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ORIGINAL ARTICLE

Continuous attractors of discrete-time recurrent neural networks

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Abstract This paper studies the continuous attractors of discrete-time recurrent neural networks. Networks in discrete time can directly provide algorithms for efficient implementation in digital hardware. Continuous attractors of neural networks have been used to store and manipulate continuous stimuli for animals. A continuous attractor is defined as a connected set of stable equilibrium points. It forms a lower dimensional manifold in the original state space. Under some conditions, the complete analytical expressions for the continuous attractors of discrete-time linear recurrent neural networks as well as discrete-time linear-threshold recurrent neural networks are derived. Examples are employed to illustrate the theory.

Keywords Continuous attractors · Discrete-time · Linear-threshold recurrent neural networks · Connected

1 Introduction

Generally, a recurrent neural network (RNN) can possess more than one and even infinite stable equilibrium points. These points may be isolated [1] or connected [2]. When the stable equilibrium points are distributed in a connected way, the network can exhibit the persistent activity. The set of stable connected equilibrium points is defined as a continuous attractor [3]. Continuous attractors have been used to describe the encoding of continuous stimuli such as the eye position [4, 5], head direction [6], moving direction [7, 8], path integrator [9–11], cognitive map [12] and population decoding [13, 14]. In these studies, there are two different categories of RNNs. In the first category, the networks have a finite number of neurons [4, 15]. The second sort of networks has infinite neurons [6, 16, 17]. These two categories of networks are different in dynamics from a mathematical point of view. This brief studies continuous attractors of the networks that belong to the first category.

So far, most studied models of continuous attractors are continuous-time recurrent neural networks (CRNNs) [4, 15, 18]. Few results of continuous attractors of discrete-time recurrent neural networks (DRNNs) have been reported. Generally speaking, dynamics of a discrete-time networks are different from the corresponding continuous-time networks [19, 20]. DRNNs possess advantages for direct computer simulations over digital simulations of the corresponding continuous-time models, and they can be easily implemented using digital hardware. Discrete-time neural networks have been widely studied in recent years by many authors such as image processing [21–23], associative memory [24–26], digital filters [27], some optimization problems [19, 28–30], winner-take-all [31, 32], etc.. In this paper, continuous attractors of two kinds of DRNNs are studied: discrete-time linear RNNs and discrete-time linear-threshold (LT) RNNs. Detailed sufficient conditions for the two kinds of DRNNs to possess continuous attractors are obtained. These conditions are different from the conditions of the CRNNs in [15].

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In order to find out the essence of continuous attractor of DRNNs, we use the simplest linear model to study the continuous attractor. Continuous attractors also exist in some nonlinear recurrent neural networks. It is not easy to explore the properties of continuous attractors of nonlinear neural models. Linear-threshold neural networks are some biologically plausible nonlinear network models [22, 33, 34]. Network's switching behavior from one partition to another was conducted in [33]. Periodic orbit that is one kind of connected attractors is presented in [34]. Necessary and sufficient conditions for complete convergence, existence of permitted and forbidden sets, as well as conditional multiattractivity of discrete-time LT networks, are given in [22]. The main contribution from this paper is the establishment of detailed sufficient conditions for the discrete-time LT networks to possess continuous attractors. This paper and [22] both study the set of stable equilibrium points of discrete-time LT networks, but they focus on different dynamical properties. The sets of stable equilibrium points in this paper are connected, which are called continuous attractors, while the sets of stable equilibrium points in [22] are disconnected. The continuous attractors theory in this paper is the complementarity of stability study in [22]. Moreover, in order to illustrate the differences of the continuous attractors of the linear and LT networks, comparative results of the two kinds of networks with the same connection matrices and external inputs are given. Two cases can be happened: both networks possess continuous attractors but their continuous attractors are different, and linear network does not possess continuous attractor, but the corresponding LT network possess continuous attractors.

This paper is organized as follows. Preliminaries are given in Sect. 2. The main results on the continuous attractors of linear DRNNs are given in Sect. 3. The continuous attractors of LT DRNNs are presented in Sect. 4. Comparative results and discussions of linear networks and LT networks are given in Sect. 5. Finally, conclusions are given in Sect. 6.

