1 1st order correction for the driven normal modes

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1.1 Mode space

The *mode space* is a space spanning mode evolutions of the system. It's defined as $\vec{u} \in \mathbf{R}^N$ where the modevector $\vec{u}$ has the physical significance of the following set of holonomic constraints:

$$\vec{\theta}(t) = \vec{u}e^{i\omega(\vec{u})t}$$

This set of constraints shall be referred to as the *mode constraint*.

1.2 Natural modal frequency

Every mode has its natural frequency, which minimizes the constraint forces. By applying the mode constraint to the Lagrangian, we reduce it to a single coordinate (let's say θ_1), and its associated E-L equation is the equation of a simple harmonic oscillator:

$$\theta_1 \frac{g}{u_1^2} \sum_{i=1}^{N} l_1 u_i^2 \mu_i + \ddot{\theta}_1 \frac{1}{u_1^2} \sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} u_j l_j \right)^2 = 0$$

This recovers the expected solution in the form $\theta_1 \propto e^{i\omega_0(\vec{u})t}$, where

$$\omega_0(\vec{u}) = \left[\frac{g \sum\limits_{i=1}^{N} l_i u_i^2 \mu_i}{\sum\limits_{i=1}^{N} m_i \left(\sum\limits_{j=1}^{i} u_j l_j \right)^2} \right]^{\frac{1}{2}}$$

We can verify that $\omega_0(c\vec{u}) = \omega(\vec{u})$, hence in the mode space, points on a line passing through the origin all have constant associated frequency. Lines through the origin are isofrequenths.

For a normal mode $\vec{v}_n$, we can verify that $\omega_0(\vec{v}_n) = \omega_n$, and the function recovers the associated normal mode frequency.

1.3 Constraint forces

For holonomic constraints, we have

$$\tau_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i}$$

We anticipate the force associated with the mode and a frequency to also be a mode with a matching frequency:

$$\tau_i = Q_i e^{i\omega t}$$

Remember that Lagrangian perturbed to 2nd order about the equilibrium:

$$L = \frac{1}{2} \sum_{x=1}^{N} m_x \left(\sum_{y=1}^{x} \dot{\theta}_y l_y \right)^2 - \frac{1}{2} g \sum_{x=1}^{N} l_x \theta_x^2 \mu_x$$

By solving the equation and utilizing $\sum_{x=i}^{N} \sum_{y=1}^{x} A_{xy} = \sum_{y=1}^{N} \sum_{x=\max(i,y)}^{N} A_{xy}$ we obtain

$$\tau_i = l_i \left[g\theta_i \mu_i + \sum_{x=1}^{N} \ddot{\theta}_x l_x \mu_{\max(i,x)} \right]$$

Now we substitute in the mode constraint and recover

$$\tau_i(\omega, \vec{u}) = l_i e^{i\omega t} \left[g u_i \mu_i - \omega^2 \sum_{x=1}^{N} u_x l_x \mu_{\max(i,x)} \right]$$

from which we recover the force mode

$$Q_i(\omega, \vec{u}) = l_i \left[g u_i \mu_i - \omega^2 \sum_{x=1}^{N} u_x l_x \mu_{\max(i,x)} \right]$$

We can verify that for a normal mode $\vec{v}_n$, the constraint forces vanish: $Q_i(\vec{v}_n) = 0$. This recovers the original E-L equations and is therefore consistent with their solutions.

1.4 Mode gradient of constraint forces

The mode gradient of the i-th constraint force is a vector such that

$$\nabla_u Q_i(\vec{u}) = \begin{pmatrix} \frac{\partial Q_i}{\partial u_1} \\ \frac{\partial Q_i}{\partial u_2} \\ \vdots \\ \frac{\partial Q_i}{\partial u_N} \end{pmatrix}$$

We can see that

$$\frac{\partial Q_i}{\partial u_j} = l_i \left(g\mu_i \delta_{ij} - \omega(\vec{u})^2 l_j \mu_{\max(i,j)} \right)$$

We see that all dependence on c vanishes (since $\omega(c\vec{v}_n) = \omega(\vec{v}_n)$), hence **the mode gradient will be constant on the isofrequenth** (that makes great sense from the perspective of physical interpretation of modes - we require scale symmetry).

