Literature Review of Calabi-Yau Manifolds

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Abstract

The aim of this overview is to outline notions that build up to the description of Calabi-Yau manifolds; with that, literature that generalizes this notion will be discussed. To start off, a differential geometric approach via holonomy will be taken and a specific kind of complex manifold will be introduced, a Kähler manifold, of which a Calabi-Yau manifold is a specific kind determined by holonomy. Therewith, the final aim of this paper is to discuss generalizing the idea of a Calabi-Yau manifold, as done in the work of Hitchin [Hit02][Hit10] and Gualtieri [Gua03], his recent PhD student.

1 Kählerian Preliminaries

First off, orienting with background notions, out of which Kähler manifolds arise. From this, the notion of holonomy will be introduced as to segue into specializing considerations to Calabi-Yau manifolds.

1.1 Definition Let M be a real manifold, and let $X, Y \in \mathfrak{X}(M)$, the set of smooth vector fields on M.

An *almost complex structure* on M is a $(1,1)$-tensor J on the tangent bundle of M such that

$$J_a^b J_b^c = -\delta_a^c.$$

To each such structure there is an associated *Nijenhuis tensor*, a $(1,2)$-tensor,

$$N_{ab}^c X^a Y^b = ([X,Y] + J([JX,Y] + [X,JY]) - [JX,JY])^c,$$

where $[\cdot,\cdot]$ is the Lie bracket.

A metric g on M is called *Hermitian* if

$$g(X,Y) = g(JX,JY).$$

Associated to this Hermitian metric is a *Hermitian form* ω defined as

$$\omega(X,Y) = g(JX,Y),$$

so that it recovers the metric thusly

$$\omega(X,JY) = g(JX,JY) = g(X,Y).$$

If $N \equiv 0$ for a given J on an M, then (M,J) is called a *complex manifold*; the reasoning for this is that, when N vanishes for a given J, then J is called *integrable*, meaning it is possible[1] to find complex biholomorphic coordinate charts for M.

[GHJ03, §§I.4.1-2][MS95, §4]

A case of these complex manifolds, is a Kähler manifold gotten via the following definition.

[1] [GHJ03, §I.4.1][MS95, §4.2] The vanishing is a necessary and sufficient condition by the Newlander-Nirenberg theorem. [Joy00, §4.1] [Mor07, §7.4 & §8.1]

1.2 Definition A complex manifold (M,g,J) is a called *Kähler* if g is a Hermitian metric and if any of the following equivalent[2] conditions hold: where ω is the Hermitian form associated to (M,g,J), as defined in definition (1.1),

i.) $\mathrm{d}\omega = 0$

ii.) $\nabla J = 0$

iii.) $\nabla\omega = 0$,

where ∇ is the Levi-Civita connection associated to g. If such an M is of real dimension $2n$, i.e. of complex dimension n, then M is called a *Kähler n-fold*.

[GHJ03, §I.4.2][Joy00, §4.4]

Here it is worth noting that, for a real manifold (M,g) with an almost complex structure J, if $\nabla J = 0$ for the Levi-Civita connection of g, then the associated Nijenhuis tensor acts on any $X, Y \in \mathfrak{X}(M)$ as

$$\begin{aligned}
N^c_{ab}X^aY^b &= ([X,Y] + J([JX,Y]+[X,JY]) - [JX,JY])^c \\
&= \left(\nabla_X Y - \nabla_Y X + J\nabla_{JX}Y - J\nabla_Y(JX) + J\nabla_X(JY) - J\nabla_{JY}X - \nabla_{JX}(JY) + \nabla_{JY}(JX)\right)^c && \text{since } \nabla \text{ is torsion-free} \\
&= (\nabla_X Y - \nabla_Y X \\
&\qquad + J\nabla_{JX}Y - J(\nabla_Y J)X - J^2\nabla_Y X \\
&\qquad + J(\nabla_X J)Y + J^2\nabla_X Y - J\nabla_{JY}X \\
&\qquad - (\nabla_{JX}J)\,Y - J\nabla_{JX}Y + (\nabla_{JY}J)\,X + J\nabla_{JY}X)^c && \text{via the Leibniz rule, by natural extension of action of } \nabla \text{ to higher-order tensors} \\
&= \left(-J(\nabla_Y J)X + J(\nabla_X J)Y - (\nabla_{JX}J)\,Y + (\nabla_{JY}J)\,X\right)^c && J^2 = -\mathbb{I} \\
&= 0 && \text{by assumption.}
\end{aligned}$$

Also, note that, if (M,J) is Kähler, then ω is closed by definition, is nondegenerate by the nondegeneracy of the metric g on M, and is anti-symmetric,

$$\omega(X,Y) = g(JX,Y) = g\left(JX, -J^2Y\right) = g(X,-JY) = g(JY,-X) = -\omega(Y,X);$$

this shows that ω is symplectic, making (M,ω) also a symplectic manifold.

