Construction of $U(1) \otimes SL(M, \mathbb{R})$ fermionic coherent states on the particle-preserving dynamical group.

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Abstract

The dynamical group of a fermionic system with M modes which preserves total particle number is identified as $G = U(1) \otimes SL(M, \mathbb{R})$. A reference state $|\phi_0\rangle$ is constructed as a member of the full occupancy basis by partitioning the modes into π_1 (S occupied modes) and π_0 ($M - S$ unoccupied modes). The quotient space of $(G, |\phi_0\rangle)$ is shown to be generated by $\hat{f}_i^\dagger \hat{f}_j$, where $i \in \pi_0, j \in \pi_1$, and a generalised coherent state $|Z\rangle$ belonging to this quotient space is decomposed into the full occupancy basis. The overlap element $\langle Z_a | Z_b \rangle$ is shown to be $\frac{\det\left(I_S + Z_a^\dagger Z_b\right)}{\sqrt{\det\left(I_S + Z_a^\dagger Z_a\right) \det\left(I_S + Z_b^\dagger Z_b\right)}}$. The action of the "transition operator" $\hat{f}_i^\dagger \hat{f}_j$ (for arbitrary i, j) on $|Z\rangle$ is expressed in the full occupancy basis, and the expression for a general two-body-interacting total-particle-preserving Hamiltonian matrix element $\langle Z_a | \hat{H} | Z_b \rangle$ is given. The time complexity of calculating said quantities is discussed.

Contents

1 Construction of $U(1) \otimes SL(M, \mathbb{R})$ coherent states

Consider a system with M modes and $S < M$ fermions, each occupying one of the modes. The Pauli exclusion principle forbids more than one fermion in a single mode, and thus the full Hilbert space is spanned by a full occupancy basis where each element is parametrised by a permuted sequence of S ones and $M - S$ zeroes. The size of this basis is $\binom{M}{S}$.

1.1 Dynamical group and its operator algebra

The basis of the system as described above can be transversed by applying the transition operators

$$\hat{T}_{ij} = \hat{f}_i^\dagger \hat{f}_j \tag{1}$$

Using the commutator identities in Eq. 69 we obtain

$$\begin{aligned}
\left[\hat{T}_{ij}, \hat{T}_{i'j'}\right] &= \left[\hat{f}_i^\dagger \hat{f}_j, \hat{f}_{i'}^\dagger \hat{f}_{j'}\right] \\
&= \left[\hat{f}_i^\dagger, \hat{f}_{i'}^\dagger\right] \hat{f}_j \hat{f}_{j'} + \hat{f}_{i'}^\dagger \left[\hat{f}_i^\dagger, \hat{f}_{j'}\right] \hat{f}_j + \hat{f}_i^\dagger \left[\hat{f}_j, \hat{f}_{i'}^\dagger\right] \hat{f}_{j'} + \hat{f}_{i'}^\dagger \hat{f}_i^\dagger \left[\hat{f}_j, \hat{f}_{j'}\right] \\
&= 2\hat{f}_i^\dagger \hat{f}_{i'}^\dagger \hat{f}_j \hat{f}_{j'} + \hat{f}_{i'}^\dagger (2\hat{f}_i^\dagger \hat{f}_{j'} - \delta_{ij'}) \hat{f}_j + \hat{f}_i^\dagger (\delta_{i'j} - 2\hat{f}_{i'}^\dagger \hat{f}_j) \hat{f}_{j'} + 2\hat{f}_{i'}^\dagger \hat{f}_i^\dagger \hat{f}_j \hat{f}_{j'} \\
&= \hat{f}_i^\dagger \hat{f}_{j'} \delta_{i'j} - \hat{f}_{i'}^\dagger \hat{f}_j \delta_{ij'} = \hat{T}_{ij'} \delta_{i'j} - \hat{T}_{i'j} \delta_{ij'}
\end{aligned} \tag{2}$$

We see that $\hat{T}_{ij}$ form the Lie algebra of $U(M)$, however, since our total particle number is preserved, we choose to perform a basis transformation like so:

$$\begin{aligned}
\hat{S} &= \sum_{i=1}^{M} \hat{T}_{ii} \\
\hat{H}_i &= \hat{T}_{i+1,i+1} - \hat{T}_{ii} \quad \text{for} \quad i = 1, 2 \ldots M-1 \\
\hat{E}_{ij} &= \hat{T}_{ij} \quad \text{for} \quad i > j \\
\hat{E}_{ij}^\dagger &= \hat{T}_{ji} \quad \text{for} \quad i > j
\end{aligned}$$

This set of operators forms the basis to the Lie algebra of $U(1) \otimes SL(M, \mathbb{R})$, which we will identify as our dynamical group G. Other choices of G would lead to equivalent coherent state constructions: namely, not transforming the basis at all would simply yield $G = U(M)$, as done in [1, Sec. 4.3.4]; taking a complex linear transformation $\hat{T}_{ij} + \hat{T}_{ji}$ and $i(\hat{T}_{ij} - \hat{T}_{ji})$ would form the basis to the Lie algebra of $U(1) \otimes SU(M)$, which is equal to our choice under complexification of its algebra.

1.2 Reference state and quotient space

For the reference state we partition the set of modes like so:

$$\pi_1 = \{1, 2 \ldots S\} \qquad \pi_0 = \{S+1, S+2 \ldots M\} \tag{3}$$

and then

$$|\phi_0\rangle = |\pi_1\rangle = |n_1, n_2 \ldots n_M\rangle \quad \text{where} \quad n_i = \begin{cases} 1 & \text{for } i \in \pi_1 \\ 0 & \text{for } i \in \pi_0 \end{cases} \tag{4}$$

Then the action of a transition operator on the reference state is

$$\hat{T}_{ij} |\phi_0\rangle = \begin{cases} (-1)^{S+j} |\pi_1 - \{j\} + \{i\}\rangle & \text{if } i \in \pi_0, j \in \pi_1 \\ n_i |\phi_0\rangle & \text{if } i = j, j \in \pi_1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Therefore the exponential map of the first option transverses the quotient space, and an unnormalised coherent state is formed as

$$|Z\} = \exp\left(\sum_{i\in\pi_0}\sum_{j\in\pi_1}(-1)^{S+j}Z_{ij}\hat{T}_{ij}\right)|\phi_0\rangle \tag{6}$$

We see that the general coherent state has $S(M-S)$ complex parameters[1]. The normalised state $|Z\rangle$ can be constructed like so:

$$|Z\rangle = N(Z)\,|Z\} \quad \text{where} \quad N(Z) = \frac{1}{\sqrt{\{Z|Z\}}} \tag{7}$$

Note that, for the sake of simplicity of notation, the row indices on Z are shifted by S, i.e. Z_{ij} labels the $(i-S)$-th row, j-th column of the $(M-S,S)$ matrix Z.

