Chapter 6: Symplectic Representations & the Torelli group

Farb & Margalit, *A primer on mapping class groups (v 5.0)*

Michael Robert Jimenez

Mapping Class Groups
reading seminar SS24

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Overview

With the aim of better understanding $\mathrm{Mod}(S_g)$, we will look at its representation on $\mathrm{H}_1(S_g;\mathbb{Z})$

$$\mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(\mathrm{H}_1(S_g;\mathbb{Z})) \cong \mathrm{Aut}\left(\mathbb{Z}^{2g}\right) \cong \mathrm{GL}(2g,\mathbb{Z})$$

so that our understanding of $\mathrm{Aut}(\mathrm{H}_1(S_g;\mathbb{Z}))$ may be transferred into a “first approximation” of $\mathrm{Mod}(S_g)$. Note that we can already factor this map through the inclusion $\mathrm{SL}(2g,\mathbb{Z}) \hookrightarrow \mathrm{GL}(2g,\mathbb{Z})$, since we already know that $\mathrm{Mod}(S_g)$ is orientation-preserving and preserves the lattice of $\mathrm{H}_1(S_g;\mathbb{Z})$.

In this chapter (§6.1), we will see that $\mathrm{H}_1(S_g;\mathbb{Z})$ has a symplectic structure via the algebraic intersection number, which is preserved by Ψ. This means the above map factors further through $\mathrm{Sp}(S_g,\mathbb{Z}) \hookrightarrow \mathrm{SL}(2g,\mathbb{Z})$, so we are thusly the representation can be written as

$$\Psi\colon\ \mathrm{Mod}(S_g) \longrightarrow \mathrm{Sp}(S_g,\mathbb{Z}) \quad .$$

We will show that this representation is surjective (§6.3.2), and look at some special subgroups with special properties (§6.4), ending up at the Torelli group (§6.5) and the Johnson homomorphism (§6.6). Because of the inclusion inducing an isomorphism on the first homology, we will make remarks about the related representations

$$\mathrm{Mod}(S_{g,1}) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}) \qquad \mathrm{Mod}\left(S_g^1\right) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}) \quad .$$

6.1 Symplectic Structure on $\mathrm{H}_1(S_g; \mathbb{Z})$

6.1.1 Symplectic Background — 1 / 3

Definition

A *symplectic structure* on a vector space V is a non-degenerate, alternating, bilinear form ω on V: $\omega \in \bigwedge^2 V^*$ the second exterior power of V^*, the dual to V.

Remark

It can be shown that non-degeneracy forces the $\dim V$ to be even. Moreover, it can be shown that, up to change of basis (a symplectomorphism in the simplest sense), there is only one such a form on a vector space.

Here, let $\{x_j, y_j\}_{j=1}^{2g}$ be a basis for the vector space $\mathbb{R}^{2g}$, then the standard symplectic structure on $\mathbb{R}^{2g}$ is given by

$$\omega = \sum_j \mathrm{d}x_j \wedge \mathrm{d}y_j := \sum_j (\mathrm{d}x_j \otimes \mathrm{d}y_j - \mathrm{d}x_j \otimes \mathrm{d}y_j) \quad .$$

6.1.1 Symplectic Background — 2 / 3

Definition

The *symplectic linear group* $\mathrm{Sp}(2g, \mathbb{R})$ is the group of linear transformations that preserve the standard symplectic form:

$$\mathrm{Sp}(2g, \mathbb{R}) = \{A \in \mathrm{GL}(2g, \mathbb{R}) \,|\, A^* \omega = \omega\} \quad \text{or} \quad \mathrm{Sp}(2g, \mathbb{R}) = \{A \in \mathrm{GL}(2g, \mathbb{R}) \,|\, A^\intercal J A = J\}$$

where J is the standard complex structure on $\mathbb{R}^{2g}$

$$J := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad .$$

Of course, $\mathrm{Sp}(2g, \mathbb{Z}) := \mathrm{Sp}(2g, \mathbb{R}) \cap \mathrm{GL}(2g, \mathbb{Z})$.

Remark

In the case $g = 1$, it can be shown that the symplectic linear group is simply the special linear group: $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. This will be considered an extraordinary case. Note that $\mathrm{SL}(n, \mathbb{R})$ generated by elementary matrices.

6.1.1 Symplectic Background — 3 / 3

For $g = 2$, there are the so-called *Burkhardt generators* for $\mathrm{Sp}(4, \mathbb{Z})$:

- *transvection*, $(x_1, y_1, x_2, y_2) \longmapsto (x_1 + y_1, y_1, x_2, y_2)$
- *factor rotation*, $(x_1, y_1, x_2, y_2) \longmapsto (-y_1, x_1, x_2, y_2)$
- *factor mix*, $(x_1, y_1, x_2, y_2) \longmapsto (x_1 - y_2, y_1, x_2 - y_1, y_2)$
- *factor swap*, $(x_1, y_1, x_2, y_2) \longmapsto (x_2, y_2, x_1, y_1)$.

For $g > 2$, it is possible to augment these generators with further factor swaps, namely swaps for adjacent factors $2j - 1$ and $2j$ for each $1 \leq j \leq g$, to obtain generators for $\mathrm{Sp}(2g, \mathbb{Z})$.

Theorem 6.1

$\mathrm{Sp}(2g, \mathbb{Z})$ is generated by a (finite) collection of *elementary symplectic matrices*.

6.1.2 Symplectic Structure on $\mathrm{H}_1(S_g;\mathbb{Z})$ — 1 / 2

Consider the following ordered basis for $\mathrm{H}_1(S_g;\mathbb{Z})$:

$$\{[a_j],[b_j]\}_{j=1}^{g}$$

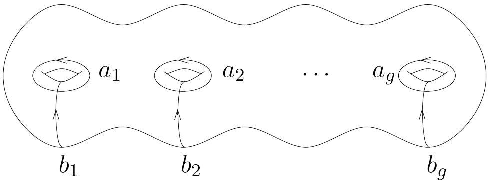


Figure 6.1 The "standard" geometric symplectic basis for $H_1(S_g;\mathbb{Z})$.

It will be shown that the algebraic intersection number (cf. §1.2.3) is a symplectic form on $\mathrm{H}_1(S_g;\mathbb{Z})$:

$$\hat{\imath}\colon \mathrm{H}_1(S_g;\mathbb{Z}) \times \mathrm{H}_1(S_g;\mathbb{Z}) \longrightarrow \mathbb{Z} \quad ,$$

which is this signed intersection number of two homology classes and was already noted to be bilinear and alternating (anti-symmetric).

6.1.2 Symplectic Structure on $\mathrm{H}_1(S_g;\mathbb{Z})$ — 2 / 2

Evaluating the algebraic intersection number on the aforementioned basis,

$$\hat{i}([a_j],[a_k]) = 0 = \hat{i}([b_j],[b_k])$$

$$\hat{i}([a_j],[b_k]) = \delta_{jk} \quad ,$$

which shows this operation is non-degenerate and is thusly a *symplectic basis*. F&M go on to call this a *geometric* symplectic basis because this basis is an instantiation of a collection of non-separating, oriented, simple, closed (n.o.s.c.) curves realizing the geometric intersection number, as discussed in §1.2.3.

