Appendix A
Spinor Representation

A.1
Definition of a Group

A group G is a set of elements with a definite operation that satisfy the following conditions.

1. Closure: For any two elements A, B, a product AB exists and also belongs to the group.
2. Associativity: For any three elements, $(AB)C = A(BC)$.
3. Identity element: A unit element 1 which does not cause anything to happen exists. For all elements A, $1A = A1 = A$.
4. Inverse element: For any element A, an inverse A^{-1} exists and $A^{-1}A = AA^{-1} = 1$.

If $AB = BA$, the set is called commutative or abelian group.

Example A.1

Integers $\mathbb{Z}$ with addition "+" as the operation and zero as an identity operation. Inverse is subtraction.

Example A.2

Rational numbers $\mathbb{Q}$ without zero with operation multiplication "$\times$" and the number 1 as identity element. Zero has to be excluded, because it does not have an inverse element. Other examples are parallel movement or rotation in two-dimensional space (a plane).

If $AB \neq BA$, the set is called noncommutative or nonabelian group.

Example A.3

Space rotation $O(3)$ which is a member of more general group $O(N)$ which keeps the length of the N-dimensional real variables space $r = \sqrt{\sum_{i=1}^{N} x_i^2}$ invariant.

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Group Representation The operations of a group are abstract. By using the group representation they are transformed to a format that is calculable. A familiar representation is a collection of matrices $U(g)$ that act on a vector space $|\Psi\rangle \in V$, such that for all group elements $g \in G$, there exists a corresponding matrix $U(g)$ that satisfies

$$\begin{aligned} U(A)\,U(B) &= U(AB) \\ U(A^{-1}) &= U^{-1}(A) \\ U(1) &= \mathbf{1} \quad \text{(Unit matrix)} \end{aligned} \tag{A.1}$$

The collection of operand vectors is called the representation space, and the dimensionality n of the vectors is called the dimension of the space.

Note, hermitian matrices do not form a group unless it is commutative.

Example A.4

$U(N)$ is a transformation that keeps the metric of the N-dimensional complex variables space $\ell^2 = \sum_{i=1}^{N} |u_i|^2$ invariant. If we express $u_i's$ as a vector $\psi^{\mathrm{T}} = (u_1, u_2, \cdots, u_N)$, then $N \times N$ unitary matrices are the representation of the $U(N)$ group elements. When $\det U = 1$, it is referred to as the special unitary group $SU(N)$.

Subgroup When a subset of a group makes a group by itself under the same multiplication (transformation) law, it is called subgroup.

Subgroup of $SU(N)$: Any $SU(N)$ group has an abelian discrete subgroup H for which

$$\text{An element } h \in H\,, \quad h = e^{i2\pi k/N} \quad k = 0, 1, 2, \cdots N-1 \tag{A.2}$$

ordinarily designated as Z_N. It is the "center" of the group whose elements commute with all other elements of the group.

A.1.1
Lie Group

When the group element is an analytical function of continuous parameters, it is called a **Lie group**. Any element of the Lie group can be expressed as

$$A(\theta_1, \theta_2, \cdots, \theta_n) = \exp\left(i \sum_{i=1}^{n} \theta_i F_i\right) \tag{A.3}$$

where $n = N^2 - 1$ for the $SU(N)$ group. F_i's satisfy commutation relations called Lie algebra.

$$[F_i, F_j] = i f_{ijk} F_k \tag{A.4}$$

F_i's are called generators of the group, f_{ijk} structure constants. Since the whole representation matrices are analytic functions smoothly connected to the unit matrix in the limit of $\theta_i \to 0$, the Lie algebra Eq. (A.4) completely determines the local structure of the group.[1] f_{ijk} can be made totally antisymmetric in the indices ijk. Determination of the generator is not unique, because different F_i' can be made from any independent linear combination of F_i.[2] We are concerned with a compact Lie group in which the range of parameters is finite and closed within a finite volume in the space. For example, the rotation in a two-dimensional plane, $0 \leq \theta \leq 2\pi$ and is compact. In the Lorentz group, $-1 < v < 1$ and is finite but not closed. If another parameter rapidity is used $-\infty < \eta < \infty$ and hence, the group is not compact.

