The aim of this overview is to outline notions that build up to the description of Calabi-Yau manifolds; with that, literature that generalizes this notion will be discussed. To start off, a differential geometric approach via holonomy will be taken and a specific kind of complex manifold will be introduced, a Kähler manifold, of which a Calabi-Yau manifold is a specific kind determined by holonomy. Therewith, the final aim of this paper is to discuss generalizing the idea of a Calabi-Yau manifold, as done in the work of Hitchin [Hit02][Hit10] and Gualtieri [Gua03], his recent PhD student.

# 1 Kählerian Preliminaries

1.1 Definition Let $M$ be a real manifold, and let $X, Y \in \mathfrak{X}(M)$, the set of smooth vector fields on $M$.

An *almost complex structure* on $M$ is a $(1,1)$-tensor $J$ on the tangent bundle of $M$ such that $J_a^b J_b^c = -\delta_a^c$. To each such structure there is an associated *Nijenhuis tensor*, a $(1,2)$-tensor, $N(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$, where $[\cdot,\cdot]$ is the Lie bracket. A metric $g$ on $M$ is called *Hermitian* if $g(X,Y) = g(JX,JY)$. Associated to this Hermitian metric is a *Hermitian form* $\omega$ defined as $\omega(X,Y) = g(JX,Y)$, so that it recovers the metric thusly $\omega(X,JY) = g(JX,JY) = g(X,Y)$.

If $N \equiv 0$ for a given $J$ on an $M$, then $(M,J)$ is called a *complex manifold*; the reasoning for this is that, when $N$ vanishes for a given $J$, then $J$ is called *integrable*, meaning it is possible[1] to find complex biholomorphic coordinate charts for $M$. [GHJ03, §§I.4.1-2][MS95, §4]

1.2 Definition A complex manifold $(M,g,J)$ is a called *Kähler* if $g$ is a Hermitian metric and if any of the following equivalent[2] conditions hold: where $\omega$ is the Hermitian form associated to $(M,g,J)$, as defined in definition (1.1),
$\mathrm{d}\omega = 0, \quad \nabla J = 0, \quad \nabla\omega = 0,$
where $\nabla$ is the Levi-Civita connection associated to $g$. If such an $M$ is of real dimension $2n$, i.e. of complex dimension $n$, then $M$ is called a *Kähler $n$-fold*. [GHJ03, §I.4.2][Joy00, §4.4]

Also, note that, if $(M,J)$ is Kähler, then $\omega$ is closed by definition, is nondegenerate by the nondegeneracy of the metric $g$ on $M$, and is anti-symmetric,

$$\omega(X,Y) = g(JX,Y) = g\left(JX, -J^2Y\right) = g(X,-JY) = g(JY,-X) = -\omega(Y,X);$$

this shows that $\omega$ is symplectic, making $(M,\omega)$ also a symplectic manifold.

1.5 Theorem Let $T$ is a smooth section of $\bigotimes^r TM \otimes \bigotimes^s T^*M$, i.e. $T \in \Gamma\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$, and suppose $T$ is such that it is constant, i.e. $\nabla T = 0$. Then, at every $p \in M$, $T|_p$ is stabilized by all elements of $\mathrm{Hol}_p(\nabla)$. Also, if some $\tau \in \bigotimes^r T_pM \otimes \bigotimes^s T_p^*M$ is stabilized by all of $\mathrm{Hol}_p(\nabla)$, then $\tau$ can be extended to a tensor $T \in \Gamma\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$ such that $\nabla T = 0$. [GHJ03, §I.2.3][Joy00, §2.5]

With this, on a Riemannian manifold $(M,g)$ of dimension $2n$ with Levi-Civita connection $\nabla$, it follows that since $\nabla g = 0$, $\mathrm{Hol}\, g$ must stabilize $g$; in particular, this means that $\mathrm{Hol}(g)$ is isomorphic to a subgroup of $\mathrm{O}_{2n}(\mathbb{R})$, i.e. $\mathrm{SO}_{2n}(\mathbb{R})$ up to conjugation. Moreover, for a Kähler manifold $(M,g,J)$ with Kähler form $\omega$, it follows from definiton that $\mathrm{Hol}(g)$ must also preserve $J$ and $\omega$, forcing $\mathrm{Hol}(g) \subset \mathrm{U}_n(\mathbb{C})$.

# 2 Calabi-Yau Introduction

This section will, in particular, work to an equivalent definition (see remark (2.3)) of a Calabi-Yau manifold, which motivated the construction of a generalized Calabi-Yau manifold by Hitchin [Hit02].

