

1 The Lagrangian

The multipendulum is described by two static parameters \vec{m}, \vec{l} , two dynamic parameters $\vec{\theta}, \dot{\vec{\theta}}$ (which define the phase space), the number of segments N and gravitational acceleration $\vec{g} = -g\hat{y}$.

The vector displacement between the $(n-1)$ -th and n -th segment is

$$\vec{r}_n - \vec{r}_{n-1} = l_n \begin{pmatrix} \sin \theta_n \\ -\cos \theta_n \\ 0 \end{pmatrix}$$

The position of the n -th segment is then

$$\vec{r}_n = \sum_{i=1}^n l_i \begin{pmatrix} \sin \theta_i \\ -\cos \theta_i \\ 0 \end{pmatrix} = \hat{x} \sum_{i=1}^n l_i \sin \theta_i - \hat{y} \sum_{i=1}^n l_i \cos \theta_i$$

Similarly, the velocity of the n -th segment relative to the $(n-1)$ -th segment is

$$\vec{v}_n - \vec{v}_{n-1} = (\dot{\theta}_n \hat{z}) \times (\vec{r}_n - \vec{r}_{n-1}) = \dot{\theta}_n l_n \begin{pmatrix} \cos \theta_n \\ \sin \theta_n \\ 0 \end{pmatrix}$$

Hence the velocity of the n -th segment is then

$$\vec{v}_n = \sum_{i=1}^n \dot{\theta}_i l_i \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \\ 0 \end{pmatrix} = \hat{x} \sum_{i=1}^n \dot{\theta}_i l_i \cos \theta_i + \hat{y} \sum_{i=1}^n \dot{\theta}_i l_i \sin \theta_i$$

And the square magnitude of that is

$$v_n^2 = \left(\sum_{i=1}^n \dot{\theta}_i l_i \cos \theta_i \right)^2 + \left(\sum_{i=1}^n \dot{\theta}_i l_i \sin \theta_i \right)^2$$

The potential energy of the n -th segment due to the uniform gravitational field is

$$U_n = -m_n \vec{r}_n \cdot \vec{g} = -m_n g \sum_{i=1}^n l_i \cos \theta_i$$

Hence the total gravitational potential energy is

$$U_g = \sum_{i=1}^N U_i = - \sum_{i=1}^N m_i g \sum_{j=1}^i l_j \cos \theta_j$$

And the potential energy due to the torque $\tau_i(t)$ driving the i -th segment is

$$U_\tau = - \sum_{i=1}^N \tau_i(t) \theta_i$$

Since this force-inducing potential is not a function of $\vec{\theta}$, we can absorb it into the Lagrangian. Hence the total potential energy is

$$U = - \sum_{i=1}^N \tau_i(t) \theta_i - \sum_{i=1}^N m_i g \sum_{j=1}^i l_j \cos \theta_j$$

The kinetic energy of the n -th segment is

$$T_n = \frac{1}{2} m_n v_n^2 = \frac{1}{2} m_n \left(\left(\sum_{i=1}^n \dot{\theta}_i l_i \cos \theta_i \right)^2 + \left(\sum_{i=1}^n \dot{\theta}_i l_i \sin \theta_i \right)^2 \right)$$

Hence the total kinetic energy is

$$T = \sum_{i=1}^N T_i = \sum_{i=1}^N \frac{1}{2} m_i \left(\left(\sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right)^2 + \left(\sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right)^2 \right)$$

This yields the Lagrangian of the system:

$$L = T - U = \sum_{i=1}^N \tau_i(t) \theta_i + \sum_{i=1}^N m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right)^2 + \left(\sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right)^2 \right) + g \sum_{j=1}^i l_j \cos \theta_j \right]$$

It is meaningful to define a term L_n as

$$L_n = T_n - U_n = m_n \left[\frac{1}{2} \left(\left(\sum_{i=1}^n \dot{\theta}_i l_i \cos \theta_i \right)^2 + \left(\sum_{i=1}^n \dot{\theta}_i l_i \sin \theta_i \right)^2 \right) + g \sum_{i=1}^n l_i \cos \theta_i \right]$$

so that

$$L = \sum_{i=1}^N \tau_i(t) \theta_i + \sum_{i=1}^N L_i$$

2 The Euler-Lagrange equation

2.1 The $\frac{\partial L}{\partial \theta_n}$ term

First, we wish to find the partial derivative $\frac{\partial L}{\partial \theta_n}$.

