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Quantum Mechanics for Nuclear Structure, Volume 2
An intermediate level view
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Chapter 1

Representation of rotations, angular momentum and spin

The various representations of rotations in physical space, (3, $\mathbb{R}$) and Hilbert space (n, $\mathbb{C}$) are developed in detail. This leads to an in-depth treatment of the representation of states of well-defined spin and angular momentum in quantum systems. The peculiarities of the physics of spin-$\frac{1}{2}$ systems (spinors) are outlined. The tensorial character of representations is implicit in the treatment. The Schwinger and Bargmann representations are introduced in some detail; and this leads to $SU(2)$ coherent states (which are important for more advanced group representation theory).

Concepts: Euler angles; matrix representations; Pauli spin matrices; ket rotations; $SU(2)$ and $SO(3)$ tensor representations; Schwinger representation; spherical harmonics as Cartesian tensors; spin-$\frac{1}{2}$ neutron interferometry; Bargmann space; measure of a space; $SU(2)$ coherent states; non-unitary representations.

Angular momentum and spin are dynamical variables that are fundamental to finite systems in quantum mechanics, i.e. for molecules, atoms, nuclei and hadrons. To fully handle the quantum mechanics of these systems, the mathematical *representation* of rotations is fundamental. Some elements of these issues in quantum mechanics are introduced in Volume 1. Namely, the concept of a group, the use of matrices, the distinction between rotations in physical space, (3, $\mathbb{R}$), and Hilbert space is presented in chapter 10; and the basic quantization of spin and angular momentum, using algebraic methods, is presented in chapter 11. Further, the facility with which these methods *reduce* the solution of central force problems in quantum mechanics to simple algebraic problems in terms of a single (radial) degree of freedom is presented in chapter 12.

The mathematical representation of rotations is a rich paradigm for the whole of quantum mechanics. In this chapter, a wide range of mathematical *tools* is introduced. Matrix algebra and the algebra of polynomials in real and complex

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variables feature prominently. The peculiar physics of spin-$\frac{1}{2}$ particles and spinors is presented. But, the primary aim is to initiate a *language* that is suitable for the theoretical formulation of finite many-body quantum systems. Group theory and Lie algebras are implicit in the material presented in this chapter: the groups $SO(2)$, $SO(3)$ and $SU(2)$ feature prominently in their behind-the-scene role. The road into many-body systems necessitates more complicated groups such as $SU(3)$: some of the material in this chapter is intended to 'pave' this road.

1.1 Rotations in $(3, \mathbb{R})$

One way to describe a rotation in $(3, \mathbb{R})$ is in terms of rotation in a plane through a specified angle[1]. This is defined in terms of an axis of rotation $\hat{n}$ and an angle ϕ. The axis $\hat{n}$ is perpendicular to the plane defined by the initial and final orientations of the vectors $\vec{V}$, $\vec{V}'$: $R(\phi)\vec{V} = \vec{V}'$. The difficulty lies in ascertaining the direction of $\hat{n}$. Although this *'axis-angle'* parameterisation or *Darboux* parameterisation is simple in principle, it is difficult to use in practice.

The most widely used practical parameterisation of rotations in $(3, \mathbb{R})$ is in terms of *Euler* rotations. Consider a space-fixed coordinate frame $Oxyz$ and a body-fixed coordinate frame $O\bar{x}\bar{y}\bar{z}$. The orientation of an object can be specified by the rotation R that rotates the $O\bar{x}\bar{y}\bar{z}$ frame into the $Oxyz$ frame. This can be done in three steps as illustrated in figure 1.1.

Figure 1.1 depicts the following:

$$R(\alpha, \beta, \gamma) = R_z(\gamma)\ R_Y(\beta)\ R_{\bar{z}}(\alpha). \tag{1.1}$$

Note the order of the three rotations. The problem is that these three rotations are about axes belonging to three different frames of reference. The three rotations on the right-hand side of equation (1.1) can be restated in terms of a single frame of reference using *similarity transformations*, specifically

$$R_z(\gamma) = R_Y(\beta)\ R_{\bar{z}}(\gamma)\ R_Y^{-1}(\beta) \tag{1.2}$$

and

$$R_Y(\beta) = R_{\bar{z}}(\alpha)\ R_{\bar{y}}(\beta)\ R_{\bar{z}}^{-1}(\alpha). \tag{1.3}$$

Thus,

$$R(\alpha,\beta,\gamma) = R_Y(\beta)\ R_{\bar{z}}(\gamma)\ \overset{I}{\cancel{R_Y^{-1}(\beta) R_Y(\beta)}}\ R_{\bar{z}}(\alpha), \tag{1.4}$$

$$\therefore\ R(\alpha, \beta, \gamma) = R_Y(\beta)\ R_{\bar{z}}(\gamma)\ R_{\bar{z}}(\alpha); \tag{1.5}$$

and, since $R_{\bar{z}}(\gamma)$ and $R_{\bar{z}}(\alpha)$ commute,

[1] It should be noted that rotations in $(3, \mathbb{R})$ can be elegantly represented using *quaternions*. Use of quaternions avoids gimbal lock; they are used for programming robots and computer games.

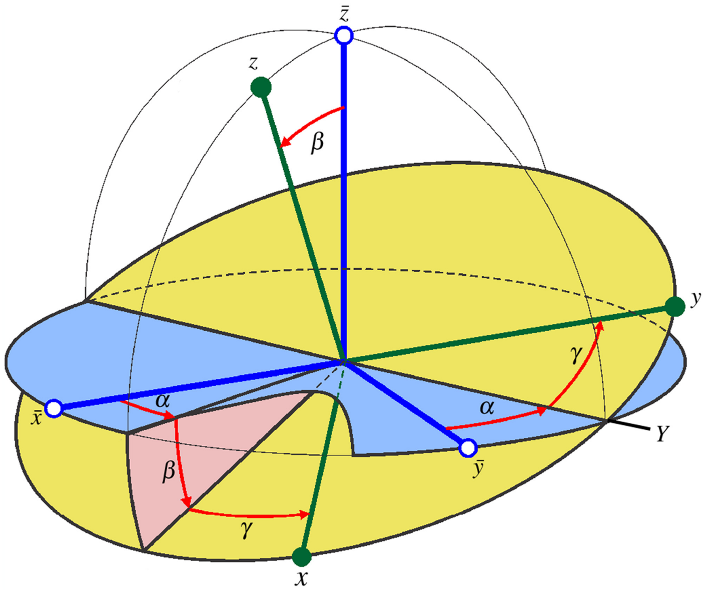

Figure 1.1. The Euler angles (α, β, γ) defined in terms of a three-step sequence of rotations that take an intrinsic or body-fixed frame $O\bar{x}\bar{y}\bar{z}$ into a space-fixed frame $Oxyz$. Note that the axes of rotation are: $O\bar{z}$; the line of intersection of the $O\bar{x}\bar{y}$ and Oxy planes, OY; and Oz. In Step I, $\bar{y}$ rotates to Y and $\bar{z}$ remains fixed; in Step II, $\bar{z}$ rotates to z and Y remains fixed; in Step III, Y rotates to y and z remains fixed. Further, note that the ranges of the angles are: $0 \leqslant \alpha < 2\pi$, $0 \leqslant \beta < \pi$, $0 \leqslant \gamma < 2\pi$. This results in an ambiguity for the rotation $\beta = 0$, $(\alpha, 0, \gamma) \equiv (\alpha', 0, \gamma')$ if $\alpha + \gamma = \alpha' + \gamma'$: this is referred to as '*gimbal lock*' (where 'gimbal' refers to the rotation device or mechanical operator). This figure is adapted from that found on the Easyspin website.

$$\therefore R(\alpha, \beta, \gamma) = R_Y(\beta)\ R_{\bar{z}}(\alpha)\ R_{\bar{z}}(\gamma), \tag{1.6}$$

$$\therefore R(\alpha, \beta, \gamma) = R_{\bar{z}}(\alpha)\ R_{\bar{y}}(\beta)\ \cancelto{I}{R_{\bar{z}}^{-1}(\alpha) R_{\bar{z}}(\alpha)}\ R_{\bar{z}}(\gamma), \tag{1.7}$$

$$\therefore R(\alpha, \beta, \gamma) = R_{\bar{z}}(\alpha)\ R_{\bar{y}}(\beta)\ R_{\bar{z}}(\gamma). \tag{1.8}$$

Note the new order of the three rotations (cf. equation (1.1)).

1.2 Matrix representations of spin and angular momentum operators

The matrix elements of $\hat{J}_z, \hat{J}_\pm$ in the $\{|jm\rangle; j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; m = +j, +j-1, \ldots, -j\}$ basis are (cf. Volume 1, chapter 11):

$$\langle j'm'|\hat{J}_z|jm\rangle = m\hbar\delta_{j'j}\delta_{m'm}, \tag{1.9}$$

$$\langle j'm'|\hat{J}_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,\hbar\delta_{j'j}\delta_{m'm\pm 1}. \tag{1.10}$$

Matrix elements of $\hat{J}_x$ and $\hat{J}_y$ follow from:

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \tag{1.11}$$

$$\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-), \tag{1.12}$$

where, recall $\hat{J}_\pm := \hat{J}_x \pm i\hat{J}_y$. Thus, the matrix representations of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ in a $\{|jm\rangle\}$ basis are:

$$\hat{J}_x \leftrightarrow \frac{\hbar}{2}\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & \\ \hline 0 & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & 0 & & \\ \hline 0 & 0 & \begin{array}{ccc} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{array} & 0 & \\ \hline 0 & 0 & 0 & \begin{array}{cccc} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{array} & \\ \hline & & & & \ddots \end{array}\right), \tag{1.13}$$

$$\hat{J}_y \leftrightarrow \frac{\hbar}{2i}\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & \\ \hline 0 & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & 0 & & \\ \hline 0 & 0 & \begin{array}{ccc} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{array} & 0 & \\ \hline 0 & 0 & 0 & \begin{array}{cccc} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{array} & \\ \hline & & & & \ddots \end{array}\right), \tag{1.14}$$

$$\hat{J}_Z \leftrightarrow \frac{\hbar}{2}\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & \\ \hline 0 & \begin{matrix}1 & 0\\ 0 & -1\end{matrix} & 0 & & \\ \hline 0 & 0 & \begin{matrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2\end{matrix} & 0 & \\ \hline 0 & 0 & 0 & \begin{matrix}3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -3\end{matrix} & \\ \hline & & & & \ddots \end{array}\right). \tag{1.15}$$

Note the 'block-diagonal' form of $\hat{J}_x$ and $\hat{J}_y$. These blocks correspond to $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The matrix representation of $\hat{J}_z$ is diagonal with eigenvalues $0; \frac{1}{2}\hbar, -\frac{1}{2}\hbar; \hbar, 0, -\hbar; \frac{3}{2}\hbar, \frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar; \ldots$. It is normal practice to *reduce* these (infinite) matrices by breaking apart the blocks to give finite dimensional matrices. Thus, e.g.

$j = \frac{1}{2}$:

$$\hat{J}_x^{\left(\frac{1}{2}\right)} \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \hat{J}_y^{\left(\frac{1}{2}\right)} \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \hat{J}_z^{\left(\frac{1}{2}\right)} \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; \tag{1.16}$$

$j = 1$:

$$\begin{aligned}
\hat{J}_x^{(1)} &\leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{2} & 0\\ \sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 0\end{pmatrix},\\
\hat{J}_y^{(1)} &\leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -\sqrt{2}i & 0\\ \sqrt{2}i & 0 & -\sqrt{2}i\\ 0 & \sqrt{2}i & 0\end{pmatrix},\\
\hat{J}_z^{(1)} &\leftrightarrow \frac{\hbar}{2}\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2\end{pmatrix}.
\end{aligned} \tag{1.17}$$

The terminology:

$$\hat{J}_x^{\left(\frac{1}{2}\right)} \coloneqq \hat{S}_x, \qquad \hat{J}_y^{\left(\frac{1}{2}\right)} \coloneqq \hat{S}_y, \qquad \hat{J}_z^{\left(\frac{1}{2}\right)} \coloneqq \hat{S}_z, \tag{1.18}$$

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.19}$$

(cf. equation (1.16)), where σ_x, σ_y, σ_z are the *Pauli spin matrices*, is in common use.

1.3 The Pauli spin matrices

The Pauli spin matrices, σ_x, σ_y, σ_z, possess a number of useful properties. We redefine them by σ_j, σ_k, σ_l, $(j, k, l) = (x, y, z)$. Then

$$\sigma_j^2 = \sigma_k^2 = \sigma_l^2 = \hat{I}, \tag{1.20}$$

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 0, \qquad \text{for } j \neq k, \tag{1.21}$$

i.e.

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}\hat{I}, \tag{1.22}$$

where '{, }' is an anticommutator bracket (also written '$[,]_+$'). Further,

$$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l, \tag{1.23}$$

where

$$\varepsilon_{jkl} \equiv \varepsilon_{klj} \equiv \varepsilon_{ljk} \equiv 1; \qquad \varepsilon_{kjl} \equiv \varepsilon_{jlk} \equiv \varepsilon_{lkj} \equiv -1. \tag{1.24}$$

From equations (1.22) and (1.23)

$$\sigma_j\sigma_k = -\sigma_k\sigma_j = i\sigma_l. \tag{1.25}$$

Also,

$$\sigma_j^\dagger = \sigma_j, \tag{1.26}$$

$$\det(\sigma_j) = -1, \tag{1.27}$$

$$\mathrm{tr}(\sigma_j) = 0. \tag{1.28}$$

For the three-dimensional Cartesian vector $\vec{a}$, $\vec{\sigma} \cdot \vec{a}$ is a 2×2 matrix[2]:

$$\vec{\sigma} \cdot \vec{a} := \sigma_x a_x + \sigma_y a_y + \sigma_z a_z, \tag{1.29}$$

$$\therefore \vec{\sigma} \cdot \vec{a} = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}. \tag{1.30}$$

[2] The notation $\vec{\sigma}$, i.e. $(\sigma_x, \sigma_y, \sigma_z)$, viewed as a vector when the components are matrices, needs to be adopted as a powerful language. This will lead to the concept of *vector operators* in chapter 3; therein, the concept is formally developed. (See also Volume 1, equation (7.72).)

