Where Do Potentials Come From?

Fucking magnets, how do they work?

Abstract

This paper shows that it's possible to create a numerical model of source charges that are not eternal (i.e. they pop into existence at some time) through a dynamic way of thinking about electric scalar potentials. The electric scalar potential—and thus electric field—created by these charges does not go to infinity at the origin (this is good), but otherwise meets all of the criteria set out by previous experimental, and theoretical treatments (this is also good). This model also forces us to ask interesting questions about how the electric field created by such a charge radiates outward. That is, why don't charges interact with their own potential, and, "well, what about the magnetic vector potential?"

Growing up as an electrical engineer, they made us spend a lot of time analyzing systems for stability. You know, to make sure that they didn't explode or make robots go crazy and hit a guy off of an assembly line. Through that kind of rote activity, I was infused with a distaste for infinities, singularities, poles, or any other sources of instability. When I started reading into physics however (specifically electrodynamics and quantum mechanics), infinities and infinitesimals pop up everywhere. For example, point charges are singular points, and even with vacuum polarization they create a potential that shoots up to infinity pretty quickly. While there are a bunch of different other examples, I'm most interested with the electrical engineering related ones.

Does this even make any sense; where do these infinitely infinitesimal potentials come from? I have no idea, but maybe if we start with the basics we can work our way through some math on a computer and gain some additional insight. So, let's start with electricity, and see if that helps get us anywhere.

When you're first taught how to solve problems in electrostatics, the lesson usually starts by talking about the history of the electrostatic force. That is, how through experiment Coulomb found that the force between two charged particles follows an inverse square law, similar to that of gravity:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 \mathbf{r}^2}\,\hat{\mathbf{r}}$$

Where each q is a charge, and **r** is the distance vector between the charges. To make it possible to visualize what this force could actually mean, we're then taught that the force on a single particle can be described by an electric force field acting on that particle,

$$\mathbf{F} = q_1\mathbf{E}$$

Where the electric field, **E**, is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{\mathbf{r}^2}\,\hat{\mathbf{r}}$$

Which they say is created by the charge q_2, extending through all of space along $\hat{\mathbf{r}}$, the unit vector pointing away from the location of the charge, but now **r** is the vector from the location of the charge q_2. Then, if you have multiple charges or a distribution of charge, you can invoke superposition to add up all of the fields and then figure out the force that would be caused by every charge on every other charge. This is actually a pretty neat way of thinking about things, as it allows us to visualize problems in electrostatics quite easily and solve problems with things like the method of images.

I’ve always had an issue with this though: if charges create fields, and you add up all of the fields to figure out the net force on any given particle, why do you have to ignore the field created by that particle? Practically—say when you’re trying to teach yourself numerical methods via MATLAB—this means you have to keep track of a lot of different field configurations, since each charge has a different perspective from all of the others. If I remember correctly, this fact is usually explained away as, “we only deal with test charges,” or, “charges are infinitely small points,” so of course they don’t interact with their own field. Wait what; why? But I digress.

When you get upgraded to using Maxwell's equations for electrostatics, you're told that since static electric fields have no curl (or circulation) you can actually think of them in terms of something called the electric scalar potential, where the fields are the negative gradient of the potential:

$$\mathbf{E} = -\nabla\phi$$

Where we use ϕ to represent the potential. Meaning (with *q* = 1; electric constant equal to one) that the potential of a single point charge at the origin is given by

$$\phi = \frac{1}{4\pi|\mathbf{r}|}$$

The nice thing about this approach is that if you multiply this potential by a charge, you can talk about how this is the work required to bring that charge in from infinity in the presence of this potential. Then, through superposition you can talk about the potential energy of a configuration of charges or a charge distribution. This is the energy stored by this configuration of charges; it's capacity to do some work.

But here's the thing, where do these potentials come from? Even the concept of potential energy only makes sense when you have *at least* two charges. One creates the potential felt by the other charge (or vice versa) and we can measure this potential energy by measuring the forces on the charges. However, from quantum mechanics we know that potentials are fundamental; we can measure their effect on the world through something like the Bohm-Aharonov effect. And indeed, looking at the equations of quantum mechanics (e.g. Schrodinger's equation) you can see right in them that it's the potentials that have a direct effect on wavefunctions, fully determining the stationary states of simple quantum systems.

