40.002 OPTIMIZATION

An Introduction to Optimization

Michael Hoon

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1 Introduction to Linear Programming

An optimization problem is defined by:

- **Decision variables**: elements under the control of the decision maker
- **A (single) objective function**: a function of the decision variables that we want to optimize, corresponding to a criterion for measuring maximize
- **Constraints**: restrictions that define which values of the decision variables are allowed.

We want to find the **minimum** or **maximum** of a function of one or many variables subject to a set of **constraints**:

$$\begin{aligned} &\min\ f(x_1, \dots x_n) \\ &\ni (x_1, \dots x_n) \in \chi \subseteq \mathbb{R}^n \end{aligned} \tag{1}$$

where the decision variables are vectors $x_1, \dots x_n$, the objective function is $f(x_1, \dots x_n)$ and the constraints are defined by the set $\chi \subseteq \mathbb{R}^n$. A vector $\mathbf{x}^*$ is called *optimal*, or a *solution* of the problem, if it has the **smallest objective value** among all vectors that satisfy the constraints.

1.1 Linear Programming

A linear program is a class of optimisation problem in which the objective and all constraint functions are **linear**, given by:

$$\begin{aligned} &\min\ \mathbf{c}^\top \mathbf{x} \\ &\ni \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \text{and } \mathbf{x} \geq 0 \end{aligned} \tag{2}$$

where the decision vector is $\mathbf{x}$ (n variables), linear objective function: $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} = c_1 x_1 + \cdots + c_n x_n$, and the linear constraints are $\chi = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ (m constraints) [1]. Note that matrix $\mathbf{A}_{(m \times n)}$ is of $m \times n$ dimension.

1.1.1 Inequality Transformations

We have matrix $\mathbf{A}$ given by:

$$\mathbf{A} = \begin{pmatrix} - & \mathbf{a_1}^\top & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{a_m}^\top & - \end{pmatrix}$$

- An equality constraint $\mathbf{a_i}^\top \mathbf{x} = b_i$ is equivalent to two equality constraints $\mathbf{a_i}^\top \mathbf{x} \leq b_i$ and $\mathbf{a_i}^\top \mathbf{x} \geq b_i$
- An inequality constraint $\mathbf{a_i}^\top \mathbf{x} \leq b_i$ is equivalent to the inequality constraint $-\mathbf{a_i}^\top \mathbf{x} \geq -b_i$ (Note the negatives applied to both sides of the inequality).
- Constraints such as $x_j \geq 0,\ x_j \leq 0$ can be expressed in the form $\mathbf{a_i}^\top \mathbf{x} \geq b_i$ by appropriately choosing $\mathbf{a}_i,\ b_i$.

Note that there is no simple analytic formula for the solution of a linear program, but there are a variety of effective methods for solving them, including Dantzig's simplex method, and the more recent interior-point methods. We cannot give the exact number of arithmetic operations required to solve a linear program, but we can establish rigorous bounds on the number of operations required to solve a linear program using an interior-point method (in practice, this is of the order $n^2 m$, assuming $m \geq n$).

[1] Note: vector inequalities are interpreted componentwise.

1.1.2 Terminology

Definition 1.1. We now introduce some terminology for geometric linear programming:

- A **linear function** $f : \mathbb{R}^n \to \mathbb{R}$ is a function of the form:

$$f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n, \quad a_n \in \mathbb{R}$$

- A **hyperplane** in $\mathbb{R}^n$ is the set of points satisfying a single linear equation:

$$a_1x_1 + \dots + a_nx_n = b, \quad a_n \in \mathbb{R}$$

- A **halfspace** in $\mathbb{R}^n$ is the set of points satisfying a single linear inequality:

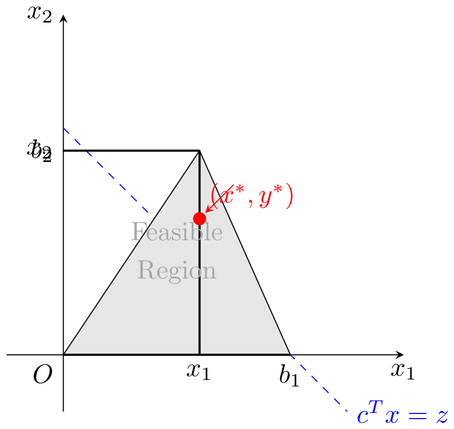
$$a_1x_1 + \dots + a_nx_n \geq b, \quad a_n, b \in \mathbb{R}$$

- A **polyhedron** is a set that can be described by a finite number of halfspaces. A **polytope** is a **bounded** polyhedron.
- An assignment of values to the decision variables is a **feasible solution** if it satisfies all the constraints (infeasible otherwise). The set of all feasible solutions is the **feasible region.**
- An **optimal solution** is a feasible solution that achieves the **best possible objective function value**. For a minimisation problem, x^* is optimal **iff** $\mathbf{c}^\top \mathbf{x}^\top \leq \mathbf{c}^\top \mathbf{x}$ for all feasible $\mathbf{x}$.
- We call $\mathbf{c}^\top x^*$ the **optimal objective value**.
- $\forall K \in \mathbb{R}$ we can find a feasible solution $\mathbf{x}$ such that $\mathbf{c}^\top \mathbf{x} \leq K$, then the linear program in **minimisation** form has **unbounded** cost. The optimum cost is then $-\infty$. In this case, we can find a feasible $\mathbf{x}$ and direction $\mathbf{d}$ such that $\mathbf{x} + t\mathbf{d}$ is feasible $\forall t \geq 0$ and $\mathbf{c}^\top d < 0$.

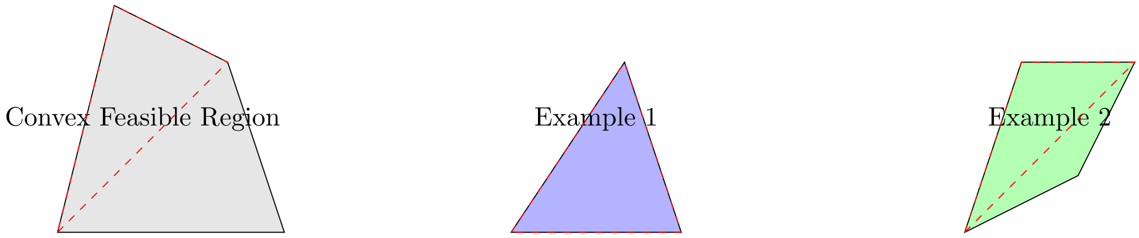
1.2 Geometric Definition

In a simple two-dimensional space with the equation $x_1 + x_2 = z$, this function can be represented by a line. The decision variables are x_1 ad x_2, and this line represents all possible combinations of x_1 and x_2 that yield the same objective value z.

Each constraint is a linear inequality, which creates a boundary in the solution space. The feasible region is now the polygon formed by the intersection of all these constraint boundaries.



The feasible region is often convex, meaning that if points A and B are inside the region, the line segment connecting A and B are also inside the region. Some examples are shown below:



1.3 Feasible Regions

> **Theorem 1.2.** The feasible region of a linear program is **convex**.

Mathematically, a set $\chi \subseteq \mathbb{R}^n$ is convex if

$$\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in \chi, \ \forall \mathbf{x}, \mathbf{y} \in \chi, \ \forall \lambda \in [0,1]$$

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