Complex Geometry Seminar: Kähler Identities

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INTENT §1 Recalling some operators, and results, from the previous talk, but, instead, having them act on the exterior bundles, and their sections, of some comlex manifold X – in particular one with a Kähler structure.

§2 Computing the various Kähler Identities, which involve the aforementioned operators on Kähler X.

MOTIVATION These identities are very useful in proving Hodge's theorem, and the Hodge decomposition theorem, for compact Kähler X – namely, yielding

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong \mathscr{H}^k X \cong \bigoplus_{p+q=k} \mathscr{H}^{p,q} X \cong \bigoplus_{p+q=k} H^{p,q}_{\mathrm{Dolb}} X,$$

where $\mathscr{H}^k$ are the harmonic k-forms and $\mathscr{H}^{p,q}$ the harmonic (p,q)-forms.(Cannas da Silva, §17.1) Also, they are helpful in computing the cohomology of X.

NOTATION Let X be a complex manifold, i.e. there is an induced almost complex structure I on X, which is integrable (so that $\mathrm{d} = \partial + \bar{\partial}$); let $\dim_{\mathbb{C}} X = n$. Furthermore, X will be Hermitian, i.e. it is part of the pair (X, g), where g is a Riemannian metric on X that is compatible with I, $g(I\cdot, I\cdot) = g(\cdot,\cdot)$; this structure comes with a fundamental form $\omega(\cdot,\cdot) := g(I\cdot,\cdot)$, which happens to be a real (1,1)-form, as shown two talks ago.

1 Operator Recall

1.1 Linear Recall

1.1 DEFINITION i.) *Lefschetz operator*

$$\begin{aligned} L : \textstyle\bigwedge^k X &\longrightarrow \textstyle\bigwedge^{k+2} X \\ \alpha &\longmapsto \omega \wedge \alpha = \alpha \wedge \omega \qquad \text{since } \omega \text{ is of even degree,} \end{aligned}$$

where $\bigwedge^k X$ is the bundle of k-forms on (X, g).

ii.) *Hodge $*$-operator* associated to g with a given orientation on (X, g)

$$* : \textstyle\bigwedge^k X \longrightarrow \textstyle\bigwedge^{2n-k} X.$$

It was shown in a previous talk that, on k-forms, $*^2 = (-1)^{k(2n-k)} = (-1)^k$.

iii.) *dual Lefschetz operator*

$$\Lambda := *^{-1} L * : \textstyle\bigwedge^k X \longrightarrow \textstyle\bigwedge^{k-2} X,$$

which is dual (adjoint) to L with respect to g; it was shown in §1.2 that it is of this form.

Of course, these operators can be extended $\mathbb{C}$-linearly to the complexified bundles $\bigwedge^k_{\mathbb{C}} X$; these extended operators will be denoted by the same symbols, and discussion about $\bigwedge^\bullet X$ will extend to use on the complexification. Once this is done, it can be shown (as it was in Huybrechts, §1.2), that these operators have a certain bidegree, or act on the bidegree in a certain way:

$$L : \bigwedge^{p,q} X \longrightarrow \bigwedge^{p+1,q+1} X$$
$$* : \bigwedge^{p,q} X \longrightarrow \bigwedge^{n-q,n-p} X$$
$$\Lambda : \bigwedge^{p,q} X \longrightarrow \bigwedge^{p-1,q-1} X.$$

Also, the following operator can be defined, which come of use:

$$\mathbf{I} := \sum_{p,q=0}^{n} \mathrm{i}^{p-q} \Pi^{p,q},$$

where $\Pi^{p,q} : \bigwedge^{\bullet} X \longrightarrow \bigwedge^{p,q} X$, and, similarly, $\Pi^k : \bigwedge^{\bullet} X \longrightarrow \bigwedge^{k} X$.