2 Frequency-mode Space

Frequency-mode space is an extension of mode space, where a vector $\vec{s}$ is defined as

$$\vec{s} = \begin{pmatrix} \omega \\ \vec{u} \end{pmatrix} = \begin{pmatrix} \omega \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

It is a $\mathbb{R}^{N+1}$ space.

The normal modes in this space are defined as

$$\vec{w}_n = \begin{pmatrix} \omega_n \\ \vec{v}_n \end{pmatrix}$$

We can also redefine the constraint force vector field as

$$\vec{Q}(\omega, \vec{u}) = \vec{Q}(\vec{s}) \quad \text{where} \quad s_\omega = \omega, \vec{s}_{\vec{u}} = \vec{u}$$

To preserve the scaling properties of ω_0 and $\vec{Q}$, we define a scalar multiple of a frequency-mode vector such that its associated frequency stays invariant:

$$c\vec{s} = \begin{pmatrix} \omega \\ c\vec{u} \end{pmatrix} \quad \text{for} \quad c \neq 0$$

2.1 Gradient of constraint forces

The full gradient of the i-th constraint force takes into account the dependence on ω as well:

$$\nabla Q_i(\vec{s}) = \begin{pmatrix} \frac{\partial Q_i}{\partial \omega} \\ \nabla_u Q_i \end{pmatrix} = \begin{pmatrix} \frac{\partial Q_i}{\partial \omega} \\ \frac{\partial Q_i}{\partial u_1} \\ \frac{\partial Q_i}{\partial u_2} \\ \vdots \\ \frac{\partial Q_i}{\partial u_N} \end{pmatrix}$$

We see that

$$\frac{\partial Q_i}{\partial \omega} = -2 l_i \omega \sum_{x=1}^{N} u_x l_x \mu_{\max(i,x)}$$

2.2 Perturbation in frequency-mode space

We have $\vec{Q}(\vec{w}_n) = 0$ Hence, in the vicinity of normal modes, we can do a first-order approximation:

$$Q_i(c\vec{w}_n + \delta\vec{s}) = \delta\vec{s} \cdot \nabla Q_i(c\vec{w}_n)$$

We now define a matrix $P_{(N)\times(N+1)}$ so that its i-th row is the ∇Q_i vector:

$$P_{ij}(\vec{s}) = \begin{cases} \frac{\partial Q_i}{\partial \omega}(\vec{s}) & j = 1 \\ \frac{\partial Q_i}{\partial u_{j-1}}(\vec{s}) & \text{otherwise} \end{cases}$$

Then we see that

$$P(c\vec{w})\delta\vec{s} = \begin{pmatrix} \delta\vec{s} \cdot \nabla Q_1(c\vec{w}_n) \\ \delta\vec{s} \cdot \nabla Q_2(c\vec{w}_n) \\ \vdots \\ \delta\vec{s} \cdot \nabla Q_N(c\vec{w}_n) \end{pmatrix} = \vec{Q}(c\vec{w} + \delta\vec{s})$$

It shall also be useful to define a reduced square matrix $P^*_{(N)\times(N)}$, which only encompasses the mode gradients of constraint forces:

$$P^u_{ij} = P_{i(j+1)}$$

2.3 Linear dependence and reduced matrices

Since we can scale the position along the normal mode in mode space using c, we can define $\delta\vec{u}$ to be perpendicular to $\vec{v}_n$ without losing a single degree of freedom. This allows us to express δu_N as a function of $\vec{v}_n$ and δu_i, where $i = 1, 2 \ldots N-1$:

$$\begin{aligned}
\delta\vec{u}\cdot\vec{v}_n &= 0 && (1)\\
\sum_{i=1}^{N} \delta u_i v_{ni} &= 0 && (2)\\
\delta u_N v_{nN} + \sum_{i=1}^{N-1} \delta u_i v_{ni} &= 0 && (3)\\
\delta u_N &= -\frac{1}{v_{nN}} \sum_{i=1}^{N-1} \delta u_i v_{ni} && (4)\\
\delta u_N &= \sum_{i=1}^{N-1} d_i \delta u_i \quad \text{where} \quad d_i = -\frac{v_{ni}}{v_{nN}} && (5)
\end{aligned}$$

