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MAT983

In order to get to cut loci and the injectivity radius of a manifold, it is necessary to recall some concepts from last semester: normal neighborhoods, Jacobi fields, and conjugate points; along with these definitions, a few earlier results will be used.

Recall

Def. Let $V \subset T_pM$ be a set on which $\exp_p$ is a diffeomorphism. Then, the set $\exp_p(V) \subset M$ is called a *normal neighborhood* of p.
A *totally normal neighborhood* of M is a normal neighborhood which is a normal neighborhood of each of its points.

Def. Let $\gamma : [0, a] \longrightarrow M$ be a geodesic on M. A vector field J along γ is called a *Jacobi field* iff it satisfies the Jacobi equation,

$$\frac{D^2 J}{dt^2} + R\left(\gamma'(t), J(t)\right)\gamma'(t) = 0,$$

where $R(X, Y) : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ is the Riemannian curvature of M, with $X, Y \in \mathcal{X}(M)$, the set of all C^∞ vector fields on M.

Def. A point $\gamma(t_0)$, $t_0 \in (0, a]$, is called *conjugate* to $\gamma(0)$ along γ iff there is a non-identically-zero Jacobi field J along γ with $J(0) = 0 = J(t_0)$.
The set of all such points is called the *conjugate locus* of p, $C(p)$.

Using these concepts in an example with a surface $S^2 \subset \mathbb{R}^3$ below:

Ex. Consider the unit sphere $S^2 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$ with the normal parametrization given by

$$\begin{aligned} f : [0, 2\pi) \times [0, \pi] &\longrightarrow S^2 \\ (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) &\longmapsto (x_1, x_2, x_3) \end{aligned}$$

Consider the geodesic

$$\gamma(t) = (\cos t, \sin t, 0) \quad \text{with} \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

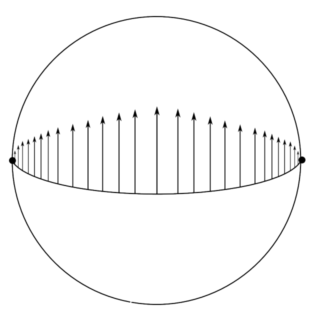
which is the image $f\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \frac{\pi}{2}\right)$. This is a geodesic since it is part of a great circle connecting the antipodal points $(0, 1, 0)$ and $(0, -1, 0)$. It follows that

$$\gamma'(t) = (-\sin t, \cos t, 0),$$

showing that γ is a normalized geodesic, $|\gamma'| = 1$. Now, let $g(s)$ be a parallel transport along γ of the vector $g(0) \in T_{\gamma(\pi/2)}S^2$ with $|g(0)| = 1$ and $\langle g(0), \gamma'(\pi/2)\rangle = 0$. Claim that

$$J(t) = g(t)\sin t \quad \text{for} \quad t \in [0, \pi]$$

is a Jacobi field along γ.



To show this, it suffices to show that J satisfies the Jacobi equation. According to `Chavel, Thm. II.1.1 (pg.59)` and `DoCarmo, Ch.4 Lemma 3.4 (pg.96)`, for a two-dimensional manifold, it is possible to write the curvature as

$$R(\gamma', J)\gamma' = \mathcal{K}(\gamma(t))(\langle\gamma',\gamma'\rangle J - \langle J,\gamma'\rangle\gamma'),$$

where $\mathcal{K}$ is the Gaußian curvature on the manifold. Note that S^2 is a manifold of constant Gaußian curvature; thus, $\mathcal{K}(\gamma(t)) = 1$ for all $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Checking to see if J satisfies the Jacobi equation, using what was just mentioned,

$$\begin{aligned}\frac{D^2}{dt^2}(g(t)\sin t) + \underbrace{\langle\gamma',\gamma'\rangle}_{\substack{1\\ \text{since } \gamma \text{ normalized}}} J - \langle J,\gamma'\rangle\gamma' &= \frac{D}{dt}\left(g(t)\cos t + \sin t \underbrace{\frac{D}{dt}g(t)}_{\substack{0\\ \text{since } g \text{ is a parallel}\\ \text{transport}}}\right) + J - \underbrace{\langle g,\gamma'\rangle}_{\substack{0\\ \text{by construction}}}\gamma'\sin t\\ &= \underbrace{-g(t)\sin t}_{-J} + J\\ &= 0,\end{aligned}$$

showing that J satisfies the Jacobi equation.