6.2 Euclidean Algorithm for Simple Closed Curves

6.2 Euclidean Algorithm for Simple Closed Curves — 1 / 5

In this section, the objective is to "strengthen the dictionary between the algebraic and the topological aspects of $\mathrm{H}_1(S_g; \mathbb{Z})$", which amounts to the following proposition.

Proposition 6.2

A non-zero (i.e. non-separating) element of $\mathrm{H}_1(S_g; \mathbb{Z})$ is represented by a s.c. curve iff it is primitive (i.e. $v \in \mathrm{H}_1(S_g; \mathbb{Z})$ such that $v \neq mw$ for any $w \in \mathrm{H}_1(S_g; \mathbb{Z})$ and $n > 1$).

Outline of Proof

The only-if-direction of the proof follows quickly from the change-of-coördinates argument introduced in Ch. 1. For the if-direction, the idea of the proof is to appeal to the $g = 1$ case, apply it locally in the $g > 1$ case, and then iteratively "surgering" the curves to obtain a single n.o.s.c. curve at the end.

6.2 Euclidean Algorithm for Simple Closed Curves — 2 / 5

Proof

$\underline{g = 1 \text{ Case}}$: This was proven in Theorem 1.5, which showed this result for $\pi_1 \mathbb{T}^2 \cong \mathrm{H}_1(S_1; \mathbb{Z})$, where $S_1 := \mathbb{T}^2$: homotopy classes of o.s.c. curves are 1-to-1 with primitive elements.

$\underline{g > 1 \text{ Case}}$: ($\Rightarrow$) Let $\gamma \subseteq S_g$ be a non-separating, oriented, simply, closed curve. Then, by the change-of-coördinates argument, there is a $\varphi \in \mathrm{Homeo}^+(S_g)$ such that $\varphi(\gamma)$ is some primitive curve, e.g., a_1 as in the aforementioned symplectic basis $\{[a_j], [b_j]\}_j$. This means that γ is a member of some basis (i.e. the φ preïmages of the symplectic basis) and is thusly primitive.

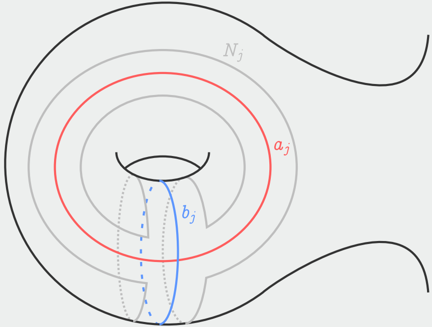
($\Leftarrow$) As before, let $\{[a_j], [b_j]\}_j$ be the (symplectic) basis for $\mathrm{H}_1(S_g; \mathbb{Z})$. Also, let $x = (v_1, w_1, \ldots, v_g, w_g) \in \mathrm{H}_1(S_g; \mathbb{Z})$ be some primitive element, where $v_j, w_j \geq 0$ for all j w.l.o.g., up to choosing an orientation of the basis.

6.2 Euclidean Algorithm for Simple Closed Curves — 3 / 5

Proof (continued)

Now for each j, locally look in the $S_{1,1}$-neighborhoods N_j about $\{[a_j], [b_j]\}$. In each N_j, it is possible to apply the $g = 1$ case since the inclusion $N_j \hookrightarrow S_1$ is an isomorphism on the first homology: we can find a n.o.s.c. curve $\gamma_j \subseteq N_j$ such that

$$\gcd(v_j, w_j)[\gamma_j] = v_j[a_j] + w_j[b_j] \in \mathrm{H}_1(S_g; \mathbb{Z}) \quad .$$



This obtains a representative for x as $\sum_j \gcd(v_j, w_j)[\gamma_j]$, wherein the $[\gamma_j]$ are disjoint n.o.s.c. curves. Therewith, the objective now is to combine ("surger") these curves into a single homologous n.o.s.c. curve.

6.2 Euclidean Algorithm for Simple Closed Curves — 4 / 5

Proof (continued)

Here is the Euclidean-algorithm like process comes into play: consider the curves in two of the $\{N_j\}_j$, e.g. $j = 1, 2$; these are collections of $n_j := \gcd(v_j, w_j)$ copies of $[\gamma_j]$.

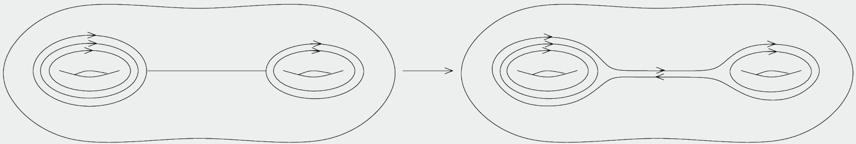


Figure 6.2 Surgering two oriented simple closed curves along an arc.

Since these curves are non-separating, we can connect them with an oriented strip that is trivial in $\mathrm{H}_1(S_g; \mathbb{Z})$, combining them into a single n.o.s.c. curve homologous to the original two. Performing this operation repetitively behaves precisely like the Euclidean algorithm: after the first iteration there are two collections of curves, one with the $|n_1 - n_2|$ combined curves and the other with $\min(n_1, n_2)$ remaining original curves; reïterate on these two collections to obtain once again a collection with the difference and the other with the minimum, until one is empty; at the end, the remaining number is $\gcd(n_1, n_2)$.

6.2 Euclidean Algorithm for Simple Closed Curves — 5 / 5

Proof (continued)

Iterating then across the rest of the $1 \leq j \leq g$, obtains a homologous n.o.s.c. curve, and furthermore, only one because $(v_1, w_1, \ldots, v_g, w_g) \in \mathrm{H}_1(S_g; \mathbb{Z})$ is primitive,

$$\gcd(n_j)_j = \gcd(v_j, w_j)_j = 1 \quad .$$

■

Remark

This result also holds for $S_{g,1}$ and S_g^1 since the inclusion into S_g induces an isomorphism on the first homology, a fact used in the above proof.

6.3 Mapping Class as Symplectic Automorphisms

6.3.1 Action of a Dehn Twist on Homology — 1 / 2

Using the fact that Dehn twists generate $\mathrm{Mod}(S_g)$, it will be helpful to first understand what the representation Ψ does to them.

Proposition 6.3

Let $[a]$, $[b]$ be isotopy classes of n.o.s.c. curves in S_g. Then, for $k \geq 0$,

$$\Psi\left(T_b^k\right)([a]) = [a] + k\,\hat{i}(a, b)\,[b] \quad .$$

Proof

Using the linearity of Ψ, it is possible to prove this by simply showing this on a basis. Taking the symplectic basis $\{a_j, b_j\}_j$ for $\mathrm{H}_1(S_g; \mathbb{Z})$ from earlier, we can assume w.l.o.g. that $[b] = [b_1]$ using the change-of-coördinates argument: there is a homeomorphism φ such that $\varphi(b) = b_1$.