This leads to the important identity:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\hat{I} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}). \tag{1.31}$$

This can be obtained from equations (1.22) and (1.23):

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_j \sigma_j a_j \sum_k \sigma_k b_k. \tag{1.32}$$

$$\therefore (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_j\sum_k\left(\frac{1}{2}\{\sigma_j, \sigma_k\} + \frac{1}{2}[\sigma_j, \sigma_k]\right)a_j b_k, \tag{1.33}$$

$$\therefore (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_j\sum_k(\delta_{jk} + i\varepsilon_{jkl}\sigma_l)a_j b_k, \tag{1.34}$$

$$\therefore (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\hat{I} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}). \tag{1.35}$$

If the components of $\vec{a}$ are real then

$$(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2\hat{I}, \tag{1.36}$$

where $|\vec{a}|$ is the magnitude of the vector $\vec{a}$.

1.4 Matrix representations of rotations in ket space

We are now in a position to obtain matrix representations of rotation operators in ket space. For a rotation about an axis $\hat{n}$ through an angle ϕ (cf. Volume 1, chapter 10),

$$\mathcal{D}(R) = \mathcal{D}(\hat{n}, \phi) = \exp\left\{-i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi\right\}, \tag{1.37}$$

the matrix elements of $\mathcal{D}(\hat{n}, \phi)$ are

$$\langle jm'|\exp\left\{-i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi\right\}|jm\rangle \coloneqq \mathcal{D}^{(j)}_{m'm}(\hat{n}, \phi), \tag{1.38}$$

where $j' = j$ is explicitly incorporated: this is because

$$\hat{J}^2\mathcal{D}(R)|jm\rangle = \mathcal{D}(R)\hat{J}^2|jm\rangle, \tag{1.39}$$

$$\therefore \hat{J}^2\{\mathcal{D}(R)|jm\rangle\} = j(j+1)\hbar^2\{\mathcal{D}(R)|jm\rangle\}, \tag{1.40}$$

which follows from the general relationship $[\hat{A}, \exp\{\hat{A}\}] = 0$. This is sensible because rotations cannot change the length of a vector. Thus, the matrix representation of $\mathcal{D}(R)$ has the form:

$$\mathcal{D}(R) \leftrightarrow \left(\begin{array}{c|c|c|c|c} \mathcal{D}^{(0)} & 0 & 0 & 0 & \\ \hline 0 & \mathcal{D}^{\left(\frac{1}{2}\right)} & 0 & 0 & \\ \hline 0 & 0 & \mathcal{D}^{(1)} & 0 & \\ \hline 0 & 0 & 0 & \mathcal{D}^{\left(\frac{3}{2}\right)} & \\ \hline & & & & \ddots \end{array}\right) \tag{1.41}$$

and we can discuss the $\mathcal{D}^{(j)}$ individually.

The Euler angle parameterisation leads to a simplification when one considers a matrix representation:

$$\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle jm'|e^{\frac{-i\hat{J}_z\alpha}{\hbar}} \; e^{\frac{-i\hat{J}_y\beta}{\hbar}} \; e^{\frac{-i\hat{J}_z\gamma}{\hbar}}|jm\rangle; \tag{1.42}$$

but,

$$\langle jm'|e^{\frac{-i\hat{J}_z\alpha}{\hbar}} = \langle jm'|e^{-im'\alpha}, \tag{1.43}$$

$$\therefore \mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma) = e^{-i(m'\alpha+m\gamma)}\langle jm'|e^{\frac{-i\hat{J}_y\beta}{\hbar}}|jm\rangle, \tag{1.44}$$

i.e. only the '$\hat{J}_y$' rotation is non-trivial. We define

$$d^{(j)}_{m'm}(\beta) := \langle jm'|e^{\frac{-i\hat{J}_y\beta}{\hbar}}|jm\rangle. \tag{1.45}$$

The $\mathcal{D}^{(j)}_{m'm}(R)$'s ($R = \hat{n}, \phi$ or α, β, γ) are called *Wigner functions*. They tell us how much of $|jm\rangle$ rotates into $|jm'\rangle$ under the action of R:

$$\mathcal{D}(R)|jm\rangle = \sum_{m'}|jm'\rangle\langle jm'|\mathcal{D}(R)|jm\rangle, \tag{1.46}$$

where the completeness relation has been used.

We are now in a position to obtain explicit matrix representations of $\mathcal{D}(R)$, the so-called *Wigner matrices*:

$\mathcal{D}^{(0)}$: This is trivial. It is the 1×1 matrix (1).

$\mathcal{D}^{(\frac{1}{2})}$: This is a 2×2 matrix and can be evaluated from the properties of the Pauli spin matrices. Consider

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi) = \exp\left\{-i\frac{\vec{J}^{\left(\frac{1}{2}\right)} \cdot \hat{n}}{\hbar}\phi\right\} = \exp\left\{-i\frac{\vec{\sigma} \cdot \hat{n}\phi}{2}\right\}. \tag{1.47}$$

Then, expanding the exponential:

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi) = \hat{I} - i\frac{\phi}{2}\vec{\sigma} \cdot \hat{n} - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2(\vec{\sigma} \cdot \hat{n})^2 + \frac{i}{3!}\left(\frac{\phi}{2}\right)^3(\vec{\sigma} \cdot \hat{n})^3 + \cdots. \tag{1.48}$$

But, from equation (1.36),

$$(\vec{\sigma} \cdot \hat{n})^m = \hat{I}, \qquad m \text{ even}, \tag{1.49}$$

$$(\vec{\sigma} \cdot \hat{n})^m = (\vec{\sigma} \cdot \hat{n}), \qquad m \text{ odd}, \tag{1.50}$$

$$\therefore \mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi) = \hat{I}\left\{1 - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \cdots\right\} - i\vec{\sigma} \cdot \hat{n}\left\{\frac{\phi}{2} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \cdots\right\}, \tag{1.51}$$

$$\therefore \mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi) = \hat{I} \cos\frac{\phi}{2} - i\vec{\sigma} \cdot \hat{n} \sin\frac{\phi}{2}. \tag{1.52}$$

Explicitly,

$$\therefore \mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi) = \begin{pmatrix} \cos\frac{\phi}{2} - in_z \sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + in_z \sin\frac{\phi}{2} \end{pmatrix} \tag{1.53}$$

for an axis-angle parameterisation.
For an Euler angle parameterisation

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha, \beta, \gamma) = \mathcal{D}_z^{\left(\frac{1}{2}\right)}(\alpha)\mathcal{D}_y^{\left(\frac{1}{2}\right)}(\beta)\mathcal{D}_z^{\left(\frac{1}{2}\right)}(\gamma), \tag{1.54}$$

then using equation (1.52):

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\gamma}{2}} & 0 \\ 0 & e^{\frac{i\gamma}{2}} \end{pmatrix}, \tag{1.55}$$

$$\therefore \mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}} \cos\frac{\beta}{2} & -e^{-\frac{i(\alpha-\gamma)}{2}} \sin\frac{\beta}{2} \\ e^{\frac{i(\alpha-\gamma)}{2}} \sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}} \cos\frac{\beta}{2} \end{pmatrix}. \tag{1.56}$$

Note that $\mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, \phi)$ and $\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha, \beta, \gamma)$ fulfil the unitary unimodular or special unitary form $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, cf. Volume 1, equation (10.42), and herein section 5.10.2.

$\mathcal{D}^{(1)}$: This is a 3×3 matrix. It can be evaluated using a series expansion if we use its Euler angle parameterisation. From

$$\mathcal{D}^{(1)}_{m'm}(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} d^{(1)}_{m'm}(\beta), \tag{1.57}$$

expanding the exponential in $d^{(1)}$:

$$e^{-\frac{i\hat{J}_y^{(1)}\beta}{\hbar}} = \hat{I} - \frac{i\beta}{\hbar}\hat{J}_y^{(1)} - \frac{1}{2!}\frac{\beta^2}{\hbar^2}\left(\hat{J}_y^{(1)}\right)^2 + \frac{i}{3!}\frac{\beta^3}{\hbar^3}\left(\hat{J}_y^{(1)}\right)^3 + \cdots. \tag{1.58}$$

This is greatly simplified by the following identity

$$\begin{aligned}\frac{\left(\hat{J}_y^{(1)}\right)^3}{\hbar^3} = \frac{1}{8}&\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} \\ &\times\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix},\end{aligned} \tag{1.59}$$

$$\therefore \frac{\left(\hat{J}_y^{(1)}\right)^3}{\hbar^3} = \frac{1}{8}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \tag{1.60}$$

$$\therefore \frac{\left(\hat{J}_y^{(1)}\right)^3}{\hbar^3} = \frac{1}{8}\begin{pmatrix} 0 & -4\sqrt{2}i & 0 \\ 4\sqrt{2}i & 0 & -4\sqrt{2}i \\ 0 & 4\sqrt{2}i & 0 \end{pmatrix}, \tag{1.61}$$

$$\therefore \frac{\left(\hat{J}_y^{(1)}\right)^3}{\hbar^3} = \frac{\hat{J}_y^{(1)}}{\hbar}. \tag{1.62}$$

Then, equation (1.58) reduces to

$$e^{-\frac{i\hat{J}_y^{(1)}\beta}{\hbar}} = \hat{I} + \frac{\hat{J}_y^{(1)}}{\hbar}\left\{-i\beta + \frac{i\beta^3}{3!} + \cdots\right\} + \frac{\left(\hat{J}_y^{(1)}\right)^2}{\hbar^2}\left\{\frac{-\beta^2}{2!} + \cdots\right\}, \tag{1.63}$$

$$\therefore e^{-\frac{i\hat{J}_y^{(1)}\beta}{\hbar}} = \hat{I} - \frac{i\hat{J}_y^{(1)}}{\hbar}\sin\beta + \frac{\left(\hat{J}_y^{(1)}\right)^2}{\hbar^2}(\cos\beta - 1). \tag{1.64}$$

Thus,

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & \frac{-1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & \frac{-1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}. \tag{1.65}$$

To evaluate $\mathcal{D}^{(1)}(\hat{n}, \phi)$ and $\mathcal{D}^{(j)}(\hat{n}, \phi)$ or $\mathcal{D}^{(j)}(\alpha, \beta, \gamma)$ with $j > 1$, we must develop the theory of tensor bases of representation in ket space.

1.5 Tensor representations for $SU(2)$

Consider the general $SU(2)$ transformation (cf. Volume 1, chapter 10)

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}, \tag{1.66}$$

where the 2×2 matrix may, for example, have the form given by equation (1.53) or equation (1.56). Then, defining

$$q_1 := u_1^2, \qquad q_2 := \sqrt{2}\, u_1 u_2, \qquad q_3 := u_3^2, \tag{1.67}$$

under the transformation, equation (1.66), we obtain

$$q_1' = (u_1')^2 = (au_1 + bu_2)^2 = a^2u_1^2 + 2abu_1u_2 + b^2u_2^2, \tag{1.68}$$

$$\therefore\ q_1' = a^2q_1 + \sqrt{2}\, abq_2 + b^2q_3; \tag{1.69}$$

and similarly,

$$q_2' = -\sqrt{2}\, ab^*q_1 + (aa^* - bb^*)q_2 + \sqrt{2}\, ba^*q_3, \tag{1.70}$$

$$q_3' = (b^*)^2q_1 - \sqrt{2}\, a^*b^*q_2 + (a^*)^2q_3; \tag{1.71}$$

whence

$$\begin{pmatrix} a^2 & \sqrt{2}\, ab & b^2 \\ -\sqrt{2}\, ab^* & (aa^* - bb^*) & \sqrt{2}\, ba^* \\ (b^*)^2 & -\sqrt{2}\, a^*b^* & (a^*)^2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix}. \tag{1.72}$$

This is still a representation of an $SU(2)$ transformation: there are no new parameters. However, it is a 3×3 matrix representation of $SU(2)$. From the Euler angle parameterisation, equation (1.56),

$$a = \exp\left\{-\frac{i(\alpha + \gamma)}{2}\right\}\cos\frac{\beta}{2}, \qquad b = -\exp\left\{-\frac{i(\alpha - \gamma)}{2}\right\}\sin\frac{\beta}{2}; \tag{1.73}$$

and substitution of these values of a and b into the matrix in equation (1.72) will yield $\mathcal{D}^{(1)}(\alpha, \beta, \gamma)$, the β-dependent part of which is given by equation (1.65).

The process can be iterated by defining

$$p_1 := u_1^3, \qquad p_2 := \sqrt{3}\, u_1^2u_2, \qquad p_3 := \sqrt{3}\, u_1u_2^2, \qquad p_4 := u_2^3, \tag{1.74}$$

which will yield a 4×4 matrix representation of $SU(2)$, i.e. an expression for $\mathcal{D}^{(\frac{3}{2})}(R)$.

Expressions for $\mathcal{D}^{(j)}(\alpha, \beta, \gamma)$ can be obtained by this process, for any j, together with the values of a and b given in equation (1.73). Likewise, $\mathcal{D}^{(j)}(\hat{n}, \phi)$ can be obtained using (cf. equation (1.53))

$$a = \cos\frac{\phi}{2} - in_z \sin\frac{\phi}{2}, \qquad b = (-in_x - n_y)\sin\frac{\phi}{2}, \tag{1.75}$$

where, recall, the constraint $n_x^2 + n_y^2 + n_z^2 = 1$ ensures that (n_x, n_y, n_z, ϕ) corresponds to three free parameters. To reiterate: $SU(2)$ is a three-parameter group.

The representation associated with the two-component spinor (u_1, u_2) is called the *fundamental representation*. The representation associated with the three-component entity (q_1, q_2, q_3) is a *rank-2 $SU(2)$ tensor*, i.e. it is constituted from quadratic combinations of the fundamental representation. In turn, (p_1, p_2, p_3, p_4) is a rank-3 $SU(2)$ tensor.

1.6 Tensor representations for *SO*(3)

Consider the general $SO(3)$ transformation of the vector $\vec{V}$

$$V_{i'} = \sum_i R_{i'i} V_i, \tag{1.76}$$

where the V_i are the Cartesian components of the vector and the $R_{i'i}$ are the elements of the 3×3 matrix R that effects the orthogonal transformation. This can be generalised to

$$T_{i'j'k'\cdots} = \sum_i \sum_j \sum_{k\cdots} R_{i'i} R_{j'j} R_{k'k} \cdots T_{ijk\cdots}, \tag{1.77}$$

where the $T_{ijk\cdots}$ are the Cartesian components of a tensor, the rank of which is equal to the number of indices and the $R_{i'i}$ are, as before, elements of the 3×3 matrix R. Details are clarified in the following.