2.1 Definition A *Calabi-Yau manifold*, or *Calabi-Yau $n$-fold*, is a compact Kähler manifold $(M,g,J)$ of real dimension $2n$ with $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$. [nb.[3]]

2.2 Proposition Let $(M,g,J)$ be a compact Kähler $n$-fold such that $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$. Then, $M$ admits a non-zero constant holomorphic form $\Omega \in \Gamma(\Lambda^{n,0}M)$, which is unique up to multiplication by $\mathrm{e}^{\mathrm{i}\theta}$, for $\theta \in \mathbb{R}$, and is such that

$$\omega^n = \left(\frac{\mathrm{i}}{2}\right)^n n!(-1)^{\frac{n(n-1)}{2}}\,\Omega \wedge \overline{\Omega};$$

this form $\Omega$ is called the *holomorphic volume form* on $M$. Calling $\Lambda^{n,0}M =: K_M$ the *canonical bundle*, the existence of $\Omega$ implies that a Calabi-Yau manifold has a *trivial* canonical bundle by the definition of being a trivial bundle. [GHJ03, §I.4.5][Joy00, §6.1]

Conversely, if the compact Kähler $n$-fold $(M,g,J)$ has a trivial canonical bundle (i.e. there exists such an $\Omega$), then it follows that $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$.

2.3 Remark From this proposition, it follows that there is an equivalent definition of a Calabi-Yau $n$-fold (cf. definition (2.1)): a compact Kähler $n$-fold $(M,g,J)$, which has a trivial canonical bundle. ◀

# 3 Generalized Calabi-Yau

In this section, the idea of generalized geometry from Hitchin [Hit02][Hit10], and Gualtieri [Gua03], will be introduced. Specifically, the sights will be set on the notion of a generalized Calabi-Yau manifold as discussed in Hitchin [Hit02]. As mentioned, generalizing a Calabi-Yau manifold comes from using the equivalent definition in remark (2.3).

Baring contrast to previous considerations involving the tangent bundle, in generalized geometry, considerations are with respect to the bundle $(TM \oplus T^*M) \otimes \mathbb{C}$. With this, the metric is replaced with an indefinite metric, which extends over $\mathbb{C}$, from the natural action of sections of $T^*M$ on sections of $TM$. Then, the use of the Lie bracket on sections of $TM$ translates to the analogous use of the Courant bracket on $TM \oplus T^*M$.

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[1] [GHJ03, §I.4.1][MS95, §4.2] The vanishing is a necessary and sufficient condition by the Newlander-Nirenberg theorem. [Joy00, §4.1] [Mor07, §7.4 & §8.1]
[2] [GHJ03, §I.4.2][Mor07, §11.2]
[3] This differs in from some of the referenced material, cf. [GHJ03, §I.4.5][Joy00, §6.1][Mor07, §21.2], where $\mathrm{Hol}(g) \cong \mathrm{SU}_n(\mathbb{C})$. The reason for this is as was suggested in the introduction to this section: this definition is equivalent to one which motivated Hitchin [Hit02]. As should be seen, this is done without affect to the results which were gleaned from those referenced works.

3.1 Definition Let $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$, smooth sections of $TM \oplus T^*M$. Then, the *Courant bracket* of $X+\alpha, Y+\beta$ is

$$[\![X+\alpha, Y+\beta]\!] := [X,Y] + \mathfrak{L}_X\beta - \mathfrak{L}_Y\alpha - \frac{1}{2}\mathrm{d}\left(\iota_X\beta - \iota_Y\alpha\right),$$

where $[\cdot,\cdot]$, and $\mathfrak{L}$, are the usual Lie bracket, and derivative, respectively, and $\iota$ is the usual interior product.

On the bundle $TM \oplus T^*M$, the natrual indefinite metric is given as follows, for $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$,

$$\langle X+\alpha, Y+\beta\rangle := \frac{1}{2}\left(\beta(X) + \alpha(Y)\right) = \frac{1}{2}\left(\iota_X\beta + \iota_Y\alpha\right).$$

[Hit02, §3.1][Gua03, §3.2 & §2.2][Hit10, §§1.1-2]

