Proposition: Let $M \subset \mathbb{R}^3$ be a compact surface and $f : M \to \mathbb{R}$ a morse function on M. Define $A :=$ # critical points (of f) with index 0, $B :=$ # critical points with index 1, and $C :=$ # critical points with index 2. Then, the Euler characteristic χ_M of M is given by $\chi_M = A - B + C$.

We start by giving a list of definitions and proving some useful lemmas.

Definition 1 (Simplicial 1,2-Complex)**.** A **simplicial 1-complex** is a topological space consisting of points and line segments, identified by verticies and along edges.

A **simplicial 2-complex** is a topological space consisting of points, line segments, and triangles identified along verticies and edges.

Examples and Non-examples

Example 1. The following are all simplicial 2-complexes:

1. A point
2. A line segment between two points
3. A triangle consisting of three line segments connecting three points

Example 2. The following are **NOT** simplicial 2-complexes

1.

Definition 2 (Euler Characteristic)**.** The **Euler characteristic** χ of a simplicial 2-complex is given by the formula

$$\chi = V - E + F$$

where V is the number of verticies, E is the number of edges, and F the number of triangles (faces).

We will be using the following two theorems without including proofs.

Theorem 1. If P_1, P_2 are homeomorphic simplicial 2-complexes, then $\chi(P_1) = \chi(P_2)$

Theorem 2. Every compact n-manifold M is homeomorphic to a simplicial n-complex P. We can define $\chi(M) := \chi(P)$.

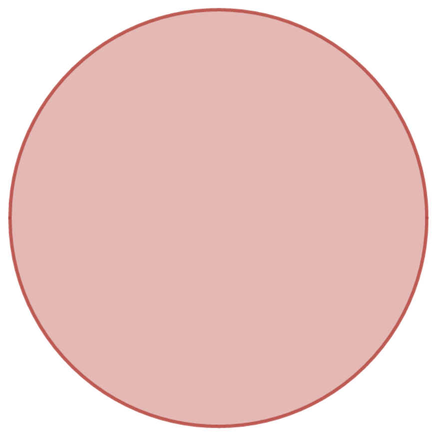
Theorem 3. Let P, Q be two disjoint simplicial n-complexes, then we have that

$$\chi(P \sqcup Q) = \chi(P) + \chi(Q).$$

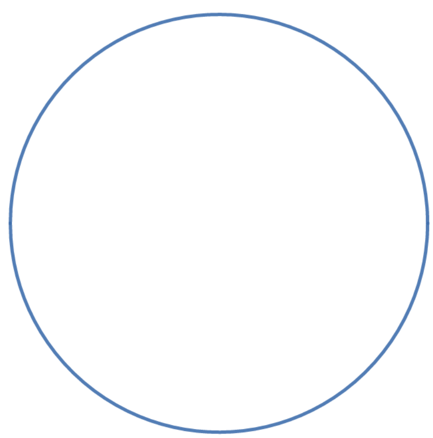
Proof. Let $\{V_1, E_1, F_1\} \subset \mathbb{N}^3$ be the number of verticies, edges, and faces of P, $\{V_2, E_2, F_2\}$ that of Q. Then, since $Q \cap P = \emptyset$, we have that the total number of verticies of $Q \sqcup P$ is $V_1 + V_2$, and the same goes for # edges and faces. Thus we get $\chi(P \sqcup Q) = V_1 + V_2 - (E_1 + E_2) + F_1 + F_2 = V_1 - E_1 + F_1 + V_2 - E_2 + F_2 = \chi(P) + \chi(Q)$. $\square$

In the context of Morse theory on surfaces, we can look for simplicial 2-complexes homeomorphic to 0,1,2-handles and their "attachment boundaries" in order to extract information about their Euler characteristics.

1. **0-Handle**
   A 0-handle H_0 is homeomorphic to the 2-disk, $H_0 \cong D^2 \times D^0$, with attachment boundary $\partial_{\mathcal{A}} H_0 \cong \partial D^2 \times D^0 \cong S^1$. The 2-disk is homeomorphic to a (filled) triangle, so $\chi(H_0) = 3 - 3 + 1 = 1$. Similarly, S^1 is homeomorphic to three verticies connected by 3 edges, ie. the perimeter of a triangle, so $\chi(\partial_{\mathcal{A}} H_0) = 3 - 3 + 0 = 0$.