2 Preliminaries

The general form of DRNNs is:

$$x(k+1) = f(x(k) + b) \tag{1}$$

for $k \geq 0$, the state of the network at time k is described by $x(k) \in R^n, b \in R^n$ denotes the external input. $f : R^n \to R^n$ is a function.

In order to define the continuous attractor, some classical definitions of Lyapunov stability are given.

A point $x^* \in R^n$ is called an equilibrium point of (1), if it satisfies

$x^* = f(x^* + b)$.

Next, the definition of stability for an equilibrium point is given.

Definition 1 An equilibrium point x^* is called stable, if given any constant $\epsilon > 0$, there exists a constant $\delta > 0$ such that $\|x(0) - x^*\| \leq \delta$ implies that $\|x(k) - x^*\| \leq \epsilon$ for all $k \geq 0$. An equilibrium point is called unstable if it is not stable.

Definition 2 Set D is a connected if and only if Ø and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \bigcup B$ and A, B are disjoint open subsets of D, then either A = Ø or B = Ø.

A connected set is a set that cannot be partitioned into two nonempty subsets that are open in the relative topology induced on the set. Equivalently, it is a set that cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

With the stability definition of points and the definition of connected set, the definition of continuous attractor is given as follows naturally:

Definition 3 A set of equilibrium points C is called a continuous attractor if it is a connected set and each point $x^* \in C$ is stable.

Definition 4 A vector x is said to be positive, denoted by $x > 0$, if each element of x is positive. Similarly, a vector x is negative, denoted by $x < 0$, if each element of x is negative.

In the next two sections, continuous attractors of both linear DRNNs and LT DRNNs will be studied.

3 Discrete-time linear recurrent neural networks

The simplest discrete-time network model is completely linear:

$$x(k+1) = Wx(k) + b \tag{2}$$

for $k \geq 0$, where the strength of the connections is given by W, we assume that $W = (W_{ij})_{n \times n}$ is symmetric.

Since W is a symmetric matrix, it possesses an orthonormal eigensystem. Let $\lambda_i (i = 1, \cdots, n)$ be all the eigenvalues of W ordered by $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. Let the multiplicity of λ_1 be m and denote by V_{λ_1} the eigensubspace associated with the eigenvalue λ_1. Suppose that $S_i (i = 1, \cdots, n)$ compose an orthonormal basis in R^n such that each S_i is an eigenvector of W belonging to λ_i.

Suppose that $b = \sum_{i=1}^{n} \tilde{b}_i S_i$.

Given any initial state of the network $x(0) \in R^n$, let $x(k)$ be the trajectory starting from $x(0)$, then $x(k)$ can be represented as $x(k) = \sum_{i=1}^{n} z_i(k) S_i$ for $k \geq 0$, where $z_i(k)(i = 1, \cdots, n)$ are some functions. It is clear that $x(0) = \sum_{i=1}^{n} z_i(0) S_i$.

The following theorem gives sufficient conditions for network (2) to possess continuous attractor.

Theorem 1 *Suppose* $\lambda_1 = 1, |\lambda_i| \leq 1 (i = m+1, \cdots, p)$, *and* $b \perp V_{\lambda_1}$. *Then,*

$$C = \left\{ \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1 - \lambda_j} S_j \middle| c_i \in R (1 \leq i \leq m) \right\}$$

is a continuous attractor of network (2).

Proof Since $b \perp V_{\lambda_1}$, then $\tilde{b}_1 = \cdots = \tilde{b}_m = 0$.

Firstly, it is easy to see that C is a connected set.

Next, we prove that each point of C is an equilibrium point. Given any $x^* \in C$, there exist constants $c_i (i = 1, \cdots, m)$ such that

$$x^* = \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1 - \lambda_j} S_j.$$

Then,

$$\begin{aligned} Wx^* + b &= \sum_{i=1}^{m} c_i W S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1 - \lambda_j} W S_j + \sum_{i=1}^{n} \tilde{b}_i S_i \\ &= \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{\lambda_j \tilde{b}_j}{1 - \lambda_j} S_j + \sum_{i=1}^{n} \tilde{b}_i S_i \\ &= \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1 - \lambda_j} S_j \\ &= x^*. \end{aligned}$$

So, x^* is an equilibrium point of (2).

