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Dynamics of iterated-map neural networks

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We analyze a discrete-time neural network with continuous state variables updated in parallel. We show that for symmetric connections, the only attractors are fixed points and period-two limit cycles. We also present a global stability criterion which guarantees only fixed-point attractors by placing limits on the gain (maximum slope) of the sigmoid nonlinearity. The iterated-map network has the same fixed points as a continuous-time analog electronic neural network and converges to an attractor after a small number of iterations of the map.

Recent progress in the design and analysis of neural-network models suggests that high-dimensional nonlinear dynamical systems can be used to perform rapid parallel computation.[1,2] However, large networks implemented as software or as digital hardware can be disappointingly slow, and networks implemented with analog hardware can have stability problems.[3,4] The study of the dynamics of spin glasses[5] and other complex high-dimensional systems is similarly hampered by these computational limitations.

The design of neural networks is a tradeoff between speed and stability. For example, it is well known that a symmetrically connected neural network evolving in continuous time[6,7] or with discrete-time sequential updating of state variables (and zero self-connection)[8] will always converge to a fixed-point attractor. This fixed-point-only condition is not assured for discrete-time networks with parallel updating of state variables.[9,10] Systems designed to perform parallel updating operate quickly, but can produce unwanted sustained oscillation.

In this Rapid Communication we analyze a deterministic analog neural network with discrete-time parallel dynamics, and present a simple global stability criterion which guarantees that all attractors are fixed points. This iterated-map network is designed to have the same fixed points as the continuous-time analog neural network described by Hopfield.[6] Stability analyses of similar parallel-update systems have been presented for discrete (± 1) state variables[10] (cellular automata) and for continuous state variables with a linear then hard saturation function.[11] Stability results for these systems emerge as special cases of the results presented here.

The continuous-time evolution of an analog neural network can be described by the following system of coupled nonlinear differential equations:

$$C_i du_i(t)/dt = -u_i(t)/R_i + \sum_j T_{ij} f_j(u_j(t)) + I_i \,. \quad (1)$$

In the electronic circuit interpretation of (1),[6] each state variable $u_i(t)$ $(i=1,\ldots,N)$ represents the input voltage of a saturable amplifier with transfer function $f_i(u_i)$, input capacitance C_i, and fixed external input current I_i. R_i is the parallel resistance at the input of amplifier i. The nonlinear function $f_i(u_i)$, assumed throughout to be a monotone sigmoid function, is often modeled as $\tanh(\beta_i u_i)$, where β_i is the gain of the amplifier before saturation. We also assume throughout that $\mathbf{T}=[T_{ij}]$ is a real symmetric matrix. The attractors of (1) are fixed points whose components u_i^* satisfy $u_i^*/R_i = \sum_j T_{ij} f_j(u_j^*) + I_i$. Because of the nonlinearity, many fixed-point solutions may exist.

The iterated-map version of (1) is given by

$$u_i(t+1)/R_i = \sum_j T_{ij} f_j(u_j(t)) + I_i \,. \quad (2)$$

As in (1), $\mathbf{T}=[T_{ij}]$ is a symmetric matrix and the nonlinear functions $f_i(u_i)$ are all sigmoidal—though possibly different functions for each i. Specific properties of $f_i(u_i)$ required for the analysis will be discussed below. An important property of (2) is that its fixed points, defined by the condition $u_i^*(t+1)=u_i^*(t)$ for all i, are identical to the fixed points of (1).

We will now prove two properties of the iterated-map system (2). First, we show that the only attractors of the iterated-map system (2) are fixed points and period-two limit cycles. Second, we show that the iterated-map system (2) has only fixed-point attractors when the matrix $\mathbf{T}+(\mathbf{RB})^{-1}$ is positive definite, where $\mathbf{R}=[R_i\delta_{ij}]$, $\mathbf{B}=[\beta_i\delta_{ij}]$, and the gain β_i is defined as the maximum slope of the sigmoid $f_i(u_i)$. A sufficient condition for $\mathbf{T}+(\mathbf{RB})^{-1}$ to be positive definite, and thus for the iterated map (2) to possess only fixed-point attractors, can be stated as a simple stability criterion which places a limit on the maximum gain:

$$(R_i\beta_i) < |1/\lambda_{\min}| \text{ for all } i \implies \text{only fixed-point attractors}\,. \quad (3)$$