6.3.1 Action of a Dehn Twist on Homology — 2 / 2

Proof (continued)

Therewith, for some other element of the basis $c \in \{a_j, b_j\}_j$

$$\Psi\left(T_{b_1}^k\right)([c]) := \left[T_{b_1}^k(c)\right] = \begin{cases} [a_1] + k\,[b_1] & c = a_1 \\ [c] & \text{else} \end{cases} .$$

Therefore, for an arbitrary o.s.c. curve a, $\operatorname{coeff}_{[a_1]}[\varphi(a)] = \hat{i}(\varphi(a), b_1)$, making

$$\Psi\left(T_b^k\right)([a]) = \Psi\left(T_{b_1}^k\right)([\varphi(a)]) = \underbrace{[\varphi(a)]}_{=[a]} + k\,\underbrace{\hat{i}(\varphi(a), b_1)[b_1]}_{=\hat{i}(a,b)[b]} . \quad \blacksquare$$

Remark

- It folows that $\Psi(T_a) = \Psi(T_b)$ iff $[a] = [b]$, which is a weaker version of Fact 3.6, $T_a = T_b$ iff $a \simeq b$ as isotopy classes.
- If $[a] = 0$, then $\Psi(T_a)$ is trivial.

6.3.2 Surjectivity of the Symplectic Rep. — 1 / 7

Proposition 6.4

For $g \geq 1$, there is a surjection

$$\Psi : \operatorname{Mod}(S_g) \twoheadrightarrow \operatorname{Sp}(2g, \mathbb{Z}) \quad .$$

Proof

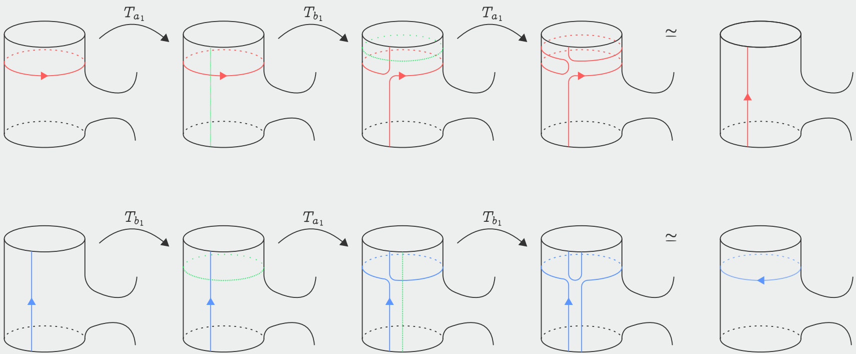
$g = 1$ Case: From Theorem 2.5 and an earlier remark, $\operatorname{Mod}(S_1) \cong \operatorname{SL}(2, \mathbb{Z}) \cong \operatorname{Sp}(2, \mathbb{Z})$.
$g > 1$ Case: The idea of the proof is to find Dehn twists that have the Burkhardt generators as images.

- Transvection: This is simply Proposition 6.3: if $(x_j, y_j)_j \in \operatorname{H}_1(S_g; \mathbb{Z})$ is an element in the symplectic basis $\{[a_j], [b_j]\}_j$, then $\Psi(T_{b_1})([a_1]) = [a_1] + [b_1]$, which is $\Psi(T_{b_1}) : (x_j, y_j)_j \longmapsto (x_1 + y_1, y_2, \ldots)$ by linearity.

6.3.2 Surjectivity of the Symplectic Rep. — 2 / 7

Proof (continued)

- Factor rotation: Looking at the genus-1 subsurface of S_g about the first factor $\{a_1, b_1\}$ of the symplectic basis, consider the sequence of Dehn twists $T_{b_1}T_{a_1}T_{b_1} = T_{a_1}T_{b_1}T_{a_1}$ (the braid relation from Proposition 3.11 on the first factor) on the first factor of the basis (everywhere else it is identity).



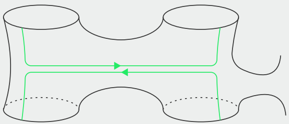
6.3.2 Surjectivity of the Symplectic Rep. — 3 / 7

Proof (continued)

- (Factor rotation:) Under the representation, this is

$$\begin{aligned}
\Psi(T_{a_1}T_{b_1}T_{a_1})([a_1]) = \Psi(T_{a_1}T_{b_1})([a_1]) &= \Psi(T_{a_1})([b_1]+[a_1]) \\
&= -[a_1]+[b_1]+[a_1] = [b_1] \\
\Psi(T_{b_1}T_{a_1}T_{b_1})([b_1]) = \Psi(T_{b_1}T_{a_1})([b_1]) &= \Psi(T_{b_1})(-[a_1]+[b_1]) \\
&= -([b_1]+[a_1])+[b_1] = -[a_1] \quad ,
\end{aligned}$$

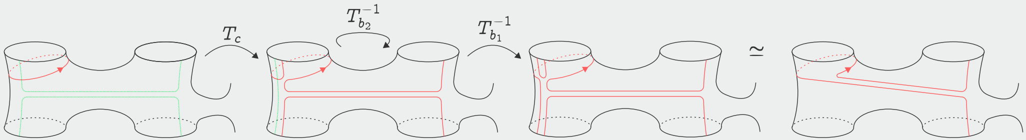
  which is, of course, the factor rotation.
- Factor mix: As this generator involves the first two factors $\{a_1, b_1, a_2, b_2\}$, look at the genus-2 subsurface of S_g around those factors and consider the sequence of Dehn twists $T_{b_1}^{-1}T_{b_2}^{-1}T_c$ where $[c] = [b_2] - [b_1]$.



6.3.2 Surjectivity of the Symplectic Rep. — 4 / 7

Proof (continued)

- (Factor mix:) It is easy to see this sequence leave $[b_1]$ and $[b_2]$ fixed, so looking at action of the representation on $[a_1]$ and $[a_2]$,



$$
\begin{aligned}
\Psi\left(T_{b_1}^{-1}T_{b_2}^{-1}T_c\right)([a_1]) &= \Psi\left(T_{b_1}^{-1}T_{b_2}^{-1}\right)\left(-\left([b_2]-[b_1]\right)+[a_1]\right)\\
&= \Psi\left(T_{b_1}^{-1}\right)\left(-[b_2]+[b_1]+[a_1]\right)\\
&= -[b_1]-[b_2]+[b_1]+[a_1] = -[b_2]+[a_1]\\
\Psi\left(T_{b_1}^{-1}T_{b_2}^{-1}T_c\right)([a_2]) &= \Psi\left(T_{b_1}^{-1}T_{b_2}^{-1}\right)\left([b_2]-[b_1]+[a_2]\right)\\
&= \Psi\left(T_{b_1}^{-1}\right)\left(-[b_2]+[b_2]-[b_1]+[a_2]\right)\\
&= -[b_2]+[b_2]-[b_1]+[a_2] = -[b_1]+[a_2]
\end{aligned}
$$

6.3.2 Surjectivity of the Symplectic Rep. — 5 / 7

Proof (continued)

- Factor swap: Consider the sequence of Dehn twists $\left(T_{a_{j+1}} T_{b_{j+1}} T_{d_j} T_{a_j} T_{b_j}\right)^3$, where $[d_j] = [a_{j+1}] + [b_j]$, for each $1 \leq j < g$: it suffices to look at this for $j = 1$

$$
\begin{aligned}
\Psi\left(T_{a_2} T_{b_2} T_{d_1} T_{a_1} T_{b_1}\right)([a_1]) &= \Psi\left(T_{a_2} T_{b_2} T_{d_1} T_{a_1}\right)([b_1] + [a_1]) \\
&= \Psi\left(T_{a_2} T_{b_2} T_{d_1}\right)\underbrace{(-[a_1] + [b_1] + [a_1])}_{=[b_1]} \\
&= [b_1] && \hat{\imath}(b_1, d_1) = 0 \\
& && \hat{\imath}(b_1, b_2) = 0 \\
& && \hat{\imath}(b_1, a_2) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Psi\left(T_{a_2} T_{b_2} T_{d_1} T_{a_1} T_{b_1}\right)([b_1]) &= \Psi\left(T_{a_2} T_{b_2} T_{d_1}\right)(-[a_1] + [b_1]) && \hat{\imath}(b_1, b_1) = 0 \\
&= \Psi\left(T_{a_2} T_{b_2} T_{d_1}\right)\underbrace{(-([a_2] + [b_1] + [a_1]) + [b_1])}_{=-[a_2]-[a_1]} \\
&= \Psi\left(T_{a_2}\right)(-([b_2] + [a_2]) - [a_1]) \\
&= -(-[a_2] + [b_2]) - [a_2] - [a_1] = -[b_2] - [a_1]
\end{aligned}
$$