1.3 Decomposition into the full occupancy basis

In this subsection we decompose a coherent state $|Z\}$ into the full occupancy basis; i.e. we wish to find the overlap

$$\langle n_1, n_2 \dots n_M|Z\rangle \quad \text{for any sequence of} \quad n_i \in \{0,1\} \quad \text{where} \quad \sum_{i=1}^{M} n_i = S \tag{8}$$

1.3.1 General approach to CS decomposition

This method is based on the approach in [3, App. E]. Suppose we have a reference state $|\phi_0\rangle$, and the quotient space of the dynamical group of some system is transversed by the exponential map of the operator $\hat{D}(z)$, so that

$$|z\} = e^{\hat{D}(z)}\,|\phi_0\rangle \tag{9}$$

and $\hat{D}(z)$ is a linear combination of transition operators which all destroy the vacuum state[2]. Define $\hat{\phi}_0$ as such an operator so that

$$\hat{\phi}_0\,|\text{vac.}\rangle = |\phi_0\rangle \tag{10}$$

Then we can write

$$|z\} = e^{\hat{D}(z)}\hat{\phi}_0 e^{-\hat{D}(z)}\,|\text{vac.}\rangle \tag{11}$$

using the fact that $\hat{D}(z)\,|\text{vac.}\rangle = 0$. We now express the operator product using Hadamard's lemma:

$$e^{\hat{D}(z)}\hat{\phi}_0 e^{-\hat{D}(z)} = \sum_{r=0}^{\infty}\frac{1}{r!}\left[\hat{D}(z),\hat{\phi}_0\right]_r \tag{12}$$

[1]There is no further reduction of degrees of freedom due to normalisation or fixing the global phase, as it can be trivially seen that, in the decomposition into the full occupancy basis, the reference state always appears with coefficient 1. A trivial but interesting consequence is that normalisation, which projects an element of the space of $|Z\}$ onto the space of $|Z\}$ (by definition equivalent to the space of all normalised vectors in the full Hilbert space) is bijective.

[2]This is characteristic of dynamical groups which preserve total particle number.

where

$$\left[\hat{A},\hat{B}\right]_r = \left[\hat{A},\left[\hat{A},\hat{B}\right]_{r-1}\right] \quad \text{and} \quad \left[\hat{A},\hat{B}\right]_0 = \hat{B} \tag{13}$$

are the repeated commutators. The unnormalised coherent state is obtained by acting with this sum on the vacuum state.

1.3.2 Repeated commutators of the fermionic coherent states

In our case, we identify

$$\hat{D}(Z) = \sum_{i\in\pi_0}\sum_{j\in\pi_1}(-1)^{S+j}Z_{ij}\hat{T}_{ij} \quad \text{and} \quad \hat{\phi}_0 = \hat{f}^\dagger_{\langle\pi_1\rangle} \tag{14}$$

To find the repeated commutators, we first observe

$$\begin{aligned}
\left[\hat{T}_{ij},\hat{f}^\dagger_{\langle\sigma\rangle}\right] &= \sum_{k=1}^{|\langle\sigma\rangle|}\hat{f}^\dagger_{\langle\sigma\rangle_1}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{k-1}}\left[\hat{T}_{ij},\hat{f}^\dagger_{\langle\sigma\rangle_k}\right]\hat{f}^\dagger_{\langle\sigma\rangle_{k+1}}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{|\langle\sigma\rangle|}}\\
&= \sum_{k=1}^{|\langle\sigma\rangle|}\hat{f}^\dagger_{\langle\sigma\rangle_1}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{k-1}}\left(\hat{f}^\dagger_i\delta_{j,\langle\sigma\rangle_k}\right)\hat{f}^\dagger_{\langle\sigma\rangle_{k+1}}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{|\langle\sigma\rangle|}}\\
&= \begin{cases}\hat{f}^\dagger_{\langle\sigma\rangle_1}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{k-1}}\hat{f}^\dagger_i\hat{f}^\dagger_{\langle\sigma\rangle_{k+1}}\dots\hat{f}^\dagger_{\langle\sigma\rangle_{|\langle\sigma\rangle|}} \text{ where } \langle\sigma\rangle_k = j & \text{if } j\in\langle\sigma\rangle\\ 0 & \text{if } j\notin\langle\sigma\rangle\end{cases}
\end{aligned} \tag{15}$$

When calculating $\left[\hat{D}(Z),\hat{\phi}_0\right]$, the expression is simplified due to the trivial nature of the creation operator product sequence, with every $j\in\langle\pi_1\rangle$ and every $i>S$:

$$\left[\hat{T}_{ij},\hat{f}^\dagger_{\langle\pi_1\rangle}\right] = \hat{f}^\dagger_1\dots\hat{f}^\dagger_{j-1}\hat{f}^\dagger_i\hat{f}^\dagger_{j+1}\dots\hat{f}^\dagger_S \quad \text{where} \quad i\in\pi_0, j\in\pi_1 \tag{16}$$

Therefore the full commutator is

$$\left[\hat{D}(Z),\hat{\phi}_0\right]_1 = \sum_{i\in\pi_0}\sum_{j\in\pi_1}(-1)^{S+j}Z_{ij}\hat{f}^\dagger_1\dots\hat{f}^\dagger_{j-1}\hat{f}^\dagger_i\hat{f}^\dagger_{j+1}\dots\hat{f}^\dagger_S \tag{17}$$

To find the second commutator, we commute $\hat{D}(Z)$ with the result of Eq. 17:

$$\left[\hat{D}(Z),\hat{\phi}_0\right]_2 = \sum_{i\in\pi_0}\sum_{j\in\pi_1}\sum_{i'\in\pi_0}\sum_{j'\in\pi_1}(-1)^{2S+j+j'}Z_{ij}Z_{i'j'}\left[T_{i'j'},\hat{f}^\dagger_1\dots\hat{f}^\dagger_{j-1}\hat{f}^\dagger_i\hat{f}^\dagger_{j+1}\dots\hat{f}^\dagger_S\right] \tag{18}$$

The result is zero if $j=j'$ (since the commutator vanishes) or if $i=i'$ (since there would be two creation operators acting on the i-th mode). Hence we can restrict the sum domains for j' as $\pi_1-\{j\}$ and for i' as $\pi_0-\{i\}$, respectively. Then, the index pairs i,i' and j,j' are all 2-element (not necessarily ascending) sequences on π_0 and π_1, respectively. Therefore, the expression can be rewritten by summing the summand over all *ascending* subsequences $\langle a\rangle \in \Gamma_2\langle\pi_0\rangle, \langle b\rangle \in \Gamma_2\langle\pi_1\rangle$, and also summing over all of their permutations. By induction it is obvious that the x-th commutator can be expressed as

$$\left[\hat{D}(Z),\hat{\phi}_0\right]_x = \sum_{\langle a\rangle\in\Gamma_x\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_x\langle\pi_1\rangle}(-1)^{Sx+\sum\langle b\rangle}\sum_{P_a\in P^x}\sum_{P_b\in P^x}\left(\prod_{u=1}^{x} Z_{P_a\langle a\rangle_u,P_b\langle b\rangle_u}\right)\hat{f}^\dagger_{\upsilon(P_a\langle a\rangle,P_b\langle b\rangle)} \tag{19}$$

where the creation operator product is over a (not necessarily ascending) sequence $\upsilon(P_a\langle a\rangle, P_b\langle b\rangle)$ which is constructed by taking the ascending sequence $\langle\pi_1\rangle$ and then replacing element $P_b\langle b\rangle_i$ by element $P_a\langle a\rangle_i$ for $i = 1, 2 \ldots x$. A trivial but important corollary of this construction is that the sign of any further permutation of either $P_a\langle a\rangle$ or $P_b\langle b\rangle$ equals the sign of the corresponding permutation of the entire υ sequence. Before proceeding, let us find the monotonic ordering of $\hat{f}^\dagger_{\upsilon(P_a\langle a\rangle,P_b\langle b\rangle)}$ using Eq. 71. First, since $\mathrm{sgn}(P) = \mathrm{sgn}(P^{-1})$, we undo the permutations on $\langle a\rangle, \langle b\rangle$, obtaining:

$$\hat{f}^\dagger_{\upsilon(P_a\langle a\rangle,P_b\langle b\rangle)} = \mathrm{sgn}(P_a)\mathrm{sgn}(P_b)\hat{f}^\dagger_{\upsilon(\langle a\rangle,\langle b\rangle)} \tag{20}$$