3.2 Proposition Define the action of $X+\alpha \in \Gamma(TM \oplus T^*M)$ on $\varphi \in \Gamma(\Lambda^\bullet T^*M)$ to be $(X+\alpha)\cdot\varphi = \iota_X\varphi + \alpha\wedge\varphi \in \Gamma(\Lambda^\bullet T^*M)$. Then, with this action and with the relation $(X+\alpha)^2 = \langle X+\alpha, X+\alpha\rangle$, $TM \oplus T^*M$ has a Clifford algebra represention $\mathrm{C}\ell(TM \oplus T^*M)$ on $\Lambda^\bullet T^*M$. [Hit02, §3.2][Gua03, §2.3][Hit10, §2.1]

3.3 Remark Therewith, it can be shown that the natural choice[4] of *spinors* for $\mathrm{C}\ell(TM \oplus T^*M)$ is the exterior algebra $\Lambda^\bullet T^*M$. For $\dim_{\mathbb{R}} TM = m = 2n$, this corresponds to the spin respresentation

$$S = \Lambda^\bullet T^*M \otimes (\Lambda^n T^*M)^{-\frac{1}{2}}.$$

Letting $\Lambda^+ T^*M, \Lambda^- T^*M$ be, respectively, the even- and odd-form parts of $\Lambda^\bullet T^*M$, this representation also has a(n) (invariant[5]) bilinear form: for $\varphi, \psi \in \Gamma(\Lambda^\bullet T^*M)$,

$$(\varphi,\psi) := \sum_{j=0}^{n} (-1)^j \left(\varphi_{2j}\wedge\psi_{m-2j} + \varphi_{2j+1}\wedge\psi_{m-2j-1}\right) \in \Gamma(\Lambda^n T^*M),$$

where a subscript $k$ denotes taking the part of $\varphi, \psi$, which is a form of real degree $k$. Note that, in this construction, the degrees summed over is doubled when the bundle is complexfied, i.e. summed to $\dim_{\mathbb{R}}(TM\otimes\mathbb{C}) = 2\dim_{\mathbb{R}} TM = 2m$, as to account for the new vector space dimension. ◀

3.4 Definition Let $\varphi$ be a spinor, and let its annihilator be $E_\varphi := \{X+\alpha \in \Gamma(TM\oplus T^*M) \,|\, (X+\alpha)\cdot\varphi = 0\}$. Note that, via the multiplication rule[6] for Clifford algebras,

$$2\langle X+\alpha, Y+\beta\rangle\cdot\varphi = \left((X+\alpha)(Y+\beta) + (Y+\beta)(X+\alpha)\right)\cdot\varphi = 0,$$

for any $X+\alpha, Y+\beta \in E_\varphi$. Since $\varphi$ is non-trivial, this implies that $\langle X+\alpha, Y+\beta\rangle = 0$ for any such $X+\alpha, Y+\beta$, which makes, by definition, $E_\varphi$ *isotropic* with respect to $\langle\cdot,\cdot\rangle$.

The spinor $\varphi$ is called *pure* if $\dim_{\mathbb{R}} E_\varphi = \dim_{\mathbb{R}} M$. [Hit02, §3.3][Gua03, §2.5]

3.5 Definition Let $M$ be a smooth manifold of dimension $2n$, with the indefinite metric $\langle\cdot,\cdot\rangle$, as defined in definition (3.1), on its bundle $TM\oplus T^*M$. Then, a *generalized complex structure* on $M$ is a subbundle $E \subset (TM\oplus T^*M)\otimes\mathbb{C}$ such that: (1.) $E \oplus \overline{E} = (TM\oplus T^*M)\otimes\mathbb{C}$, i.e. $\dim E = 2n$, (2.) $\Gamma(E)$ is closed under the Courant bracket $[\![\cdot,\cdot]\!]$, (3.) $E$ is isotropic with respect to $\langle\cdot,\cdot\rangle$. [Hit02, §4.1][Gua03, §4.2][nb.[7]]

3.6 Definition A *generalized Calabi-Yau manifold* is a smooth manifold $M$ of real dimension $2n$ with a closed form $\varphi$ in $\Gamma(\Lambda^+ T^*M)$ or $\Gamma(\Lambda^- T^*M)$, which is a (complex) pure spinor for $\mathrm{C}\ell(TM\oplus T^*M)$ such that the bilinear form $(\varphi,\overline{\varphi}) \neq 0$. [Hit02, §4.1]

3.7 Proposition Let $M$ with $\varphi$ be a generalized Calabi-Yau manifold of real dimension $2n$. Then, the annihilator subbundle $E_\varphi \subset (TM\oplus T^*M)\otimes\mathbb{C}$ is a generalized complex structure on $M$. [Hit02, §4.1]