The simplest Cartesian tensor is of rank 2 and is often called a *dyad* or *dyadic*. It is formed from two Cartesian vectors, e.g.

$$\vec{U} = (U_1, U_2, U_3), \qquad \vec{V} = (V_1, V_2, V_3), \tag{1.78}$$

$$T_{ij} = U_i V_j, \tag{1.79}$$

of which there are nine components. Thus,

$$T_{i'j'} = \sum_i \sum_j R_{i'i} R_{j'j} T_{ij}, \tag{1.80}$$

where the T_{ij} are the nine components of the dyadic and the $R_{i'i}R_{j'j}$ are the elements of the 9×9 matrix that effects the underlying transformation. We note that there are still only three parameters involved in describing this ($SO(3)$) transformation.

The nine components of **T** are *reducible*, i.e. they can be expressed as linear combinations that form subsets which transform among themselves under rotations. They are:

$$U_1V_1 + U_2V_2 + U_3V_3 = \vec{U} \cdot \vec{V} := T, \tag{1.81}$$

$$\frac{1}{2}(U_2V_3 - U_3V_2) := A_1, \tag{1.82}$$

$$\frac{1}{2}(U_3V_1 - U_1V_3) := A_2, \tag{1.83}$$

$$\frac{1}{2}(U_1V_2 - U_2V_1) := A_3, \tag{1.84}$$

i.e.

$$\frac{1}{2}(U_iV_j - U_jV_i) := A_k\varepsilon_{ijk}. \tag{1.85}$$

Further,

$$\frac{1}{2}(U_1V_2 + U_2V_1) := S_{12}, \tag{1.86}$$

$$\frac{1}{2}(U_2V_3 + U_3V_2) := S_{23}, \tag{1.87}$$

$$\frac{1}{2}(U_3V_1 + U_1V_3) := S_{31}, \tag{1.88}$$

$$U_1V_1 - \frac{1}{3}T := S_{11}, \tag{1.89}$$

$$U_2V_2 - \frac{1}{3}T := S_{22}, \tag{1.90}$$

i.e.

$$\frac{1}{2}(U_iV_j + U_jV_i) - \frac{1}{3}T\delta_{ij} = S_{ij}. \tag{1.91}$$

The combination $U_3V_3 - \frac{1}{3}T$ is excluded because

$$U_3V_3 - \frac{1}{3}T = -\left\{\left(U_1V_1 - \frac{1}{3}T\right) + \left(U_2V_2 - \frac{1}{3}T\right)\right\}. \tag{1.92}$$

T is a scalar product which is invariant under rotations. The A_k are the three independent components of an antisymmetric tensor: by antisymmetric we mean that they change sign under exchange of the indices. The S_{ij} are the five independent components of a traceless second-rank tensor. The trace, $\sum_{i=1}^{3} S_{ii}$ is evidently zero from equation (1.92). Note that

$$U_i V_j = \frac{\vec{U} \cdot \vec{V}}{3}\delta_{ij} + \frac{(U_i V_j - U_j V_i)}{2} + \left(\frac{U_i V_j + U_j V_i}{2} - \frac{\vec{U} \cdot \vec{V}}{3}\delta_{ij}\right), \tag{1.93}$$

i.e.

$$U_i V_j = \frac{T}{3}\delta_{ij} + A_k \varepsilon_{ijk} + \delta_{ij}. \tag{1.94}$$

Consider a rotation around the z-axis through an angle γ:

$$\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} U_1' \\ U_2' \\ U_3' \end{pmatrix}, \tag{1.95}$$

whence

$$U_1' = U_1 \cos\gamma - U_2 \sin\gamma, \qquad U_2' = U_1 \sin\gamma + U_2 \cos\gamma, \qquad U_3' = U_3; \tag{1.96}$$

and similarly

$$V_1' = V_1 \cos\gamma - V_2 \sin\gamma, \qquad V_2' = V_1 \sin\gamma + V_2 \cos\gamma, \qquad V_3' = V_3. \tag{1.97}$$

Then for the A_k,

$$\begin{aligned} A_1' &= \frac{1}{2}(U_2' V_3' - U_3' V_2') \\ &= \frac{1}{2}\{(U_1 \sin\gamma + U_2 \cos\gamma)V_3 - U_3(V_1 \sin\gamma + V_2 \cos\gamma)\} \\ &= A_1 \cos\gamma - A_2 \sin\gamma; \end{aligned} \tag{1.98}$$

similarly

$$A_2' = A_1 \sin\gamma + A_2 \cos\gamma \tag{1.99}$$

and

$$A_3' = A_3, \tag{1.100}$$

i.e.

$$\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix}. \tag{1.101}$$

Evidently, we can write

$$\vec{A} = \frac{1}{2}\vec{U} \times \vec{V}. \tag{1.102}$$

Further, for the S_{ij},

$$\begin{aligned} S'_{12} &= \frac{1}{2}\{(U_1 \cos\gamma - U_2 \sin\gamma)(V_1 \sin\gamma + V_2 \cos\gamma) \\ &\quad + (U_1 \sin\gamma + U_2 \cos\gamma)(V_1 \cos\gamma - V_2 \sin\gamma)\} \\ &= (U_1V_1 - U_2V_2)\sin\gamma\cos\gamma + \frac{1}{2}(U_1V_2 + U_2V_1)(\cos^2\gamma - \sin^2\gamma) \\ &= (S_{11} - S_{22})\sin\gamma\cos\gamma + S_{12}(\cos^2\gamma - \sin^2\gamma) \\ &= \frac{1}{2}(S_{11} - S_{22})\sin 2\gamma + S_{12}\cos 2\gamma; \end{aligned} \tag{1.103}$$

similarly,

$$S'_{23} = S_{23}\cos\gamma + S_{31}\sin\gamma, \tag{1.104}$$

$$S'_{31} = -S_{23}\sin\gamma + S_{31}\cos\gamma, \tag{1.105}$$

$$S'_{11} = -S_{12}\sin 2\gamma + \frac{1}{2}S_{11}(1 + \cos 2\gamma) + \frac{1}{2}S_{22}(1 - \cos 2\gamma), \tag{1.106}$$

and

$$S'_{22} = S_{12}\sin 2\gamma + \frac{1}{2}S_{11}(1 - \cos 2\gamma) + \frac{1}{2}S_{22}(1 + \cos 2\gamma), \tag{1.107}$$

i.e.

$$\begin{pmatrix} \cos 2\gamma & 0 & 0 & \frac{1}{2}\sin 2\gamma & -\frac{1}{2}\sin 2\gamma \\ 0 & \cos\gamma & \sin\gamma & 0 & 0 \\ 0 & -\sin\gamma & \cos\gamma & 0 & 0 \\ -\sin 2\gamma & 0 & 0 & \frac{1}{2}(1 + \cos 2\gamma) & \frac{1}{2}(1 - \cos 2\gamma) \\ \sin 2\gamma & 0 & 0 & \frac{1}{2}(1 - \cos 2\gamma) & \frac{1}{2}(1 + \cos 2\gamma) \end{pmatrix} \begin{pmatrix} S_{12} \\ S_{23} \\ S_{31} \\ S_{11} \\ S_{22} \end{pmatrix} = \begin{pmatrix} S'_{12} \\ S'_{23} \\ S'_{31} \\ S'_{11} \\ S'_{22} \end{pmatrix}. \tag{1.108}$$

Similar equations can be obtained for $\{S_{12}, S_{23}, S_{31}, S_{11}, S_{22}\}$ for rotations about the x and y axes.

The $\{A_k\}$, T, and the $\{S_{ij}\}$ transform separately under rotations. Dyadics are said to possess a *reducible* structure with respect to rotations.

1.7 The Schwinger representations for $SU(2)$

Representations of $SU(2)$ can be constructed using a method due to Schwinger. Consider

$$\left| j = \frac{1}{2}, m = \frac{1}{2} \right\rangle := a_+^\dagger|0\rangle, \qquad \left| j = \frac{1}{2}, m = -\frac{1}{2} \right\rangle := a_-^\dagger|0\rangle; \tag{1.109}$$

i.e. $a_+^\dagger$ creates a state (particle in a state) of spin-$\frac{1}{2}$ up and $a_-^\dagger$ creates a state of spin-$\frac{1}{2}$ down by action on the 'vacuum' $|0\rangle$ (which has no particles), where

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad \{i, j\} = \{+, -\}. \tag{1.110}$$

Then, defining

$$\hat{J}_+ := a_+^\dagger a_-, \qquad \hat{J}_- := a_-^\dagger a_+, \qquad \hat{J}_0 := \frac{(a_+^\dagger a_+ - a_-^\dagger a_-)}{2}, \tag{1.111}$$

it follows that

$$[\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \tag{1.112}$$

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \tag{1.113}$$

which define the structure developed for angular momentum and spin (here, $\hbar \equiv 1$).

Although the elementary building blocks in the Schwinger representation have spin-$\frac{1}{2}$, they should not be regarded as fermions. These spin-$\frac{1}{2}$ 'objects' are designed to be combined to produce any desired value of total spin: the number of spin-$\frac{1}{2}$'s needed to produce a total spin of j will be $2j$. These building blocks can be regarded as bosons. They can be visualised in terms of a two-dimensional harmonic oscillator:

$$[a_+, a_+^\dagger] = 1, \qquad [a_-, a_-^\dagger] = 1, \tag{1.114}$$

$$\hat{N}_+ = a_+^\dagger a_+, \qquad \hat{N}_- = a_-^\dagger a_-, \tag{1.115}$$

$$|n_+\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}|0\rangle, \qquad |n_-\rangle = \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}|0\rangle, \tag{1.116}$$

$$\hat{N}_+|n_+\rangle = n_+|n_+\rangle, \qquad \hat{N}_-|n_-\rangle = n_-|n_-\rangle, \tag{1.117}$$

$$a_+^\dagger|n_+\rangle = \sqrt{n_+ + 1}\,|n_+ + 1\rangle, \qquad a_-^\dagger|n_-\rangle = \sqrt{n_- + 1}\,|n_- + 1\rangle, \tag{1.118}$$

$$a_+|n_+\rangle = \sqrt{n_+}\,|n_+ - 1\rangle, \qquad a_-|n_-\rangle = \sqrt{n_-}\,|n_- - 1\rangle, \tag{1.119}$$

$$a_+|0\rangle = 0, \qquad a_-|0\rangle = 0. \tag{1.120}$$

The states $|n_+\rangle$, $|n_-\rangle$ can be written in the combined form $|n_+ n_-\rangle$ which, from

$$[a_-, a_+^\dagger] = [a_-, a_+] = [a_-^\dagger, a_+^\dagger] = [a_-^\dagger, a_+] = 0, \tag{1.121}$$

obey

$$|n_+n_-\rangle = \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!n_-!}}|00\rangle, \tag{1.122}$$

$$\hat{N}_+|n_+n_-\rangle = n_+|n_+n_-\rangle, \qquad \hat{N}_-|n_+n_-\rangle = n_-|n_+n_-\rangle, \tag{1.123}$$

$$a_+^\dagger|n_+n_-\rangle = \sqrt{n_+ + 1}\,|n_+ + 1, n_-\rangle, \qquad a_-^\dagger|n_+n_-\rangle = \sqrt{n_- + 1}\,|n_+, n_- + 1\rangle, \tag{1.124}$$

$$a_+|n_+n_-\rangle = \sqrt{n_+}\,|n_+ - 1, n_-\rangle, \qquad a_-|n_+n_-\rangle = \sqrt{n_-}\,|n_+, n_- - 1\rangle, \tag{1.125}$$

$$a_+|00\rangle = 0, \qquad a_-|00\rangle = 0. \tag{1.126}$$

Then, from equations (1.111) and (1.115), i.e.

$$\hat{J}_+ = a_+^\dagger a_-, \qquad \hat{J}_- = a_-^\dagger a_+, \qquad \hat{J}_0 = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{1}{2}(\hat{N}_+ - \hat{N}_-), \tag{1.127}$$

together with

$$\hat{N} := \hat{N}_+ + \hat{N}_- = a_+^\dagger a_+ + a_-^\dagger a_- \tag{1.128}$$

and

$$\hat{J}^2 := \hat{J}_0^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+), \tag{1.129}$$

we obtain:

$$\hat{J}_+|n_+n_-\rangle = \sqrt{n_-(n_+ + 1)}\,|n_+ + 1, n_- - 1\rangle, \tag{1.130}$$

$$\hat{J}_-|n_+n_-\rangle = \sqrt{n_+(n_- + 1)}\,|n_+ - 1, n_- + 1\rangle, \tag{1.131}$$

$$\hat{J}_0|n_+n_-\rangle = \frac{1}{2}(n_+ - n_-)|n_+n_-\rangle, \tag{1.132}$$

$$\hat{N}|n_+n_-\rangle = (n_+ + n_-)|n_+n_-\rangle, \tag{1.133}$$

and

$$\begin{aligned}
\hat{J}^2|n_+n_-\rangle &= \hat{J}_0^2|n_+n_-\rangle + \frac{1}{2}\hat{J}_+\hat{J}_-|n_+n_-\rangle + \frac{1}{2}\hat{J}_-\hat{J}_+|n_+n_-\rangle \\
&= \frac{1}{4}(n_+ - n_-)^2|n_+n_-\rangle + \frac{1}{2}n_+(n_- + 1)|n_+n_-\rangle \\
&\quad + \frac{1}{2}n_-(n_+ + 1)|n_+n_-\rangle \\
&= \left(\frac{n_+ + n_-}{2}\right)\left(\frac{n_+ + n_-}{2} + 1\right)|n_+n_-\rangle \\
&= \frac{n}{2}\left(\frac{n}{2} + 1\right)|n_+n_-\rangle,
\end{aligned} \tag{1.134}$$

where

$$n = n_+ + n_-. \tag{1.135}$$

Evidently, by making the associations

$$n_+ \leftrightarrow j + m, \qquad n_- \leftrightarrow j - m, \tag{1.136}$$

we obtain

$$n = 2j; \tag{1.137}$$

and from equations (1.130)–(1.132) and (1.134)

$$\hat{J}_+|n_+n_-\rangle = \sqrt{(j-m)(j+m+1)}\,|n_+ + 1, n_- - 1\rangle, \tag{1.138}$$

$$\hat{J}_-|n_+n_-\rangle = \sqrt{(j+m)(j-m+1)}\,|n_+ - 1, n_- + 1\rangle, \tag{1.139}$$

$$\hat{J}_0|n_+n_-\rangle = m|n_+n_-\rangle, \tag{1.140}$$

$$\hat{J}^2|n_+n_-\rangle = j(j+1)|n_+n_-\rangle, \tag{1.141}$$

respectively. Thus, by comparison with

$$\hat{J}_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\,|j\ m+1\rangle, \tag{1.142}$$

$$\hat{J}_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\,|j\ m-1\rangle, \tag{1.143}$$

$$\hat{J}_0|jm\rangle = m|jm\rangle, \tag{1.144}$$

$$\hat{J}^2|jm\rangle = j(j+1)|jm\rangle, \tag{1.145}$$

we can assert that

$$|n_+n_-\rangle \equiv |jm\rangle, \tag{1.146}$$

and from equation (1.122)

$$|jm\rangle \equiv \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|00\rangle. \tag{1.147}$$