1.2 Corollary (Huybrechts, 1.2.28) For any $\alpha \in \bigwedge^k X$,

$$\left[L^j, \Lambda\right] \alpha = j\,(k-n+j-1)\, L^{j-1}\alpha.$$

Proof Working through, recursively and using $[L, \Lambda] = H := \sum_{k=0}^{2n} (k-n)\Pi^k$ from the previous talk,

$$\begin{aligned}
\left[L^j, \Lambda\right] \alpha &= L^j \Lambda \alpha - \Lambda L^j \alpha \\
&= L L^{j-1} \Lambda \alpha - L\Lambda L^{j-1}\alpha + L\Lambda L^{j-1}\alpha - \Lambda L L^{j-1}\alpha \\
&= L\left(L^{j-1}\Lambda\alpha - L\Lambda L^{j-1}\alpha\right) + [L,\Lambda]\left(L^{j-1}\alpha\right) \\
&\vdots \quad \text{doing this recursively } (j-2)\text{-times} \\
&= L^{j-1}[L,\Lambda]\,\alpha + \sum_{m=1}^{j-1} L^{m-1}[L,\Lambda]\,L^{j-m}\alpha \\
&= (k-n)\,L^{j-1}\alpha + \sum_{m=1}^{j-1} \left(2\,(j-m) + k - n\right) L^{j-1}\alpha \qquad \text{using definition of } H \text{ and } L^{j-m}\alpha \in \mathscr{A}^{2(j-m)+k-n}X \\
&= \left(j\,(k-n) + j\,(j-1)\right) L^{j-1}\alpha \\
&= j\,(k-n+j-1)\, L^{j-1}\alpha.
\end{aligned}$$

■

1.3 Corollary (Huybrechts, 3.1.2) Lefschetz decomposition for bundles, going off the proposition (1.2.30) from the previous talk, which was proved using $\mathfrak{sl}(2)$-representation theory.

a.) If (X, g) is a Hermitian manifold, then there is the following decomposition

$$\bigwedge^k X = \bigoplus_{i \geq 0} L^i \left(P^{k-2i} X\right),$$

where $P^{k-2i}X := \ker\left(\Lambda : \bigwedge^{k-2i} X \longrightarrow \bigwedge^{k-2i-2} X\right)$ is the set of primitive $(k-2i)$-forms on X.

b.) For $k \leq n$, the following map is injective

$$L^{n-k} : P^k X \longrightarrow \bigwedge^{2n-k} X.$$

c.) For $k \leq n$, the primitive k-forms are $P^k X = \{\alpha \in \mathscr{A}^k X : L^{n-k+1}\alpha = 0\}$. For $k > n$, $P^k X = 0$. check this

1.4 Corollary (Huybrechts, 1.2.31) Shown in the previous talk using clever induction on the dimension of the vector space; here it is, of course, mentioned as being for bundles.
For $\alpha \in P^k X$,

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbf{I}\alpha.$$

1.2 Differential Recall

1.5 Definition i.) The adjoint operator to $\mathrm{d}: \mathscr{A}^k X \longrightarrow \mathscr{A}^{k+1}X$, the *adjoint differential*

$$\mathrm{d}^* := -*\mathrm{d}*: \mathscr{A}^k X \longrightarrow \mathscr{A}^{k-1}X,$$

where the qualification "adjoint" comes it being the adjoint to d with respect to the bilinear form $(\alpha,\beta) := \int_X \alpha \wedge *\beta$, for compactly-supported $\alpha,\beta \in \mathscr{A}^k X$.

ii.) In a similar fashion, define the "adjoints" (to be shown) to ∂ and $\bar{\partial}$,

$$\partial^* := -*\bar{\partial}* \qquad \text{and} \qquad \bar{\partial}^* := -*\partial*.$$

iii.) With these, there are the associated Laplacians,

$$\begin{aligned}\Delta &:= \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^* \\ \Delta_{\partial} &:= \partial^*\partial + \partial\partial^* \\ \Delta_{\bar{\partial}} &:= \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*.\end{aligned}$$

Looking at the bidegrees,

$$\begin{aligned}\partial &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p+1,q}X & \partial^* &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p-1,q}X \\ \bar{\partial} &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q+1}X & \bar{\partial}^* &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q-1}X \\ \mathrm{d} &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p+1,q}X \oplus \mathscr{A}^{p,q+1}X & \mathrm{d}^* &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p-1,q}X \oplus \mathscr{A}^{p,q-1}X,\end{aligned}$$

where the $\mathrm{d} = \partial + \bar{\partial}$ since I is integrable, and

$$\begin{aligned}\Delta &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q} \\ \Delta_{\partial} &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q} \\ \Delta_{\bar{\partial}} &: \mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q}.\end{aligned}$$

1.6 Lemma (Huybrechts, 3.1.4) Let (X,g) be a Hermitian manifold, then $\mathrm{d}^* = \partial^* + \bar{\partial}^*$ along with $\partial^{*2} = 0 = \bar{\partial}^{*2}$ and $\bar{\partial}^*\partial^* = -\partial^*\bar{\partial}^*$.