We now wish to find the sign of the permutation which turns $\upsilon(\langle a\rangle, \langle b\rangle)$ into an ascending sequence. This permutation can be expressed as a composition of cyclic permutations, where each brings a particular element $\langle a\rangle_i$ (which is the $\langle b\rangle_i$-th element of υ) right after the largest element smaller or equal to S, starting with $\langle a\rangle_x$ and working backwards. For the i-th element of the replacement subsequence, $\langle a\rangle_i$, the old index of the element is $\langle b\rangle_i$, and the new index of the element is $S-(x-i)$, where $x-i$ counts the number of elements which were shifted before $\langle a\rangle_i$, thus lowering the target index by 1 each. Thus the sign of the corresponding cyclic permutation is $(-1)^{S+1+i-x-\langle b\rangle_i}$. Permuting all the elements of the replacement subsequence until υ is ascending yields the total sign

$$\mathrm{sgn}(\upsilon\to\langle\upsilon\rangle) = (-1)^{\sum_{i=1}^{x} S+1+i-x-\langle b\rangle_i} = (-1)^{\frac{1}{2}x(2S-1-x)-\sum\langle b\rangle} \tag{21}$$

Collecting the monotone-ordered creation operator product sequences, we obtain

$$\left[\hat{D}(Z),\hat{\phi}_0\right]_x =$$
$$(-1)^{\frac{1}{2}x(x+1)}\sum_{\langle a\rangle\in\Gamma_x\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_x\langle\pi_1\rangle}\hat{f}^\dagger_{\langle\pi_1-\{b\}+\{a\}\rangle}\sum_{P_a\in P^x}\sum_{P_b\in P^x}\mathrm{sgn}(P_a)\mathrm{sgn}(P_b)\left(\prod_{u=1}^{x} Z_{P_a\langle a\rangle_u,P_b\langle b\rangle_u}\right) \tag{22}$$

Now: since the product of scalar matrix elements $Z_{P_a\langle a\rangle_u,P_b\langle b\rangle_u}$ is commutative, we may permute the order in which the product is taken without altering the result:

$$\prod_{u=1}^{x} Z_{P_a\langle a\rangle_u,P_b\langle b\rangle_u} = \prod_{u=1}^{x} Z_{P_cP_a\langle a\rangle_u,P_cP_b\langle b\rangle_u} \quad \text{for any} \quad P_c\in P^x \tag{23}$$

Choosing $P_c = P_a^{-1}$ and noting that $\mathrm{sgn}(P_a) = \mathrm{sgn}(P_a^{-1})$ we obtain

$$\sum_{P_a\in P^x}\sum_{P_b\in P^x}\mathrm{sgn}(P_a)\mathrm{sgn}(P_b)\left(\prod_{u=1}^{x} Z_{P_a\langle a\rangle_u,P_b\langle b\rangle_u}\right) = \sum_{P_a\in P^x}\sum_{P_b\in P^x}\mathrm{sgn}(P_a^{-1}P_b)\left(\prod_{u=1}^{x} Z_{\langle a\rangle_u,P_a^{-1}P_b\langle b\rangle_u}\right) \tag{24}$$

Since P^x forms a group under composition, there is a unique element $P_a^{-1}P_b \in P^x$ for all $P_b \in P^x$, and therefore summing over all P_b is equivalent to summing over all $P_a^{-1}P_b$. Denoting $P_a^{-1}P_b = P_c$ we obtain

$$\sum_{P_a\in P^x}\sum_{P_b\in P^x}\mathrm{sgn}(P_a^{-1}P_b)\left(\prod_{u=1}^{x} Z_{\langle a\rangle_u, P_a^{-1}P_b\langle b\rangle_u}\right) = \sum_{P_a\in P^x}\sum_{P_c\in P^x}\mathrm{sgn}(P_c)\left(\prod_{u=1}^{x} Z_{\langle a\rangle_u, P_c\langle b\rangle_u}\right) \tag{25}$$

The summand is independent on P_a, turning the sum over P_a into a constant factor of $|P^x| = x!$. The rest of the expression can be identified as the determinant of a submatrix of Z equal to $Z_{\langle a\rangle,\langle b\rangle}$. Hence the commutator becomes

$$\left[\hat{D}(Z), \hat{\phi}_0\right]_x = (-1)^{\frac{1}{2}x(x+1)} x! \sum_{\langle a\rangle\in\Gamma_x\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_x\langle\pi_1\rangle} \det\big(Z_{\langle a\rangle,\langle b\rangle}\big)\hat{f}^\dagger_{\langle\pi_1-\{b\}+\{a\}\rangle} \tag{26}$$

1.3.3 The unnormalised coherent state

Substituing Eq. 26 into Eq. 12 and acting on the vacuum state, for which we invoke Eq. 72, allows us to write down the unnormalised fermionic coherent state:

$$|Z\} = \sum_{r=0}^{\min(S,M-S)} (-1)^{\frac{1}{2}r(r+1)} \sum_{\langle a\rangle\in\Gamma_r\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_r\langle\pi_1\rangle} \det\big(Z_{\langle a\rangle,\langle b\rangle}\big)\,|\pi_1-\{b\}+\{a\}\rangle \tag{27}$$

where we used the fact that $\left[\hat{D}(Z), \hat{\phi}_0\right]_r = 0$ if r is bigger than either S or $M-S$, as there are no subsequences of π_1 and π_0 of size r.

2 Action of the fermionic creation and annihilation operators on the coherent state

2.1 Reductions of coherent states

Let us define $\eta_x(S)$ as the number of elements of S smaller than x. Then the action of the creation and annihilation operator on an occupancy basis state is

$$\hat{f}_i^\dagger|\sigma\rangle = \begin{cases}(-1)^{\eta_i(\sigma)}|\sigma+\{i\}\rangle & \text{if } i\notin\sigma\\ 0 & \text{if } i\in\sigma\end{cases} \tag{28}$$

$$\hat{f}_i|\sigma\rangle = \begin{cases}(-1)^{\eta_i(\sigma)}|\sigma-\{i\}\rangle & \text{if } i\in\sigma\\ 0 & \text{if } i\notin\sigma\end{cases} \tag{29}$$

