Seminar on the h-Cobordism Theorem:
Existence of Morse Functions

Michael R. Jimenez

May 2, 2014

Abstract

This before you are notes for a talk (to be) given in a seminar on the h-Cobordism Theorem based on the text by J. Milnor of similar name, [Mil65]. Specifically, the notes focus on chapter 2 of the text, from ¶2.5 until its end; the topic is the existence of Morse functions on arbitrary cobordism $(W; V_0, V_1)$ with a smooth structure.

0 Recall

As the name suggests, this section will cover the needed context, and definitions, for the later discussion; this basically amounts to assuming [Mil65, §1], while quickly going over the beginning of [Mil65, §2] until ¶2.5 and including a lemma about compact refinements of coverings of paracompact spaces.

0.1 Definition (Milnor, ¶2.1, ¶2.3)

Let W be a compact smooth n-dimensional manifold in the triad $(W; V_0, V_1)$, with $V_0 \sqcup V_1 = \partial W$. Also, let $\mathbf{R}^n_+ := \left\{ \left(x^i\right)_i \in \mathbf{R}^n \middle| x^n \geq 0 \right\}$.

A *Morse function* is a smooth real function on a manifold without degenerate critical points. Specifically, a Morse function $f : W \longrightarrow I := [0,1]$ on W in a triad, where $\partial W \neq \varnothing$, is defined to be a smooth function, the critical points of which lie in $W \smallsetminus \partial W$ and are nondegenerate, and

$$f^{-1}(0) = V_0 \quad \text{and} \quad f^{-1}(1) = V_1.$$

When a function does not have degenerate critical points on some set, then it is simply called *suitable* there; naturally, this means Morse functions are exactly those functions which meet the necessary boundary conditions and are suitable on all their domain.

(A point p of a domain of a real function f is *critical* iff all of the first derivatives vanish, i.e. $\mathrm{T}_p f = 0$; a *critical value* is the image of such a point. It is *nondegenerate* iff the determinant of the Hessian of the function is non-zero, i.e. $\det\left(\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p \right)_{i,j} \neq 0$; this means $\left(\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p \right)_{i,j}$ is not singular.)

Then, recall the Morse lemma.

0.2 Lemma (Milnor, ¶2.2)

If p is a nondegenerate point critical point of an f, then there is a coordinate neighborhood about p such that f can be written locally as

$$f\left(x^i\right)_i = f(p) - \sum_{i=1}^{\lambda} \left(x^i\right)^2 + \sum_{i=\lambda+1}^{n} \left(x^i\right)^2,$$

where λ is the called the *index* of p.

As a consequence, it follows that the critical points of Morse functions on such W are isolated, forcing there to be finitely many of them by compactness. Therewith, the following definition becomes viable.

0.3 Definition (Milnor, ¶2.4)

The *Morse number* μ of the triad $(W; V_0, V_1)$ is the minimum, over all Morse functions on W, of the number of critical points.

Next, a lemma regarding compact refinements of coverings, which is used frequently in the later lemmas after recalling that compact Hausdorff implies normal paracompact — the former is the context of objects for this talk.

0.4 Lemma (Munkres [Mun66], ¶1.4)

Let $\{U_i\}_i$ be a covering of a normal paracompact space. Then, there is a locally-finite covering by compact sets $\{K_i\}_i$ such that $K_i \subset U_i$ for all i.

Lastly, there is a remark which will become pertinent when handling manifolds with boundary.

0.5 Remark

A smooth function on $\mathbf{R}^n_+$ is defined here to be one which extends smoothly to a neighborhood of $\mathbf{R}^n_+$ canonically embedded in $\mathbf{R}^n$.

Note that results pertaining to open sets of $\mathbf{R}^n_+$, which does not contain $\partial\mathbf{R}^n_+$, carry over to open sets of $\mathbf{R}^n$ because there is a natural homeomorphism $\mathbf{R}^n \cong \mathbf{R}^n_+ \smallsetminus \partial\mathbf{R}^n_+$. ◀

1 Existence

This section is the remainder of the notes, and it will be in a direction towards proving the existence of Morse functions, as stated in the following theorem of Morse.

1.1 Theorem (Milnor, ¶2.5)

For every triad $(W; V_0, V_1)$, there is a Morse function $f : W \longrightarrow I$.

$\mathscr{C}^2$-topology

The proof will be broken down into a series of lemmas, in which the heart of the approach will be to work with the so-called $\mathscr{C}^2$-topology on the space of real $\mathscr{C}^2$-functions $\mathscr{F}(W, \mathbf{R})$. The definition of this topology follows, after which the proof (lemma ¶1.4) of its independence of covering, and charts, is given via lemma ¶1.3.

1.2 Definition

To define the $\mathscr{C}^2$-topology on $\mathscr{F}(W, \mathbf{R})$, neighborhoods of an arbitrary element $f \in \mathscr{F}(W, \mathbf{R})$ will be defined.

