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MAT984

Now they we have a notion of curvature, we would like to get a grasp on how curvature affects geodesics. Doing so involves using a construction that we have used before:

<u>Recall</u>

From the proof of Gauß's Lemma, for $p \in M$ and $v \in T_pM$ such that $\exp_p v$ is defined, consider any $w \in T_v(T_pM)$, then

$$\left(d\exp_p\right)_v w = \frac{\partial f}{\partial s}(1,0),$$

where f is a parametrized surface, $f : A \longrightarrow M$, such that

$$f(t,s) = \exp_p(tv(s)),\ 0 \le t \le 1,\ -\varepsilon < s < \varepsilon$$

and $v(s)$ is a curve in T_pM with $v(0) = v$ and $v'(0) = w$.
Also, make note that $f(t, s_0)$, fixed s_0, is a geodesic by the nature of the $\exp_p$ map.

It is convenient to study $\left|\left(d\exp_p\right)_v w\right|$ because this expression describes the spreading for the geodesics $f(t,s)$ for each s depending on w. We can extend this expression to

$$\frac{\partial f}{\partial s}(t,0) = \frac{\partial}{\partial s}\exp_p(tv(s))\bigg|_{t=t,s=0} = \left(d\exp_p\right)_{tv(0)}\left(t\,\frac{\partial}{\partial s}v(s)\bigg|_{s=0}\right) = \left(d\exp_p\right)_{tv}(tw),$$

which is along the geodesic $\gamma(t) = \exp_p(tv) = \gamma(1,p,tv) = \gamma(t,p,v)$.

Using the fact that for fixed s, f is a geodesic, it follows from definition that $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$. Coincidentally (read: something I am not proving), there is the identity from `Ch. 4 Lemma 4.1` allows us to write

$$\begin{aligned} 0 &= \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \\ 0 &= \underbrace{\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s}}_{\text{symmetry}} + \underbrace{R\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}}_{\substack{\text{identity of}\\ \text{curvature } R}} \end{aligned}$$

let $J(t) = \frac{\partial f}{\partial s}(t,0)$

$$\Longrightarrow \boxed{\frac{D^2 J}{dt^2} + R\left(\gamma'(t), J(t)\right)\gamma'(t) = 0}.$$

This is the Jacobi equation.

<u>Def.</u>

Given a geodesic $\gamma : [0,a] \longrightarrow M$, a vector field along γ, J, is a Jacobi field if it satisfies the Jacobi equation $\forall t \in [0,a]$.

Now, consider $\dim M = n$. Let the basis of a Jacobi field J be $\{e_1(t), \ldots, e_n(t)\}$. Rewriting the terms of the Jacobi equation in this basis, for $i, j \in \{1, \ldots, n\}$,

$$J(t) = \sum_i f_i(t) e_i(t) \implies \frac{D^2 J}{dt^2} = \sum_i f_i'' e_i$$

and

$$\begin{aligned} R(\gamma', J)\gamma' &= \sum_j \langle R(\gamma', J)\gamma', e_j\rangle e_j && \text{breaking it up until appropriate components} \\ &= \sum_{i,j} f_i \underbrace{\langle R(\gamma', e_i)\gamma', e_j\rangle}_{a_{ij}} e_j \\ R(\gamma', J)\gamma' &= \sum_{i,j} f_i a_{ij} e_j. \end{aligned}$$

Thus, the Jacobi equation is the system of n second-order differential equations: for $j \in \{1, \ldots, n\}$,

$$f_j'' + \sum_i a_{ij} f_i = 0.$$

This shows that there are $2n$ possible Jacobi field along γ, and that J is uniquely determined given the initial conditions $J(0)$ and $\frac{DJ}{dt}(0)$.

The following proposition will state that this construction for the Jacobi field of a geodesic is the only way of constructing a Jacobi field.

Prop.

Given a geodesic $\gamma : [0, a] \longrightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$ with a Jacobi field J along it such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w \in T_{av}(T_p M)$.

Construct a curve $v(s)$ in $T_p M$ such that $v(0) = av$ and $v'(0) = w$. Let $f(t, s) = \exp_p\left(\frac{t}{a} v(s)\right)$, and define the Jacobi field $\overline{J}(t) = \frac{\partial f}{\partial s}(t, 0) = \left(d\exp_p\right)_{tv}(tw)$.

Then, $\overline{J} = J$.

Proof

Evaluating $\overline{J}$:

$$\overline{J}(0) = \left(d\exp_p\right)_0(0) = 0.$$

Looking at the covariant derivative, when $s = 0$,

$$\begin{aligned} \frac{D\overline{J}}{dt} = \frac{D}{dt}\frac{\partial f}{\partial s} &= \frac{D}{\partial t}\left(d\exp_p\right)_{tv}(tw) \\ &= \frac{D}{\partial t}\left(t\left(d\exp_p\right)_{tv}(w)\right) && \text{by linearity} \\ \frac{D\overline{J}}{dt} &= \left(d\exp_p\right)_{tv}(w) + t\frac{D}{\partial t}\left(d\exp_p\right)_{tv}(w), \end{aligned}$$

and, from that, when $t = 0$,

$$\frac{D\overline{J}}{dt}(0) = \left(d\exp_p\right)_0(w) = w.$$

However, this makes $J(0) = \overline{J}(0) = 0$ and $\frac{DJ}{dt}(0) = \frac{D\overline{J}}{dt}(0) = w$, which forces $\underline{J(t) = \overline{J}(t)}$ by uniqueness that was discussed before.

