Characteristics Classes Seminar:

§11 Computations on Smooth Manifolds

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INTENT This talk will go over material from the first half of §11 from Milnor's book *Characteristic Classes*. Here, the sectioning (and numbering of statments) corresponds to those in the book.

NOTATION Unless otherwise states, throughout the talk, $M = M^n$ is a Riemannian manifold smoothly embedded in a Riemannian manifold $A = A^{n+k}$. In most cases, M is assumed further to be closed in A – meaning without boundary.

Associated to M: τ_M the tangent bundle, $\nu_{M \subset A}$ the normal bundle of M as it sits in A.

The Normal Bundle

Starting off, there is a theorem from Geometry, which will not be proved here. However, a certain map on the normal bundle, which can be used in the proof, will be introduced.

11.1 THEOREM There exists an open neighborhood of $M \subset$ – called the *tubular neighborhood* of M – which is diffeomorphic to $E(\nu_{M \subset A})$ via a map which takes M to the zero section of $E(\nu_{M \subset A})$.

In the proof of this, there is the following map.

DEFINITION Let $E_\varepsilon(\nu_{M \subset A})$ be gotten from $E(\nu_{M \subset A})$ by taking an ε-neighborhood of 0 in each fiber.

Then,

$$\exp : E_\varepsilon(\nu_{M \subset A}) \longrightarrow A$$

is the map that takes (x, v) to $\gamma_{x,v}(1)$, where $\gamma_{x,v}$ is the geodesic starting at x with initial tangent vector v. Note that this map depends on the manifold it is considered in; for example, $\exp_M$ is not necessarily the same as $\exp_A$.

This map comes with the following property, which will also not be proved here.

ASSERTION For sufficiently small $\varepsilon > 0$, exp diffeomorphically maps $E_\varepsilon(\nu_{M \subset A})$ onto a neighborhood $N_\varepsilon \subset A$.

This neighborhood will be a tubular neighborhood of $M \subset A$.

Onto the first result here.

11.2 COROLLARY If M is closed in A, then, over any coefficient ring, $H^\bullet(A, A \setminus M) \cong H^\bullet(E(\nu_{M \subset A}), E^\times(\nu_{M \subset A}))$, where $E^\times(\nu_{M \subset A})$ is $E(\nu_{M \subset A})$ without its zero section.

PROOF This is done by using exicison, and the assertion about the exp-map. Given the context: $E = E(\nu_{M \subset A})$ and similary for E_ε, $E^\times$, and $E_\varepsilon^\times$.

As before, let N_ε be the tublar neighborhood of $M \subset A$; then, since $\overline{A \setminus N_\varepsilon} \subset (A \setminus M)^\circ$,

$$H^\bullet(A, A \setminus M) \cong H^\bullet(N_\varepsilon, N_\varepsilon \setminus M).$$

Using the assertion about exp, it follows that

$$\exp : (E_\varepsilon, E_\varepsilon^\times) \longrightarrow (N_\varepsilon, N_\varepsilon \setminus M)$$

induces that isomorphism on cohomology

$$\exp^* := H^\bullet \exp : H^\bullet(N_\varepsilon, N_\varepsilon \setminus M) \longrightarrow H^\bullet(E_\varepsilon, E_\varepsilon^\times).$$

Also, by excision,

$$H^\bullet(E_\varepsilon, E_\varepsilon^\times) \cong H^\bullet(E, E^\times).$$

Composing the isomorphisms yields the desired result:

$$H^\bullet(A, A \setminus M) \cong H^\bullet(N_\varepsilon, N_\varepsilon \setminus M) \cong H^\bullet(E_\varepsilon, E_\varepsilon^\times) \cong H^\bullet(E, E^\times).$$

■

REMARK Since the set of Riemannian metrics on M is convex, it follows that the Levi-Civita connection is defined up to homotopy of the metric and, in turn, so is the exp-map. This means that corollary **11.2** does not depend on exp being a result in cohomology.

