How a, why a Fukaya?

Fukaya Talk I

Michael Robert Jimenez

Low-Dimensional Topology Seminar:
Fukaya Categories

Talk Outline

- Floer setting
- Operations on Floer complexes, the μ^k on the $CF(\cdot, \cdot)$
- A_∞-relations on those operations
- Definition of Fukaya categories

Floer Setting

Symplectic Context

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Let (M, ω) be a symplectic manifold, wherein L_j are "suitable" Lagrangian submanifolds.

Here, "suitable" loosely means that the L_j should:

- intersect transversely, so that the moduli spaces to be considered are actually manifolds
- not bound a holomorphic disk, so that there are indeed chain complexes, i.e. $\partial^2 \stackrel{!}{=} 0$.

Some of the subtleties of this "suitability" will be discussed later.

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Floer Chain Complexes

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Recall that $\mathrm{CF}(L_j, L_\ell)$ is the *Floer complex* associated to the Lagrangians L_j, L_ℓ, which is a free Λ-module, and Λ is a Novikov ring over some field $\mathbb{K}$.

After picking an ω-compatible almost-complex structure J on M, it is possible to define the *Floer* differential on $\mathrm{CF}(L_j, L_\ell)$ by counting pseudo-holomorphic maps u from a strip (or a disk with two marked points) into M with boundary in $L_j \cup L_\ell$, meeting the intersection points $p, q \in \mathcal{X}(L_j, L_\ell)$. These u are such that $\overline{\partial}_J u = 0$, having finite symplectic area $\int u^*\omega < \infty$.

Figure 2. A pseudo-holomorphic strip contributing to the Floer differential on $CF(L_0, L_1)$ [Auroux, pg. 4]

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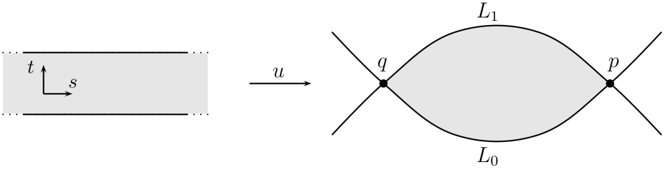

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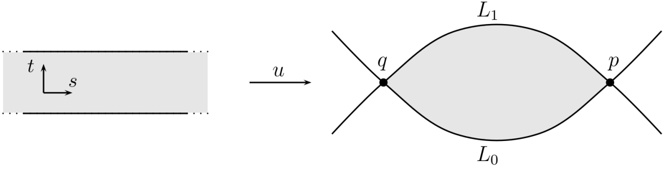

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Floer Differential and Product

Floer differential, a.k.a. μ^1

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One operation on the Floer complexes has already been introduced, namely the Floer differential

$$\partial\colon \ \mathrm{CF}\left(L_0, L_1\right) \longrightarrow \mathrm{CF}\left(L_0, L_1\right) \quad ,$$

which is a Λ-linear maps taking $p \in \mathcal{X}\left(L_0, L_1\right)$ to

$$\mu^1\left(p\right) := \partial p := \sum_{\substack{q \in \mathcal{X}(L_0, L_1) \\ [u]\colon\ \mathrm{ind}[u]=1}} \#\mathcal{M}\left(p, q; [u], J\right)\ T^{\omega[u]} q \quad ,$$

wherein $\#\mathcal{M}\left(p, q; [u], J\right) \in \mathbb{Z}$ (or respectively $\mathbb{Z}/2$) is the signed (or unsigned) count of points in the moduli space of pseudo-holomorphic strips connecting p to q in the homotopy class $[u]$, and $\omega\left[u\right] := \int u^*\omega$.

Floer product, a.k.a. μ^2

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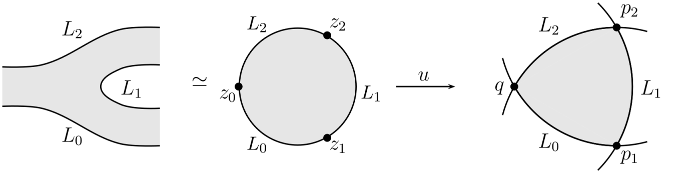
Incrementing the number of intersecting Lagrangians, enables a product to be defined: this would be a map

$$\mathsf{CF}(L_1, L_2) \otimes \mathsf{CF}(L_0, L_1) \longrightarrow \mathsf{CF}(L_0, L_2)$$

and the expected behavior is that is takes two intersection points $p_1 \in \mathcal{X}(L_0, L_1)$, $p_2 \in \mathcal{X}(L_1, L_2)$ returns a weighted sum of the count of pseudo-holomoprhic maps of disks with 3 marked points to M with boundary on $L_0 \cup L_1 \cup L_2$, meeting p_1, p_2, q.