$H_0 \cong D^2$



$\partial_{\mathcal{A}} H_0 \cong S^1$

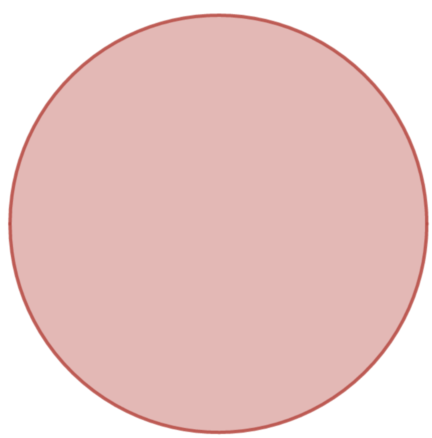
2. **1-Handle**
   A 1-handle H_1 is homeomorphic to the cartesian product of two intervals, $H_1 \cong D^1 \times D^1$ with attachment boundary $\partial_{\mathcal{A}} H_1 \cong \partial D^1 \times D^1 \cong D^1 \sqcup D^1$. H_1 is also homeomorphic to the 2-disk, but another way to determine its Euler

characteristic is by considering it as two (congruent, equilateral) triangles with one coincident edge. This gives $\chi(H_1) = 4 - 5 + 2 = 1$. The attachment boundary ∂H_1 is just given by two disjoint sets of two verticies and one edge connecting them, giving us $\chi(\partial_{\mathcal{A}} H_1) = 4 - 2 + 0 = 2$.

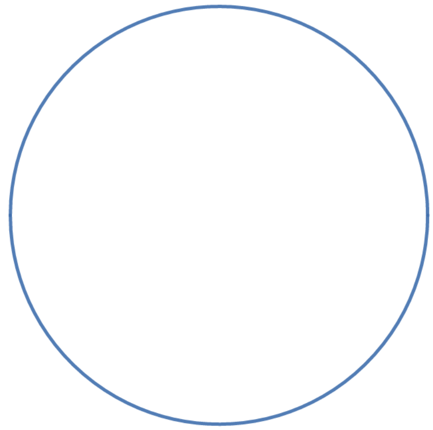
$H_1 \cong D^1 \times D^1$

$\partial_{\mathcal{A}} H_1 \cong D^1 \sqcup D^1$

3. **2-Handle**
   A 2-handle H_2 is homeomorphic to the 2-disk, $H_2 \cong D^0 \times D^2 \cong D^2$, with attachment boundary $\partial_{\mathcal{A}} H_2 \cong S^1$. The 2-disk is homeomorphic to a (filled) triangle, so $\chi(H_0) = 3 - 3 + 1 = 1$. Similarly, S^1 is homeomorphic to three verticies connected by 3 edges, ie. the perimeter of a triangle, so $\chi(\partial_{\mathcal{A}} H_0) = 3 - 3 + 0 = 0$.



$H_2 \cong D^2$



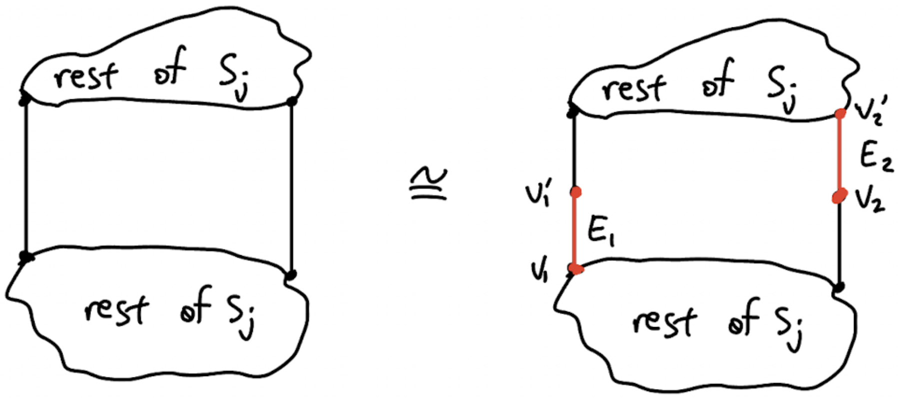
$\partial_{\mathcal{A}} H_2 \cong S^1$

Now we can give the proof of the main result:

Proof. Let $M \subset \mathbb{R}^3$ be a compact manifold which exhibits a Morse function $f : M \to \mathbb{R}$. Let $\{i_1, i_2, ..., i_n\} \in \{0, 1, 2\}^n$ denote the Morse indicies of the n (non-degenerate) critical points $\{p_1, p_2, ..., p_n\}$ of f. By the handle decomposition theorem, we know that M is homeomorphic to the union of A 0-handles, B 1-handles, and C 2-handles, where $A = \#\{i_j : i_j = 0\}, B = \#\{i_j : i_j = 1\}, C = \#\{i_j : i_j = 2\}$. To prove our desired result, we will split into cases of observing what happens to the Euler characteristic upon attaching a 0,1, or 2-handle. In each of the following cases, we will let $M_p = \{x \in M : f(x) \leq f(p)\}$, i.e. the sublevel set of M at p. Note that the boundary $\partial M_p = \{x \in M : f(x) = f(p)\}$.

1. $i_j = 0$
   In the case that our morse index is 0, we are simply taking the disjoint union $M_{p_{i_j}} \sqcup H_0$, so we can use our additivity property to get that $\chi(M \sqcup H_0) = \chi(M) + \chi(H_0) = \chi(M) + 1$.

2. $i_j = 1$
   Attaching a 1 handle is slightly more complicated. First, $M_{p_{i_j}}$ is homeomorphic to some simplicial 2-complex $\mathcal{S}_M$ with V_M verticies, E_M edges, and F_M faces. Furthermore $\partial M_{p_{i_j}}$ is homeomorphic to some simplicial 1-complex $\mathcal{S}_j$ with V_j verticies and E_j edges, where $\mathcal{S}_j \subseteq \mathcal{S}_M$. We claim that we can assume that there are two distinct edges E_1, E_2 such that their endpoints (verticies called, say V_1, V_1' and V_2, V_2') are also distinct. We are free to assume this since the simplicial complex $\mathcal{S}_j'$ created by adding verticies at the midpoints of each edge and "splitting" the original edge into two edges is homeomorphic $\mathcal{S}_j$.



Now that we have a subset of $\mathcal{S}_j$ which is homeomorphic to the attachment boundary $\partial_{\mathcal{A}} H_1$, we can attach the handle to the boundary by identifying the respective edges and verticies. Concretely, our 1-handle H_1 is homeomorphic to a simplicial 2-complex with $V_H = 4, E_H = 5, F_H = 2$ with attachment boundary $\partial_{\mathcal{A}} H_1$ homeomorphic to a simplicial 1-complex with $V_\partial = 4, E_\partial = 2$. Upon attaching, we identify the V_∂ with $\{V_1, V_1', V_2, V_2'\}$ and E_∂ with $\{E_1, E_2\}$ and we can recount the total number of verticies, edges, and faces to get the resulting Euler characteristic. Clearly, the number of faces is just the sum of the two parts, since we don't identify any faces, i.e. $F_{M \cup H_1} = F_M + F_H = F_M + 2$. Next, since we are now identifying 4 verticies of H_1 with those on S_j, we get that $V_{M \cup H_1} = V_M + V_H - 4 = V_M + 4 - 4 = V_M$ to account for double counting. Finally, we get something similar for the number of edges, namely $E_{M \cup H_1} = E_M + E_H - 2 = E_M + 5 - 2 = E_M + 3$. To get the Euler characteristic, we apply our formula to get $\chi(M \cup H_1) = V_M - E_M - 3 + F_M + 2 = V_M - E_M + F_M - 1 = \chi(M) - 1$

3. $i_j = 2$
   In order to even attach a 2-handle to $M_{p_{i_j}}$, we necessarily require that there be a connected component of $\partial M_{p_{i_j}}$ which is homeomorphic to S^1. Because the attachment boundary of H_2 is exacly S^1 (which is homeomorphic to a triangle), we get that $V_{M \cup H_2} = V_M + V_H - V_H = V_M$, since all verticies of H_2 are on its boundary. A similar statement holds for the edges of H_2, so we get $E_{M \cup H_2} = E_M + E_H - E_H = E_M$. As for faces, since we are only identifying the edges and verticies as subsets of the simplicial 2-complexes, the resulting number of faces is just the sum of the two parts, $F_{M \cup H_2} = F_M + F_H = F_M + 1$. Counting, we get $\chi(M \cup H_2) = V_M - E_M + F_M + 1 = \chi(M) + 1$.

From the above results, we see that we add 1 to our Euler characteristic everytime we attach a 0 or 2-handle, and subtract 1 everytime we attach a 1-handle, so

$$\chi(M) = A - B + C.$$

$\square$