Homework 2

Question 5

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

(b) $p \rightarrow (q \wedge r)$

$\underline{\qquad\qquad \neg q}$

$\therefore \neg p$

1	$p \rightarrow (q \wedge r)$	Hypothesis
2	$p \rightarrow q$	Simplification, 1
3	$\neg q$	Hypothesis
4	$\neg p$	Modus tollens, 2, 3

(e) $p \vee q$

$\neg p \vee r$

$\underline{\qquad\qquad \neg q}$

$\therefore r$

1	$p \vee q$	Hypothesis
2	$\neg p \vee r$	Hypothesis
3	$q \vee r$	Resolution, 1, 2
4	$\neg q$	Hypothesis
5	r	Disjunctive syllogism, 3, 4

Table 1.12.1: Rules of inference known to be valid arguments.

Rule of inference	Name
p $p \rightarrow q$ $\therefore q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	Modus tollens
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification

Rule of inference	Name
p q $\therefore p \wedge q$	Conjunction
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

2. Exercise 1.12.3, section c

$p \lor q$

$\underline{\neg p}$

$\therefore q$

1	$p \lor q$	Hypothesis
2	$\neg\neg p \lor q$	Double negation, 1
3	$\neg p \to q$	Conditional identity, 2
4	$\neg p$	Hypothesis
5	q	Modus ponens, 3, 4

Table 1.5.1: Laws of propositional logic.

1.	Idempotent laws:	$p \lor p \equiv p$	$p \land p \equiv p$
2.	Associative laws:	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
3.	Commutative laws:	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
4.	Distributive laws:	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
5.	Identity laws:	$p \lor F \equiv p$	$p \land T \equiv p$
6.	Domination laws:	$p \land F \equiv F$	$p \lor T \equiv T$
7.	Double negation law:	$\neg\neg p \equiv p$	
8.	Complement laws:	$p \land \neg p \equiv F$ $\neg T \equiv F$	$p \lor \neg p \equiv T$ $\neg F \equiv T$
9.	De Morgan's laws:	$\neg(p \lor q) \equiv \neg p \land \neg q$	$\neg(p \land q) \equiv \neg p \lor \neg q$
10.	Absorption laws:	$p \lor (p \land q) \equiv p$	$p \land (p \lor q) \equiv p$
11.	Conditional identities:	$p \to q \equiv \neg p \lor q$	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

3. Exercise 1.12.5, sections c, d

(c) I will buy a new car and a new house only if I get a job.

I am not going to get a job.

∴ I will not buy a new car.

Assign variable names to each of the individual propositions:

c: I will buy a new car.

h: I will buy a new house.

j: I get a job.

Then the above argument can be expressed as:

$c \wedge h \to j$

$\neg j$

$\therefore \neg c$

Consider the case when c=T, h=F, j=F, then $c \wedge h = F, (c \wedge h \to j) = T, \neg j = T$

But $\neg c = F$

Therefore the original argument is **invalid**.

(d) I will buy a new car and a new house only if I get a job.

I am not going to get a job.

I will buy a new house.

∴ I will not buy a new car.

Assign the same variable names as (c) to each of the individual propositions, Then the above argument can be expressed as:

$c \wedge h \to j$

$\neg j$

h

$\therefore \neg c$

Proof:

1	$c \wedge h \to j$	Hypothesis
2	$\neg j$	Hypothesis
3	$\neg(c \wedge h)$	Modus tollens, 1, 2
4	$\neg c \vee \neg h$	De Morgan's Law, 3
5	h	Hypothesis
6	$\neg\neg h$	Double negation, 5
7	$\neg c$	Disjunction syllogism, 4, 6

Therefore the original argument is **valid**.

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

The given argument is invalid if we consider the truth value for domain {a, b} as follows:

	P	Q
a	F	T
b	F	F

2. Exercise 1.13.5, sections d, e

(d) Every student who missed class got a detention.

Penelope is a student in the class.

Penelope did not miss class.

∴ Penelope did not get a detention.

Suppose that the predicates P and Q are defined as follows:

P(x): x missed a class.

Q(x): x got a detention.

Then the above argument can be expressed as:

$\forall x, P(x) \rightarrow Q(x)$

$Penelope\ is\ a\ particular\ element$

$\neg P(Penelope)$

$\therefore \neg Q(Penelope)$

This argument is invalid. Assume another student called Bill is also in the class, with truth table as follows:

	P	Q
Bill	T	T
Penelope	F	T

Then Bill's case will support the first hypothesis, Penelope's case will support the second and the third hypothesis, but Penelope's case will make the conclusion false.

(e) Every student who missed class or got a detention did not get an A.

Penelope is a student in the class.

Penelope got an A.

∴ Penelope did not get a detention.

Suppose that the predicates P, Q and R are defined as follows:

P(x): x missed a class.

Q(x): x got a detention.

R(x): x got an A.