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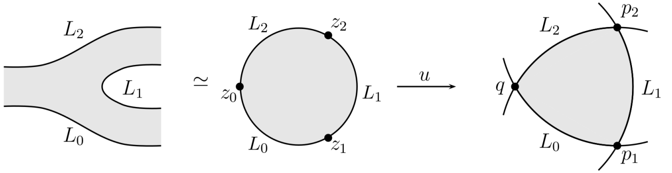

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FLOER PRODUCT, A.K.A. μ^2

To this end, let $\mathcal{M}(p_1, p_2, q; [u], J)$ be the moduli space of such (finite-energy) maps.

The dimension of this space is the index of the linearized Cauchy-Riemann operator $\mathrm{D}_{\bar{\partial}_j, u}$, which is expressible in terms of the Maslov index, just as in the case with just two Lagrangians:

- obtain a closed loop in $\mathrm{LGr}(M)$ with Maslov index $\operatorname{ind} u$ by going around the boundary of u, taking canonical short paths at p_1, p_2, q
- this is done by choosing a map $\alpha\colon \ \mathrm{LGr}(M) \longrightarrow U(1) \cong \mathbb{S}^1$ (since $\mathrm{LGr}(n) \cong U(n)/O(n)$, this is like a fibrewise $\det^2$), a trivialization of the square of the canonical bundle, or equivalently, $2c_1(\mathrm{T}M) = [0]$
- the L_j should have vanishing Maslov class $\mu_L \in \mathrm{H}^1(L, \mathbb{Z})$, which is the same the homotopy class of $\alpha|_L$.

The last two points are particularly so that there are $\mathbb{Z}$-gradings on the complexes.

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This obtains

$$\text{ind}\, u = \deg q - \deg p_1 - \deg p_2 \quad ,$$

which is not symmetric in q, p_2.

The reason for this is due to the difference in the gradings of $\mathrm{CF}(L_0, L_2)$ and $\mathrm{CF}(L_2, L_0)$, of which both are generators. In this way, the complexes $\mathrm{CF}(L_0, L_2)$, $\mathrm{CF}(L_2, L_0)$ are dual to each other, with dual differentials, given suitable almost-complex structures and perturbations of the L_j.

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A few notes on the moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$:

- It is smooth manifold if the L_j intersect transversely.
- In general, the transversality the L_j needs to be achieved using domain-dependent almost-complex structures and Hamiltonian perturbations.
- If char $\mathbb{K} \neq 2$, then its first Stiefel-Whitney $w_1(\mathcal{M})$ depends on the $w_2(L_j)$ and the Maslov classes μ_{L_j}. So an easy way to get its orientability is to have the $w_2(L_j)$, μ_{L_j} vanish. (see Seidel)
- Further, if the $w_1(L_j)$ also vanish, then the L_j admit spin structures.

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With those assumptions (particularly transversality), it is possible to make the following definition.

Definition [Auroux, 2.2]

The *Floer product* is the Λ-linear map

$$\mu^2 \colon \; \mathrm{CF}\,(L_1, L_2) \otimes \mathrm{CF}\,(L_0, L_1) \longrightarrow \mathrm{CF}\,(L_0, L_2)$$

defined as

$$\mu^2\,(p_2, p_1) := p_2 \cdot p_1 := \sum_{\substack{q \in \mathcal{X}(L_0, L_2) \\ [u]\colon\ \mathrm{ind}[u]=0}} \#\mathcal{M}\,(p_1, p_2, q; [u]\,, J)\ T^{\omega[u]} q \quad ,$$

in similar fashion to the Floer differential $\partial = \mu^1$.

Floer Product, a.k.a. μ^2

The Floer product satisfies the Leibniz rule.