Now, note that $J(0) = 0 = J(\pi)$; by definition, this means that $\gamma\left(\frac{3\pi}{2}\right) = (0,-1,0)$ is conjugate to $\gamma\left(\frac{\pi}{2}\right) = (0,1,0)$ along γ by means of J. (`DoCarmo, `*`Differential Geometry`*`, §5-5 (pg.362)`) ⧓

The primary motivation leading to the concept of cut loci, and the injectivity radius, is to find the largest neighborhood of point of a manifold such that all geodesics starting from that point are minimizing. To do this, it is natural to relay through normal neighborhoods as is seen in what follows. Consider a complete Riemannian manifold M with a normalized geodesic $\gamma : [0,\infty) \longrightarrow M$ so that $\gamma(0) = p \in M$. (In this regard, here, from now onward, M will be a complete Riemannian manifold, and all geodesics will be normalized, unless stated otherwise.) By `DoCarmo, Ch.3 Prop. 2.9 (pg.65)`, it follows that there is a small enough neighborhood of p which is a normal neighborhood. Using the fact that γ is a normalized geodesic, this implies via `DoCarmo, Ch.3 Prop. 3.6 (pg.70)` that there is a $t \in (0,\infty)$ small enough so that $d(\gamma(0),\gamma(t)) = t$, namely that $\gamma([0,t])$ is minimizing. For any p, it is obvious that the set of t such that $\gamma([0,t])$ is minimizing is an interval in $\mathbb{R}_{\geq 0}$ with an (inclusive) endpoint at 0. Now, it is to be shown that the right endpoint of this interval is also inclusive, if the interval is bounded. Assume that every $t \in [0,t_0)$, $t_0 < \infty$, is such that $\gamma([0,t])$ is minimizing. It follows that for any $0 < \varepsilon \leq t_0$, it is the case that $d(\gamma(0),\gamma(t_0-\varepsilon)) = t_0 - \varepsilon$; in the limit, using the continuity of γ and of the distance functioin $d(\cdot,\cdot)$, this yields

$$d(\gamma(0),\gamma(t_0)) = \lim_{\varepsilon\to 0} d(\gamma(0),\gamma(t_0-\varepsilon)) = \lim_{\varepsilon\to 0}(t_0-\varepsilon) = t_0,$$

which, by definition, makes γ minimizing up to t_0 at least. From this, it follows that the set of t such that this condition is satisfied is either of the form $[0,t_0]$ or $[0,\infty)$.

<u>DEF.</u> In the case when the set of all suitable t is of the form $[0,t_0]$, the point $\gamma(t_0)$ is called the *cut point* of $\gamma(0) = p$ along γ. If the set is of the other form, then it is said that p has no cut point along γ.

<u>DEF.</u> Let $C_m(p)$ be the union of all cut points of $p \in M$ along any geodesic starting at p; call it the *cut loci.*

Observe that, if M is compact, then all normal neighborhoods of $p \in M$ must be bounded; it follows from what was shown before that any $p \in M$ has a cut point along any geodesic starting at p, since the set of suitable t cannot have the form $[0,\infty)$, which is unbounded.

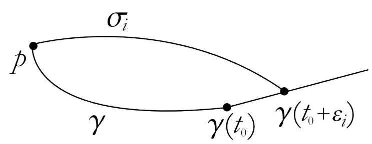
<u>PROP.</u> Let $\gamma(t_0)$ be a cut point of $\gamma(0) = p$ along γ then at least one of the following holds:

(i) $\gamma(t_0)$ is the first conjugate point of p along γ,

(ii) there is a geodesic $\sigma \neq \gamma$ joining p and $\gamma(t_0)$ such that $\operatorname{len}(\sigma) = \operatorname{len}(\gamma)$, where $\operatorname{len}(\alpha)$ is the length of the geodesic α.

Conversely, if (i) or (ii) holds, then there is a $\widetilde{t} \in (0, t_0]$ such that $\gamma\left(\widetilde{t}\right)$ us a cut point of p along γ. (`DoCarmo, Ch.13 Prop. 2.2 (pg.267)`)

Proof Let t_0 be as per the assumptions of the proposition. Construct the sequence $\{t_0 + \varepsilon_i\}$, where $\varepsilon_i > 0$ and $\{\varepsilon_i\} \to 0$. Using the Hopf-Rinow Theorem (`Chavel, Thm. I.7.1 (pg.26)`), since M is complete and connected, there are minimizing geodesics σ_i connecting p and $\gamma(t_0 + \varepsilon_i)$; let each σ_i be normalized. Now, use these geodesics to construct the sequence $\{\sigma_i'(0)\} \subset T_pM$ of corresponding tangent (unit) vectors of each σ_i at p. Also, let ε_i' be such that $\sigma_i(t_0 + \varepsilon_i') = \gamma(t_0 + \varepsilon_i)$. It is possible to bound these ε_i': using the triangle inequality it follows that



$$\begin{aligned}
\operatorname{len}(\sigma_i) &\leq \operatorname{len}(\gamma) + \varepsilon_i \\
t_0 + \varepsilon_i' &\leq t_0 + \varepsilon_i \\
\varepsilon_i' &\leq \varepsilon_i
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{len}(\sigma_i) + \varepsilon_i &\geq \operatorname{len}(\gamma) \\
t_0 + \varepsilon_i' + \varepsilon_i &\geq t_0 \\
-\varepsilon_i' &\leq \varepsilon_i,
\end{aligned}$$

which together imply that $|\varepsilon_i'| \leq \varepsilon_i$. Since $\gamma(t_0)$ is a cut point by assumption, it follows that $\varepsilon_i' \neq \varepsilon_i$, making the inequality stricter: $-\varepsilon_i \leq \varepsilon_i' < \varepsilon_i$.