Therefore, the action of the creation operator on an unnormalised fermionic coherent state is

$$i\in\pi_1 \quad : \quad \hat{f}_i^\dagger|Z\} = \qquad kkk \tag{30}$$

And, the action of the annihilation operator on an unnormalised fermionic coherent state is

$$i \in \pi_1 \quad : \quad \hat{f}_i \, |Z\} = \sum_{r=0}^{\min(S-1,M-S)} (-1)^{\frac{1}{2}r(r+1)} \sum_{\langle a\rangle \in \Gamma_r\langle \pi_0\rangle} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1 - \{i\}\rangle} \det\big(Z_{\langle a\rangle,\langle b\rangle}\big)(-1)^{i-1-\eta_i(\langle b\rangle)} \, |\pi_1 - \{b\} + \{a\} - \{i\}\rangle \tag{31}$$

$$i \in \pi_0 \quad : \quad \hat{f}_i \, |Z\} = \sum_{r=0}^{\min(S,M-S)} (-1)^{\frac{1}{2}r(r+1)} \sum_{\langle a\rangle \in \Gamma_{r-1}\langle \pi_0 - \{i\}\rangle} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1\rangle} \det\big(Z_{\langle a\rangle \oplus \{i\},\langle b\rangle}\big)(-1)^{S-r-\eta_i(\langle a\rangle)} \, |\pi_1 - \{b\} + \{a\}\rangle \tag{32}$$

2.1.1 Reduction of coherent state in π_1

Consider this: setting all the elements in the i-th column of Z does a single thing: it removes all terms from the occupancy basis decomposition which have $n_i = 0$ when $i \in \pi_1$. Formally, if we let $R_i^{\pi_1}$ be an (S, S) matrix such that

$$R_i^{\pi_1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{S-i} \end{pmatrix} \tag{33}$$

then

$$|ZR_i^{\pi_1}\} = \sum_{r=0}^{\min(S,M-S)} (-1)^{\frac{1}{2}r(r+1)} \sum_{\langle a\rangle \in \Gamma_r\langle \pi_0\rangle} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1 - \{i\}\rangle} \det\big(Z_{\langle a\rangle,\langle b\rangle}\big) \, |\pi_1 - \{b\} + \{a\}\rangle \tag{34}$$

A direct consequence of Eq. 34 is that

$$\hat{f}_i \, |Z\} = \hat{f}_i \, |ZR_i^{\pi_1}\} \quad \text{if} \quad i \in \pi_1 \tag{35}$$

and the action of $\hat{f}_i$ on an occupancy basis state $|\pi_1 - \{b\} + \{a\}\rangle$ yields $(-1)^{i-1-\eta_i(\langle b\rangle)}$, where $\eta_i(\langle b\rangle)$ is the number of elements in b smaller than i. We can absorb this extra factor into the corresponding minor $\det\big(Z_{\langle a\rangle,\langle b\rangle}\big)$ by flipping the sign of the first $i-1$ columns in Z, which is equivalent to multiplying Z by a matrix $Q_i^{\pi_1}$ from the right, where

$$Q_i^{\pi_1} = \begin{pmatrix} -I_{i-1} & 0 \\ 0 & I_{S+1-i} \end{pmatrix} \tag{36}$$

Finally, we can express the action of the annihilation operator acting on one of the first S modes as

$$\hat{f}_i \, |Z\} = (-1)^{i-1} \left| (ZQ_i^{\pi_1} R_i^{\pi_1})^{-}_{\{i\}} \right\} \tag{37}$$

where the CS label with superscript $-$ and subscript $\{i\}$ means the occupancy of the i-th mode is "forced" to be zero. This equation can be generalised to any sequence of n annihilation operators:

$$\hat{f}_{P\langle\sigma\rangle} \, |Z\} = \operatorname{sgn}(P)(-1)^{-n+\sum\langle\sigma\rangle} \left| (ZQ^{\pi_1}_{\langle\sigma\rangle} R^{\pi_1}_{\langle\sigma\rangle})^{-}_{\langle\sigma\rangle} \right\} \tag{38}$$

where the sequence in the subscript of R, Q implies product over all elements in $\langle\sigma\rangle$.

2.1.2 Reduction of coherent state in π_0

An analogous argument to that in Sec. 2.1.1 can be constructed: that setting all elements in the i-th row of Z removes all terms from the occupancy basis decomposition which have $n_i = 1$ when $i \in \pi_0$. From this, the action of the creation operator acting on a mode in π_0 satisfies the property

$$\hat{f}_i^\dagger |Z\} = \hat{f}_i^\dagger |R_i^{\pi_0} Z\} \quad \text{where} \quad R_i^{\pi_0} = \begin{pmatrix} I_{i-1-S} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{M-i} \end{pmatrix} \quad \text{if} \quad i \in \pi_0 \tag{39}$$

Then the action of the creation operator on the occupancy basis states in the remaining terms becomes

$$\hat{f}_i^\dagger |\pi_1 - \{b\} + \{a\}\rangle = (-1)^{S-|\langle b\rangle|+\eta_i(a)} |\pi_1 - \{b\} + \{a\} + \{i\}\rangle \tag{40}$$

Since $|\langle a\rangle| = |\langle b\rangle|$, we can absorb a part of the sign prefactor into the corresponding minor of Z by transforming Z like so:

$$\det\big(Z_{\langle a\rangle,\langle b\rangle}\big) = (-1)^{|\langle b\rangle|-\eta_i(a)} \det\big((Q_i^{\pi_0} Z)_{\langle a\rangle,\langle b\rangle}\big) \quad \text{where} \quad Q_i^{\pi_0} = \begin{pmatrix} I_{i-S} & 0 \\ 0 & -I_{M-i} \end{pmatrix} \tag{41}$$

Therefore

$$\hat{f}_i |Z\} = (-1)^S \left|(R_i^{\pi_0} Q_i^{\pi_0} Z)^+_{\{i\}}\right\} \tag{42}$$

where the CS label with superscript $+$ and subscript $\{i\}$ means the occupancy of the i-th mode is "forced" to be one. This equation can be generalised to any sequence of n creation operators:

$$\hat{f}^\dagger_{P\langle\sigma\rangle} |Z\} = \text{sgn}(P)(-1)^{Sn} \left|(R^{\pi_0}_{\langle\sigma\rangle} Q^{\pi_0}_{\langle\sigma\rangle} Z)^+_{\langle\sigma\rangle}\right\} \tag{43}$$

where the sequence in the subscript of R, Q implies product over all elements in $\langle\sigma\rangle$.

2.2 Action of the transition operator

The transition operator $\hat{T}_{ij} = \hat{f}_i^\dagger \hat{f}_j$ not only generates the quotient space of $(U(1) \otimes SL(M,\mathbb{R}), |\phi_0\rangle)$ when restricting the domain of i, j, but, in general, constitues any S-preserving operator. This can be readily seen from the fact that an arbitrary sequence $\hat{f}^\dagger_{a_1} \dots \hat{f}^\dagger_{a_X} \hat{f}_{b_1} \dots \hat{f}_{b_Y}$ can commute with $\hat{N} = \sum_{m=1}^M \hat{f}_m^\dagger \hat{f}_m$ only if $X = Y$, i.e. the numbers of creation and annihilation operators are equal. Using Eq. 2, we have

$$\left[\hat{T}_{ij}, \hat{N}\right] = \sum_{m=1}^M \left[\hat{T}_{ij}, \hat{T}_{mm}\right] = \sum_{m=1}^M \left(\hat{T}_{im}\delta_{jm} - \hat{T}_{mj}\delta_{im}\right) = \hat{T}_{ij} - \hat{T}_{ij} = 0 \tag{44}$$

and we see that any operator which commutes with $\hat{N}$ (other than the identity) can be expressed as a sum of products of $\hat{T}_{ij}$. Therefore, the action of $\hat{T}_{ij}$ on a coherent state $|Z\rangle$ is of great interest to us.

