# Predicting Resonant Behaviour of Undamped Multipendulums by Perturbing Normal Modes With Respect to Constraint Forces

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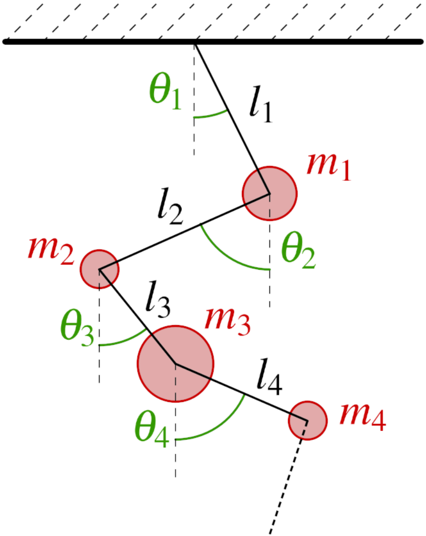
# Introduction and Motivation

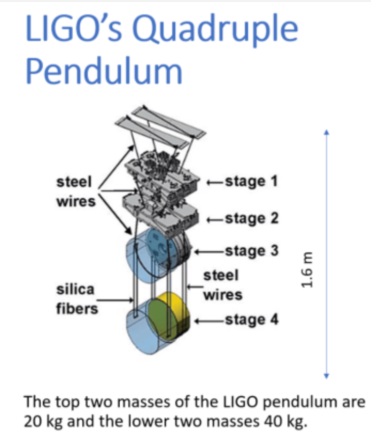
*Every pendulum has its mode.*

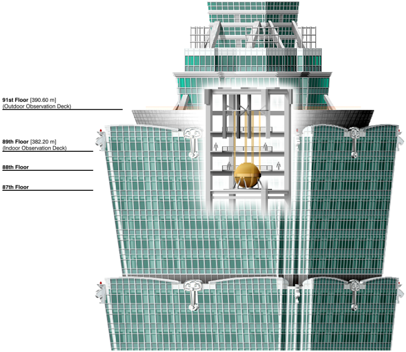
*Jacob J. Edginton, 2023*



# Introduction and Motivation

## Real-life Examples of a Multipendulum

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# Motion of a Multipendulum

## Physical Descriptors

1. The potential energy:

$$U = -\sum_{i=1}^{N} \tau_i(t)\theta_i - \sum_{i=1}^{N} m_i g \sum_{j=1}^{i} l_j \cos\theta_j$$

   where $\tau_i(t)$ is the external torque acting on the $i$-th segment

2. The kinetic energy

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left( \left( \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left( \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right)$$

3. The Lagrangian

$$L = T - U$$

# Motion of a Multipendulum

## Euler-Lagrange Equations and RK-Compatible Formulation

1. The Euler-Lagrange equations:

$$\sum_{i=n}^{N} m_i \left[ g \sin\theta_n + \sum_{j=1}^{i} l_j \left[ \ddot{\theta}_j \cos(\theta_n - \theta_j) + \dot{\theta}_j^2 \sin(\theta_n - \theta_j) \right] \right] = F_n(t)$$

   for $n = 1, 2, \ldots N$, where $F_n(t) = l_n^{-1} \tau_n(t)$

2. We define the following quantities:

$$S_a = F_a(t) - \sum_{i=a}^{N} m_i \left[ g \sin\theta_a + \sum_{j=1}^{i} l_j \dot{\theta}_j^2 \sin(\theta_a - \theta_j) \right]$$

$$M^*_{ab} = \cos(\theta_a - \theta_b)\mu_{ab}, \quad M^*_n \text{ being } M^* \text{ with } n\text{-th col. set to } \vec{S}$$

$$\text{then} \quad \ddot{\theta}_n(\vec{\theta}, \dot{\vec{\theta}}, t) = \frac{\det(M^*_n)}{l_n \det(M^*)}$$

# Normal Modes of a Multipendulum Under Free Conditions

## Why Care About Normal Modes?

1. Mode: all components have the same frequency.
2. Normal modes - emerge by perturbing around an equilibrium
3. Normal modes *tend* to match resonant frequencies
4. System with many coordinates: ambiguous definition of resonance.
5. Maximised mechanical energy