Similarly, since $\vec{Q}(c\vec{v}_n) = 0$, we have

$$\nabla Q_i(c\vec{v}_n) \perp \vec{v}_n$$

Since there's N force gradients spanning $N-1$ dimensions, these vectors must be linearly dependent (equivalently: P^u is singular). We can find the linear dependence between the columns of P^u, which we define like so:

$$\vec{p}^{\,u}_j = \begin{pmatrix} \frac{\partial Q_1}{\partial u_j} \\ \frac{\partial Q_2}{\partial u_j} \\ \vdots \\ \frac{\partial Q_{N-1}}{\partial u_j} \end{pmatrix} = \frac{\partial \vec{Q}}{\partial u_j}$$

(We can omit the bottom row of P^u from the calculation of linear dependence, since the transpose of a singular matrix is also singular).

Then, from the linear dependence, we have:

$$\vec{p}^{\,u}_N = \sum_{j=1}^{N-1} c_j \vec{p}^{\,u}_j$$

We can write this in matrix form, if we define a reduced square matrix $P^*_{(N-1)\times(N-1)}$ as

$$P^*_{ij} = P^u_{ij} = P_{i(j+1)}$$

The equation then becomes

$$P^*\vec{c} = \vec{p}^{\,u}_N$$

The solution to this equation is

$$\vec{c} = (P^*)^{-1}\vec{p}^{\,u}_N$$

This is not a directly useful result, but provides insight into the nature of P.

2.4 Finding the displacement for a general force mode

Consider now the reduced displacement

$$\delta\vec{s}^* = \begin{pmatrix} \delta\omega \\ \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta u_{N-1} \end{pmatrix}$$

and a matrix $R_{(N)\times(N)}$ defined like so:

$$R_{ij}(\vec{s}) = \begin{cases} \frac{\partial Q_i}{\partial \omega}(\vec{s}) & j = 1 \\ \frac{\partial Q_i}{\partial u_{j-1}}(\vec{s}) + d_{j-1}\frac{\partial Q_i}{\partial u_N}(\vec{s}) & \text{otherwise} \end{cases}$$

Multiplying the matrix R with the reduced displacement $\delta\vec{s}^*$, we obtain a vector $\vec{\gamma}$, whose i-th element is equal to

$$\begin{aligned}
\gamma_i &= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=2}^{N} \delta u_{j-1}\left(\frac{\partial Q_i}{\partial u_{j-1}} + d_{j-1}\frac{\partial Q_i}{\partial u_N}\right) \\
&= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=1}^{N-1} \delta u_j\left(\frac{\partial Q_i}{\partial u_j} + d_j\frac{\partial Q_i}{\partial u_N}\right) \\
&= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=1}^{N-1} \delta u_j\frac{\partial Q_i}{\partial u_j} + \sum_{j=1}^{N-1} \delta u_j d_j\frac{\partial Q_i}{\partial u_N} \\
&= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=1}^{N-1} \delta u_j\frac{\partial Q_i}{\partial u_j} + \frac{\partial Q_i}{\partial u_N}\sum_{j=1}^{N-1} d_j\delta u_j \\
&= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=1}^{N-1} \delta u_j\frac{\partial Q_i}{\partial u_j} + \frac{\partial Q_i}{\partial u_N}\delta u_N \\
&= \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=1}^{N} \delta u_j\frac{\partial Q_i}{\partial u_j} \\
&= \delta\vec{s}\cdot\nabla Q_i \\
&= Q_i(c\vec{w}_n + \delta\vec{s})
\end{aligned}$$

Hence $\vec{\gamma} \equiv \vec{Q}(c\vec{w}_n + \delta\vec{s})$, and we can now find the reduced displacement like so:

$$\begin{aligned}
R\delta\vec{s}^* &= \vec{Q}(c\vec{w}_n + \delta\vec{s}) \\
\delta\vec{s}^* &= R^{-1}\vec{Q}(c\vec{w}_n + \delta\vec{s})
\end{aligned}$$