Proposition: A Morse function $f : M \to \mathbb{R}$ defined on a closed surface M has only a finite number of critical points.

Proof. Consider the set $S = \bigcup p_i$, of all critical points on M. We claim that $\bar{S} = S$, that is, S is closed. Let $x \in \bar{S}$. Since S is dense in $\bar{S}$, for any $\varepsilon > 0$, we can find some p_i such that $d(p_i, x) < \varepsilon$. By looking at the sequence of p_{i_n} such that $d(p_{i_n}, x) \leq \frac{1}{2^n}$, we can find a sequence of critical points converging to x. We claim this means x must be a critical point and thus in S. Indeed, since f is differentiable with continuous derivative, $\frac{\partial f}{\partial x^i}(p_{i_n}) = 0$ for all p_{i_n} implies that $\frac{\partial f}{\partial x^i}(x) = 0$. Now that we know S is a closed subset of a compact space, S itself must be compact. Clearly the U_i form an open cover of S, so since S is compact, there is some finite subset $\{p_{\alpha_1}, ..., p_{\alpha_n}\}$ of critical points such that the U_{α_i} cover S. However, we claim now that f can only have n critical points in S. Indeed, by the Morse lemma, each U_{α_i} has a local coordinate system (x, y) on which $M = \pm x^2 \pm y^2 + c$, so there is exactly one critical point in each U_{α_i}, meaning the total number of critical points is equal to the total number in the finite subcover, n. □

Proposition: Let $f : M \to \mathbb{R}$ be a Morse function and let $[a, b]$ be a real interval. If f has no critical values in $[a, b]$, then $M_{[a,b]} \cong f^{-1}(a) \times [0, 1]$.

Proof. Let X be a gradient-like vector field for f. Since f has no critical values in $[a, b]$, we can consider a new vector field Y on $M_{[a,b]}$ defined by $Y = \frac{X}{X \cdot f}$. Intuitively, we are normalizing X such that the integral curves of Y are unit speed under the "metric" defined by f. We can verify this. Let $p \in f^{-1}(a)$ (in fact, this p can be chosen to be anywhere on $M_{[a,b]}$) and let $c_p(t)$ denote the integral curve of Y through p. Notice, $\frac{d}{dt} f(c_p(t)) = \frac{d}{dt} c_p(t) \cdot f = Y|_{c(t)} \cdot f = \frac{X \cdot f}{X \cdot f} = 1$, so we can write $f(c_p(t)) = t + C$ for some constant C. Since we defined $c_p(0) = p$ and $f(p) = a$, we have that $f(c_p(0)) = a$ and we get that $f(c_p(t)) = t + a$, so $f(c_p(b - a)) = b - a + a = b$. What we have just shown is that each integral curve of Y starting at a point on $f^{-1}(a)$ reaches some point in $f^{-1}(b)$. By the uniqueness of integral curves, each point on $f^{-1}(a)$ reaches exactly one point on $f^{-1}(b)$. In fact, we claim that the points in $f^{-1}(b)$ are in bijection with the points in $f^{-1}(a)$. Injectivity has already been shown, and to see surjectivity, we can repeat the steps above except with points starting at $f^{-1}(b)$ with integral curves of $Y = -\frac{X}{X \cdot f}$. What we get in this case is that each integral curve starting at a point on $f^{-1}(b)$ reaches a point on $f^{-1}(a)$, so the points on $f^{-1}(a)$ are in bijection with the points on $f^{-1}(b)$ by considering the family of integral curves defined by $Y = \pm \frac{X}{X \cdot f}$.

Now consider the map $h : f^{-1}(a) \times [0, b - a] \to M_{[a,b]}$ defined by $h(p, t) = c_p(t)$. We claim this is a diffeomorphism. First, we will show h is surjective. Indeed, for any point $p \in M_{[a,b]}$, we can consider the integral curve $c_p(t)$ of $Y = \frac{X}{X \cdot f}$. By the analysis above, there is some t' such that $c_p(t') \in f^{-1}(b)$, so there is some $q \in f^{-1}(a)$ such that $c_q(t_0) = p$ for some t_0. Injectivity follows from the uniqueness of integral curves.

injectivity needs work (periodic integral curve), can clean up surjectivity by proving lemma

injectivity: f composed with integral curve is montonic

□