6.3.2 Surjectivity of the Symplectic Rep. — 6 / 7

Proof (continued)

- (Factor swap:) and similarly,

$$\Psi\left(T_{a_2}T_{b_2}T_{d_1}T_{a_1}T_{b_1}\right)([a_2]) = [b_2] \quad \text{and} \quad \Psi\left(T_{a_2}T_{b_2}T_{d_1}T_{a_1}T_{b_1}\right)([b_2]) = -[b_1]-[a_2]$$

$$\begin{tikzcd}
 & {[a_1]} \arrow[r, mapsto] & {[b_1]} \arrow[dr, mapsto] & \\
-[b_1]-[a_2] \arrow[ur, mapsto] & & & -[b_2]-[a_1] \arrow[dl, mapsto] \\
 & {[b_2]} \arrow[ul, mapsto] & {[a_2]} \arrow[l, mapsto] &
\end{tikzcd} .$$

Thusly, after three applications of $T_{a_2}T_{b_2}T_{d_1}T_{a_1}T_{b_1}$, $[a_1] \longmapsto [a_2]$, $[b_1] \longmapsto [b_2]$, $[a_2] \longmapsto [a_1]$, and $[b_2] \longmapsto [b_1]$, which is the wanted factor swap. This is analogous for the other j. ■

6.3.2 Surjectivity of the Symplectic Rep. — 7 / 7

Remark

Induced by the inclusions $S_{g,1} \hookrightarrow S_g$ and $S_g^1 \hookrightarrow S_g$, there are the surjections $\mathrm{Mod}\left(S_{g,1}\right) \twoheadrightarrow \mathrm{Mod}\left(S_g\right)$ and $\mathrm{Mod}\left(S_g^1\right) \twoheadrightarrow \mathrm{Mod}\left(S_g\right)$. Thus, the corresponding representations are also surjective via composition: $\mathrm{Mod}\left(S_{g,1}\right) \twoheadrightarrow \mathrm{Sp}\left(2g, \mathbb{Z}\right)$ and $\mathrm{Mod}\left(S_g^1\right) \twoheadrightarrow \mathrm{Sp}\left(2g, \mathbb{Z}\right)$.

6.3.3 Minimality of Humphries Generating Set — 1 / 1

Proposition 6.5

For $g \geq 2$, $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ cannot be generated by fewer than $2g+1$ transvections.

For $g = 1$, this is not true, since $\mathrm{Sp}(2g, \mathbb{Z}) \cong \mathrm{SL}(2g, \mathbb{Z})$, which has only two generators.

Corollary 6.6

For $g \geq 2$, the generating set for $\mathrm{Mod}(S_g)$ must consist of at least $2g+1$ Dehn twists. From Theorem 4.14, this means that the *Humphries generating set* is minimal among such.

6.4 Congruence and Torsion-free subgroups, and Residual Finiteness

6.4 Congruence and Torsion-free subgroups, and Residual Finiteness — 1 / 1

To study the topology of a space with an infinite fundamental group, it is helpful to look at its torsion-free subgroups that have finite index. As such, this section outlines some properties of $\mathrm{Mod}(S_g)$, namely that it has torsion-free subgroups of finite index and it is residually finite.

6.4.1 Congruence Subgroups of $\mathrm{Sp}(2g,\mathbb{Z})$ — 1 / 1

The aforementioned subgroups are the following.

Definition

The *level-m congruence subgroup* $\mathrm{Sp}(2g,\mathbb{Z})[m]$ of $\mathrm{Sp}(2g,\mathbb{Z})$ is the kernel of the following reduction homomorphism.

$$\mathrm{Sp}(2g,\mathbb{Z})[m] := \ker\left(\mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}/m\mathbb{Z})\right) \quad .$$

Proposition 6.7

For $g \geq 1$ and $m \geq 3$, $\mathrm{Sp}(2g,\mathbb{Z})[m]$ is torsion-free.

The proof of this proposition is simply a calculation.

Remark

- For $m = 2$, $\mathrm{Sp}(2g,\mathbb{Z})[m]$ is of course not torsion-free because of, e.g., $-\mathbb{1}_{2g}$.
- This also holds for $\mathrm{SL}(n,\mathbb{Z})[m]$ with an analogous proof.

6.4.2 Congruence Subgroups of $\mathrm{Mod}(S_g)$ — 1 / 2

Definition

For $g \geq 1$ and $m \geq 2$, the *level-m congruence subgroup* $\mathrm{Mod}(S_g)[m]$ of $\mathrm{Mod}(S_g)$ is

$$\mathrm{Mod}(S_g)[m] := \Psi^{-1}(\mathrm{Sp}(2g, \mathbb{Z})[m]) \quad .$$

Remark

Since $\mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$ is finite, it follows that $\mathrm{Mod}(S_g)[m]$ must have finite index in $\mathrm{Mod}(S_g)$.

Theorem 6.8

For $g \geq 1$, non-trivial elements of $\mathrm{Mod}(S_g)$ that have finite order must have non-trivial image under Ψ.

This will be proved as an application of the Lefschetz fixed point theorem in §7.1.2.

6.4.2 Congruence Subgroups of $\mathrm{Mod}(S_g)$ — 2 / 2

Theorem 6.9

For $g \geq 1$ and $m \geq 3$, $\mathrm{Mod}(S_g)[m]$ is torsion-free.

Proof

Assume for the sake of contradiction that there is a non-trivial, finite-order $\varphi \in \mathrm{Mod}(S_g)[m] \subseteq \mathrm{Mod}(S_g)$. Since $\mathrm{Sp}(2g, \mathbb{Z})[m]$ is torsion-free, it must be that $\Psi(\varphi)$ is trivial: this contradicts Theorem 6.8. ∎

6.4.3 Residual Finiteness — 1 / 2

In combinatorial group theory, the notion of "residual finiteness" is an important one. This subsection will quickly go over this as it pertains to $\operatorname{Mod}(S_g)$.

Definition

There are numerous equivalent qualifications for the *residual finiteness* of a group G:

- For each non-trivial $g \in G$, there is a finite-index subgroup $H < G$ such that $g \notin H$.
- For each non-trivial $g \in G$, there is a finite-index normal subgroup $N \triangleleft G$ such that $g \notin N$.
- For each non-trivial $g \in G$, there is a finite quotient $\varphi : G \twoheadrightarrow B$ such that $\varphi(g) \neq 1$.
- The intersection of all finite-index subgroups of G is empty.
- The intersection of all finite-index normal subgroups of G is empty.
- There is an injective map of G into its *profinite completion* $\hat{G} := \varprojlim G/H$, where H is across all finite-index normal subgroups of G.