With this, let $u \in H^k(E(\nu_{M \subset A}), E^\times(\nu_{M \subset A}); \mathbb{Z}/2)$ be the fundamental cohomology class (8.1); then, by what was just shown, there is a corresponding canonical cohomology class

$$u' \in H^k(A, A \setminus M; \mathbb{Z}/2).$$

Similarly, in the case when $\nu_{M \subset A}$ is orientable (9.1):

$$u' \in H^k(A, A \setminus M; \mathbb{Z}) \quad \text{corresponds to} \quad u \in H^k\left(E(\nu_{M \subset A}), E^\times(\nu_{M \subset A}); \mathbb{Z}\right).$$

Now, it is possible to (in the relelvant cases) relate $u'|_M$ to the Stiefel-Whitney class, or Euler class, of $\nu_{M \subset A}$.

11.3 THEOREM If M is closed in A, then the following compositions of restriction homomorphisms (pg. 97-98)

$$H^k(A, A \setminus M) \longrightarrow H^k A \longrightarrow H^k M$$

maps u' to: the Stiefel-Whitney class $w_k(\nu_{M \subset A})$ when over $\mathbb{Z}/2$; the Euler class $e(\nu_{M \subset A})$ when over $\mathbb{Z}$.

PROOF This involves basically one of the Thom isomorphisms, along with the defintions of the classes at hand. Again, using context, $E = E(\nu_{M \subset A})$, etc.

First, doing this with $\mathbb{Z}/2$ coefficients. Let $s : M \hookrightarrow E$ be the inclusion of the zero section; this induces an isomorphism $H^\bullet E \longrightarrow H^\bullet M$ by contraction. Then, with the Thom isomorphism, there are the compositions

$$H^k(E, E^\times) \longrightarrow H^k E \xrightarrow{s^*} \underbrace{H^k M \xrightarrow{\pi^*} H^k E \xrightarrow{\cup u} H^{2k}(E, E^\times)}_{\substack{\varphi \\ \text{Thom isomorphism}}}.$$

Following u through these, one obtains

$$
\begin{aligned}
\varphi s^*\left(u|_E\right) = \left(\pi^* s^*\left(u|_E\right)\right) \cup u = \left(u|_E\right) \cup u & \qquad \pi^* s^* = (s\pi)^* \simeq (\mathbf{I}_E)^* = \mathbf{I}_{H^k E} \\
= u \cup u & \qquad \text{by definition of the cup product} \\
= \mathrm{Sq}^k(u) & \qquad \text{by an axiom of the Steenrod squaring operation (pg. 90-91) with } u \in H^k\left(E, E^\times\right) \\
\iff s^*\left(u|_E\right) = \varphi^{-1} \mathrm{Sq}^k(u) & \\
= \varphi^{-1} \mathrm{Sq}^k \varphi(1) & \qquad \text{by the uniqueness of } u \\
= w_k\left(\nu_{M\subset A}\right) & \qquad \text{by definition of the Stiefel-Whitney class (pg. 91).}
\end{aligned}
$$

Now, since $(E, E^\times)$ is diffeomorphic to $(N_\varepsilon, N_\varepsilon \setminus M)$ via the exp-map and dilating, there are induced isomorphisms

$$H^k\left(E, E^\times\right) \cong H^k\left(N_\varepsilon, N_\varepsilon \setminus M\right)$$

and, before or after the restrictions,

$$H^k E \cong H^k N_\varepsilon.$$

From this, it follows corresponding diagram commutes, showing that image $u'|_{(N_\varepsilon, N_\varepsilon\setminus M)}$ of u in $H^k(N_\varepsilon, N_\varepsilon \setminus M)$ gets mapped to $w_k(\nu_{M\subset A})$ through

$$
\begin{tikzcd}
H^k\left(N_\varepsilon, N_\varepsilon \setminus M\right) \arrow[r] & H^k N_\varepsilon \arrow[r] & H^k M \\
H^k\left(E, E^\times\right) \arrow[u] & &
\end{tikzcd}
.
$$

Finally, using the following commutative diagram of restrictions

$$
\begin{tikzcd}
H^k\left(A, A \setminus M\right) \arrow[r] \arrow[d, "\wr", "\text{excision}"'] & H^k A \arrow[d] \\
H^k\left(N_\varepsilon, N_\varepsilon \setminus M\right) \arrow[r] & H^k M
\end{tikzcd}
,
$$

the result is obtained.