By construction, $\{\sigma_i'(0)\}$ is contained in the compact unit-sphere about $\gamma'(0)$ in T_pM; it follows that there is at least a subsequence that converges to some $\sigma'(0)$ in that sphere. As such, let this sequence be that subsequence without loss of generality. Let σ be the corresponding normalized geodesic starting at p with initial velocity $\sigma'(0) \in T_pM$. Now, it is needed to show that σ ends at $\gamma(t_0)$:

$$\begin{aligned}
\gamma(t_0) &= \lim_{i\to\infty} \gamma(t_0 + \varepsilon_i) && \text{by continuity of } \gamma \\
&= \lim_{i\to\infty} \sigma_i(t_0 + \varepsilon_i') && \text{by construction} \\
&= \lim_{i\to\infty} \exp_p\left((t_0 + \varepsilon_i')\,\sigma_i'(0)\right) && \text{by definition of } \exp_p\text{, using the fact} \\
& && \text{that } \sigma_i \text{ is normalized} \\
&= \exp_p\left(t_0\sigma'(0)\right) && \text{by continuity of } \exp_p\text{, and} \\
& && \lim_{i\to\infty}\varepsilon_i = 0 \text{ making } \lim_{i\to\infty}\varepsilon_i' = 0 \\
& && \text{by } -\varepsilon_i \leq \varepsilon_i' < \varepsilon_i \\
&= \sigma(t_0) && \text{by construction of the normalized} \\
& && \text{geodesic } \sigma.
\end{aligned}$$

Thus, by convergence, σ joins p and $\gamma(t_0)$, and $\operatorname{len}(\sigma) = \operatorname{len}(\gamma)$. Therefore, if $\sigma \neq \gamma$, then <u>(ii) is satisfied</u>.

In order to show (i), assume $\sigma = \gamma$. By `DoCarmo, Ch.5 Prop. 3.5 (pg.117)`, the point $\gamma(t_0)$ is conjugate to p along γ iff $t_0\gamma'(0)$ is a critical point of the map $\exp_p$. In light of this, assume, for the sake of contradiction, that $t_0\gamma'(0)$ is not a singular point of $d\exp_p$. From this, it follows by the Inverse Mapping Theorem that there is a neighborhood U of $t_0\gamma'(0)$ on which $\exp_p$ is a diffeomorphism onto a neighborhood of M. Consider now the fact from construction that was mentioned before: there is an ε_j' such that $-\varepsilon_j \leq \varepsilon_j' < \varepsilon_j$ and $\sigma_j\left(t_0 + \varepsilon_j'\right) = \gamma(t_0 + \varepsilon_j)$. Using that fact that $\{\sigma_j'(0)\} \to \sigma'(0) = \gamma'(0)$, and taking ε_j sufficiently small (j large enough), it follows that $(t_0 + \varepsilon_j')\,\sigma_j'(0), (t_0 + \varepsilon_j)\,\gamma'(0) \in U$. Now,

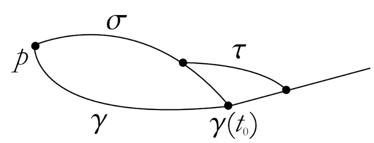
$$\exp_p (t_0 + \varepsilon_j)\,\gamma'(0) = \gamma(t_0 + \varepsilon_j) = \sigma_j(t_0 + \varepsilon_j') = \exp_p (t_0 + \varepsilon_j')\,\sigma_j'(0).$$

Because $\exp_p$ is a diffeomorphism (bijective), this shows that $(t_0 + \varepsilon_j)\,\gamma'(0) = (t_0 + \varepsilon_j')\,\sigma_j'(0)$, which in turn implies that $\varepsilon_j = \varepsilon_j'$, contradicting the choice of $\varepsilon_j' < \varepsilon_j$. Therefore, (i) is satisfied: $\gamma(t_0)$ is the first conjugate point to p along γ.

As for the converse:

(i) By `DoCarmo, Ch.11 Cor. 2.9 (pg.248)`, a geodesic is not minimizing after the first conjugate point. Thus, the cut point of p along γ must be $\gamma(\widetilde{t})$ for some $\widetilde{t} \leq t_0$.