Finally, we prove that each equilibrium point of C is stable.

Given any $\epsilon > 0$, choose a constant $\delta = \epsilon$, if

$$\begin{aligned} \|x(0) - x^*\| &= \left\| \sum_{i=1}^{m} (z_i(0) - c_i) S_i + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1 - \lambda_j} \right) S_j \right\| \\ &= \sqrt{ \sum_{i=1}^{m} (z_i(0) - c_i)^2 + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1 - \lambda_j} \right)^2 } \\ &\leq \delta, \end{aligned}$$

then, we will prove that $\|x(k) - x^*\| \leq \epsilon$ for all $k \geq 0$.

Since the multiplicity of the largest eigenvalue $\lambda_1 = 1$ is m, it follows from (2) that

$$z_i(k+1) = \begin{cases} z_i(k), & 1 \leq i \leq m \\ \lambda_i z_i(k) + \tilde{b}_i, & m+1 \leq i \leq n \end{cases}$$

for $k \geq 0$. Solving this equation, it gives that

$$z_i(k) = \begin{cases} z_i(0), & 1 \leq i \leq m \\ \lambda_i^k z_i(0) + \frac{1 - \lambda_i^k}{1 - \lambda_i} \tilde{b}_i, & m+1 \leq i \leq n \end{cases}$$

for $k \geq 0$. Thus,

$$x(k) = \sum_{i=1}^{m} z_i(0) S_i + \sum_{j=m+1}^{n} \left(\lambda_j^k z_j(0) + \frac{1 - \lambda_j^k}{1 - \lambda_j} \tilde{b}_j \right) S_j \tag{3}$$

for $k \geq 0$. So we have

$$\begin{aligned} \|x(k) - x^*\| &= \left\| \sum_{i=1}^{m} (z_i(0) - c_i) S_i + \sum_{j=m+1}^{n} \lambda_j^k z_j(0) S_j \right. \\ &\quad \left. + \sum_{j=m+1}^{n} \left(\frac{1 - \lambda_j^k}{1 - \lambda_j} - \frac{1}{1 - \lambda_j} \right) \tilde{b}_j S_j \right\| \\ &= \sqrt{ \sum_{i=1}^{m} (z_i(0) - c_i)^2 + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1 - \lambda_j} \right)^2 \lambda_j^{2k} } \\ &\leq \sqrt{ \sum_{i=1}^{m} (z_i(0) - c_i)^2 + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1 - \lambda_j} \right)^2 } \\ &\leq \delta \\ &= \epsilon \end{aligned}$$

for all $k \geq 0$. By Definition 1, it shows that x^* is stable.

By Definition 3, the set C is a connected set, and each equilibrium point of C is stable. Then, C is a continuous attractor of the network (2). The proof is complete.

From the discussion above, we can see that the linear neural network (2) possesses a continuous attractor if the connection matrix W and the external input b satisfy the following conditions: the largest eigenvalue of the connection matrix W is unity and the external input b is orthogonal to the eigensubspace associated with the largest eigenvalue. The conditions will be used in the design of discrete-time linear RNNs.

Consider a three-dimensional linear DRNN with:

$$W = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}.$$

The largest eigenvalue of W is $\lambda_1 = 1$ with multiplicity 2, and another eigenvalue is -0.5. Moreover, $b \perp V_{\lambda_1}$. By Theorem 1, the network possesses a continuous attractor

$$C = \left\{ c_1 S_1 + c_2 S_2 + \frac{2}{3} S_3 \middle| c_1, c_2 \in R \right\},$$

where

$$S_1 = \begin{bmatrix} 0.2895 \\ -0.5164 \\ 0.8059 \end{bmatrix}, S_2 = \begin{bmatrix} 0.7634 \\ 0.6325 \\ 0.1310 \end{bmatrix}, S_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}.$$

S_1 and S_2 are two orthonormal eigenvectors of W belonging to the largest eigenvalue 1, S_3 is the unit eigenvector of W belonging to the rest eigenvalue -0.5. S_1, S_2, S_3 compose an orthonormal basis and determine a plane in Fig. 1. This plane is the continuous attractor.