$\square$

An immediate corollary of this is as follows:

<u>Cor.</u>

Given geodesic $\gamma : [0, a] \longrightarrow M$, then Jacobi field J that is along γ with $J(0) = 0$ is given by

$$J(t) = \left(d\exp_p\right)_{t\gamma'(0)}\left(t\frac{DJ}{dt}(0)\right).$$

Finally, the relationship between geodesic spreading, $|J(t)|$, and curvature is shown in the following proposition and its corollaries.

<u>Prop.</u>

As per usual, for $p \in M$, let $\gamma : [0, a] \longrightarrow M$ be a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Also, let $w \in T_v(T_pM)$ such that $|w| = 1$, and let J be the Jacobi field along γ be defined by

$$J(t) = \left(d\exp_p\right)_{tv}(tw), \text{ for } 0 \leq t \leq a.$$

Then, the Taylor expansion of $\langle J(t), J(t)\rangle = |J(t)|^2$ about $t = 0$ is

$$|J(t)|^2 = t^2 - \frac{t^4}{3}\langle R(v, w)v, w\rangle + r(t),$$

such that $\lim\limits_{t \to 0}\frac{r(t)}{t^4} = 0$.

<u>Proof</u>

Let $\frac{DJ}{dt} = J'$, $\frac{D^2J}{dt^2} = J''$, etc. Calculating the first few coefficients using the facts shown before that $J(0) = 0$ and $J'(0) = w$

$$\begin{aligned}
\langle J, J\rangle(0) &= 0 \\
\langle J, J\rangle'(0) &= 2\left\langle J, J'\right\rangle(0) = 0 \\
\langle J, J\rangle''(0) &= \underbrace{2\left\langle J'', J\right\rangle(0)}_{0} + \underbrace{2\left\langle J', J'\right\rangle(0)}_{2|w|} = 2
\end{aligned}$$

From the Jacobi equation and the definition of the curvature R, it follows that

$$J''(t) = -R\left(\gamma'(t), J(t)\right)\gamma'(t) \implies J''(0) = 0.$$

This makes

$$\langle J, J\rangle''' (0) = \underbrace{2\left\langle J''', J\right\rangle (0)}_{0} + \underbrace{6\left\langle J'', J'\right\rangle (0)}_{0} = 0$$

Now, a clever trick: using a compatible metric $\langle \cdot, \cdot \rangle$, it follows that, for any vector field W along γ,

$$\begin{aligned}
\frac{d}{dt}\left\langle R\left(\gamma', W\right)\gamma', J\right\rangle = \underbrace{\frac{d}{dt}\left\langle R\left(\gamma', J\right)\gamma', W\right\rangle}_{\substack{\text{identity on the} \\ \text{curvature } R}} &= \left\langle \frac{D}{dt}\left(R\left(\gamma', J\right)\gamma'\right), W\right\rangle + \left\langle R\left(\gamma', J\right)\gamma', W'\right\rangle \\
\iff \left\langle \frac{D}{dt}\left(R\left(\gamma', J\right)\gamma'\right), W\right\rangle &= \frac{d}{dt}\left\langle R\left(\gamma', W\right)\gamma', J\right\rangle - \left\langle R\left(\gamma', J\right)\gamma', W'\right\rangle \\
\left\langle \frac{D}{dt}\left(R\left(\gamma', J\right)\gamma'\right), W\right\rangle &= \left\langle \frac{D}{dt}\left(R\left(\gamma', W\right)\gamma'\right), J\right\rangle + \left\langle R\left(\gamma', W\right)\gamma', J'\right\rangle \\
&\quad - \left\langle R\left(\gamma', J\right)\gamma', W'\right\rangle
\end{aligned}$$

so that when $t = 0 \implies J(0) = 0$, this yields

$$\begin{aligned}
\left\langle \frac{D}{dt}\left(R\left(\gamma', J\right)\gamma'\right), W\right\rangle (0) = \underbrace{\left\langle R\left(\gamma', W\right)\gamma', J'\right\rangle (0)}_{\text{identity on } R} &= \left\langle R\left(\gamma', J'\right)\gamma', W\right\rangle (0) \\
\iff \frac{D}{dt}\left(R\left(\gamma', J\right)\gamma'\right)(0) &= \left(R\left(\gamma', J'\right)\gamma'\right)(0) \\
\iff -\frac{D}{dt}\left(J''\right)(0) &= \left(R\left(\gamma', J'\right)\gamma'\right)(0) \qquad \text{from the Jacobi equation} \\
\iff J'''(0) &= -\left(R\left(\gamma', J'\right)\gamma'\right)(0)
\end{aligned}$$