When the cohomology is over $\mathbb{Z}$, i.e. $\nu_{M\subset A}$ is orientable. Then, there is

$$H^k\left(E, E^\times\right) \longrightarrow H^k E \xrightarrow{s^*} H^k M \xrightarrow{\pi^*} H^k E,$$

which outputs u as

$$\pi^* s^*\left(u|_E\right) = u|_E$$

as before, showing that

$$s^*\left(u|_E\right) = \left(\pi^*\right)^{-1}\left(u|_E\right) = e\left(\nu_{M\subset A}\right)$$

by definition (pg. 98). ■

With this, now there is a relationship between $u'|_A \in H^k A$, and Stiefel-Whitney class or Euler class: if it is zero, then the corresponding class in $H^k M$ after restriction is, of course, zero.

Definition The class $u'|_A \in H^k A$ is called the *dual cohomology class* of M of codimension k in A.

11.4 COROLLARY If M is embedded in $A = \mathbb{R}^{n+k}$, $k > 0$, then $w_k\left(\nu_{M\subset\mathbb{R}^{n+k}}\right) = 0$ in the case where the coefficients are in $\mathbb{Z}/2$, and $e\left(\nu_{M\subset\mathbb{R}^{n+k}}\right) = 0$ in the case where the coefficients are in $\mathbb{Z}$, i.e. $\nu_{M\subset\mathbb{R}^{n+k}}$ is orientable.

PROOF Since $\mathbb{R}^{n+k}$ is contractible, then $0 \cong H^k\mathbb{R}^{n+k} \ni u'|_{\mathbb{R}^{n+k}}$, which remains so after restriction to H^kM. ■

The Tangent Bundle

To get to discussing the diagonal cohomology class in the next section, this section will look at the the diagonal embedding of M as ΔM in $M \times M$, namely given be the map from M to $M \times M$ taking x to (x, x). Note that this map is not an isometry but it is up to dilation.

11.5 LEMMA The normal bundle $\nu_{M\subset M\times M}$ is isomorphic to τ_M.

PROOF Let the total space of τ_M be $\mathrm{D}M$, and similarly for $\tau_{M\times M}$ as $\mathrm{D}\left(M \times M\right)$. Then, for each $x \in M$, canonically the fiber $\mathrm{D}_x\left(M \times M\right) \cong \mathrm{D}_xM \times \mathrm{D}_xM$.

Now, $(u, v) \in \mathrm{D}_xM \times \mathrm{D}_xM$ is tangent to ΔM iff $u = v$, as evident through the embedding map. From this, it follows that (u, v) is normal of ΔM iff $u + v = 0$: $(t, t) \cdot (u, v) = 0$ iff $t \cdot (u + v) = 0$.

Thus, the diffeomorphism from $\mathrm{D}M$ to $E\left(\nu_{M\subset M\times M}\right)$ is fiberwise defined as

$$\begin{aligned} \mathrm{D}_xM &\longrightarrow E\left(\nu_{M\subset M\times M}\right)|_x \\ (x, v) &\longmapsto \left((x, x), (-v, v)\right). \end{aligned}$$

■

Attention will soon be focused on (potentially-necessarily) orientable τ_M; with that, here is a lemma relating the orientation of M and that of its tangent bundle but with a full omitted proof.

11.6 LEMMA Any orientation on τ_M induces an orientation on M, and vice versa.

SKETCH Use the definition of orientation in regards to a preferred generator in the n-th homology. Then, use corollary **11.2** but for homology. Finally, show the result locally using the charts on M to bring it to $\mathbb{R}^n$.

From now on, it will be assume that τ_M is orientable, or that the coeffecients are in $\mathbb{Z}/2$. Now, as shown in corollary **11.2**, there is a fundamental cohomology class $u' \in H^n\left(M \times M, M \times M \setminus \Delta M\right)$, since $M \cong \Delta M$ is of codimension n. Then, by theorem **11.3**, it follows that the restriction of u' to $\Delta M \cong M$ is

$$w_n\left(\nu_{M\subset M\times M}\right) = w_n\left(\tau_M\right) \quad \text{or} \quad e\left(\nu_{M\subset M\times M}\right) = e\left(\tau_M\right),$$

wherein the equalities are by lemma **11.5**.