In Eq. (3) R_i and β_i are positive real values and $\lambda_{\min}<0$ is the most negative eigenvalue of the real symmetric matrix $\mathbf{T}$. If $\mathbf{T}$ has no negative eigenvalues then $\mathbf{T}+(\mathbf{RB})^{-1}$ is always positive definite and (2) has only fixed-point attractors for all values of β_i and R_i. When the stability criterion (3) is satisfied the iterated map (2) works as a fast, globally stable neural network. An example of the rapid convergence of the iterated map (2) to the identical fixed point as the differential system (1) is shown in Fig. 1(a), where the gain $\beta=2.1$ satisfies the stability criterion (3). In Fig. 1(b), the gain $\beta=4$ exceeds the stability limit, pro-

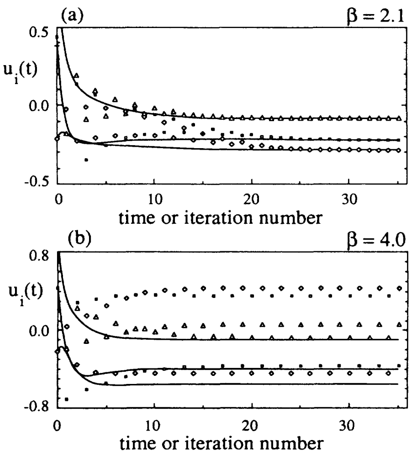

FIG. 1. Comparison of continuous-time dynamics of (1) with iterated-map dynamics of (2) for identical initial states. The systems are size $N=10$, but only three state variables $u_i(t)$ are plotted. T_{ij} is a random symmetric ± 1 matrix with $T_{ii}=0$; $R_i^{-1}=\sum_j |T_{ij}|$, $f_i(u_i)=\tanh(\beta u_i)$, and $R_iC_i=1$ in (1) for all i. The stability criterion (3) applied to this iterated map requires $\beta<2.29$ for fixed-point-only dynamics. (a) At $\beta=2.1$, the two systems converge to the same fixed point. (b) At $\beta=4$, the iterated map enters a period-two limit cycle while the continuous-time system converges to a fixed point.

ducing oscillation in the iterated map while the corresponding differential system remains stable.

To prove these two results, it is useful to change the form of (2) using a new set of variables $x_i, i=1,\ldots,N$ defined by $\sum_j T_{ij}x_j(t) \equiv u_i(t)/R_i - I_i$. Equation (2) can be written in terms of the x_i as

$$x_i(t+1)=F_i\left[\sum_j T_{ij}x_j(t)+I_i\right], \tag{4}$$

where $F_i(z)=f_i(R_iz)$, with z a dummy argument. The two forms of the iterated-map system, Eqs. (2) and (4), are equivalent.[12] For a special choice of the sigmoid function $F_i(z)=\tanh(\beta z)$ the fixed points of (4) are the solutions of the mean-field Ising model, where β is the inverse temperature, I_i is the local applied magnetic field, and $x_i(t)$ is the magnetization at site i at time t.[13] The hyperbolic tangent nonlinearity also provides a realistic transfer function model of saturable amplifiers used in hardware implementations of neural networks.[14] We will use the more general form $F_i(z)$, which allows a variety of nonlinear functions.

We now show that the attractors of (4) are either fixed points or period-two limit cycles by constructing a Liapunov function $E(t)$:

$$E(t)=-\sum_{i,j}T_{ij}x_i(t)x_j(t-1)-\sum_i I_i[x_i(t)+x_i(t-1)]+\sum_i[G_i(x_i(t))+G_i(x_i(t-1))], \tag{5a}$$

where

$$G_i(x_i)\equiv\int_0^{x_i}F_i^{-1}(z)\,dz. \tag{5b}$$

Two conditions on $F_i(z)$ ensure that $E(t)$ is a Liapunov function: (i) $F_i(z)$ is a monotone increasing function for all i. (ii) $|F_i(z)|\le A_i|z|^{\alpha_i}$ for all z for some constants $0\le\alpha_i<1$ and $A_i<\infty$. Note: $F_i(z)$ does not need to be symmetric about $z=0$; $F_i(z)$ can be locally concave up or concave down but must increase in magnitude slower than linear at large z: $F_i(z)$ does not need to saturate for large z; $F_i(z)$ can be a different function for each i.

