Topology & Geometry Seminar WS24-25

Lecture 9: Fillability for Confoliations

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0. Symplectic Overview

The notion of fillability inherently involves manifolds with a certain structure, namely a *symplectic* structure. To that end, we will define what this means.

0.1 Definition [G, 1.4.2]
A *symplectic form* on a manifold W of (necessarily) even dimension $2n$ is a differential 2-form ω that is closed ($\mathrm{d}\,\omega = 0$) and non-degenerate, i.e. the $2n$-form $\omega^{\wedge n}$ does not vanish (i.e. it is a volume form). Therewith, a *symplectic manifold* is the pair (W, ω).

0.2 Remark
As with contact manifolds, there is a symplectic Darboux's theorem [CdS, theorem 8.1] which states that locally all symplectic forms are the same, symplectomorphic to the standard form

$$\mathrm{d}\,x_1 \wedge \mathrm{d}\,y_1 + \cdots + \mathrm{d}\,x_n \wedge \mathrm{d}\,y_n \quad .$$

Compare this to the corresponding result [G, 2.5.1] for contact structures,

$$\mathrm{d}\,z + x_1\,\mathrm{d}\,y_1 + \cdots + x_{n-1}\,\mathrm{d}\,y_{n-1} \quad .$$ ◂

An accompanying object in the bridge between symplectic and contact structures is the following special vector field.

0.3 Definition [G, 1.4.5]
A *Liouville vector field* on a symplectic manifold (W, ω) is a vector field Y on W satisfying $\mathcal{L}_Y\,\omega = \omega$.

0.4 Remark
In this case, ω is necessarily exact by Cartan's magic formula and the closedness of ω,

$$\omega = \mathcal{L}_Y\,\omega = \iota_Y\,\mathrm{d}\,\omega + \mathrm{d}\,\iota_Y\,\omega = \mathrm{d}\,\iota_Y\,\omega \quad ,$$

and the 1-form $\alpha \coloneqq \iota_Y\,\omega$ is a contact form on any hypersurface M transverse to Y, following from the previous statement

$$\alpha \wedge (\mathrm{d}\,\alpha)^{\wedge(n-1)} = (\iota_Y\,\omega) \wedge \omega^{\wedge(n-1)} = \frac{1}{n}\,\iota_Y\,\omega^{\wedge n}$$

is non-zero when restricted to any hypersurface transverse to Y, using the fact that $\omega^{\wedge n}$ is a volume form. ◂

An rudimentary example of the symplectic manifold is the following.

0.5 Example [G, 1.4.8]
The disk $\mathbf{D}^4$ with the symplectic form

$$\omega = \mathrm{d}\,x_1 \wedge \mathrm{d}\,y_1 + \mathrm{d}\,x_2 \wedge \mathrm{d}\,y_2$$

induces a contact structure on S^3 with the contact form

$$\alpha = \iota_Y \omega = \frac{1}{2}(x_1\,\mathrm{d}\,y_1 - y_1\,\mathrm{d}\,x_1 + x_2\,\mathrm{d}\,y_2 - y_2\,\mathrm{d}\,x_2) \quad ,$$

wherein the Liouville vector field is

$$Y = \frac{1}{2} r\,\partial_r = \frac{1}{2}(x_1\,\partial_{x_1} + y_1\,\partial_{y_1} + x_2\,\partial_{x_2} + y_2\,\partial_{y_2}) \quad .$$ ⋈

1. Fillability for $n = 2$

Across the literature, the notion of a *filling* is defined slightly differently: this is also the case for [G] and [E-T]. In [G, 1.7.4 & 5.1.1] for example, the notions of *weak* and *strong* symplectic fillings are defined. However, for this talk, we will be following [E-T]. To that end, we first need to define what a *confoliation* is, as the title of [E-T] suggests: this is an amalgamated class of objects of both *con*tact structures and *foliations*.

