Lemma 1 *Let $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ be basis function partners for an irreducible representation Γ_n in group G. Then $\exists \hat{R} \in G$ such that $\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle \neq 0$.*

Proof. From the definition of the basis, we have

$$\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle = D^{(\Gamma_n)}(\hat{R})_{AB}$$

We will prove the lemma by contradiction. Assume $\forall \hat{R} \in G$ we have $\langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle = 0$. Then $\forall \hat{R} \in G : D^{(\Gamma_n)}(\hat{R})_{AB} = 0$.

Then, following the approach of Dresselhaus, we define the projection operator $\hat{P}^{(\Gamma_n)}_{AB}$ like so:

$$\begin{aligned} \hat{P}^{(\Gamma_n)}_{AB} |\Gamma_n B\rangle &= |\Gamma_n A\rangle \\ \hat{P}^{(\Gamma_n)}_{AB} |\Psi\rangle &= 0 \quad \text{for} \quad \langle \Gamma_n B|\Psi\rangle = 0 \end{aligned}$$

From the orthogonality of basis functions we see immediately that $[\hat{H}, \hat{P}^{(\Gamma_n)}_{AB}]$, hence it can be expressed as the linear combination of the elements of G:

$$\begin{aligned} \hat{P}^{(\Gamma_n)}_{AB} &= \sum_R A_{AB}(\hat{R})\hat{R} \\ \langle \Gamma_n A|\hat{P}^{(\Gamma_n)}_{AB}|\Gamma_n B\rangle &= \langle \Gamma_n A|\Gamma_n A\rangle = \sum_R A_{AB}(\hat{R}) \langle \Gamma_n A|\hat{R}|\Gamma_n B\rangle \\ 1 &= \sum_R A_{AB}(\hat{R}) D^{(\Gamma_n)}(\hat{R})_{AB} \end{aligned}$$

We have Schur's Wonderful Orthogonality Theorem:

$$\sum_R D^{(\Gamma_n)}(\hat{R})^*_{AB} D^{(\Gamma_n)}(\hat{R})_{AB} = \frac{|G|}{l_n}$$

where l_n is the dimension of Γ_n. Hence we identify

$$A_{AB}(\hat{R}) = \frac{l_n}{|G|} D^{(\Gamma_n)}(\hat{R})^*_{AB}$$

Then

$$\hat{P}^{(\Gamma_n)}_{AB} = \frac{l_n}{|G|} \sum_R D^{(\Gamma_n)}(\hat{R})^*_{AB} \hat{R} = 0$$

But $0\,|\Gamma_n B\rangle = 0 \neq |\Gamma_n A\rangle$, which is a contradiction. This finishes the proof–either $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ aren't basis partners of Γ_n or Γ_n is reducible, with $|\Gamma_n A\rangle, |\Gamma_n B\rangle$ belonging to different its subspaces.