With eyes set on obtaining a definition of a Calabi-Yau manifold, next, the idea of holomony will be introduced.

1.3 Definition Let ∇ be the connection on some vector bundle V over the manifold M. For some $p \in M$, let a loop at p be a piecewise-smooth closed curves $\gamma : [0,1] \longrightarrow M$. Then, for each $x \in V_p$, there is a unique section $X(t)$ of $V|_\gamma$ such that $\nabla_{\dot\gamma(t)}X(t) = 0$ for each $t \in [0,1]$; this section is called the *parallel transport* of x along γ. Given that γ is a loop, parallel transport along γ yields a linear map $P_\gamma : V_p \longrightarrow V_p$; in particular $P_\gamma \in \mathrm{GL}\left(V_p\right)$, or $\mathrm{GL}_d(\mathbb{R})$ if the fiber $V_p \cong \mathbb{R}^d$.

Now, let the *holonomy group* at p of ∇ be

$$\mathrm{Hol}_p(\nabla) := \left\{P_\gamma \,\middle|\, \gamma \text{ is a loop at } p\right\}.$$

Note that this forms a group in a natural way: obtaining inverses of P_γ by traveling γ in reverse, and obtaining compositions $P_{\gamma_1} \circ P_{\gamma_2}$ by concatenating γ_1 and γ_2.

[GHJ03, §I.2.3]

Albeit a product of the definition, needing to specify $p \in M$ to define a holonomy group is not entirely fruitful: given that M is connected, connecting p to q by some piecewise-smooth curve α, obtains $\mathrm{Hol}_q(\nabla) = P_\alpha \mathrm{Hol}_p(\nabla) P_\alpha^{-1}$, i.e. it is possible to talk about $\mathrm{Hol}(\nabla)$ as being defined up to conjugation in $\mathrm{GL}_d(\mathbb{R})$, for example, if the fibers of V are isomorphic to $\mathbb{R}^d$.

Of particular importance here, is the case when the considered vector bundle over a Riemannian (M,g) is its tangent bundle TM.

[2] [GHJ03, §I.4.2][Mor07, §11.2]

1.4 Definition For a Riemannian manifold (M, g), let $\mathrm{Hol}_p(g)$ be $\mathrm{Hol}_p(\nabla)$ for the Levi-Civita connection ∇ associated to g. As discussed in just after definition (1.3), this can be talked about as $\mathrm{Hol}(g)$, being defined up to conjugation.

[GHJ03, §§I.2.4-5]

Here, ∇ is taken to be the Levi-Civita connection of g, which is such that $\nabla g = 0$. In order to understand what kind of restrictions this places on $\mathrm{Hol}(g)$, comes the following theorem.

1.5 Theorem Let M be a manifold with ∇ being a connection on TM. As elements of $\mathrm{Hol}_p(\nabla)$ act on the fibers T_pM of TM over $M \ni p$, it follows that they extend to act naturally on higher-order tensors, e.g. in $\bigotimes^r T_pM \otimes \bigotimes^s T_p^*M$.

Let T is a smooth section of $\bigotimes^r TM \otimes \bigotimes^s T^*M$, i.e. $T \in \Gamma\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$, and suppose T is such that it is constant, i.e. $\nabla T = 0$. Then, at every $p \in M$, $T|_p$ is stabilized by all elements of $\mathrm{Hol}_p(\nabla)$.

Also, if some $\tau \in \bigotimes^r T_pM \otimes \bigotimes^s T_p^*M$ is stabilized by all of $\mathrm{Hol}_p(\nabla)$, then τ can be extended to a tensor $T \in \Gamma\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$ such that $\nabla T = 0$.

[GHJ03, §I.2.3][Joy00, §2.5]

Proof (Sketched.)

For the first part, assume $T|_p$ is not fixed by some element of $\mathrm{Hol}_p(\nabla)$. Then, by construction of Hol, it follows that there is some piecewise-smooth curve γ along which T is not constant with respect to ∇, i.e. $\nabla\, T|_\gamma \neq 0$, contradiction.