Proof First, note that, since $\varphi$ is pure, $E_\varphi$ is isotropic, satisfying condition (3.) of being a generalized complex structure. Furthermore, since $(\varphi,\overline{\varphi}) \neq 0$, it follows that, by the definition of the form $(\cdot,\cdot)$, that $E_\varphi \cap E_{\overline{\varphi}} = 0$, which, from the definition of an annihilator subbundle, makes $E_\varphi \cap \overline{E_\varphi} = E_\varphi \cap E_{\overline{\varphi}} = 0$. This with the fact that $\dim_{\mathbb{R}} E_\varphi = 2n = \dim_{\mathbb{R}} M$, since $\varphi$ is pure, makes $E_\varphi \oplus \overline{E_\varphi} = (TM\oplus T^*M)\otimes\mathbb{C}$, satisfying condition (1.) of being a generalized complex structure.

Lastly, it is needed to show that $E_\varphi$ satisfies condition (2.). Let $X+\alpha, Y+\beta \in E_\varphi$, and consider the identity

$$\iota_{[X,Y]}\varphi = \mathfrak{L}_X\left(\iota_Y\varphi\right) - \iota_Y\mathfrak{L}_X\varphi = -\mathfrak{L}_X\beta\wedge\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi.$$

Using antisymmetry of the Lie bracket $[\cdot,\cdot]$ and what was just shown, it follows

$$\iota_{[X,Y]}\varphi = \frac{1}{2}\left(\iota_{[X,Y]}\varphi - \iota_{[Y,X]}\varphi\right) = \left(\mathfrak{L}_Y\alpha - \mathfrak{L}_X\beta - \frac{1}{2}\left(\mathrm{d}\iota_Y\alpha - \mathrm{d}\iota_X\beta\right)\right)\wedge\varphi \iff [\![X+\alpha, Y+\beta]\!]\cdot\varphi = 0,$$

showing that $[\![X+\alpha, Y+\beta]\!] \in E_\varphi$ by definition, which makes sections of $E_\varphi$ closed under the Courant bracket; thus, condition (2.) is satisfied.

Therefore, $E_\varphi$ is a generalized complex structure on $M$ for such a $\varphi$. ■

3.8 Example Let $(M,g,J)$ be a Calabi-Yau $n$-fold, for $m = 2n$, as defined in definition (2.1); it follows from proposition (2.2), that there is an associated holomorphic volume form $\Omega \in \Gamma(\Lambda^{n,0}M)$ on $M$, which is naturally closed. Looking at $E_\Omega$, it contains elements $X+\alpha$ such that $X \in \Gamma(TM\otimes\mathbb{C})$ of type $(0,1)$, since then $\iota_X\Omega = 0$, and $\alpha \in \Gamma(T^*M\otimes\mathbb{C})$ of type $(1,0)$, since then $\alpha\wedge\varphi = 0$, whence $(X+\alpha)\cdot\varphi = 0$ by definition. This, of course, makes $\dim_{\mathbb{R}} E_\Omega = 2n = \dim_{\mathbb{R}} M$, and, thusly, $\Omega$ is a pure spinor. Lastly, looking the necessary bilinear form

$$\left(\Omega,\overline{\Omega}\right) = (-1)^n\,\Omega\wedge\overline{\Omega},$$

since $\Omega$ is a form of real degree $2n$, so $\Omega_{2n} = \Omega$ and $\overline{\Omega}_{2n} = \overline{\Omega}$, and the real dimension of the complexified bundle is $4n$, so $\overline{\Omega}_{4n-2n} = \overline{\Omega}$. Moreover, this product is nonzero by the construction of $\Omega$: $\left(\Omega,\overline{\Omega}\right) \neq 0$. Therefore, it follows from proposition (3.7), that $M$ with such a form $\Omega = \varphi$ has a generalized complex structure given by $E_\Omega$, making $M$ also a generalized Calabi-Yau manifold. ⧓

[4] For slightly-differing mentions of this, see [Hit02, §3.2] and [Hit10, §2.1]; this is discussed more so in [Gua03, §2.3 & §2.8]. The general references for spin are [Jos08, §1.11] and [LM89, §§I.1-8].

[5] [Hit02, §3.2][Gua03, §2.4]

[6] [Jos08, §1.11][Gua03, §2.5]

[7] From this definition, it follows that an endomorphism $\mathscr{J}$ on $\Gamma(TM\oplus T^*M)$ can be defined, analogous to the one for a regular complex structure, cf. definition (1.1).

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