We notice that L_i is a function of $\theta_1, \theta_2, \dots, \theta_i$ and $\dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_i$, in other words, it is invariant to θ_n iff $n > i$. Hence

$$\frac{\partial L_i}{\partial \theta_n} = 0 \quad \text{if } n > i$$

Now we can use this to simplify the expression

$$\frac{\partial L}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} \sum_{i=1}^N \tau_i(t) \theta_i + \frac{\partial}{\partial \theta_n} \sum_{i=1}^N L_i = \tau_n(t) + \sum_{i=1}^N \frac{\partial L_i}{\partial \theta_n} = \tau(t)_n + \sum_{i=n}^N \frac{\partial L_i}{\partial \theta_n}$$

where for each sum term in the last expression $i \geq n$, hence all the terms should be nonzero. For these terms we have

$$\begin{aligned}\frac{\partial L_i}{\partial \theta_n} &= \frac{\partial}{\partial \theta_n} m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right)^2 + \left(\sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right)^2 \right) + g \sum_{j=1}^i l_j \cos \theta_j \right] \\ &= m_i \left[\frac{1}{2} \left(\frac{\partial A^2}{\partial \theta_n} + \frac{\partial B^2}{\partial \theta_n} \right) + g \frac{\partial C}{\partial \theta_n} \right] = m_i \left[A \frac{\partial A}{\partial \theta_n} + B \frac{\partial B}{\partial \theta_n} + g \frac{\partial C}{\partial \theta_n} \right]\end{aligned}$$

where A, B, C are the sums

$$A = \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j; \quad B = \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j; \quad C = \sum_{j=1}^i l_j \cos \theta_j$$

Since $i \geq n$, these sums will have exactly one term for which $j = n$, for which the partial derivative $\frac{\partial}{\partial \theta_n}$ is trivial; for all other terms, it is zero. Hence

$$\frac{\partial A}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} \dot{\theta}_n l_n \cos \theta_n = -\dot{\theta}_n l_n \sin \theta_n; \quad \frac{\partial B}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} \dot{\theta}_n l_n \sin \theta_n = \dot{\theta}_n l_n \cos \theta_n; \quad \frac{\partial C}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} l_n \cos \theta_n = -l_n \sin \theta_n$$

This yields

$$\frac{\partial L_i}{\partial \theta_n} = -m_i l_n \left[\dot{\theta}_n \left(\sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j - \cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right) + g \sin \theta_n \right]$$

From which we have

$$\begin{aligned}\frac{\partial L}{\partial \theta_n} &= \tau_n(t) - l_n \sum_{i=n}^N m_i \left[\dot{\theta}_n \left(\sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j - \cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right) + g \sin \theta_n \right] \\ &= \tau_n(t) - l_n \sum_{i=n}^N m_i \left[g \sin \theta_n + \dot{\theta}_n \sum_{j=1}^i \dot{\theta}_j l_j (\sin \theta_n \cos \theta_j - \cos \theta_n \sin \theta_j) \right]\end{aligned}$$

Using the angle subtraction identity $\sin \theta_n \cos \theta_j - \cos \theta_n \sin \theta_j = \sin(\theta_n - \theta_j)$ yields

$$\frac{\partial L}{\partial \theta_n} = \tau_n(t) - l_n \sum_{i=n}^N m_i \left[g \sin \theta_n + \dot{\theta}_n \sum_{j=1}^i \dot{\theta}_j l_j \sin(\theta_n - \theta_j) \right]$$

2.2 The $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_n}$ term

First, we need to find $\frac{\partial L}{\partial \dot{\theta}_n}$. We can use approach analogous to subsection 2.1:

$$\frac{\partial L}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \sum_{i=1}^N \tau_i(t) \theta_i + \frac{\partial}{\partial \dot{\theta}_n} \sum_{i=1}^N L_i = \sum_{i=1}^N \frac{\partial L_i}{\partial \dot{\theta}_n} = \sum_{i=n}^N \frac{\partial L_i}{\partial \dot{\theta}_n}$$

where

$$\begin{aligned}\frac{\partial L_i}{\partial \dot{\theta}_n} &= \frac{\partial}{\partial \dot{\theta}_n} m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right)^2 + \left(\sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right)^2 \right) + g \sum_{j=1}^i l_j \cos \theta_j \right] \\ &= m_i \left[\frac{1}{2} \left(\frac{\partial A^2}{\partial \dot{\theta}_n} + \frac{\partial B^2}{\partial \dot{\theta}_n} \right) + g \frac{\partial C}{\partial \dot{\theta}_n} \right] = m_i \left[A \frac{\partial A}{\partial \dot{\theta}_n} + B \frac{\partial B}{\partial \dot{\theta}_n} + g \frac{\partial C}{\partial \dot{\theta}_n} \right]\end{aligned}$$

where once again

$$A = \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j; \quad B = \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j; \quad C = \sum_{j=1}^i l_j \cos \theta_j$$

and the new derivatives are

$$\frac{\partial A}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \dot{\theta}_n l_n \cos \theta_n = l_n \cos \theta_n; \quad \frac{\partial B}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \dot{\theta}_n l_n \sin \theta_n = l_n \sin \theta_n; \quad \frac{\partial C}{\partial \dot{\theta}_n} = 0$$