# Normal Modes of a Multipendulum Under Free Conditions

## Finding the Normal Modes - The Natural Lagrangian

1. The only stable equilibrium: $\theta_i = 0, i = 1, 2 \ldots N$
2. The Lagrangian perturber to the $2^{\mathrm{nd}}$ order about the equilibrium:

$$L = \frac{1}{2} \sum_{i=1}^{N} m_i \left( \sum_{j=1}^{i} \dot{\theta}_j l_j \right)^2 - \frac{1}{2} g \sum_{i=1}^{N} l_i \theta_i^2 \mu_i$$

3. We find the natural coordinates as:

$$\begin{aligned} q_1 &= \sqrt{m_1} l_1 \theta_1 \\ q_n &= \sqrt{\frac{m_n}{m_{n-1}}} q_{n-1} + \sqrt{m_m} l_n \theta_n \end{aligned} \qquad \begin{aligned} \theta_1 &= \frac{q_1}{l_1 \sqrt{m_1}} \\ \theta_n &= \frac{1}{l_n} \left( \frac{q_n}{\sqrt{m_n}} - \frac{q_{n-1}}{\sqrt{m_{n-1}}} \right) \end{aligned}$$

# Normal Modes of a Multipendulum Under Free Conditions

## Finding the Normal Modes - The *k*-matrix

1. The $k$-matrix elements are now found to be

$$k_{ij} = \frac{\partial^2 V}{\partial q^i \partial q^j} = \begin{cases} \frac{g}{l_N} & i = j = N \\ -\frac{g}{l_N}\sqrt{\frac{m_N}{m_{N-1}}} & i = N-1, j = N \\ -\frac{g}{l_{j+1}}\frac{\mu_{j+1}}{\sqrt{m_j m_{j+1}}} & i = j+1, j \neq N \\ \frac{g}{m_j}\left(\frac{\mu_j}{l_j} + \frac{\mu_{j+1}}{l_{j+1}}\right) & i = j \neq N \\ -\frac{g}{l_j}\frac{\mu_j}{\sqrt{m_{j-1} m_j}} & i = j-1, j \neq N \\ 0 & \text{otherwise} \end{cases}$$

# Normal Modes of a Multipendulum Under Free Conditions

## Finding the Normal Modes - Back to Original Coordinates

1. The normal modes are formed from the eigenvectors $v_n$ of $k$ and their associated eigenvalues $\lambda_n$ like so:

$$\vec{q}_n(t) = \vec{v}_n e^{i\omega_n t}, \omega_n = \sqrt{\lambda_n}$$

2. Changing back to $\theta$ coordinates:

$$\begin{aligned}
\theta_{n,1}(t) &= \frac{v_{n,1}}{l_1\sqrt{m_1}} e^{i\omega_n t} \\
\theta_{n,i}(t) &= \frac{1}{l_i}\left(\frac{v_{n,i}}{\sqrt{m_i}} - \frac{v_{n,i-1}}{\sqrt{m_{i-1}}}\right) e^{i\omega_n t}
\end{aligned}$$

where $n$ is the index of the normal mode and $i$ is the index of the segment

# Normal Modes of a Multipendulum Under Free Conditions

## Comparison of Normal Mode Frequencies to Resonant Frequencies

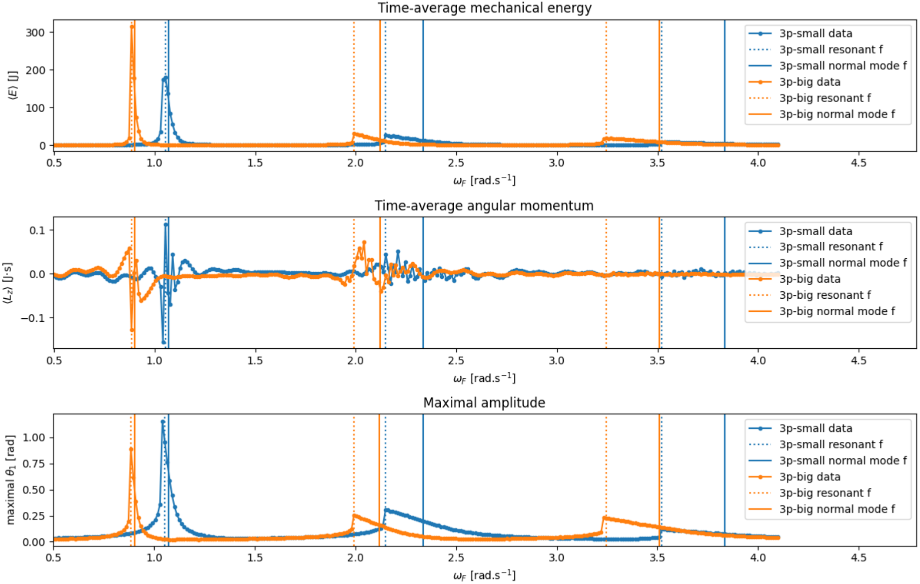
1. We choose a random small driving force mode - qualitative picture
2. Two triple pendulums:

