Construction of Sym(M) fermionic coherent states on the particle-preserving dynamical group.

Michal Horanský

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Abstract

The dynamical group of a fermionic system with M modes which preserves total particle number is identified as $G = \mathrm{Sym}(M)$. A reference state $|\phi_0\rangle$ is constructed as a member of the full occupancy basis by partitioning the modes into π_1 (S occupied modes) and π_0 ($M - S$ unoccupied modes). The quotient space of $(G, |\phi_0\rangle)$ is shown to be generated by $\hat{f}_i^\dagger \hat{f}_j$, where $i \in \pi_0, j \in \pi_1$, and a generalised coherent state $|Z\rangle$ belonging to this quotient space is decomposed into the full occupancy basis. The overlap element $\langle Z_a|Z_b\rangle$ is expressed as a sum of the coefficients of the characteristic polynomial of $Z_a^\dagger Z_b$ with non-trivially alternating signs. The action of the "transposition operator" $\hat{f}_i^\dagger \hat{f}_j$ (for arbitrary i, j) on $|Z\rangle$ is expressed in the full occupancy basis, and the expression for a general two-body-interacting total-particle-preserving Hamiltonian matrix element $\langle Z_a|\hat{H}|Z_b\rangle$ is given. The time complexity of calculating said quantities is discussed.

1 Construction of Sym(M) coherent states

1.1 General approach to decomposition into the full occupancy basis

1.2 Dynamical group, reference state, and quotient space

1.3 Sym(M) decomposition into the full occupancy basis

2 Properties of Sym(M) coherent states

2.1 Overlap of two coherent states

2.2 Action of the transposition operator

The transposition operator $\hat{T}_{ij} = \hat{f}_i^\dagger \hat{f}_j$ not only generates the quotient space of $(\mathrm{Sym}(M), |\phi_0\rangle)$ when restricting the domain of i, j, but, in general, constitues any S-preserving operator. This can be readily seen from the fact that an arbitrary sequence $\hat{f}_{a_1}^\dagger \ldots \hat{f}_{a_X}^\dagger \hat{f}_{b_1} \ldots \hat{f}_{b_Y}$ can commute with $\hat{N} = \sum_{m=1}^{M} \hat{f}_m^\dagger \hat{f}_m$ only if $X = Y$, i.e. the numbers of creation and annihilation operators are equal. Since we have

$$\begin{aligned}
\left[\hat{T}_{ij}, \hat{N}\right] &= \sum_{m=1}^{M} \left[\hat{f}_i^\dagger \hat{f}_j, \hat{f}_m^\dagger \hat{f}_m\right] \\
&= \sum_{m=1}^{M} \left(\left[\hat{f}_i^\dagger, \hat{f}_m^\dagger\right] \hat{f}_j \hat{f}_m + \hat{f}_m^\dagger \left[\hat{f}_i^\dagger, \hat{f}_m\right] \hat{f}_j + \hat{f}_i^\dagger \left[\hat{f}_j, \hat{f}_m^\dagger\right] \hat{f}_m + \hat{f}_m^\dagger \hat{f}_i^\dagger \left[\hat{f}_j, \hat{f}_m\right]\right) \\
&= \sum_{m=1}^{M} \left(2\hat{f}_i^\dagger \hat{f}_m^\dagger \hat{f}_j \hat{f}_m + \hat{f}_m^\dagger (2\hat{f}_i^\dagger \hat{f}_m - \delta_{im}) \hat{f}_j + \hat{f}_i^\dagger (\delta_{jm} - 2\hat{f}_m^\dagger \hat{f}_j) \hat{f}_m + 2\hat{f}_m^\dagger \hat{f}_i^\dagger \hat{f}_j \hat{f}_m\right) \\
&= \sum_{m=1}^{M} \left(\hat{f}_i^\dagger \hat{f}_m \delta_{jm} - \hat{f}_m^\dagger \hat{f}_j \delta_{im}\right) = \hat{f}_i^\dagger \hat{f}_j - \hat{f}_i^\dagger \hat{f}_j = 0 \qquad (1)
\end{aligned}$$

we see that any operator which commutes with $\hat{N}$ (other than the identity) can be expressed as a sum of products of $\hat{T}_{ij}$. Therefore, the action of $\hat{T}_{ij}$ on a coherent state $|Z\rangle$ is of great interest to us.

2.3 Matrix element of the quadratic S-preserving Hamiltonian

For an S-preserving Hamiltonian, the one-body interaction can be expressed as $V_{\alpha,\beta}^{(1)} \hat{f}_\alpha^\dagger \hat{f}_\beta$, and two-body interaction as $\frac{1}{2} V_{\alpha,\beta,\gamma,\delta}^{(2)} \hat{f}_\alpha^\dagger \hat{f}_\beta^\dagger \hat{f}_\gamma \hat{f}_\delta$, where

- $V^{(1)}$ is Hermitian
- $V^{(2)}$ is anti-symmetric w.r.t. exchange of the first or second pair of indices, and Hermitian w.r.t. exchange of the two pairs of indices.