Let $\{U_i\}_{i=1}^m$ be a finite subcover of W with charts $\varphi_i : U_i \longrightarrow \mathbf{R}^n_+$ for each i, and let $f_i := f \circ \varphi_i^{-1}$ for any i and any $f \in \mathscr{F}(W, \mathbf{R})$. Then, for any $\delta > 0$, define the δ-neighborhood of f as

$$N_\delta(f) := \left\{ g \in \mathscr{F}(W, \mathbf{R}) \mid \|f - g\|_{\mathscr{C}^2} < \delta \right\},$$

where $\|f - g\|_{\mathscr{C}^2} < \delta$ iff, for all $1 \le i \le m$ and $1 \le j, k \le n$,

$$|f_i - g_i| < \delta, \quad \left| \frac{\partial f_i}{\partial x^j} - \frac{\partial g_i}{\partial x^j} \right| < \delta, \quad \text{and} \quad \left| \frac{\partial^2 f_i}{\partial x^j \partial x^k} - \frac{\partial^2 g_i}{\partial x^j \partial x^k} \right| < \delta,$$

on all of $\operatorname{im}\varphi_i$.

This next lemma will be used to show that this topology is independent of choice of covering and charts.

1.3 Lemma (Milnor, "Lemma C")

Let $h : U \longrightarrow U'$ be a diffeomorphism of open subsets of $\mathbf{R}^n_+$, which maps a compact set $K \subset U$ onto $K' \subset U'$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that, if a map $f : U' \to \mathbf{R}$ is such that

$$\|f\|_{\mathscr{C}^2} < \delta,$$

on all of K', then

$$\|f \circ h\|_{\mathscr{C}^2} < \varepsilon,$$

on all of K. (Note in regards to $\|\cdot\|_{\mathscr{C}^2}$: the charts are simply the identity.)

Proof Here, the proof speaks for itself in its simplicity.

Let $\varepsilon > 0$ be given. First note that, when $|f| < \delta_0$ on K', then $|f \circ h| < \delta_0$ on K'; so, letting $\delta_0 < \varepsilon$ is sufficient for the first inequalities in each row. Next, by the chain rule, it follows

$$\left| \frac{\partial f \circ h}{\partial x^i} \right| = \left| \frac{\partial f \circ h}{\partial h^j} \frac{\partial h^j}{\partial x^i} \right|.$$

This means that, if

$$\left| \frac{\partial f}{\partial x^i} \right| < \delta_1$$

on K', then

$$\left|\frac{\partial f \circ h}{\partial x^i}\right| < \delta_1 \sigma_1$$

on K, where $\sigma_1 > 0$ is the bound of the derivative of h guaranteed by it being a diffeomorphism on the compact set K. Similarly, but also using the product rule,

$$\left|\frac{\partial^2 f \circ h}{\partial x^i \partial x^j}\right| = \left|\frac{\partial^2 f \circ h}{\partial x^i \partial h^k}\frac{\partial h^k}{\partial x^j} + \frac{\partial f \circ h}{\partial h^k}\frac{\partial^2 h^k}{\partial x^i \partial x^j}\right|;$$

so, if the previous inequalities hold and

$$\left|\frac{\partial^2 f}{\partial x^i \partial x^j}\right| < \delta_2,$$

then

$$\left|\frac{\partial^2 f \circ h}{\partial x^i \partial x^j}\right| < \delta_2 \sigma_1 + \delta_1 \sigma_2,$$

where $\sigma_2 > 0$ is the bound on the derivatives of h guaranteed by it being a diffeomorphism on the compact set K.

Finally, let $\delta = \max\{\delta_0, \delta_1, \delta_2\}$ and $\sigma = \max\{\sigma_1, \sigma_2\}$, and pick $\delta < \min\left\{\frac{\varepsilon}{2\sigma}, \varepsilon\right\}$ so that, on K,

$$|f \circ h| < \delta_0 \leq \delta < \varepsilon,$$

$$\left|\frac{\partial f \circ h}{\partial x^i}\right| < \delta_1 \sigma_1 \leq \delta\sigma < \varepsilon,$$

and

$$\left|\frac{\partial^2 f \circ h}{\partial x^i \partial x^j}\right| < \delta_2 \sigma_1 + \delta_1 \sigma_2 \leq 2\delta\sigma < \varepsilon. \qquad \blacksquare$$

And, finally, the lemma stating the independence.

1.4 Lemma

The $\mathscr{C}^2$-topology as defined in definition ¶1.2 is independent of the covering, and corresponding charts, of W.

Proof Let $\mathscr{T}$ be such a topology as defined by a cover of compact sets $\{K_i\}_{i=1}^m$ having neighborhoods $N_\delta(\cdot)$ with charts φ_i and norm $\|\cdot\|_{\mathscr{C}^2}$, and similarly let $\mathscr{T}'$ be a "primed" version defined by $\{K'_{i'}\}_{i'=1}^{m'}$ having $N'_{\delta'}(\cdot)$ with $\varphi'_{i'}$ and $\|\cdot\|'_{\mathscr{C}^2}$. It will be shown that $\mathscr{T} \supseteq \mathscr{T}'$, from which it will follow by the symmetry of the problem that the other inclusion also holds, showing that the topologies are actually equivalent.