$$\mathrm{3p-small} \quad : \quad \vec{m} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \mathrm{kg}, \vec{l} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \mathrm{m}$$

$$\mathrm{3p-big} \quad : \quad \vec{m} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \mathrm{kg}, \vec{l} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \mathrm{m}$$

# Normal Modes of a Multipendulum Under Free Conditions

## Comparison of Normal Mode Frequencies to Resonant Frequencies - Motional Spectrum



# A Driven Multipendulum

## Driving Force Equations - The Impact on Resonance

1. Driving force in the form

$$\vec{\tau}(t) = \vec{Q}e^{-i\omega_\tau t}$$

   where $\vec{Q}$ is the "force mode"

2. Separate the potential term
3. Updated E-L equation:

$$\tau_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

4. Resonant frequencies depend on the driving force

# THE MODE HYPOTHESIS

## The trajectory of a driven multipendulum at resonance is a mode, which is perturbed from an associated normal mode.

1. We can take an arbitrary mode trajectory and interpret the driving force as a constraint force
2. The holonomic mode constraint:

$$\vec{\theta}(t) = \vec{u}e^{i\omega(\vec{u})t}$$

3. Does the system possess a scale?

# Mode Space

## Definition of Mode Space - Natural Modal Frequency - Isofrequenths

1. Space $\mathbb{R}^N$ spanning all possible $\vec{u}$ modes
2. Every mode has its natural frequency (scalar field $\omega_0(\vec{u})$) and constraint force (vector field $\vec{Q}(\vec{u})$)
3. By reducing the perturbed Lagrangian to $\theta_1$ we get

$$\omega_0(\vec{u}) = \left[ \frac{g \sum_{i=1}^{N} l_i u_i^2 \mu_i}{\sum_{i=1}^{N} m_i \left( \sum_{j=1}^{i} u_j l_j \right)^2} \right]^{\frac{1}{2}}$$

4. For a normal mode $\vec{v}_n$ with an associated frequency $\omega_n$, we recover $\omega_0(\vec{v}_n) = \omega_n$
5. $\omega_0(\vec{u})$ is *scale-invariant* ($\omega_0(c\vec{u}) = \omega(\vec{u})$), hence isofrequenths are lines passing through the origin

# Mode Space

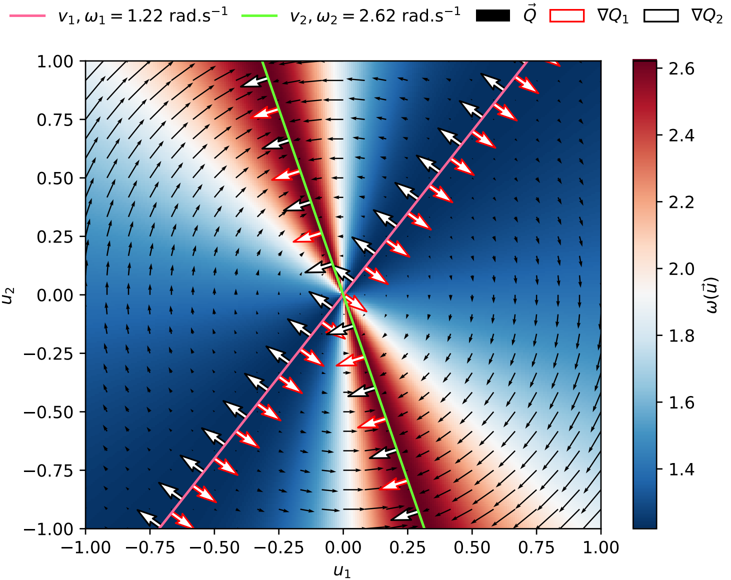
## Constraint Force Modes

1. *Constraint force*: a force needed to be introduced for a specific trajectory (given by constraints) to be allowed
2. By equating it to the driving force, we can directly search for steady-state solutions to the E-L equations
3. From the E-L equation for a mode:

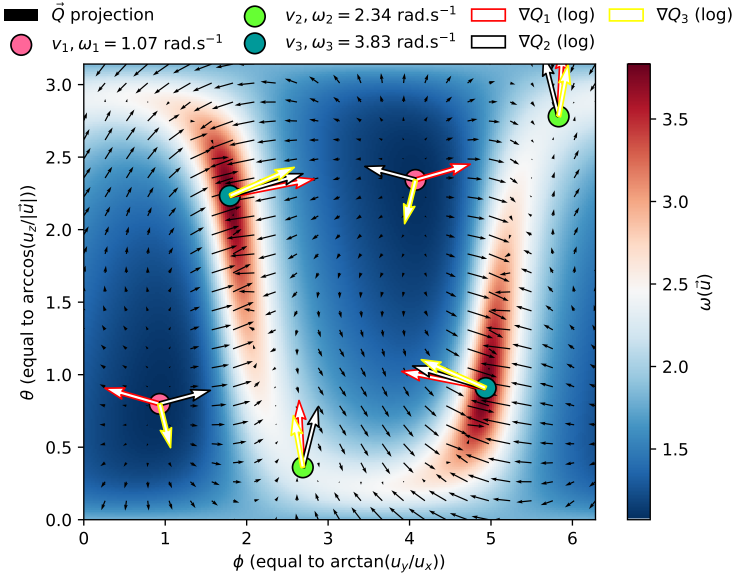
$$Q_i(\omega, \vec{u}) = l_i \left[ g u_i \mu_i - \omega(\vec{u})^2 \sum_{x=1}^{N} u_x l_x \mu_{\max(i,x)} \right]$$

   We can check numerically this satisfies $\vec{Q}(\omega_n, \vec{v}_n) = 0$
4. $\vec{Q}(\vec{u})$ scales linearly: $\vec{Q}(c\vec{u}) = c\vec{Q}(\vec{u})$

# Mode Space - Double Pendulum



# Mode Space - Triple Pendulum (Projection Onto a Sphere)



# Frequency-Mode Space



1. Space $\mathbb{R}^{N+1}$ in which a vector $\vec{s}$ represents

$$\vec{s} = \begin{pmatrix} \omega \\ \vec{u} \end{pmatrix} = \begin{pmatrix} \omega \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

2. It gives us one more degree of freedom in reference to $\vec{Q}$
3. The constraint force vector field translates well

$$\vec{Q}(\omega, \vec{u}) = \vec{Q}(\vec{s}) \quad \text{where} \quad s_\omega = \omega, \vec{s}_{\vec{u}} = \vec{u}$$

4. Scaling convention

$$c\vec{s} = \begin{pmatrix} \omega \\ c\vec{u} \end{pmatrix} \quad \text{for} \quad c \neq 0$$

preserves mode scaling properties and isofrequenths

# Perturbation of Normal Modes in Frequency-Mode Space

## A General Perturbation - Linear Dependence of Elements

1. Normal mode in F-M space $= \vec{w}_n$. For a perturbation $\delta\vec{s}$:

$$Q_i(c\vec{w}_n + \delta\vec{s}) = Q_i(c\vec{w}_n) + \delta\vec{s}\cdot\nabla Q_i\Big|_{c\vec{w}_n} = \delta\vec{s}\cdot\nabla Q_i\Big|_{c\vec{w}_n}$$

2. We control $c$, so we choose $\delta\vec{s}_{\vec{u}} \perp \vec{v}_n$, from which we have

$$\delta s_{\vec{u}N} = \sum_{i=1}^{N-1} d_i \delta s_{\vec{u}i} \quad \text{where} \quad d_i = -\frac{v_{ni}}{v_{nN}}$$

   We define a reduced displacement

$$\delta\vec{s}^* = \begin{pmatrix} \delta\omega \\ \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta u_{N-1} \end{pmatrix}$$

3. $\delta\vec{s}^*$ has $N$ dimensions, which gives us $N$ degrees of freedom plus the scale $c$.