Hence

$$\frac{\partial L_i}{\partial \dot{\theta}_n} = m_i l_n \left[\cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right]$$

and then

$$\frac{\partial L}{\partial \dot{\theta}_n} = l_n \sum_{i=n}^N m_i \left[\cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right]$$

Now we want to calculate the time derivative of this expression. For this, we will use the product rule:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_n} &= l_n \frac{d}{dt} \sum_{i=n}^N m_i \left[\cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right] \\ &= l_n \sum_{i=n}^N m_i \left[\frac{d}{dt} \left(\cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right) + \frac{d}{dt} \left(\sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right) \right] \\ &= l_n \sum_{i=n}^N m_i \left[\frac{d \cos \theta_n}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \cos \theta_n \frac{d}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j \right. \\ &\quad \left. + \frac{d \sin \theta_n}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j + \sin \theta_n \frac{d}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j \right] \\ &= l_n \sum_{i=n}^N m_i \left[\frac{d \cos \theta_n}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \cos \theta_n \sum_{j=1}^i l_j \left(\frac{d \dot{\theta}_j}{dt} \cos \theta_j + \dot{\theta}_j \frac{d \cos \theta_j}{dt} \right) \right. \\ &\quad \left. + \frac{d \sin \theta_n}{dt} \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j + \sin \theta_n \sum_{j=1}^i l_j \left(\frac{d \dot{\theta}_j}{dt} \sin \theta_j + \dot{\theta}_j \frac{d \sin \theta_j}{dt} \right) \right] \\ &= l_n \sum_{i=n}^N m_i \left[-\dot{\theta}_n \sin \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \cos \theta_j + \cos \theta_n \sum_{j=1}^i l_j (\ddot{\theta}_j \cos \theta_j - \dot{\theta}_j^2 \sin \theta_j) \right. \\ &\quad \left. + \dot{\theta}_n \cos \theta_n \sum_{j=1}^i \dot{\theta}_j l_j \sin \theta_j + \sin \theta_n \sum_{j=1}^i l_j (\ddot{\theta}_j \sin \theta_j + \dot{\theta}_j^2 \cos \theta_j) \right] \\ &= l_n \sum_{i=n}^N m_i \sum_{j=1}^i l_j [\cos \theta_n (\ddot{\theta}_j \cos \theta_j + \dot{\theta}_j (\dot{\theta}_n - \dot{\theta}_j) \sin \theta_j) + \sin \theta_n (\ddot{\theta}_j \sin \theta_j - \dot{\theta}_j (\dot{\theta}_n - \dot{\theta}_j) \cos \theta_j)] \\ &= l_n \sum_{i=n}^N m_i \sum_{j=1}^i l_j [\ddot{\theta}_j (\cos \theta_n \cos \theta_j + \sin \theta_n \sin \theta_j) + \dot{\theta}_j (\dot{\theta}_n - \dot{\theta}_j) (\cos \theta_n \sin \theta_j - \sin \theta_n \cos \theta_j)] \\ &= l_n \sum_{i=n}^N m_i \sum_{j=1}^i l_j [\ddot{\theta}_j \cos(\theta_n - \theta_j) - \dot{\theta}_j (\dot{\theta}_n - \dot{\theta}_j) \sin(\theta_n - \theta_j)] \end{aligned}$$

2.3 The resulting Euler-Lagrange equation

The E-L equation is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_n} - \frac{\partial L}{\partial \theta_n} = 0$$

Plugging our obtained expressions for these terms yields a set of N second-order differential equations:

$$l_n \sum_{i=n}^N m_i \left[g \sin \theta_n + \sum_{j=1}^i l_j [\ddot{\theta}_j \cos(\theta_n - \theta_j) - \dot{\theta}_j(\dot{\theta}_n - \dot{\theta}_j) \sin(\theta_n - \theta_j) + \dot{\theta}_n \dot{\theta}_j l_j \sin(\theta_n - \theta_j)] \right] = \tau_n(t) \quad \text{for } n = 1, 2, \dots, N$$

which simplifies to

$$l_n \sum_{i=n}^N m_i \left[g \sin \theta_n + \sum_{j=1}^i l_j [\ddot{\theta}_j \cos(\theta_n - \theta_j) + \dot{\theta}_j^2 \sin(\theta_n - \theta_j)] \right] = \tau_n(t) \quad \text{for } n = 1, 2, \dots, N$$