3 Overlaps of $U(1) \otimes SL(M, \mathbb{R})$ coherent states

3.1 Overlap of two coherent states

Consider two unnormalised fermionic coherent states $|Z_a\}, |Z_b\}$. To calculate their overlap, we use Eq. 27:

$$\{Z_a|Z_b\} = \sum_{r=0}^{\min(S,M-S)} \sum_{\langle a\rangle \in \Gamma_r\langle \pi_0\rangle} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1\rangle} \det\big((Z_a)_{\langle a\rangle,\langle b\rangle}\big)^* \det\big((Z_b)_{\langle a\rangle,\langle b\rangle}\big) \tag{45}$$

where we used the fact that the full occupancy basis is orthonormal. Let us express Z_a using its Hermitian conjugate, using the fact that $\det(A)^* = \det(A^*)$:

$$\{Z_a|Z_b\} = \sum_{r=0}^{\min(S,M-S)} \sum_{\langle a\rangle \in \Gamma_r\langle \pi_0\rangle} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1\rangle} \det\Big((Z_a^\dagger)_{\langle b\rangle,\langle a\rangle}\Big) \det\big((Z_b)_{\langle a\rangle,\langle b\rangle}\big) \tag{46}$$

Invoking the Cauchy-Binet formula for both possible reductions yields

$$\{Z_a|Z_b\} = \sum_{r=0}^{\min(S,M-S)} \sum_{\langle b\rangle \in \Gamma_r\langle \pi_1\rangle} \det\Big((Z_a^\dagger Z_b)_{\langle b\rangle,\langle b\rangle}\Big) = \sum_{r=0}^{\min(S,M-S)} \sum_{\langle a\rangle \in \Gamma_r\langle \pi_0\rangle} \det\Big((Z_b Z_a^\dagger)_{\langle a\rangle,\langle a\rangle}\Big) \tag{47}$$

We identify the determinant of a submatrix with row indices equal to column indices as a principal minor of the square matrix $Z_a^\dagger Z_b$ (or $Z_b Z_a^\dagger$). The sum over all k-order principal minors of a square matrix $A_{(n\times n)}$, here denoted T_k^A, is easily calculable from the characteristic polynomial of A defined as $p_A(\lambda) = \det(\lambda I_n - A)$, using the well-known formula

$$\det(\lambda I_n - A) = \sum_{k=0}^{n} \lambda^{n-k} (-1)^k T_k^A \tag{48}$$

Evaluating the characteristic polynomial at $\lambda = -1$ yields

$$\det(-I_n - A) = \sum_{k=0}^{n} (-1)^n T_k^A \quad \text{or, to standardise the sign,} \quad \det(I_n + A) = \sum_{k=0}^{n} T_k^A \tag{49}$$

Since the overlap is a sum of all principal minors of all orders, it can be evaluated as

$$\{Z_a|Z_b\} = \det\Big(I_S + Z_a^\dagger Z_b\Big) = \det\Big(I_{M-S} + Z_b Z_a^\dagger\Big) \tag{50}$$

This also gives us the normalisation function $N(Z)$:

$$N(Z) = \frac{1}{\sqrt{\det(I_S + Z^\dagger Z)}} = \frac{1}{\sqrt{\det(I_{M-S} + Z Z^\dagger)}} \tag{51}$$

3.2 Standardisation of transition operator sequences in overlap integrals

Consider the overlap integral of a general normal-ordered product sequence of creation and annihilation operators:

$$\langle Z_a|\hat{f}^\dagger_{P_1\langle\rho\rangle}\hat{f}_{P_2\langle\sigma\rangle}|Z_b\rangle \tag{52}$$

The rest of Sec. 3 is dedicated to calculating this overlap integral. We can observe certain general rules which apply to the expression, which allow us to rewrite it as a composition of simpler problems, each of which is tackled below.

First, the overlap integral can be conceptualised as the overlap of $\hat{f}_{P_1\langle\rho\rangle}|Z_a\rangle$ and $\hat{f}_{P_2\langle\sigma\rangle}|Z_b\rangle$. Since each annihilation operator lowers the total particle number by one, this overlap vanishes if $|\langle\rho\rangle|\neq|\langle\sigma\rangle|$.

Second, due to the normal ordering, the overlap vanishes if any index repeats more than once in either $\langle\sigma\rangle$ or $\langle\rho\rangle$. This guarantees that strictly ascending sequences $\langle\rho\rangle,\langle\sigma\rangle$ exist, and can be constructed by undoing the permutations P_1,P_2.

3.3 Overlap of two π_1-reduced coherent states

Now: consider the overlap between two reduced coherent states, $\hat{f}_i|Z_a\}$ and $\hat{f}_j|Z_b\}$ where $i,j\in\pi_1$. By Eq. 38, we have

$$\{Z_a|\hat{f}^\dagger_i\hat{f}_j|Z_b\}=(-1)^{i+j}\left\{(Z_aQ_i^{\pi_1}R_i^{\pi_1})^-_{\{i\}}\middle|(Z_bQ_j^{\pi_1}R_j^{\pi_1})^-_{\{j\}}\right\} \tag{53}$$

3.3.1 Disjoint π_1-reduction overlap

In this subsubsection, we treat the case $i\neq j$.

The overlap can be expanded as

$$\begin{aligned}
&\left\{(Z_aQ_i^{\pi_1}R_i^{\pi_1})^-_{\{i\}}\middle|(Z_bQ_j^{\pi_1}R_j^{\pi_1})^-_{\{j\}}\right\}\\
&=\sum_{r=0}^{\min(S-1,M-S)}\sum_{\langle a\rangle\in\Gamma_r\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_r\langle\pi_1-\{i\}\rangle}\sum_{\langle b'\rangle\in\Gamma_r\langle\pi_1-\{j\}\rangle}\\
&\qquad\qquad\det\Big((R_i^{\pi_1}Q_i^{\pi_1}Z_a^\dagger)_{\langle b\rangle,\langle a\rangle}\Big)\det\Big((Z_bQ_j^{\pi_1}R_j^{\pi_1})_{\langle a\rangle,\langle b'\rangle}\Big)\delta_{b+\{i\},b'+\{j\}}\\
&=\sum_{r=0}^{\min(S-1,M-S)}\sum_{\langle a\rangle\in\Gamma_r\langle\pi_0\rangle}\sum_{\langle b\rangle\in\Gamma_{r-1}\langle\pi_1-\{i,j\}\rangle}\det\Big((R_i^{\pi_1}Q_i^{\pi_1}Z_a^\dagger)_{\langle b\rangle\cup\{j\},\langle a\rangle}\Big)\det\Big((Z_bQ_j^{\pi_1}R_j^{\pi_1})_{\langle a\rangle,\langle b\rangle\cup\{i\}}\Big)
\end{aligned}$$

Taking an arbitrary minor with one column (row) specified to be at index m is equivalent to taking an arbitrary minor of the original matrix transformed in a specific way with one column (row) specified to be at index one. That transformation entails multiplying the first $m-1$ columns (rows) by -1 and then bringing the m-th column (row) to index one. Also, since the $Z_a^\dagger$ (Z_b) minor cannot contain row i

