40.017 PROBABILITY & STATISTICS

Homework 4

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Section 2

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Question 1

Let X represent the fabric durability. $\sigma = 3500$, $n = 25$, $s_x = 4270$. Given that X is normally distributed, we conduct a 90% Confidence Interval (CI) for X. Since we are comparing variances, we use the χ^2 distribution. We set up the hypothesis:

$$H_0 : \sigma = 3500$$
$$H_1 : \sigma > 3500$$

We construct a one-sided CI for σ^2:

$$\begin{aligned}\left[\frac{(n-1)s_x^2}{\chi^2_{n-1,;\alpha}}, \infty\right) &= \left[\frac{24(4270)^2}{\chi^2_{24,0.1}}, \infty\right) \\ &= \left[\frac{24(4270)^2}{\chi^2_{24,0.1}}, \infty\right) \\ &= \left[\frac{24(4270)^2}{33.19629}, \infty\right) \\ &= [13181882.674, \infty)\end{aligned}$$

Then the CI for σ is given by the square root of the previous CI:

$$[3630.68, \infty)$$

$\because$ σ is not in the CI, we have sufficient evidence to reject H_0 at the 10% significance level and conclude that σ is greater than 3500.

Question 2

(a)

We have a sample of $n = 26$. Let X be the seeded clouds and Y be the unseeded clouds. We construct the hypothesis:

$$H_0 : \mu_X - \mu_Y = 0$$
$$H_1 : \mu_X - \mu_Y > 0$$

This is an **independent samples design** as the clouds were not matched in any particular fixed method.

(b)

Now, let A, B be the natural logarithm of the rainfall of seeded and unseeded clouds respectively. We conduct a hypothesis test for the *transformed* data:

$$H_0 : \mu_A - \mu_B = 0$$
$$H_1 : \mu_A - \mu_B > 0$$

$\because$ A, B are approximately normal, then we use the 2-sample t-test, and calculate the t-statistic:

$$\begin{aligned} t &= \frac{\bar{A} - \bar{B} - 0}{\sqrt{(s_{\bar{A}}^2 + s_{\bar{B}}^2)/n}} \\ &= \frac{5.134187 - 3.990406 - 0}{\sqrt{(1.599514^2 + 1.641847^2)/26}} \\ &= 2.544369 \end{aligned}$$

We now calculate the p-value:

$$\begin{aligned} \mathbb{P}(T_{2n-2} \geq 2.544369) &= 0.0070413 \\ &\approx \boxed{0.00704} \end{aligned}$$

Since the p-value $< \alpha = 0.05$, we have significant evidence at the 5% level to reject H_0, thus A has a higher true mean rainfall than B. A screenshot of the excel calculations is in Figure below.

unseeded	seeded		ln_unseeded	ln_sedeed		b_bar	a_bar
1202.6	2745.6		7.092241159	7.917754909		3.9904056	5.1341868
830.1	1697.8		6.721546175	7.437088574			
372.4	1656		5.919968545	7.412160335		**b_sdv**	**a_sdv**
345.5	978		5.844992643	6.88550967		1.6418475	1.5995136
321.2	703.4		5.772063982	6.55592572			
244.3	489.1		5.498396978	6.192566968		**t**	**p-value**
163	430		5.093750201	6.063785209		2.5443693	0.0070413
147.8	334.1		4.995860009	5.811440349			
95	302.8		4.553876892	5.713072522		**n**	
87	274.7		4.465908119	5.615679593		26	
81.2	274.7		4.396915247	5.615679593			
68.5	255		4.226833745	5.541263545			
47.3	242.5		3.856510295	5.49100171			
41.1	200.7		3.716008122	5.301811256			
36.6	198.6		3.60004824	5.291292752			
29	129.6		3.36729583	4.864452784			
28.6	119		3.353406718	4.779123493			
26.3	118.3		3.269568939	4.773223771			
26.1	115.3		3.261935314	4.747537427			
24.4	92.4		3.194583132	4.526126979			
21.7	40.6		3.077312261	3.703768067			
17.3	32.7		2.850706502	3.487375078			
11.5	31.4		2.442347035	3.446807893			
4.9	17.5		1.589235205	2.862200881			
4.9	7.7		1.589235205	2.041220329			
1	4.1		0	1.410986974			

Figure 1: Excel Calculations for Q2

Question 3

We have the sample size $n = 30$. The aim of the treatment is to gain weight, and we let μ be the true mean weight gain:

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu > 0 \end{aligned}$$

We can conduct a signed rank test, and let W_+ represent the positive ranks. Since n is large, W_+ is approximately normal, with

$$\begin{aligned} \mathbb{E}(W_+) &= \frac{n(n+1)}{4} \\ &= \frac{30 \times 31}{4} \\ &= 232.5 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(W_+) &= \frac{n(n+1)(2n+1)}{24} \\ &= 2363.75 \end{aligned}$$

Thus, we have:

$$W_+ \sim \mathcal{N}(232.5, 2363.75)$$

To calculate the p-value, we use the sum of positive ranks, calculated in excel as $w_+ = 347.5$:

$$\begin{aligned} \mathbb{P}(W_+ \geq 347.5) &= \Phi\left(-\frac{347.5 - 0.5 - 232.5}{\sqrt{2363.75}}\right) \\ &= \Phi\left(-2.35507\right) \\ &\approx \boxed{0.00925} \end{aligned}$$

$\because$ p-value $= 0.00925 < \alpha = 0.01$, we have sufficient evidence to reject H_0 at the 1% significance level, and conclude that the **true mean weight gain is indeed more than 0**. A screenshot of the excel calculations is in Figure 2 below.