6.4.3 Residual Finiteness — 2 / 2

Proposition 6.10

For $n \geq 2$, the group $\mathrm{SL}(n, \mathbb{Z})$ is residually-finite. This implies that, for $g \geq 1$, $\mathrm{Sp}(2g, \mathbb{Z})$ is residually finite, because subgroups of residually-finite group are themselves residually-finite.

Theorem 6.11

For compact surfaces S, $\mathrm{Mod}(S)$ is residually-finite.

Comments on the proof

For non-hyperbolic surfaces, the proof is claimed to be trivial or easy. The hyperbolic case remains, for which a geometric approach is used using geodesics on S, which won't work for punctured surfaces without modification.

6.5 The Torelli Group

6.5 The Torelli Group — 1 / 1

In this section, we will define and discuss the "Torelli group", which has connections to 3-manifolds and algebraic geometry.

Definition

The *Torelli group* $\mathcal{I}(S_g) \subseteq \operatorname{Mod}(S_g)$ is defined by the following short-exact sequence:

$$1 \longrightarrow \mathcal{I}(S_g) \hookrightarrow \operatorname{Mod}(S_g) \xrightarrow{\Psi} \operatorname{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \quad .$$

By construction, this means that $\mathcal{I}(S_g)$ are elements of $\operatorname{Mod}(S_g)$ that act trivially by Ψ. There are analogous definitions for $\mathcal{I}(S_{g,1})$ and $\mathcal{I}\left(S_g^1\right)$.

Remark

There is a connection to integral homology 3-spheres: Consider a handlebody decomposition of $\mathbb{S}^3 \cong H \cup_\varphi H'$, where $\varphi\colon \partial H \xrightarrow{\sim} \partial H'$. Taking a homeomorphism ψ of ∂H, the manifold $M_\psi := H \cup_{\varphi\circ\psi} H'$ depends only on the isotopy class of ψ. Moreover, the homology of M_ψ depends only on $\Psi([\psi]) \in \operatorname{Sp}(2g, \mathbb{Z})$; thus, if $[\psi] \in \mathcal{I}(\partial H)$, then M_ψ is an integral homology 3-sphere. In fact, all integral homology 3-spheres arise in this fashion.

6.5.1 Torelli Groups are Torsion-free — 1 / 1

The following theorem is a consequence of Theorem 6.8 from the definition of $\mathcal{I}(S_g)$.

Theorem 6.12

For $g \geq 1$, $\mathcal{I}(S_g) \subseteq \mathrm{Mod}(S_g)$ is torsion-free.

Remark

Similarly, $\mathcal{I}(S_{g,1})$ is also torsion-free. As for S_g^1, the entirety of $\mathrm{Mod}\left(S_g^1\right) \supset \mathcal{I}\left(S_g^1\right)$ is torsion-free, which will be shown in Corollary 7.3.

6.5.2 Examples of Elements — 1 / 5

Here a few examples of elements of $\mathcal{I}(S_g)$ will be presented. These will be looked at later in terms of what generates $\mathcal{I}(S_g)$.

- Dehn twists about separating curves: The group generated by such Dehn twists on S_g is called $\mathcal{K}(S_g)$. This is a subgroup of $\mathcal{I}(S_g)$ because, for each separating curve γ and some s.c. curve c on S_g, Proposition 6.3 gives

$$\Psi(T_\gamma)([c]) = [c] + \hat{i}(c,\gamma)[\gamma] \quad ,$$

wherein $[\gamma] = 0$ because it is a boundary by virtue of being separating. The obstruction to an element of $\mathcal{I}(S_g)$ being in $\mathcal{K}(S_g)$ is given by the Johnson homomorphism, which will be discussed in §6.6.

6.5.2 Examples of Elements — 2 / 5

- Bounding-pair maps: A *bounding-pair* γ_1, γ_2 on S_g is a pair of disjoint, homologous, n.o.s.c. curves. Their corresponding mapping class is given by the sequence of Dehn twists $T_{\gamma_1} T_{\gamma_2}^{-1}$, which is in $\mathcal{I}(S_g)$ by Proposition 6.3.
  Recall that the kernel of the forgetful map $\mathrm{Mod}(S_{g,1}) \longrightarrow \mathrm{Mod}(S_g)$ is generated by the bounding-pair maps (Theorem 4.6 and Fact 4.7).
- Fake bounding-pair maps: In fact, for any two homologous curves γ_1, γ_2, the mapping class of $T_{\gamma_1} T_{\gamma_2}^{-1}$ is in $\mathcal{I}(S_g)$. A case of this is the homologous pair $\gamma_1, T_{\gamma_2}(\gamma_1)$ with $\hat{i}(\gamma_1, \gamma_2) = 0$: this has trivial action on $\mathrm{H}_1(S_g; \mathbb{Z})$

$$T_{\gamma_1} T_{T_{\gamma_2}(\gamma_1)}^{-1} = T_{\gamma_1} T_{\gamma_2} T_{\gamma_1}^{-1} T_{\gamma_2}^{-1} = [T_{\gamma_1}, T_{\gamma_2}] \quad .$$

Seeing this is just a simple computation,

$$\begin{aligned}
\Psi\left(T_{\gamma_1} T_{T_{\gamma_2}(\gamma_1)}^{-1}\right)([\alpha]) &= \Psi(T_{\gamma_1})\left([\alpha] - \hat{i}(\alpha, T_{\gamma_2}(\gamma_1))[T_{\gamma_2}(\gamma_1)]\right) \\
&= [\alpha] + \hat{i}(\alpha, \gamma_1)[\gamma_1] - \hat{i}(\alpha, T_{\gamma_2}(\gamma_1))\left([T_{\gamma_2}(\gamma_1)] + \hat{i}\left(T_{\gamma_2(\gamma_1)}, \gamma_1\right)[\gamma_1]\right) \\
&= [\alpha] \qquad\qquad \text{because } [T_{\gamma_2}(\gamma_1)] = [\gamma_1].
\end{aligned}$$

6.5.2 Examples of Elements — 3 / 5

- Point pushes: Following from Fact 4.7, for each $[\alpha] \in \pi_1(S_g, p)$, the point-push map $\mathrm{Push} \colon \pi_1(S_g, p) \longrightarrow \mathrm{Mod}(S_{g,1})$ is written as a sequence of Dehn twists of homologous curves γ_1, γ_2:

$$\mathrm{Push}([\alpha]) = T_{\gamma_1} T_{\gamma_2}^{-1} \quad ,$$

which is a corresponds to a bounding-pair map. Thus,

$$\mathrm{Push}(\pi_1(S_g, p)) \subseteq \mathcal{I}(S_{g,1}) \quad .$$

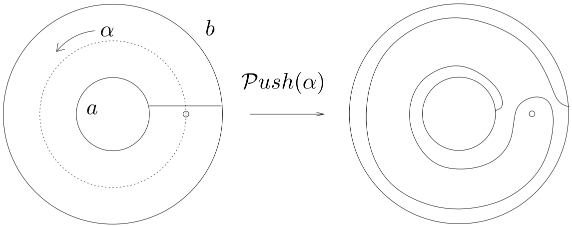


Figure 4.4 The point pushing map $\mathcal{P}ush$ from the Birman exact sequence.