Proposition [Auroux, 2.3]

Assume the elements of $\pi_2(M, L_j)$ have zero symplectic area, so that bubbling does not occur. Then, the Floer product satisfies, with suitable signs (to be clarified later),

$$\partial(p_2 \cdot p_1) = \pm\partial p_2 \cdot p_1 \pm p_2 \cdot \partial p_1 \quad .$$

Therewith, there is a well-defined product on the cohomology $\mathrm{HF}(L_1, L_2) \otimes \mathrm{HF}(L_0, L_1) \longrightarrow \mathrm{HF}(L_0, L_2)$, which is independent of the almost-complex structure, Hamiltonian perturbations, and is associative (but not on the chains, seen later).

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Sketch of Proof

Look at the moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$ with ind $u = 1$; this is a 1-dimensional smooth manifold, which admits, by Gromov compactness, a compactificaton $\overline{\mathcal{M}}(p_1, p_2, q; [u], J)$.

Because there is no bubbling, the boundary of $\overline{\mathcal{M}}$, the only phenomenon that can occur at the boundary is strip breaking. And since transversality ensures that disk have index ≥ 0 and nonconstant strips index ≥ 1, it follows that the boundary can only consist of the 3 configurations of an index-0 disk with an index-1 strip.

Figure 6. The ends of a 1-dimensional moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$.

[Auroux, pg. 16]

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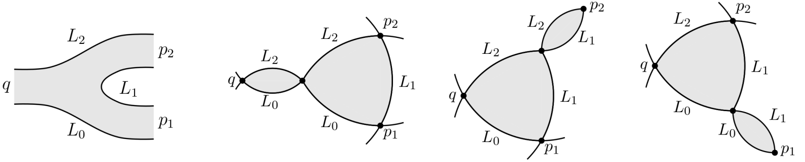

Figure 6. The ends of a 1-dimensional moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$.

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Floer Product, a.k.a. μ^2

(Continuation)

By a gluing theorem, all these degenerations must occur on the ends of this index-1 stratum, and $\overline{\mathcal{M}}$ is 1-dimensional compact manifold with boundary. Furthermore, the orientations agree up to a sign, and depend only on the deg p_j. Lastly, as the signed total number of boundary points of $\overline{\mathcal{M}}$ is zero, the Leibniz rule from the proposition is obtained.

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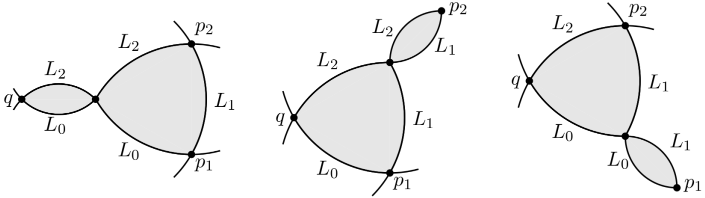

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Remarks on Transversality and Compatibility

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Transversality can be achieved by choosing domain-dependent almost-complex structures and Hamiltonian perturbations, whereby suitable choice is needed near punctures so that the mentioned Leibniz rule holds.

Start by doing the following:

- Fix neighborhoods of the punctures $Z = \{z_0, z_1, z_2\}$ in the disk $\mathbb{D}^2_Z := \mathbb{D}^2 \setminus Z$ with biholomorphisms with half-strips $\mathbb{R}_- \times [0,1]$, $\mathbb{R}_+ \times [0,1]$, with coördinates (s,t).
- Choose a smooth family of almost-complex structures $J(z)$ and Hamiltonians $H(z)$, $z \in \mathbb{D}^2_Z$.

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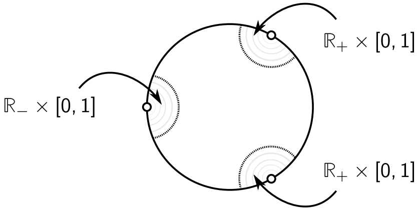

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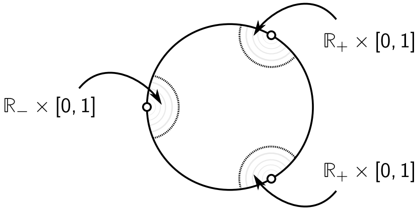

Remarks on Transversality and Compatibility

Then:

- Perturb the Cauchy-Riemann equation accordingly to

$$\left(\mathrm{d}u - X_{H(z)} \otimes \beta\right)^{0,1}_{J(z)} = 0 \quad ,$$

  wherein u is the map from the disk into M as before, $X_{H(z)}$ is the vector field corresponding to $H(z)$, and $\beta \in \Omega^1\left(\mathbb{D}^2_Z\right)$ is a 1-form such that $\beta|_{\partial \mathbb{D}^2_Z} = 0$ and $\beta = \mathrm{d}t$ in the half-strip neighborhoods.