Since M is also oriented by lemma **11.6**, it follows by definition that there is, for each $x \in M$, preferred generator $\mu_x \in H_n\left(M, M \setminus \{x\}\right)$; with this, it is possible to obtain a preferred generator $u_x \in H^n\left(M, M \setminus \{x\}\right)$ by Kronecker index $\langle u_x, \mu_x\rangle = 1$.

The next result will relate u' and u_x via the following map

$$\begin{aligned} j_x : \left(M, M \setminus \{x\}\right) &\longrightarrow \left(M \times M, M \times M \setminus \Delta M\right) \\ y &\longmapsto (x, y). \end{aligned}$$

11.7 Lemma The class $u' \in H^n(M \times M, M \times M \setminus \Delta M)$ is uniquely characterized by the property that its image $j_x^*(u')$ is equal to the preferred generator $u_x \in H_n(M, M \setminus \{x\})$ for any $x \in M$.

Proof This is proved using the definition and uniqueness (Thom isomorphism, 10.4) of the fundamental cohomology class $u \in H^n(E(\nu_{M \subset M \times M}), E^\times(\nu_{M \subset M \times M}))$, and the isomorhisms from corollary **11.2** and lemma **11.5**; of course with a certain homotopy in mind. Let $E = E(\nu_{M \subset M \times M})$, etc.

Let U_ε be an ε-neighborhood of zero in $\mathrm{D}_x M$, for a *sufficiently*[1] small $\varepsilon > 0$; then, let f be the isomorphism from lemma **11.5** and V_ε the image of U_ε under $f|_{(U_\varepsilon, U_\varepsilon^\times)}$. Consider the map

$$\sigma : (U_\varepsilon, U_\varepsilon^\times) \xrightarrow{f|_{(U_\varepsilon, U_\varepsilon^\times)}} (V_\varepsilon, V_\varepsilon^\times) \xrightarrow{\exp_{M \times M}} (M \times M, M \times M \setminus \Delta M).$$

First, note that it follows from the defining property of the fundamental cohomology class, namely that it is the unique class which restricts to the preferred generator in each fiber, that the isomorphism

$$f^* : H^n(E, E^\times) \longrightarrow H^n(\mathrm{D}M, \mathrm{D}^\times M)$$

must map the fundamental cohomology classes of the domain to that of the codomain (pg. 92) – say, respectively, u as before and $\overline{u} := f^*(u)$. With that in mind, again by the definition of $\overline{u}$, it must be that its restriction to a fiber

$$\overline{u}|_{(U_\varepsilon, U_\varepsilon^\times)} = u'_x|_{(U_\varepsilon, U_\varepsilon^\times)} \in H^n(U_\varepsilon, U_\varepsilon^\times) \cong H^n(\mathrm{D}_x M, \mathrm{D}_x^\times M),$$

where $u'_x \in H^n(\mathrm{D}_x M, \mathrm{D}_x^\times M)$ is the preferred generator corresponding to the preferred generator $u_x \in H^n(M, M \setminus \{x\})$ via the isomorphism from corollary **11.2** with $\{x\} \subset M$ being an embedded closed subset. Therewith, taking into account the restrictions,

$$\begin{aligned}
\sigma^*(u') &= \left(f|_{(U_\varepsilon, U_\varepsilon^\times)}\right)^* \exp^*_{M \times M}(u') \\
&= \left(f|_{(U_\varepsilon, U_\varepsilon^\times)}\right)^* \left(u|_{(V_\varepsilon, V_\varepsilon^\times)}\right) && \text{as defined via corollary } \mathbf{11.2}, \text{ where } M \cong \Delta M \subset M \times M =: A \\
&= u'_x|_{(U_\varepsilon, U_\varepsilon^\times)} && \text{as just discussed.}
\end{aligned}$$