(column j), we can account for these additional restrictions by removing the associated row (column). Therefore the unnormalised braket is equal to

$$= \sum_{r=0}^{\min(S-1,M-S)} \sum_{\langle a\rangle\in\Gamma_r\langle\pi_0\rangle} \sum_{\langle b\rangle\in\Gamma_{r-1}\langle\pi_1'-\{1\}\rangle} \det\Big((P_{j\to 1}Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)^{(i)}_{\langle b\rangle\cup\{1\},\langle a\rangle}\Big)\det\Big((P_{i\to 1}Z_bQ^{\pi_1}_{\{i,j\}})^{(j)}_{\langle a\rangle,\langle b\rangle\cup\{1\}}\Big) \tag{54}$$

where $Q^{\pi_1}_{\{i,j\}}$ is understood to represent the product $Q^{\pi_1}_iQ^{\pi_1}_j$, π_1' is π_1 once-reduced, omitting the final element S, and the superscript in parentheses for $Z_a^\dagger$ (Z_b) means the i-th row (j-th column) is removed. The determinant product over minors with a constraining index we utilise the modified Cauchy-Binet formula, which is stated and proven in [4, App. C], and which yields

$$= -\sum_{r=0}^{\min(S-1,M-S)} \sum_{\langle a\rangle\in\Gamma_r\langle\pi_0\rangle} \det\begin{pmatrix} 0 & (Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)_{j,\langle a\rangle} \\ (Z_bQ^{\pi_1}_{\{i,j\}})_{\langle a\rangle,i} & (Z_bQ^{\pi_1}_{\{i,j\}})^{(i,j)}_{\langle a\rangle,\langle\pi_1''\rangle}(Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)^{(i,j)}_{\langle\pi_1''\rangle,\langle a\rangle}\end{pmatrix} \tag{55}$$

The determinant is the principal minor of a matrix

$$M = \begin{pmatrix} 0 & (Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)_{j,\langle\pi_0\rangle} \\ (Z_bQ^{\pi_1}_{\{i,j\}})_{\langle\pi_0\rangle,i} & (Z_bQ^{\pi_1}_{\{i,j\}})^{(i,j)}(Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)^{(i,j)}\end{pmatrix} = \begin{pmatrix} 0 & (Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)_{j,\langle\pi_0\rangle} \\ (Z_bQ^{\pi_1}_{\{i,j\}})_{\langle\pi_0\rangle,i} & (Z_b)^{(i,j)}(Z_a^\dagger)^{(i,j)}\end{pmatrix} \tag{56}$$

determined by indices $\langle a\rangle\cup\{1\}$. The sum over all such principal minors of rank $r+1$ is equivalent to the sum over all principal minors of rank $r+1$ minus the sum of all principal minors of the lower-right block of rank r. Since we take the sum of this over all values of r on which either of the aforementioned sums is non-zero, the result becomes

$$\begin{aligned}\{Z_a|\hat f_i^\dagger \hat f_j|Z_b\} &= (-1)^{i+j}\Big[\det\Big(I+(Z_b)^{(i,j)}(Z_a^\dagger)^{(i,j)}\Big)-\det(I+M)\Big] \\ &= (-1)^{i+j+1}\det\begin{pmatrix} 0 & (Q^{\pi_1}_{\{i,j\}}Z_a^\dagger)_{j,\langle\pi_0\rangle} \\ (Z_bQ^{\pi_1}_{\{i,j\}})_{\langle\pi_0\rangle,i} & I+(Z_b)^{(i,j)}(Z_a^\dagger)^{(i,j)}\end{pmatrix}\end{aligned} \tag{57}$$

where we used the Laplace expansion of $\det(I+M)$ for the final step. We also notice that whether $i>j$ or $j<i$, the overall effect of the two $Q^{\pi_1}_{\{i,j\}}$ terms is introducing a single sign flip to the determinant. Therefore we have

$$\{Z_a|\hat f_i^\dagger \hat f_j|Z_b\} = (-1)^{i+j}\det\begin{pmatrix} 0 & (Z_a^\dagger)_{j,\langle\pi_0\rangle} \\ (Z_b)_{\langle\pi_0\rangle,i} & I+(Z_b)^{(i,j)}(Z_a^\dagger)^{(i,j)}\end{pmatrix} \tag{58}$$

Eq. 58 can be trivially generalised for two disjoint sequences $\hat f_{\langle a\rangle}, \hat f_{\langle b\rangle}$ of equal length acting on $|Z_a\}, |Z_b\}$ respectively. The overlap becomes

$$\{Z_a|\hat f^\dagger_{\langle a\rangle-}\hat f_{\langle b\rangle}|Z_b\} = (-1)^{\sum_i\langle a\rangle_i+\langle b\rangle_i}\det\begin{pmatrix} 0 & (Z_a^\dagger)_{\langle b\rangle,\langle\pi_0\rangle} \\ (Z_b)_{\langle\pi_0\rangle,\langle a\rangle} & I+(Z_b)^{(\langle a\rangle\cup\langle b\rangle)}(Z_a^\dagger)^{(\langle a\rangle\cup\langle b\rangle)}\end{pmatrix} \tag{59}$$

Permuting the order of the index sequences is reflected by permuting the rows (columns) of the upper-right (lower-left) block, effecting a sign change equivalent to the permutation sign.

3.3.2 Repeated π_1-reduction overlap

Going back to Eq. 53, let us investigate the case $i = j$. Since the mode occupancy reduction occurs on the same mode for both the bra and the ket, the occupancy basis overlap restricts $\langle a\rangle = \langle a'\rangle, \langle b\rangle = \langle b'\rangle$. After removing the i-th (j-th) column from Z_a (Z_b), the sum reduces to an analogy of Eq. 46 with the respective columns removed, from which the same process yields

$$\{Z_a|\hat{f}_i^\dagger \hat{f}_i|Z_b\} = (-1)^{2i} \det\Big(I + Z_b^{(i)}(Z_a^\dagger)^{(i)}\Big) = \det\Big(I + Z_b^{(i)}(Z_a^\dagger)^{(i)}\Big) \tag{60}$$

This process trivially generalises to any sequence of annihilation operators acting on two coherent states, so that their overlap may be written as

$$\{Z_a|(\hat{f}_{\langle\sigma\rangle})^\dagger \hat{f}_{\langle\sigma\rangle}|Z_b\} = \{Z_a|\hat{f}^\dagger_{\langle\sigma\rangle^-} \hat{f}_{\langle\sigma\rangle}|Z_b\} = \det\Big(I + Z_b^{(\sigma)}(Z_a^\dagger)^{(\sigma)}\Big) \tag{61}$$

where $\langle\sigma\rangle^-$ is the descending sequence over the indices in σ.