Proof First off, since X is Hermitian, it is required that X is a complex manifold, which, in turn (Huybrechts, 2.6.15), means that the induced almost complex structure I on X is integrable, i.e. $\mathrm{d} = \partial + \bar{\partial}$.

Using this and the fact that $\mathrm{d}^2 = 0$, it follows that $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. With this, on some $\alpha \in \mathscr{A}^k X$,

$$\begin{aligned}\partial^{*2}\alpha &= \left(*\bar{\partial}*\right)\left(*\bar{\partial}*\right)\alpha \\ &= (-1)^{2n-k+1} * \bar{\partial}^2 * \alpha \qquad \text{since } \bar{\partial}*\alpha \in \mathscr{A}^{2n-k+1} \\ &= 0, \\ \bar{\partial}^{*2}\alpha &= (-1)^{2n-k+1} * \partial^2 * \alpha \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}-*\partial\bar{\partial}*\alpha &= *\bar{\partial}\partial*\alpha \\ (-1)^{2n-k}(-*\partial*)\left(-*\bar{\partial}*\right)\alpha &= (-1)^{2n-k-1}\left(-*\bar{\partial}*\right)(-*\partial*)\alpha \\ -\bar{\partial}^*\partial^*\alpha &= \partial^*\bar{\partial}^*\alpha.\end{aligned}$$

■

One last operator, which will be useful in simiplifying some notation in the proof of the identities, is the following.

1.7 Definition Consider the following

$$\mathrm{d}^c := \mathbf{I}^{-1}\mathrm{d}\mathbf{I} \qquad \text{and} \qquad \mathrm{d}^{c*} := -*\mathrm{d}*.$$

Defined in this way, it is such that:

a.) for $\alpha \in \mathscr{A}^{p,q}X$,

$$\begin{aligned} \mathrm{i}\mathrm{d}\mathbf{I}\alpha &= \mathrm{i}^{p-q+1}\mathrm{d}\alpha \\ &= \mathrm{i}^{p-q+1}\left(\partial + \bar{\partial}\right)\alpha \\ &= \mathrm{i}^{p+1-q}\underbrace{\partial\alpha}_{\in \mathscr{A}^{p+1,q}X} + \mathrm{i}^{p-(q+1)}(\mathrm{i})^2\underbrace{\bar{\partial}\alpha}_{\in \mathscr{A}^{p,q+1}X} \qquad \text{using definition of } \mathbf{I} \text{ to determine the coefficients} \\ &= \mathbf{I}\left(\partial - \bar{\partial}\right)\alpha \\ \iff \mathrm{d}^c = \mathbf{I}^{-1}\mathrm{d}\mathbf{I} &= -\mathrm{i}\left(\partial - \bar{\partial}\right) \end{aligned}$$

since α was arbitrary.

b.) and, using the identities on $\mathrm{d}, \partial, \bar{\partial}$,

$$\mathrm{d}\mathrm{d}^c = -\mathrm{i}\left(\partial + \bar{\partial}\right)\left(\partial - \bar{\partial}\right) = 2\mathrm{i}\partial\bar{\partial} = -2\mathrm{i}\bar{\partial}\partial = \mathrm{i}\left(\partial - \bar{\partial}\right)\left(\partial + \bar{\partial}\right) = -\mathrm{d}^c\mathrm{d}.$$

2 Kähler Endpoint

Recalling the qualification of being Kähler with the next definition.

2.1 Definition A Hermitian manifold (X, g) has a *Kähler structure*, i.e. it is a *Kähler manifold*, if the fundamental form ω associated to g and I, is closed, $\mathrm{d}\omega = 0$. In this case, the ω is called the *Kähler form*, and g a *Kähler metric*.

Finally, onto the Kähler identities with the following proposition.

2.2 Proposition Let (X, g) be a Kähler manifold. Then,

i.) $[\partial, L] = 0 = \left[\bar{\partial}, L\right]$ and $\left[\bar{\partial}^*, \Lambda\right] = 0 = [\partial^*, \Lambda]$

ii.) $\left[L, \bar{\partial}^*\right] = -\mathrm{i}\partial$, $[L, \partial^*] = \mathrm{i}\bar{\partial}$ and $\left[\Lambda, \bar{\partial}\right] = -\mathrm{i}\partial^*$, $[\Lambda, \partial] = \mathrm{i}\bar{\partial}^*$

iii.) $2\Delta_\partial = \Delta = 2\Delta_{\bar{\partial}}$ and Δ commutes with $*, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L, \Lambda$.