1.1 Definition [E-T, §3.2]
A (*positive*, resp. *negative*) *confoliation* is a pair (M, ξ), where $\xi = \ker \alpha$ for some 1-form α on M such that

$$\alpha \wedge \mathrm{d}\,\alpha \geq 0 \quad \text{resp.} \quad \alpha \wedge \mathrm{d}\,\alpha \leq 0 \quad .$$

Therewith, we can now define what a filling is in the [E-T] context.

1.2 Definition [E-T, §3.2]
Let (M, ξ) be a confoliation, and let ω be a closed 2-form on M. Then, ω is said to *dominate* ξ iff $\omega|_\xi \neq 0$.

A compact symplectic 4-manifold (W, ω) is called a *symplectic filling* of positive confoliation (M, ξ) iff $\omega|_M$ dominates ξ and $\partial W = M$ as oriented manifolds. In this case, (M, ξ) is said to be *symplectically fillable*.

A slightly weaker notion is that of a *symplectic semi-filling*, or correspondingly that (M, ξ) is *symplectically semi-fillable*, which is when M is only a connected component of a (disconnected) symplectically-fillable confoliated 3-manifold — that it is "symplectically cobordant" to another symplectically semi-fillable, but possibly disconnected, 3-manifold.

With that, we obtain examples of symplectically semi-fillable manifolds from the following proposition.

1.3 Proposition [E-T, 3.2.2]
Taut foliations are symplectically semi-fillable.

Proof From previous lectures, a (coöriented) foliation (M, ξ) is taut when there is a map $\check{\varphi} : \mathrm{S}^1 \longrightarrow M$ such that at each point of its image is transverse to ξ while intersecting all leaves of the foliation. Following [C, §4.4], we will first show that this implies (this is actually an equivalence [C, 4.29]) that there is a vector field transverse to ξ whose flow preserves a canonical volume form on (M, ξ) relative to its leaves. To get to the proof from [E-T], we will then show that is equivalent to the existence of a 2-form dominating ξ.

Following [C], the map $\check{\varphi}$ can be extended to a "tubular neighborhood" immersion $\varphi : \mathbf{D}^2 \times \mathrm{S}^1 \longrightarrow M$, such that, for each $p \in \mathbf{D}^2$, $\varphi(p, -)$ is transverse to ξ. In this way, because φ intersects each leaf transversely, we can think of homotopies of φ bringing any $x \in M$ into its image, namely that there is a loop that goes through x and each leaf transversely. Moreover, it is possible (akin to the Whitney embedding theorem) to perturb φ to a smooth embedding keeping these properties because of the aforementioned transversality.

Now, since M is compact, we can find a finite cover of M by such open tubular neighborhoods $\varphi(\mathrm{D}2 \times \mathrm{S}^1)$ given by the homotopies of φ; call the smooth embeddings that give rise to this cover $\varphi_j : \mathbf{D}^2 \times \mathrm{S}^1 \longrightarrow M$, wherein the disks on each leaf corresponding to $t \in \mathrm{S}^1$ are, of course, $\{\varphi_j(-, t)\}$.

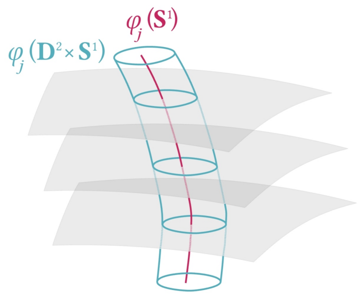

Figure 1 The "tubular neighborhood" $\varphi_j(\mathbf{D}^2 \times \mathrm{S}^1)$ of $\varphi_j(\mathrm{S}^1)$ intersecting leaves of the foliation given by ξ on M.