(ii) By `DoCarmo, Ch.3 Thm. 3.7 (pg.72)`, there exists a totally normal neighborhood of $\gamma(t_0)$; using this fact, take a small enough $\varepsilon > 0$ such that $\sigma(t_0 - \varepsilon), \gamma(t_0 + \varepsilon)$ lie in this neighborhood of $\gamma(t_0)$. Now, by the remark of that theorem (`DoCarmo, Ch.3 Rem. 3.8 (pg.72)`), there is a unique minimizing geodesic connecting $\sigma(t_0 - \varepsilon)$ and $\gamma(t_0 + \varepsilon)$; call it τ, and let $\widehat{\tau}$ be the geodesic from p to $\gamma(t_0 + \varepsilon)$ travelling along σ up to $\sigma(t_0 - \varepsilon)$ then τ.



By the triangle inequality, it follows that

$$\operatorname{len}(\tau) \leq 2\varepsilon. \qquad \texttt{[1]}$$

Now, note that, if $\operatorname{len}(\tau) = 2\varepsilon$, then it follows from uniqueness of τ that τ has the image $\sigma([t_0 - \varepsilon, t_0]) \cup \gamma([t_0, t_0 + \varepsilon])$, which in turn implies that $\gamma'(t_0)$ is parallel to $\sigma'(t_0)$ by parallel transport property of geodesics. This means that, either:

— $\gamma'(t_0) = -\sigma'(t_0)$, which implies that γ and σ form a geodesic loop at p. This means that $\gamma(t_0 + \varepsilon) = \sigma(t_0 - \varepsilon)$, implying that $\operatorname{len}(\tau) = 0$, contradiction.

— $\gamma'(t_0) = \sigma'(t_0)$, which implies that γ and σ are the same geodesics, since a geodesic is uniquely determined by its initial position and tangent vector; this contradicts the initial assumption that $\sigma \neq \gamma$.

Thus, `[1]` can be made a strict inequality. As such, it follows that

$$\begin{aligned}
\operatorname{len}(\widehat{\tau}) &= \operatorname{len}(\sigma)_0^{t_0 - \varepsilon} + \operatorname{len}(\tau) \\
&= t_0 - \varepsilon + \operatorname{len}(\tau) && \text{since } \sigma \text{ is normalized} \\
&< t_0 - \varepsilon + 2\varepsilon && \text{by the triangle inequality, } \texttt{[1]} \\
&= t_0 + \varepsilon \\
\implies \operatorname{len}(\widehat{\tau}) &< \operatorname{len}(\gamma)_0^{t_0 + \varepsilon}.
\end{aligned}$$

Since this can be done for all such $\varepsilon > 0$ small enough, it follows that γ cannot be minimizing beyond $\gamma(t_0)$, meaning that, for some $0 < \widetilde{t} \leq t_0$, $\gamma(\widetilde{t})$ is a cut point of p along γ.

The proof is now complete. ■

COR. If q is a cut point of p along γ, then p is a cut point of q along γ^-, γ in reverse. In particular, this means that $q \in C_m(p)$ iff $p \in C_m(q)$. (`DoCarmo, Ch.13 Cor. 2.7 (pg.271)`)

Proof It follows from the previous proposition that q is the first conjugate point of p along γ, or there is a $\sigma \neq \gamma$ such that $\operatorname{len}(\sigma) = \operatorname{len}(\gamma)$ joining p and q; both of these statements are, by their nature, reflexive, meaning that it is possible to use the converse in the proposition with regards to the cut points of q. As such, this means that there is a $0 < \widetilde{t} \leq t_0$ such that $\gamma^-(\widetilde{t})$ is a cut point of q along γ^-. Using the fact that $q \in C_m(p)$, if the suitable $\widetilde{t} < t_0$, then the minimality of γ near p would be contradicted. Thus, it must be that $\widetilde{t} = t_0$. Therefore, $\gamma^-(t_0) = p$ is the cut point of q along γ^-. ■

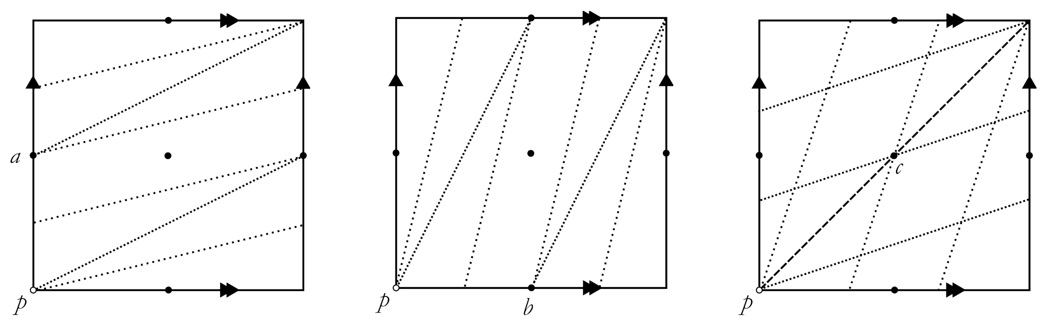
Cor. If $q \in M \setminus C_m(p)$, then there is a unique minimizing geodesic joining p and q. (`DoCarmo, Ch.13 Cor. 2.8 (pg.271)`)