The change in $E(t)$ between times t and $t+1$, defined as $\Delta E(t)\equiv E(t+1)-E(t)$, can be found from (4) and (5a) and the symmetry $T_{ij}=T_{ji}$:

$$\Delta E(t)=-\sum_i F_i^{-1}(x_i(t+1))\Delta_2x_i(t)+\sum_i[G_i(x_i(t+1))-G_i(x_i(t-1))], \tag{6}$$

where $\Delta_2x_i(t)\equiv x_i(t+1)-x_i(t-1)$ is the change in x_i over *two* time steps. For $G_i(x)$ concave up for all values of x we can write the following inequality [see Fig. (2)]:

$$G_i(x_i(t+1))-G_i(x_i(t-1))\le G_i'(x_i(t+1))\Delta_2x_i(t), \tag{7}$$

where $G_i'(x_i(t+1))$ is the derivative of $G_i(x)$ at the point $x=x_i(t+1)$. The case of equality in (7) only occurs when $\Delta_2x_i(t)=0$. The requirement that $G_i(x)$ be concave up is not very restrictive, it only requires that $F_i(z)$ be single valued for all z. Inserting the inequality (7) into (6) gives

$$\Delta E(t)\le\sum_i[G_i'(x_i(t+1))-F_i^{-1}(x_i(t+1))]\Delta_2x_i(t).$$

The difference in the square brackets equals zero by Eq. (5b) giving our first result:

$$\Delta E(t)\le 0;\quad \Delta E(t)=0 \Longrightarrow \Delta_2x_i(t)=0. \tag{8}$$

Thus, $E(t)$ is a Liapunov function for the iterated map (4). The attractors of (4), where $\Delta E(t)=0$, satisfy $\Delta_2x_i(t)=0$ or $x_i(t+1)=x_i(t-1)$. Therefore, all attractors of (4), or equivalently of (2), are either two-cycles or fixed points.

We now show that by lowering all gains below a certain value which depends on the minimum eigenvalue of the connection matrix, all stable limit cycles disappear and only fixed-point attractors remain. We consider a second function $L(t)$ which is very similar to $E(t)$:

$$L(t)=-\tfrac{1}{2}\sum_{i,j}T_{ij}x_i(t)x_j(t)-\sum_i I_ix_i(t)+\sum_i G_i(x_i(t)), \tag{9}$$

with $G_i(x_i(t))$ as defined in (5b). The change in $L(t)$ between times t and $t+1$, defined as $\Delta L(t)\equiv L(t+1)-L(t)$, can be found using (9) and (4) and the symmetry $T_{ij}=T_{ji}$.

$$\Delta L(t)=-\tfrac{1}{2}\sum_{i,j}T_{ij}\Delta x_i(t)\Delta x_j(t)-\sum_i F_i^{-1}(x_i(t+1))\Delta x_i(t)+\sum_i[G_i(x_i(t+1))-G_i(x_i(t))], \tag{10}$$

where $\Delta x_i(t)\equiv x_i(t+1)-x_i(t)$. Note that $\Delta x_i(t)$ is the change in *one* time step.

We now construct an inequality similar to Eq. (7), but including a quadratic term along with the linear term that appears on the right-hand side of (7). Choosing the coefficient of the quadratic term to be the minimum curvature of $G_i(x)$, given by $(R_i\beta_i)^{-1} = \min_x[d^2G_i(x)/dx^2]$, yields the following inequality, as illustrated in Fig. 2:

$$G_i(x_i(t+1)) - G_i(x_i(t)) \le G_i'(x_i(t+1))\Delta x_i(t) - \tfrac{1}{2}(R_i\beta_i)^{-1}[\Delta x_i(t)]^2. \tag{11}$$

Equations (10) and (11), and the equality $G_i'(x_i) = F_i^{-1}(x_i)$ from (5b), yield

$$\Delta L(t) \le -\tfrac{1}{2}\sum_{i,j}[T_{ij} + \delta_{ij}(R_i\beta_i)^{-1}]\Delta x_i(t)\Delta x_j(t), \tag{12}$$

where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. If the matrix $\mathbf{T}+(\mathbf{RB})^{-1}$, which appears in component form in (12), is positive definite then

$$\Delta L(t) \le 0;\ \Delta L(t) = 0 \Longrightarrow \Delta x_i(t) = 0. \tag{13}$$

Thus under the condition that $\mathbf{T}+(\mathbf{RB})^{-1}$ is a positive definite matrix, $L(t)$ is a Liapunov function of the iterated-map system (4), and $\Delta x_i(t) = 0$ for all i on all attractors, that is, all attractors are fixed points. A sufficient condition for $\mathbf{T}+(\mathbf{RB})^{-1}$ to be positive definite is $R_i\beta_i < |1/\lambda_{\min}|$ for all i, which appears as Eq. (3). If $\mathbf{T}$ has no negative eigenvalues then $\mathbf{T}+(\mathbf{RB})^{-1}$ is always positive definite.