Looking further at σ^*, as to express it in terms of j_x^*. First off, as remarked earlier, M is in $M \times M$ up a dilation; in particular, this means that there is a homotopy $\exp_{M \times M} \simeq \exp_M \times \exp_M$ manipulating this dilation. This is, of course, to say that $\exp^*_{M \times M} = (\exp_M \times \exp_M)^*$, whereby we can homotope σ to a slightly different map and still keep the discussion from before on the table. Consider that such map,

$$\begin{aligned}
\rho : (U_\varepsilon, U_\varepsilon^\times) &\longrightarrow (M \times M, M \times M \setminus \Delta M) \\
v &\longmapsto (\exp_M \times \exp_M)\, f|_{(U_\varepsilon, U_\varepsilon^\times)}(v) = (\exp_M(x, -v), \exp_M(x, v)),
\end{aligned}$$

which uses the fact that $\mathrm{D}_x(M \times M) \cong \mathrm{D}_x M \times \mathrm{D}_x M$. But it does not stop there: consider another homotopic map,

$$\begin{aligned}
\eta : (U_\varepsilon, U_\varepsilon^\times) &\longrightarrow (M \times M, M \times M \setminus \Delta M) \\
v &\longmapsto (x, \exp_M(x, v)),
\end{aligned}$$

which is $h(\cdot, 0)$ of the following homotopy

$$\begin{aligned}
h : (U_\varepsilon, U_\varepsilon^\times) \times [0,1] &\longrightarrow (M \times M, M \times M \setminus \Delta M) \\
(v, t) &\longmapsto (\exp_M(x, -tv), \exp_M(x, v)),
\end{aligned}$$

since $\exp_M(x, 0) = x$. Finally, note that $\eta = j_x \circ \exp_M$ on $(U_\varepsilon, U_\varepsilon^\times)$ so that, by homotopy equivalence, there is the following commutative diagram

[1] Hiding some geometric details in this assumption: conjugate points and completeness.

$$\begin{tikzcd}
H^n(M \times M, M \times M \setminus \Delta M) \arrow[rr, "\rho^* = \sigma^*"] \arrow[dr, "j_x^*"'] & & H^n(U_\varepsilon, U_\varepsilon^\times) \cong H^n(\mathrm{D}_x M, \mathrm{D}_x^\times M) \\
& H^n(M, M \setminus \{x\}) \arrow[ur, "\exp_M"'] &
\end{tikzcd},$$

Thus,

$$\exp_M^* j_x^*(u') = u_x'|_{(U_\varepsilon, U_\varepsilon^\times)},$$

which, by the isomorphism from corollary **11.2**, shows that

$$j_x^*(u') = u_x.$$

■

The Diagonal Cohomology Class in $H^n(M \times M)$

As assumed in the previous section, M will be orientable or the coefficients will be in $\mathbb{Z}/2$, so that we have the fundamental class $u' \in H^n(M \times M, M \times M \setminus \Delta M)$. With that, comes the next definition.

DEFINITION Let $u'' := u'|_{M \times M} \in H^n(M \times M)$ be the *diagonal cohomology class* in $H^n(M \times M)$.

This class has the following property: it is "concentrated" along the diagonal of $M \times M$.

11.8 LEMMA For any $a \in H^\bullet M$,

$$(a \times 1) \cup u' = (1 \times a) \cup u',$$

or, after restriction to $M \times M$,

$$(a \times 1) \cup u'' = (1 \times a) \cup u'',$$

PROOF This is proved by considering the canoncial projection maps, first, in a tubular neighborhood of the diagonal, then, bringing it back across the restrictions, using excision.

