Lemma 1 *Consider an operator $\hat{A}$ which is a function of some simultaneously diagonalizable operators $\hat{a}_1, \hat{a}_2 \dots \hat{a}_n$ with respective eigenstates $|a_1, b_1\rangle, |a_2, b_2\rangle \dots |a_n, b_n\rangle$, where the degeneracy of quantum number a_m of the operator $\hat{a}_m$ is $d_m(a_m)$. Then the Hilbert space spanned by the eigenstates of $\hat{A}$ is a subspace of the Hilbert space spanned by vectors $|a_1, a_2 \dots a_n, B\rangle$, where $B = 1, 2 \cdots \prod_{i=1}^{n} d_i(a_i)$.*

Proof. Suppose

$$\hat{A} = A(\hat{a}_1, \hat{a}_2 \dots \hat{a}_n)$$

Then the vector $|a_1, a_2 \dots a_n, B\rangle$ is an eigenstate of $\hat{A}$:

$$\hat{A} |a_1, a_2 \dots a_n, B\rangle = A(a_1, a_2 \dots a_n) |a_1, a_2 \dots a_n, B\rangle$$

And since $\hat{A}$ is simultaneously diagonalisable with all of the operators $\hat{a}_1, \hat{a}_2 \dots \hat{a}_n$, any of its eigenstates must be an eigenstate of each of these operators, hence it must be expressable in this form. To each sequence $b_1, b_2 \dots b_n$ we assign one unique degeneracy quantum number B, hence the Hilbert space spanned by $|a_1, a_2 \dots a_n, B\rangle$ must contain all the eigenstates of $\hat{A}$. However, this doesn't have to be the smallest such Hilbert space, if there exist two different sequences a_m, a'_m for which $A(a_m) = A(a'_m)$. Then the corresponding basis vectors in the full Hilbert space may be equal, in which case by erasing the duplicates we obtain a subspace of the full Hilbert space. By removing basis vectors so that all basis vectors are unique, we obtain the smallest Hilbert space containing all eigenstates of $\hat{A}$.

Lemma 1 *Consider an operator $\hat{a}$ acting on particle 1 with eigenstates $|a, b\rangle$ spanning a Hilbert space $\mathcal{H}_a$, where $\hat{a} |a, b\rangle = f(a) |a, b\rangle$ and b separates the degenerate eigenstates with equal eigenvalues of $\hat{a}$. Consider a coupling operator $\hat{A} = \hat{a}_1 + \hat{a}_2 + \hat{C}$, where the eigenvalues of C are a function of the eigenvalues a_1, b_1, a_2, b_2. When we couple two states associated with quantum numbers a_1, a_2, the eigenstates of the coupling operator $\hat{A}$ can be either labelled by $|a_3, b_3\rangle$, where these vectors span $\mathcal{H}_a$, or by $|a_3, b_1, b_2\rangle$, which is a Hilbert space constructed from $\mathcal{H}_a$ by a specific procedure.*

Proof. If the degeneracy of $|a\rangle$ is d_a, then there are d_a values of b allowed for that value of a. We shall relabel these eigenstates by a canonized degeneracy label c, where

$$c = -\frac{d_a - 1}{2}, -\frac{d_a - 1}{2} + 1 \dots \frac{d_a - 1}{2} - 1, \frac{d_a - 1}{2}$$

Then the operator $\hat{c}$ shall retrieve c as its eigenvalue:

$$\hat{c} |a, c\rangle = c |a, c\rangle$$

Then we construct a vector operator $\hat{\vec{a}}$ so that $\hat{\vec{a}} |a, c\rangle = \vec{a} |a, c\rangle$, where $\vec{a}$ satisfies the following conditions:

$$\begin{aligned} \vec{a} &\in \mathbb{R}^2 \\ \vec{a} \cdot \vec{a} &= f(a) \\ \vec{a} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= c \end{aligned}$$

From this we see that

$$\hat{\vec{a}} \cdot \hat{\vec{a}} = \hat{a}$$

and the quantum numbers a and c are determined by the magnitude and direction of the vector $\vec{a}$, respectively. Hence each eigenstate corresponds to a vector $\vec{a}$ and thus the operator $\hat{C}$ which describes the interaction between *two* particles must be a function of $\hat{\vec{a}}_1$ and $\hat{\vec{a}}_2$:

$$\hat{C} = C(\hat{\vec{a}}_1, \hat{\vec{a}}_2)$$

This is a scalar operator, and hence can be written out like so[1]:

$$\hat{C} = C\left(\hat{\vec{a}}_1 \cdot \hat{\vec{a}}_1, \hat{\vec{a}}_2 \cdot \hat{\vec{a}}_2, \hat{\vec{a}}_1 \cdot \hat{\vec{a}}_2, \hat{\vec{a}}_1 \cdot \vec{u}, \hat{\vec{a}}_2 \cdot \vec{u}\right), \quad \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can write out the most general form of $\hat{C}$:

$$\hat{C} = C\left(\hat{a}_1, \hat{a}_2, \hat{c}_1, \hat{c}_2, \hat{\vec{a}}_1 \cdot \hat{\vec{a}}_2\right)$$