Proof i.) By means of linearity (of the bracket and the operators), and bidegrees, proving this can be reduced to showing $[\mathrm{d}, L] = 0$:

$$[\mathrm{d}, L] = 0 \iff \left[\left(\partial + \bar{\partial}\right), L\right] = 0 \iff \underbrace{[\partial, L]}_{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p+2,q+1}X} + \overbrace{\left[\bar{\partial}, L\right]}^{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p+1,q+2}X} = 0 \iff [\partial, L] = 0 = \left[\bar{\partial}, L\right].$$

Showing this: for any $\alpha \in \mathscr{A}^k X$,

$$\begin{aligned} [\mathrm{d}, L]\,\alpha &= \mathrm{d}(\omega \wedge \alpha) - \omega \wedge \mathrm{d}\alpha \\ &= \mathrm{d}\alpha \wedge \omega + (-1)^k \alpha \wedge \mathrm{d}\omega - \omega \wedge \mathrm{d}\alpha \\ &= 0 \qquad \text{since } \omega \text{ is Kähler.} \end{aligned}$$

And, similarly: for any $\alpha \in \mathscr{A}^k X$,

$$
\begin{aligned}
[\mathrm{d}^*, \Lambda]\,\alpha &= -*\mathrm{d}*^{-1} L*\alpha - *^{-1} L*\circ(-*\mathrm{d}*)\,\alpha \\
&= -*\mathrm{d}L*\alpha + (-1)^{2n-k+1} *^{-1} L\mathrm{d}*\alpha && \text{$\mathrm{d}*\alpha \in \mathscr{A}^{2n-k+1}$, $*^2\alpha = (-1)^{2n-k+1}\,\alpha$} \\
&= -*\mathrm{d}L*\alpha + (-1)^{-2} *L\mathrm{d}*\alpha && \text{$L\mathrm{d}*\alpha \in \mathscr{A}^{2n-k+1+2}X$, $*^{-1}L\mathrm{d}*\alpha = (-1)^{-2n+k-3}*L\mathrm{d}*\alpha$} \\
&= -*[\mathrm{d}, L]*\alpha \\
&= 0 \\
\iff \left[\bar{\partial}^*, \Lambda\right] &= 0 = \left[\partial^*, \Lambda\right] && \text{using linearity and bidegree decomposition.}
\end{aligned}
$$

ii.) First, it will be shown that $[\Lambda, \mathrm{d}] = -\mathrm{d}^{c*}$, whereby the wanted identities involving Λ can be yielded using linearity and bidegree decomposition; the identities involving L will be shown using these.

Before beginning, there is a possible reduction: with the fact that d and L commute, as just shown in part (i.), and the computation of $\left[L^j, \Lambda\right]$ from corollary (1.2), it is helpful to use the Lefschetz decomposition in order to reduce the computation of $[\Lambda, \mathrm{d}]$ on arbitrary α to $[\Lambda, \mathrm{d}]\, L^j\alpha$ for primitive α of appropriate degree.

Starting off, for any k and for any $\alpha \in P^k X = \left\{\alpha \in \mathscr{A}^k X : L^{n-k+1}\alpha = 0\right\}$ by corollary (1.3), $\mathrm{d}\alpha$ will be computed with help of the Lefschetz decomposition: check this

$$\mathrm{d}\alpha = \alpha_0 + L\alpha_1 + \cdots$$

with $\alpha_j \in P^{k+1-2j}X$. As shown before,

$$L^{n-k+1}\mathrm{d}\alpha = \mathrm{d}L^{n-k+1}\alpha = 0,$$

where the first equality is from part (i.). Therewith, since it is a direct-sum decomposition, this means

$$L^{n-k+1+j}\alpha_j = 0,$$

for each $j \geq 0$.