Then

$$\hat{H} = V^{(1)}_{\alpha,\beta}\hat{f}^\dagger_\alpha \hat{f}_\beta + \frac{1}{2}V^{(2)}_{\alpha,\beta,\gamma,\delta}\hat{f}^\dagger_\alpha \hat{f}^\dagger_\beta \hat{f}_\gamma \hat{f}_\delta \tag{2}$$

A Notation in this article

- $\langle S\rangle$: A sequence constructed from the elements of set $S \subset \mathbb{N}^+$ such that $\langle S\rangle_i < \langle S\rangle_j \iff i < j$. Such sequence shall be referred to as ascending. If S is a number, the sequence is explicitly $\langle 1, 2 \dots S\rangle$.
- $\Gamma_n\langle S\rangle$: Set of all subsequences of length n of sequence $\langle S\rangle$.
- $\langle S_1\rangle \oplus \langle S_2\rangle$: An ascending sequence constructed from ascending sequences $\langle S_1\rangle$ and $\langle S_2\rangle$ with no common elements, such that it contains every element from $\langle S_1\rangle$ and $\langle S_2\rangle$. The length of $\langle S\rangle$ shall be denoted as $|\langle S\rangle|$.
- $M_{\langle S_1\rangle,\langle S_2\rangle}$ where M is a matrix: This denotes a matrix M' such that $M'_{ij} = M_{\langle S_1\rangle_i,\langle S_2\rangle_j}$, which is a submatrix of M.
- $|\langle S\rangle\rangle$: An element of the full occupancy basis where the i-th mode is occupied iff $i \in \langle S\rangle$.
- $\hat{f}^\dagger_{\langle S\rangle}$: A product of $N = |\langle S\rangle|$ fermionic creation operators $\hat{f}^\dagger_{\langle S\rangle_1} \dots \hat{f}^\dagger_{\langle S\rangle_N}$. An analogous construction can be defined for a sequence of annihilation operators.
- P^k: The set of permutations of k elements. For an element $P \in P^k$ and an ascending sequence $\langle S\rangle$, we denote $P\langle S\rangle_i$ the i-the element of the (not necessarily ascending) sequence constructed by permuting $\langle S\rangle$ by P.
- $\upharpoonleft \hat{f}^\dagger_{\sigma_1} \dots \hat{f}^\dagger_{\sigma_n} \upharpoonright$: The *monotonic ordering* of a product of fermionic creation (or annihilation) operators. The result is the product of the same set of operators $\hat{f}^\dagger_{\rho_1} \dots \hat{f}^\dagger_{\rho_n} \equiv \hat{f}^\dagger_{\langle\rho\rangle}, \{\rho\} = \{\sigma\}$ such that their indices are in an ascending order. Equivalently, for any permutation $P \in P^n$, we have $\upharpoonleft \hat{f}^\dagger_{P\langle\rho\rangle} \upharpoonright = \hat{f}^\dagger_{\langle\rho\rangle}$. *Note:* If applied to a sequence of both creation and annihilation operators, the monotonic ordering first applies a normal ordering, and then is applied to the creation and annihilation operators separately.

B Properties of fermionic creation and annihilation operators

Commutators and such

C Invalidity of the boson-analogous construction

The $SU(M)$ bosonic coherent state with S particles can be expressed as

$$|z\rangle = N(z)\left(\sum_{m=1}^{M} z_m \hat{b}_m^\dagger\right)^S |\text{vac.}\rangle \tag{3}$$

where $N(z)$ is some real-valued normalisation function. Let us create a "naive" fermionic coherent state with S particles by replacing the bosonic creation operators by their fermionic counterparts:

$$|z\rangle = N(z)\left(\sum_{m=1}^{M} z_m \hat{f}_m^\dagger\right)^S |\text{vac.}\rangle \tag{4}$$

Expanding the multinomial product, we see that all terms with repeated creation operators $\hat{f}_i^\dagger \hat{f}_i^\dagger$ vanish, yielding

$$|z\rangle = N(z) \sum_{\langle a\rangle \in \Gamma^S\langle M\rangle} \left(\prod_{i=1}^{S} z_{\langle a\rangle_i}\right) \sum_{P\in P^S} \hat{f}_{P\langle a\rangle}^\dagger = N(z) \sum_{\langle a\rangle \in \Gamma^S\langle M\rangle} \left(\prod_{i=1}^{S} z_{\langle a\rangle_i}\right) \hat{f}_{\langle a\rangle}^\dagger \sum_{P\in P^S} \text{sgn}(P)$$

However, since $\text{sgn}(P)$ is an irreducible representation of the permutation group on P^S, it is orthogonal to the trivial representation (for $S > 1$), and hence its sum over all group elements vanishes. Hence

1. For $S = 0, 1$, the naive construction is equivalent to the construction in this article up to a meaningless transformation of the z parameter.
2. For $S > 1$, the naive construction vanishes.

References