1 1st order correction for the driven normal modes

LETSGOOO

1.1 Mode space

The *mode space* is a space spanning mode evolutions of the system. It's defined as $\vec{u} \in \mathbf{R}^N$ where the modevector $\vec{u}$ has the physical significance of the following set of holonomic constraints:

$$\vec{\theta}(t) = \vec{u}e^{i\omega(\vec{u})t}$$

This set of constraints shall be referred to as the *mode constraint*.

1.2 Modal frequency

Every mode has its frequency. By applying the mode constraint to the Lagrangian, we reduce it to a single coordinate (let's say θ_1), and its associated E-L equation is the equation of a simple harmonic oscillator:

$$\theta_1 \frac{g}{u_1^2} \sum_{i=1}^{N} l_1 u_i^2 \mu_i + \ddot{\theta}_1 \frac{1}{u_1^2} \sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} u_j l_j \right)^2 = 0$$

This recovers the expected solution in the form $\theta_1 \propto e^{i\omega(\vec{u})t}$, where

$$\omega(\vec{u}) = \left[\frac{g \sum_{i=1}^{N} l_i u_i^2 \mu_i}{\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} u_j l_j \right)^2} \right]^{\frac{1}{2}}$$

We can verify that $\omega(c\vec{u}) = \omega(\vec{u})$, hence in the mode space, points on a line passing through the origin all have constant associated frequency. Lines through the origin are isofrequenths.

For a normal mode $\vec{v}_n$, we can verify that $\omega(\vec{v}_n) = \omega_n$, and the function recovers the associated normal mode frequency.

1.3 Constraint forces

For holonomic constraints, we have

$$Q_i = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i}$$

Remember that Lagrangian perturbed to 2nd order about the equilibrium:

$$L = \frac{1}{2} \sum_{x=1}^{N} m_x \left(\sum_{y=1}^{x} \dot{\theta}_y l_y \right)^2 - \frac{1}{2} g \sum_{x=1}^{N} l_x \theta_x^2 \mu_x$$

By solving the equation and utilizing $\sum\limits_{x=i}^{N}\sum\limits_{y=1}^{x} A_{xy} = \sum\limits_{y=1}^{N}\sum\limits_{x=\max(i,y)}^{N} A_{xy}$ we obtain

$$Q_i = l_i \left[g\theta_i\mu_i + \sum_{x=1}^{N} \ddot{\theta}_x l_x \mu_{\max(i,x)} \right]$$

Now we substitute in the mode constraint and recover

$$Q_i(\vec{u}) = l_i e^{i\omega(\vec{u})t} \left[g u_i \mu_i - \sum_{x=1}^{N} \omega(\vec{u})^2 u_x l_x \mu_{\max(i,x)} \right]$$

We can verify that for a normal mode $\vec{v}_n$, the constraint forces vanish: $Q_i(\vec{v}_n) = 0$. This recovers the original E-L equations and is therefore consistent with their solutions.

1.4 Gradient of constraint forces

The gradient of the i-th constraint force is a vector such that

$$\nabla Q_i(\vec{u}) = \begin{pmatrix} \frac{\partial Q_i}{\partial u_1} \\ \frac{\partial Q_i}{\partial u_2} \\ \vdots \\ \frac{\partial Q_i}{\partial u_N} \end{pmatrix}$$

We can see that

$$\frac{\partial Q_i}{\partial u_j} = l_i \left(g\mu_i\delta_{ij} - \omega(\vec{u})^2 l_j \mu_{\max(i,j)} \right)$$

We see that all dependence on c vanishes (since $\omega(c\vec{v}_n) = \omega(\vec{v}_n)$), hence **the gradient will be constant on the isofrequenth** (that makes great sense from the perspective of physical interpretation of modes - we require scale symmetry).