The second part is simply done by parallel transport, as ∇ is extended to all tensor-types. ■

With this, on a Riemannian manifold (M, g) of dimension $2n$ with Levi-Civita connection ∇, it follows that since $\nabla g = 0$, $\mathrm{Hol}\, g$ must stabilize g; in particular, this means that $\mathrm{Hol}(g)$ is isomorphic to a subgroup of $\mathrm{O}_{2n}(\mathbb{R})$, i.e. $\mathrm{SO}_{2n}(\mathbb{R})$ up to conjugation. Moreover, for a Kähler manifold (M, g, J) with Kähler form ω, it follows from definiton that $\mathrm{Hol}(g)$ must also preserve J and ω, forcing $\mathrm{Hol}(g) \subset \mathrm{U}_n(\mathbb{C})$.

2 Calabi-Yau Introduction

Starting off, this section introduces a Calabi-Yau manifold, via the discussion of holomony ending the previous section. Out of this, some properties of a Calabi-Yau manifold will be mentioned; these properties can serve as alternative (albeit, not necessarily equivalent, see remark (2.5)) definitions. This section will, in particular, work to an equivalent definition (see remark (2.3)) of a Calabi-Yau manifold, which motivated the construction of a generalized Calabi-Yau manifold by Hitchin [Hit02].

2.1 Definition A *Calabi-Yau manifold*, or *Calabi-Yau n-fold*, is a compact Kähler manifold (M, g, J) of real dimension $2n$ with $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$.

[nb.[3]]

With that said, there are a few more properties of Calabi-Yau manifolds, which themselves could, in some cases, serve as alternative definitions, pretaining to its canonical bundle, and curvature. [Bou07, §3][Joy00, §6]

2.2 Proposition Let (M, g, J) be a compact Kähler n-fold such that $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$. Then, M admits a non-zero constant holomorphic form $\Omega \in \Gamma(\Lambda^{n,0}M)$, which is unique up to multiplication by $e^{i\theta}$, for $\theta \in \mathbb{R}$, and is such that

$$\omega^n = \left(\frac{i}{2}\right)^n n!(-1)^{\frac{n(n-1)}{2}}\, \Omega \wedge \overline{\Omega};$$

[3] This differs in from some of the referenced material, cf. [GHJ03, §I.4.5][Joy00, §6.1][Mor07, §21.2], where $\mathrm{Hol}(g) \cong \mathrm{SU}_n(\mathbb{C})$. The reason for this is as was suggested in the introduction to this section: this definition is equivalent to one which motivated Hitchin [Hit02]. As should be seen, this is done without affect to the results which were gleaned from those referenced works.

this form Ω is called the *holomorphic volume form* on M. Calling $\Lambda^{n,0}M =: K_M$ the *canonical bundle*, the existence of Ω implies that a Calabi-Yau manifold has a *trivial* canonical bundle by the definition of being a trivial bundle.

[GHJ03, §I.4.5][Joy00, §6.1]

Conversely, if the compact Kähler n-fold (M, g, J) has a trivial canonical bundle (i.e. there exists such an Ω), then it follows that $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$.

Proof By definition, a Kähler manifold is a complex manifold; it follows that the complexified tangent bundle has fibers over each point $p \in M$, which are isomorphic to $\mathbb{C}^{2n}$ with coordinates $\{z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n\}$ and the standard metric g_0. With the assumptions on $\mathrm{Hol}(g)$, the following forms are preserved

$$\omega_0 = \frac{\mathrm{i}}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \cdots + \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n)$$

and

$$\Omega_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

along with the preservation of the standard metric

$$g_0 = \mathrm{d}z_1 \otimes \mathrm{d}\overline{z}_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}\overline{z}_n$$

via theorem (1.5), by definition of $\mathrm{Hol}(g)$. Note that the complex structure is

$$J = \begin{pmatrix} \mathrm{i}\,\mathbb{I}^n & 0 \\ 0 & -\mathrm{i}\,\mathbb{I}^n \end{pmatrix}$$

so that

$$\begin{aligned} g_0(JX, JY) &= (\mathrm{d}z_1 \circ J \otimes \mathrm{d}\overline{z}_1 \circ J + \cdots + \mathrm{d}z_n \circ J \otimes \mathrm{d}\overline{z}_n \circ J)(X, Y) \\ &= (\mathrm{i}\mathrm{d}z_1 \otimes -\mathrm{i}\mathrm{d}\overline{z}_1 + \cdots + \mathrm{i}\mathrm{d}z_n \otimes -\mathrm{i}\mathrm{d}\overline{z}_n)(X, Y) \\ &= g_0(X, Y) && \text{by linearity} \end{aligned}$$