3.3.3 Compound π_1-reduction overlap

Restricting the operator indices to π_1, the general overlap integral can be constructed as

$$\langle Z_a|\hat{f}^\dagger_{P_1\langle\rho\cup\tau\rangle} \hat{f}_{P_2\langle\sigma\cup\tau\rangle}|Z_b\rangle \tag{62}$$

where ρ, σ, τ are disjoint subsets of π_1. To solve this overlap integral we permute the sequence like so:

$$\langle Z_a|\hat{f}^\dagger_{\langle\tau\rangle^-} \hat{f}^\dagger_{\langle\rho\rangle^-} \hat{f}_{\langle\sigma\rangle} \hat{f}_{\langle\tau\rangle}|Z_b\rangle \tag{63}$$

The point here is to keep the sign factor of the repeated π_1-reduction trivial by mimicking Eq. 61. Afterwards, the disjoint π_1-reduction has a "non-trivial" sign factor given by the sum of $\langle\rho\rangle$ and $\langle\sigma\rangle$, and permutations within these sequences are reflected in permutations of the rows or columns of the off-diagonal blocks in Eq. 59. Since both the disjoint and the repeated π_1-reduction transforms Z in the overlap expression only with regard to a specific column, a sequence of such transformations can be composed trivially. The full overlap integral therefore becomes

$$\langle Z_a|\hat{f}^\dagger_{\langle\tau\rangle^-} \hat{f}^\dagger_{\langle\rho\rangle^-} \hat{f}_{\langle\sigma\rangle} \hat{f}_{\langle\tau\rangle}|Z_b\rangle = (-1)^{\sum_i \langle\rho\rangle_i + \langle\sigma\rangle_i} \det\begin{pmatrix} 0 & (Z_a^\dagger)_{\langle\sigma\rangle,\langle\pi_0\rangle} \\ (Z_b)_{\langle\pi_0\rangle,\langle\rho\rangle} & I + (Z_b)^{(\rho\cup\sigma\cup\tau)}(Z_a^\dagger)^{(\rho\cup\sigma\cup\tau)} \end{pmatrix} \tag{64}$$

3.4 Matrix element of the quadratic S-preserving Hamiltonian

For an S-preserving Hamiltonian, the one-body interaction can be expressed as $V^{(1)}_{\alpha,\beta}\hat{f}^\dagger_\alpha \hat{f}_\beta$, and two-body interaction as $\frac{1}{2}V^{(2)}_{\alpha,\beta,\gamma,\delta}\hat{f}^\dagger_\alpha \hat{f}^\dagger_\beta \hat{f}_\gamma \hat{f}_\delta$, where

- $V^{(1)}$ is Hermitian
- $V^{(2)}$ is anti-symmetric w.r.t. exchange of the first or second pair of indices, and Hermitian w.r.t. exchange of the two pairs of indices.

Then

$$\hat{H} = V^{(1)}_{\alpha,\beta}\hat{f}^\dagger_\alpha \hat{f}_\beta + \frac{1}{2}V^{(2)}_{\alpha,\beta,\gamma,\delta}\hat{f}^\dagger_\alpha \hat{f}^\dagger_\beta \hat{f}_\gamma \hat{f}_\delta \tag{65}$$

A Notation in this article

- $\langle S\rangle$: A sequence constructed from the elements of set $S \subset \mathbb{N}^+$ such that $\langle S\rangle_i < \langle S\rangle_j \iff i < j$. Such sequence shall be referred to as ascending. If S is a number, the sequence is explicitly $\langle 1, 2 \dots S\rangle$. The length of $\langle S\rangle$ shall be denoted as $|\langle S\rangle|$.
- $\langle S\rangle^-$: An object analogous to $\langle S\rangle$, except the sequence is strictly descending.
- $\Gamma_n\langle S\rangle$: Set of all subsequences of length n of sequence $\langle S\rangle$.
- $\langle S_1\rangle \oplus \langle S_2\rangle$: An ascending sequence constructed from ascending sequences $\langle S_1\rangle$ and $\langle S_2\rangle$ with no common elements, such that it contains every element from $\langle S_1\rangle$ and $\langle S_2\rangle$.
- Two sequences are said to be disjoint if the sets upon which they are built are disjoint; that is, they share no common index.
- $M_{\langle S_1\rangle,\langle S_2\rangle}$ where M is a matrix: This denotes a matrix M' such that $M'_{ij} = M_{\langle S_1\rangle_i,\langle S_2\rangle_j}$, which is a submatrix of M.
- $|\langle S\rangle\rangle$: An element of the full occupancy basis where the i-th mode is occupied iff $i \in \langle S\rangle$.
- $\hat{f}^\dagger_{\langle S\rangle}$: A product of $N = |\langle S\rangle|$ fermionic creation operators $\hat{f}^\dagger_{\langle S\rangle_1} \dots \hat{f}^\dagger_{\langle S\rangle_N}$. An analogous construction can be defined for a sequence of annihilation operators.
- P^k: The set of permutations of k elements. For an element $P \in P^k$ and an ascending sequence $\langle S\rangle$, we denote $P\langle S\rangle_i$ the i-the element of the (not necessarily ascending) sequence constructed by permuting $\langle S\rangle$ by P.
- $\upharpoonleft \hat{f}^\dagger_{\sigma_1} \dots \hat{f}^\dagger_{\sigma_n} \upharpoonright$: The *monotonic ordering* of a product of fermionic creation (or annihilation) operators. The result is the product of the same set of operators $\hat{f}^\dagger_{\rho_1} \dots \hat{f}^\dagger_{\rho_n} \equiv \hat{f}^\dagger_{\langle\rho\rangle}, \{\rho\} = \{\sigma\}$ such that their indices are in an ascending order. Equivalently, for any permutation $P \in P^n$, we have $\upharpoonleft \hat{f}^\dagger_{P\langle\rho\rangle} \upharpoonright = \hat{f}^\dagger_{\langle\rho\rangle}$. *Note:* If applied to a sequence of both creation and annihilation operators, the monotonic ordering first applies a normal ordering, and then is applied to the creation and annihilation operators separately.
- I_n denotes the identity matrix of order n.
- $\hat{I}$ denotes the identity operator.
- $\eta_x(S)$ is the number of elements in S smaller than x.
- η^y_x is equal to one if $x > y$ and to zero otherwise. It can be seen that $\eta_x(S) = \sum_{s\in S} \eta^s_x$.

B Properties of fermionic creation and annihilation operators

Let $|n_1, n_2 \dots n_M\rangle$ denote an element of the full-occupancy basis for which n_i particles occupy the i-th mode (for fermions $n_i \in \{0, 1\}$). Then the action of the fermionic creation and annihilation operator on the i-th mode is [2, Eq. 3.47]

$$
\begin{aligned}
\hat{f}_i^\dagger \left|n_1, n_2 \ldots n_M\right\rangle &= (-1)^{\sum_{j<i} n_j} \sqrt{1-n_i} \left|n_1 \ldots n_{i-1}, n_i+1, n_{i+1} \ldots n_M\right\rangle && (66)\\
\hat{f}_i \left|n_1, n_2 \ldots n_M\right\rangle &= (-1)^{\sum_{j<i} n_j} \sqrt{n_i} \left|n_1 \ldots n_{i-1}, n_i-1, n_{i+1} \ldots n_M\right\rangle && (67)
\end{aligned}
$$

The operators obey well-known anti-commutator identities:

$$\left\{\hat{f}_i, \hat{f}_j\right\} = 0 \qquad \left\{\hat{f}_i^\dagger, \hat{f}_j^\dagger\right\} = 0 \qquad \left\{\hat{f}_i, \hat{f}_j^\dagger\right\} = \delta_{ij} \tag{68}$$

Using the identity $\left[\hat{A}, \hat{B}\right] = 2\hat{A}\hat{B} - \left\{\hat{A}, \hat{B}\right\} = \left\{\hat{A}, \hat{B}\right\} - 2\hat{B}\hat{A}$, we can express the corresponding commutator relations:

$$\left[\hat{f}_i, \hat{f}_j\right] = 2\hat{f}_i\hat{f}_j \qquad \left[\hat{f}_i^\dagger, \hat{f}_j^\dagger\right] = 2\hat{f}_i^\dagger\hat{f}_j^\dagger \qquad \left[\hat{f}_i, \hat{f}_j^\dagger\right] = \delta_{ij} - 2\hat{f}_j^\dagger\hat{f}_i \qquad \left[\hat{f}_i^\dagger, \hat{f}_j\right] = 2\hat{f}_i^\dagger\hat{f}_j - \delta_{ij} \tag{69}$$