Two special cases of note are: $m = +j$, i.e.

$$|jj\rangle \equiv \frac{(a_+^\dagger)^{2j}}{\sqrt{(2j)!}}|00\rangle; \tag{1.148}$$

and $m = -j$, i.e.

$$|j, -j\rangle \equiv \frac{(a_-^\dagger)^{2j}}{\sqrt{(2j)!}}|00\rangle. \tag{1.149}$$

Consider then the rotation of the states $|j = \frac{1}{2}, m = \frac{1}{2}\rangle \equiv |\frac{1}{2}, \frac{1}{2}\rangle$ and $|j = \frac{1}{2}, m = -\frac{1}{2}\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$:

$$\mathcal{D}_y(\beta)\left|\frac{1}{2}, \frac{1}{2}\right\rangle \leftrightarrow \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix}, \tag{1.150}$$

$$\mathcal{D}_y(\beta)\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \leftrightarrow \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{pmatrix}; \tag{1.151}$$

i.e.

$$\mathcal{D}_y(\beta)\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \cos\frac{\beta}{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sin\frac{\beta}{2}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \tag{1.152}$$

$$\mathcal{D}_y(\beta)\left(\frac{1}{2}, -\frac{1}{2}\right) = -\sin\frac{\beta}{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \cos\frac{\beta}{2}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle. \tag{1.153}$$

Then from

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = a_+^\dagger|0\rangle, \qquad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = a_-^\dagger|0\rangle, \tag{1.154}$$

we have

$$\mathcal{D}_y(\beta)\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \mathcal{D}_y(\beta)a_+^\dagger\mathcal{D}_y^{-1}(\beta)\mathcal{D}_y(\beta)|0\rangle, \tag{1.155}$$

$$\mathcal{D}_y(\beta)\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \mathcal{D}_y(\beta)a_-^\dagger\mathcal{D}_y^{-1}(\beta)\mathcal{D}_y(\beta)|0\rangle; \tag{1.156}$$

whence

$$\mathcal{D}_y(\beta)a_+^\dagger\mathcal{D}_y^{-1}(\beta) \equiv a_+^{\dagger'} = \cos\frac{\beta}{2}a_+^\dagger + \sin\frac{\beta}{2}a_-^\dagger, \tag{1.157}$$

$$\mathcal{D}_y(\beta)a_-^\dagger\mathcal{D}_y^{-1}(\beta) \equiv a_-^{\dagger'} = -\sin\frac{\beta}{2}a_+^\dagger + \cos\frac{\beta}{2}a_-^\dagger. \tag{1.158}$$

Thus,

$$\mathcal{D}_y(\beta)|jm\rangle := \frac{(a_+^{\dagger\prime})^{j+m}(a_-^{\dagger\prime})^{j-m}}{\sqrt{(j+m)!(j-m)!}}|00\rangle. \tag{1.159}$$

$$\therefore \mathcal{D}_y(\beta)|jm\rangle = \frac{\left(\cos\frac{\beta}{2}a_+^\dagger + \sin\frac{\beta}{2}a_-^\dagger\right)^{j+m}\left(-\sin\frac{\beta}{2}a_+^\dagger + \cos\frac{\beta}{2}a_-^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|00\rangle. \tag{1.160}$$

The right-hand side of equation (1.160) can be expanded using the binomial theorem:

$$\begin{aligned}\mathcal{D}_y(\beta)|jm\rangle = {}& \frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_l \frac{(j+m)!}{l!(j+m-l)!}\left(a_+^\dagger\cos\frac{\beta}{2}\right)^l\left(a_-^\dagger\sin\frac{\beta}{2}\right)^{j+m-l} \\ & \times\sum_k \frac{(j-m)!}{k!(j-m-k)!}(-a_+^\dagger\sin\frac{\beta}{2})^k\left(a_-^\dagger\cos\frac{\beta}{2}\right)^{j-m-k}|00\rangle.\end{aligned} \tag{1.161}$$

$$\begin{aligned}\therefore \mathcal{D}_y(\beta)|jm\rangle = {}& \sqrt{(j+m)!(j-m)!}\sum_{l,k}(-1)^k\frac{\left(\cos\frac{\beta}{2}\right)^{j-m+l-k}\left(\sin\frac{\beta}{2}\right)^{j+m-l+k}}{l!(j+m-l)!k!(j-m-k)!} \\ & \times(a_+^\dagger)^{l+k}(a_-^\dagger)^{2j-l-k}|00\rangle,\end{aligned} \tag{1.162}$$

and comparing with

$$\mathcal{D}_y(\beta)|jm\rangle = \sum_{m'}|jm'\rangle d^{(j)}_{m'm}(\beta), \tag{1.163}$$

i.e.

$$\mathcal{D}_y(\beta)|jm\rangle = \sum_{m'}d^{(j)}_{m'm}(\beta)\frac{(a_+^\dagger)^{j+m'}(a_-^\dagger)^{j-m'}}{\sqrt{(j+m')!(j-m')!}}|00\rangle, \tag{1.164}$$

equating coefficients of powers of $a_+^\dagger$ in equations (1.162) and (1.164),

$$l + k = j + m'. \tag{1.165}$$

Then, for a particular choice of m',

$$l = j + m' - k \tag{1.166}$$

and

$$\begin{aligned}d^{(j)}_{m'm}(\beta) = {}& \sum_{\substack{k \\ \text{(no negative factorials)}}}(-1)^k\frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m'-k)!(m-m'+k)!k!(j-m-k)!} \\ & \times\left(\cos\frac{\beta}{2}\right)^{2j-2k+m'-m}\times\left(\sin\frac{\beta}{2}\right)^{2k+m-m'}.\end{aligned} \tag{1.167}$$

1.8 A spinor function basis for $SU(2)$

The Schwinger representation and its associated basis leads directly to a *spinor function basis* for $SU(2)$:

$$\left| j = \frac{1}{2}, m = \frac{1}{2} \right\rangle \leftrightarrow u, \qquad \left| j = \frac{1}{2}, m = -\frac{1}{2} \right\rangle \leftrightarrow v, \tag{1.168}$$

where u and v are independent functions. We require u and v to satisfy (again, $\hbar \equiv 1$)

$$\hat{J}_0 u = \frac{1}{2}u, \; \hat{J}_0 v = -\frac{1}{2}v, \tag{1.169}$$

$$\hat{J}_+ u = 0, \quad \hat{J}_+ v = u, \tag{1.170}$$

$$\hat{J}_- u = v, \quad \hat{J}_- v = 0. \tag{1.171}$$

Thus, we deduce the *realisation*

$$\hat{J}_0 \leftrightarrow \frac{1}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \right), \tag{1.172}$$

$$\hat{J}_+ \leftrightarrow u\frac{\partial}{\partial v}, \tag{1.173}$$

$$\hat{J}_- \leftrightarrow v\frac{\partial}{\partial u}. \tag{1.174}$$

It follows from equations (1.148), (1.149) and (1.147) that

$$|jj\rangle := \frac{u^{2j}}{\sqrt{(2j)!}}, \qquad |j, -j\rangle := \frac{v^{2j}}{\sqrt{(2j)!}}, \tag{1.175}$$

and

$$|jm\rangle := \frac{u^{j+m}v^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \tag{1.176}$$

It should be noted that $\{u, v\}$ are elements of a complex function space which is formally developed in section 1.17 under the title of the *Bargmann representation*, i.e. this function space is known as *Bargmann space*. These bases are *irreducible* (unlike Cartesian tensors).

1.9 A spherical harmonic basis for $SO(3)$

The use of spinor functions as a basis for $SU(2)$ and the relations for $\hat{J}_0$, $\hat{J}_\pm$ given in equations (1.172)–(1.174) lead to the consideration of a functional representation of the $|lm\rangle$ for ($\hbar \equiv 1$)

$$\hat{L}_0 = \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \leftrightarrow -ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x}; \tag{1.177}$$

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y + i\hat{z}\hat{p}_x - i\hat{x}\hat{p}_z, \tag{1.178}$$

$$\therefore \hat{L}_+ \leftrightarrow -iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}; \tag{1.179}$$

$$\hat{L}_- \leftrightarrow -iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} - z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}; \tag{1.180}$$

where the postion representation has been used. Evidently, $\hat{L}_0$, $\hat{L}_\pm$ in the form given by equations (1.177), (1.179) and (1.180) leave the degree of a polynomial in x, y and z unchanged. Therefore, we consider the space of homogeneous polynomials in x, y and z, i.e.

$$f(x, y, z) = (ax + by + cz)^l, \tag{1.181}$$

where a, b, and c are complex numbers.

We start with the homogeneous polynomials $\phi_{lm=-l}(\vec{r})$, $\vec{r} := (x, y, z)$, that satisfy the so-called '*lowest weight*' conditions:

$$\hat{L}_0\phi_{l,-l}(\vec{r}) = -l\phi_{l,-l}(\vec{r}) \tag{1.182}$$

and

$$\hat{L}_-\phi_{l,-l}(\vec{r}) = 0. \tag{1.183}$$

Then, consider

$$\begin{aligned}\hat{L}_0(ax + by + cz)^l &= \left(-ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x}\right)(ax + by + cz)^l \\ &= -ixl(ax + by + cz)^{l-1}b + iyl(ax + by + cz)^{l-1}a \\ &= l(ax + by + cz)^{l-1}(-ibx + iay),\end{aligned} \tag{1.184}$$

and the right-hand side fulfils equation (1.182), i.e.

$$\hat{L}_0(ax + by + cz)^l = -l(ax + by + cz)^l, \tag{1.185}$$

provided $a = 1$, $b = -i$, $c = 0$. Thus,

$$\phi_{l,-l}(\vec{r}) = (x - iy)^l. \tag{1.186}$$

Evidently,

$$\begin{aligned}\hat{L}_-\phi_{l,-l}(\vec{r}) &= \left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} - z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)(x - iy)^l \\ &= izl(-i)(x - iy)^{l-1} - zl(x - iy)^{l-1} \\ &= 0.\end{aligned} \tag{1.187}$$

We can construct the $\phi_{lm}(\vec{r})$ using ($\hbar \equiv 1$)

$$\hat{L}_{+}\phi_{lm}(\vec{r}) = \sqrt{(l-m)(l+m+1)}\,\phi_{l,m+1}(\vec{r}). \tag{1.188}$$

For $l = 1$: from

$$\phi_{1,-1}(\vec{r}) = x - iy, \tag{1.189}$$

$$\begin{aligned}\hat{L}_{+}\phi_{1,-1}(\vec{r}) &= \left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)(x - iy)\\ &= iz(-i) + z\\ &= 2z\\ &:= \sqrt{2}\,\phi_{1,0}(\vec{r});\end{aligned} \tag{1.190}$$

$$\therefore\ \phi_{1,0}(\vec{r}) = \sqrt{2}\,z. \tag{1.191}$$

Then,

$$\begin{aligned}\hat{L}_{+}\phi_{1,0}(\vec{r}) &= \sqrt{2}\left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)z\\ &= \sqrt{2}(-iy - x)\\ &= -\sqrt{2}(x + iy)\\ &:= \sqrt{2}\,\phi_{1,1}(\vec{r});\end{aligned} \tag{1.192}$$

$$\therefore\ \phi_{1,1}(\vec{r}) = -(x + iy). \tag{1.193}$$

For $l = 2$: from

$$\phi_{2,-2}(\vec{r}) = (x - iy)^2, \tag{1.194}$$

$$\begin{aligned}\hat{L}_{+}\phi_{2,-2}(\vec{r}) &= \left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)(x - iy)^2\\ &= iz2(x - iy)(-i) + z2(x - iy)\\ &= 4z(x - iy)\\ &:= 2\phi_{2,-1}(\vec{r});\end{aligned} \tag{1.195}$$

$$\therefore\ \phi_{2,-1}(\vec{r}) = 2z(x - iy). \tag{1.196}$$

Then,

$$\begin{aligned}
\hat{L}_+\phi_{2,-1}(\vec{r}) &= \left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)2z(x - iy) \\
&= -iy2(x - iy) + iz2z(-i) + z2z - x2(x - iy) \\
&= -2(x - iy)(x + iy) + 4z^2 \\
&= -2(x^2 + y^2) + 4z^2 \\
&:= \sqrt{6}\,\phi_{2,0}(\vec{r});
\end{aligned} \tag{1.197}$$

$$\therefore \phi_{2,0}(\vec{r}) = \sqrt{\frac{2}{3}}(-x^2 - y^2 + 2z^2). \tag{1.198}$$

Similarly,

$$\phi_{2,1}(\vec{r}) = -2z(x + iy), \tag{1.199}$$

$$\phi_{2,2}(\vec{r}) = (x + iy)^2. \tag{1.200}$$

The functions $\phi_{lm}(\vec{r})$ are proportional to the spherical harmonics, $Y_{lm}(\theta, \phi)$ (see table 1.1). This follows from the relationship between Cartesian coordinates and spherical polar coordinates:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{1.201}$$

whence

$$\begin{aligned}
\phi_{1,\pm1} &= \mp(x \pm iy) \\
&= \mp r\sin\theta e^{\pm i\phi} \\
&= r\sqrt{\frac{8\pi}{3}}\,Y_{1,\pm1}(\theta, \phi).
\end{aligned} \tag{1.202}$$

Similarly,

$$\begin{aligned}
\phi_{1,0}(\vec{r}) &= \sqrt{2}\,r\cos\theta \\
&= r\sqrt{\frac{8\pi}{3}}\,Y_{1,0}(\theta, \phi).
\end{aligned} \tag{1.203}$$

$$\begin{aligned}
\phi_{2,\pm2}(\vec{r}) &= r^2\sin^2\theta e^{\pm 2i\phi} \\
&= r^2\sqrt{\frac{32\pi}{15}}\,Y_{2,\pm2}(\theta, \phi).
\end{aligned} \tag{1.204}$$

$$\begin{aligned}
\phi_{2,\pm1}(\vec{r}) &= \mp 2r^2\cos\theta\sin\theta e^{\pm i\phi} \\
&= r^2\sqrt{\frac{32\pi}{15}}\,Y_{2,\pm1}(\theta, \phi).
\end{aligned} \tag{1.205}$$