For some $f \in \mathscr{F}(W, \mathbf{R})$, consider $N_\delta(f)$ and $N'_{\delta'}(f)$; an appropriate $\delta > 0$ will be found so that $N_\delta(f) \subseteq N'_{\delta'}(f)$. First note, that, for every i, there is an i' such that $K_i \cap K'_{i'} \neq \varnothing$, by the fact that the K_i and $K'_{i'}$ both cover W. Next note, that $\varphi'_{i'} \circ \varphi_i^{-1}$, for appropriate i' and i, is a diffeomorphism from a compact set $\varphi_i(K_i \cap K'_i) \subset \mathbf{R}^n_+$ to $\varphi'_{i'}(K_i \cap K'_{i'}) \subset \mathbf{R}^n_+$, by nature of them being charts. Finally note, that, for any $f \in \mathscr{F}(W, \mathbf{R})$,

$$f_i \circ \left(\varphi_i \circ \varphi'^{-1}_{i'}\right) = f \circ \varphi'^{-1}_{i'} =: f_{i'},$$

so

$$\left.\begin{aligned} \left|(f_i - g_i) \circ \left(\varphi_i \circ \varphi'^{-1}_{i'}\right)\right| &< \varepsilon \\ \left|\frac{\partial (f_i - g_i) \circ \left(\varphi_i \circ \varphi'^{-1}_{i'}\right)}{\partial x^j}\right| &< \varepsilon \\ \left|\frac{\partial^2 (f_i - g_i) \circ \left(\varphi_i \circ \varphi'^{-1}_{i'}\right)}{\partial x^j \partial x^k}\right| &< \varepsilon \end{aligned}\right\} \iff \|f - g\|'_{\mathscr{C}^2} < \varepsilon$$

by definition of $\|\cdot\|'_{\mathscr{C}^2}$ via the linearity of the derivatives. Thus, it follows from lemma ¶1.3, that, for each $\varepsilon > 0$, there is a $\delta > 0$ such that, when $\|f - g\|_{\mathscr{C}^2} < \delta$, then $\|f - g\|'_{\mathscr{C}^2} < \varepsilon$. Therefore, after letting $\varepsilon \leq \delta'$, it follows $N_\delta(f) \subseteq N'_{\delta'}(f)$, finishing the proof of the lemma as stated. $\blacksquare$

Focusing on $(W;\varnothing,\varnothing)$

Attention will first be focused to the case when W is closed, i.e. $(W;\varnothing,\varnothing)$; the motivation for this comes with the next lemma showing existence of a function without critical points on the boundary of W.

1.5 Lemma (Milnor, ¶2.6)
There is a function $f : W \longrightarrow I$ in a triad such that $f^{-1}(0) = V_0 \neq \varnothing$ and $f^{-1}(1) = V_1 \neq \varnothing$, and such that there are no critical points in some neighborhood of ∂W.

Proof The construction will be done explicitly: first locally with respect to some cover of W, and then glued together with a partition of unity subordinate to that cover.

Let $\{U_i\}_{i=1}^m$ be a finite subcover of W as guaranteed by its compactness, and let it be, without loss of generality, such that no U_i intersects both V_0 and V_1, for ease of construction. From the definition of W, there are charts $\varphi_i : U_i \longrightarrow \mathbf{R}_+^n$. Also, for ease of construction, assume the charts for $U_i \cap \partial W \neq \varnothing$ are maps to the n-disk in $\mathbf{R}_+^n$.

Now, if U_i does not intersection ∂W, then let $f_i \equiv \frac{1}{2}$. If U_i intersects V_j, $j \in \{0,1\}$, then let

$$f_i = j + (-1)^j \operatorname{pr}_n \varphi_i,$$

where $\operatorname{pr}_n : \mathbf{R}_+^n \longrightarrow \mathbf{R}_+$ is the projection to the n-coordinate. Note that this satisfies, for $p \in U_i \cap \partial W \neq \varnothing$,

$$f_i(p) = \begin{cases} \operatorname{pr}_n \varphi_i(p) = 0 & p \in U_i \cap V_0 \neq \varnothing \\ 1 - \operatorname{pr}_n \varphi_i(p) = 1 & p \in U_i \cap V_1 \neq \varnothing \end{cases}.$$

Next, extend the f_i trivially to $\widetilde{f}_i$ defined on all of W:

$$\widetilde{f}_i(p) := \begin{cases} f_i(p) & p \in U_i \\ 0 & p \in W \smallsetminus U_i \end{cases},$$

which is no longer continuous.

Finally, in order to define f in terms of these "local" $\widetilde{f}_i$, note that there is a partition of unity on W subordinate to $\{U_i\}_{i=1}^m$ by compactness, and thereby paracompactness, of W; call it $\{\sigma_i\}_{i=1}^m$. Then, define

$$f := \sum_{i=1}^m \sigma_i \cdot \widetilde{f}_i,$$

which is then a smooth function $W \longrightarrow \mathbf{R}_+$. Moreover, note that, by the construction of the f_i and definition of the σ_i, it follows that, for $p \in W \smallsetminus \partial W$,

$$f(p) = \sum_i \sigma_i(p) \widetilde{f}_i(p),$$

$$\forall i : 0 \leq \sigma_i(p) \leq 1 \quad \text{and} \quad 0 < \widetilde{f}_i(p) < 1,$$
$$\exists i : \sigma_i(p) \neq 0,$$

and thus,

$$f(p) \leq \left(\max_i \widetilde{f}_i(p)\right) \underbrace{\sum_i \sigma_i(p)}_{1} < 1 \quad \text{and} \quad f(p) \geq \left(\min_i \widetilde{f}_i(p)\right) \underbrace{\sum_i \sigma_i(p)}_{1} > 0,$$

i.e. $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$ — also using what was noted earlier.