A.2 $SU(2)$

We define a spinor as a base vector of $SU(2)$ group representation in two-dimensional complex variable space.

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{A.6}$$

Then representation matrices are expressed by 2×2 unitary matrices with unit determinant. ξ transforms under $SU(2)$

$$\xi \to \xi' = \boldsymbol{U}\xi \quad \text{or} \quad \begin{bmatrix} \xi_1' \\ \xi_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{A.7}$$

The condition for $\boldsymbol{U}$ is

$$\mathbf{U}^{-1} = \mathbf{U}^\dagger\,, \qquad \text{and } \det \boldsymbol{U} = 1 \tag{A.8}$$

which constrains the $\boldsymbol{U}$ matrix to be of the form

$$\boldsymbol{U} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad |a|^2 + |b|^2 = 1 \tag{A.9}$$

Therefore, the number of independent parameters in $SU(2)$ is three. Equation (A.9) means

$$\begin{aligned} \xi_1' &= a\xi_1 + b\xi_2 \\ \xi_2' &= -b^*\xi_1 + a^*\xi_2 \end{aligned} \tag{A.10}$$

1) An example of global structure is "connectedness". For example $SU(2)$ and $O(3)$ have the same structure constants, but $SU(2)$ is simply connected and $O(3)$ is doubly connected and there is a one-to-two correspondence between the two.

2) For instance, instead of angular momentum J_i, we can use $J_\pm = J_x \pm iJ_y$ which satisfies

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm \tag{A.5}$$

Taking the complex conjugate of the equations and rearranging, we have

$$\begin{aligned}(-\xi_2^*)' &= \quad a\,(-\xi_2^*) + b\,(\xi_1^*)\\ (\xi_1^*)' &= -b^*\,(-\xi_2^*) + a^*\,(\xi_1^*)\end{aligned} \tag{A.11}$$

Therefore

$$\xi^c \equiv \begin{bmatrix}-\xi_2^*\\ \xi_1^*\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\xi_1^*\\ \xi_2^*\end{bmatrix} = -i\sigma_2\xi^* \tag{A.12}$$

transforms in the same way as ξ.

As there are three traceless hermitian matrices in $SU(2)$, which are taken as Pauli matrices, a general form of the unitary matrix with unit determinant takes a form of

$$\boldsymbol{U} = \exp\left[-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}\right] = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\, i\boldsymbol{\sigma}\cdot\mathbf{n} \tag{A.13}$$

$$\boldsymbol{U}^\dagger = \exp\left[i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}\right] = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\, i\boldsymbol{\sigma}\cdot\mathbf{n} \tag{A.14}$$

$s_i = \sigma_i/2$ are generators of the SU(2) group and satisfy

$$[s_i, s_j] = i\varepsilon_{ijk}s_k \tag{A.15}$$

which is the same commutation relation as that of angular momentum. To see connections with the $O(3)$ (three-dimensional rotation) group, let us take a look at the transformation properties of a product of ξ's, $H_{ij} = \xi_i\xi_j^*$. Taking H_{ij} as components of a hermitian matrix, it can be expanded in terms of Pauli matrices.

$$\boldsymbol{H} = a\mathbf{1} + \mathbf{b}\cdot\boldsymbol{\sigma} = a\mathbf{1} + \begin{bmatrix} b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{bmatrix} = \boldsymbol{A} + \boldsymbol{B} \tag{A.16}$$

The matrix $\boldsymbol{H}$ transforms under $SU(2)$

$$\boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^\dagger = \boldsymbol{A}' + \boldsymbol{B}' = a'\mathbf{1} + \mathbf{b}'\cdot\boldsymbol{\sigma} \tag{A.17}$$

$$\boldsymbol{A}' = a'\mathbf{1} = \boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\dagger = \boldsymbol{U}a\mathbf{1}\boldsymbol{U}^\dagger = a\mathbf{1} \quad \therefore \quad a' = a \tag{A.18}$$