and

$$\begin{aligned} g_0(JX, Y) &= \frac{1}{2}\left(g_0(JX, Y) + g_0(Y, JX)\right) && \text{using symmetry} \\ &= \frac{1}{2}(\mathrm{i}\mathrm{d}z_1 \otimes \mathrm{d}\overline{z}_1 + \cdots + \mathrm{i}\mathrm{d}z_n \otimes \mathrm{d}\overline{z}_n)(X, Y) + \frac{1}{2}(\mathrm{d}z_1 \otimes -\mathrm{i}\mathrm{d}\overline{z}_1 + \cdots + \mathrm{d}z_n \otimes -\mathrm{i}\mathrm{d}\overline{z}_n)(Y, X) \\ &= \frac{\mathrm{i}}{2}(\mathrm{d}z_1 \otimes \mathrm{d}\overline{z}_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}\overline{z}_n)(X, Y) - \frac{\mathrm{i}}{2}(\mathrm{d}\overline{z}_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}\overline{z}_n \otimes \mathrm{d}z_n)(X, Y) \\ &= \omega_0(X, Y). \end{aligned}$$

Now, computing

$$\begin{aligned} \omega_0^n &= \left(\frac{\mathrm{i}}{2}\right)^n \bigwedge^n \sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \\ &= \left(\frac{\mathrm{i}}{2}\right)^n \sum_{\substack{\text{perm.} \\ \{j_1, \ldots, j_n\} \in \{1, \ldots, n\}}} \bigwedge_{k=1}^n \mathrm{d}z_{j_k} \wedge \mathrm{d}\overline{z}_{j_k} \\ &= \left(\frac{\mathrm{i}}{2}\right)^n n! \bigwedge_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \\ &= \left(\frac{\mathrm{i}}{2}\right)^n n! (-1)^{\frac{n(n-1)}{2}} \bigwedge_{j=1}^n \mathrm{d}z_j \wedge \bigwedge_{k=1}^n \mathrm{d}\overline{z}_k \\ &= \left(\frac{\mathrm{i}}{2}\right)^n n! (-1)^{\frac{n(n-1)}{2}} \bigwedge_{j=1}^n \mathrm{d}z_j \wedge \overline{\bigwedge_{k=1}^n \mathrm{d}z_k} \end{aligned}$$

$$= \left(\frac{\mathrm{i}}{2}\right)^n n!(-1)^{\frac{n(n-1)}{2}} \Omega_0 \wedge \overline{\Omega_0}.$$

Therefore, using theorem (1.5), it follows that preserved g_0, ω_0, Ω_0 extend to constant tensors g, ω, and some Ω, respectively; furthermore, it is still naturally the case that, at each point

$$\omega^n = \left(\frac{\mathrm{i}}{2}\right)^n n!(-1)^{\frac{n(n-1)}{2}} \Omega \wedge \overline{\Omega}.$$

Lastly, the uniqueness of Ω up to multiplication by $\mathrm{e}^{\mathrm{i}\theta}$ for $\theta \in R$, follows from that fact that elements $A \in \mathrm{SU}_n(\mathbb{C})$ are, by definition, such that $\det A = 1$ and from the fact that $\Omega \in \Gamma(\Lambda^{n,0}M)$: A cannot scale the local basis, on which Ω acts; it can at most rotate the basis, which corresponds to a multiplication by $\mathrm{e}^{\mathrm{i}\theta}$, for $\theta \in \mathbb{R}$, on Ω.

As for the stated converse, it follows that Ω is a constant tensor with respect to the Levi-Civita connection ∇ of g; by theorem (1.5), this means that $\mathrm{Hol}(g)$ must stabilize Ω, which, in turn, naturally forces $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$. ■

2.3 Remark From this proposition, it follows that there is an equivalent definition of a Calabi-Yau n-fold (cf. definition (2.1)): a compact Kähler n-fold (M, g, J), which has a trivial canonical bundle. ◄

Now, with the next proposition, it follows that the metric of a Calabi-Yau n-fold, as defined, is Ricci-flat.

2.4 Proposition Let (M, g, J) be a Kähler n-fold. If $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$, then g is Ricci-flat. Conversely, if such a Kähler manifold is Ricci-flat (i.e. its metric is Ricci-flat) and is also simply-connected, then $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$.