Using this:

$$\begin{aligned}
\langle J, J\rangle'''' (0) &= \underbrace{2\left\langle J'''', J\right\rangle (0)}_{0} + \underbrace{6\left\langle J'', J''\right\rangle (0)}_{0} + 8\left\langle J''', J'\right\rangle (0) \\
&= -8\left\langle R\left(\gamma', J'\right)\gamma', J'\right\rangle (0) \\
\langle J, J\rangle'''' (0) &= -8\left\langle R(v, w)\, v, w\right\rangle (0).
\end{aligned}$$

By definition of the Taylor expansion around $t = 0$, it follows from what was shown that

$$|J(t)|^2 = \langle J, J\rangle (t) = \langle J, J\rangle (0) + t\langle J, J\rangle' (0) + \frac{t^2}{2!}\langle J, J\rangle'' (0) + \frac{t^3}{3!}\langle J, J\rangle''' (0) + \frac{t^4}{4!}\langle J, J\rangle'''' (0) + \dots$$

$$\underline{|J(t)|^2 = t^2 - \frac{t^4}{3}\langle R(v, w)\, v, w\rangle (0) + r(t)}$$

where $r(t)$ are the higher order terms so that $\underline{\lim_{t\to 0} \frac{r(t)}{t^4} = 0}$.

$\square$

Recall

From the previous section about curvature, for a two-dimensional subspace $\sigma \subset T_pM$, the sectional curvature, which is invariant under choice of basis $\{v, w\}$ of σ, is defined at point p as

$$K(p,\sigma) = \frac{\langle R(v,w)\,v, w\rangle}{|v \wedge w|^2},$$

where

$$|v \wedge w|^2 = |v|^2\,|w|^2 - \langle v, w\rangle^2 .$$

It is possible to rewrite the expression gotten in the previous proposition in terms the sectional curvature. The following corollary states this fact.

Cor.

Given a geodesic $\gamma : [0, \ell] \longrightarrow M$ that is parametrized by arc length, namely $|\gamma'(0)| = |v| = 1$, such that $\langle v, w\rangle = 0$ so that $\{v, w\}$ is a basis of $\sigma \subset T_pM$, it is possible to write

$$|J(t)|^2 = t^2 - \frac{t^4}{3} K(p,\sigma) + r(t)$$

with $\lim\limits_{t\to 0} \dfrac{r(t)}{t^4} = 0$.

Proof

Since $\langle v, w\rangle = 0$, it follows that

$$K(p,\sigma) = \frac{\langle R(v,w)\,v, w\rangle}{|v \wedge w|^2} = \langle R(v,w)\,v, w\rangle\,,$$

it follows from the previous proposition that

$$\underline{|J(t)|^2 = t^2 - \frac{t^4}{3} K(p,\sigma) + r(t),}$$

where $\underline{\lim\limits_{t\to 0} \dfrac{r(t)}{t^4} = 0.}$

□

In a similar fashion to the previous proposition and corollary, rewriting the Taylor expansion of $|J(t)|$ around $t = 0$ yields a similar expression.

Cor.

Given the same conditions as the previous corollary, it is possible to write

$$|J(t)| = t - \frac{t^3}{6} K(p,\sigma) + \widetilde{r}(t)\,,$$

where $\lim\limits_{t\to 0} \dfrac{\widetilde{r}(t)}{t^3} = 0$.

To conclude, returning the motivating construction used at the start that is now a bit modified:

> Let there be a parametrized surface
>
> $$f(t,s) = \exp_p(tv(s)),\ 0 \leq t \leq \delta,\ -\varepsilon < s < \varepsilon,$$
>
> where δ is chosen small enough such that f is defined and $v(s)$ is a curve in T_pM such that $|v(s)| = 1$, $v(0) = v, v'(0) = w \in T_v(T_pM)$ and $|w| = 1$.

Consider the rays in T_pM formed by the map $\vec{\rho}_s : t \longmapsto tv(s)$ as t runs across the interval $[0,\delta]$ for fixed s. These rays deviate (separate) from $\vec{\rho}_0$ at a rate of

$$\left| t\,\frac{\partial}{\partial s} v(s)\Big|_{s=0} \right| = \left| tv'(0) \right| = |tw| = t.$$

Looking at the corresponding geodesics in M of these rays under the map of f, their rate of separation, $|J(t)|$, is given by the previous corollary

$$|J(t)| = t - \frac{t^3}{6} K(p,\sigma) + \widetilde{r}(t).$$

From this expression, it is easy to see that, locally, the rate of separation of the geodesics in M can be approximated, and that:

— If the sectional curvature is positive, $K(p,\sigma) > 0$, the rate of separation of the geodesics in M is slower than that of the separation of the rays in T_pM.

— If $K(p,\sigma) < 0$, the rate of separation of the geodesics in M is faster than the rate of separation of the rays in T_pM.