If you skip a few years of physics lessons and take a look into how they treat the coulomb potential in quantum field theory[1] you'll get a wacky logical progression like:

$$\phi = \frac{\delta(\mathbf{r})}{\Box} = \int \frac{1}{(2\pi)^3}\frac{\mathrm{e}^{\jmath\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2}\,\mathrm{d}^3\mathbf{k} = \frac{1}{4\pi|\mathbf{r}|}$$

Which, while I won't go through the full derivation, is a pretty clever use of Fourier analysis and complex analysis of the wave equation. The $\Box^{-1}$ is the inverse of the d'Alembertian, the wave operator or the box operator (I like how the technical term is literally box). All this really means is that if you have a wave equation with a source term that is an eternal, infinitesimal point particle (given by the Dirac delta function), then this particle will create a potential identical to what's found through experiment and mathematical finagling.

[1] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge, UK: Cambridge University Press, 2014. Section 3.4.2, "Coulomb Potential," pp. 39.

The key words here are things like eternal, and infinitesimal. That's all well and good for the quantum field theory guys, but how can charges and potentials be eternal if quantum mechanics also says that they're popping in and out of existence every moment? How can a charge create a potential that exists over all time and space if things can't go faster than the speed of light and a charge has only existed for a few moments? If we go further into classical electrodynamics (as opposed to statics), we can learn about how potentials are affected by moving *charges*, but again, those charges are assumed to be eternal. What happens if in one moment there is no charge, and in the next there is?

I think we can gain some insight to this if we start again with Maxwell's equations, but first simplified using the electric scalar potential and magnetic vector potential. Without going through the whole derivation, we know that since the divergence of the magnetic field is necessarily zero, we can define it to be the curl of a vector potential. We can then use this result to show that the electric field can be defined in terms of the gradient of a scalar potential (as before), and the time derivative of the vector potential. If we then mix and match electric and magnetic fields, and invoke the Lorenz gauge condition:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

We can reduce Maxwell's equations to two wave equations of the potentials. One wave equation for the electric scalar potential (ϕ) where the sources are charges, and one for the magnetic vector potential (**A**) where the sources are currents. The two potentials are then coupled via the Lorenz gauge condition above. A method to find the electric field and magnetic field from the potentials also naturally emerges from this definition. To start, if we're only talking about scalar potentials and charges, let's first focus on one wave equation:

$$\Box\phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = f(\mathbf{r}, t)$$

Where ϕ is the potential as before, *f* is a charge distribution and we've expanded out our box to what it looks like in standard vector calculus notation. Also, to keep things easy, we'll say the electric constant is just one. There's also an identical equation for the magnetic vector potential, **A**, that's really three equations, but since we're only talking charges and have no currents, we won't go do any additional work for now. Keep in mind that the solutions to a wave equation with no sources are just... waves. Something like spherical harmonics oscillating, radiating and moving around—either retarded or advanced. In order to solve the wave equation when there are sources, we can take advantage of Green's functions (or fundamental solutions), that is, a solution to the equation when the source is just a single point, given by a Dirac delta function. This means we're trying to solve the equation:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}g(\mathbf{r}, t) - \nabla^2 g(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t)$$

Where *g* is the Green's function, and our source is such that it's a single point in both space and time. Because everything we're dealing with is linear, once we find the Green's function, we can then invoke superposition and construct a general solution by the convolution of our Green's function with our source charge distribution of interest. Thankfully, all the hard work has been done for us already via classical electrodynamics[2] which gives us the Green's function of the wave equation:

[2] John David Jackson. *Classical Electrodynamics, Third Addition.* USA: John Wiley & Sons, 1999. Section 6.4, "Green Functions for the Wave Equation," pp. 243.

$$g(\mathbf{r},t) = \frac{\delta\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

This represents an infinitely thin spherical shell radiating outward from the point source at a speed *c*. To find this it's actually a pretty neat derivation using Fourier transforms to then solve the Helmholtz equation, and then transform back to get the radiating delta function. From a different perspective, we can literally say that this is the impulse response of the wave equation, and since the wave equation is linear and time invariant, that helps us justify finding the general solution using convolution. Now switching back to our wave equation for the electric potential after simplifying Maxwell's equations:

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = f(\mathbf{r},t)$$

Our general solution for the potential ϕ can then be given by the convolution of *f* and *g*:

$$\phi = g(\mathbf{r},t) * f(\mathbf{r},t)$$

Which is great, but how does that help us? The whole point of this exercise is that we'd like to know how to get the standard form of a potential, but starting from a charge that isn't eternal, or maybe one that isn't even infinitesimal. In order to represent a charge that is not eternal but comes into existence at a certain moment in time at some point **r** = 0, we can take advantage of a couple of handy functions, the Dirac delta and the Heaviside step function. This means that at a fundamental level, our charge distribution can be represented by:

$$f(\mathbf{r},t) = \delta(\mathbf{r})H(t)$$

Where *H* is the Heaviside step function, given by the piecewise formula:

$$H(t) = \begin{cases} 0, \ t < 0, \\ \frac{1}{2}, \ t = 0, \\ 1, \ t > 0. \end{cases}$$

So, for all times *t* less than zero, the Heaviside step function is zero, but right at *t* = 0 it turns on. What this means for our charge distribution, *f*, is that for all times less than zero, there is no source to our wave equation, but right at time *t* = 0, a charge appears at **r** = 0, and then stays there for all time. Thankfully, we've already done all the hard work figuring out what ϕ can now be:

$$\phi = \frac{\delta\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|} * \delta(\mathbf{r})H(t) = \frac{H\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

Which instead of an infinitely thin shell, is now an expanding sphere of potential. Remember, this can probably only be interpreted as a potential energy in the presence of another charge, but regardless, our charge is a source for something and is constantly radiating it to slowly form what we think of as potential (or rather, an electric field).

Since we're dealing with spherical coordinates here, this can be kind of hard to visualize. Since the Heaviside step function can pretty much be only on or off, we can think about it in terms of time steps and radial distances from the origin (where the charge has been placed). For example, when *t* = 0, *H*(*t* - *r/c*) would be one for all values of r less than zero. And since r can't be less than zero, our charge distribution is just zero. Now, when $t > 0$, $H(t - r/c)$ would be one for all $r < ct$; meaning as time increases the step function travels outward. At the same time as it moves outward, it attenuates by a factor of $1/4\pi r$ as shown in Fig. 1 below. Where in this case we set c = 10 just so that we could get some nice round numbers.

Everything else is arbitrary units, as this is just to prove the concept instead of act as a rigorous solution. Also, we have to keep in mind that the results we're using are strictly for the three-dimensional wave equation (1D and 2D get different and interesting results based on their fundamental solutions). So even though our graph only shows the x-axis, we have to imagine this potential flowing out from the origin in all three dimensions. Also noting that as time progresses, the curve matches the theoretical result given by assuming eternal charges. What this means is that if a charge instantly appears at some point, it creates a potential that is continually radiating outward, and whose shape approaches the theoretical, eternal result, as *t* approaches infinity.

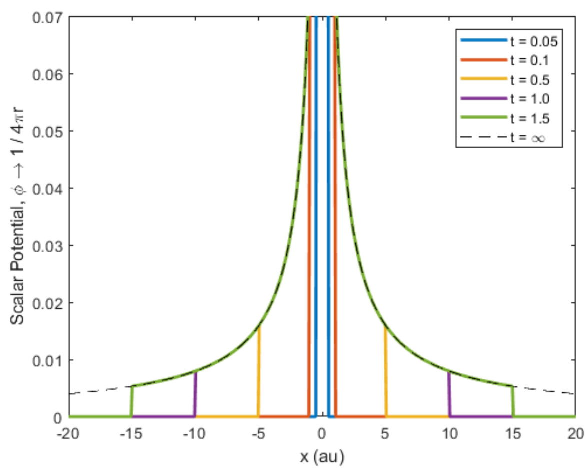

Fig. 1: Electric scalar potential created by a point charge coming into existence at t = 0.

