, we have Figure 1 with corner points labelled and the feasible region shaded:

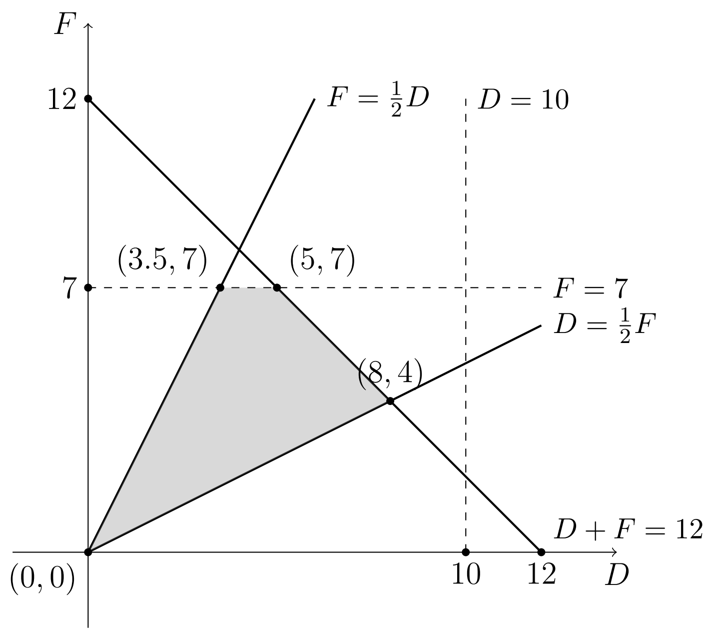

Figure 1: Feasible Region of LP

c) Solving graphically, the optimal value of F and D that maximises returns are $D = 5$, $F = 7$ from Figure 1, for a total (maximum) return of:

$$\boxed{0.11(5) + 0.17(7) = \$1.74 \text{ million}}$$

To prove that this is the global maximum, we brute-force attempt to find the returns of the other corner points:

For $(3.5, 7)$, we have:

$$0.11(3.5) + 0.17(7) = \$1.575 \text{ million} < \$1.74 \text{ million}$$

For $(8, 4)$, we have:

$$0.11(8) + 0.17(4) = \$1.56 \text{ million} < \$1.74 \text{ million}$$

And thus \$1.74 million is indeed the global maximum return for the fund. The mutual fund manager should divide \$5 million into Domestic stocks and \$7 million into Foreign stocks.

Question 2

a) Given the LP constraints

$$\begin{aligned} -x_1 + x_2 &\geq 1 \\ x_1 &\geq 0 \end{aligned}$$

We have the feasible region shaded and corner points labelled:

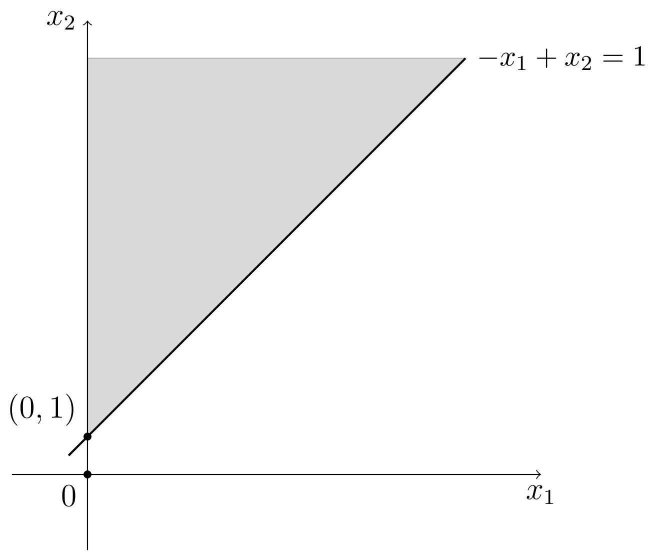

Figure 2: Feasible Region of LP

b) For a minimisation problem, we have the complete LP:

$$\begin{aligned} \min_{x_1, x_2} \quad & \alpha x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \geq 1 \\ & x_1 \geq 0 \end{aligned}$$

i) For the optimal solution to be unique, we need $\alpha > -1$, and the unique optimum is at the point $(0, 1)$. For any value of α within this range, the optimum solution will always converge to this minimum point.

ii) For there to be multiple optimal solutions with finite optimal objective values, we need $\alpha = -1$, and this results in slope $= 1$. The line now lies on $-x_1 + x_2 = 1$ on Figure 2, with multiple optimal solutions and optimal objective value of 1.

iii) For there to be an unbounded optimal objective value, we need $\alpha < -1$, resulting in a slope $= 1$.