Lastly, it is needed to check that f does not have critical points in a neighborhood of ∂W. Let $p_j \in V_j$ for $j \in \{0,1\}$, and let $\varphi_i : U_i \longrightarrow \mathbf{R}_+^n$ act as $p \longmapsto \left(x^i(p)\right)_i$. The differential is

$$\mathrm{T}_{p_j} f = \left(\sum_i \left.\frac{\partial \sigma_i}{\partial x^k}\right|_{p_j} f_i(p_j) + \sum_i \sigma_i(p_j) \left.\frac{\partial f_i}{\partial x^k}\right|_{p_j} \right)_{k=1}^n,$$

where, for all i,

$$f_i(p_j) = j \quad \text{and} \quad \left.\frac{\partial f_i}{\partial x^n}\right|_{p_j} = \left.\frac{\partial}{\partial x^n}\right|_{p_j} \left(j + (-1)^j \operatorname{pr}_n \varphi_i\right) = (-1)^j.$$

Notice that the n-th coordinate of $\mathrm{T}_{p_j} f$ is

$$
\begin{aligned}
j \sum_i \left.\frac{\partial \sigma_i}{\partial x^n}\right|_{p_j} + (-1)^j \underbrace{\sum_i \sigma_i\left(p_j\right)}_{1} &= j \left.\frac{\partial}{\partial x^n}\right|_{p_j} \underbrace{\sum_i \sigma_i}_{\equiv 1} + (-1)^j \\
&= (-1)^j \\
&\implies \mathrm{T}_{p_j} f \neq 0.
\end{aligned}
$$

Hence, there are no critical points of f on ∂W, and furthermore, no critical points in a neighborhood of ∂W by the continuity of f. ∎

Making/finding suitable functions

To start off, there is a lemma is of Morse that says that there are "plenty" of (linear) functions, which can be subtracted from an f to resolve degenerate critical points, making the resulting function suitable.

1.6 Lemma (Milnor, "Lemma A")
Let $U \subseteq \mathbf{R}^n_+ \subset \mathbf{R}^n$ be open, and let $f : U \longrightarrow \mathbf{R}$ be a $\mathscr{C}^2$-map. Then, outside a set of measure zero in $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}) \cong \mathbf{R}^n$, there is an $L \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$ such that $f - L$ is suitable on U.

Proof The idea of the proof is to consider a graph to $L \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$ over U defined by those L such that $f - L$ achieves a critical point,

$$\mathrm{T}\left(f - L\right) = 0 \iff L = \mathrm{T}L = \mathrm{T}f,$$

and then, use Sard's theorem on the projection of the graph to $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$ to show the conclusion.

First, look at the set which forms that graph:

$$M := \left\{(x, L) \in U \times \mathrm{Hom}_{\mathbf{R}}\left(\mathbf{R}^n, \mathbf{R}\right) \middle| \mathrm{T}_x\left(f - L\right) = 0\right\},$$

i.e. points of the form $\left(x, \mathrm{T}_x f\right)$. Now, notice the clever bit: the degenerate points of $f - L$ are now exactly those which are the critical points of the projection

$$
\begin{aligned}
\mathrm{pr}_2 : M &\longrightarrow \mathrm{Hom}_{\mathbf{R}}\left(\mathbf{R}^n, \mathbf{R}\right) \\
\left(x, \mathrm{T}_x f\right) &\longmapsto \mathrm{T}_x f
\end{aligned},
$$

since

$$\mathrm{T}_{(x, \mathrm{T}f_x)}\, \mathrm{pr}_2 = \left(\underbrace{0 \quad \dots \quad 0}_{n \times n} \quad \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right)_x \quad \text{is singular iff} \quad \left.\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right|_x \quad \text{is singular.}$$

Finally, before using Sard's, note that pr_2 is a $\mathscr{C}^1$-map, since f is $\mathscr{C}^2$, where $1 \geq \max(1 - n + 1, 1)$. Thus, by Sard's, the critical points of pr_2 are a set of measure zero in M. Moreover, by the continuity of pr_2, this means the critical values of pr_2 are also of measure zero (in $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$); these are precisely those maps $L : \mathbf{R}^n \longrightarrow \mathbf{R}$ for which $f - L$ has a degenerate critical point. Therefore, for $L \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$ outside that set of measure zero, $f - L$ must be suitable on U. ∎

Next is a lemma that says that functions "near enough" (i.e. in the $\|\cdot\|_{\mathscr{C}^2}$-sense) to suitable functions, are suitable themselves.

1.7 Lemma (Milnor, "Lemma B")
Let K be a compact subset of an open subset U of $\mathbf{R}^n_+$, and let $f : U \longrightarrow \mathbf{R}$ be $\mathscr{C}^2$-map suitable on K. Then, there is a $\delta > 0$ small enough such that, if a $\mathscr{C}^2$-map $g : U \longrightarrow \mathbf{R}$ is, at all points in K, such that

$$\left|\frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i}\right| < \delta \tag{1}$$

$$\left|\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 g}{\partial x^i \partial x^j}\right| < \delta, \tag{2}$$

for all $i, j \in \{1, \dots, n\}$, then g is also suitable on K. Moreover, if f does not have any critical points on K, then neither does g.