It shows the coefficient of the unit matrix is invariant. To extract a, we take trace of both sides of Eq. (A.18) and obtain

$$2a = \mathrm{Tr}[\boldsymbol{H}] = \xi_1\xi_1^\dagger + \xi_2\xi_2^\dagger = |\xi_1|^2 + |\xi_2|^2 = \xi^\dagger\xi \tag{A.19}$$

We see that $SU(2)$ keeps the norm of the spinor invariant. Considering that the unitary transformation keeps the value of determinant invariant,

$$\det\mathbf{B}' = \det\mathbf{U}\mathbf{B}\mathbf{U}^\dagger = \det\mathbf{B} \quad \therefore \quad b_x'^2 + b_y'^2 + b_z'^2 = b_x^2 + b_y^2 + b_z^2 \tag{A.20}$$

This means the set of three parameters $\boldsymbol{b} = (b_x, b_y, b_z)$ defined in Eq. (A.17) behaves as a vector in three-dimensional real space, i.e. a vector in $O(3)$. We shall show explicitly that components of $\boldsymbol{b}$ exactly follow the relation Eq. (3.3).

To extract b_x from Eq. (A.17), we multiply both sides of the equation by σ_x and take trace. Using $\mathrm{Tr}[\sigma_x^2] = 2$, $\mathrm{Tr}[\sigma_x] = \mathrm{Tr}[\sigma_x \sigma_y] = \mathrm{Tr}[\sigma_x \sigma_z] = 0$, we obtain

$$b_x = \frac{1}{2}\mathrm{Tr}[\sigma_x \mathbf{H}] = \frac{1}{2}(\sigma_{x,ij})\xi_j \xi_i^\dagger = \xi_i^\dagger(\sigma_{x,ij})\xi_j = \xi^\dagger \sigma_x \xi \tag{A.21a}$$

similarly

$$b_y = \xi^\dagger \sigma_y \xi , \quad b_z = \xi^\dagger \sigma_z \xi \tag{A.21b}$$

Therefore we can write

$$\boldsymbol{b} = \xi^\dagger \boldsymbol{\sigma} \xi \tag{A.22}$$

Let us show that $\boldsymbol{b}$ indeed behaves as a vector in three-dimensional real space. Operation of $SU(2)$ transforms $\xi \to \xi' = \boldsymbol{U}\xi$. Rotation on axis z by θ transforms

$$b_x \to b_x' = \xi'^\dagger \sigma_x \xi' = \xi^\dagger \mathbf{U}^\dagger \sigma_x \mathbf{U} \xi \tag{A.23}$$

$$= \xi^\dagger \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\, i\sigma_z\right) \sigma_x \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\, i\sigma_z\right) \xi \tag{A.24}$$

Inserting $\sigma_z \sigma_x \sigma_z = -\sigma_x,\ \sigma_z \sigma_x = i\sigma_y$, we obtain

$$b_x' = \cos\theta\, b_x - \sin\theta\, b_y \tag{A.25a}$$

Similarly

$$b_y' = \sin\theta\, b_x + \cos\theta\, b_y \tag{A.25b}$$

$$b_z' = b_z \tag{A.25c}$$

Thus, $\boldsymbol{b}$ behaves exactly as a vector in $O(3)$. We may define a spinor whose bi-linear form (A.22) behaves like a vector. Loosely speaking, it is like a square root of the vector.

Composites of Spinors Let us consider products of two independent spinors ξ and η, which may be thought as representing two particles of spin 1/2,

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} , \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{A.26a}$$

$$s = \frac{1}{\sqrt{2}}(\xi_1 \eta_2 - \xi_2 \eta_1) , \quad \boldsymbol{v} = \begin{cases} v_x = & -\frac{1}{\sqrt{2}}(\xi_1 \eta_1 - \xi_2 \eta_2) \\ v_y = & \frac{1}{\sqrt{2}i}(\xi_1 \eta_1 + \xi_2 \eta_2) \\ v_z = & \frac{1}{\sqrt{2}}(\xi_1 \eta_2 + \xi_2 \eta_1) \end{cases} \tag{A.26b}$$

It is easy to show that s and $\boldsymbol{v}$ behave like a scalar and a vector, namely as spin 0 and 1 particles.