Now, the conditional injectivity of L will be used to determine which of the such α_j are non-trivial. Recall that part (b.) of corollary (1.3) states that L^{n-m} is injective on $P^m X$ if $m \leq n$. With this, $L^{n-(k+1-2j)}$ is injective on $P^{k+1-2j}X \ni \alpha_j$ if $k+1-2j \leq n$; if j is not large enough so that the inequality holds, then $k+1-2j > n$, which means $P^{k+1-2j}X = 0$ anyway, i.e. α_{k+1-2j} is trivial. Thus, $L^{n-k+1+j}$ is injective when

$$n-k+1+j \leq n-(k+1-2j) \iff 2 \leq j,$$

which means $\alpha_j = 0$ for $j \geq 2$ and

$$\mathrm{d}\alpha = \alpha_0 + L\alpha_1.$$

Computing now with this,

$$
\begin{aligned}
[\Lambda, \mathrm{d}]\, L^j\alpha &= \Lambda L^j(\alpha_0 + L\alpha_1) - \mathrm{d}\Lambda L^j\alpha && \text{part (i.) and as just shown} \\
&= \left[\Lambda, L^j\right]\alpha_0 + \left[\Lambda, L^{j+1}\right]\alpha_1 - \mathrm{d}\left[\Lambda, L^j\right]\alpha && \text{$\alpha_0, \alpha_1, \alpha$ primtive} \\
&= -j\,(k+1-n+j-1)\,L^{j-1}\alpha_0 - (j+1)\,(k-1-n+j)\,L^j\alpha_1 \\
&\qquad + j\,(k-n+j-1)\,L^{j-1}\mathrm{d}\alpha && \text{corollary (1.2) and part (i.)} \\
&= -j\,(k+1-n+j-1)\,L^{j-1}\alpha_0 - (j+1)\,(k-1-n+j)\,L^j\alpha_1 \\
&\qquad + j\,(k-n+j-1)\,L^{j-1}(\alpha_0 + L\alpha_1) && \text{as just shown} \\
&= -j\,L^{j-1}\alpha_0 - (k-n+j-1)\,L^j\alpha_1.
\end{aligned}
$$

Finally, to show that right-hand side of the wanted equality,

$$-\mathrm{d}^{c*}L^j\alpha = *\mathbf{I}^{-1}\mathbf{d}\mathbf{I}*L^j\alpha$$

$$= C_{n,k,j} * \mathbf{I}^{-1}\mathrm{d}\mathbf{I}L^{n-k-j}\mathbf{I}\alpha$$ corollary (1.4), with $C_{n,k,j}$ the coefficient;

note that $\mathbf{I} := \sum_{p,q=0}^{n} \mathrm{i}^{p-q}\Pi^{p,q}$ commutes with L, since L has bidegree $(1,1)$: $\mathrm{i}^{p+1-(q+1)} = \mathrm{i}^{p-q}$; then,

$$= C_{n,k,j} * \mathbf{I}^{-1}L^{n-k-j}\mathrm{d}\mathbf{I}^2\alpha$$ using commutative relations

$$= (-1)^k C_{n,k,j} * \mathbf{I}^{-1}L^{n-k-j}\mathrm{d}\alpha$$ by definition of $\mathbf{I}$ since $\left(\mathrm{i}^{p-q}\right)^2 = (-1)^{p-q} = (-1)^k$

$$= (-1)^k C_{n,k,j} * \mathbf{I}^{-1}L^{n-k-j}(\alpha_0 + L\alpha_1)$$

note that $*$ and $\mathbf{I}^{-1} = \sum_{p,q=0}^{n} \mathrm{i}^{q-p}\Pi^{p,q}$ commute since $*$ acts as $(p,q) \longmapsto (n-q, n-p)$ on the bidegree and $\mathrm{i}^{q-p} = \mathrm{i}^{n-p-(n-q)}$; then,

$$\begin{aligned}
&= (-1)^k C_{n,k,j}\mathbf{I}^{-1} * L^{n-k-j}(\alpha_0 + L\alpha_1) \\
&= (-1)^k C_{n,k,j} C_{n,k+1,n-k-j}\mathbf{I}^{-1}L^{n-(k+1)-(n-k-j)}\mathbf{I}\alpha_0 \\
&\qquad + (-1)^k C_{n,k,j} C_{n,k-1,n-k-j+1}\mathbf{I}^{-1}L^{n-(k-1)-(n-k-j+1)}\mathbf{I}\alpha_1
\end{aligned}$$ corollary (1.4), where $\alpha_0 \in \mathscr{A}^{k+1}X$, $\alpha_1 \in \mathscr{A}^{k-1}X$

$$\begin{aligned}
&= (-1)^k C_{n,k,j} C_{n,k+1,n-k-j} L^{j-1}\alpha_0 \\
&\qquad + (-1)^k C_{n,k,j} C_{n,k-1,n-k-j+1} L^{j}\alpha_1
\end{aligned}$$ commutative relation of $\mathbf{I}$ and L.