Proof This is proved by simply forming a series of inequalities stemming off the fact that f is suitable on K iff, on K,

$$|\mathrm{T}f| + \left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right| > 0.$$

As such, it suffices to find a similar inequality: one with g instead of f. To do this, bounds on the corresponding terms in each will be found.

Let $\varepsilon > 0$ be the minimum of the previous expression over K; this inherently uses the compactness of K since the minimum is the attained infimum over K. Then, assuming (1) for some $\delta > 0$ to be set later,

$$\left.\begin{aligned} |\mathrm{T}f| - |\mathrm{T}g| &\le |\mathrm{T}f - \mathrm{T}g| \\ -|\mathrm{T}f| + |\mathrm{T}g| &\le |\mathrm{T}f - \mathrm{T}g| \end{aligned}\right\} \implies ||\mathrm{T}f| - |\mathrm{T}g|| \le |\mathrm{T}f - \mathrm{T}g|,$$

so

$$\begin{aligned} ||\mathrm{T}f| - |\mathrm{T}g|| \le |\mathrm{T}f - \mathrm{T}g| &= \left(\sum_i \left(\frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i}\right)^2\right)^{\frac{1}{2}} \\ &\le \sum_i \left|\frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i}\right| \\ &< n\delta. \end{aligned}$$

Similarly, assuming (2),

$$\begin{aligned} \left|\left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right| - \left|\det\left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right)_{i,j}\right|\right| &\le \left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j} - \det\left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right)_{i,j}\right| \\ &< C\delta, \end{aligned}$$

for some $C > 0$. Thus,

$$|\mathrm{T}g| + \left|\det\left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right)_{i,j}\right| > |\mathrm{T}f| - n\delta + \left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right| - C\delta.$$

Finally, for the punchline inequality, let

$$\delta < \frac{\varepsilon}{2C'},$$

where $C' = \max\{n, C\}$, so that it falls together as

$$|\mathrm{T}f| - n\delta + \left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right| - C\delta > |\mathrm{T}f| - \frac{\varepsilon}{2} + \left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}\right| - \frac{\varepsilon}{2} \ge 0,$$

where the last inequality follows from the definition of ε. Therewith, it follows that such a g is like f: suitable on K.

Lastly, the other conclusion follows immediately from (1) for a small enough $\delta > 0$, via the continuity of the derivatives of both f, and g, on the compact K. ■

Proof for $(W; \varnothing, \varnothing)$

1.8 Theorem (Milnor, ¶2.7)
If W be a closed manifold, then the Morse functions on W are an open dense subset of $\mathscr{F}(W, \mathbf{R})$ as per the $\mathscr{C}^2$-topology.

Proof Let $\{U_i\}_i^m$ be a finite covering of W with corresponding charts $\varphi_i : U_i \longrightarrow \mathbf{R}^n$; as usual, via lemma ¶0.4, let $\{K_i\}_i$ be the compact subordinate refinement. This result comes from use of lemma ¶1.6, and lemma ¶1.7, in induction over the cover $\{K_i\}_i$. In particular, to show it is dense, the method will be to use lemma ¶1.6 on some K_i to find a nearby suitable function there, and then, inductively use lemma ¶1.7 to find suitable functions nearby that, until it is Morse.

First, it will be shown that the subset is open. Let $f : W \longrightarrow \mathbf{R}$ be a Morse function. It follows from lemma ¶1.7 that there is a $\mathscr{C}^2$-neighborhood N_i of f in $\mathscr{F}(W, \mathbf{R})$, which contains functions that are suitable on K_i. Thus, the open $N_f = \bigcap_i N_i$ contains functions which are suitable on $\bigcup_i K_i = W$, i.e. which are Morse.

Therewith, the Morse functions must be an open set of $\mathscr{F}(W,\mathbf{R})$ as it is the union of such open neighborhoods of its elements.

Last, it is to be shown that the subset is dense; for this, it will suffice to show that there is a Morse function in any neighborhood of any element in $\mathscr{F}(W,\mathbf{R})$. Let N be any neighborhood of some $f \in \mathscr{F}(W,\mathbf{R})$; without loss of generality, it maybe assumed it is of the form $N_\delta(f) = N$ as used to define the $\mathscr{C}^2$-topology in definition ¶1.2.

Now comes the first inductive step. Let $\sigma : W \longrightarrow I$ be a bump function that is unit on K_1 while having its support on the neighborhood U_1 of K_1. With that, it follows from lemma ¶1.6 that there is a linear map $L : \mathbf{R}^n \longrightarrow \mathbf{R}$ such that

$$f' := f - \sigma \cdot L \circ \varphi_1$$

is suitable on K_1. The perturbation is bounded on K_1 as

$$\left| \sigma \sum_{i=1}^{n} l_i x^i \right| \leq \left| \sum_{i=1}^{n} l_i x^i \right|$$

since $0 \leq \sigma \leq 1$, and there are similar inequalities for the derivatives. By the compactness of K_i and the smoothness of σ, it follows that, for small enough $|l_i|$, $\|f - f'\|_{\mathscr{C}^2} < \delta$ for any $\delta > 0$. The existence of an appropriate L meeting these constraints is guaranteed by the fact that such L are almost everywhere in $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R})$ by lemma ¶1.6. Thus, there is an $f' \in N$ which is suitable on K_1. Then, by lemma ¶1.7, there is a small-enough neighborhood N' of f', which can be assumed to be in N without loss of generality, such that all functions in N' are suitable on K_1.