Requiring $|\lambda_i| \leq 1 (i = m+1, \cdots, p)$ is an important condition. This is different from the continuous-time recurrent neural networks [15]. For CRNNs, if W is symmetric, then $\lambda_1 = 1$ and $b \perp V_{\lambda_1}$ are sufficient for the existing of the continuous attractor. However, for DRNNs, $\lambda_1 = 1$ and $b \perp V_{\lambda_1}$ are not sufficient to guarantee the linear DRNNs to have continuous attractor. This can be illustrated by the following three-dimensional linear DRNN with:

$$W = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

The largest eigenvalue of W is $\lambda_1 = 1$ with multiplicity 2. Moreover, $b \perp V_{\lambda_1}$. However, the other eigenvalue is -2. The network does not possess continuous attractor. Generally, discrete-time networks have different dynamics from their corresponding continuous-time networks.

4 Discrete-time linear-threshold recurrent neural networks

In the last section, the representation of continuous attractors of linear networks is obtained, but most biologically realistic models are not linear. It is not easy to explore the properties of continuous attractors of nonlinear RNNs. In this section, we study linear-threshold DRNNs which is biologically plausible.

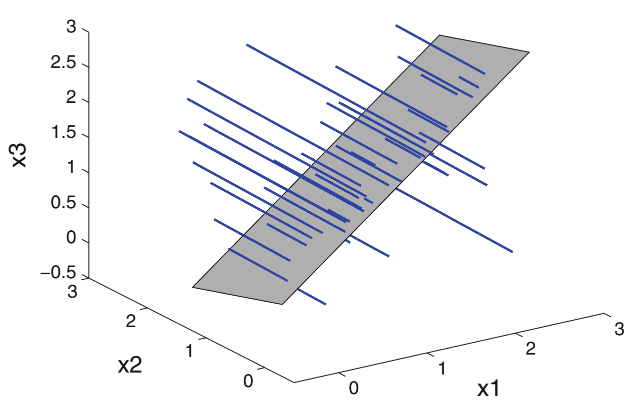

Fig. 1 Continuous attractor of discrete-time linear recurrent neural network. The plane is the continuous attractor. Forty trajectories starting from randomly selected initial points are attracted to the attractor

The model of discrete-time linear-threshold RNNs is described as follows:

$$x(k+1) = W\sigma(x(k)) + b \tag{4}$$

for $k \geq 0$. $\sigma(s)$ is a rectification nonlinear function:

$$\sigma(s) = \max\{0, s\}, \quad s \in R.$$

$W = (W_{ij})_{n \times n}$ is assumed to be symmetric. we desired to tuning W and b to derive conditions that can guarantee that a linear-threshold network will possess a continuous attractor.

Let $P \subseteq \{1, 2, \cdots, n\}$ be an index set with p elements and $Z = \{1, 2, \cdots, n\} - P$. W_P is a submatrix of W by removing from W all rows and columns not indexed by P. W_Z is constructed from W all rows and columns not indexed by Z. Then, the network (4) can be rewritten as

$$\begin{cases} x_P(k+1) = W_P \cdot \sigma(x_P(k)) + W_{PZ} \cdot \sigma(x_Z(k)) + b_P \\ x_Z(k+1) = W_{ZP} \cdot \sigma(x_P(k)) + W_Z \cdot \sigma(x_Z(k)) + b_Z \end{cases} \tag{5}$$

for $k \geq 0$, W_{PZ} is a matrix constructed from W by removing from W all rows not indexed by P and all columns not indexed by Z, W_{ZP} is a matrix constructed from W by removing from W all rows not indexed by Z and all columns not indexed by P.

Since the matrix W_P is the submatrix of W and W is a symmetric matrix, it is easy to check that W_P is also symmetric. Then there exists an orthonormal basis composed by eigenvectors of W_P. Let $\lambda_i^P (i = 1, \cdots, p)$ be all the eigenvalues of W_P ordered by $\lambda_1^P \geq \lambda_2^P \cdots \geq \lambda_p^P$. Suppose that the multiplicity of λ_1^P is m $(m \leq p)$, clearly, $\lambda_1^P = \lambda_2^P = \cdots = \lambda_m^P$. Denote by $V_{\lambda_1^P}^P$ the eigensubspace associated with the largest eigenvalue λ_1^P. Let $S_i^P \in V_{\lambda_1^P}^P (i = 1, \cdots, p)$ be unit eigenvectors that are mutually orthogonal. These eigenvectors form a basis of $V_{\lambda_1^P}^P$. Suppose that $b_P = \sum_{i=1}^p \tilde{b}_i^P \cdot S_i^P$.