Question 3

a) Since Z is a discrete random variable that takes values in the set $\{1, 2, \ldots, 10\}$, and $p_i = \text{Prob}(Z = i)\ \forall i \in \{1, 2, \ldots, 10\}$, with sum and non-negative constraints, we can formulate the LP:

$$\begin{aligned} \max_{p_i} \quad & \sum_{i=5}^{10} p_i \\ \text{s.t.} \quad & \sum_{i=1}^{10} p_i = 1 \\ & \sum_{i=1}^{10} i \cdot p_i = 4 \\ & p_i \geq 0 \end{aligned}$$

We want to optimise the probability distribution of Z such that the likelihood of Z being 5, 6, 7, 8, 9, or 10 is maximised. The first and last constraints come from the fact that the probabilities must sum to one and be non-negative. The second constraint arises from the average value (and hence the expectation) $\mathbb{E}(X) = \sum_{i=1}^{10} i \cdot p_i = 4$.

Using the following *JuMP* code and GLPK solver, we have the final probability distribution of Z being:

$$\boxed{p = \{0.25, 0, 0, 0, 0.75, 0, 0, 0, 0, 0\}}$$

Question 4

For the minimisation problem:

$$
\begin{aligned}
\min \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 && (1) \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 && (2) \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 && (3) \\
& x_1 \leq 0 && (4) \\
& x_2,\ x_3 \geq 0 && (5)
\end{aligned}
$$

We can rearrange the LP into the form:

$$
\begin{aligned}
\max \quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}
$$

Note that to transform the LP from a minimisation to maximisation problem, we multiply the objective function by -1:

$$\max_{\mathbf{x} \in \chi} f(x_1, \ldots, x_2) = -\min_{\mathbf{x} \in \chi} -f(x_1, \ldots, x_n) \implies \min\ -x_1 + x_2$$

For constraint (3), we separate into two cases:

$$
\begin{aligned}
-x_1 - x_2 + 2x_3 + x_4 \geq 6 \\
x_1 + x_2 - 2x_3 - x_4 \leq -6
\end{aligned}
$$

where:

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & -1 & -4 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -3 \\ 6 \\ -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

After transformation to the maximisation problem, in both LPs, the optimum solution remains the exact same. However, since we multiply the objective value by -1, turning it into a maximisation problem means that the optimal objective value now differs from the minimisation problem by -1, i.e. its negative value.

Question 5

The given optimisation problem is non-linear in nature, due to the absolute value function $f(z)$ in the objective function. We linearise the problem by rewriting $f(x)$ as a piecewise linear function:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \max\left(0,\ \mathbf{a}_i^\top \mathbf{x} - b_i - 5,\ -\mathbf{a}_i^\top \mathbf{x} - b_i - 5\right) \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

We let y_i be auxiliary variables for each i, and z_i be the argument of the function. Enforcing additional constraints to the problem, we now formulate an appropriate LP:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & z_i = \mathbf{a}_i^\top \mathbf{x} - b_i, \quad i = 1, \ldots, m \\ & y_i \geq z_i - 5, \quad i = 1, \ldots, m \\ & y_i \geq -z_i - 5, \quad i = 1, \ldots, m \\ & y_i \geq 0, \quad i = 1, \ldots, m \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Note that $y_i \in \mathbb{R}^n$ as well. The constraints ensure that we are obtaining the minimum value of all i's as the optimal solution. Graphically, refer to Figure 3 below for $f(z) = \max(0,\ z-5,\ -z-5)$:

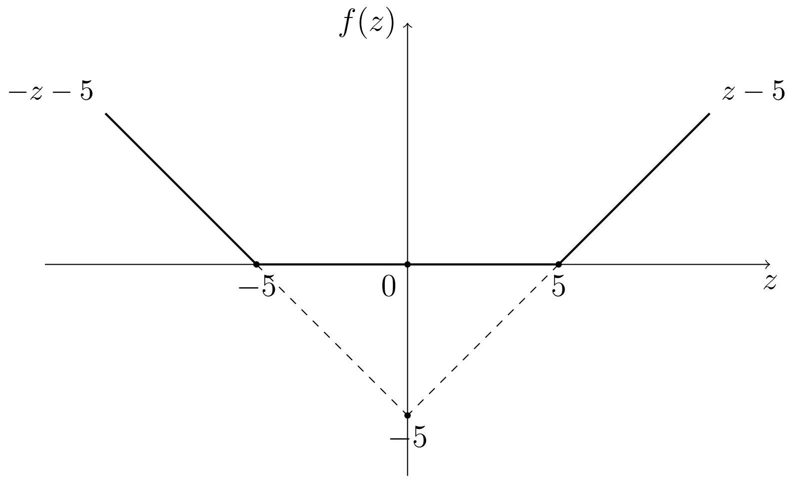

Figure 3: Graph of Piecewise Function

Question 6

We are given that $f(\mathbf{x})$ is convex, and we need to show that $\forall t \in \mathbb{R}$, the set of $\mathbf{x}$ (denoted by χ_t) that satisfies $f(\mathbf{x}) \leq t$ is a convex set. Geometrically, for any 2 points $\mathbf{x}_1$ and $\mathbf{x}_2$ in χ_t, the line segment containing them must also be in χ_t. i.e., if $\mathbf{x}_1, \mathbf{x}_2 \in \chi_t$, then $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \chi_t,\ \forall\lambda \in [0,1]$.