Looking at the coefficients, to ensure that the wanted equality is obtained:

$$\begin{aligned}
(-1)^k C_{n,k,j} C_{n,k+1,n-k-j} &= (-1)^k \left((-1)^{\frac{(k+1)k}{2}} \frac{j!}{(n-k-j)!} \right) \left((-1)^{\frac{(k+2)(k+1)}{2}} \frac{(n-k-j)!}{(j-1)!} \right) \\
&= -j,
\end{aligned}$$

since

$$\frac{2k + k^2 + k + k^2 + 3k + 2}{2} = k^2 + 3k + 1 \equiv 1 \bmod 2,$$

and

$$\begin{aligned}
(-1)^k C_{n,k,j} C_{n,k-1,n-k-j+1} &= (-1)^k \left((-1)^{\frac{(k+1)k}{2}} \frac{j!}{(n-k-j)!} \right) \left((-1)^{\frac{k(k-1)}{2}} \frac{(n-k-j+1)!}{j!} \right) \\
&= (n-k-j+1)
\end{aligned}$$

since

$$\frac{2k + k^2 + k + k^2 - k}{2} = k^2 + k \equiv 0 \bmod 2.$$

Therefore, since $L^j\alpha$ is arbitrary by the Lefschetz decomposition,

$$\begin{aligned}
[\Lambda, \mathrm{d}] &= -\mathrm{d}^{c*} \\
&= -*\mathrm{i}\left(\partial - \bar{\partial}\right)* \qquad \text{definition (1.7)} \\
&= -\mathrm{i}\bar{\partial}^* + \mathrm{i}\partial^*
\end{aligned}$$

$$\iff \underbrace{[\Lambda, \partial]}_{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q-1}X} + \overbrace{\left[\Lambda, \bar{\partial}\right]}^{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p-1,q}X} = -\mathrm{i}\bar{\partial}^* + \mathrm{i}\partial^*$$ using integrability and linearity

$$\iff [\Lambda, \partial] = -\mathrm{i}\bar{\partial}^* \quad \text{and} \quad \left[\Lambda, \bar{\partial}\right] = \mathrm{i}\partial^*$$ bidegree decomposition.

Onto the identities involving L, via bidegree decomposition: for some $\alpha \in \mathscr{A}^k X$,

$$[L, \mathrm{d}^*]\,\alpha = -L*\mathrm{d}*\alpha + *\mathrm{d}*L\alpha$$

$$
\begin{aligned}
&= -L*\mathrm{d}*\alpha + (-1)^{k+2} *\mathrm{d}*^{-1} L\alpha && *^2 = (-1)^{k+2} \\
& && \iff * = (-1)^{k+2} *^{-1} \\
& && \text{on } (k+2)\text{-forms} \\
&= -**^{-1} L*\mathrm{d}*\alpha + *\mathrm{d}*^{-1} L**\alpha \\
&= -*[\Lambda, \mathrm{d}]*\alpha \\
&= \mathrm{d}^c\alpha \\
&= -\mathrm{i}\partial\alpha + \mathrm{i}\bar{\partial}\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\iff \underbrace{\left[L, \bar{\partial}^*\right]}_{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p+1,q}X} + \overbrace{\left[L, \partial^*\right]}^{\mathscr{A}^{p,q}X \longrightarrow \mathscr{A}^{p,q+1}X} = -\mathrm{i}\partial + \mathrm{i}\bar{\partial} \\
&\iff \left[L, \bar{\partial}^*\right] = -\mathrm{i}\partial \qquad \text{and} \qquad \left[L, \partial^*\right] = \mathrm{i}\bar{\partial}.
\end{aligned}
$$

An alternative proof of this is given in Griffiths and Harris: it uses the fact (Huybrechts 1.3.12, Moroianu §11.3, Ballmann §4) on complex manifolds that g is Kähler iff $\mathrm{d}\omega = 0$ iff g osculates to the standard Hermitian metric to the second-order iff there are normal holomorphic coordinates at each point of X (i.e. it looks like $\mathbb{C}^n$ locally).