Next, to start constructing the "canonical" volume form of (M, ξ): look first at $\overline{\mathbf{D}^2}$, and let $\check{\omega}$ be a 2-form which is a bump function, being positive on $\mathring{\mathbf{D}}^2$ and vanishing on $\partial \mathbf{D}^2$. Pulling this back by the projection $\mathrm{pr}_1 : \mathbf{D}^2 \times \mathrm{S}^1 \twoheadrightarrow \mathbf{D}^2$, obtains a 2-form $\omega \coloneqq \mathrm{pr}_1^* \check{\omega}$, which is closed because it is a top form on $\mathbf{D}^2$ and pullback commutes with the differential. Therewith, we can obtain a closed 2-form on $\varphi_j\left(\mathbf{D}^2 \times \mathbf{S}^1\right) \subset M$

$$
\begin{gathered}
M \xleftarrow{\varphi_j} \mathbf{D}^2 \times \mathbf{S}^2 \xrightarrow{\mathrm{pr}_1} \mathbf{D}^2 \\
(\varphi_j^{-1})^* \circ \mathrm{pr}_1^* \check{\omega} : \underset{\mathrm{T}\, M \times \mathrm{T}\, M}{\underset{\cap}{\mathrm{T}\left(\varphi_j\left(\mathbf{D}^2 \times \mathbf{S}^1\right)\right) \times \mathrm{T}\left(\varphi_j\left(\mathbf{D}^2 \times \mathbf{S}^1\right)\right)}} \longrightarrow \mathrm{T}\left(\mathbf{D}^2 \times \mathbf{S}^1\right) \times \mathrm{T}\left(\mathbf{D}^2 \times \mathbf{S}^1\right) \longrightarrow \mathrm{T}\,\mathbf{D}^2 \times \mathrm{T}\,\mathbf{D}^2 \longrightarrow \mathbf{R} \quad ,
\end{gathered}
$$

which collectively across j gives in turn a (smooth) closed 2-form on M

$$
\omega := \sum_j (\varphi_j^{-1})^* \circ \mathrm{pr}_1^* \check{\omega} \quad .
$$

Note that ω is, by construction, then smooth and strictly positive on ξ, since ξ is coöriented. It is also easy to see that $\ker \omega$ is a 1-dimensional distribution on M, because $\dim M = 3$ and ω is a 2-form, and it is transverse to ξ by construction.

Let α by a 1-form on M such that $\xi = \ker \alpha$, and pick a (smooth) vector field X such that $\iota_X\, \alpha \equiv 1$, simply a "normal" to ξ normalized to α. Thusly,

$$
\Omega = \omega \wedge \alpha
$$

becomes a volume form, canonical in its form relative to the leaves of (M, ξ), and

$$
\begin{aligned}
\mathcal{L}_X\, \Omega &= \mathcal{L}_X\, \omega \wedge \alpha + \omega \wedge \mathcal{L}_X\, \alpha \\
&= \iota_x \underbrace{\cancel{\mathrm{d}\,\omega}}_{\because\ \text{closed}} + \mathrm{d} \overbrace{\cancel{\iota_X\, \omega}}^{\because\ X \in \ker \omega} + \omega \wedge \underbrace{\cancel{\mathcal{L}_X\, \alpha}}_{\because\ \iota_X\, \alpha \equiv 1} \\
&= 0 \quad ,
\end{aligned}
$$

so the flow of X preserves the volume form Ω on M. Therefore, tautness in the sense of the existence of the appropriate $\check{\varphi} : \mathbf{S}^1 \longrightarrow M$ implies that there is a vector field X on M that is transverse to ξ and preserves a volume form $\Omega := \omega \wedge \alpha$ on M.

Moving on to [E-T], the existence of such an X is equivalent to the existence of a dominating 2-form ω. For the sufficient direction of the equivalence: if there is such an X, then it follows from the logic above that $\omega := \iota_X\, \Omega$ dominates ξ because Ω is a non-degenerate top form, making ω closed

$$
0 = \mathcal{L}_X\, \Omega = \mathrm{d}\, \iota_X\, \Omega = \mathrm{d}\, \omega \quad ,
$$

and $X \pitchfork \xi$. For the necessary direction: if ω dominates $\xi = \ker \alpha$, then following the argumentation above it is possible to pick a vector field X transverse to ξ such that $\iota_X\, \omega \equiv 0$ and $\iota_X\, \alpha \equiv 1$, which makes $\mathcal{L}_X\,(\omega \wedge \alpha) = 0$, so X preserves the volume form $\omega \wedge \alpha$.