Proof The existence of a minimizing geodesic follows from the Hopf-Rinow Theorem (`Chavel, Thm. I.7.1 (pg.26)`). Assume that there are at least two minimizing geodesics, σ and γ, connecting p and q. Using the converse in the proposition, it follows that the cut point of p along γ occurs at some $\widetilde{t} \leq t_0$. If $\widetilde{t} < t_0$, the minimality of γ will be contradicted. Also, if $\widetilde{t} = t_0$, the assumption that $q \in M \setminus C_m(p)$ would be contradicted. Thus, there is a unique minimizing geodesic joining p and q. ■

From this corollary, it follows immediately that $\exp_p$ is bijective on $M \setminus C_m(p)$, making it a diffeomorphism there. As such, the following examples use this to find the cut locus of a manifold.

Ex. Referring back to the first example of S^2. It is easy to see that any point on S^2 has its antipodal point as both its conjugate locus (by the first example) and its cut locus (the punctured S^2 is diffeomorphic to $\mathbb{R}^2$, consider the stereographic projection as the exponential map). ⋈

Ex. Consider the flat torus, $\overline{\mathbb{T}}^2$, a square with the usual identifications on the edges (denoted by arrows). Since it is flat, its curvature is, by definition zero; from this it follows that $\overline{\mathbb{T}}^2$ does not have any conjugate points. Thus, by the previous proposition, all the cut points of some point of $\overline{\mathbb{T}}^2$ must be the midpoint of a geodesic loop, if they exist. Using the fact that the tangent space at any point of $\overline{\mathbb{T}}^2$ is a universal covering of $\overline{\mathbb{T}}^2$, just as in the Euclidean case, it follows that exp map is the identity map and, hence, a diffeomorphism. Looking at loops at $p \in \overline{\mathbb{T}}^2$ (some depicted below), it is easy to see that the only cut points of p are a, b, and c – simple examples of their loops lying on the edges (for a and b) and on the diagonal (for c).



Place $\overline{\mathbb{T}}^2$, as an unit square, on the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ with p at the origin; note that all points of $\mathbb{Z}^2$ are equivalent to p under the identification of $\overline{\mathbb{T}}^2$. Now, it follows that all geodesic loops at p are lines at the origin with rational slope, since these are the only lines that intersect more than one point of $\mathbb{Z}^2$ after a finite length. From these observations, it follows that the midpoints of these loops are the cut points, and they must lie at the half integers $\frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$, which are, up to equivalence, a, b, and c, as mentioned. ⋈

In particular, this means that there is an open ball $B_r(p) \subset M \setminus C_m(p)$ on which $\exp_p$ is injective (bijective) whenever r is at most the distance from p to $C_m(p)$. Motivated by this, the following definition arises.

Def. Let the *injectivity radius* of M be

$$i(M) = \inf_{p \in M} d(p, C_m(p)).$$

With this, it is possible to proceed to the main theorems which has a refined and stricter result than the previous proposition.

Thm. (Klingenberg) Let $p \in M$, and let $q \in C_m(p)$ such that

$$d(p,q) = d(p, C_m(p)),$$

namely q is one of the closest points to p that lies in $C_m(p)$. Assume that q is not the first conjugate point of p along a minimizing geodesic. Then, q is the midpoint a unique geodesic loop at p.

More specifically, if M is compact and the sectional curvatures of $p \in M$ are bounded, $\mathcal{K}(p) \leq \delta$, then

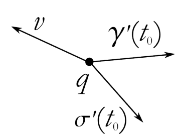
$$i(M) \geq \min\left\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell(M)}{2}\right\},$$

where $\ell(M)$ is the length of the shortest geodesic loop in M. (`Chavel, Thm. III.2.4 (pg.118), DoCarmo, Ch.13 Prop. 2.12 (pg.274)`)

Proof Let γ be a minimizing geodesic joining p and q, with $\operatorname{len}(\gamma) = t_0$. Given the assumptions of the theorem, it follows from the previous proposition, that it must be the case that there are two distinct geodesics of the same length joining p and q; namely, there is a $\sigma \neq \gamma$ with $\operatorname{len}(\sigma) = \operatorname{len}(\gamma)$, making it also a minimizing geodesic.