- Choose a smooth family of almost-complex structures $J(z)$ and Hamiltonians $H(z)$, $z \in \mathbb{D}^2_Z$. And in each half-strip neighborhood, these both just depend on the t-coördinate.

REMARKS ON TRANSVERSALITY AND COMPATIBILITY

Then:

- Perturb the Cauchy-Riemann equation accordingly to

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Remarks on Transversality and Compatibility

Finally:

- For $0 \leq j < \ell \leq 2$, let $H_{j\ell}$, $J_{j\ell}$ be the t-dependent Hamiltonians and almost-complex structures on the half-strip neighborhoods whose boundaries map to L_j and L_ℓ. These are patched together
- The solutions to the perturbed Cauchy-Riemann equation converge now not to the punctures, but rather a time-1 flow of $H_{j\ell}$ from L_j to L_ℓ, which are the generators of the perturbed Floer complex $\mathrm{CF}(L_j, L_\ell; H_{j\ell}, J_{j\ell})$.
- This makes the Floer product a map

$$\mathrm{CF}(L_1, L_2; H_{12}, J_{12}) \otimes \mathrm{CF}(L_0, L_1; H_{01}, J_{01}) \longrightarrow \mathrm{CF}(L_0, L_2; H_{02}, J_{02})$$

and the proposition holds with respect to the perturbed Floer differential.

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Higher Operations and A_∞-Relations

Higher Operations, μ^k

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The ethos to constructing the higher operations μ^k on the Floer complexes is the same as for the construction of μ^2, just with more Lagrangians:

- given suitable Lagrangians $L_0, \ldots, L_k$, and let $p_j \in \mathcal{X}(L_{j-1}, L_j)$, $q \in \mathcal{X}(L_0, L_k)$;
- the coefficient of q in $\mu^k(p_1, \ldots, p_k) \in \mathrm{CF}(L_0, L_k)$ should be a weighted count of the number of pseudo-holomorphic maps from disk $\mathbb{D}^2_Z$ with k boundary punctures $Z = \{z_0, \ldots, z_k\}$ to M with boundary on $L_0 \cup \cdots \cup L_k$, meeting the $p_1, \ldots, p_k, q$.

Given an almost-complex structure J and a homotopy class $[u]$ of such a map, the moduli space of such maps is $\mathcal{M}(p_1, \ldots, p_k, q; [u], J)$, up to $\mathrm{Aut}(\mathbb{D}^2)$.

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However, there is a bit more flexibility here than the cases $k < 3$. The reason for this is the "size" of $\mathcal{M}_{0,k+1}$, the moduli space of conformal structures on the $(k+1)$-boundary-punctured disk, the domain of the considered maps for μ^k.

For $k \geq 3$, after modding out by $\mathrm{Aut}\left(\mathbb{D}^2\right)$, there are $(k+1) - 3$ degrees of freedom for selecting the boundary punctures. This means that the expected dimension of $\mathcal{M}_{0,k+1}$ is $k-2$.

It will turn out that this observation will be needed to obtain the A_∞-relations on the μ^k, as this flexibility adds allows for more degenerations of the maps, from the domain side.

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For fixed conformal structure on $\mathbb{D}^2_Z$, the index of the linearized Cauchy-Riemann operator is given by the Maslov index, as before for $k < 3$. Using that and the previous remarks, it is possible to determine

$$\begin{aligned}\dim \mathcal{M}\left(p_1, \ldots, p_k, q; [u], J\right) &= \dim \mathcal{M}_{0,k+1} + \operatorname{ind}[u] \\ &= k - 2 + \operatorname{ind}[u] \quad .\end{aligned}$$

Putting this together:

Definition [Auroux, 2.4]

Let μ^k be the following Λ-linear operation

$$\mu^k\colon \ \mathrm{CF}\left(L_{k-1}, L_k\right) \otimes \cdots \otimes \mathrm{CF}\left(L_0, L_1\right) \longrightarrow \mathrm{CF}\left(L_0, L_k\right)$$

$$\mu^k\left(p_1, \ldots, p_k\right) := \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u]\colon\ \operatorname{ind}[u] = 2-k}} \#\mathcal{M}\left(p_1, \ldots, p_k, q; [u], J\right)\ T^{\omega[u]} q \quad .$$

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HIGHER OPERATIONS, μ^k

As with the lower operations, in generality, there needs to be the introduction of domain-dependent almost-complex structures and Hamiltonian perturbations as to achieve transversality.

