$$\left[\hat{a}_i\hat{a}_j\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k\hat{a}_i^\dagger\hat{a}_k, \, \hat{a}_k^\dagger\hat{a}_i\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_k^\dagger\hat{a}_j^\dagger\right]$$

$$
\begin{aligned}
&= 4\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{i,k} + 30\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\delta_{k,i} \\
&\quad + 51\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,i}\delta_{i,k} + 462\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\delta_{j,i}\delta_{k,i} \\
&\quad + 14\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{i,k} + 111\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\delta_{k,i} \\
&\quad + 1358\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\delta_{j,k}\delta_{j,i} + 4\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\delta_{i,k} \\
&\quad + 42\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\delta_{k,i} + 1408\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_k\delta_{j,k}\delta_{j,i} + 56\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\delta_{k,i} \\
&\quad + 218\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_k\delta_{k,i} + 100\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_k\delta_{k,i} + 384\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k\delta_{j,k}\delta_{j,i} \\
&\quad - 2\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k - 91\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,i} \\
&\quad - 15\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k + 16\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\hat{a}_k\delta_{j,k} \\
&\quad + 128\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,k} - 405\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,i} \\
&\quad - 18\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k + 206\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_k\hat{a}_k\delta_{j,k} - 522\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_k\hat{a}_k\delta_{j,i} \\
&\quad - 144\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k\hat{a}_k\delta_{i,j} - 6\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k - 33\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k \\
&\quad + 0\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,k} + 48\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,k} - 30\hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_k\hat{a}_k \\
&\quad + 114\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k\hat{a}_k\delta_{j,k} - 5\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{i,j} \\
&\quad + 4\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,k} + 28\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k\delta_{j,k} \\
&\quad + 42\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_k\hat{a}_k\delta_{j,k} - \hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k\hat{a}_k \\
&\quad - 2\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\hat{a}_k\hat{a}_k + 4\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k \\
&\quad + 16\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_j\hat{a}_k + 28\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_j\hat{a}_k\delta_{j,i} \\
&\quad + 188\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_k\delta_{j,i} + 20\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k \\
&\quad + 80\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_k + 296\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i\hat{a}_j\hat{a}_k\delta_{j,i} + 72\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i\hat{a}_k\delta_{j,i} \\
&\quad + 8\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\hat{a}_k + 8\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_k\delta_{j,k} + 40\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_k \\
&\quad + 40\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_k\delta_{j,k} + 12\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_j\hat{a}_k + 48\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j\hat{a}_k \\
&\quad + 24\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_k + 24\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k\delta_{j,k} + 0\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_k\delta_{j,i} + 2\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_j\hat{a}_j \\
&\quad + 8\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_i\hat{a}_j + 10\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j + 40\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j \\
&\quad + 14\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\hat{a}_j\hat{a}_j\delta_{i,j} + 94\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j\delta_{j,i} + 148\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j\delta_{j,i} \\
&\quad + 4\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i\hat{a}_j + 20\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_j + 18\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_j\hat{a}_j\hat{a}_j\delta_{k,i} \\
&\quad + 72\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j\hat{a}_j\delta_{k,i} + 6\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_j\hat{a}_j\hat{a}_j + 24\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j \\
&\quad + 36\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j\delta_{k,i} + 12\hat{a}_i^\dagger\delta_{k,i}\delta_{j,k} + 12\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_j + 36\hat{a}_j^\dagger\hat{a}_i^\dagger\hat{a}_j\delta_{j,i}
\end{aligned}
$$