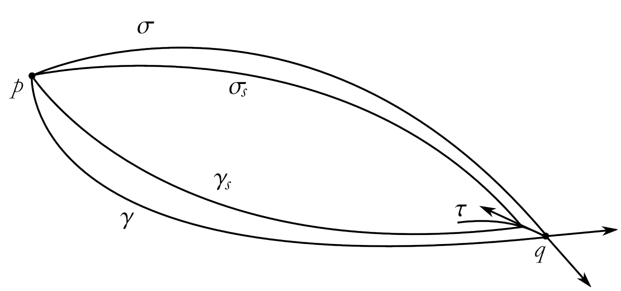
Assume, for the sake of contradiction, that γ and σ do not form a geodesic loop, namely $\gamma'(t_0) \neq -\sigma'(t_0)$. From this, it follows that there is a $v \in T_qM$ such that

$$\langle v, \gamma'(t_0)\rangle < 0 \quad \text{and} \quad \langle v, \sigma'(t_0)\rangle < 0.$$



As mentioned in the proof of the previous proposition, since q is not a conjugate point of p, it follows that there is a neighborhood $U_\gamma \subset T_qM$ of $\gamma'(t_0) = t_0\gamma'(0)$ on which $\exp_p$ is a diffeomorphism. Let $\tau : (-\varepsilon, \varepsilon) \longrightarrow \exp_p(U_\gamma) \subset M$ be the geodesic at $\tau(0) = q$ with $\tau'(0) = v$; the pre-image of τ is u_γ in U_γ, $\exp_p u_\gamma(s) = \tau(s)$ for $s \in (-\varepsilon, \varepsilon)$.

From this, it is possible to define a (not necessarily normalized) variation of γ: $\gamma_s(t) = \exp_p \frac{t}{t_0} u_\gamma(s)$ for $t \in [0, t_0]$, which has one endpoint fixed at p. In a similar manner, define a variation of σ: $\sigma_s(t) = \exp_p \frac{t}{t_0} u_\sigma(s)$.



The first variation of these are, for $\alpha = \gamma, \sigma$, and the variation field along γ called $V(t)$,

$$E'_\alpha(0) = \langle V, \alpha'\rangle|_0^{t_0} - \int_0^{t_0} \underbrace{\left\langle V, \frac{D}{dt}\alpha'\right\rangle}_{\substack{0 \\ \text{since } \frac{D}{dt}\alpha' = 0 \text{ by} \\ \text{parallel transport}}} dt$$

$$= \langle V(t_0), \alpha'(t_0)\rangle - \underbrace{\langle V(0), \alpha'(0)\rangle}_{\substack{0 \\ \text{since } V(0) = 0 \text{ by} \\ \text{construction}}}$$

given by the formula in `Chavel, Thm.II.4.1 (pg.74)`. Explicitly, since $V(t_0) = v$ by construction,

$$E'_\gamma(0) = \langle v, \gamma'(t_0)\rangle < 0 \quad \text{and} \quad E'_\sigma(0) = \langle v, \sigma'(t_0)\rangle < 0.$$

This implies that, for small enough s, it is the case that $\text{len}(\gamma_s) < \text{len}(\gamma)$ and $\text{len}(\sigma_s) < \text{len}(\sigma)$. Now, in cases,

when $\text{len}(\gamma_s) = \text{len}(\sigma_s)$:

It follows from the previous proposition that there is a cut point of p along γ_s at $\gamma_s(\widetilde{t})$ for some $t \in (0, t_0]$, meaning, by construction,

$$d\left(p, \gamma_s\left(\widetilde{t}\right)\right) \leq \text{len}(\gamma_s) < \text{len}(\gamma) = d(p, C_m(p)),$$

contradicting the initial definition of q.

when $\text{len}(\gamma_s) < \text{len}(\sigma_s)$:

It follows that σ_s is not minimizing; from this it follows that the cut point of p along σ_s is at $\sigma_s\left(\widetilde{t}\right)$ for some $t \in (0, t_0)$, meaning, by construction,

$$d\left(p, \sigma_s\left(\widetilde{t}\right)\right) < \text{len}(\sigma_s) < \text{len}(\sigma) = d(p, C_m(p)),$$

contradicting the initial definition of q.

when $\text{len}(\gamma_s) > \text{len}(\sigma_s)$:

Similar to the previous case, contradiction.

Thus, γ and σ form a geodesic loop at p. Since the choice of these geodesics was arbitrary, it must be the case that any two distinct minimizing geodesics joining p and q must form a loop; in particular, having found a loop, any third minimizing geodesic from p to q forms a loop with one of the geodesics in the original loop, forcing it to be in the original loop by the parallel tangent vector argumentation. <u>Therefore, γ and σ must form a unique geodesic loop at p, and, since $\text{len}(\gamma) = \text{len}(\sigma)$, it follows that q is the midpoint of this loop.</u>

In the case, where M is compact, the cut locus of any point in M is bounded, as mentioned earlier. Let p and q be as in the initial assumptions of the theorem, and let $t_{p,q}$ be the length of a minimizing geodesic joining them. There are two cases:

— If q is not a conjugate point of p along some geodesic, it follows that q is the midpoint of some geodesic loop; this makes the distance $d(p,q) = t_{p,q}$ equal to half the length of the geodesic loop.