[GHJ03, §I.4.5][Joy00, §6.1][Bou07, §3.1]

The proof uses the fact that, if $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$, the canonical bundle is trivial by proposition (2.2) and, thus, flat with respect to the connection induced ∇^K upon it by the Levi-Civita connection ∇ of g. Furthermore, it uses the fact that the curvature of ∇^K is some 2-form (specifically a $(1,1)$-form) with coefficients determined exactly by the Ricci curvature tensor associated to the connection of g. Hence, $\mathrm{Hol}(g) \subset \mathrm{SU}_n(\mathbb{C})$ forces g to be Ricci-flat, i.e. the Ricci tensor associated to ∇ vanishes.

2.5 Remark From here, it follows that defining Calabi-Yau n-fold as a compact Kähler n-fold (M, g, J), where g is Ricci-flat, is not equivalent to definition (2.1); in particular, by the proposition, it follows that equivalence is only guarenteed if M is assumed to be simply-connected. ◄

3 Generalized Calabi-Yau

In this section, the idea of generalized geometry from Hitchin [Hit02][Hit10], and Gualtieri [Gua03], will be introduced. Specifically, the sights will be set on the notion of a generalized Calabi-Yau manifold as discussed in Hitchin [Hit02]. As mentioned, generalizing a Calabi-Yau manifold comes from using the equivalent definition in remark (2.3).

Baring contrast to previous considerations involving the tangent bundle, in generalized geometry, considerations are with respect to the bundle $(TM \oplus T^*M) \otimes \mathbb{C}$. With this, the metric is replaced with an indefinite metric, which extends over $\mathbb{C}$, from the natural action of sections of T^*M on sections of TM. Then, the use of the Lie bracket on sections of TM translates to the analogous use of the Courant bracket on $TM \oplus T^*M$.

3.1 Definition Let $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$, smooth sections of $TM \oplus T^*M$. Then, the *Courant bracket* of $X+\alpha, Y+\beta$ is

$$[\![X+\alpha, Y+\beta]\!] := [X, Y] + \mathfrak{L}_X\beta - \mathfrak{L}_Y\alpha - \frac{1}{2}\mathrm{d}\left(\iota_X\beta - \iota_Y\alpha\right),$$

where $[\cdot,\cdot]$, and $\mathfrak{L}$, are the usual Lie bracket, and derivative, respectively, and ι is the usual interior product.

On the bundle $TM\oplus T^*M$, the natrual indefinite metric is given as follows, for $X+\alpha, Y+\beta\in\Gamma(TM\oplus T^*M)$,

$$\langle X+\alpha, Y+\beta\rangle := \frac{1}{2}\left(\beta(X)+\alpha(Y)\right)=\frac{1}{2}\left(\iota_X\beta+\iota_Y\alpha\right).$$

[Hit02, §3.1][Gua03, §3.2 & §2.2][Hit10, §§1.1-2]

With that, it will be shown next that $TM\oplus T^*M$ has a Clifford algebra representation on $\Lambda^\bullet T^*M$.

3.2 Proposition Define the action of $X+\alpha\in\Gamma(TM\oplus T^*M)$ on $\varphi\in\Gamma(\Lambda^\bullet T^*M)$ to be

$$(X+\alpha)\cdot\varphi=\iota_X\varphi+\alpha\wedge\varphi\in\Gamma\left(\Lambda^\bullet T^*M\right).$$

Then, with this action and with the relation $(X+\alpha)^2=\langle X+\alpha, X+\alpha\rangle$, $TM\oplus T^*M$ has a Clifford algebra represention $C\ell(TM\oplus T^*M)$ on $\Lambda^\bullet T^*M$.

[Hit02, §3.2][Gua03, §2.3][Hit10, §2.1]

Proof Looking at

$$\begin{aligned}
(X+\alpha)^2\cdot\varphi &= (X+\alpha)\left(\iota_X\varphi+\alpha\wedge\varphi\right) && \text{by definition}\\
&= \iota_X\left(\iota_X\varphi+\alpha\wedge\varphi\right)+\alpha\wedge\left(\iota_X\varphi+\alpha\wedge\varphi\right) && \text{by definition}\\
&= \underbrace{\iota_X\left(\iota_X\varphi\right)}_{\substack{0\\ \text{with antisymmetry}}}+(\iota_X\alpha)\varphi-\alpha\wedge\iota_X\varphi+\alpha\wedge\iota_X\varphi+\underbrace{\alpha\wedge\alpha\wedge\varphi}_{\substack{0\\ \text{with antisymmetry}}} && \text{since } \alpha\in\Gamma(T^*M)\\
&= (\iota_X\alpha)\varphi\\
&= \langle X+\alpha, X+\alpha\rangle\,\varphi && \text{by definition,}
\end{aligned}$$

showing that $\Lambda^\bullet T^*M$ is a module for the Clifford algebra of $TM\oplus T^*M$. ■