6.5.2 Examples of Elements — 4 / 5

- (Points pushes:) As for the elements of $\mathcal{I}\left(S_g^1\right)$: From the inclusion $S_g^1 \hookrightarrow S_{g,1}$, there is an induced isomorphism $\mathrm{H}_1\left(S_g^1;\mathbb{Z}\right) \xrightarrow{\sim} \mathrm{H}_1\left(S_{g,1};\mathbb{Z}\right)$, which in turn, by the boundary-capping homomorphism $\mathrm{Mod}\left(S_g^1\right) \twoheadrightarrow \mathrm{Mod}\left(S_{g,1}\right)$ (Proposition 3.19) induces a surjection $\mathcal{I}\left(S_g^1\right) \twoheadrightarrow \mathcal{I}\left(S_{g,1}\right)$. Thus, using the fact that Dehn twists about separating curves are in $\mathcal{I}\left(S_{g,1}\right)$,

$$1 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{I}\left(S_g^1\right) \twoheadrightarrow \mathcal{I}\left(S_{g,1}\right) \longrightarrow 1 \quad ,$$

  where $\mathbb{Z}$ is from the Dehn twists about the boundary of S_g^1.
- Handle pushes: From §4.2, there is the following commutative diagram

$$\begin{tikzcd} \pi_1\left(\mathrm{UT}\,S_g\right) \arrow[r, hook, "\varphi"] \arrow[d, two heads, "\pi_1\,\mathrm{pr}"'] & \mathrm{Mod}\left(S_g^1\right) \arrow[d, two heads, "\mathrm{Cap}"] \\ \pi_1\,S_g \arrow[r, hook, "\mathrm{Push}"] & \mathrm{Mod}\left(S_{g,1}\right) \end{tikzcd}$$

$$\left.\begin{aligned} \mathrm{Push}\left(\pi_1\,S_g\right) &\subseteq \mathcal{I}\left(S_{g,1}\right) \\ \varphi\left(\ker \pi_1\,\mathrm{pr}\right) &\subseteq \mathcal{I}\left(S_g^1\right) \\ \mathrm{Cap}\left(\mathcal{I}\left(S_g^1\right)\right) &= \mathcal{I}\left(S_{g,1}\right) \end{aligned}\right\} \Longrightarrow \varphi\left(\pi_1\left(\mathrm{UT}\,S_g\right)\right) \subseteq \mathcal{I}\left(S_g^1\right) \quad .$$

6.5.2 Examples of Elements — 5 / 5

- (Handle pushes:) The inclusion $S_g^1 \hookrightarrow S_{g+1}$ induces an injective homomorphism $\mathrm{Mod}\left(S_g^1\right) \hookrightarrow \mathrm{Mod}\left(S_{g+1}\right)$, which then restricts to the injective homomorphism $\mathcal{I}\left(S_g^1\right) \hookrightarrow \mathcal{I}\left(S_{g+1}\right)$. Using the above, this means

$$\pi_1\left(\mathrm{UT}\, S_g\right) \xhookrightarrow{\varphi} \mathcal{I}\left(S_g^1\right) \hookrightarrow \mathcal{I}\left(S_{g+1}\right) \quad ,$$

these are the "handle pushes".

6.5.3 A Birman Exact Sequence for Torelli Group — 1 / 1

The comments made about point- and handle-push mapping classes in the previous subsection the following can be concluded, namely the *Birman short-exact sequences* for $\mathcal{I}(S_g)$.

Proposition 6.13

For $g \geq 2$, the forgetful map $S_{g,1} \longrightarrow S_g$ induces the following short-exact sequence

$$1 \longrightarrow \pi_1 S_g \hookrightarrow \mathcal{I}(S_{g,1}) \twoheadrightarrow \mathcal{I}(S_g) \longrightarrow 1$$

and the boundary-capping map $S_g^1 \longrightarrow S_g$ induces the following short-exact sequence

$$1 \longrightarrow \pi_1 (\mathrm{UT}\, S_g) \hookrightarrow \mathcal{I}\left(S_g^1\right) \twoheadrightarrow \mathcal{I}(S_g) \longrightarrow 1 \quad .$$

6.5.4 The Action on Simple Closed Curves — 1 / 1

To understand the orbits of s.c. curves in S_g up to the action of $\mathcal{I}(S_g)$, it is helpful to have a change-of-coördinates-type argument for separating s.c. curves.

Proposition 6.14

Let c, c' be isotopy classes of s.c. curves in S_g. If c, c' are separating, then they are $\mathcal{I}(S_g)$-equivalent iff they induce the same splitting of $\mathrm{H}_1(S_g;\mathbb{Z})$. If they are non-separating, then they are $\mathcal{I}(S_g)$-equivalent iff they are homologous up to sign, $[c] = \pm [c] \in \mathrm{H}_1(S_g;\mathbb{Z})$.

Remark

- A separating s.c. curve splits $\mathrm{H}_1(S_g;\mathbb{Z})$ into $\hat{\imath}$-orthogonal components.
- For such curves that are also oriented, the proposition also holds with a slight modification: for separating oriented curves, an “oriented” splitting of $\mathrm{H}_1(S_g;\mathbb{Z})$ is obtained; for non-separating, they are of the same sign in $\mathrm{H}_1(S_g;\mathbb{Z})$.
- The proposition also holds for S_g^1 and $S_{g,1}$.

6.5.5 Generators for the Torelli Group — 1 / 3

The inspiration for finding generators for $\mathcal{I}(S_g)$ comes from the following fact from group theory: Consider the following short-exact sequence of groups

$$1 \longrightarrow K \hookrightarrow E \overset{\varphi}{\twoheadrightarrow} Q \longrightarrow 1 \quad ,$$

where $\{e_j\}$ are generators for E. Then, Q has a presentation with generators $\{\varphi(e_j)\}$ with relations $\{r_j\}$ written as words of those generators. These relations can be lifted to E to obtain relations $\{\hat{r}_j\}$, which are in turn a normal generating set (i.e. generated by conjugacy classes in E) of K.

This idea was applied to the following short-exact sequence involving $\mathcal{I}(S_g)$

$$1 \longrightarrow \mathcal{I}(S_g) \hookrightarrow \mathrm{Mod}(S_g) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1$$

by Birman [22, 175] using a finite presentation of $\mathrm{Sp}(2g, \mathbb{Z})$ to obtain generators for $\mathcal{I}(S_g)$. Her student Powell then shows that these generators are actually sequences of Dehn twists about separating curves and bounding-pair maps, thusly showing that $\mathcal{I}(S_g)$ is generated by infinitely-many such maps.

6.5.5 Generators for the Torelli Group — 2 / 3

For $g \geq 3$, Johnson [107] showed that there are actually finitely-many generators needed, namely without Dehn twists about separating s.c. curves because they can be written as a sequence of bounding-pair maps via the "Lantern relation"

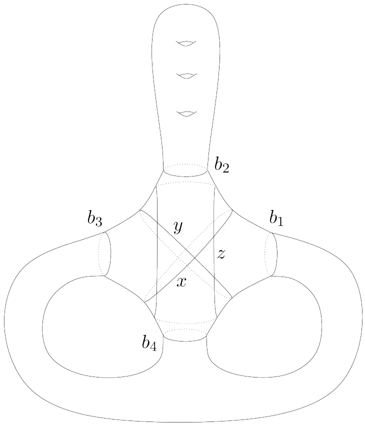


Figure 6.4 A lantern showing how to write the twist about the separating simple closed curve a as a product of bounding pair maps.

with the bounding pairs (x, b_3), (y, b_1), and (z, b_4)

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4} \quad ,$$

which is

$$\left(T_x T_{b_3}^{-1}\right)\left(T_y T_{b_1}^{-1}\right)\left(T_z T_{b_4}^{-1}\right) = T_{b_2}$$

since the T_{b_j} commute with all other curves in the figure.