# Perturbation of Normal Modes in Frequency-Mode Space

## Force Mode Gradient

1. We see the gradient is

$$\nabla Q_i(\vec{s}) = \begin{pmatrix} \frac{\partial Q_i}{\partial \omega} \\ \nabla_u Q_i \end{pmatrix} = \begin{bmatrix} \frac{\partial Q_i}{\partial \omega} \\ \frac{\partial Q_i}{\partial u_1} \\ \frac{\partial Q_i}{\partial u_2} \\ \vdots \\ \frac{\partial Q_i}{\partial u_N} \end{bmatrix}$$

2. By direct differentiation:

$$\begin{aligned} \frac{\partial Q_i}{\partial \omega} &= -2 l_i \omega \sum_{x=1}^{N} u_x l_x \mu_{\max(i,x)} \\ \frac{\partial Q_i}{\partial u_j} &= l_i \left( g \mu_i \delta_{ij} - \omega(\vec{u})^2 l_j \mu_{\max(i,j)} \right) \end{aligned}$$

3. Since $\nabla_u Q_i \perp \vec{v}_n$, we need $\delta s_\omega$ to span all possible $\vec{Q}$

# Perturbation of Normal Modes in Frequency-Mode Space

## The Displacement for an Arbitrary Force Mode

1. Consider a matrix $R_{(N)\times(N)}$ defined like so:

$$R_{ij}(\vec{s}) = \begin{cases} \frac{\partial Q_i}{\partial \omega}(\vec{s}) & j = 1 \\ \frac{\partial Q_i}{\partial u_{j-1}}(\vec{s}) + d_{j-1}\frac{\partial Q_i}{\partial u_N}(\vec{s}) & \text{otherwise} \end{cases}$$

2. Multiply $R$ by the reduced displacement:

$$(R\delta\vec{s}^*)_i = \delta\omega\frac{\partial Q_i}{\partial \omega} + \sum_{j=2}^{N}\delta u_{j-1}\left(\frac{\partial Q_i}{\partial u_{j-1}} + d_{j-1}\frac{\partial Q_i}{\partial u_N}\right) = \delta\vec{s}\cdot\nabla Q_i$$

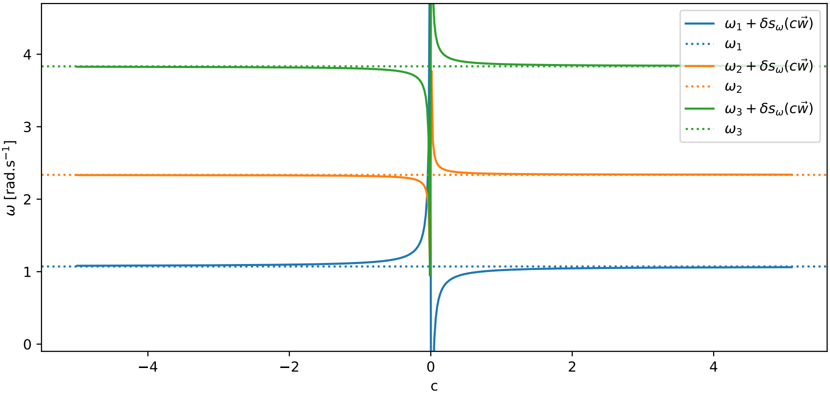
3. By equating the perturbed constraint force mode to our imposed force mode, we get:

$$\delta\vec{s}^*(\vec{Q}) = R^{-1}\vec{Q}(c\vec{w}_n + \delta\vec{s})$$

and the corrected frequency is $\omega(\vec{Q}) = \omega_n + \delta\vec{s}^*_\omega(\vec{Q})$

# Perturbation of Normal Modes in Frequency-Mode Space

## Perturbed Modal Frequency Scale Variance - *C the Scaleless?*




1. The problem of $c$: if $\omega(\vec{Q})$ isn't scale-invariant, how do we choose the value of modal frequency?
2. As of now, we aren't able to predict the resonant frequency using this model

# Conclusions and Further Research

1. We can calculate the normal modes to approximate resonance without considering the specific force mode
2. For a given force mode, we can find allowed mode trajectories in the vicinities of normal mode isofrequenths
3. *Either* the system possesses a scale, *or* the mode hypothesis is false (as not all allowed modes are resonant)
4. Immediate directions of further research:
   4.1 Find the scale of the perturbed mode trajectory whose frequency matches resonance - is it somehow special?
   4.2 Can we derive the scale theoretically? Maybe only one scale satisfies a certain condition (average work done on the steady state...)
   4.3 Maybe *all* measured trajectories around the normal mode are modes? Is mechanical energy maximised for a certain scale?
   4.4 Empirically search for mode behaviour in simulation (problem with slowly decaying transient)

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Thank you for your attention