In this section, detailed sufficient condition for the LT network to have continuous attractor is given, and the expression of the continuous attractor of LT network is obtained.

Theorem 2 *Suppose there exists an index set P such that* $\lambda_1^P = 1, |\lambda_i^P| \leq 1 (i = m+1, \cdots, p)$, *and* $b_P \perp V_{\lambda_1^P}^P$. *If it holds that*

$$\begin{cases} \sum_{i=1}^{m} c_i S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P > 0 \\ W_{ZP} \cdot \left[\sum_{i=1}^{m} c_i \cdot S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P \right] + b_Z < 0 \end{cases} \tag{6}$$

for any constants $c_i > 0 (i = 1, \cdots, m)$, *then, the network* (4) *possesses a continuous attractor:*

$$C = \begin{bmatrix} \sum_{i=1}^{m} c_i S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P \\ W_{ZP}\left[\sum_{i=1}^{m} c_i \cdot S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P\right] + b_Z \end{bmatrix},$$

where $c_i > 0 (i = 1, \cdots, m)$.

Proof Since $b_P \perp V_{\lambda_1^P}^P$, then $\tilde{b}_1^P = \cdots = \tilde{b}_m^P = 0$. Clearly, C is a connected set.

Given any $x^* \in C$, there exist constants $c_i > 0 (i = 1, \cdots, m)$ such that

$$\begin{cases} x_P^* = \sum_{i=1}^{m} c_i S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P > 0, \\ x_Z^* = W_{ZP} \cdot \left[\sum_{i=1}^{m} c_i \cdot S_i^P + \sum_{j=m+1}^{p} \frac{\tilde{b}_j^P}{1-\lambda_j^P} S_j^P\right] + b_Z < 0. \end{cases}$$

It is easy to show that x^* is an equilibrium point.

Next, we prove that each $x^* \in C$ is stable.

Given a constant ε such that

$$0 < \epsilon \le \min\left\{x_i^*(i \in P), -x_j^*(j \in Z)\right\},$$

define a neighborhood B_ϵ of x^* by

$$B_\epsilon = \left\{x \in R^n \middle| \left\|x_P - x_P^*\right\| < \frac{\epsilon}{2}, \|x_Z - x_Z^*\| < \|W_{ZP}\| \cdot \epsilon\right\}. \tag{7}$$

It can be proven that each trajectory starting from B_ϵ will stay there forever. We will show this by counterproof.

To declare this point, construct two functions as follows:

$$\begin{cases} V_P(k) = \left\|x_P(k) - x_P^*\right\|^2 \\ V_Z(k) = \left\|x_Z(k) - x_Z^*\right\|^2 \end{cases}$$

for $k \ge 0$.

From (5) it holds that

$$V_P(k+1) - V_P(k) = \left\|W_P\sigma(x_P(k)) + W_{PZ}\sigma(x_Z(k)) + b_P - x_P^*\right\|^2 - \left\|x_P(k) - x_P^*\right\|^2 \tag{8}$$

and

$$V_Z(k+1) - V_Z(k) = \left\|W_{ZP}\sigma(x_P(k)) + W_Z\sigma(x_Z(k)) + b_Z - x_Z^*\right\|^2 - \left\|x_Z(k) - x_Z^*\right\|^2 \tag{9}$$

for $k \ge 0$.

If $x(0) \in B_\epsilon$ and the trajectory $x(k)$ starting from $x(0)$ cannot stay in B_ϵ for all $k \ge 0$, then, two cases can happen.

Case 1. There exists a $k_1 > 0$ such that

$$\begin{cases} V_P(k_1) \ge \left(\frac{\epsilon}{2}\right)^2, & \\ V_P(k) < \left(\frac{\epsilon}{2}\right)^2, & 0 \le k < k_1, \\ V_Z(k) \le (\|W_{ZP}\| \cdot \epsilon)^2, & 0 \le k \le k_1. \end{cases}$$

However, since $\lambda_1^P = 1$ and $\left|\lambda_i^P\right| < 1 (i = m+1, \cdots, p)$, then

$$\max_{1 \le i \le p}\left\{\left|\lambda_i^P\right|\right\} = 1.$$

Clearly,

$$\max_{1 \le i \le p}\left\{\left(\lambda_i^P\right)^2\right\} = 1.$$

Moreover, since $x_P^* = W_P \cdot x_P^* + b_P$, then $x_P^* - b_P = W_P \cdot x_P^*$.