There's a pretty big issue with this picture though. It seems that the *moment* the charge turns on, the potential immediately jumps up to infinity because at the point where the charge is, you're dividing by zero which is... not ideal. Remember, singularities make me really uncomfortable. I mean, an infinite spike of potential in an infinitely small timeframe just doesn't sound right. Is there another way we can approach things to keep the infinities at bay? What if instead of just a point charge at the origin we switch to a slightly more complicated source distribution, one that still turns on at $t = 0$, something like:

$$f(\mathbf{r},t) = \varphi(\mathbf{r},t)H(t)$$

Where φ is some kind of smooth function (infinitely differentiable) with compact support (goes to zero outside a small window), normalized to some kind of number that makes sense for whatever you're trying to accomplish. For us that means it's normalized to one (because we have one charge) just like the Dirac delta function: it's a distribution that's normalized to one. When we follow through with the convolution of this new source term with the Green's function (or fundamental solution) we get something like:

$$\phi = \frac{H\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|} * \varphi(\mathbf{r},t)$$

If φ is smooth, more or less with compact support, and properly normalized, the convolution of φ with any function will be very close to that function; indeed, outside of the range of φ it will be identical to the original function. This is quite literally what happens when you find the convolution of something with a Dirac delta function: the function stays the same, except shifted to wherever the delta happens to be. To pretend we're going to keep things simple, we'll use as our charge distribution, something close to a Dirac delta, but a little bit better defined, like a properly normalized Gaussian:

$$\varphi(\mathbf{r},t) = \frac{1}{\sqrt{(2\pi)^3}\sigma^3}\, \mathrm{e}^{-\frac{\mathbf{r}^2}{2\sigma^2}}$$

Remembering that we're doing things in 3D, so that's why we've normalized φ that way. For this distribution, as the standard deviation gets smaller, it gets thinner and thinner, taller and taller. Indeed, in the limit where σ --> 0, a normalized Gaussian approaches the Dirac delta function: infinite only at a singular point, but with an integral defined to be one. If we set our standard deviation to one and see how our potential evolves over the same time range as we used previously, we get figure 2 below.

Note: we have to do this calculation in 3D on our computer, as the fundamental solution for the wave equation in 1D is just the Heaviside function, and we'd get weird results if we did a 1D convolution with gaussians and deltas. That's also why Fig. 2 has some noticeable error (e.g. right around x = 0), this is just an artefact of the need to turn the resolution down to do a 3D convolution without crashing my computer.

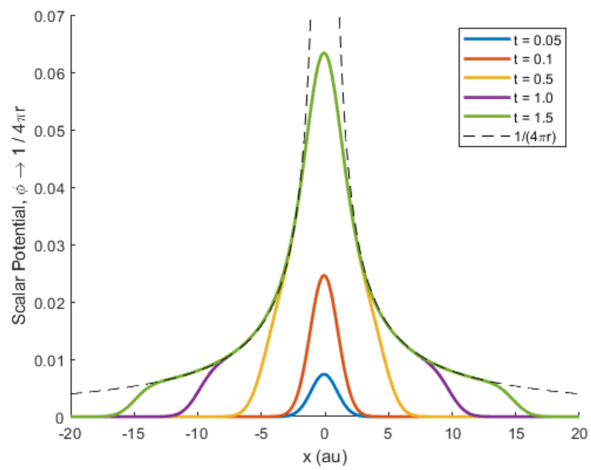

Fig. 2: Same as Fig. 1 but with Gaussian charge distribution.

This is remarkably similar to what we saw when our charge distribution was just a single point charge at the origin. At larger times, the potential evolves to approach what we would expect for a standard potential—except for small radii. Also, the wave fronts are now no longer vertical lines, but a bit smoother because of the convolution with the Gaussian distribution. What's going on here? We see that as the charge comes into existence, it slowly builds up to a maximum, and then the potential radiates out from there. Why doesn't it just blow up to infinity right at the start? In short, the initial infinity is perfectly balanced by the initial infinitesimal as the charge comes into existence. Remember, all of these equations so far are only really undefined right at the origin, but if you set the problem up correctly, you can change it so that in the limit where r approaches zero, things are well behaved. The easiest way to do this is just via Green's functions or fundamental solutions or the impulse response of the system.

Now, what happens if we shift around some of the parameters? If we change σ, yes, the potential does get higher and skinnier as sigma approaches zero, but the potential still approaches the theoretical limit of 1/4πr as it radiates outward, eventually matching the potential of the eternal point charge. Changing the speed of the wave or the temporal resolution (i.e. time step) only affects how fast the potential waves radiate outward but doesn't change the overall shape. What about if we change the *spatial* resolution? The only effect there is the accuracy of the final result of the simulation itself near $r = 0$, and how quickly it takes to run.