Weight gains (kg)	ABS Diff	Rank	Positive Ranks	Negative Ranks	Sum of Positive Rank	Sample Size
-1.9	1.9	19.5		19.5	347.5	30
-1.7	1.7	18		18		
-1.3	1.3	15.5		15.5		
-1.2	1.2	13		13		
-1.2	1.2	13		13		
-0.9	0.9	9		9		
-0.9	0.9	9		9		
-0.9	0.9	9		9		
-0.5	0.5	5.5		5.5		
-0.4	0.4	2.5		2.5		
-0.4	0.4	2.5		2.5		
-0.3	0.3	1		1		
0.5	0.5	5.5	5.5			
0.5	0.5	5.5	5.5			
0.5	0.5	5.5	5.5			
1	1	11	11			
1.2	1.2	13	13			
1.3	1.3	15.5	15.5			
1.4	1.4	17	17			
1.9	1.9	19.5	19.5			
2.5	2.5	21	21			
2.7	2.7	22	22			
3.3	3.3	23	23			
3.6	3.6	24	24			
3.7	3.7	25	25			
4.1	4.1	26	26			
4.2	4.2	27	27			
4.3	4.3	28.5	28.5			
4.3	4.3	28.5	28.5			
4.9	4.9	30	30			

Figure 2: Excel calculations for Q3

Question 4

If we want to be within 0.1 percentage points of the true proportion with 95% confidence, the margin of error is $E = 0.001$, and $\alpha = 0.05$. We are also given that at most 1% of people will sign up, so we have $\hat{p} = 0.01$. To determine the number of mail offers that the company should send out, we use the formula from Week 10 Class 1:

$$\begin{aligned} n &= \left(\frac{z_{\alpha/2}}{E}\right)^2 \cdot \hat{p} \cdot (1-\hat{p}) \\ &= \left(\frac{1.95996}{(0.001)}\right)^2 \cdot (0.01) \cdot (0.99) \\ &= 38030.28769 \\ &\approx \boxed{38031} \qquad \text{(nearest integer)} \end{aligned}$$

Thus they should send out a total of 38031 mail offers.

Question 5

(a)

Let p be the true proportion of bashes. We construct the hypothesis:

$$H_0 : p = 0.17$$
$$H_1 : p \neq 0.17$$

for $\alpha = 0.05$. Calculations of n, $\hat{p}$ are done in excel in Figure 3 below. Since n is large, p is approximately normal with

$$\mathbb{E}(\hat{p}) = p = 0.17$$

and

$$\text{Var}(\hat{p}) = \frac{p(1-p)}{n} = \frac{0.17 \times (1-0.17)}{690+130} = 0.00017207$$

$\therefore \hat{p} \sim \mathcal{N}(0.17, 0.00017207)$. We calculate the 2-sided p-value:

$$\begin{aligned} 2 \times \mathbb{P}(\hat{p} \leq 0.1585366) &= 2 \times \mathbb{P}\left(\frac{\hat{p} - p'}{\sqrt{p'(1-p')/n}}\right) \\ &= 2 \times \mathbb{P}\left(\frac{0.1585366 - 0.17}{\sqrt{0.17(1-0.17)/(690+130)}}\right) \\ &\approx \boxed{0.382} \end{aligned}$$

Since the p-value $= 0.382 > \alpha = 0.05$, we do not have sufficient evidence to reject H_0 at the 95% significance level, and conclude that the true proportion of bashes is indeed $p = 0.17$.

(b)

From the excel attached in Figure 3 below, we have calculated there are $n = 690$ normal attacks, and $m = 130$ bashes, with total number of runs 231. Since m and n are large, then the distribution is approximately normal with:

$$\begin{aligned} \mu &= \frac{2mn}{m+n} + 1 \\ &= \frac{2(130)(690)}{130+690} + 1 \\ &= 219.780 \end{aligned}$$

and

$$\begin{aligned} \sigma^2 &= \frac{(\mu-1)(\mu-2)}{m+n-1} \\ &= \frac{(219.780-1)(219.780-2)}{130+690-1} \\ &= 58.17597 \end{aligned}$$

and $\therefore X \sim \mathcal{N}(219.780, 58.17597)$. We use a continuity correction for a 2-sided test and calculate the p-value:

$$2 \times \mathbb{P}\left(Z > \frac{232 - 0.5 - 219.78049}{\sqrt{58.175972}}\right) = 0.124411 \approx \boxed{0.124}$$

$\because$ p-value $= 0.124 > \alpha = 0.05$, we do not have sufficient evidence to reject H_0 at the 95% significance level, and conclude that the bashes do occur randomly.