— If q is a conjugate point of p along some geodesic, it follows from the Morse-Schönberg Theorem (`Chavel, Thm. II.6.3 (pg.86)`), that

$$t_{p,q} \geq \frac{\pi}{\sqrt{\delta}}.$$

By definition of the injectivity radius, $i(\cdot)$, it follows that the infimum of each of these if taken over all $p \in M$; taking the minimum of both of those infimums, thus, yields the lower bound

$$\underline{i(M) \geq \min\left\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell(M)}{2}\right\}}.$$

This completes the proof. ∎

With a few more restrictions, this bound can be improved as evidenced in the following proposition.

<u>PROP.</u> If the section curvature, $\mathcal{K}$, on a compact orientable even-dimensional Riemannian manifold, M, is such that $0 < \mathcal{K} \leq 1$, then $i(M) \geq \pi$. (`DoCarmo, Ch.13 Prop. 3.4 (pg.281)`)

Proof It follows from that fact that M is compact (hence, bounded) that there are $p, q \in M$ such that $q \in C_m(p)$ and $d(p,q) = i(M)$. Assume for the sake of contradiction that $d(p,q) < \pi$.

Letting $\delta = 1$, it follows from the Morse-Schönberg Theorem (`Chavel, Thm. II.6.3 (pg.86)`) that, if q is conjugate to p along some geodesic, then

$$d(p,q) \geq \frac{\pi}{\sqrt{\delta}} = \pi,$$

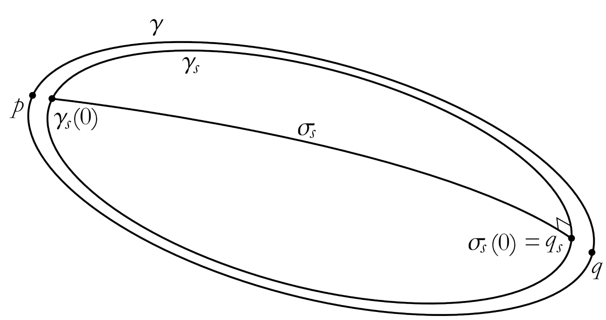
which contradicts the assumption. Thus, q cannot be conjugate to p along any geodesic.

Now, with this, it follows from the previous theorem, the Klingenberg Lemma, that q is the midpoint of a unique geodesic loop, γ, at p, which is such that $\operatorname{len}(\gamma) = t_0 < 2\pi$. Also, note that, by definition, the parallel transport along γ is an orientation-preserving ($T_{\gamma(t)}M$ is even-dimensional, orientable) orthogonal (preserves basis) linear transformation on $(\gamma'(t))^{\perp} \subset T_{\gamma(t)}M$ – intuitively, this is because only vectors perpendicular to $\gamma'(t)$ are acted upon by the parallel transport. Inasmuch as this, following from `DoCarmo, Ch.9 Lemma 3.8 (pg.203)`, it is the case that the parallel transport along γ leaves invariant a vector v, which is orthogonal to γ; call the corresponding vector field $V(t)$ gotten by parallel transport of v along γ. Since γ is a geodesic, it follows that the first variation is 0 by `DoCarmo, Ch.9 Prop. 2.5 (pg.196)`. Calculating the second variation induced by $V(t)$ with the formula in `DoCarmo, Ch.9 Prop. 2.8 (pg.197)`

$$\begin{aligned}
E''_V(0) &= -2\int_0^{t_0} \left\langle V, \frac{D^2V}{dt^2} + R(\gamma', V)\gamma' \right\rangle \, dt - \sum_{i=1}^{k} \left\langle V(t_i), \frac{DV}{dt}\left(t_i^+\right) - \frac{DV}{dt}\left(t_i^-\right) \right\rangle \\
&= -2\int_0^{t_0} \langle V, R(\gamma', V)\gamma' \rangle \, dt && \tfrac{DV}{dt} = 0 = \tfrac{D^2V}{dt} \text{ by parallel transport} \\
&= -2\int_0^{t_0} \Big(\underbrace{\langle V, V\rangle \langle \gamma', \gamma'\rangle}_{>0} - 2\underbrace{\langle \gamma', V\rangle}_{\substack{0 \\ \text{by orthogonality}}} \Big) \, dt && \text{by } \texttt{DoCarmo, Ch.4 Lemma 3.4 (pg.96)} \\
&< 0.
\end{aligned}$$

Thus, there is a variation of γ, call it γ_s with $s \in [0, \varepsilon]$, such that $\operatorname{len}(\gamma) > \operatorname{len}(\gamma_s)$ for all $s \neq 0$.