3.3 Remark Therewith, it can be shown that the natural choice[4] of *spinors* for $C\ell(TM\oplus T^*M)$ is the exterior algebra $\Lambda^\bullet T^*M$. For $\dim_{\mathbb{R}} TM=m=2n$, this corresponds to the spin respresentation

$$S=\Lambda^\bullet T^*M\otimes\left(\Lambda^n T^*M\right)^{-\frac{1}{2}}.$$

Letting $\Lambda^+T^*M, \Lambda^-T^*M$ be, respectively, the even- and odd-form parts of $\Lambda^\bullet T^*M$, this representation also has a(n) (invariant[5]) bilinear form: for $\varphi,\psi\in\Gamma(\Lambda^\bullet T^*M)$,

$$(\varphi,\psi) := \sum_{j=0}^{n}(-1)^j\left(\varphi_{2j}\wedge\psi_{m-2j}+\varphi_{2j+1}\wedge\psi_{m-2j-1}\right)\in\Gamma\left(\Lambda^n T^*M\right),$$

where a subscript k denotes taking the part of φ,ψ, which is a form of real degree k. Note that, in this construction, the degrees summed over is doubled when the bundle is complexfied, i.e. summed to $\dim_{\mathbb{R}}(TM\otimes\mathbb{C})=2\dim_{\mathbb{R}} TM=2m$, as to account for the new vector space dimension. ◀

3.4 Definition Let φ be a spinor, and let its annihilator be

$$E_\varphi := \left\{X+\alpha\in\Gamma\left(TM\oplus T^*M\right)\middle|(X+\alpha)\cdot\varphi=0\right\}.$$

Note that, via the multiplication rule[6] for Clifford algebras,

$$2\langle X+\alpha, Y+\beta\rangle\cdot\varphi=\left((X+\alpha)\left(Y+\beta\right)+\left(Y+\beta\right)(X+\alpha)\right)\cdot\varphi=0,$$

for any $X+\alpha, Y+\beta\in E_\varphi$. Since φ is non-trivial, this implies that $\langle X+\alpha, Y+\beta\rangle=0$ for any such $X+\alpha, Y+\beta$, which makes, by definition, E_φ *isotropic* with respect to $\langle\cdot,\cdot\rangle$.

The spinor φ is called *pure* if $\dim_{\mathbb{R}} E_\varphi=\dim_{\mathbb{R}} M$.

[Hit02, §3.3][Gua03, §2.5]

[4]For slightly-differing mentions of this, see [Hit02, §3.2] and [Hit10, §2.1]; this is discussed more so in [Gua03, §2.3 & §2.8]. The general references for spin are [Jos08, §1.11] and [LM89, §§I.1-8].

[5][Hit02, §3.2][Gua03, §2.4]

[6][Jos08, §1.11][Gua03, §2.5]

Finally, with that in the arsenal, the notion of a generalized complex structure can be defined.

3.5 Definition Let M be a smooth manifold of dimension $2n$, with the indefinite metric $\langle\cdot,\cdot\rangle$, as defined in definition (3.1), on its bundle $TM \oplus T^*M$. Then, a *generalized complex structure* on M is a subbundle $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ such that:

1.) $E \oplus \overline{E} = (TM \oplus T^*M) \otimes \mathbb{C}$, i.e. $\dim E = 2n$

2.) $\Gamma(E)$ is closed under the Courant bracket $[\![\cdot,\cdot]\!]$

3.) E is isotropic with respect to $\langle\cdot,\cdot\rangle$.

[Hit02, §4.1][Gua03, §4.2][nb.[7]]

Note the similarities[8] with a complex manifold M with a complexified tangent bundle $TM \otimes \mathbb{C}$: the decomposition in (1.) corresponds to $T^{1,0}M \oplus \overline{T^{1,0}M} = T^{1,0}M \oplus T^{0,1}M = TM \otimes \mathbb{C}$; the condition (2.) corresponds the integrability condition of complex structure on TM, i.e. $T^{1,0}M$ is closed with respect to the Lie bracket.

Finally, comes the notion of a generalized Calabi-Yau manifold, with the next definition.

3.6 Definition A *generalized Calabi-Yau manifold* is a smooth manifold M of real dimension $2n$ with a closed form φ in $\Gamma(\Lambda^+ T^*M)$ or $\Gamma(\Lambda^- T^*M)$, which is a (complex) pure spinor for $C\ell(TM \oplus T^*M)$ such that the bilinear form $(\varphi, \overline{\varphi}) \neq 0$.