To finish, induction goes over the remaining K_i: linearly perturb $f^{(j)}$ to $f^{(j+1)}$ by an $L^{(j)}$ multiplied with a $\sigma^{(j)}$, so that $f^{(j+1)}$ is suitable on K_{j+1}, and on $\bigcup_{i=1}^{j} K_i$ by staying in $N^{(j)}$; then, find a smaller neighborhood $N^{(j+1)}$ of such suitable functions. At the end of this, there is a function $f^{(m)}$ suitable on $\bigcup_i K_i = W$, i.e. it is Morse, which is contained in a neighborhood $N^{(m)}$ of similarly suitable, i.e. Morse, functions. Therefore, the set of Morse functions on W is an open dense subset of $\mathscr{F}(W,\mathbf{R})$. ■

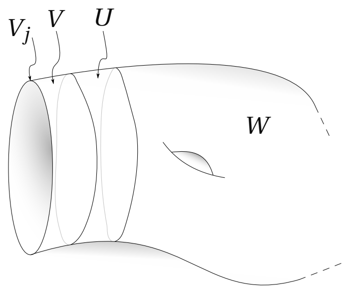
General existence

Proof of theorem ¶1.1

As discussed earlier, this proof will start by using lemma ¶1.5 to get a Morse function, with which to start off; this function will then be perturbed as it was in the proof of the closed case in theorem ¶1.8.

Using lemma ¶1.5, it follows that there is a function $f : W \longrightarrow I$ on the triad $(W; V_0, V_1)$, which is suitable on some neighborhood of $\partial W = V_0 \sqcup V_1$ — more specifically, it is without critical points there — and is such that $f^{-1}(j) = V_j$ for $j \in \{0, 1\}$. Since these qualities are wanted in the final Morse function, as per the prescription of definition ¶0.1, it is need to ensure that the perturbations as done in the proof of the closed case does not affect f in some neighborhood of ∂W.

To this end, let U be an open neighborhood of ∂W which is without critical points, and let V be an open neighborhood of ∂W such that $\overline{V} \subset U$; the subset $U \smallsetminus \overline{V} \subset W$ will serve as a mediating border between the critical-point-free domain of f, which already meets the needed boundary conditions, and the rest of the interior of W. This border will be used to find a cover of W which is easier to manage.



Let $\{U_i\}_{i=1}^m$ be a finite subcover of W, which is such that each U_i is contained in $W \smallsetminus \overline{V}$ or U. Then, let $\{K_i\}_{i=1}^m$ be one of its subordinate compact refinements via lemma ¶0.4. Using this, it follows that there are compact sets K_i which lie in U; the union K of such sets is also compact. Thus, by lemma ¶1.7, there is a neighborhood $N_\delta(f) \subseteq \mathscr{F}(W,\mathbf{R})$ containing functions which also do not have critical points in K. Furthermore, by the construction of f, it follows that $f|_{W \smallsetminus V}$ has an image contained in $(\varepsilon, 1-\varepsilon)$ for some $\varepsilon > 0$. Thus, upon picking a $0 < \delta' < \varepsilon$, there is an $N_{\delta'}(f) \subseteq N_\delta(f)$ such that, for each $g \in N_{\delta'}(f)$, the image of $g|_{W \smallsetminus V}$ is also contained in $(\varepsilon, 1-\varepsilon)$, so that it does not map to the boundary of I.

Next, assume that the $\{U_i\}_{i=1}^{m'}$ are the coordinate neighborhoods contained in $W \smallsetminus \overline{V}$, and let $N = N_{\delta'}(f)$. Using the method from the proof of theorem ¶1.8, on $f \in N$ suitable on K, it follows that there is a function $f^{(m')}$, which is suitable on $K \cup \bigcup_{i=1}^{m'} K_i = W$, and contained in a neighborhood $N^{(m')}$ of similarly suitable functions. Moreover, note that, since $N^{(m')} \subseteq N$, it follows that such functions are also without critical points in $K \supset \partial W$, and in particular, $f^{(m')}\Big|_V = f|_V$ since the perturbations occur only on $W \smallsetminus \overline{V}$. Therefore, $f^{(m')}$ is a Morse function on W in the triad $(W; V_0, V_1)$. Note that the other functions in $N^{(m')}$ maybe not meeting the boundary conditions since there is an allowed deviation to outside of $I \subset \mathbf{R}$. ■

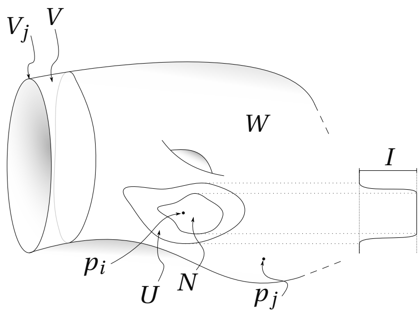
1.9 Remark (Milnor, pg.17)
In theorem ¶1.8, more was shown than in proof of theorem ¶1.1, namely that the set of Morse functions was dense in $\mathscr{F}(W, \mathbf{R})$ when W is closed. A similar result holds for the general case with maps with the imposed boundary conditions in the space $\mathscr{F}((W; V_0, V_1), (I; \{0\}, \{1\}))$ with the norm defined analogously to the one defined in definition ¶1.2 ◄

Ironing out a Morse function

To end this talk, a few results from the end of §2 of Milnor will be discussed — particularly, the result about "ironing out" a Morse function's critical points across the level sets will be of use later, in decomposing cobordisms. This is done precisely in the following lemma.