What if we take a slightly different approach and see how these things evolve from the bottom up? The easiest way to do that is—starting with our theoretical treatment above—we simulate things as time marches on through numerical integration of the wave equation itself. Starting with our wave equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = f(\mathbf{r}, t)$$

And our source term, the three-dimensional charge distribution with a standard deviation of one, given by:

$$f(\mathbf{r}, t) = \frac{1}{\sqrt{(2\pi)^3}}\,\mathrm{e}^{-\frac{1}{2}\mathbf{r}^2}H(t)$$

In order to solve the wave equation numerically, we use a very simple discretization scheme (Euler's forward, explicit method) to help us figure out how to evolve our solution in time:

$$\frac{1}{(c\Delta t)^2}(u^{t+1} - 2u^t + u^{t-1}) = \nabla^2 u^t + f$$

Where u is the discretized version of our potential ϕ, c is the speed as before, and Δt is the temporal resolution, i.e. how far we march forward in time each step of our numerical simulation. In this instance, we're not going to manually discretize the Laplacian since we're using an explicit method and don't need to get fancy re-arranging things to solve linear algebra problems. Also, MATLAB has a very handy Laplacian function (del2) that makes implementing our simulation in three dimensions very easy. Since we're in effect modeling a wave that starts at the origin and radiates outward, the only complication would be at the boundaries. We'll stop our simulation before the potential radiates that far, so we don't have to worry about that.

Re-arranging things to solve for the potential at the next time step, which is based on the potential of the previous two time steps and the source term, we get:

$$u^{t+1} = 2u^t - u^{t-1} + (c\Delta t)^2(\nabla^2 u^t + f)$$

And we implement the Heaviside function by setting the value of the first set of the two previous values of u to be zero:

$$u^0 = 0,\ u^{-1} = 0$$

Since we're dealing with a very basic explicit method, we have to be careful of stability. That means, we have to keep our simulation parameters within the CFL condition:

$$\frac{c^2(\Delta t)^2}{(\Delta x)^2} < 1$$

Where Δx is the spatial resolution. This doesn't enter directly into our equation, but it is used when calculating the Laplacian (i.e. hidden behind it), so it still matters for stability. We also have to keep in mind that the MATLAB implementation of the Laplacian, del2, divides the end results by twice the number of dimensions (six in our case) so we have to update our code accordingly. Starting off at $t = 0$ and simulating over the same timeframe as the theoretical results, we get figures 3 and 4 below.

The first figure I put in just to prove that I actually simulated things in 3D and didn't just fudge it in 1D. It shows three slices of the full 3D figure; one on the x-y plane, y-z plane, and x-z plane. The full code for the simulation can be found in the appendix. Also, if you were to directly simulate the wave equation in one or two dimensions you'd see different fundamental solutions emerge.

The second figure is way more interesting even though it's very similar to the previous theoretical results. The potential slowly grows from zero, reaches a maximum (given by the normalization constant), and then radiates outward from there. The best part is, no matter how we change the parameters, the picture doesn't change. If you change the standard deviation, the width of the initial potentials of course does change, as does the maximum, but adjusting the resolution (spatial or temporal) has no effect. In this way, we've confirmed that using the Green's function for the wave equation yields the same result as if you allow the equation to evolve, "naturally,": you can keep the potential completely constrained to conform to the shape of the charge distribution itself. No infinities necessary.

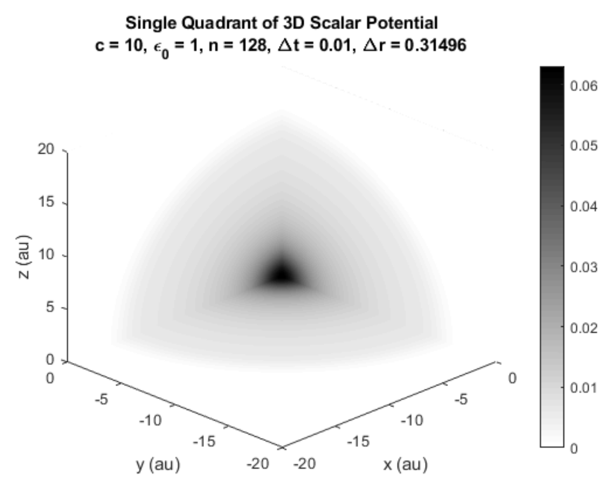

Fig. 3: x-y, y-z, z-x slices of 3D scalar potential.