6.5.5 Generators for the Torelli Group — 3 / 3

These pairs break up S_g into two components, the smaller of the two genera of those components is called the *genus* of the pair. Johnson showed that $\mathcal{I}(S_g)$ is generated by genus-1 bounding-pair maps. By the change-of-coördinates argument via Proposition 6.14, this means that $\mathcal{I}(S_g)$ is normally generated by a single genus-1 bounding-pair map.

Theorem 6.15

For $g \geq 3$, the Torelli group $\mathcal{I}(S_g)$ is generated by finitely-many bounding-pair maps.

Remark

- $\mathcal{I}(S_2)$ is not finitely-generated: it is an infinitely-generated free group with one Dehn-twist generator for each orbit of the action of $\mathcal{I}(S_2)$ on the set of separating s.c. curves in S_2. Since there are no bounding pairs in S_2, $\mathcal{I}(S_2)$ is generated solely by Dehn twists about separating curves, namely $\mathcal{I}(S_2) = \mathcal{K}(S_2)$.
- For $S_{g,1}$, S_g^1, it follows from the theorem and Proposition 6.13 that $\mathcal{I}(S_{g,1})$ is generated by finitely-many bounding-pair maps and that $\mathcal{I}\left(S_g^1\right)$ is generated by those and Dehn twists about the boundary curve.

6.6 The Johnson Homomorphism

6.6.1 Construction — 1 / 4

Here, we will primarily focus on S_g^1 because $\pi_1 S_g^1$ is free, which is essential to the construction of the homomorphism in question. The *Johnson homomorphism* is the following map

$$\tau \colon \mathcal{I}\left(S_g^1\right) \longrightarrow \mathrm{H}_1\left(S_g^1; \mathbb{Z}\right)^{\wedge 3} \cong \mathrm{H}_1\left(S_g; \mathbb{Z}\right)^{\wedge 3} \quad ,$$

where the isomorphism comes from the inclusion. Deriving this map will involve quite a bit of "algebraic juggling".

Let $g \geq 2$ so that we may use the Birman sequences (Proposition 6.13), and let

$$\Gamma := \pi_1 S_g^1 \cong \mathbb{Z}^{2g} \quad \text{and} \quad \Gamma' := [\Gamma, \Gamma] \quad .$$

Note that, by definition, $\mathcal{I}\left(S_g^1\right)$ acts trivially on $\Gamma/\Gamma' = \mathrm{H}_1\left(S_g^1; \mathbb{Z}\right)$.
Johnson's idea was to look at the action of $\mathcal{I}\left(S_g^1\right)$ on $\Gamma/\left[\Gamma, \Gamma'\right]$, which is Γ modulo the "next term in its lower central series".

6.6.1 Construction — 2 / 4

Consider the following short-exact sequence

$$1 \longrightarrow \underbrace{\Gamma'/\left[\Gamma,\Gamma'\right]}_{\substack{\| \\ N}} \longrightarrow \underbrace{\Gamma/\left[\Gamma,\Gamma'\right]}_{\substack{\| \\ E}} \longrightarrow \underbrace{\Gamma/\Gamma'}_{\substack{\| \\ H}} \longrightarrow 1$$

and define the map

$$\hat{\tau}\colon \mathcal{I}\left(S_g^1\right) \longrightarrow \mathrm{Hom}\left(H,N\right) \quad \text{such that} \quad \hat{\tau}\left(\varphi\right)\left(h\right) = \hat{\varphi}\left(\tilde{h}\right)\tilde{h}^{-1} \quad ,$$

where $\tilde{h} \in E$ is a lift of $h \in H$. Then, the earnest juggling:

$$\begin{aligned} \mathrm{Hom}\left(H,N\right) &\cong \mathrm{Hom}\left(H,H^{\wedge 2}\right) && \text{Sp}\left(2g,\mathbb{Z}\right)\text{-module isomorphism } H^{\wedge 2} \ni a\wedge b \overset{\sim}{\longmapsto} [\tilde{a},\tilde{b}] \in N \\ &\cong H^{*} \otimes H^{\wedge 2} && \\ &\cong H \otimes H^{\wedge 2} && \text{using the isomorphism induced by the symplectic structure } \hat{i}. \end{aligned}$$

Lastly, there is the inclusion $H^{\wedge 3} \hookrightarrow H \otimes H^{\wedge 2}$, making the promised τ, which is $\hat{\tau}$ composed with those isomorphisms, will be (Proposition 6.16) so that $\mathrm{im}\,\tau \subset H^{\wedge 3}$.

6.6.1 Construction — 3 / 4

Remark

- For $S_{g,1}$, because the isotopy class of ∂S_g^1 is trivial under τ, as with all separating curves as we will soon see, τ factors as the following, using the boundary-capping homomorphism for the surjection

$$\begin{tikzcd} \mathcal{I}\left(S_g^1\right) \arrow[r] \arrow[dr, two heads] & H^{\wedge 3} \\ & \mathcal{I}\left(S_{g,1}\right) \arrow[u] \end{tikzcd}$$

- For S_g, the Johnson homomorphism is

$$\tau\colon \mathcal{I}\left(S_g\right) \twoheadrightarrow H^{\wedge 3}/H \quad \text{wherein} \quad H \hookrightarrow H^{\wedge 3}\,, \quad a \longmapsto \left(\sum_j a_j \wedge b_j\right) \wedge a \quad ,$$

using the symplectic basis for the inclusion. This comes from a Birman exact sequence relating $\mathcal{I}\left(S_{g,1}\right)$ and $\mathcal{I}\left(S_g\right)$ along with the factoring of the previous remark. The reason for quotienting by H is to account "for the fact that there is no preferred side of a bounding pair" on S_g.

6.6.1 Construction — 4 / 4

Interpretation of τ via Mapping Tori

For $\varphi \in \mathcal{I}(S_{g,1})$, we will find a corresponding element of $H^{\wedge 3}$: let

$$M_\varphi := \frac{S_g \times [0,1]}{(x,0) \sim (\varphi(x),1)}$$

and note that, since $\mathcal{I}(S_{g,1})$ acts trivially on $\mathrm{H}_1(S_{g,1};\mathbb{Z})$ by definition, it follows that $\mathrm{H}_1(M_\varphi;\mathbb{Z}) \cong \mathrm{H}_1\left(S_g \times \mathbb{S}^1;\mathbb{Z}\right)$. Also, note that the projection $S_g \times \mathbb{S}^1 \twoheadrightarrow S_g$ induces $\mathrm{H}_1\left(S_g \times \mathbb{S}^1;\mathbb{Z}\right) \longrightarrow \mathrm{H}_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Therewith, precomposing with the abelianization homomorphism obtains

$$\pi_1 M_\varphi \twoheadrightarrow \mathrm{H}_1(M_\varphi;\mathbb{Z}) \longrightarrow \mathbb{Z}^{2g} \cong \pi_1 \mathbb{T}^{2g} \quad .$$

Now, since M_φ is $\mathrm{K}(\pi_1 M_\varphi, 1)$ (that it has a contractible universal cover), it follows that the above composition is induced by a based map of $M_\varphi \longrightarrow \mathbb{T}^{2g}$. In turn, this induces a map $\Phi\colon \mathrm{H}_3(M_\varphi;\mathbb{Z}) \longrightarrow \mathrm{H}_3\left(\mathbb{T}^{2g};\mathbb{Z}\right) \cong H^{\wedge 3}$, with which we obtain the sought image $\tau(\varphi) = \Phi([M_\varphi])$.

