Lemma 1 *The following two statements are equivalent:*

(P1) If $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ *are partners in a basis for some representation* Γ_n *of a group* G*, then there exists an irreducible representation* $\Gamma^i_m \in \{\Gamma^i(G)\}$ *for which they form partners in its basis.*

(P2) $\exists \hat{R} \in G$ *such that* $\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle \neq 0$

Proof. We will first prove $(P1) \to (P2)$ and then $(P2) \to (P1)$. From the definition of the basis, we have

$$\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle = D^{(\Gamma_n)}(\hat{R})_{AB}$$

We will prove $(P1) \to (P2)$ by contradiction. Assume $\forall \hat{R} \in G$ we have $\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle = 0$. Then $\forall \hat{R} \in G : D^{(\Gamma_n)}(\hat{R})_{AB} = 0$.

Then, following the approach of Dresselhaus, we define the projection operator $\hat{P}^{(\Gamma_n)}_{AB}$ like so:

$$\begin{aligned} \hat{P}^{(\Gamma_n)}_{AB} |\Gamma_n B\rangle &= |\Gamma_n A\rangle \\ \hat{P}^{(\Gamma_n)}_{AB} |\Psi\rangle &= 0 \quad \text{for} \quad \langle \Gamma_n B|\Psi\rangle = 0 \end{aligned}$$

From the orthogonality of basis functions we see immediately that $[\hat{H}, \hat{P}^{(\Gamma_n)}_{AB}]$, hence it can be expressed as the linear combination of the elements of G:

$$\begin{aligned} \hat{P}^{(\Gamma_n)}_{AB} &= \sum_R A_{AB}(\hat{R})\hat{R} \\ \langle \Gamma_n A|\hat{P}^{(\Gamma_n)}_{AB}|\Gamma_n B\rangle &= \langle \Gamma_n A|\Gamma_n A\rangle = \sum_R A_{AB}(\hat{R}) \langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle \\ 1 &= \sum_R A_{AB}(\hat{R}) D^{(\Gamma_n)}(\hat{R})_{AB} \end{aligned}$$

We have Schur's Wonderful Orthogonality Theorem:

$$\sum_R D^{(\Gamma_n)}(\hat{R})^*_{AB} D^{(\Gamma_n)}(\hat{R})_{AB} = \frac{|G|}{l_n}$$

where l_n is the dimension of Γ_n. Hence we identify

$$A_{AB}(\hat{R}) = \frac{l_n}{|G|} D^{(\Gamma_n)}(\hat{R})^*_{AB}$$

Then

$$\hat{P}^{(\Gamma_n)}_{AB} = \frac{l_n}{|G|} \sum_R D^{(\Gamma_n)}(\hat{R})^*_{AB} \hat{R} = 0$$

But $0\,|\Gamma_n B\rangle = 0 \neq |\Gamma_n A\rangle$, which is a contradiction. This proves $(P1) \to (P2)$.

Now we prove $(P2) \to (P1)$ in almost the same fashion: we consider the representation Γ_n of which $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ are partner basis vectors. From $(P2)$ we see that $\hat{P}^{(\Gamma_n)}_{AB} \neq 0$, since all elements $\hat{R} \in G$ are linearly independent in its group-element space. Now:

either Γ_n is itself an irrep of G or it can be decomposed into irreps of G. In the former case, the proof is finished; in the latter case:

$$D^{(\Gamma_n)}(\hat{R}) = \bigoplus_a c_a D^{\left(\Gamma_a^i\right)}(\hat{R})$$

such that $D^{(\Gamma_n)}(\hat{R})_{XY} = 0$ if X, Y don't belong to the same subspace of Γ_n–this is the definition of the block-diagonal form. However, if this were the case for A, B, then $\hat{P}_{AB}^{(\Gamma_n)} = 0$, which contradicts $(P2)$. Hence $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ belong to the same subspace of Γ_n and form partners in the basis of one of its constituent irreducible representations, Γ_m^i. QED.