Data for 'bashes' and normal attacks					
5	1				
B	1	For instance, '5 B 2 B 3' means:			
2	1	5 consecutive normal (non-bash) attacks, followed by a bash, followed by 2 consecutive normal attacks,			
B	1	followed by a bash, followed by 3 consecutive normal attacks.			
3	1				
B	1	**Sum of Normal**		**Sum of Bashes**	
9	1	690		130	
B	1				
4	1	**mu**		**sigmasq**	
B	1	219.7805		58.17597	
3	1				
B	1				
B	0			**Total Runs**	
6	1			232	
B	1				
3	1				
B	1			**p**	**var**
4	1			0.158537	0.000163

Figure 3: Excel calculations for Q5

Question 6

We construct a table:

i	**O**	**A**	**B**	**AB**
n_i	18	10	5	2
p_i	0.3	0.4	0.2	0.1
e_i	$0.3 \times 0.35 = 10.5$	$0.4 \times 0.35 = 14$	$0.2 \times 35 = 7$	$0.1 \times 35 = 3.5$

Table 1: Table of values for each Blood Type

From Table 1, we find that the sample size is 35. We first construct the hypothesis:

$$\begin{aligned} &H_0 : (p_{\mathrm{O}} = 0.3) \cap (p_{\mathrm{A}} = 0.4) \cap (p_{\mathrm{B}} = 0.2) \cap (p_{\mathrm{AB}} = 0.1) \\ &H_1 : (p_{\mathrm{O}} \neq 0.3) \cup (p_{\mathrm{A}} \neq 0.4) \cup (p_{\mathrm{B}} \neq 0.2) \cup (p_{\mathrm{AB}} \neq 0.1) \qquad \text{(otherwise)} \end{aligned}$$

We now conduct a χ^2 test with $\alpha = 0.05$. To find the test statistic for 4 categories:

$$\begin{aligned} \chi^2 &= \sum_{i=1}^{4} \frac{(n_i - e_i)^2}{e_i} \\ &= \frac{(18 - 10.5)^2}{10.5} + \frac{(10 - 14)^2}{14} + \frac{(5 - 7)^2}{7} + \frac{(2 - 3.5)^2}{3.5} \\ &= \boxed{7.714285} \end{aligned}$$

Since we have 4 Blood Types (categories), $m = 4$, and $m - 1 = 3$. With $\alpha = 0.05$, the critical value is

$$\chi^2_{3,\ 0.05} = 7.81445$$

$\because \chi^2 = 7.714285 < 7.81445$, we do not have sufficient evidence to reject H_0 at the 5% significance level, and we thus conclude that the prime ministers' blood types are **not** significantly different from what one would expect from the probability proportions.

Question 7

(a)

The missing entries are given in Figure 4 below. The values are calculated via excel with the inverse normal distribution, with the probability that a value falls into each bin is $\frac{1}{8}$.

Raw data						
37.319		sample mean:	47.24521			
37.518		sample SD:	3.978996			
37.636						
37.832		bin	*from:*	*to:*	observed	expected
38.105		*1*	-1000	42.668	114	100
38.177		*2*	42.668	44.561	99	100
38.245		*3*	44.561	45.977	89	100
38.275		*4*	45.977	47.245	95	100
38.542		*5*	47.245	48.513	94	100
38.875		*6*	48.513	49.929	84	100
38.984		*7*	49.929	51.822	129	100
38.988		*8*	51.822	1000	96	100
39.128						
39.269				total:	800	800

Figure 4: Missing Entries

(b)

Since we are estimating both μ and σ, we lose **two extra degrees of freedom**. Since each $e_i \geq 5$, we also do not combine any of the tables. Thus, the degrees of freedom is now $8-3=5$. We set up the hypothesis:

$$\begin{aligned} &H_0 : p_0 = f(0|\mu, \sigma^2) \ \cap p_1 = f(1|\mu, \sigma^2) \cap \dots && \text{(where } p_i \text{ represents the } i^{\text{th}} \text{ bin)} \\ &H_1 : p_0 \neq f(0|\mu, \sigma^2) \ \cup p_1 \neq f(1|\mu, \sigma^2) \cup \dots && \text{(otherwise)} \end{aligned}$$

$\forall i \in \{1, 2, \dots, 8\}$. We now calculate the χ^2 test statistic:

$$\begin{aligned} \chi^2 &= \sum_{i=1}^{8} \frac{(n_i - e_i)^2}{e_i} \\ &= \frac{(114-100)^2}{100} + \frac{(99-100)^2}{100} + \frac{(89-100)^2}{100} + \frac{(95-100)^2}{100} + \frac{(94-100)^2}{100} + \frac{(84-100)^2}{100} \\ &+ \frac{(129-100)^2}{100} + \frac{(96-100)^2}{100} \\ &= 14.92 \end{aligned}$$

With 5 degrees of freedom at $\alpha = 0.05$, we have the critical value:

$$\chi^2_{5,0.05} = 11.07031$$

Since the test statistic $\chi^2 = 14.92 > 11.07031$, we have sufficient evidence to reject the null hypothesis and conclude that **a normal distribution does not fit the data well.**