[Hit02, §4.1]

Now, it will be shown that this definition does, as it suggests, yield a generalized complex structure on M.

3.7 Proposition Let M with φ be a generalized Calabi-Yau manifold of real dimension $2n$. Then, the annihilator subbundle $E_\varphi \subset (TM \oplus T^*M) \otimes \mathbb{C}$ is a generalized complex structure on M.

[Hit02, §4.1]

Proof First, note that, since φ is pure, E_φ is isotropic, satisfying condition (3.) of being a generalized complex structure. Furthermore, since $(\varphi, \overline{\varphi}) \neq 0$, it follows that, by the definition of the form $(\cdot,\cdot)$, that

$$E_\varphi \cap E_{\overline{\varphi}} = 0,$$

which, from the definition of an annihilator subbundle, makes

$$E_\varphi \cap \overline{E_\varphi} = E_\varphi \cap E_{\overline{\varphi}} = 0.$$

This with the fact that $\dim_{\mathbb{R}} E_\varphi = 2n = \dim_{\mathbb{R}} M$, since φ is pure, makes

$$E_\varphi \oplus \overline{E_\varphi} = (TM \oplus T^*M) \otimes \mathbb{C},$$

satisfying condition (1.) of being a generalized complex structure.

Lastly, it is needed to show that E_φ satisfies condition (2.). Let $X+\alpha, Y+\beta \in E_\varphi$, and consider the identity

$$\begin{aligned}
\iota_{[X,Y]}\varphi &= \mathfrak{L}_X(\iota_Y\varphi) - \iota_Y \mathfrak{L}_X \varphi \\
&= \mathfrak{L}_X \underbrace{\left((Y+\beta)\cdot\varphi - \beta\wedge\varphi\right)}_{\substack{\text{by action of } (Y+\beta) \\ \text{on } \varphi}} - \iota_Y \underbrace{\left(\mathrm{d}\iota_X\varphi + \iota_X \mathrm{d}\varphi\right)}_{\substack{\text{Cartan's magic} \\ \text{formula}}} \\
&= -\mathfrak{L}_X(\beta\wedge\varphi) - \iota_Y \mathrm{d}(\iota_X\varphi) && \text{using linearity, and } (Y+\beta)\cdot\varphi = 0 = \mathrm{d}\varphi
\end{aligned}$$

[7] From this definition, it follows that an endomorphism $\mathcal{J}$ on $\Gamma(TM \oplus T^*M)$ can be defined, analogous to the one for a regular complex structure, cf. definition (1.1).

[8] cf. [CdS01, §15.2][Mor07, §7.4]

$$
\begin{aligned}
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathfrak{L}_X\varphi - \iota_Y \mathrm{d}\underbrace{\big((X+\alpha)\cdot\varphi - \alpha\wedge\varphi\big)}_{\substack{\text{by action of } (X+\alpha)\\ \text{on } \varphi}} && \text{Leibniz rule of } \mathfrak{L}_X \text{ across } \wedge\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathfrak{L}_X\varphi + \iota_Y \mathrm{d}\left(\alpha\wedge\varphi\right) && \text{using linearity, and } (X+\alpha)\cdot\varphi = 0\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathfrak{L}_X\varphi + \iota_Y\left(\mathrm{d}\alpha\wedge\varphi\right) + \iota_Y\underbrace{\left(\alpha\wedge\mathrm{d}\varphi\right)}_{0} && \text{Leibniz rule of } d\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathfrak{L}_X\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi + \mathrm{d}\alpha\wedge\iota_Y\varphi && \text{property of interior product}\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathfrak{L}_X\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi + \mathrm{d}\alpha\wedge\underbrace{\left(-\beta\wedge\varphi\right)}_{\substack{\text{by the vanishing of}\\ \text{the action of } (Y+\beta)\\ \text{on } \varphi}}\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathrm{d}\left(\iota_X\varphi\right) + (\iota_Y\mathrm{d}\alpha)\wedge\varphi - \mathrm{d}\alpha\wedge\beta\wedge\varphi\\
&= -\mathfrak{L}_X\beta\wedge\varphi - \beta\wedge\mathrm{d}\left(-\alpha\wedge\varphi\right) + (\iota_Y\mathrm{d}\alpha)\wedge\varphi - \mathrm{d}\alpha\wedge\beta\wedge\varphi\\
&= -\mathfrak{L}_X\beta\wedge\varphi + \beta\wedge\mathrm{d}\alpha\wedge\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi - \mathrm{d}\alpha\wedge\beta\wedge\varphi\\
&= -\mathfrak{L}_X\beta\wedge\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi && \text{since } \mathrm{d}\alpha \text{ is a 2-form.}
\end{aligned}
$$

