1 Description

The following is a reformulation of the toy problem we presented during the meeting on 6 Oct. It regards the questions we have on the degeneracy of systems constructed as tensor products of two or more Hilbert spaces.

2 Toy problem: two-particle system

Suppose we have a Hamiltonian $\hat{H}^0$ acting on particle A in a Hilbert space $\mathcal{H}_A$. Let $\hat{H}^0$ be chosen such that it has no symmetry operators in $\mathcal{H}_A$ that commute with it. We can then see that, up to possible accidental degeneracy, there are no degeneracies for the eigenstates of $\hat{H}^0$ and we can label these eigenstates in $\mathcal{H}_A$ sequentially, ordered by the magnitude of their corresponding energies: $|1\rangle_{\mathcal{H}_A}, |2\rangle_{\mathcal{H}_A}, |3\rangle_{\mathcal{H}_A} \dots |m\rangle_{\mathcal{H}_A} \dots$, with energies $E_1, E_2, E_3 \dots E_m \dots$.

Now suppose we have another particle, particle B, which is distinguishable from particle A, which exists in Hiblert space $\mathcal{H}_B$ that is identical to $\mathcal{H}_A$, and within which the Hamiltonian of particle B is $\hat{H}^0$, identical to the Hamiltonian of particle A. Then the eigenstates and energies of this particle are $|1\rangle_{\mathcal{H}_B}, |2\rangle_{\mathcal{H}_B}, |3\rangle_{\mathcal{H}_B} \dots |n\rangle_{\mathcal{H}_B} \dots$ and $E_1, E_2, E_3 \dots E_n \dots$, respectively.

Now, suppose we consider both of these particles simultaneously. The full Hilbert space is then $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Since we introduce no interaction term, the full Hamiltonian can be understood as the sum of the two Hamiltonians acting on each respective particle; written rigorously, this becomes

$$\hat{H}_{\mathcal{H}_A \otimes \mathcal{H}_B} = \hat{H}^0_{\mathcal{H}_A} \otimes \hat{\mathbb{I}}_{\mathcal{H}_B} + \hat{\mathbb{I}}_{\mathcal{H}_A} \otimes \hat{H}^0_{\mathcal{H}_B}$$

where $\hat{\mathbb{I}}$ is the identity operator.

Now: notice that the tensor products of the original eigenstates of the respective particles will form the new eigenstates of the full system. If we label $|mn\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B} = |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B}$, then

$$\begin{aligned}
\hat{H}_{\mathcal{H}_A \otimes \mathcal{H}_B} |mn\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B} &= \left(\hat{H}^0_{\mathcal{H}_A} \otimes \hat{\mathbb{I}}_{\mathcal{H}_B} + \hat{\mathbb{I}}_{\mathcal{H}_A} \otimes \hat{H}^0_{\mathcal{H}_B}\right) |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B} \\
&= \hat{H}^0_{\mathcal{H}_A} \otimes \hat{\mathbb{I}}_{\mathcal{H}_B} |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B} + \hat{\mathbb{I}}_{\mathcal{H}_A} \otimes \hat{H}^0_{\mathcal{H}_B} |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B} \\
&= E_m |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B} + E_n |m\rangle_{\mathcal{H}_A} \otimes |n\rangle_{\mathcal{H}_B} \\
&= (E_m + E_n) |mn\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}
\end{aligned}$$

Now, since particles A and B are distinguishable, the states $|mn\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}$ and $|nm\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}$ are two separate states, but they both correspond to the energy level of $E_m + E_n$. Hence, there is a 2-fold degeneracy for each state $|mn\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}$ where $m \neq n$ (states where $m = n$ are non-degenerate).

We would expect to be able to label these energy states by an irrep of the full Hamiltonian with dimensionality of 2. However, we first need to construct the point group of symmetries of $\hat{H}_{\mathcal{H}_A \otimes \mathcal{H}_B}$. From the specific construction of this Hamiltonian, we see that the only symmetry operation that commutes with $\hat{H}_{\mathcal{H}_A \otimes \mathcal{H}_B}$ other than the identity

$\hat{P}_{\mathbb{I}} = \hat{\mathbb{I}}_{\mathcal{H}_A \otimes \mathcal{H}_B}$ is the exchange operator $\hat{P}_X$ defined such that

$$\hat{P}_X \left| mn \right\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B} = \left| nm \right\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}$$

We see that this operator transforms one degenerate eigenstate into its non-equivalent partner, hence this is not an accidental degeneracy.

However, since the group of the Hamiltonian is of order 2, it must be isomorphic to C_2, which has the trivial character table:

C_2	$\{\hat{P}_{\mathbb{I}}\}$	$\{\hat{P}_X\}$
Γ_1	1	1
Γ_2	1	-1

We see that there is no irrep with dimension 2 with which we could label the 2-fold degenerate states $\left| mn \right\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}, m \neq n$.