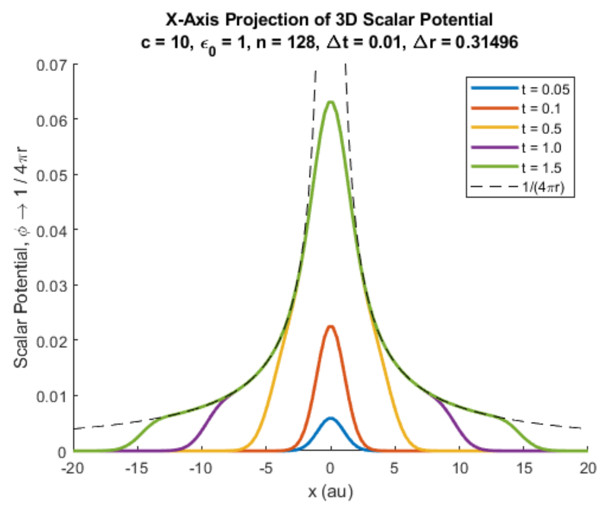

Fig. 4: Projection of Fig. 4 onto the x-axis over time, t.

What are the consequences of all this? What we've found is that the electric scalar potential is created by charges (and in a similar way, the magnetic vector potential with currents), but the potential is constantly radiating outward from the central charge. That is, if the charge were suddenly turned off again, the waves would stop radiating, and the potential would eventually dissipate completely. This means that potentials aren't actually this static thing, they are a, "wave," that is constantly being created by the charge and radiating outward.

If the charge is a single, infinitely small point charge, that's why it can't interact with its own potential (contrary to the implications of quantum mechanics e.g. Schrodinger's equation with a potential), as the moment the potential is created by the charge, it immediately radiates away at the speed of light. However, if the charge is a *distribution*, it stands to reason that each part of the distribution creates and radiates its own potential, so that all other parts can be affected by it.

To close things out, what about the electric and magnetic fields?

First off, let's start with the magnetic field, as that's easy since one of Maxwell's equations is:

$$\nabla \cdot \mathbf{B} = 0$$

Which literally says that the magnetic field can't have a strict radial symmetry, which would seem to be demanded by the way we've set up the problem. Remember, there are no magnetic monopoles (as far as we can tell), so we know that the field needs to loop around; and, it simply can't do that when all we have is a point sitting in the middle of nowhere. Even if we did something slightly more clever than have a founding assumption of, "no currents means no magnetic field," by e.g. defining the current to be the time rate of change of the charge distribution we'd still have issues. That is, with,

$$\mathbf{J} = \delta(\mathbf{r})\frac{\partial}{\partial t}H(t) = \delta(\mathbf{r})\delta(t)$$

We have one major problem: this current doesn't go anywhere, and current is supposed to have a direction. I mean, it's supposed to be a vector and what we have here is no vector. Now, if the charge were moving, then yes, we would get a magnetic field, and we can actually find that out by playing around with standard classical electrodynamics that we know and love along with the results above. But for our purposes here: no currents, no magnetic fields.

Now, going back to Maxwell's equations using the Lorenz gauge, we have:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

To now properly figure out the electric field, this means that we need to find what the magnetic vector potential, **A**, is. However, since there are no currents, the first step is to take advantage of the Lorenz gauge condition (since that's how we found the wave equation), which is of course:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

If we take the theoretical result for the radiating potential in terms of the Heaviside function, we can get an idea of what the magnetic field might look like. Starting with our definition for the potential of a charge turned on and left on at $t = 0$,

$$\phi = \frac{H\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

We can find the time derivative of this—using the definition of the Heaviside function—to then get:

$$\frac{\partial \phi}{\partial t} = \frac{\delta\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

If we then re-arrange things in our Lorenz gauge condition, this gives us:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

We then expand the divergence in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} = -\frac{1}{c^2}\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

Oh brother, looking at this gives me a headache. I used to wonder why people loved covariant versions of everything and tensor calculus, but then when you have to spend more than a glance looking at vector calculus in all its glory, well... the covariant formulation of things isn't that bad once you memorize all of the rules.