It follows from (8) that

$$\begin{aligned} V_P(k_1) - V_P(k_1-1) &= \|W_P \cdot \sigma(x_P(k_1-1)) + W_{PZ} \cdot \sigma(x_Z(k_1-1)) \\ &\quad + b_P - x_P^*\|^2 - \|x_P(k_1-1) - x_P^*\|^2 \\ &= \left\|W_P \cdot x_P(k_1-1) + b_P - x_P^*\right\|^2 \\ &\quad - \left\|x_P(k_1-1) - x_P^*\right\|^2 \\ &= \left\|W_P \cdot \left(x_P(k_1-1) - x_P^*\right)\right\|^2 \\ &\quad - \left\|x_P(k_1-1) - x_P^*\right\|^2 \\ &\le \left[x_P(k_1-1) - x_P^*\right]^T \cdot W_P^2 \\ &\quad \cdot \left[x_P(k_1-1) - x_P^*\right] \\ &\quad - \left\|x_P(k_1-1) - x_P^*\right\|^2 \\ &\le \left[\max_{1 \le i \le p}\left\{\left(\lambda_i^P\right)^2\right\} - 1\right] \\ &\quad \times \left\|x_P(k_1-1) - x_P^*\right\|^2 \\ &= 0. \end{aligned}$$

This is a contradiction.

Case 2. There exists a $k_2 > 0$ such that

$$\begin{cases} V_Z(k_2) \ge (\|W_{ZP}\| \cdot \epsilon)^2, & \\ V_Z(k) < (\|W_{ZP}\| \cdot \epsilon)^2, & 0 \le k < k_2 \\ V_P(k) \le \left(\frac{\epsilon}{2}\right)^2, & 0 \le k \le k_2. \end{cases}$$

Thus, it must hold that $V_Z(k_2) > \left(\frac{\|W_{ZP}\| \cdot \epsilon}{2}\right)^2$. However, from (9), it follows that

$$\begin{aligned} V_Z(k_2) &= \left\|x_Z(k_2) - x_Z^*\right\|^2 \\ &= \left\|W_{ZP}\sigma(x_P(k_2-1)) + W_Z\sigma(x_Z(k_2-1)) + b_Z - x_Z^*\right\|^2 \\ &= \left\|W_{ZP}(x_P(k_2-1) - x_P^*)\right\|^2 \\ &= \|W_{ZP}\|^2 \left\|x_P(k_2-1) - x_P^*\right\|^2 \\ &\le \left(\frac{\|W_{ZP}\| \cdot \epsilon}{2}\right)^2. \end{aligned}$$

This is a contradiction.

It means that for any $x(0) \in B_\epsilon$, it holds that $x(k) \in B_\epsilon$ for all $k \ge 0$. Clearly, it shows that x^* is stable.

By Definition 3, the set C is a connected set and each equilibrium point of C is stable. Then, C is a continuous attractor of the network (4). The proof is complete.

It can be seen from Theorem 2 that submatrix W_P plays an important role. These conditions are given in terms of the eigenvectors of the eigensubspace $V^P_{\lambda_1^P}$, thus, the continuous attractors can be easily constructed from the conditions of Theorem 2.

Let us consider an example of three-dimensional discrete-time linear-threshold network

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sigma(x(k)) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{10}$$

for $k \geq 0$. Denote

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Let $P = \{1, 2\}$, then $Z = \{3\}$ and

$$W_P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, W_{ZP} = [0 \quad 0],$$

and

$$b_P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b_Z = -1.$$

The largest eigenvalue of W_P is $\lambda_1^P = 1$ with multiplicity 2. Moreover, $b_P \perp V^P_{\lambda_1^P}$.

$$S_1^P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, S_2^P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$\begin{cases} c_1 \cdot S_1^P + c_2 \cdot S_2^P = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} > 0, \\ W_{ZP} \cdot \left[c_1 \cdot S_1^P + c_2 \cdot S_2^P \right] + b_Z = -1 < 0 \end{cases}$$

for any $c_1, c_2 > 0$. By Theorem 2,

$$C = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ -1 \end{bmatrix} \middle| c_1, c_2 > 0 \right\}$$

is the continuous attractor.