The means of doing so continues with the ethos used for μ^3, but the degenerations in the domain of the maps need to be taken into account appropriately, especially so that there is consistency with disks with fewer punctures. As a result, the needed H, J depend on the conformal structure of the domain.

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A_∞-Relations

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As with the lower operation, to understand the algebraic properties of μ^k it is needed to look at the boundary of the compactification of the 1-dimensional moduli space of pseudo-holomorphic maps.

Where there differs from those cases is that the degenerations of the domain can now contribute to that boundary, which was hinted at earlier with the moduli space $\mathcal{M}_{0,k+1}$.

That moduli space also admits a compactification, which is a $(k-2)$-dimensional polytope $\overline{\mathcal{M}}_{0,k+1}$ called the *Stasheff associahedron*.

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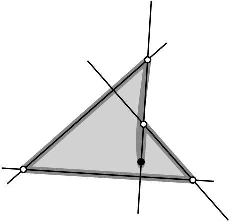
A_∞-Relations

The top-dimensional facets of this polytope $\overline{\mathcal{M}}_{0,k+1}$ are degenerations of the disk into two disks of at least two punctures, and the faces with higher codimension correspond to more disks in the degeneration, all with at least two punctures.



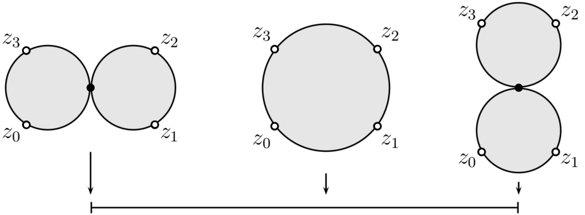


Figure 7. The 1-dimensional associahedron $\overline{\mathcal{M}}_{0,4}$.
[Auroux, pg. 20]

A_∞-Relations

Putting together those domain degenerations with the strip breaking, as for the cases $k < 3$, obtains:

Proposition [Auroux, 2.7]

If the elements of $\pi_2(M, L_j)$ have zero symplectic area, then μ^k satisfies the A_∞-*relations*,

$$\sum_{j=1}^{k} \sum_{\ell=0}^{k-j} (-1)^{\star} \mu^{k+1-j} \Big(p_k, \ldots, p_{\ell+j+1}, \mu^j \left(p_{\ell+j}, \ldots, p_{\ell+1} \right), p_\ell, \ldots, p_1 \Big) = 0$$

wherein $\star := \ell + \deg p_1 + \cdots + \deg p_j$.

A_∞-Relations

This yields:

- for $k = 1$,
$$\mu^1\left(\mu^1(p_1)\right) = 0 \quad \text{, which is} \quad \partial^2 p_1 = 0 \quad ;$$
- for $k = 2$,
$$(-1)^\star \mu^2\left(p_2, \mu^1(p_1)\right) + (-1)^\star \mu^2\left(\mu^1(p_2), p_1\right) + (-1)^\star \mu^1\left(\mu^2\left(p^2, p^1\right)\right) = 0 \quad ;$$
which is the Leibniz rule
$$(-1)^\star p_2 \cdot \partial p_1 + (-1)^\star \partial p_2 \cdot p_1 + (-1)^\star \partial\left(p^2 \cdot p^1\right) = 0 \quad ;$$

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A_∞-Relations

- for $k = 3$,

$$\begin{aligned}
&(-1)^\star \mu^3\left(p_3, p_2, \mu^1(p_1)\right) + (-1)^\star \mu^3\left(p_3, \mu^1(p_2), p_1\right) \\
&+ (-1)^\star \mu^3\left(\mu^1(p_3), p_2, p_1\right) + (-1)^\star \mu^2\left(p_3, \mu^2(p_2, p_1)\right) \\
&+ (-1)^\star \mu^2\left(\mu^2(p_3, p_2), p_1\right) + (-1)^\star \mu^1\left(\mu^3(p_3, p_2, p_1)\right) = 0 \quad ,
\end{aligned}$$

which is the failure of the associativity of the Floer product

$$\begin{aligned}
&(-1)^\star \mu^3(p_3, p_2, \partial p_1) + (-1)^\star \mu^3(p_3, \partial p_2, p_1) \\
&+ (-1)^\star \mu^3(\partial p_3, p_2, p_1) + (-1)^\star p_3 \cdot (p_2 \cdot p_1) \\
&+ (-1)^\star (p_3 \cdot p_2) \cdot p_1 + (-1)^\star \partial\mu^3(p_3, p_2, p_1) = 0 \quad .
\end{aligned}$$