Finally, moving on to the proof of the main statement: let (M, ξ) be a taut foliation with $\xi = \ker \alpha$, and let ω be the corresponding dominating 2-form. Consider the cylinder $M \times [0,1]$ over M and the 2-form $\tilde{\omega} = \mathrm{pr}_1^* \omega + \varepsilon\, \mathrm{d}\,(t\alpha)$ on it, where $\mathrm{pr}_1 : M \times [0,1] \twoheadrightarrow M$, $t \in [0,1]$, and $\varepsilon > 0$ small enough. The claim is that $\tilde{\omega}$ dominates ξ on $M \times \{0\}$ and $M \times \{1\}$:

— it is non-degenerate for each $t \in [0,1]$ because of the non-degeneracy of ω

$$
\tilde{\omega} \wedge \tilde{\omega} = \mathrm{pr}_1^* \omega \wedge \mathrm{pr}_1^* \omega + 2\varepsilon \mathrm{pr}_1^* \omega \wedge \mathrm{d}\,(t\alpha) + \varepsilon^2\, \mathrm{d}\,(t\alpha) \wedge \mathrm{d}\,(t\alpha) = \varepsilon \mathrm{pr}_1^* \omega \wedge \mathrm{d}\, t \wedge \alpha \quad ,
$$

since $\omega \wedge \alpha$ is a volume form on M and $\alpha \wedge \mathrm{d}\, \alpha = 0$

— it is closed for each $t \in [0,1]$

$$
\mathrm{d}\, \tilde{\omega} = \mathrm{d}\, \mathrm{pr}_1^* \omega + \varepsilon\, \mathrm{d}^2\,(t\alpha) = \mathrm{pr}_1^*\, \mathrm{d}\, \omega = 0
$$

because ω is closed and $\mathrm{d}^2 = 0$

— it does not vanish on ξ on the boundary of the cylinder over M

$$
\begin{aligned}
&\text{at } t = 0 \qquad \tilde{\omega}|_\xi = (\mathrm{pr}_1^* \omega)|_\xi = \omega|_\xi \neq 0 \\
&\text{at } t = 1 \qquad \tilde{\omega}|_\xi = (\mathrm{pr}_1^* \omega)|_\xi + \varepsilon\, (\mathrm{d}\, \alpha)|_\xi \neq 0
\end{aligned}
$$

for small enough $\varepsilon > 0$ because ω already dominates ξ and the smoothness of α, since $\ker \alpha = \xi$.

Hence, the taut foliation (M,ξ) is symplectically semi-fillable, with its symplectic semi-filling being the cylinder over itself with the symplectic form $\tilde{\omega}$ and with $M \subset \partial(M \times [0,1])$ agreeing with the orientation of $M \times \{1\}$. ■

We can also say something about fillability for contact structures. First, recall [G, §4.5] (cf. [E-T, §3.1]) from the previous lecture that an *overtwisted* contact structure (M,ξ) is one for which it is possible to embed a disk into M such that its interior is transverse to ξ while its boundary is tangent to it; going along with this, was the complementary notion of being *tight*, which is not overtwisted. With that, we can state the following result.

1.6 Theorem [E-T, 3.2.4]
Symplectically semi-fillable contact structures are necessarily tight.

To finish off, a remark looping Liouville vector fields into the discussion of fillability, an example from before, and an example in a canonical symplectic manifold.

1.7 Remark
As discussed in remark ¶0.4, a Liouville vector field Y of (W,ω) affords contact structures to its hypersurfaces that are transverse to Y. This sounds similar to the existence of a transverse vector field X, as above, which preserves a volume form. However, the existence of a Liouville vector field is a stronger condition: of course, if such a Y exists, then (M,ξ) such that $M \subset \partial W$ and $\xi \coloneqq \ker \iota_Y\, \omega|_{\mathrm{T}M}$ satisfies $\omega|_\xi \neq 0$ since $\xi \subset Y^{\perp\omega}$ by definition and ω is non-degenerate.
In this way, [G, 1.7.4 & 5.1.1] defines a *weak* and a *strong* fillability. ◂