The plane in Fig. 2 is the continuous attractor of discrete-time LT recurrent neural network (10). This plane is located in the fifth octant, and it is just a part of the plane $x_3 = -1$. The open circles that are the terminal points of the trajectories are located on this plane, so all the trajectories converge to the continuous attractor.

Requiring $\left|\lambda_i^P\right| \leq 1 (i = m+1, \cdots, p)$ is also an important condition for LT DRNNs.

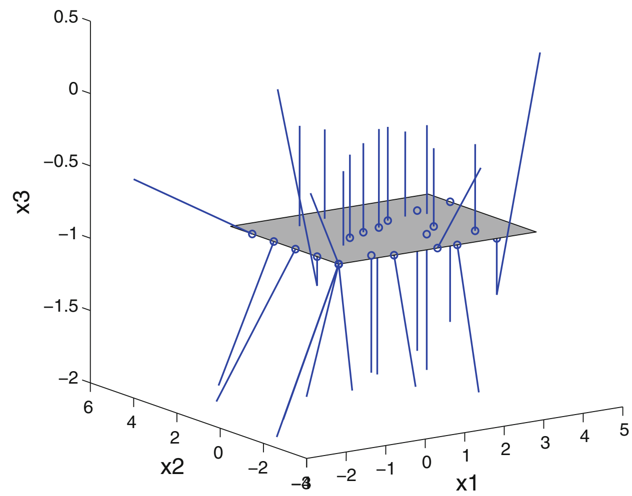

Fig. 2 Continuous attractor of discrete-time LT recurrent neural network (10). The plane is the continuous attractor. Trajectories are attracted to the continuous attractor

5 Comparative results and discussions

Continuous attractors of linear DRNNs and LT DRNNs have been studied in the above sections. LT DRNN is a kind of nonlinear network with unsaturating linear-threshold activation function:

$$\sigma(s) = \begin{cases} s, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

When $\sigma(s) = s$, LT RNN is reduced to linear RNN, while when $\sigma(s) = 0$, LT RNN is different from linear RNN. Due to the activation function σ, the two networks are different from each other. In fact, the dynamics and analyzing methods of LT network are more complex than linear neural networks. The whole synaptic connection matrix contributes to the existence of the continuous attractor in linear network, while the submatrices of the connection matrix are used to study the continuous attractor of LT networks. In this section, we will illustrate the differences of the continuous attractors of the two kinds of networks with the same connection matrix and external input by some examples.

Case 1. Linear network and LT network both possess continuous attractors, but their continuous attractors are different.

The linear DRNN

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{11}$$

and LT DRNN

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma(x(k)) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{12}$$

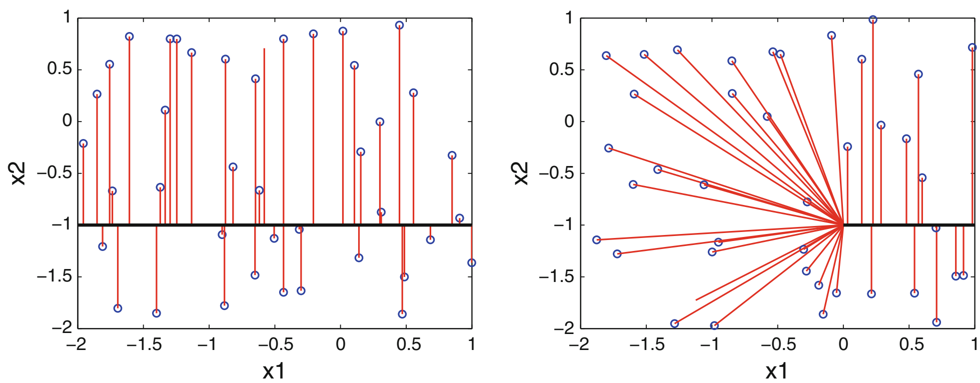

Fig. 3 Continuous attractors of linear network (11) (*left*) and LT networks (12) (*right*). The *black bold lines* in the figure are the continuous attractors. The *open circles* are randomly selected initial points, forty trajectories from these initial points are attracted to the attractors respectively