1.10 Lemma (Milnor, ¶2.8)
Let f be a Morse function on W^n in a triad, which has critical points $\{p_i\}_{i=1}^{m}$. Then, there is another Morse function g which is a perturbation of f, having the same critical points but each on different levels, i.e. $g(p_i) \neq g(p_j)$, for any distinct i, j.

Proof A method will be prescribed here to perturb f locally around a critical point, as needed to "iron out" its image.
Assume that there are distinct p_i, and p_j, such that $f(p_i) = f(p_j)$. Since f is Morse, there is a neighborhood $V \supset \partial W$, on which f is without critical points. By the Hausdorff, and normal, properties of the compact W, there is a neighborhood U of p_i such that $\overline{U}$ contains no other critical points of f, and is disjoint from V. Let $N \subset U$ be a neighborhood of p_i; then, let $\sigma : W \longrightarrow I$ be a bump function with support in U such that $\sigma(N) = 1$.



For some $\varepsilon' > 0$ yet to be tuned, it follows that

$$(f + \varepsilon'\sigma)(p_i) = f(p_i) + \varepsilon' \neq f(p_j) = (f + \varepsilon'\sigma)(p_j),$$

while it is still the case that they are critical points,

$$\mathrm{T}_{p_i}(f + \varepsilon'\sigma) = \mathrm{T}_{p_i} f + \varepsilon' \mathrm{T}_{p_i}\sigma = \mathrm{T}_{p_i} f = 0 = \mathrm{T}_{p_j} f = \mathrm{T}_{p_j} f + \varepsilon' \mathrm{T}_{p_j}\sigma = \mathrm{T}_{p_j}(f + \varepsilon'\sigma),$$

since $\mathrm{T}\sigma|_N = 0 = \mathrm{T}\sigma|_{W \smallsetminus U}$ by construction; likewise, note that those critical points are also not made degenerate, since the second-derivatives of σ also vanish there. Now, note that $f(W \smallsetminus V) \subset (\delta, 1 - \delta)$ for some $\delta > 0$ by continuity; thus, setting $\varepsilon' < \delta$ maintains the boundary conditions for $f + \varepsilon'\sigma$.

Next, it is needed to check that there are no new critical points; since σ is constant on N and $W \smallsetminus U$, it is just needed to check on $U \smallsetminus N$. It is already the case that $\mathrm{T}f|_{\overline{U \smallsetminus N}}$ is non-zero. This will be done by finding an appropriate $\varepsilon_k > 0$ in a coordinate neighborhood V_k, and then, across neighborhoods via chart transition; in this, let $\{V_k\}_{k=1}^{N}$ be a finite subcover of $\overline{U \smallsetminus N}$, as given by restriction of charts of W.

First, in some chart $\varphi_k : V_k \longrightarrow \mathbf{R}^n$, the differential can be written explicitly as

$$\left(\frac{\partial f + \varepsilon'\sigma}{\partial x^i}\right)_{i=1}^{n} = \left(\frac{\partial f}{\partial x^i} + \varepsilon' \frac{\partial \sigma}{\partial x^i}\right)_{i=1}^{n},$$

where, as stated before,

$$\left(\frac{\partial f}{\partial x^i}\right)_{i=1}^n \neq 0.$$

Each of these differentials can be viewed as elements of $\operatorname{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}) \cong \mathbf{R}^n$, namely as vectors; in this way, it is possible to consider the magnitude of the differentials, in the $\mathbf{R}^n$-sense. Using this, it is possible to then pick an ε_k, where $0 < \varepsilon_k < \varepsilon'$ such that

$$\inf_{p \in V_k} \left\| \left(\frac{\partial f}{\partial x^i}\right)_{i=1}^n \right\| > \varepsilon_k \sup_{p \in V_k} \left\| \left(\frac{\partial \sigma}{\partial x^i}\right)_{i=1}^n \right\|,$$

because f is smooth and its differential is non-zero on the compact set $\overline{U \smallsetminus N} \supseteq V$; such an ε_k suffices for

$$\left(\frac{\partial f}{\partial x^i} + \varepsilon_k \frac{\partial \sigma}{\partial x^i}\right)_{i=1}^n \neq 0.$$