If $l_i = 0$, the system is identical to a multipendulum of order $N - 1$ with one segment being of mass $m_{i-1} + m_i$. As this produces a degeneracy, we may safely ignore it. Since $l_n \neq 0 \forall n$, we can divide each equation by l_n . Let $\tau_n(t) = l_n F_n(t)$. The system of equations becomes

$$\sum_{i=n}^N m_i \left[g \sin \theta_n + \sum_{j=1}^i l_j [\ddot{\theta}_j \cos(\theta_n - \theta_j) + \dot{\theta}_j^2 \sin(\theta_n - \theta_j)] \right] = F_n(t) \quad \text{for } n = 1, 2, \dots, N$$

This has been checked manually for $N = 1$ and $N = 2$.

3 Solving for second time derivative terms

To be able to numerically integrate these equations of motion, we need to express $\ddot{\theta}_i$ as $\ddot{\theta}_i(\theta_1, \theta_2, \dots, \theta_N, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_N)$. For this, we need to formulate the equations in the form

$$M \ddot{\theta} = \vec{S}$$

where M is a square matrix $N \times N$. The a -th equation of motion then becomes

$$\vec{M}_a \cdot \ddot{\theta} = S_a \iff \sum_{b=1}^N M_{ab} \ddot{\theta}_b = S_a$$

so we see that M_{ab} is the coefficient of $\ddot{\theta}_b$ in the a -th equation of motion.

Collecting terms in our equations of motion:

$$\sum_{i=a}^N m_i \left[\sum_{j=1}^i l_j \ddot{\theta}_j \cos(\theta_a - \theta_j) \right] = F_a(t) - \sum_{i=a}^N m_i \left[g \sin \theta_a + \sum_{j=1}^i l_j \dot{\theta}_j^2 \sin(\theta_a - \theta_j) \right] = S_a$$

from which we have

$$S_a = F_a(t) - \sum_{i=a}^N m_i \left[g \sin \theta_a + \sum_{j=1}^i l_j \dot{\theta}_j^2 \sin(\theta_a - \theta_j) \right]$$

Let's take a look at the sum

$$\sum_{i=a}^N m_i \left[\sum_{j=1}^i l_j \ddot{\theta}_j \cos(\theta_a - \theta_j) \right] = \sum_{i=a}^N \sum_{j=1}^i X_{ij} \quad \text{where} \quad X_{ij} = m_i l_j \ddot{\theta}_j \cos(\theta_a - \theta_j)$$

We now want to swap the order of the summation (into j first, i second), but we need to consider which pairs of indices i, j are present in the sum. For $j < a$, i ranges from a to N . For $j \geq a$, i ranges from j to N . This allows us to rewrite the sum as

$$\sum_{i=a}^N \sum_{j=1}^i X_{ij} = \sum_{j=1}^N \sum_{i=\max(a,j)}^N X_{ij} = \sum_{j=1}^N l_j \ddot{\theta}_j \cos(\theta_a - \theta_j) \sum_{i=\max(a,j)}^N m_i \quad \text{where} \quad \max(a, j) = \begin{cases} a, & \text{if } a > j \\ j, & \text{if } a \leq j \end{cases}$$

From this we see that

$$M_{ab} = l_b \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^N m_i$$

And by applying Cramer's rule, we obtain our desired solution:

$$\ddot{\theta}_n = \frac{\det(M_n)}{\det(M)}$$

where M_n is M with the n -th column by \vec{S} .

We can simplify the expression further still. Notice that M has a property that $M_{ab} \propto l_b$; in other words, we can write M as

$$M = M^* \begin{pmatrix} l_1 & & & \\ & l_2 & & \\ & & \ddots & \\ & & & l_N \end{pmatrix} \quad \text{where} \quad M_{ab}^* = \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^N m_i$$

Then obviously

$$\det(M) = \det(M^*) \prod_{i=1}^N l_i$$

You can convince yourself that

$$\det(M_n) = \det(M_n^*) \prod_{i=1, i \neq n}^N l_i$$

Then we can rewrite the result in terms of the simpler matrix M^* :

$$\ddot{\theta}_n = \frac{\det(M_n^*)}{l_n \det(M^*)}$$

Now: consider the matrix element M_{ba}^* :

$$M_{ba}^* = \cos(\theta_b - \theta_a) \sum_{i=\max(b,a)}^N m_i = \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^N m_i = M_{ab}^*$$

Hence M^* is symmetric.