Table 1.1. The spherical harmonics, $Y_{lm}(\theta, \phi)$, $m = l, l-1, l-2, \ldots, 1, 0, -1, \ldots, -l+1, -l$, for $l = 0, 1, 2$, and 3. They are normalized for $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$.

l	m	$Y_{lm}(\theta, \phi)$
0	0	$\frac{1}{\sqrt{4\pi}}$
1	0	$\sqrt{\frac{3}{4\pi}}\cos\theta$
1	±1	$\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$
2	0	$\sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$
2	$\pm$ 1	$\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\cos\theta\sin\theta$
2	±2	$\sqrt{\frac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta$
3	0	$\sqrt{\frac{63}{16\pi}}\left(\frac{5}{3}\cos^3\theta - \cos\theta\right)$
3	±1	$\mp\sqrt{\frac{21}{64\pi}}e^{\pm i\phi}(5\cos^2\theta - 1)\sin\theta$
3	±2	$\sqrt{\frac{105}{32\pi}}e^{\pm 2i\phi}\sin^2\theta\cos\theta$
3	±3	$\mp\sqrt{\frac{35}{64\pi}}e^{\pm 3i\phi}\sin^3\theta$

$$\begin{aligned}\phi_{2,0}(\vec{r}) &= r^2\sqrt{\frac{2}{3}}(3\cos^2\theta - 1)\\ &= r^2\sqrt{\frac{32\pi}{15}}\,Y_{2,0}(\theta, \phi).\end{aligned} \tag{1.206}$$

In general,

$$\phi_{l,\pm l}(\vec{r}) = (\mp1)^l(r\sin\theta\cos\phi \pm ir\sin\theta\sin\phi)^l, \tag{1.207}$$

i.e.

$$\phi_{l,\pm l}(\vec{r}) = (\mp1)^l r^l \sin^l\theta e^{\pm il\phi}. \tag{1.208}$$

The spherical harmonics $Y_{l,m=\pm l}(\theta, \phi)$ are:

$$Y_{l,\pm l}(\theta, \phi) = \sqrt{\frac{(2l+1)!!}{4\pi(2l)!!}}\,e^{\pm il\phi}\sin^l\theta, \tag{1.209}$$

where $(2l)!! := (2l)(2l-2)(2l-4)\ldots2$ or 1 and

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta |Y_{l,\pm l}(\theta, \phi)|^2 = 1. \tag{1.210}$$

Thus,

$$\phi_{l,\pm l}(\vec{r}) = r^l \sqrt{\frac{4\pi(2l)!!}{(2l+1)!!}} Y_{l,\pm l}(\theta, \phi). \tag{1.211}$$

It then follows from

$$\phi_{lm}(\vec{r}) = \sqrt{\frac{(l-m)!}{(2l)!(l+m)!}} (\hat{L}_+)^{l+m}(x - iy)^l, \tag{1.212}$$

which is obtained by repeated application of equations (1.186)–(1.188), that a general spherical harmonic is given by

$$Y_{lm}(\theta, \phi) = \frac{1}{2^l l!} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \frac{1}{r^l} (\hat{L}_+)^{l+m}(x - iy)^l. \tag{1.213}$$

This leads to the general expression for spherical harmonics:

$$Y_{lm}(\theta, \phi) = \frac{1}{2^l l!} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi}(-\sin\theta)^m \left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{l+m} (\cos^2\theta - 1)^l. \tag{1.214}$$

The spherical harmonics are related to the Legendre polynomials, P_l by:

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_{l,m=0}(\theta, \phi). \tag{1.215}$$

1.10 Spherical harmonics and wave functions

Spherical harmonics naturally arise when using three-dimensional position wave functions in quantum mechanics. Thus, for the position eigenkets $|\vec{r}\rangle$:

$$|\alpha\rangle = \int \mathrm{d}\vec{r} |\vec{r}\rangle\langle\vec{r}|\alpha\rangle, \tag{1.216}$$

the position wave function $\Psi_\alpha(\vec{r})$ is the amplitude $\langle\vec{r}|\alpha\rangle$ and $\Psi_\alpha(\vec{r})$ is often expressed in spherical polar coordinates:

$$\Psi_\alpha(\vec{r}) = R_\alpha(r)\Omega_\alpha(\theta, \phi). \tag{1.217}$$

The functions $\Omega_\alpha(\theta, \phi)$ are then expanded in terms of spherical harmonics

$$\Omega_\alpha(\theta, \phi) = \sum_{lm} c_{\alpha lm} Y_{lm}(\theta, \phi). \tag{1.218}$$

Within the above framework, we can define direction eigenkets $|\hat{n}\rangle$, $\hat{n} = \frac{\vec{r}}{r}$:

$$|\alpha\rangle = \int \mathrm{d}\hat{n}|\hat{n}\rangle\langle\hat{n}|\alpha\rangle; \tag{1.219}$$

and for

$$|lm\rangle = \int \mathrm{d}\hat{n}|\hat{n}\rangle\langle\hat{n}|lm\rangle, \tag{1.220}$$

$$\langle\hat{n}|lm\rangle = Y_{lm}(\theta, \phi) = Y_{lm}(\hat{n}), \tag{1.221}$$

i.e. $Y_{lm}(\theta, \phi)$ is the amplitude for the state $|lm\rangle$ to be found in the direction $\hat{n}$ specified by θ and ϕ.

1.11 Spherical harmonics and rotation matrices

Spherical harmonics can be related to (the elements of) rotation matrices because of their connection to direction eigenkets:

$$|\hat{n}\rangle = \sum_{lm}|lm\rangle\langle lm|\hat{n}\rangle = \sum_{lm} Y^*_{lm}(\theta, \phi)|lm\rangle. \tag{1.222}$$

To see this, consider

$$|\hat{n}\rangle = \mathcal{D}(R)|\hat{z}\rangle, \tag{1.223}$$

i.e. $|\hat{n}\rangle$ is obtained by the rotation of $|\hat{z}\rangle$. Evidently,

$$\mathcal{D}(R) = \mathcal{D}(\alpha = \phi, \beta = \theta, \gamma = 0) \tag{1.224}$$

will do the job. Then for equation (1.223), from the completeness relation:

$$|\hat{n}\rangle = \sum_{lm}\mathcal{D}(R)|lm\rangle\langle lm|\hat{z}\rangle, \tag{1.225}$$

$$\begin{aligned}\therefore \langle l'm'|\hat{n}\rangle &= \sum_{lm}\langle l'm'|\mathcal{D}(R)|lm\rangle\langle lm|\hat{z}\rangle \\ &= \mathcal{D}^{(l')}_{m'm}(\alpha = 0, \beta = \theta, \gamma = 0)\langle l'm|\hat{z}\rangle.\end{aligned} \tag{1.226}$$

But, $\langle l'm|\hat{z}\rangle$ is just $Y^*_{l'm}(\theta = 0, \phi)$ and $Y_{l'm}(\theta = 0, \phi) = 0$ for $m \neq 0$: this is seen by inspection of table 1.1. Thus,

$$\begin{aligned}\langle l'm|\hat{z}\rangle &= Y^*_{l'm}(\theta = 0, \phi)\delta_{m0} \\ &= \sqrt{\frac{2l' + 1}{4\pi}}\, P_{l'}(\cos\theta)|_{\theta=0}\delta_{m0} \\ &= \sqrt{\frac{2l' + 1}{4\pi}}\,\delta_{m0},\end{aligned} \tag{1.227}$$

where the $P_{l'}(\cos\theta)$ are the Legendre polynomials given by equation (1.215). Hence, from equations (1.226), (1.223) and (1.227):

$$Y^*_{l'm'}(\theta, \phi) = \mathcal{D}^{(l')}_{m'0}(\alpha = \phi, \beta = \theta, \gamma = 0)\sqrt{\frac{2l' + 1}{4\pi}}, \tag{1.228}$$

or

$$\mathcal{D}^{(l)}_{m0}(\alpha, \beta, \gamma = 0) = \sqrt{\frac{4\pi}{2l + 1}}\, Y^*_{lm}(\theta, \phi)|_{\theta=\beta,\phi=\alpha}; \tag{1.229}$$

and for $m = 0$

$$\mathcal{D}^{(l)}_{00}(\alpha, \beta, \gamma) = d^{(l)}_{00}(\beta), \tag{1.230}$$

and

$$\therefore d^{(l)}_{00}(\beta) = P_l(\cos\theta)|_{\theta=\beta}. \tag{1.231}$$

Theorem 1.11.1. *The addition theorem for spherical harmonics,*

$$P_l(\cos\theta) = \sum_m \frac{4\pi}{2l + 1} Y_{lm}(\theta_2, \phi_2)\, Y^*_{lm}(\theta_1, \phi_1), \tag{1.232}$$

where θ is defined by

$$\cos\theta := \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2). \tag{1.233}$$

Proof. Consider

$$\langle l0|\mathcal{D}(\phi, \theta, 0)|l0\rangle = \langle l0|\mathcal{D}(\phi_2, \theta_2, 0)\mathcal{D}(\phi_1, \theta_1, 0)|l0\rangle, \tag{1.234}$$

where the group properties of rotations in ket space have been used. Then, from the completeness relation

$$\langle l0|\mathcal{D}(\phi, \theta, 0)|l0\rangle = \sum_m \langle l0|\mathcal{D}(\phi_2, \theta_2, 0)|lm\rangle\langle lm|\mathcal{D}(\phi_1, \theta_1, 0)|l0\rangle, \tag{1.235}$$

$$\therefore \mathcal{D}^{(l)}_{00}(\phi, \theta, 0) = \sum_m \mathcal{D}^{(l)}_{0m}(\phi_2, \theta_2, 0)\mathcal{D}^{(l)}_{m0}(\phi_1, \theta_1, 0), \tag{1.236}$$

and from equations (1.229) and (1.231),

$$P_l(\cos\theta) = \sum_m \frac{4\pi}{2l + 1} Y_{lm}(\theta_2, \phi_2)\, Y^*_{lm}(\theta_1, \phi_1). \tag{1.237}$$

□

1.12 Properties of the rotation matrices

The rotation matrices $\mathcal{D}^{(j)}(\alpha, \beta, \gamma)$ are unitary. Thus, their matrix elements $\mathcal{D}^{(j)}_{mm'}(\alpha, \beta, \gamma)$ obey:

$$\mathcal{D}^{(j)}_{m'm}(-\gamma, -\beta, -\alpha) = \mathcal{D}^{(j)*}_{mm'}(\alpha, \beta, \gamma), \tag{1.238}$$

$$\sum_m \mathcal{D}^{(j)*}_{mm'}(\alpha, \beta, \gamma)\mathcal{D}^{(j)}_{mm''}(\alpha, \beta, \gamma) = \delta_{m'm''}, \tag{1.239}$$

$$\sum_m \mathcal{D}^{(j)*}_{m'm}(\alpha, \beta, \gamma)\mathcal{D}^{(j)}_{m''m}(\alpha, \beta, \gamma) = \delta_{m'm''}. \tag{1.240}$$

The reduced rotation matrices $d^{(j)}(\beta)$ are real. Thus, their matrix elements, $d^{(j)}_{mm'}(\beta)$, from equation (1.238), obey:

$$d^{(j)}_{m'm}(-\beta) = d^{(j)}_{mm'}(\beta). \tag{1.241}$$

From the general expression for the matrix elements of $d^{(j)}_{mm'}(\beta)$, equation (1.167), it follows that

$$(-1)^{m'-m} d^{(j)}_{-m',-m}(\beta) = d^{(j)}_{m'm}(\beta) = (-1)^{m'-m} d^{(j)}_{mm'}(\beta), \tag{1.242}$$

and hence

$$\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma) = (-1)^{m'-m}\mathcal{D}^{(j)*}_{-m',-m}(\alpha, \beta, \gamma). \tag{1.243}$$

1.13 The rotation of $\langle jm|$

The rotation of $\langle jm|$ involves an important phase factor. From the rotation of $|jm\rangle$ by $\mathcal{D}^{(j)}(\alpha, \beta, \gamma)$:

$$\mathcal{D}(\alpha, \beta, \gamma)|jm\rangle = \sum_{m'} \mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma)|jm'\rangle, \tag{1.244}$$

$$\therefore \langle jm|\mathcal{D}^{\dagger}(\alpha, \beta, \gamma) = \sum_{m'} \mathcal{D}^{(j)*}_{m'm}(\alpha, \beta, \gamma)\langle jm'|. \tag{1.245}$$

Then, from the complex conjugate of equation (1.243):

$$\langle jm|\mathcal{D}^{\dagger}(\alpha, \beta, \gamma) = \sum_{m'} (-1)^{m'-m}\mathcal{D}^{(j)}_{-m',-m}(\alpha, \beta, \gamma)\langle jm'|, \tag{1.246}$$

and replacing $-m' \leftrightarrow m'$, $-m \leftrightarrow m$, and noting that the sum is over $m' = -j, -j+1, \dots, j-1, j$ and so is unaffected,

$$\therefore \langle j, -m|\mathcal{D}^{\dagger}(\alpha, \beta, \gamma) = \sum_{m'} (-1)^{-m'+m}\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma)\langle j, -m'|, \tag{1.247}$$

$$\therefore (-1)^{-m}\langle j, -m|\mathcal{D}^{\dagger}(\alpha, \beta, \gamma) = \sum_{m'}(-1)^{-m'}\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma)\langle j, -m'|, \tag{1.248}$$

i.e. $(-1)^{-m}\langle j, -m|$ transforms like $|jm\rangle$. It is conventional to multiply both sides of equation (1.248) by $(-1)^{j}$ and then $(-1)^{j-m}\langle j, -m|$ transforms like $|jm\rangle$, and the phase is real.