From the above arguments, we realize that $SU(2)$ operation on ξ and that of $O(3)$ on vector (x, y, z) are closely related. Writing down explicitly operations on spin part

$$\mathbf{U} = e^{-i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\, i\boldsymbol{\sigma}\cdot\mathbf{n} \Leftrightarrow \mathbf{R} = e^{-i\mathbf{J}\cdot\boldsymbol{\theta}} = \cos\theta - \sin\theta\, i\mathbf{S}\cdot\mathbf{n} \tag{A.27}$$

we see that while in $SU(2)$, 4π rotation takes the spinor back to where it started, it takes 2π rotation in $O(3)$. Between the range $2\pi \sim 4\pi$, $\boldsymbol{U} \to -\boldsymbol{U}$ corresponds to the same operation in $O(3)$. The correspondence is $1 : 2$. This is the reason that $O(3)$ realizes rotations only on integer spin particles, while $SU(2)$ is capable of handling those of half integer spin.

There are many reasons to believe that $SU(2)$ is active in our real world. For instance, we know that quarks and leptons, the most fundamental particles of matter have spin 1/2. A direct evidence has been obtained by a neutron interference experiment using a monolith crystal which has explicitly shown that it takes 4π to go back to the starting point [86, 103]. To confirm it personally, the reader may try a simple experiment shown in Figure A.1 in the boxed pragraph.

Confirm for yourself that 720° rotation brings you back to where you were, but 360° rotation does not.

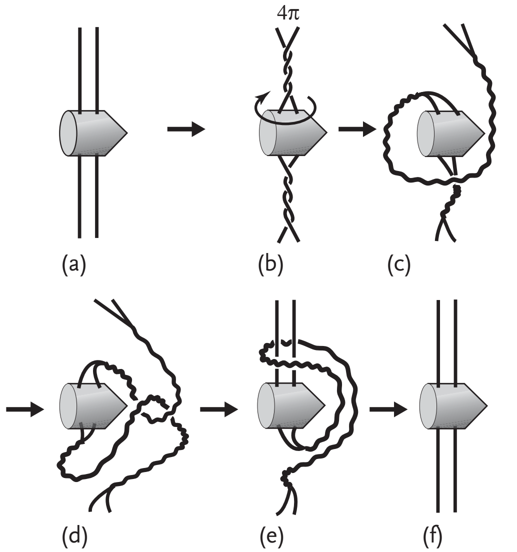

Figure A.1 Illustration of $SU(2)$ topology. After a rotation of 4π, the strings can be disentangled without rotating the tube, but it is not possible to do so with 2π rotation. (a) No rotation. (b) The tube is rotated through an angle 4π. The configuration is topologically equivalent to (a). (c)–(f) show how to undo the twists of the two strings.

A.3 Lorentz Operator for Spin 1/2 Particle

Referring to Eq. (3.77) and Eq. (3.74) in the text, combinations of spin operator $\boldsymbol{S}$ and boost operator $\boldsymbol{K}$

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{S} + i\boldsymbol{K}), \quad \boldsymbol{B} = \frac{1}{2}(\boldsymbol{S} - i\boldsymbol{K}) \tag{A.28}$$

satisfy the following equalities

$$[A_i, A_j] = i\varepsilon_{ijk} A_k \tag{A.29a}$$

$$[B_i, B_j] = i\varepsilon_{ijk} B_k \tag{A.29b}$$

$$[A_i, B_j] = 0 \tag{A.29c}$$

Both $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy commutation relations of $SU(2)$, and they commute. In other words, the Lorentz group can be expressed as a direct product of two groups $SU(2) \times SU(2)$, and hence its representations can be specified by a couple of spins (s_A, s_B). Let us consider the most interesting case where one has spin 1/2 and the other 0.