Let N_ε be a tubular neighborhood of $\Delta M \subset M \times M$. Then, let p_1, p_2 be the projection of $M \times M$ onto each M-factor:

$$\begin{aligned} p_1, p_2 &: M \times M \longrightarrow M \\ p_1 &: (x, y) \longmapsto x \\ p_2 &: (x, y) \longmapsto y. \end{aligned}$$

Now, note that $N_\varepsilon \simeq \Delta M$ by deformation retract and that $p_1|_{\Delta M} = p_2|_{\Delta M}$; from this, it follows that $p_1|_{N_\varepsilon} := p_1 i_{N_\varepsilon} \simeq p_2 i_{N_\varepsilon} =: p_2|_{N_\varepsilon}$, where $i_{N_\varepsilon} : N_\varepsilon \hookrightarrow M \times M$ is the inclusion. Thus, the compositions for $i = 1, 2$

$$H^j M \xrightarrow{p_i^*} H^j(M \times M) \xrightarrow{i_{N_\varepsilon}^*} H^j N_\varepsilon$$

are identical by homotopy invariance; in particular, following $a \in H^j M$ through yields

$$(a \times 1)|_{N_\varepsilon} = i_{N_\varepsilon}^* p_1^*(a) = i_{N_\varepsilon}^* p_2^*(a) = (1 \times a)|_{N_\varepsilon}.$$

Finally, to get the result, consider the following commutative diagram:

$$\begin{tikzcd}
H^j(M \times M) \arrow[r, "i_{N_\varepsilon}^*"] \arrow[d, "\cup u'"] & H^j N_\varepsilon \arrow[d, "\cup u'|_{(N_\varepsilon, N_\varepsilon \setminus \Delta M)}"] \\
H^{j+n}(M \times M, M \times M \setminus \Delta M) \arrow[r, "\sim", "\text{excision}"'] & H^{j+n}(N_\varepsilon, N_\varepsilon \setminus \Delta M)
\end{tikzcd},$$

since any $b \in H^j(M \times M)$ has the image

$$(b \cup u')|_{(N_\varepsilon, N_\varepsilon \setminus \Delta M)} = b|_{N_\varepsilon} \cup u'|_{(N_\varepsilon, N_\varepsilon \setminus \Delta M)} \in H^{j+n}(N_\varepsilon, N_\varepsilon \setminus \Delta M),$$

following either route, by the properties of the cup product. Hence,

$$((a \times 1) \cup u')|_{(N_\varepsilon, N_\varepsilon \setminus \Delta M)} = ((1 \times a) \cup u')|_{(N_\varepsilon, N_\varepsilon \setminus \Delta M)},$$

which is the wanted result

$$(a \times 1) \cup u' = (1 \times a) \cup u'$$

after following it back through the exicison isomorphism. Following back further through a the restriction $H^{j+n}(M \times M, M \times M \setminus \Delta M) \longrightarrow H^{j+n}(M \times M)$, yields

$$(a \times 1) \cup u'' = (1 \times a) \cup u''$$

by the properties of the cup product and the definition of $u'' := u'|_{M \times M}$. ∎

To finish off the section, there will be a lemma relating the (unique) fundamental homology class $\mu \in H_n M$ for an also-compact M, to u'' via a binary operator called the slant product. Attention will be turned to the case where the coefficients lie in some field Λ.

Definition Let X, Y be finite complexes.[2] Then, the *slant product* is a map

$$\begin{aligned} / : H^{p+q}(X \times Y) \otimes H_q Y &\longrightarrow H^p X \\ (a \times b) \otimes \alpha &\longmapsto (a \times b)/\alpha = a \langle b, \alpha \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker index. It has the characterizing property that for some $\alpha \in H_q Y$

$$((a \times 1) \cup c)/\alpha = a \cup (c/\alpha) \in H^{p+r} X$$

for any $a \in H^p(X)$ and $c \in H^{r+q}(X \times Y)$.

Now, onto the aforementioned result.

11.9 Lemma Let M be a compact Riemannian manifold so that there is the (unique) fundamental homology class $\mu \in H_n M$. Then, the diagonal cohomology class $u'' \in H^n(M \times M)$ and the fundamental homology class μ of M are related by the identity

$$u''/\mu = 1 \in H^0 M.$$

Proof This proved by looking at the expression $(u''/\mu)|_{\{x\}}$, and recalling the map j_x from lemma **11.7**. Note that since M is a compact Riemannian manifold, it is a finite complex.