1.8 Example
Actually example ¶0.5 is an example of a (strong *à la* [G, 1.7.4]) symplectic filling: $(\mathbf{D}^4,\omega)$ is a symplectic filling of $(\mathrm{S}^3, \ker \iota_Y\, \omega)$. ⋈

1.9 Example
Using example ¶0.5 again, we will construct other examples of symplectic fillings from the canonical example [CdS, §§2.2-3] [G, 1.2.3 & §1.4] of symplectic manifold from the cotangent bundle T^*Q of any (smooth) manifold Q.

$$\begin{tikzcd}
\mathrm{T}^*Q \arrow[d, two heads, "\mathrm{pr}_1"] & \mathrm{T}\,\mathrm{T}^*Q \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[d, two heads, "\mathrm{T}\,\mathrm{pr}_1"] & \mathrm{T}^*\mathrm{T}^*Q \arrow[d, "\mathrm{T}^*\mathrm{pr}_1"] \\
Q & \mathrm{T}\,Q \arrow[l, two heads, "\mathrm{pr}_1"] & \mathrm{T}^*Q
\end{tikzcd}$$

Consider following *tautological* 1-form in $\mathrm{T}^*_{(q,p)}\,\mathrm{T}^*Q$, where q-coördinates are on Q and p-coördinates are for the fibre $\mathrm{T}^*_q\,Q$

$$\alpha_{(q,p)} \coloneqq \left(\mathrm{T}_{(q,p)}\,\mathrm{pr}_1\right)^*\, p \quad .$$

Expanding this further in coördinates

$$\alpha_{(q,p)} = \sum_j p_j \,\mathrm{d}\, q_j \quad ,$$

wherein with a conflation of notation $\mathrm{d}\,q_j \circ \mathrm{T}_{(q,p)}\,\mathrm{pr}_1 \rightsquigarrow \mathrm{d}\,q_j$, lifting $\mathrm{d}\,q_j \in \mathrm{T}^*_q\,Q$ naturally into $\mathrm{T}^*_{(q,p)}\,\mathrm{T}^*Q$.
Setting $\omega \coloneqq -\,\mathrm{d}\,\alpha = \sum_j \mathrm{d}\,q_j \wedge \mathrm{d}\,p_j$ obtains the so-called *canonical* symplectic structure on T^*Q: it is closed $\mathrm{d}\,\omega = -\,\mathrm{d}^2\,\alpha = 0$ and non-degenerate $\omega^{\wedge n} = \bigwedge_j \mathrm{d}\,q_j \wedge \mathrm{d}\,p_j \neq 0$.
Bringing this back to our context ($n = 2$), for a surface Σ, $(\mathrm{T}^*\Sigma,\omega)$ is a symplectic manifold:

$$\omega \coloneqq \mathrm{d}\,q_1 \wedge \mathrm{d}\,p_1 + \mathrm{d}\,q_2 \wedge \mathrm{d}\,p_2$$

is a symplectic form, as discussed, and

$$Y = p_1\partial_{p_1} + p_2\partial_{p_2}$$

is a Liouville vector field in the fibres above Σ

$$\iota_Y\,\omega = -p_1\,\mathrm{d}\,q_1 - p_2\,\mathrm{d}\,q_2 = -\alpha \qquad \Longrightarrow \qquad \mathcal{L}_Y\,\omega = \mathrm{d}\,\iota_Y\,\omega = -\,\mathrm{d}\,\alpha = \omega \quad .$$

Thus, if we consider a disk bundle $\mathrm{DT}^*\Sigma$ (the fibres above Σ are unit disks), then the corresponding sphere bundle $\mathrm{ST}^*\Sigma$ (the boundaries spheres of those disks) is symplectically fillable by $\mathrm{DT}^*\Sigma$. Moreover, any hypersurface (in the fibres above Σ) transverse to Y (a radial vector field) is a contact manifold that is symplectically filled by its interior. ⋈

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