We shall also state the Wick contraction $\hat{A}\hat{B} - :\hat{A}\hat{B}:$ for every pairing of fermionic operators:

$$\overbracket{\hat{f}_i\hat{f}_j} = 0 \qquad \overbracket{\hat{f}_i^\dagger\hat{f}_j^\dagger} = 0 \qquad \overbracket{\hat{f}_i^\dagger\hat{f}_j} = 0 \qquad \overbracket{\hat{f}_i\hat{f}_j^\dagger} = \delta_{ij} \tag{70}$$

A corollary of Eq. 68 is that exchanging two neighbouring creation or two annihilation operators in a product sequence flips the sign of the product. Since any permutation P of a sequence can be constructed from composing pairwise swaps $(i \leftrightarrow i+1)$, we have

$$\hat{f}_{\langle\sigma\rangle}^\dagger = \operatorname{sgn}(P)\hat{f}_{P\langle\sigma\rangle}^\dagger \qquad \hat{f}_{\langle\sigma\rangle} = \operatorname{sgn}(P)\hat{f}_{P\langle\sigma\rangle} \tag{71}$$

This is useful, since the action of a creation operator sequence with ascending indices on the vacuum state has a trivial sign:

$$\hat{f}_{\langle\sigma\rangle}^\dagger \left|\text{vac.}\right\rangle = \left|\sigma\right\rangle \quad \text{therefore} \quad \hat{f}_{P\langle\sigma\rangle}^\dagger \left|\text{vac.}\right\rangle = \operatorname{sgn}(P)\left|\sigma\right\rangle \tag{72}$$

C Invalidity of the boson-analogous construction

The $SU(M)$ bosonic coherent state with S particles can be expressed as

$$\left|z\right\rangle = N(z)\left(\sum_{m=1}^{M} z_m \hat{b}_m^\dagger\right)^S \left|\text{vac.}\right\rangle \tag{73}$$

where $N(z)$ is some real-valued normalisation function. Let us create a "naive" fermionic coherent state with S particles by replacing the bosonic creation operators by their fermionic counterparts:

$$\left|z\right\rangle = N(z)\left(\sum_{m=1}^{M} z_m \hat{f}_m^\dagger\right)^S \left|\text{vac.}\right\rangle \tag{74}$$

Expanding the multinomial product, we see that all terms with repeated creation operators $\hat{f}_i^\dagger \hat{f}_i^\dagger$ vanish, yielding

$$|z\rangle = N(z) \sum_{\langle a\rangle \in \Gamma^S\langle M\rangle} \left(\prod_{i=1}^{S} z_{\langle a\rangle_i} \right) \sum_{P\in P^S} \hat{f}^\dagger_{P\langle a\rangle} = N(z) \sum_{\langle a\rangle \in \Gamma^S\langle M\rangle} \left(\prod_{i=1}^{S} z_{\langle a\rangle_i} \right) \hat{f}^\dagger_{\langle a\rangle} \sum_{P\in P^S} \operatorname{sgn}(P)$$

However, since $\operatorname{sgn}(P)$ is an irreducible representation of the permutation group on P^S, it is orthogonal to the trivial representation (for $S > 1$), and hence its sum over all group elements vanishes. Hence

1. For $S = 0, 1$, the naive construction is equivalent to the construction in this article up to a meaningless transformation of the z parameter.
2. For $S > 1$, the naive construction vanishes.

D Determinant of the upper-left zero block matrix

Consider an $(m + n, m + n)$ matrix in the form

$$M = \begin{pmatrix} 0_{(m,m)} & A_{(m,n)} \\ B_{(n,m)} & C_{(n,n)} \end{pmatrix} \tag{75}$$

whose determinant we wish to calculate. If $m > n$, the determinant is trivially zero. Therefore, we can assume $m \leq n$ for the non-trivial case.

Consider the matrices

$$X = \begin{pmatrix} I_{(m,m)} & A_{(m,n)} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I_{(m,m)} \\ B_{(n,m)} \end{pmatrix} \tag{76}$$

Their product is

$$YX = \begin{pmatrix} I_{(m,m)} & A_{(m,n)} \\ B_{(n,m)} & (BA)_{(n,n)} \end{pmatrix} \quad \text{therefore} \quad M - YX = \begin{pmatrix} I_{(m,m)} & 0_{(m,n)} \\ 0_{(m,n)} & (C - BA)_{(n,n)} \end{pmatrix} \tag{77}$$

Then, using the matrix determinant lemma we obtain

$$\begin{aligned} \det(M) &= \det(M - YX + YX) \\ &= \det\big(I_{(m,m)} + X(M - YX)^{-1}Y\big) \det(M - YX) \end{aligned} \tag{78}$$

Since $M - YX$ is block-diagonal, its determinant and inverse can be expressed like so:

$$\det(M - YX) = (-1)^m \det(C - BA) \quad \text{and} \quad (M - YX)^{-1} = \begin{pmatrix} I_{(m,m)} & 0_{(m,n)} \\ 0_{(m,n)} & (C - BA)^{-1}_{(n,n)} \end{pmatrix} \tag{79}$$

Therefore

$$X(M - YX)^{-1}Y = A(C - BA)^{-1}B - I_{(m,m)} \tag{80}$$

We substitute these results into Eq. 78 and use the matrix determinant lemma again to expand $\det(C - BA)$:

$$\det(M) = (-1)^m \det\bigl(A(C - BA)^{-1}B\bigr) \det\bigl(I_{(m,m)} - AC^{-1}B\bigr) \det(C) \tag{81}$$

We shall also use the Woodbury matrix identity to expand the inverse of $C - BA$ and express the product of the first two determinants as the determinant of the product of their respective matrices, finally obtaining

$$\begin{aligned} \det(M) &= (-1)^m \det\bigl((AC^{-1}B + AC^{-1}B(I_{(m,m)} - AC^{-1}B)^{-1}AC^{-1}B)(I_{(m,m)} - AC^{-1}B)\bigr) \det(C) \\ &= (-1)^m \det\bigl(AC^{-1}B\bigr) \det(C) \end{aligned} \tag{82}$$

References

[1] Paldus, J. (2006). Dynamical Groups. In: Drake, G. (eds) *Springer Handbook of Atomic, Molecular, and Optical Physics*. Springer Handbooks. Springer, New York, NY. doi.org/10.1007/978-0-387-26308-3_4

[2] Greiner, W., Maruhn, J. A. (1996), *Nuclear Models*. 1st edn. Springer Berlin, Heidelberg. doi.org/10.1007/978-3-642-60970-1

[3] Buonsante, P., Penna, V. (2008), Some remarks on the coherent-state variational approach to nonlinear boson models. *J. Phys. A: Math. Theor.*, **41**, 175301. doi.org/10.1088/1751-8113/41/17/175301

[4] Chapman, A., Miyake, A. (2018), Classical simulation of quantum circuits by dynamical localization: analytic results for Pauli-observable scrambling in time-dependent disorder. Available at arxiv.org/pdf/1704.04405v2 [quant-ph]