Then the above argument can be expressed as:

$\forall x, P(x) \vee Q(x) \rightarrow \neg R(x)$

$Penelope\ is\ a\ particular\ element$

$R(Penelope)$

$\therefore \neg Q(Penelope)$

Proof:

1	$\forall x, P(x) \vee Q(x) \rightarrow \neg R(x)$	Hypothesis
2	$\forall x, \neg\big(P(x) \vee Q(x)\big) \vee \neg R(x)$	Conditional identity, 1
3	$\forall x, \big(\neg P(x) \wedge \neg Q(x)\big) \vee \neg R(x)$	De Morgan's Law, 2
4	$Penelope\ is\ a\ particular\ element$	Hypothesis
5	$\neg P(Penelope) \wedge \neg Q(Penelope) \vee \neg R(Penelope)$	Universal instantiation, 3, 4
6	$R(Penelope)$	Hypothesis
7	$\neg\neg R(Penelope)$	Double negation, 6
8	$\neg P(Penelope) \wedge \neg Q(Penelope)$	Disjunctive syllogism, 5, 7
9	$\neg Q(Penelope)$	Simplification, 8

Therefore the original argument is valid.

Question 6

Solve Exercise 2.2.1, sections d, c, from the Discrete Math zyBook:

(d) The product of two odd integers is an odd integer.

Proof:

Let a = 2k+1, b = 2m+1, where k, m are integers.

ab = (2k+1)(2m+1) = 4km+2k+2m+1 = 2(2km+k+m)+1

since 2km+k+m is an integer, thus ab is an odd number.

Therefore the original argument is valid.

(c) If x is a real number and $x \leqslant 3$, then $12 - 7x + x^2 \geqslant 0$.

Proof:

$12 - 7x + x^2 = (3-x)(4-x)$

since $x \leqslant 3$, $3-x \geqslant 0$, $4-x>0$

thus $12 - 7x + x^2 \geqslant 0$

Therefore the original argument is valid.

Question 7

Solve Exercise 2.3.1, sections d, f, g, l, from the Discrete Math zyBook:

(d) For every integer n, if n^2-2n+7 is even, then n is odd.

Proof by contrapositive.

Let n be even. We're going to prove n^2-2n+7 is odd.

Since n is even, we can write n as n=2k, where k is an integer.

Then $n^2-2n+7 = 4k^2-4k+7 = 2(2k^2-2k+3)+1$

Since $2k^2-2k+3$ is an integer, thus n^2-2n+7 is an odd number.

Therefore the original argument is valid.

(f) For every non-zero real number x, if x is irrational, then 1/x is also irrational.

Proof by contrapositive.

Let 1/x be rational,. We're going to prove x is rational.

Since 1/x be rational, then 1/x can be written as 1/x = p/q where p, q are non-zero integers.

Thus x = q/p which is also rational.

Therefore the original argument is valid.

(g) For every pair of real numbers x and y, if $x^3+xy^2 \leqslant x^2y+y^3$, then $x \leqslant y$.

Proof by contrapositive.

Let x>y. We're going to prove $x^3+xy^2>x^2y+y^3$

since x>y, then x-y>0.

since x, y cannot be zero at the same time, thus $x^2+y^2>0$.

Thus $(x-y)(x^2+y^2)>0$, $x^3+xy^2>x^2y+y^3$.

Therefore the original argument is valid.

(l) For every pair of real numbers x and y, if x+y>20, then x>10 or y>10.

Proof by contrapositive.

Let $x \leqslant 10$ and $y \leqslant 10$. Add the two inequalities together, we can get $x+y \leqslant 20$.

Therefore the original argument is valid.

Question 8

Solve Exercise 2.4.1, sections c, e, from the Discrete Math zyBook:

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

This argument claims that for three arbitrary real number a, b, c, there are:

$$\left(\frac{a+b+c}{3} \geq \mathrm{a}\right) \vee \left(\frac{a+b+c}{3} \geq \mathrm{b}\right) \vee \left(\frac{a+b+c}{3} \geq \mathrm{c}\right)$$

Proof by contradiction.

Assume

$$\frac{a+b+c}{3} < \mathrm{a}$$

$$\frac{a+b+c}{3} < \mathrm{b}$$

$$\frac{a+b+c}{3} < \mathrm{c}$$

Then add up the three inequities together, we have a+b+c<a+b+c, which is a contradiction.

Therefore the original argument is valid.

(e) There is no smallest integer.

Proof by contradiction.

Assume m is the smallest integer. Let k=m-1

Thus k<m, k is smaller than m, which contradicts the assumption.

Therefore the original argument is valid.

Question 9

Solve Exercise 2.5.1, section c, from the Discrete Math zyBook:

(c) If integers x and y have the same parity, then x + y is even.

The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof by cases.

Case 1 – both x and y are odd.

Let x=2p+1, y=2q+1, where p, q are integers.

Then x+y = 2(p+q+1)

Since p+q+1 is an integer,

Therefore x+y is even.

Case 2 – both x and y are even.

Let x=2p, y=2q, where p, q are integers.

Then x+y = 2(p+q)

Since p+q is an integer,

Therefore x+y is even.

Above all, the original argument is valid.