Although the differential system and the iterated-map system have the same fixed points, the stability criterion (3) does not guarantee that all *stable* fixed points of the differential system will also be stable in the iterated-map system. It can be shown using local stability analysis that any stable fixed point of the iterated map will also be stable in the differential system, and any unstable fixed point of the differential system will also be unstable for the iterated map. In practice we have not found a case numerically where the differential system had a stable fixed point that was not also stable for an identically configurated iterated-map system which satisfied (3).

As an application of the iterated-map neural network we construct an associative memory based on the Hebb learning rule,[15] specified by the following iterated-map system:

$$u_i(t+1)/R_i = \sum_{j=1}^{N} T_{ij}\tanh[\beta u_j(t)], \tag{14a}$$

$$T_{ij} = \sum_{\mu=1}^{m}\xi_i^\mu\xi_j^\mu;\ T_{ii}=0;\ R_i^{-1} = \sum_{j=1}^{N}|T_{ij}|. \tag{14b}$$

The dynamics of (14) are designed to send an initial state to a fixed point near the memory vector ξ^μ closest to that initial state. We take each of the m memories to be a random string of ± 1's. We assume $1 \ll m < N$, and define the ratio of memories to neurons as $m = \alpha N$ where $0<\alpha<1$. In this case, we find $\lambda_{\min} = -m$ and R_i is narrowly peaked about $N^{-1}(\pi/2m)^{1/2}$ for all i, giving a stability criterion based on (3) in terms of the gain, network size, and number of stored memories:

$$\beta < \left(\frac{2}{\pi}\right)^{1/2}\frac{N}{m^{1/2}} \approx 0.8\left(\frac{N}{\alpha}\right)^{1/2}. \tag{15}$$

Figure 3 compares the performance of (14) and its continuous-time counterpart for $N=100$ and $m=7$. Starting from the same initial state, the iterated map converges to the same fixed point as the differential equation after about 13 iterations. The gain used for both systems was $\beta=4$; from (15) the iterated map has only fixed-point attractors for $\beta < \sim 30$. As the network is made larger, the stability criterion for the iterated-map Hebb rule becomes more easily satisfied, as seen in (15).

In summary we have investigated a discrete-time analog neural network and have shown that for symmetric connections the only attractors are either fixed points or period-two limit cycles. When neuron gains β_i satisfy the stability criterion (3), only fixed-point attractors exist. These results can be used to design fast, stable neural networks implemented using parallel processing software and digital hardware and to guarantee stability in analog very large-scale integration (VLSI) implementations by using clocking techniques to synchronize analog neurons.

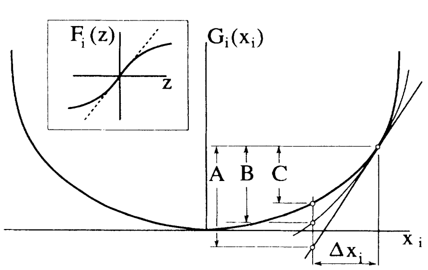

FIG. 2. Graphical representation of inequalities (7) and (11) for a typical sigmoid nonlinearity $F_i(z)$ with maximum slope β_iR_i. The concave-up function $G_i(x_i)$ is defined in (5b). A line and a parabola with second derivative $(R_i\beta_i)^{-1} = \min_x[d^2G(x)/dx^2]$ are tangent to the curve $G_i(x_i)$ at the point $[x_i(t+1), G_i(x_i(t+1))]$. Equation (7) is represented as the strict inequality $C<A$; Eq. (11) is represented as the inequality $C \le B$.

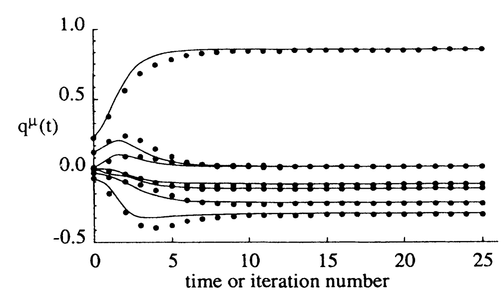

FIG. 3. Associative memory neural network (14) with $N=100$, $m=7$, and $\beta=4$ converges after ~ 13 iterations to the same fixed point as the corresponding continuous-time system, which was integrated numerically from the identical initial state. Continuous-time system is Eq. (1) with $R_iC_i=1$ and Eq. (14b). Order parameters $q^\mu(t) = (1/N)\sum_i \tanh[\beta u_i(t)]\xi_i^\mu$ measure the overlap of the output vector with each of the m memories.

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