A_∞-Relations

Reärranging, the failure of associativity of the Floer product (on the chain level) shows that the product is associative up to chain homotopy μ^3 (note how ∂ interacts with μ^3 on the right-hand side):

$$\begin{aligned}(-1)^\star\, p_3 \cdot (p_2 \cdot p_1) + (-1)^\star\, (p_3 \cdot p_2) \cdot p_1 = \\ &(-1)^\star\, \partial\mu^3\,(p_3, p_2, p_1) + (-1)^\star\, \mu^3\,(\partial p_3, p_2, p_1) \\ &+ (-1)^\star\, \mu^3\,(p_3, \partial p_2, p_1) + (-1)^\star\, \mu^3\,(p_3, p_2, \partial p_1) \quad .\end{aligned}$$

In general, the A_∞-relation gives μ^k as a chain homotopy for the lower μ^k.

A_∞-Relations

Sketch of Proof

The idea for the proof is essentially the same as for the lower cases: look at the boundary of the compactification of 1-dimensional moduli space of (perturbed) pseudo-holomorphic maps from the disk with $k+1$ boundary punctures, while taking care of transversality and compatibility with use of domain-dependent J, H.

However, as alluded to earlier, there are now more degenerations possible for $k \geq 3$, coming from the domain: this is to say that the compact $\overline{\mathcal{M}}(p_1, \ldots, p_k, q; [u], J)$ for $\operatorname{ind}[u] = 3-k$ has part of its boundary coming from the boundary of $\overline{\mathcal{M}}_{0,k+1}$, the moduli space of conformal structures of the disk with $k+1$ boundary punctures. The rest of the boundary of $\overline{\mathcal{M}}(p_1, \ldots, p_k, q; [u], J)$ comes from strip breaking, as before; these correspond to the terms involving μ^1.

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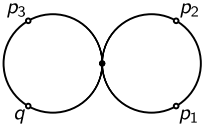
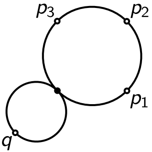
A_∞-Relations

For μ^3, the boundary of $\overline{\mathcal{M}}(p_1, p_2, p_3, q; [u], J)$ consists of the following degenerations: on the left are from the domain and on the right are from strip breaking.

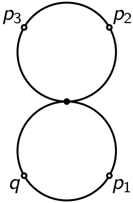
$p_3 (p_2 \cdot p_1)$

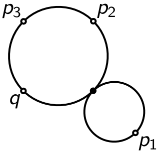




$\partial\mu^3 (p_3, p_2, p_1)$

$(p_3 \cdot p_2)\, p_1$





$\mu^3 (p_3, p_2, \partial p_1)$

etc.

Fukaya Categories

Fukaya Features

There are several formulations of the Fukaya category of a symplectic manifold, but they all have the following features:

- objects are "suitable" Lagrangian submanifold, with extra data on them
- morphisms are Floer chain complexes, with the Floer differential
- compositions of morphisms are given by the Floer product, which is associative only up to chain homotopy
- it is an A_∞-category, satisfying the A_∞-relations from the previous proposition on operations μ^k, of which μ^1 is the Floer differential and μ^2 is the Floer product.

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PIÈCE DE RÉSISTANCE

DEFINITION [AUROUX, 2.9]

Let (M, ω) be a symplectic manifold such that $2c_1(\mathrm{T}M) = [0]$. The objects of the (compact) *Fukaya category* $\mathcal{G}(M, \omega)$ are compact, closed, oriented, spin Lagrangian submanifolds $L_j \subset M$ such that the classes $\pi_2(M, L_j)$ have zero symplectic area, and their Maslov classes $\mu_{L_j} = [0] \in \mathrm{H}^1(L_j, \mathbb{Z})$. These Lagrangians have the extra data of a chosen spin structure and a graded lift to $\mathrm{LGr}(M)$.