6.6.2 Computing the Image of τ — 1 / 4

Consider the following n.o.s.c. curves, which we will use to compute the image of τ, where the curve c separates a genus-k surface from ∂S_g^1.

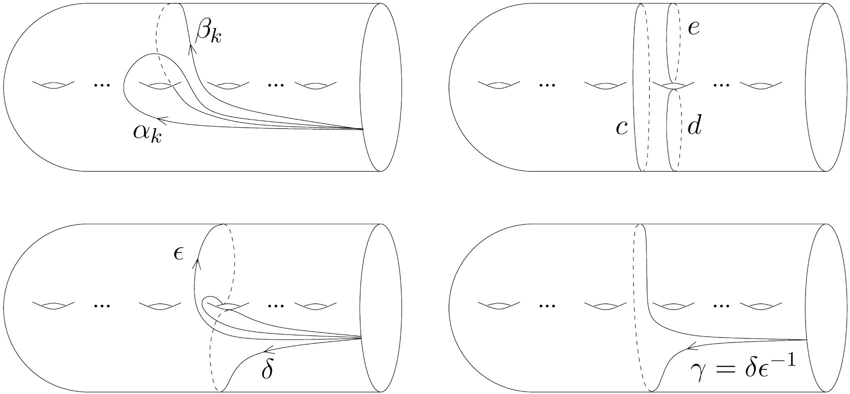


Figure 6.5 The simple closed curves c and d, and the elements of $\pi_1(S_g^1)$ used to compute $\tau(T_c)$ and $\tau(T_d T_e^{-1})$.

6.6.2 Computing the Image of τ — 2 / 4

Dehn Twists about Separating Curves

Consider the Dehn twist T_c about a standard separating curve c. For $j > k$, T_c of course acts trivially on α_j, β_j. For $x \in \{\alpha_j, \beta_j\}_{j=1}^{k}$,

$$T_c(x) = \gamma x \gamma^{-1} \quad \Longrightarrow \quad \Gamma' / \left[\Gamma, \Gamma'\right] =: N \ni \tau(T_c)(x) := T_c(x)\, x^{-1} \overset{!}{=} [\gamma, x] \quad .$$

Because γ is separating, $\gamma \in [\Gamma, \Gamma] =: \Gamma'$, which implies $[\gamma, x] \in [\Gamma, \Gamma']$, which in turn means $\tau(T_c)(x) = 0$. Therefore, $\tau(T_c) = 0$.

Using a change-of-coördinates argument for separating curves and the following *naturality property* of τ

$$\forall \varphi \in \mathcal{I}\left(S_g^1\right) \quad , \quad \forall \psi \in \mathrm{Mod}\left(S_g^1\right) \quad , \quad \tau\left(\psi \varphi \psi^{-1}\right) = \psi_* \tau(\varphi) \quad ,$$

it follows that Dehn twists about any separating s.c. curve has trivial image under τ, which is $\mathcal{K}\left(S_g^1\right) \leq \ker \tau$.

6.6.2 Computing the Image of τ — 3 / 4

Bounding-pair Maps

Consider a standard bounding-pair map $T_d T_e^{-1} =: B$, and note that $\delta\varepsilon^{-1} = \prod_{j=1}^{k} [\alpha_j, \beta_j]$ and $[\delta] = [\beta_{k+1}]$. Looking at how it behaves on $\{\alpha_j, \beta_j\}_{j=1}^{g}$,

$$\begin{array}{rcll} N \ni B(\alpha_j)\,\alpha_j^{-1} = [\delta, \alpha_j] & \longleftrightarrow & [\beta_{k+1}] \wedge [\alpha_j] \in H^{\wedge 2} & j \leq k \\ B(\beta_j)\,\beta_j^{-1} = [\delta, \beta_j] & \longleftrightarrow & [\beta_{k+1}] \wedge [\beta_j] & j \leq k \\ B(\alpha_{k+1})\,\alpha_{k+1}^{-1} = \delta\varepsilon^{-1} & \longleftrightarrow & \displaystyle\sum_{j=1}^{k} [\alpha_j] \wedge [\beta_j] & \\ B(\alpha_j)\,\alpha_j^{-1} = 1 & \longleftrightarrow & 0 & j \geq k+2 \\ B(\beta_j)\,\beta_j^{-1} = 1 & \longleftrightarrow & 0 & j \geq k+1 \quad . \end{array}$$

Using the isomorphisms and inclusion $H^{\wedge 3} \hookrightarrow H \otimes H^{\wedge 2}$ from before, we obtain

$$\tau\left(T_d T_e^{-1}\right) = \sum_{j=1}^{k} [\alpha_j] \wedge [\beta_j] \wedge [\beta_{k+1}] \in H^{\wedge 3} \quad .$$

6.6.2 Computing the Image of τ — 4 / 4

Proposition 6.16

For $g \geq 2$, $\tau\left(\mathcal{I}\left(S_g^1\right)\right) = H^{\wedge 3}$.

Idea

For $g \geq 3$, use the above images and the Burkhardt generators to show that the basis of $H^{\wedge 3}$ is in $\operatorname{im}\tau$ via the naturality property of τ. For $g = 2$, there are no bounding-pair maps and the text leaves the proof as an exercise.

6.6.3 Some Applications — 1 / 1

As shown before, $\mathcal{K}\left(S_g^1\right) \leq \ker \tau$, and it is also the case that the image of τ is infinite, so we obtain the following topological result from a "purely algebraically defined 'invariant"'.

Corollary 6.17

For $g \geq 3$, $\mathcal{K}\left(S_g^1\right)$ has infinite index in $\mathcal{I}\left(S_g^1\right)$. Moreover, no non-trivial powers bounding-pair maps lie in $\mathcal{K}\left(S_g^1\right)$, i.e. bounding-pair maps cannot be written as a sequence of Dehn twists about separating curves.

Johnson [109, 110] also proved the following stronger results.

Theorem 6.18

For $g \geq 3$, $\ker \tau = \mathcal{K}\left(S_g^1\right)$.

Theorem 6.19

For $g \geq 2$,

$$\mathrm{H}_1\left(\mathcal{I}\left(S_g^1\right);\mathbb{Z}\right) \cong H^{\wedge 3} \times (\mathbb{Z}/2\mathbb{Z})^N$$

where $N = \binom{2g}{0} + \binom{2g}{1} + \binom{2g}{2} + \binom{2g}{3}$. Thusly, τ captures exactly the torsion-free part of $\mathrm{H}_1\left(\mathcal{I}\left(S_g^1\right);\mathbb{Z}\right)$.