Consider the following diagram involving the slant product with $\mu \in H_n M$: with $i_{\{x\} \times M}$ is the canonical inclusion of $\{x\} \times M$ into $M \times M$ and similarly for the maps $i_{\{x\}}$, i_M,

$$\begin{tikzcd} H^n(M \times M) \arrow[r, "/\mu"] \arrow[d, "i^*_{\{x\} \times M} = i^*_{\{x\}} \times i^*_M"'] & H^0 M \arrow[d] \\ H^0\{x\} \times H^n M \cong H^n(\{x\} \times M) \arrow[r, "/\mu"] & H^0\{x\} \cong \Lambda \end{tikzcd},$$

[2]This definition can be generalized, but here it is stated in such a way that the proof of the becomes easier.

where the bottom-left isomorphism is gotten from the Künneth formula (given the assumptions on the finite complex M) and the dimension axiom, and the bottom-right one from the dimension axiom. In particular, notice that it is commutative:

With the assumption on M, it follows by the Künneth formula (A.6),

$$H^n(M \times M) \cong \bigoplus_{i+j=n} H^i M \otimes H^j M.$$

As such, with linearity, it suffices to consider some $a \times b \in H^i M \otimes H^j M$. However, this can be reduced further upon noticing that $(a \times b)/\mu = a\langle b, \mu\rangle = 0$ necessarily unless $i = 0$ and $j = n$, by the definition of the slant product with $\mu \in H_n M$, and, for the other map, the map i_M^* on the second factor only maps into $H^n M$ from $H^n M$.

With that, following through $a \times b \in H^0 M \otimes H^n M$ yields

$$\begin{aligned}((a \times b)/\mu)|_{\{x\}} = a|_{\{x\}}\langle b, \mu\rangle &= \left(a|_{\{x\}} \times b\right)/\mu \\ &= \left(i_{\{x\}}^*(a) \times i_M^*(b)\right)/\mu \\ &= \left(i_{\{x\}\times M}^*(a \times b)\right)/\mu.\end{aligned}$$

Therewith, following u'' through yields

$$\left(i_{\{x\}\times M}(u'')\right)/\mu = (u''/\mu)|_{\{x\}}$$

or, since u'' restricts to the preferred generator $1 \in H^0\{x\} \cong \Lambda$,

$$\left(1 \times i_{\{x\}\times M}(u'')\right)/\mu = 1\left\langle i_{\{x\}\times M}(u''), \mu\right\rangle = (u''/\mu)|_{\{x\}}.$$

Now, recall the map from lemma **11.7**, which looks strikingly similar to i_x:

$$\begin{aligned} j_x : (M, M \setminus \{x\}) &\longrightarrow (M \times M, M \times M \setminus \Delta M) \\ y &\longmapsto (x, y). \end{aligned}$$

With this, there is the commutative diagram in cohomology induced by the respective commutative diagram in pairs of topological spaces

$$\begin{tikzcd} H^n(M \times M, M \times M \setminus \Delta M) \arrow[r, "\text{restriction}"] \arrow[d, "j_x^*"'] & H^n(M \times M) \arrow[d, "i_x^*"] \\ H^n(M, M \setminus \{x\}) \arrow[r, "\text{restriction}"] & H^n M \end{tikzcd},$$

which shows that

$$i_x^*(u'') = i_x^*\left(u'|_{M\times M}\right) = (j_x^*(u'))|_M.$$

Finally, recall that μ is characterized as the unique class in $H^n M$, which maps, for each $x \in M$, to the preferred generator $\mu_x \in H^n(M, M \setminus \{x\})$ by the map induced by the natural inclusion $M \hookrightarrow (M, M \setminus \{x\})$. This means that

$$\langle (j_x^*(u'))|_M, \mu\rangle = \langle j_x^*(u'), \mu_x\rangle,$$

by moving back across the restriction. Therefore, putting this together and using lemma **11.7**, it follows that

$$(u''/\mu)|_{\{x\}} = \langle i_x^*(u''), \mu\rangle = \langle (j_x^*(u'))|_M, \mu\rangle = \langle j_x^*(u'), \mu_x\rangle = \langle u_x, \mu_x\rangle = 1,$$

which, since $x \in M$ was arbitrary, implies that $u''/\mu = 1$. ∎