Anyway, I have no idea how to solve this. One way is to stare at it, make some reasonable assumptions, and then guess. We could also just say that no currents mean no vector potential, but then our gauge condition is violated. So, let's just keep working and see what our results look like. One way to do it is to invoke radial symmetry of the problem and say that the polar and azimuthal components should definitely be zero, which would simplify things. Or, we could try guessing something of the form:

$$A_\theta = \mathrm{e}^{j\varphi} f(r)$$

$$A_\varphi = j\mathrm{e}^{j\varphi}\cos\theta\, f(r)$$

Either way, if you work through it you end up with:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = -\frac{1}{c^2}\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

Which isn't so bad to solve, but let's try working through things from another perspective first and see what happens. First, we expand out the electric field using the gradient of the potential to get:

$$\mathbf{E} = \left[\frac{H\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi\mathbf{r}^2} + \frac{1}{c}\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}\right]\hat{\mathbf{r}} - \frac{\partial\mathbf{A}}{\partial t}$$

Looking at this, it seems mostly right! As time increases, we get an electric field radiating outward that's pointing in the correct direction and has the correct radial dependence. The only difference this time is that we get a little blip along the wave front, but as *t* approaches infinity everything checks out. There's also that pesky magnetic vector potential time derivative, but we'll come back to that. Now we take a look at the curl of the magnetic field in terms of the time derivative of the electric field:

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\left[\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi\mathbf{r}^2} + \frac{1}{c}\frac{\delta'\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}\right]\hat{\mathbf{r}} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

Where we've assumed zero current and where the prime next to the Dirac delta is the time derivative. The derivative of the Dirac delta doesn't really have any meaning, but if you were to convolve it with an actual distribution or function, you would just get the derivative of that function.

From this we know that if we *were* to have a magnetic field, the curl of the magnetic field points in the radial direction, but it's only a blip along the wave front. This is good! We don't want a static electric charge to have a magnetic field at all. But since we know that electric fields that change in time require magnetic fields that change in time (and vice versa), this checks out at first glance. This also means that the magnetic field has a polar and azimuth component. Or rather, it definitely has at least one if one or the other happens to be zero.

Here's an issue though: the curl of the electric field is zero. One of Maxwell's equations we don't have directly listed above is the one that relates the curl of the electric field to the time rate of change of the magnetic field. However, since the electric field only points along the radial direction and only depends on the radial distance from the charge, the curl of the electric field is zero. This means that we have to assume that the magnetic field is zero, even at the wave front of the electric field.

But if a time changing electric field doesn't require a magnetic field... what happens? Well, all this means is that maybe instead of a connection between magnetic and electric fields, our system as we've set it up requires a direct connection between the electric scalar and magnetic vector potentials instead. We now have two equations we can use to figure this out, one simplified and re-arranged from the Lorenz gauge condition:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = -\frac{1}{c^2}\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

And one from assuming that the magnetic field is zero, so that it's curl is zero:

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} = \left[\frac{\delta\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi\mathbf{r}^2} + \frac{1}{c}\frac{\delta'\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}\right]\hat{\mathbf{r}}$$

This second one is actually pretty easy to solve via a straight up integration. Since we're going to ignore any constants of integration (because screw that—also boundary conditions: things are zero at infinity), this gives us:

$$A_r = \frac{R\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi\mathbf{r}^2} + \frac{1}{c}\frac{H\left(t-\frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|}$$

Where the *R(x)* function is the ramp function, given by:

$$R(x) = xH(x)$$

And interestingly enough, to help us visualize it:

$$R\left(t-\frac{\mathbf{r}}{c}\right) = ct\Lambda\left(\frac{\mathbf{r}}{ct}\right), \quad t \geq 0$$

Where lambda is the triangle function. This means that before $t = 0$, everything is nice and silent, but after zero when our charge pops into existence, we have an electric potential that comes into life. Moreover, this electric potential creates a magnetic vector potential, pointing in the radial direction, that radiates out from the charge as time marches on. If we carry out the derivative of A_r required by the Lorenz gauge condition, we get the same thing, meaning we did the integration right! Because we're now assuming that there is no magnetic field, this means that the curl of the magnetic vector potential is zero, and the easiest way for that to be possible is to assume the polar and azimuthal components of the vector potential are zero. At this point, our vector potential takes the form:

$$\mathbf{A} = \left[\frac{R\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi\mathbf{r}^2} + \frac{1}{c}\frac{H\left(t - \frac{\mathbf{r}}{c}\right)}{4\pi|\mathbf{r}|} \right] \hat{\mathbf{r}}$$