1.14 The rotation of the $Y_{lm}(\theta, \phi)$

The transformations of the $Y_{lm}(\theta, \phi)$ under rotation follow from equation (1.221)

$$\langle \hat{n}|lm\rangle = Y_{lm}(\theta, \phi) = Y_{lm}(\hat{n}), \tag{1.249}$$

and

$$\mathcal{D}(R)|\hat{n}\rangle = |\hat{n}'\rangle; \tag{1.250}$$

whence from

$$\mathcal{D}(R^{-1})|lm\rangle = \sum_{m'}|lm'\rangle\langle lm'|\mathcal{D}(R^{-1})|lm\rangle, \tag{1.251}$$

i.e.

$$\mathcal{D}(R^{-1})|lm\rangle = \sum_{m'}|lm'\rangle\mathcal{D}^{(l)}_{m'm}(R^{-1}), \tag{1.252}$$

then

$$\langle \hat{n}|\mathcal{D}(R^{-1})|lm\rangle = \sum_{m'}\langle \hat{n}|lm'\rangle\mathcal{D}^{(l)}_{m'm}(R^{-1}). \tag{1.253}$$

But

$$\langle \hat{n}|\mathcal{D}(R^{-1}) = \langle \hat{n}'|, \tag{1.254}$$

$$\therefore\ Y_{lm}(\hat{n}') = \sum_{m'} Y_{lm'}(\hat{n})\mathcal{D}^{(l)*}_{mm'}(R), \tag{1.255}$$

or

$$Y_{lm}(\theta_R, \phi_R) = \sum_{m'} Y_{lm'}(\theta, \phi)\mathcal{D}^{(l)*}_{mm'}(R). \tag{1.256}$$

Similarly, from equation (1.248)

$$(-1)^{-m}\, Y_{l,-m}(\theta_R, \phi_R) = \sum_{m'}(-1)^{-m'}\, Y_{l,-m'}(\theta, \phi)\mathcal{D}^{(l)*}_{mm'}(R). \tag{1.257}$$

1.15 Exercises

1.1. Explore the commutator properties of

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.258}$$

in comparison with $SO(3)$ and $SU(2)$, $(3, \mathbb{R})$ and $(2, \mathbb{C})$.

1.2. Show that

$$d^{\left(\frac{3}{2}\right)}(\beta) = \begin{pmatrix} c^3 & -\sqrt{3}\,c^2 s & \sqrt{3}\,cs^2 & -s^3 \\ \sqrt{3}\,c^2 s & c^3 - 2cs^2 & s^3 - 2c^2 s & \sqrt{3}\,cs^2 \\ \sqrt{3}\,cs^2 & -s^3 + 2c^2 s & c^3 - 2cs^2 & -\sqrt{3}\,c^2 s \\ s^3 & \sqrt{3}\,cs^2 & \sqrt{3}\,c^2 s & c^3 \end{pmatrix}, \tag{1.259}$$

where $c := \cos\frac{\beta}{2}$, $s := \sin\frac{\beta}{2}$.

1.3. Show that the results of equation (1.167) agree with equation (1.229) for $\phi = \alpha = 0$ and $j = l = 1$, 2, and 3.

1.4. Show that

$$d^{(j)}_{m'm}(\beta) = (-1)^{m'-m} d^{(j)}_{mm'}(\beta). \tag{1.260}$$

[Hint: in the binomial expansion of equation (1.160), which results in equation (1.162) and eventually equation (1.167), reverse the positions of $\cos\frac{\beta}{2}a_+^\dagger$, $\sin\frac{\beta}{2}a_-^\dagger$ and $-\sin\frac{\beta}{2}a_+^\dagger$, $\cos\frac{\beta}{2}a_-^\dagger$, i.e. express the expansion so that it contains $\left(a_+^\dagger \cos\frac{\beta}{2}\right)^{j+m-l}$, etc.]

1.5. Show that for $R = (0, \beta, 0)$ the $Y_{1\mu}(\theta, \phi)$, $\mu = 0, \pm 1$ obey equation (1.256). [Hint: express the $Y_{1\mu}(\theta, \phi)$ in terms of x, y and z (cf. equations (1.188), (1.191), (1.193), (1.202) and (1.203)), obtain $(x, y, z)_R$ using $R_y(\beta)$, and show that $d^{(1)}(\beta)$ (equation (1.65)) transforms the $Y_{1\mu}(\theta, \phi)$ into the $Y_{1\mu}(\theta_R, \phi_R)$.]

1.16 Spin-$\frac{1}{2}$ particles; neutron interferometry

The constituents of matter—electrons, protons, and neutrons—all have intrinsic spin of $\frac{1}{2}\hbar$. 'Intrinsic' is the term coined to convey the fact that the dynamics of spin does not occur in physical space. 'Spin space' is not accessible to the physicist in the sense that the spin of a particle cannot be changed: it is intrinsic to the particle. In fact, it is not known what spin is. It is only known what spin does, namely 'couple' to other spins and angular momenta such that it behaves as a $j = \frac{1}{2}$ representation of $SU(2)$.

In the absence of other particles and when its own angular momentum is zero, the quantum mechanics of a spin-$\frac{1}{2}$ particle is almost trivial. It can exist in two possible states: 'spin up' and 'spin down'. These are directional components of the spin vector and are usually defined by

$$\hat{s}_z \left| s = \frac{1}{2}, m_s = \pm\frac{1}{2} \right\rangle = \pm\frac{1}{2}\hbar \left| s = \frac{1}{2}, m_s = \pm\frac{1}{2} \right\rangle, \tag{1.261}$$

where the direction is defined to be the z-axis in $(3, \mathbb{R})$. However, there is one extraordinary property of spin-$\frac{1}{2}$ particles: a rotation through 2π does not leave their state kets unchanged! This is seen immediately from equation (1.52) for $\phi = 2\pi$, whence (using $|sm_s\rangle \leftrightarrow \chi_\pm$)

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, 2\pi)\chi_\pm = (\hat{I}\cos\pi - i\vec{\sigma}\cdot\hat{n}\sin\pi)\chi_\pm, \tag{1.262}$$

$$\therefore \mathcal{D}^{\left(\frac{1}{2}\right)}(\hat{n}, 2\pi)\chi_\pm = -\chi_\pm. \tag{1.263}$$

This property is not observable where expectation values are involved; but it has a dramatic effect on the interferometry of beams of spin-$\frac{1}{2}$ particles.

The interferometry (diffractive splitting and recombination) of particle beams is a well-established property of quantum mechanical particles. It is most elegantly illustrated using beams of neutrons. Neutrons, being electrically neutral, are not subject to stray electric fields which can obscure the interferometric properties of electrically charged particles. However, neutrons have magnetic moments and through the use of suitable magnetic fields it is possible to effect the rotation of the state of a neutron. This has been done using the experimental arrangement shown schematically in figure 1.2. A picture of the silicon crystal, which is the essential component of the interferometer is shown in figure 1.3. The neutron beam is divided and recombined in such a way that one part passes through a magnetic field B which causes the neutron state ket to undergo a phase change. The recombined beam exhibits an interference pattern which can be varied by changing B. Some results are shown in figure 1.4. (Note: by 'divided' it is meant that for each individual neutron, it is not certain which path it takes. It is not a situation where some neutrons take one path and the other neutrons take the other path.)

The phase change produced by the magnetic field is $e^{\frac{i\omega T}{2}}$, where T is the time spent by the neutrons in the magnetic field, ω is the spin-precession frequency,

$$\omega = \frac{2\mu_n B}{\hbar}, \tag{1.264}$$

μ_n is the magnetic moment of the neutron, and the magnetic field is assumed to be of uniform constant strength B. The phase change is the standard result for a magnetic field B acting for a time T on a magnetic moment μ_n, causing the spin to precess. The connection between precession and rotation is seen to follow directly from the Hamiltonian for a neutron in the magnetic field (chosen to be in the z direction)

$$\hat{H} = \omega\hat{S}_z, \tag{1.265}$$

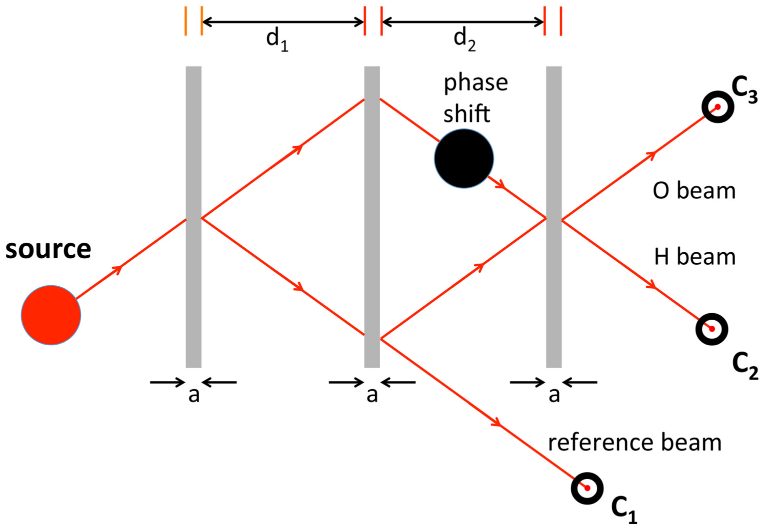

Figure 1.2. A schematic diagram of the paths of a beam of neutrons through the neutron interferometer shown in figure 1.3. The lattice planes are continuous from slab to slab and the distances a, d_1, and d_2 are machined to optical precision. The phase shift (state ket rotation) is effected in the darkened region using a magnetic field. The distances d_1 and d_2 are typically 3 cm and a is typically 0.5 cm.

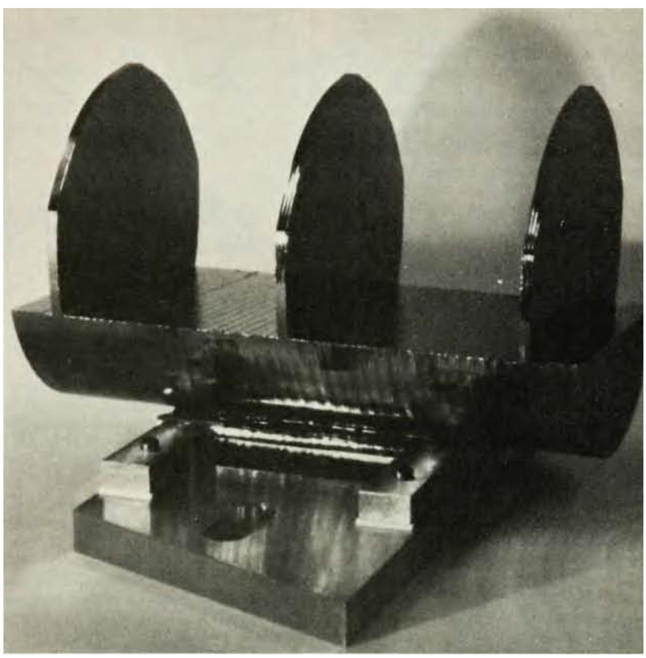

Figure 1.3. The essential component of the neutron interferometer in use at the University of Missouri. It consists of three silicon slabs machined from a single crystal of high-purity silicon to ensure alignment of crystal planes from slab to slab. (Reproduced from [1], with the permission of the American Institute of Physics.)

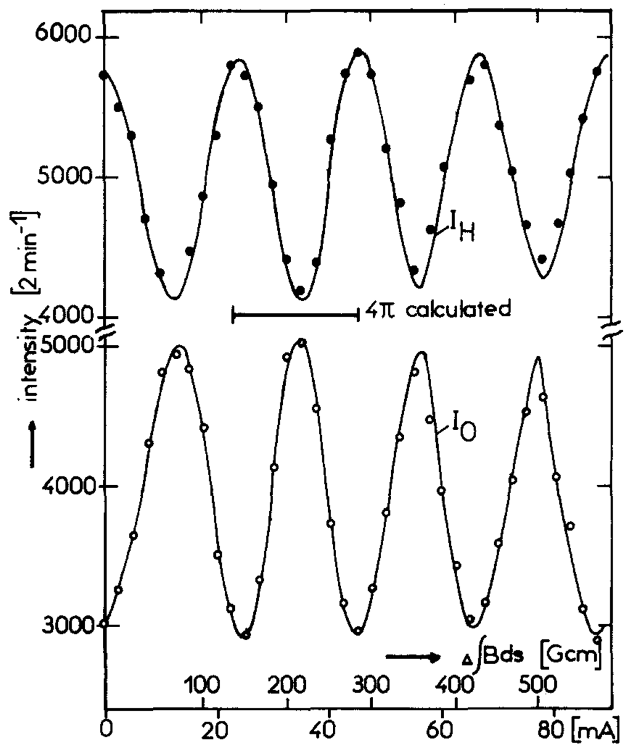

Figure 1.4. Observed neutron intensities in counts/2 min in the O beam and H beam, i.e. in counters C_3 and C_2, respectively, in figure 1.2. This is effected by changing the magnetic field action (given in Gauss cm) on the neutron beam by varying the magnet current (given in milliamps). One oscillation corresponds to a rotation of 4π not 2π. (Reprinted from [2], Copyright (1975), with permission from Elsevier.)

the time evolution operator for the system

$$U(t, 0) = \exp\left\{-\frac{i\hat{H}t}{\hbar}\right\} = \exp\left\{-\frac{i\hat{S}_z\omega t}{\hbar}\right\}, \tag{1.266}$$

and a comparison with the rotation operator about the z-axis

$$\mathcal{D}_z(\phi) = \exp\left\{-\frac{i\hat{S}_z\phi}{\hbar}\right\}, \tag{1.267}$$

i.e.

$$\phi = \omega t. \tag{1.268}$$

For a monoenergetic beam of neutrons, T is fixed. To produce the results shown in figure 1.4, B is varied (by varying the current to the magnet). The change in B necessary to yield successive maxima is given by

$$\Delta B = \frac{\hbar^2}{\mu_n \lambda m d}, \tag{1.269}$$

where λ is the de Broglie wavelength of the neutrons, m is the neutron mass, and d is the length of the path for which $B \neq 0$.

The extraordinary property of the states of spin-$\frac{1}{2}$ particles, that they must be rotated through 4π to 'bring them back to their unchanged orientation', does not parallel our experience of rotating everyday objects. Such states are called *spinors*.

1.17 The Bargmann representation

The functions

$$\chi_n(z) := \frac{z^n}{\sqrt{n!}} \tag{1.270}$$

provide an orthonormal basis for expanding functions realised on z-space (the complex plane), with scalar products defined in terms of z-space integrals with *Bargmann measure*[3], $\frac{e^{-|z|^2}}{\pi}$ [3]. This space is called *Bargmann space*.