For each loop γ_s, let q_s be the point of γ_s farthest from $\gamma_s(0)$. By the initial choice of p and q, it follows from the fact that $d(\gamma_s(0), q_s) < d(p,q)$, that there is a unique minimizing geodesic σ_s joining $q_s = \sigma_s(0)$ and $\gamma_s(0)$. Since σ_s is a minimizing geodesic for all $s \in (0, \varepsilon]$, it follows from continuity that $\lim_{s\to 0} \sigma_s \to \sigma$ is also a minimizing geodesic joining $\lim_{s \to 0} \gamma_s(0) = p$ (by construction of the variation) and $\lim_{s\to 0} q_s = q$ (using the fact that q is the distinct point of γ farthest from p).



Construct a variation of this variation: $\sigma_{s,t}$, for each $s \in [0, \varepsilon]$, is the unique minimizing geodesic joining $\gamma_s(0)$ and $\gamma_s(t)$, near q_s. The uniqueness of $\sigma_{s,t}$ for each t follows from the initial choice of p and $q \in C_m(p)$ as $d(p,q) = i(M)$, and the use of previous theorem: if it were not unique, a new choice of p and $q \in C_m(p)$ would make $d(p,q) < i(M)$, contradicting the definition of $i(M)$. Using the fact that $\sigma_{s,t}$ is a geodesic, it follows that the first variation is 0 by `DoCarmo, Ch.9 Prop. 2.5 (pg.196)`; in turn, it follows that the inner product of $\sigma'_s(0)$ and γ'_s at q_s is 0 for all $s \in [0, \varepsilon]$, making them orthogonal by definition: call the variation field Y, and using the formula from `Chavel, Thm.II.4.1 (pg.74)`,

$$
\begin{aligned}
E'(0) &= \langle Y, \sigma_s' \rangle_0^{\mathrm{len}(\sigma_s)} - \int_0^{\mathrm{len}(\sigma_s)} \underbrace{\left\langle Y, \frac{D}{dt}\sigma_s' \right\rangle}_{\substack{0 \\ \text{since } \frac{D}{dt}\sigma_s' = 0 \text{ by} \\ \text{parallel transport}}} dt \\
&= \underbrace{\langle Y(\mathrm{len}(\sigma_s)), \sigma_s'(\mathrm{len}\,\sigma_s) \rangle}_{\substack{0 \\ \text{since } Y(\mathrm{len}(\sigma_s)) = 0 \\ \text{by construction}}} - \langle Y(0), \sigma_s'(0) \rangle \\
\implies \langle Y(0), \sigma_s'(0) \rangle &= 0 \quad \text{where } Y(0) \text{ is } \gamma_s \text{ at } q_s.
\end{aligned}
$$

From continuity again, it follows that, in the limit, $\sigma'(0)$ is orthogonal to γ' at q. However, since q is not a conjugate point of p, this is a contradiction to the result of Klingenberg Lemma. Therefore, $d(p,q) = \underline{i(M) \geq \pi}$, completing the proof. ■

As a side-note, in the Euclidean case, it is easier to see the fact used in the previous proof that a minimizing geodesic connecting a point and the point farthest from it on a loop, is perpendicular to the loop at that farthest point. Consider a loop (a smooth closed curve), Γ, in Euclidean space, $\mathbb{R}^n$. Let $p \in \Gamma$, and let Γ_p be the loop with p translated to the origin to simplify without loss of generality. Now, let $q \in \Gamma_p$ be a point farthest from the origin. Parametrize Γ_p on $[a,b) \ni t$:

$$\Gamma_p(t) = (x_1(t), \ldots, x_n(t)).$$

Let $t_0 \in [a,b)$ be such that $\Gamma_p(t_0) = q$. Using the fact that a maximum corrsponds to a singularity in the first derivative, and that maximizing a non-negative function is equivalent to maximizing its square, it follows that

$$\left.\frac{d}{dt}|\Gamma_p(t)|^2\right|_{t=t_0} = 2\sum_{i=1}^{n} x_i'(t_0)\, x_i(t_0) = 0,$$

which can be written as

$$2\Gamma_p'(t_0) \cdot q = 0,$$

implying that the direction of $\Gamma_p'(t_0)$ is orthogonal to the direction of q. Noticing that the distance from the origin to q is the length of the minimizing geodesic, a line, joining the origin and q, it is easy to see that the tangent vector to this minimizing geodesic is in the direction of q. Written in the terms used in the previous proof, Γ_p' at q is orthogonal to the tangent vector at q of the minimizing geodesic joining the origin and q.