We can express the vector mixing term as

$$\hat{\vec{a}}_1 \cdot \hat{\vec{a}}_2 = \frac{1}{2}\left[\left(\hat{\vec{a}}_1 + \hat{\vec{a}}_2\right) \cdot \left(\hat{\vec{a}}_1 + \hat{\vec{a}}_2\right) - \hat{a}_1 - \hat{a}_2\right]$$

Consider a new eigenstate of $\hat{a}$ which correspods to the vector

$$\vec{a}_T = \hat{\vec{a}}_1 + \hat{\vec{a}}_2$$

Then we can write down the following expression:

$$\hat{\vec{a}}_1 \cdot \hat{\vec{a}}_2 = \frac{1}{2}\left[\hat{a}_T - \hat{a}_1 - \hat{a}_2\right]$$

Therefore $\hat{C}$ can be expressed with a different function like so:

$$\hat{C} = C'\left(\hat{a}_1, \hat{a}_2, \hat{c}_1, \hat{c}_2, \hat{a}_T\right)$$

From the definition of $\vec{a}_T$ and its relation to a_T, we see that in this expression, the quentum number a_T can only obtain values lying on the closed interval $[|a_1 - a_2|, a_1 + a_2]$ and which are possible eigenvalues of $\hat{a}$. Since $\hat{a}_1$ and $\hat{a}_2$ operate on different particles, they are simultaneously diagonalizable. By definition, $\hat{a}_T$ commutes with both of these operators, and all of these operators commute with $\hat{c}_1$ and $\hat{c}_2$. This means all of these operators are simultaneously diagonalizable, and hence by 1 the eigenstates of $\hat{A}$ span a Hilbert subspace of the Hilbert space spanned by $|a_1, a_2, a_T, c_1, c_2, c_T\rangle$. However, we can reduce this upper bound Hilbert space by quite a lot.

We couple states associated with a_1, a_2, hence these are determined and are not quantum numbers in the Hilbert space of the coupled quantum states. However, we did not specify the quantum numbers b_1, b_2 (and hence equivalently c_1, c_2), since without coupling these are degenerate in the eigenstates of $\hat{a}_1$ and $\hat{a}_2$ respectively. Now there are two options:

[1] This is because the dot product of $\vec{a}$ with an arbitrary vector in $\mathbb{R}^2$ can be written as a function of the dot product of $\vec{a}$ with some fixed vector $\vec{u}$ and the square magnitude of the vector $\vec{a} \cdot \vec{a}$.

1. C' doesn't depend on $\hat{c}_1$ and $\hat{c}_2$. Then c_1, c_2 are redundant quantum numbers and the eigenstates of $\hat{A}$ become $|a_T, b_T\rangle$, where a_T takes values of allowed eigenstates of $\hat{a}$ lying on the interval $[|a_1 - a_2|, a_1 + a_2]$.

2. C' depends on $\hat{c}_1$ or $\hat{c}_2$. Since we assume the particles to be symmetric under interchange, dependence on one of these operators imply dependence on the other one. Hence C' is dependent on $\hat{c}_1$ and $\hat{c}_2$, and c_1, c_2 become good quantum numbers. However:

$$c_T = \vec{a}_T \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\vec{a}_1 + \vec{a}_2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 + c_2$$

   , hence c_T is uniquely determined by c_1, c_2 and is no longer a quantum number. By a similar argument, specifying c_1, c_2 for specified values specifies $\vec{a}_1, \vec{a}_2$ and hence uniquely determines $\vec{a}_T$ and therefore a_T also. a_T ceases to be a quantum number and the eigenvalues of $\hat{A}$ become $|c_1, c_2\rangle$.

Note: these must have the same degeneracy–what does this impose on the conditions for when $\vec{a}$ can be constructed? (e.g. well-ordered f(a), condition on d_a etc)

Note–another property: when coupling multiple particles, the order of coupling reduction should not matter