$$(0, 1/2) \quad \boldsymbol{A} = 0 \to \mathbf{K} = +i\mathbf{S} = +i\frac{\boldsymbol{\sigma}}{2} \tag{A.30a}$$

$$(1/2, 0) \quad \boldsymbol{B} = 0 \to \mathbf{K} = -i\mathbf{S} = -i\frac{\boldsymbol{\sigma}}{2} \tag{A.30b}$$

Correspondingly there are two kinds of spinors which we denote ϕ_{L} and ϕ_{R} respectively. They transform under Lorentz operation as

$$\phi_{\mathrm{L}} \xrightarrow{L} \phi'_{\mathrm{L}} = \exp\left[-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} - \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{L}} \equiv \boldsymbol{M}\phi_{\mathrm{L}} \tag{A.31a}$$

$$\phi_{\mathrm{R}} \xrightarrow{L} \phi'_{\mathrm{R}} = \exp\left[-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} + \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{R}} \equiv \boldsymbol{N}\phi_{\mathrm{R}} \tag{A.31b}$$

which means that the two spinors behave in the same way under rotation, but transform differently under the boost operation.

Note there is no matrix T that connects each other by transformation $N = TMT^{-1}$, i.e. in mathematical language they are inequivalent. Actually they are connected by

$$N = \zeta M^* \zeta^{-1}, \quad \zeta = -i\sigma_2 \tag{A.32}$$

A.3.1 $SL(2, C)$ Group

Lorentz transformation on spinors is represented by 2×2 complex matrix with unit determinant and is called $SL(2, C)$ group.

$$\xi' = \mathbf{S}\xi; \quad \mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1 \tag{A.33}$$

The matrix $\boldsymbol{S}$ comprises 6 parameters which are related to the three angles and the three rapidities in the Lorentz transformation. To see it, we look again at the transformation properties of a product of spinor components $H_{ij} = \xi_i \xi_j^*(i, j = 1 \sim 2)$. In a similar manner as Eq. (A.16), we decompose it to two parts, one with unit matrix and the other with Pauli matrices

$$\mathbf{H} = b^0 \mathbf{1} + \mathbf{b} \cdot \boldsymbol{\sigma} = \begin{bmatrix} b^0 + b_z & b_x - i b_y \\ b_x + i b_y & b^0 - b_z \end{bmatrix} \xrightarrow{\mathrm{S}} \quad \mathbf{H}' = \mathbf{SHS}^\dagger \tag{A.34}$$

Here, as the matrix $\boldsymbol{S}$ is not unitary, $\boldsymbol{S}^\dagger \boldsymbol{S} \neq \mathbf{1}$, hence the first and the second term mix. However, the value of the determinant is invariant which leads

$$\det \mathbf{H}' = \det[\mathbf{SHS}^\dagger] = |\det \mathbf{S}|^2 \det \mathbf{H} = \det \mathbf{H} \tag{A.35}$$

$$\therefore \quad b^{0\prime 2} - b_x'^2 - b_y'^2 - b_z'^2 = b^{02} - b_x^2 - b_y^2 - b_z^2 \tag{A.36}$$

therefore $SL(2, C)$ operation has been proved to be equivalent to the Lorentz transformation.

To see the difference between ϕ_{L} and ϕ_{R}, let us see transformation properties of two vector-like quantities made of constant matrices sandwiched between the two spinors.

$$a^\mu \equiv \phi_{\mathrm{R}}^\dagger \sigma^\mu \phi_{\mathrm{R}} , \qquad a_\mu \equiv \phi_{\mathrm{R}}^\dagger \sigma_\mu \phi_{\mathrm{R}} \tag{A.37a}$$

$$b^\mu \equiv \phi_{\mathrm{L}}^\dagger \overline{\sigma}^\mu \phi_{\mathrm{L}} , \quad b_\mu \equiv \phi_{\mathrm{L}}^\dagger \overline{\sigma}_\mu \phi_{\mathrm{L}} \tag{A.37b}$$

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}) = \overline{\sigma}_\mu , \quad \overline{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma}) = \sigma_\mu \tag{A.37c}$$