The relevance of these functions to coherent states is implicit in the normalized coherent state form $|z\rangle_I$, i.e. (cf. Volume 1, section 5.5)

$$|z\rangle_I := e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(z^*)^n}{\sqrt{n!}} |n\rangle. \tag{1.271}$$

Whence, consider

$$K := \int\int \mathrm{d}z |z\rangle_{I\,I}\langle z| = \int\int \mathrm{d}z e^{-|z|^2} \sum_n \frac{(z^*)^n}{\sqrt{n!}} |n\rangle \sum_m \frac{z^m}{\sqrt{m!}} \langle m|; \tag{1.272}$$

which, for $z = re^{i\phi}$, gives

$$K = \int_0^\infty r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi e^{-r^2} \sum_{n,m} e^{i(m-n)\phi} \frac{r^{n+m}}{\sqrt{n!m!}} |n\rangle\langle m|. \tag{1.273}$$

Now,

$$\int_0^{2\pi} \mathrm{d}\phi e^{i(m-n)\phi} = 2\pi\delta_{mn}, \tag{1.274}$$

[3] The measure of a space appears in the infinitesimal volumes under integrals. For example, polar coordinates in three dimensions possess an infinitesimal volume expressed as $r^2\mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ and the measure is $r^2 \sin\theta$. Cartesian coordinates possess a trivial measure because the infinitesimal volume under an integral is $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ (this space could be said to be 'flat').

$$\begin{aligned}\therefore K &= \sum_n \int_0^\infty \mathrm{d}r\ r^{2n+1} e^{-r^2} \frac{2\pi}{n!} |n\rangle\langle n| \\ &= \sum_n \frac{\Gamma(n+1)}{2} \frac{2\pi}{n!} |n\rangle\langle n| \\ &= \pi \sum_n |n\rangle\langle n| \\ &= \pi \mathbf{I}.\end{aligned} \tag{1.275}$$

Thus, the resolution of the identity on Bargmann space is:

$$\mathbf{I} = \int\int \frac{\mathrm{d}z}{\pi} |z\rangle_{I\ I}\langle z| \coloneqq \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi} |z\rangle_{II\ II}\langle z|, \tag{1.276}$$

where $|z\rangle_{II} \leftrightarrow \chi_n(z)$, cf. equation (1.270). Then,

$$\begin{aligned}\langle \Psi_1 | \Psi_2 \rangle &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi} \langle \Psi_1 | z \rangle_{II\ II} \langle z | \Psi_2 \rangle \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi} \Psi_1^*(z) \Psi_2(z) \\ &= \int\int \mathrm{d}\mu(z) \Psi_1^*(z) \Psi_2(z),\end{aligned} \tag{1.277}$$

where

$$\Psi(z) \coloneqq {}_{II}\langle z | \Psi \rangle, \tag{1.278}$$

$$\mathrm{d}\mu(z) \coloneqq \frac{e^{-|z|^2}}{\pi} \mathrm{d}z. \tag{1.279}$$

Bargmann representations of functions are transformed into position representations of functions by the *Bargmann transformation*,

$$\Psi(x) = \int\int \mathrm{d}\mu(z) A(x, z^*) \Psi(z), \tag{1.280}$$

where

$$A(x, z^*) \coloneqq \frac{1}{\pi^{\frac{1}{4}}} \exp\left\{ -\frac{1}{2}x^2 + \sqrt{2}\, x z^* - \frac{1}{2}(z^*)^2 \right\} \tag{1.281}$$

is the *Bargmann kernel function*.

Comments:

1. The orthogonality of the $\chi_n(z)$ is evident in a polar coordinate representation which gives $(z^*)^n z^m \to e^{i(m-n)\phi}$ and $\int_0^{2\pi} \mathrm{d}\phi e^{i(m-n)\phi} = 2\pi\delta_{mn}$.
2. The normalizability of the $\chi_n(z)$ is evident from the Gaussian form of Bargmann measure which 'quenches' the scalar products for large $|z|$. (Indeed, the scalar products involve 'camouflaged' Hermite polynomials.)

3. The functions $\chi_n(z)$ are trivially generalised to tensor product functions,

$$\chi_{n_1}(z_1) \otimes \chi_{n_2}(z_2) \otimes \cdots$$

which yields functions

$$\sum_{n_1, n_2, \ldots} \alpha_{n_1, n_2, \ldots} \frac{z_1^{n_1}}{\sqrt{n_1!}} \frac{z_2^{n_2}}{\sqrt{n_2!}} \cdots$$

(cf. equations (1.147) and (1.176)).

1.17.1 Representation of operators

Consider the operator $\mathcal{O}$ and its representation, $\mathcal{O} \leftrightarrow \Gamma(\mathcal{O})$ in terms of z and $\frac{\partial}{\partial z}$, $\mathcal{O}(z, \frac{\partial}{\partial z})$ acting on z-space wave functions, $\Psi(z)$. This is similar to the procedure presented in Volume 1, chapter 8, where, e.g. the operator p_x (momentum in the x direction) was shown to have a 'position' representation $p_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}$ when acting on Cartesian-space wave functions $\Psi(x, y, z)$. The key there is to define a position eigenket basis $\{|x\rangle\}$ and arrive at statements such as $\langle x|\hat{p}_x|\Psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle = -i\hbar\frac{\partial\Psi(x)}{\partial x}$. Thus, we proceed with the $|z\rangle_{II}$ basis, ($|z\rangle_{II} \coloneqq e^{z^* a^\dagger}|0\rangle$, cf. Volume 1, section 5.5, equation (5.118))

$$\begin{aligned} \mathcal{O}|\Psi\rangle \Rightarrow \Gamma\left(\mathcal{O}\left(z, \frac{\partial}{\partial z}\right)\right)\Psi(z) &= {}_{II}\langle z|\mathcal{O}|\Psi\rangle = \langle 0|e^{za}\mathcal{O}|\Psi\rangle \\ &= \langle 0|(e^{za}\mathcal{O}e^{-za})e^{za}|\Psi\rangle \\ &= \langle 0|\left(\mathcal{O} + [za, \mathcal{O}] + \frac{1}{2}[za, [za, \mathcal{O}]] + \cdots\right)e^{za}|\Psi\rangle, \end{aligned} \tag{1.282}$$

where the Baker–Campbell–Hausdorff lemma is used (cf. Volume 1, chapter 5, equation (5.110)). Essentially all operators of relevance can be expressed in terms of a and $a^\dagger$, whence: for $\mathcal{O} = a$

$$\mathcal{O}|\Psi\rangle = \langle 0|\left(a + \cancelto{0}{[za, a]} + \cdots\right)|e\rangle^{za}|\Psi\rangle \tag{1.283}$$

and from $\frac{\partial}{\partial z}(e^{za}) = ae^{za}$

$$\Rightarrow \Gamma(a) = \frac{\partial}{\partial z}. \tag{1.284}$$

For $\mathcal{O} = a^\dagger$

$$\begin{aligned} \mathcal{O}|\Psi\rangle &= \langle 0|\left(a^\dagger + \underbrace{[za, a^\dagger]}_{z} + \frac{1}{2}\cancelto{0}{[za, \underbrace{[za, a^\dagger]}_{z}]} + \cdots\right)e^{za}|\Psi\rangle \\ &= \langle 0|ze^{za}|\Psi\rangle \Rightarrow \Gamma(a^\dagger) = z. \end{aligned} \tag{1.285}$$

Note:

1.

$$\left[\frac{\partial}{\partial z}, z\right] = 1, \text{ cf. } [a, a^\dagger] = 1. \tag{1.286}$$

2. z and $\frac{\partial}{\partial z}$ are Hermitian adjoints for scalar products defined on Bargmann measure:

$$\text{e. g. for } \Psi_a = \sum_n a_n z^n, \;\; \Psi_b = \sum_n b_n z^n, \tag{1.287}$$

$$\int\!\!\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi} \Psi_a^* \frac{\partial}{\partial z} \Psi_b = \sum_n a_n^* b_{n+1}(n+1)! = \int\!\!\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi} \Psi_b (z\Psi_a)^*. \tag{1.288}$$

1.18 Coherent states for *SU*(2)

The generalisation of the coherent state concept from the one-dimensional harmonic oscillator (Volume 1, section 5.5) to angular momentum is effected through their respective algebras: the *Heisenberg–Weyl algebra* in one dimension, *hw*(1) and *su*(2).

	hw(1)	*su*(2)
Generators	$a^\dagger$	J_+
	a	J_-
	I	J_0
Commutator relations	$[a, a^\dagger] = I$	$[J_-, J_+] = -2J_0$
	$[I, a^\dagger] = 0$	$[J_0, J_+] = +J_+$
	$[I, a] = 0$	$[J_0, J_-] = -J_-$
Lowest-weight state	$\|0\rangle$	$\|j, -j\rangle := \|-j\rangle$
	$a\|0\rangle = 0$	$J_-\|-j\rangle = 0$

Generalising the type-I coherent state from $HW(1)$ to $SU(2)$

$$|\zeta_I\rangle := \exp\{\zeta^* J_+ - \zeta J_-\}|-j\rangle, \tag{1.289}$$

for $\zeta := \frac{1}{2}\theta e^{i\phi}$,

$$e^{\zeta^* J_+ - \zeta J_-} = e^{-i\theta(J_x \sin\phi - J_y \cos\phi)} = e^{-i\theta(\vec{J}\cdot\hat{n})}, \tag{1.290}$$

where $\hat{n}$ is a unit vector in the x, y plane making an angle ϕ with the negative y-axis. This is illustrated in figure 1.5. All physically significant rotations are accommodated by this formalism (the apparent exclusion of rotations about the z-axis only excludes changes in phase, which could be introduced using $e^{-i\chi J_0}$).

The state $|\zeta\rangle_I$, $\zeta = \zeta(\theta, \phi)$, can be expressed:

$$\begin{aligned}|\zeta\rangle_I = |\theta, \phi\rangle_I &= e^{-i\theta(\vec{J}\cdot\hat{n})}|j, -j\rangle \\ &= \sum_m |jm\rangle\langle jm|e^{-i\theta(\vec{J}\cdot\hat{n})}|j, -j\rangle \\ &= \sum_m |jm\rangle\mathcal{D}^{(j)}_{m,-j}(\phi, \theta, 0)^*.\end{aligned} \tag{1.291}$$

From the orthonormality of the $\mathcal{D}$ functions, sections 1.11 and 1.12,

$$\mathbf{I} = \frac{(2j+1)}{4\pi}\int \mathrm{d}\Omega|\theta, \phi\rangle_{I\,I}\langle\theta, \phi|, \qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{1.292}$$

The states $\exp\{\zeta^* J_+ - \zeta J_-\}|j, -j\rangle$ are sometimes called 'atomic coherent' or 'Bloch' states (see, e.g. [4]).

The type-II coherent states of $HW(1)$ can be generalised to $SU(2)$:

$$|z\rangle_{II} := \exp\{(z^* J_+)\}|j, -j\rangle. \tag{1.293}$$

($|\zeta\rangle_I$ and $|z\rangle_{II}$ are no longer trivially related, hence the use of z and ζ.)

The $SU(2)$ states can be expressed in terms of the $\{|z\rangle_{II}\}$:

$$|\Psi\rangle \to \Psi(z) = {}_{II}\langle z|\Psi\rangle = \langle -j|e^{zJ_-}|\Psi\rangle := \Psi_j(z). \tag{1.294}$$

Operators are mapped into z-space realisations, $\Gamma(\mathcal{O})$ by

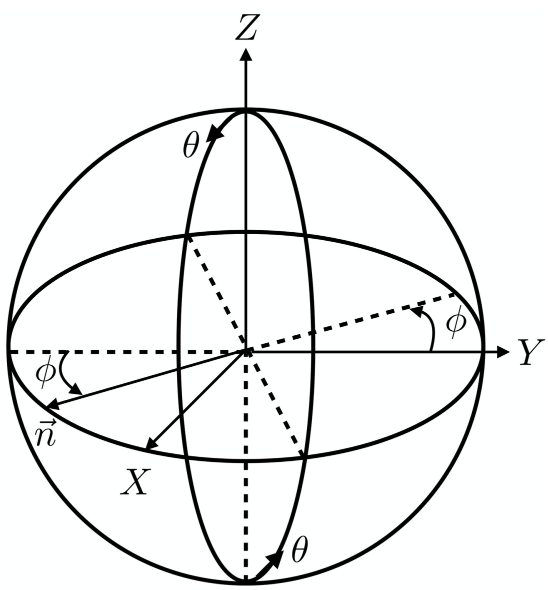

Figure 1.5. A depiction of the parameters ϕ and θ that define a type-I $SU(2)$ coherent state.

$$
\begin{aligned}
\mathcal{O}|\Psi\rangle \to \Gamma(\mathcal{O})\Psi_J(z) &= \langle z|\mathcal{O}|\Psi\rangle = \langle -j|e^{zJ_-}\mathcal{O}|\Psi\rangle \\
&= \langle -j|(e^{zJ_-}\mathcal{O}e^{-zJ_-})e^{zJ_-}|\Psi\rangle \\
&= \langle -j|(\mathcal{O} + [zJ_-, \mathcal{O}] + \frac{1}{2}[zJ_-, [zJ_-, \mathcal{O}]] + \cdots)e^{zJ_-}|\Psi\rangle.
\end{aligned} \tag{1.295}
$$

Essentially all operators of relevance can be expressed in terms of J_-, J_0, and J_+, whence: for $\mathcal{O} = J_-$

$$
\Gamma(J_-) = \langle -j|(J_- + \cancelto{0}{[zJ_-, J_-]} + \cdots)e^{z}J_-|\Psi\rangle\,, \tag{1.296}
$$

and from $\frac{\partial}{\partial z}(e^{zJ_-}) = J_- e^{zJ_-}$

$$
\Rightarrow \Gamma(J_-) = \frac{\partial}{\partial z}. \tag{1.297}
$$

For $\mathcal{O} = J_0$

$$
\Gamma(J_0) = \langle -j|\left(\underbrace{J_0}_{-j} + \underbrace{[zJ_-, J_0]}_{zJ_-} + \cancelto{0}{[zJ_-, \underbrace{[zJ_-, J_0]}_{zJ_-}]} + \cdots\right)e^{zJ_-}|\Psi\rangle\,, \tag{1.298}
$$

and from $z\frac{\partial}{\partial z}(e^{zJ_-}) = zJ_- e^{zJ_-}$

$$
\Rightarrow \Gamma(J_0) = -j + z\frac{\partial}{\partial z}. \tag{1.299}
$$

For $\mathcal{O} = J_+$

$$
\Gamma(J_+) = \langle -j|\left(\cancelto{0}{J_+} + \underbrace{[zJ_-, J_+]}_{-2zJ_0} + \frac{1}{2}[zJ_-, \underbrace{[zJ_-, J_+]}_{-2zJ_0}] + \cdots\right)e^{zJ_-}|\Psi\rangle\,, \tag{1.300}
$$

$$
\Rightarrow \Gamma(J_+) = 2jz - z^2\frac{\partial}{\partial z}. \tag{1.301}
$$

Then

$$
\Gamma(J_0)\frac{z^n}{\sqrt{n!}} = (-j + n)\frac{z^n}{\sqrt{n!}}, \qquad n = 0, 1, 2, \ldots\,, \tag{1.302}
$$

$$
\Gamma(J_+)\frac{z^n}{\sqrt{n!}} = (2j - n)\frac{z^{n+1}}{\sqrt{n!}} = (2j - n)\sqrt{n+1}\,\frac{z^{n+1}}{\sqrt{(n+1)!}}, \tag{1.303}
$$

$$
\Gamma(J_-)\frac{z^n}{\sqrt{n!}} = n\frac{z^{n-1}}{\sqrt{n!}} = \sqrt{n}\,\frac{z^{n-1}}{\sqrt{(n-1)!}}. \tag{1.304}
$$

Comments

1. Starting from the state $|j, -j\rangle$, the raising action of $\Gamma(J_+)$ terminates at $n = 2j$ (as it should for $SU(2)$).
2. The representation is non-unitary, i.e. $\Gamma(J_+)^\dagger \neq \Gamma(J_-)$ for scalar products defined on Bargmann measure (for which $(\frac{\partial}{\partial z})^\dagger = z$).
3. This type of non-unitary representation is an example of a *Dyson representation* [5].
4. The $|z\rangle_{II}$ basis is defined on an $SU(2)$ irrep labelled by j, i.e. it is a linear combination of the states $|jm\rangle$, $m = j, j-1, \ldots, -j$.