are with the same connection matrix W and external input b as follows,

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The largest eigenvalue of W is $\lambda_1 = 1$ with multiplicity 1 and W has another eigenvalue 0. Moreover, $b \perp V_{\lambda_1}$. By Theorem 1, the network (11) possesses a continuous attractor

$$C = \left\{ \left[\begin{array}{c} c \\ -1 \end{array} \right] \middle| c \in R \right\}.$$

In the LT model (12), let $P = \{1\}$, then $Z = \{2\}$ and

$W_P = 1, W_{ZP} = 0, S^P = 1, b_Z = -1$

and

$$\begin{cases} c \cdot S^P = c > 0, \\ W_{ZP} \cdot c \cdot S^P + b_Z = -1 < 0 \end{cases}$$

for any $c > 0$. By Theorem 2,

$$C = \left\{ \left[\begin{array}{c} c \\ -1 \end{array} \right] \middle| c > 0 \right\}$$

is the continuous attractor of LT model (12).

Figure 3 shows the continuous attractors of linear network (11) and LT networks (12). The black bold lines in the figure are the continuous attractors. The open circles are randomly selected initial points, forty trajectories from these initial points are attracted to the attractors respectively.

It is easy to see that the continuous attractors of the two models are different in the left part of each figure. The two continuous attractors are different because $c \in R$ in the linear network while $c > 0$ in the LT network.

Case 2. Linear network does not possess continuous attractor, but the corresponding LT network possesses continuous attractors.

For example, linear network

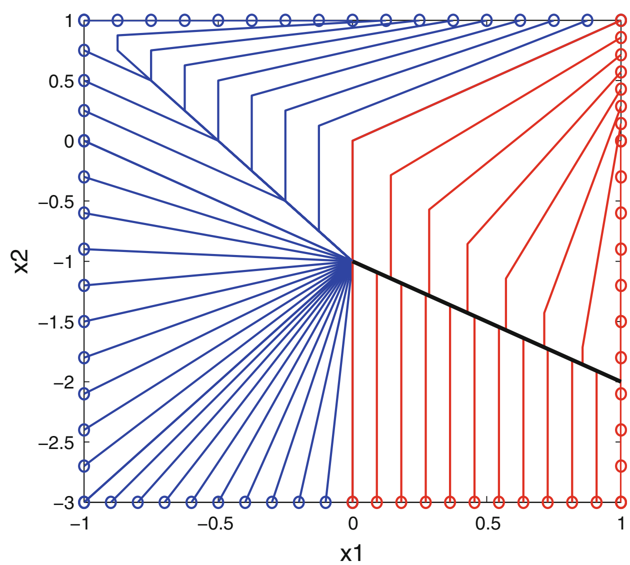

Fig. 4 Continuous attractor of discrete-time LT recurrent neural network (14). The *black bold line* in the figure is the continuous attractor. Trajectories starting from the *open circles* are attracted to the attractor

$$x(k+1) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{13}$$

has no continuous attractor, while the corresponding LT network

$$x(k+1) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \sigma(x(k)) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{14}$$

does possess continuous attractor.

Figure 4 shows the continuous attractor of the network (14). The black bold line in the figure is the continuous attractor. The open circles are initial points, trajectories from these initial points are attracted to the continuous attractor. The blue trajectories are all attracted to the same point $(0, -1)$ that is one of the endpoint of continuous attractor, while the red ones are attracted to the other points of the continuous attractor.

6 Conclusion

Continuous attractors of discrete-time linear RNNs and discrete-time linear-threshold RNNs have been studied in this paper. Explicit representations of continuous attractors are obtained by using the eigenvectors of the connection matrix. The representations can be looked as the solutions of the networks. From the study of this paper, it is possible to tune W and b to realize a perfect continuous attractor in linear-threshold networks. Linear-threshold RNNs are one of the nonlinear RNNs, the conditions that guarantee other nonlinear RNNs to possess continuous attractors remain to be resolved.

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