Proof.
$\because \mathbf{x}_1, \mathbf{x}_2 \in \chi_t$, then $f(\mathbf{x}_1) \leq t$ and $f(\mathbf{x}_2) \leq t$. Since $f(\mathbf{x})$ is convex, we use the property:

$$\begin{aligned} f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &\leq \lambda f(\mathbf{x}_1) + (1-\lambda)f(\mathbf{x}_2) \\ &\leq \lambda t + (1-\lambda)t \\ &= \lambda t + t - \lambda t \\ &= t \end{aligned}$$

$\therefore\ \forall\lambda \in [0,1],\ f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq t$, and thus $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \chi_t$, which is convex. □

An appropriate illustration of this is Figure 4 below.

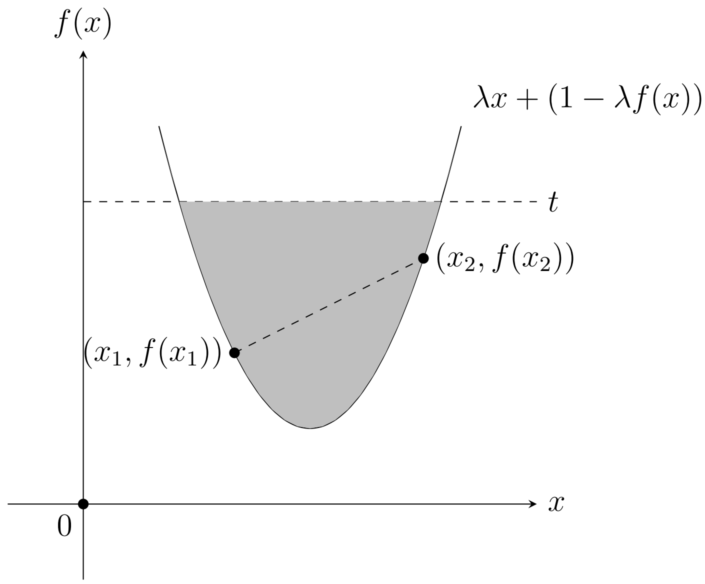

Figure 4: Illustration of Convexity

Question 7

(1)

$$
\begin{aligned}
\max \quad & 5x_1x_2 + 12x_3 \\
\text{s.t.} \quad & x_3 \leq 2 \\
& x_2 \leq 5 \\
& 3.2x_2 + 5.1x_1 = 5.1 \\
& x_1 + \log x_2 - x_3 = 0 \\
& x_1, x_2, x_3 \in \{0, 1\}
\end{aligned}
$$

For this LP, only Certainty holds, while Proportionality, Additivity, and Divisibility assumptions do not. The **logarithmic term in the constraint** does not yield a proportional contribution, so Proportionality does not hold. Similarly, the **total value of the objective function is not the sum of the contribution of each of the terms**, due to the x_1x_2 term, so Additivity does not hold. Lastly, the final constraint suggests that x_1, x_2, x_3 are **constrained to values either 0 or 1 in the set, which are only binary whole numbers, and not fractional values**. Thus Divisibility does not hold.

(2)

$$
\begin{aligned}
\max \quad & 5x_1 - x_2 - 2x_3 \\
\text{s.t.} \quad & x_3 \leq 2.9 \\
& x_2 \geq \theta \\
& x_3 + x_2 - x_1^2 = 0 \\
& 0.4\frac{x_1}{x_2} + x_3 \leq 0 \\
& x_1, x_2, x_3 \geq 0 \\
& \theta \in U(0, 1)
\end{aligned}
$$

For this LP, only Divisility holds, while Proportionality, Additivity, and Certainty do not. Rearranging the terms in constraint (4), we have $0.4x_1 \leq -x_2x_3$. Additivity does not hold in this case due to the $-x_2x_3$ term. Proportionality does not hold due to the $-x_1^2$ term in constraint (3), as it does not yield a proportional change in the contribution to the function. For Certainty, since random variables are stochastic in nature, there is no way to know for sure the value of x_2.

(3)

$$
\begin{aligned}
\max \quad & x_1 - 15x_2 - 10x_3 \\
\text{s.t.} \quad & \frac{x_3}{x_2} \leq 20 \\
& x_2 - 4x_1 \leq 0 \\
& x_3 + x_2 - x_1 = 0 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

For this LP, all 4 assumptions of Proportionality, Additivity, Divisibility, and Certainty hold.