Put

$$\mathbf{S}_\pm = \exp(\pm \eta \sigma_x / 2) , \quad S_\pm^\dagger = S_\pm \tag{A.38}$$

and define dashed quantities by

$$\begin{cases} \phi_{\mathrm{R}}'^\dagger \sigma^\mu \phi_{\mathrm{R}}' & \equiv \phi_{\mathrm{R}}^\dagger \sigma^{\mu\prime} \phi_{\mathrm{R}} \\ \phi_{\mathrm{L}}'^\dagger \overline{\sigma}_\mu \phi_{\mathrm{L}}' & \equiv \phi_{\mathrm{L}}^\dagger \overline{\sigma}_\mu{}' \phi_{\mathrm{L}} \end{cases} \tag{A.39}$$

then by the Lorentz transformation, they change as

$$\begin{cases} \sigma^\mu & \to \sigma'^\mu = S_+^\dagger \sigma^\mu S_+ \\ \overline{\sigma}_\mu & \to \overline{\sigma}_\mu' = S_-^\dagger \sigma_\mu S_- \end{cases} \tag{A.40}$$

Calculating "0,1" components under a boost in the x-direction

$$\sigma^{0\prime}(\overline{\sigma}_0{}') = \mathbf{S}_\pm^\dagger \mathbf{1} \mathbf{S}_\pm = e^{\pm \sigma_x \eta} = \cosh \eta \pm \sinh \eta \sigma_x \tag{A.41a}$$

$$\begin{aligned} \sigma^{1\prime}(\overline{\sigma}_1{}') = \mathbf{S}_\pm^\dagger \sigma_x \mathbf{S}_\pm &= \left(\cosh \frac{\eta}{2} \pm \sinh \frac{\eta}{2} \sigma_x\right) \sigma_x \left(\cosh \frac{\eta}{2} \pm \sinh \frac{\eta}{2} \sigma_x\right) \\ &= (\sinh \eta \pm \cosh \eta \sigma_x) \end{aligned} \tag{A.41b}$$

The transformation is the same as Eq. (3.66). Therefore, we have proved that a^μ, b^μ are contravariant vectors and a_μ, b_μ covariant vectors. We often refer σ^μ, $\overline{\sigma}^\mu$ as Lorentz vectors, but note that it is meaningful only if sandwiched between the two spinors. It is also clear that $\phi_{\mathrm{R}}^\dagger \phi_{\mathrm{L}}$, $\phi_{\mathrm{L}}^\dagger \phi_{\mathrm{R}}$ are Lorentz scalars.

A.3.2 Dirac Equation: Another Derivation

Here, we try to make some correspondence between the spinors and particles considering a special transformation with $\boldsymbol{\theta} = 0$. We can consider a particle with momentum $\boldsymbol{p}$ as boosted from its rest state. Then there are two distinct states of the particle expressed as

$$\phi_{\mathrm{L}}(\boldsymbol{p}) = \exp\left[-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{L}}(0) \tag{A.42a}$$

$$\phi_{\mathrm{R}}(\boldsymbol{p}) = \exp\left[+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{R}}(0) \tag{A.42b}$$

Let $\varphi_{\mathrm{L}}, \varphi_{\mathrm{R}}$ be any two-component spinors behaving like ϕ_{L}, ϕ_{R}, then $\varphi_{\mathrm{L}}^\dagger(\mathbf{1}, -\boldsymbol{\sigma})\phi_{\mathrm{L}}$, $\varphi_{\mathrm{R}}^\dagger(\mathbf{1}, \boldsymbol{\sigma})\phi_{\mathrm{R}}$ are contravariant vectors and their dot products with the energy-momentum vector $\varphi_{\mathrm{L}}^\dagger(E + \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{L}}$, $\varphi_{\mathrm{R}}^\dagger(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{R}}$ must be scalars. Considering $\varphi_{\mathrm{L}}^\dagger\phi_{\mathrm{R}}$, $\varphi_{\mathrm{R}}^\dagger\phi_{\mathrm{L}}$ are also scalars, it follows that $(E + \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{L}}$, $(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{R}}$ behave like ϕ_{R}, ϕ_{L} and hence must be proportional to them. Using a relation