iii.) First, it will be shown that $\Delta_\partial = \Delta_{\bar{\partial}}$ using part (ii.):

$$
\begin{aligned}
\Delta_\partial &= \partial\partial^* + \partial^*\partial \\
&= \mathrm{i}\partial\left[\Lambda, \bar{\partial}\right] + \mathrm{i}\left[\Lambda, \bar{\partial}\right]\partial \\
&= \mathrm{i}\left(\partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial\right) \\
&= \mathrm{i}\left([\partial, \Lambda]\bar{\partial} + \Lambda\partial\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}[\Lambda, \partial] - \bar{\partial}\partial\Lambda\right) \\
&= \mathrm{i}\left([\partial, \Lambda]\bar{\partial} - \bar{\partial}[\Lambda, \partial]\right) && \text{integrability,} \\
& && \partial\bar{\partial} = -\bar{\partial}\partial \\
&= \mathrm{i}\left(-\mathrm{i}\bar{\partial}^*\bar{\partial} - \mathrm{i}\bar{\partial}\bar{\partial}^*\right) \\
&= \Delta_{\bar{\partial}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Delta &= \left(\partial + \bar{\partial}\right)\left(\bar{\partial}^* + \partial^*\right) + \left(\bar{\partial}^* + \partial^*\right)\left(\partial + \bar{\partial}\right) \\
&= \Delta_\partial + \Delta_{\bar{\partial}} + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \bar{\partial}^*\partial + \partial^*\bar{\partial} \\
&= 2\Delta_\partial + \partial\bar{\partial}^* + \bar{\partial}^*\partial + \overline{\left(\partial\bar{\partial}^* + \bar{\partial}^*\partial\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
\partial\bar{\partial}^* &= -\mathrm{i}\partial[\Lambda, \partial] \\
&= -\mathrm{i}\partial\Lambda\partial && \text{integrability, } \partial^2 = 0 \\
&= -\mathrm{i}[\partial, \Lambda]\partial && \text{integrability, } \partial^2 = 0 \\
&= -\bar{\partial}^*\partial,
\end{aligned}
$$

so

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

Finally, some of the commutative relations: on some $\alpha \in \mathscr{A}^k X$,

$$
\begin{aligned}
\Delta*\alpha &= \left(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\right)*\alpha \\
&= -(-1)^k\,\mathrm{d}*\mathrm{d}\alpha - *\mathrm{d}*\mathrm{d}*\alpha && *^2\alpha = (-1)^k\,\alpha \\
&= -*^2\,\mathrm{d}*\mathrm{d}\alpha - *\mathrm{d}*\mathrm{d}*\alpha && *^2\mathrm{d}*\mathrm{d}\alpha = (-1)^{2n-(k+1)+1}\,\mathrm{d}*\mathrm{d}\alpha \\
& && \qquad = (-1)^k\,\mathrm{d}*\mathrm{d}\alpha \\
&= *\left(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*\right)\alpha \\
&= *\Delta\alpha;
\end{aligned}
$$

then,

$$\begin{aligned}
\Delta\partial &= 2\left(\partial\partial^* + \partial^*\partial\right)\partial \\
&= 2\partial\partial^*\partial && \text{integrability} \\
&= 2\partial\Delta_\partial && \text{integrability} \\
&= \partial\Delta
\end{aligned}$$

so, similarly,

$$\Delta\bar{\partial} = 2\bar{\partial}\bar{\partial}^*\bar{\partial} = \bar{\partial}\Delta;$$

lastly,

$$\begin{aligned}
\Delta L &= 2\left(\partial\partial^* + \partial^*\partial\right)L \\
&= 2\partial\left(L\partial^* - \mathrm{i}\bar{\partial}\right) + 2\partial^* L\partial && \text{part (i.) and (ii.)} \\
&= 2L\partial\partial^* - 2\mathrm{i}\partial\bar{\partial} + 2\left(L\partial^* - \mathrm{i}\bar{\partial}\right)\partial && \text{part (i.) and (ii.)} \\
&= 2L\left(\partial\partial^* + \partial^*\partial\right) && \text{integrability, } \partial\bar{\partial} = -\bar{\partial}\partial \\
&= L\Delta.
\end{aligned}$$

With these, it follows that Δ also commutes with $\mathrm{d} := \partial + \bar{\partial}$, $\bar{\partial}^* := -*\partial*$, $\partial^* := -*\bar{\partial}*$, $\mathrm{d}^* := -*\mathrm{d}*$, and $\Lambda = *^{-1}L*$.

■