1.19 Properties of *SU*(2) from coherent states

To obtain the properties of a quantum system which possesses an algebraic structure requires that unitary representations of the operators be found. This can be done in different ways. One way (which will not be developed here) is to change the *measure* of the z-space to enforce orthonormality. Thus, for the atomic coherent states:

$$\mathbf{I} = \int\int \mathrm{d}z \frac{(2j+1)}{\pi(1+|z|^2)^{2j+2}} |z\rangle_{I\,I}\langle z| \tag{1.305}$$

provides resolution into orthonormal (unitary) representations in z-space. A second way (which is developed here) uses a similarity transformation. This choice is made because for higher symmetry algebras changing the measure of the z-space (even if it can be found!) involves very complicated integrals.

The non-unitarity of the z-space realisation of J_-, J_0, J_+ —$\Gamma(J_-)$, $\Gamma(J_0)$, $\Gamma(J_+)$; $(\Gamma(J_+)^\dagger \neq \Gamma(J_-))$—can be converted to a unitary realisation by a similarity transformation with an operator K:

$$\gamma(J_-) = K^{-1}\Gamma(J_-)K, \tag{1.306}$$

$$\gamma(J_+) = K^{-1}\Gamma(J_+)K, \tag{1.307}$$

$$\gamma(J_0) = K^{-1}\Gamma(J_0)K, \tag{1.308}$$

where it is required that

$$\gamma(J_+) = (\gamma(J_-))^\dagger. \tag{1.309}$$

Now, from the form of $\Gamma(J_0)(=-j + z\frac{\partial}{\partial z})$, it is already Hermitian ($z^\dagger = \frac{\partial}{\partial z}$). Thus, we have the condition that K commutes with $\Gamma(J_0)$ and so K is diagonal in m. Therefore, it is sufficient for K to simply normalize each ladder step in m, $m = -j, -j+1, \ldots, +j$. Hence, for $SU(2)$ it is sufficient for K to be a diagonal matrix. (For higher symmetry groups K will be more complicated.) The '*K-matrix method*' is being *introduced* in the context of $SU(2)$, where standard methods are simpler, to 'see how it works'.

Then, from $\gamma(J_+) = K^{-1}\Gamma(J_+)K$, multiplying from the left with K and from the right with $K^\dagger$

$$K\gamma(J_+)K^\dagger = KK^{-1}\Gamma(J_+)KK^\dagger. \tag{1.310}$$

But

$$\begin{aligned}\gamma(J_+) = (\gamma(J_-))^\dagger &= (K^{-1}\Gamma(J_-)K)^\dagger = \left(K^{-1}\frac{\partial}{\partial z}K\right)^\dagger \\ &= K^\dagger z(K^{-1})^\dagger,\end{aligned} \tag{1.311}$$

$$\therefore\ K\gamma(J_+)K^\dagger = KK^\dagger z(K^{-1})^\dagger K^\dagger = KK^\dagger z(KK^{-1})^\dagger, \tag{1.312}$$

$$\therefore\ \Gamma(J_+)KK^\dagger = KK^\dagger z. \tag{1.313}$$

Thus, from the above condition that for $SU(2)$, K is diagonal with real matrix elements,

$$\therefore\ \Gamma(J_+)K^2 = K^2 z. \tag{1.314}$$

The matrix elements of $KK^\dagger$ (a real diagonal matrix for $SU(2)$, $KK^\dagger = K^2$) can be obtained by proceeding in either of two ways:

1. Take matrix elements, with Bargmann measure, between the states n, $n + 1$, viz.

$$\langle\chi_{n+1}|\Gamma(J_+)K_n^2|\chi_n\rangle = \langle\chi_{n+1}|K_{n+1}^2 z|\chi_n\rangle. \tag{1.315}$$

2. Introduce the auxiliary operator, Λ_{op} with the property

$$[\Lambda_{op}, z] = \Gamma(J_+), \tag{1.316}$$

whence

$$(\Lambda_{op}z - z\Lambda_{op})K^2 = K^2 z, \tag{1.317}$$

and then take matrix elements, with Bargmann measure, between states n, $n + 1$, see the following.

The second way is easier to use for higher symmetry algebras and is the one developed here. (In particular, the second way solves for the representation of an algebra by obtaining the ratios of matrix elements of K, which are all that is ever needed.)

The motivation for defining Λ_{op} can be seen from the analogy between $[\Lambda_{op}, z] = \Gamma(J_+)$ and $[J_0, J_+] = J_+$, recalling that for the $hw(1)$ algebra, $z = \Gamma(a^\dagger)$.

From

$$[\Lambda_{op}, z] = \Gamma(J_+) = 2jz - z^2\frac{\partial}{\partial z}, \tag{1.318}$$

the right-hand side suggests that Λ_{op} must have terms of the form $z\frac{\partial}{\partial z}$ and $z^2\frac{\partial}{\partial z^2}$, viz.

$$\left(z\frac{\partial}{\partial z}\right)(z) = z + z^2\frac{\partial}{\partial z}, \tag{1.319}$$

$$\begin{aligned}\left(z^2\frac{\partial}{\partial z^2}\right)(z) &= \left(z^2\frac{\partial}{\partial z}\right)\left(I + z\frac{\partial}{\partial z}\right) = z^2\frac{\partial}{\partial z} + \left(z^2\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z}\right)\\ &= z^2\frac{\partial}{\partial z} + z^2\frac{\partial}{\partial z} + z^3\frac{\partial}{\partial z^2},\end{aligned} \tag{1.320}$$

whence

$$\Lambda_{op} = 2jz\frac{\partial}{\partial z} - \frac{1}{2}z^2\frac{\partial}{\partial z^2} = \frac{1}{2}\left(4j - z\frac{\partial}{\partial z} + 1\right)z\frac{\partial}{\partial z}. \tag{1.321}$$

Now

$$z\frac{\partial}{\partial z}\frac{z^n}{\sqrt{n!}} = n\frac{z^n}{\sqrt{n!}}, \tag{1.322}$$

hence

$$\Lambda_{\text{eigenvalue}} \coloneqq \Lambda_n = \frac{1}{2}(4j - n + 1)n. \tag{1.323}$$

Then, from $\Gamma(J_+)K^2 = K^2 z$ and $\Gamma(J_+) = [\Lambda_{op}, z]$, taking matrix elements,

$$\langle\chi_{n+1}|(\Lambda_{op}z - z\Lambda_{op})K^2|\chi_n\rangle = \langle\chi_{n+1}|K^2 z|\chi_n\rangle, \tag{1.324}$$

$$\therefore (\Lambda_{n+1} - \Lambda_n)\langle\chi_{n+1}|z|\chi_n\rangle K_n^2 = K_{n+1}^2\langle\chi_{n+1}|z|\chi_n\rangle, \tag{1.325}$$

and so

$$\frac{K_{n+1}^2}{K_n^2} = 2j - n. \tag{1.326}$$

Starting with a normalized $|-j\rangle$, $K_0^2 = 1$ and we obtain on iteration

$$K_n^2 = \frac{(2j)!}{(2j-n)!}. \tag{1.327}$$

The matrix elements of J_0, J_+, and J_- are then straightforwardly deduced:

$$\begin{aligned}\langle m|J_0|n\rangle &= \langle \chi_m|\gamma(J_0)|\chi_n\rangle \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\chi_m^*(z)(K^{-1}\Gamma(J_0)K)\chi_n(z) \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\frac{(z^*)^m}{\sqrt{m!}}K_m^{-1}\left(-j + z\frac{\partial}{\partial z}\right)K_n\frac{z^n}{\sqrt{n!}} \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\frac{(z^*)^m}{\sqrt{m!}}\frac{K_n}{K_m}(-j + n)\frac{z^n}{\sqrt{n!}} \\ &= \delta_{m,n}\frac{K_n}{K_m}(-j + n), \qquad n = 0, 1, 2, \ldots;\end{aligned} \tag{1.328}$$

$$\begin{aligned}\langle m|J_+|n\rangle &= \langle \chi_m|\gamma(J_+)|\chi_n\rangle \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\chi_m^*(z)(KzK^{-1})\chi_n(z) \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\frac{(z^*)^m}{\sqrt{m!}}K_m z K_n^{-1}\frac{z^n}{\sqrt{n!}} \\ &= \int\int \mathrm{d}z \frac{e^{-|z|^2}}{\pi}\frac{(z^*)^m}{\sqrt{m!}}\frac{K_m}{K_n}\frac{z^{n+1}}{\sqrt{(n+1)!}}\sqrt{n+1} \\ &= \delta_{m,n+1}\frac{K_m}{K_n}\sqrt{n+1}, \qquad n = 0, 1, 2, \ldots;\end{aligned} \tag{1.329}$$

and, similarly,

$$\langle m|J_-|n\rangle = \delta_{m,n-1}\frac{K_m}{K_n}\sqrt{n}, \qquad n = 0, 1, 2, \ldots. \tag{1.330}$$

Specifically, the matrix elements of J_0, J_+, and J_- are:

$$\begin{aligned}\langle n|J_0|n\rangle &= (-j + n), \qquad n = 0, 1, 2, \ldots \\ &= -j, -j + 1, -j + 2, \ldots;\end{aligned} \tag{1.331}$$

$$\begin{aligned}\langle n+1|J_+|n\rangle &= \frac{K_{n+1}}{K_n}\sqrt{n+1} = \sqrt{\frac{(2j)!}{(2j-(n+1))!}\frac{(2j-n)!}{(2j)!}}\sqrt{n+1} \\ &= \sqrt{(2j-n)(n+1)},\end{aligned} \tag{1.332}$$

and, from $m = -j + n$, i.e. $n = m + j$

$$\langle n+1|J_+|n\rangle = \sqrt{(j-m)(j+m+1)}; \tag{1.333}$$

and, similarly,

$$\langle n-1|J_-|n\rangle = \sqrt{(2j-n+1)n}, \tag{1.334}$$

$$\therefore \langle n-1|J_-|n\rangle = \sqrt{(j-m+1)(j+m)}\,. \tag{1.335}$$

Note: for $n = 2j$, $\langle n+1|J_+|n\rangle = 0$, i.e. $n_{max} = 2j$ (as it should for $SU(2)$); and it follows also, therefore, that $2j$ = integer.

1. The associations $\frac{\partial}{\partial z} \leftrightarrow a$, $z \leftrightarrow a^\dagger$, $n \leftrightarrow a^\dagger a$ reveal that:

$$\Gamma(J_-) \leftrightarrow a, \tag{1.336}$$

$$\Gamma(J_0) \leftrightarrow -j + a^\dagger a, \tag{1.337}$$

$$\Gamma(J_+) \leftrightarrow a^\dagger(2j - a^\dagger a); \tag{1.338}$$

   and that $\Gamma(J_-)$, $\Gamma(J_+)$ can be made into an adjoint pair via

$$\Gamma(J_-) \to \gamma(J_-) := \sqrt{2j - a^\dagger a}\; a, \tag{1.339}$$

$$\Gamma(J_+) \to \gamma(J_+) := a^\dagger\sqrt{2j - a^\dagger a}\,. \tag{1.340}$$

   This type of representation is called a *Holstein–Primakoff representation* [6].
2. There is an extensive literature on boson realisations of Lie algebras (see, e.g. [7]).

The material in this section relies on a treatment adopted by K T Hecht [8].

1.20 Exercises

1.6.
   (a) Derive the result, equation (1.269).
   (b) For $d = 1.00$ cm and neutrons of de Broglie wavelength $\lambda = 1.82$ Å, show that a 4π rotation is produced by $\Delta Bd = 149$ Gauss ·cm. ($\mu_n = 9.65 \times 10^{-24}$ erg ·Gauss^{-1}, $m = 1.67 \times 10^{-27}$ kg.)

1.7. Show that substituting $\Psi(z) = \chi_n(z)$ (cf. equation (1.170)) into equation (1.280) gives

$$\Psi(x) = \frac{H_n(x)e^{\frac{-x^2}{2}}}{\sqrt{2^n n!\sqrt{\pi}}}. \tag{1.341}$$

Use the generating function for Hermite polynomials

$$e^{-s^2+2sx} = \sum_n H_n(x)\frac{s^n}{n!}, \text{ with } s = \frac{z^*}{\sqrt{2}}. \tag{1.342}$$

1.8. Show that $\Psi(z) = \sum_n c_n \frac{z^n}{\sqrt{n!}}$, where the c_n are the expansion coefficients for $\Psi(x)$ in the (orthonormal) basis defined by the one-dimensional harmonic oscillator energy eigenfunctions.

1.9. Show that

$$[\Gamma(J_-), \Gamma(J_+)] = -2\Gamma(J_0),$$
$$[\Gamma(J_0), \Gamma(J_+)] = +\Gamma(J_+),$$
$$[\Gamma(J_0), \Gamma(J_-)] = -\Gamma(J_-).$$

1.10. Show that:

$$\Gamma(J^2) := \frac{1}{2}\{\Gamma(J_-)\Gamma(J_+) + \Gamma(J_+)\Gamma(J_-)\} + \Gamma(J_0)^2 = j(j+1).$$

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