$$E + \boldsymbol{\sigma}\cdot\boldsymbol{p} = m\,(\cosh\eta + \sinh\eta\,\boldsymbol{\sigma}\cdot\boldsymbol{n}) = m\exp(\boldsymbol{\sigma}\cdot\boldsymbol{\eta}) \tag{A.43}$$

and putting the proportionality constant C,

$$\begin{aligned}(E + \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{L}}(\boldsymbol{p}) &= m\exp(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})\exp\left[-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{L}}(0)\\ &= m\exp\left[+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{L}}(0)\\ &= C\phi_{\mathrm{R}}(\boldsymbol{p}) = C\exp\left[+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\eta}\right]\phi_{\mathrm{R}}(0)\end{aligned} \tag{A.44}$$

It follows that $C = m$ and $\phi_{\mathrm{L}}(0) = \phi_{\mathrm{R}}(0)$. The latter equality is physically a simple statement that at rest there is no means to distinguish ϕ_{L} from ϕ_{R}. Applying the same argument to $(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{R}}$ and combining both results give

$$(E + \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{L}}(\boldsymbol{p}) = m\phi_{\mathrm{R}}(\boldsymbol{p}) \tag{A.45a}$$

$$(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})\phi_{\mathrm{R}}(\boldsymbol{p}) = m\phi_{\mathrm{L}}(\boldsymbol{p}) \tag{A.45b}$$

This is the celebrated Dirac equation written in terms of two-component spinors[3]. Historically, the equation was created first and its solution followed. Here, we followed the inverse logic starting from the solution and found the equation. We see

3) Equation (A.45) is written in Weyl representation. In some text books, Dirac representation is used where they are connected by

$$\psi_D = T\psi_W\,,\quad \{\boldsymbol{\alpha}, \beta\}_D = T\{\boldsymbol{\alpha}, \beta\}_W T^{-1}\,,\quad T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \tag{A.46a}$$

that it is a relation that 2-component spinors must satisfy and is the simplest one that a spin 1/2 particle respecting the Lorentz invariance has to obey.

Multiplying $(E - \boldsymbol{\sigma} \cdot \boldsymbol{p})$ on both sides of (A.45a), and using relation $(E - \boldsymbol{\sigma} \cdot \boldsymbol{p})(E + \boldsymbol{\sigma} \cdot \boldsymbol{p}) = E^2 - \boldsymbol{p}^2$ and (A.45b), we obtain

$$(E^2 - \boldsymbol{p}^2 - m^2)\phi_{\mathrm{R}} = 0\,, \quad \text{similarly} \quad (E^2 - \boldsymbol{p}^2 - m^2)\phi_{\mathrm{L}} = 0 \tag{A.47}$$

They satisfy the Einstein relation. If $\phi_{\mathrm{L}}(\boldsymbol{p})$ and $\phi_{\mathrm{R}}(\boldsymbol{p})$ are expressed as a spacetime function (i.e. take their Fourier transform), they satisfy the Klein–Gordon equation.

Problem A.1

Define

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] \tag{A.48a}$$

$$(S^{23}, S^{31}, S^{12}) = (S_1, S_2, S_3) = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{bmatrix} \tag{A.48b}$$

$$(S^{01}, S^{02}, S^{03}) = (K_1, K_2, K_3) = \frac{i}{2}\boldsymbol{\alpha} = \frac{1}{2}\begin{bmatrix} -i\boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & i\boldsymbol{\sigma} \end{bmatrix} \tag{A.48c}$$

Show that $\boldsymbol{S}$ and $\boldsymbol{K}$ satisfy the same commutation relations as Eq. (3.74), thus qualify as Lorentz operators.

Looking at Eq. (A.42), we see the plane wave solution of the Dirac equation is indeed nothing but a Lorentz boosted spin wave function at rest as it should be from our argument.