For each pair (L_j, L_ℓ), not necessary distinct, there is chosen perturbation data in a family of Hamiltonians $H_{jk} \in C^\infty([0,1] \times M, \mathbb{R})$ and a family of almost-complex structure $J_{j\ell} \in C^\infty([0,1], \mathcal{J}(M, \omega))$. And for all tuples $(L_0, \ldots, L_k)$, not necessary distinct, and all moduli spaces of conformal structures of disks with $k+1$ boundary punctures, there are chosen perturbation data in the domain-dependent Hamiltonian H and almost-complex structure J, consistent with the choices made for all the pairs (L_j, L_ℓ); this is done so that transversality is achieved. (see Seidel)

Pièce de Résistance

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(continued) [Auroux, 2.9]

With all of that, let $\hom(L_j, L_\ell) := \mathrm{CF}(L_j, L_\ell; H_{j\ell} J_{j\ell})$, the Floer chain complex. Therewith are the operations μ^k: μ^1 is the Floer differential, μ^2 is the Floer product, and the higher μ^k are given by counts of perturbed pseudo-holomorphic disks, as previously defined. These satisfies the A_∞-relations as in the last proposition, making $\mathcal{G}(M, \omega)$ a Λ-linear, $\mathbb{Z}$-graded, non-unital (but cohomologically-unital) A_∞-category.

Fukaya Flavors

There are the following flavors:

- $\mathbb{Z}/2$-grading, allows for the dropping of $2c_1(TM) = [0]$ and $\mu_{L_j} = [u]$
- $\operatorname{char} \mathbb{K} = 2$, allows for the dropping of the spin structures on the L_j
- for exact M and exact L_j, the Novikov coefficient are not needed and the condition that the elements of $\pi_2(M, L_j)$ have zero symplectic area is automatically met

It is apparent from the constructions, that the chain-level details pertaining to Fukaya categories inherently depend on the choice of perturbation data (the domain-dependent H, J). However, the categories obtain from the various choices of the perturbation data are all quasi-equivalent — namely, they are related by A_∞-functors that induce isomorphisms at the cohomological level. (see Seidel)

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Further Fukaya Facets

It is possible to turn a Fukaya category into an honest category, at the expense of losing pertinent information from the higher μ^k:

- Take the cohomology of the morphism space (the Floer complexes, with respect to the Floer differential μ^1.
- This makes μ^2 associative, as discussed earlier.
- In this cohomology category of $\mathcal{G}(M, \omega)$, $\hom(L_j, L_\ell) = \mathrm{HF}(L_j, L_\ell)$, the Floer cohomology, and the composition is given by the Floer product μ^2 on the cohomological level. This is sometimes called the *Donaldson-Fukaya category*.

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It is possible to get rid of the condition that the elements of $\pi_2(M, L_j)$ have symplectic area (without exactness), at the expense of analytic and algebraic difficulties in handling the bubbling:

- Analytically, bubbling pose problems to transversality that cannot be solved by perturbation techniques described earlier.
- Algebraically, this is resolved by resorting to a *curved* A_∞-category; this means that for each L_j, there is a element $\mu^0_{L_j} \in \hom(L_j, L_j)$, which captures the weighted count of the bubbles on L_j.
- These new $\mu^0_{L_j}$ are accounted for in the outer sum of the A_∞-relations:

$$\sum_{j=0}^{k} \sum_{\ell=0}^{k-j} (-1)^{\star} \mu^{k+1-j}\left(p_k, \ldots, p_{\ell+j+1}, \mu^j\left(p_{\ell+j}, \ldots, p_{\ell+1}\right), p_\ell, \ldots, p_1\right) = 0$$

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- This makes, for example,

$$\mu^1\left(\mu^1\left(p_1\right)\right) + (-1)^{\deg p_1}\,\mu^2\left(\mu^0_{L_1}, p_1\right) + \mu^2\left(p, \mu^0_{L_0}\right) = 0 \quad ,$$

  which is kind of like a curvature, as it is $\partial^2 p_1 \neq 0$.
- In order to not get completely bogged-down by this, the objects usually considered are *weakly unobstructed*; this means that the L_j are such that $\mu^0_{L_j}$ is a scalar multiple of the cohomological unity of $\hom\left(L_j, L_j\right)$, called the "central charge" or "superpotential".
- These considerations come from mirror symmetry.
  "Weakly-unobstructed objects of fixed central charge" form an honest A_∞-category. (see Fukaya-Oh-Ohta-Ono, *Lagrangian intersection Floer theory: anomaly and obstruction I and II*)

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Finished!
Thank you!