Is there a problem here? Remember how I don't like infinities? Well, when we look at our vector potential, we see that as time marches on it grows and grows until as *t* approaches infinity the vector potential also approaches infinity. Unlike our scalar potential it doesn't settle down into something that seems reasonable, it just keeps growing and growing. Does this make any sense? Nope! For one thing, if we plug it back into our wave equation for currents (with magnetic constant one):

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{J}$$

We get a current in the form of:

$$\mathbf{J} = \frac{\delta(t - \frac{\mathbf{r}}{c})}{2\pi\mathbf{r}^2}\,\hat{\mathbf{r}}$$

Which probably doesn't make any sense. Though it seems to say that there's a current that exists only as an outward flowing shell along the wavefront of the electric field as it radiates away from the charge.

Another issue with this is that if we assume no currents and thus no vector potential, our Lorenz gauge condition still seems to require us to have a vector potential so that things are consistent. I mean, if the vector potential were zero (or it differentiated down to zero), then the Lorenz gauge condition would imply the time derivative of our scalar potential was zero. However, this really only makes sense at $t = \infty$, that is, if our charge has existed for all time, and thus the potential is no longer changing with time.

And one more issue—my personal favourite—is that if we then take this form for the magnetic vector potential and plug it into our expression for the electric field... the electric field ends up being zero. This is not ideal!

Regardless of what the vector potential is, however, our problem as it's set up requires radial symmetry so one thing that we can say for certain is that there's no energy flow: no magnetic field means no Poynting vector, means no energy flow. To get actual energy flow, you'd need to get actual moving charges. Which makes sense! Even if our charges pop into existence, should they be giving off a burst of energy?

What does all of this mean? I think it means that even though we were able to show how the electric field evolves in time to match the long-range field given by an infinite charge, this little experiment of ours is not at all a real, physical thing. Maybe if we tried a different gauge we'd get a different result, but I like the Lorenz gauge because it's causal and gives easy wave equations to work with.

To summarize: potentials come from charges! But they don't have to be eternal, and they don't have to go up to infinity. We've also seen that if the charges are point particles, they can't possibly interact with the potential that they produce, because that potential is constantly radiating away. The only reason we get this constant potential (or electric field) as determined by experiment is because it's always being radiated away from the charge the moment it's created, and if it does so indefinitely it all adds up to the field we know and love.

When it comes to vector potentials, well, who knows. Those things are enigmatic at best, and our formulation *eventually* has the same physical consequences as the theory of eternal charges.

For some additional work it would be neat to add the quantum mechanical perspective to this to see how a wavefunction can create potentials, and simultaneously be affected by them as we've suggested. Maybe the spin of an electron can create a vector potential that behaves better than what we've made up out of thin air.

Appendix: MATLAB Code

Code for Convolution with Fundamental Solution

```
% setup workspace:
n = 128; x = linspace(-20, 20, n);
[x, y, z] = meshgrid(x, x, x);
r = sqrt(x.^2 + y.^2 + z.^2);
c = 10; dr = y(2) - y(1);

% initialize fundamental solution (g) and source(f):
f = exp(-r.^2 / 2) / sqrt((2 * pi)^3);
g = zeros(size(r));

% loop over times of interest:
a = dr^3/4/pi./r;
for t = [5, 10, 50, 100, 150]/100
    g = heaviside(t - r/c).*a;
    psi = convn(f, g, 'same');
end

      % implementation of Heaviside function:
      function H = heaviside(x)
            H = zeros(size(x));
            H(x > 0) = 1;
            H(x == 0) = 0.5;
      end
```

Code for 3D Radiating Scalar Potential

```
% setup workspace:
tmax = 150; n = 128; x = linspace(-20, 20, n);
[x, y, z] = meshgrid(x, x, x);
c = 10; dt = 0.01; dr = y(2) - y(1); s = dt^2 * c^2;

% initialize scalar potential (u) and source (f):
f = exp((-x.^2 - y.^2 - z.^2) / 2) / sqrt((2 * pi)^3);
u_now = zeros(size(x));
u_prev = zeros(size(x));
u_next = zeros(size(x));

% loop through explicit method over time of interest:
for t = 1:tmax
    u_next = 2 * u_now - u_prev + s * (6 * del2(u_now, dr, dr, dr) + f);
    u_prev = u_now;
    u_now = u_next;
end
```