Note now, that the differential of a chart transition $\varphi \circ \psi^{-1} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a non-singular matrix, since it is necessarily a diffeomorphism, and that the differential in this new chart is

$$\left(\frac{\partial f}{\partial y^i} + \varepsilon_k \frac{\partial \sigma}{\partial y^i}\right)_{i=1}^n = \left(\frac{\partial f}{\partial x^j} + \varepsilon_k \frac{\partial \sigma}{\partial x^j}\right)_{j=1}^n \cdot \left(\frac{\partial x^j}{\partial y^i}\right)_{i,j=1}^n,$$

which is similarly non-zero, since ε_k was chosen so that $\left(\frac{\partial f}{\partial x^j} + \varepsilon_k \frac{\partial \sigma}{\partial x^j}\right)_{j=1}^n \neq 0$ and $\left(\frac{\partial x^j}{\partial y^i}\right)_{i,j=1}^n$ is a non-singular matrix by means of being the differential of the chart transition. Lastly, taking the minimum of such ε_k over the finite subcover $\{V_k\}_{k=1}^N$,

$$\varepsilon = \min\{\varepsilon_k\}_{k=1}^N,$$

is sufficient so that

$$\mathrm{T}\left(f + \varepsilon\sigma\right)\big|_{\overline{U \smallsetminus N}} \neq 0,$$

ensuring that there are no critical points there by definition.

Hence, the function $f + \varepsilon\sigma$ is as needed with regards to p_i and p_j. Therewith, after repeating this method at most finitely many times, the resulting perturbed function g will not have critical points with the same critical values. ■

Decomposing cobordisms using Morse functions

1.11 Lemma (Milnor, ¶2.9 & Corollary ¶2.10)

Let $f : W \longrightarrow I$ be a Morse function in a triad, and let $x \in I \smallsetminus \partial I = (0,1)$ be a non-critical value of f. Then, $f^{-1}([0,x])$ and $f^{-1}([x,1])$ are both manifolds with boundary. Furthermore, after a possible perturbation f, it is possible to decompose W into cobordisms such that the restriction of this function to each has Morse number $\mu = 1$.

Proof The result simply follows from use of the rank theorem [Spi65, ¶2-13] used on f composed with some chart of W. Explicitly, the theorem gives a way of finding charts, which can easily be broken into charts of the new boundary $f^{-1}(x)$.

Let $x \in (0,1)$ be a non-critical value of $f : W \longrightarrow \mathbf{R}$, and let $f^{-1}(x) = M$, where $p \in M$ is arbitrary. It suffices to find charts around points in M, in which M is a new boundary, i.e. the charts take M to the boundary of $\mathbf{R}^n_+ \subset \mathbf{R}^n$.

Let U_p be a coordinate neighborhood of p, with corresponding chart $\varphi_p : U_p \longrightarrow \mathbf{R}^n$. Since x is not a critical point of f, it follows that $\mathrm{T}_p f$ is non-zero and, thus, of rank one. Therewith, the rank theorem can be used to show that there is a neighborhood $U'_p \subseteq U_p$ of p, and a smooth map $\psi_p : \varphi_p\left(U'_p\right) \longrightarrow \mathbf{R}^n$ with smooth inverse, such that

$$f \circ \varphi_p^{-1} \circ \psi_p^{-1}\left(x^i\right)_{i=1}^n = x^n \in \mathbf{R}.$$

Now, let $\tau^{\pm} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be the shifts of the n-th coordinate $x^n \longmapsto \pm x^n \mp x$, and define

$$\Phi_p^- := \tau^- \circ \psi_p \circ \varphi_p : U'_p \longrightarrow \mathbf{R}^n \quad \text{and} \quad \Phi_p^+ := \tau^+ \circ \psi_p \circ \varphi_p : U'_p \longrightarrow \mathbf{R}^n,$$

which are exactly a charts around p such that

$$\Phi_p^{\pm}\left(U'_p \cap f^{-1}(x)\right) = \left\{(x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; x^n = 0\right\}$$

while

$$\Phi_p^-\left(U_p' \cap f^{-1}([0,x])\right) \subseteq \mathbf{R}_+^n \quad \text{and} \quad \Phi_p^+\left(U_p' \cap f^{-1}([x,1])\right) \subseteq \mathbf{R}_+^n.$$

Doing this for each $p \in M$, an atlas for each of $f^{-1}([0,x])$, and $f^{-1}([x,1])$, respectively, can be obtain by taking the charts for each corresponding $p \in W \smallsetminus M$ that does not intersect M, and then taking the charts Φ_p^-, and Φ_p^+, respectively, for $p \in M$. Hence, W is decomposable into two manifolds with a shared boundary M. Furthermore, if f is a Morse function, achieving the Morse number of critical points for W in its triad, and having each of its critical point with a unique critical value, as constructed in lemma ¶1.10, then it is possible to decompose W "in between" each of its finitely-many critical points, making triads each with Morse number $\mu = 1$, by definition. ■

Outlook: Handlebody decompositions

In the coming talks for this seminar, the last two lemmas will be useful for developing the notion of the handlebody decomposition of a manifold, namely a way of finding some sort of atomos of a manifold which can be glued together to form the manifold again.

References

[Mil65] John Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965. MR 0190942 (32 #8352)

[Mun66] James R. Munkres, *Elementary differential topology*, Lectures given at Massachusetts Institute of Technology, Fall, vol. 1961, Princeton University Press, Princeton, N.J., 1966. MR 0198479 (33 #6637)

[Spi65] Michael Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965. MR 0209411 (35 #309)