Using antisymmetry of the Lie bracket $[\cdot,\cdot]$ and what was just shown, it follows

$$
\begin{aligned}
\iota_{[X,Y]}\varphi &= \frac{1}{2}\left(\iota_{[X,Y]}\varphi - \iota_{[Y,X]}\varphi\right)\\
&= \frac{1}{2}\left(-\mathfrak{L}_X\beta\wedge\varphi + (\iota_Y\mathrm{d}\alpha)\wedge\varphi + \mathfrak{L}_Y\alpha\wedge\varphi - \left(\iota_X\mathrm{d}\beta\right)\wedge\varphi\right)\\
&= \frac{1}{2}\left(-\mathfrak{L}_X\beta + \iota_Y\mathrm{d}\alpha + \mathfrak{L}_Y\alpha - \iota_X\mathrm{d}\beta\right)\wedge\varphi\\
&= \frac{1}{2}\left(\left(-\mathrm{d}\iota_X\beta - \iota_X\mathrm{d}\beta\right) + \iota_Y\mathrm{d}\alpha + \left(\mathrm{d}\iota_Y\alpha + \iota_Y\mathrm{d}\alpha\right) - \iota_X\mathrm{d}\beta\right)\wedge\varphi\\
&= \left(\iota_Y\mathrm{d}\alpha - \iota_X\mathrm{d}\beta - \frac{1}{2}\left(\mathrm{d}\iota_X\beta - \mathrm{d}\iota_Y\alpha\right)\right)\wedge\varphi\\
&= \left(\mathfrak{L}_Y\alpha - \mathrm{d}\iota_Y\alpha - \mathfrak{L}_X\beta + \mathrm{d}\iota_X\beta - \frac{1}{2}\left(\mathrm{d}\iota_X\beta - \mathrm{d}\iota_Y\alpha\right)\right)\wedge\varphi\\
&= \left(\mathfrak{L}_Y\alpha - \mathfrak{L}_X\beta - \frac{1}{2}\left(\mathrm{d}\iota_Y\alpha - \mathrm{d}\iota_X\beta\right)\right)\wedge\varphi\\
\iff [\![X+\alpha, Y+\beta]\!]\cdot\varphi &= 0 \qquad\qquad \text{by definition of the Courant bracket,}
\end{aligned}
$$

showing that $[\![X+\alpha, Y+\beta]\!] \in E_\varphi$ by definition, which makes sections of E_φ closed under the Courant bracket; thus, condition (2.) is satisfied.

Therefore, E_φ is a generalized complex structure on M for such a φ. ■

To finish off, the following example will show that a Calabi-Yau manifold is naturally a generalized Calabi-Yau manifold, justifying the motivation defining it as in definition (2.1).

3.8 Example Let (M,g,J) be a Calabi-Yau n-fold, for $m = 2n$, as defined in definition (2.1); it follows from proposition (2.2), that there is an associated holomorphic volume form $\Omega \in \Gamma\left(\Lambda^{n,0}M\right)$ on M, which is naturally closed. Looking at E_Ω, it contains elements $X+\alpha$ such that $X \in \Gamma(TM\otimes\mathbb{C})$ of type $(0,1)$, since then $\iota_X\Omega = 0$, and $\alpha \in \Gamma(T^*M\otimes\mathbb{C})$ of type $(1,0)$, since then $\alpha\wedge\varphi = 0$, whence $(X+\alpha)\cdot\varphi = 0$ by definition. This, of course, makes $\dim_{\mathbb{R}} E_\Omega = 2n = \dim_{\mathbb{R}} M$, and, thusly, Ω is a pure spinor. Lastly, looking the necessary bilinear form

$$\left(\Omega,\overline{\Omega}\right) = (-1)^n\,\Omega\wedge\overline{\Omega},$$

since Ω is a form of real degree $2n$, so $\Omega_{2n} = \Omega$ and $\overline{\Omega}_{2n} = \overline{\Omega}$, and the real dimension of the complexified bundle is $4n$, so $\overline{\Omega}_{4n-2n} = \overline{\Omega}$. Moreover, this product is nonzero by the construction of Ω: $\left(\Omega,\overline{\Omega}\right) \neq 0$. Therefore, it follows from proposition (3.7), that M with such a form $\Omega = \varphi$ has a generalized complex structure given by E_Ω, making M also a generalized Calabi-Yau manifold. ⧓

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