From Green's functions to Chord Spaces, for the Punctured 2-Disk

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Ὅπερ ἔδει ποιῆσαι.

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The path to my Masters degree — and therein, this thesis — was not geodesical, by any reasonable formulation of that qualification. However, as with all things transient, the end arises regardless of the means, which are thereby justified or not.

Two roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could

. . .

Yet knowing how way leads on to way,
I doubted if I should ever come back.

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I—
I took the one less traveled by,
And that has made all the difference.

Robert Frost
"The Road Not Taken"
Mountain Interval, 1916.

Contents

0. Introduction

Let $\mathbf{D}^2 \subset \mathbf{C} \cong \mathbf{R}^2$ be the open unit 2-disk about the origin. The aim of this thesis to understand a map from the configuration space $\mathrm{C}_n \mathbf{D}^2$ of n points in $\mathbf{D}^2$ to the so-called "chords space" $\mathrm{Ch}\,\mathbf{S}^1$, where $\mathbf{S}^1$ is the boundary $\partial\mathbf{D}^2$. This map will be factored as follows: for $n \geq 2$,

$$\begin{tikzcd} \mathrm{C}_n \mathbf{D}^2 \arrow[r, "(1)"] & \mathrm{Gr}_n \mathbf{D}^2 \arrow[rr, "(3)"] \arrow[dr, "(2)"'] & & \mathrm{Ch}_{n-1} \mathbf{S}^1 \subset \mathrm{Ch}\,\mathbf{S}^1, \\ & & \mathrm{P}_{n-1} \mathbf{D}^2 \arrow[ur, dashed] & \end{tikzcd}$$

where the maps are numbered according to the chapter, in which they will be discussed. In that factoring, $\mathrm{Gr}_n \mathbf{D}^2$ is the space of Green's functions on $\mathbf{D}^2$ with n sinks, and $\mathrm{P}_{n-1} \mathbf{D}^2$ is the space of "critical-point polynomials", which is defined in chapter 2. Each chapter will have a section first describing the corresponding codomain of the map along with properties of its elements, and then a section about the map itself.

The goal of chapter 1 is to describe the first map, from the configuration space of n points of the 2-disk $\mathbf{D}^2$ to the space of Green's functions with n sinks. In the first section §1.1, Green's functions with one, and multiple, sinks will be discussed and some of their properties will be determined. From that, a space of them $\mathrm{Gr}_n \mathbf{D}^2$ will be formed. The second section §1.2 will show that the map from the configuration space $\mathrm{C}_n \mathbf{D}^2$ of n points of $\mathbf{D}^2$ to $\mathrm{Gr}_n \mathbf{D}^2$, is a homeomorphism.

Chapter 2 will define the "critical-point polynomials" and form the space $\mathrm{P}_{n-1} \mathbf{D}^2$ of them. Several properties about these polynomials will be shown in §2.1, which will have an impact on the topology of the space, and thusly, the second map. Also in that first section, an attempt will be made at classifying such polynomials for special elements $g_{\mathcal{Z}} \in \mathrm{Gr}_n \mathbf{D}^2$, where $\mathcal{Z} \in \mathrm{C}_n \mathbf{D}^2$; the section will end with a hypothesis about further classification. The chapter ends with §2.2, which contains a hypothesis about the nature of the second map, from the space $\mathrm{Gr}_n \mathbf{D}^2$ of Green's functions with n sinks to the space of "critical-point polynomials" $\mathrm{P}_{n-1} \mathbf{D}^2$.

In chapter 3, the final codomain $\mathrm{Ch}\,\mathbf{S}^1$ of the overall map will be discussed. The construction of the space is done in §3.1 as the geometric realization of a semi-simplicial set with face maps. As to better understand the idea in mind, the section will start off with a geometric way of handling that construction. After establishing several properties and connections to the Green's functions, the last section §3.2 will define the space $\mathrm{Ch}_{n-1} \mathbf{S}^1 \subset \mathrm{Ch}\,\mathbf{S}^1$. That section ends with a discussion about the third, and final, map $\mathrm{Gr}_n \mathbf{D}^2 \longrightarrow \mathrm{Ch}_{n-1} \mathbf{S}^1 \subset \mathrm{Ch}\,\mathbf{S}^1$, including hypotheses about its properties and a potential factoring through the space $\mathrm{P}_{n-1} \mathbf{D}^2$.

At the end is chapter A, which is an auxiliary chapter with the derivation of a Green's function for the Laplacian on $\mathbf{D}^2$ from the realm of distribution theory.

0.1 The Configuration Space C_n

As opposed to including it in the following material in its own chapter, this section will serve as the starting point for discussing the aforementioned map. In particular, it will talk about the first map's domain space $\mathrm{C}_n \mathbf{D}^2$, the configuration space of n points in the open 2-disk $\mathbf{D}^2$.

0.1.1 Definition

Let $\mathrm{SP}^n := \mathrm{SP}^n \mathbf{D}^2$ be the *n-th symmetric product of the 2-disk*, in other words,

$$\mathrm{SP}^n := \mathfrak{S}_n \Big\backslash \left(\mathbf{D}^2\right)^n,$$

wherein $\mathfrak{S}_n$ is the n-th symmetric group and the action of $\mathfrak{S}_n$ on $\left(\mathbf{D}^2\right)^n$ is defined in the usual way for $\sigma \in \mathfrak{S}_n$ on $\left(\zeta_j\right)_{j=1}^n \in \left(\mathbf{D}^2\right)^n$ as

$$\sigma \,.\, \left(\zeta_j\right)_j = \left(\zeta_{\sigma^{-1}(j)}\right)_j.$$

A distinguished subspace of SP^n is $\mathrm{C}_n = \mathrm{C}_n \mathbf{D}^2$, the *(unordered) configuration space of n points the 2-disk*, which is defined as

$$\mathrm{C}_n := \mathfrak{S}_n \Big\backslash \left\{ \left(\zeta_j\right)_{j=1}^n \in \left(\mathbf{D}^2\right)^n \,\middle|\, \zeta_j \neq \zeta_k,\ \forall j \neq k \right\},$$

and is all subsets of $\mathbf{D}^2$ consisting of n distinct points. [Wes11] [AGP02, ¶5.2.1 & ¶4.6.5]

The topology on SP^n is given by the quotient topology from the projection $\pi\colon (\mathbf{D}^2)^n \twoheadrightarrow \mathrm{SP}^n$, wherein $(\mathbf{D}^2)^n$ has the usual topology inherited from $\mathbf{R}^{2n} \supset (\mathbf{D}^2)^n$. Thusly, $\mathrm{C}_n \subset \mathrm{SP}^n$ obtains the subspace topology.

For ease of discussion later on, a metric topology will be defined on C_n, which is equal to the standard subspace topology.

0.1.2 Lemma

Let $d_{\mathsf{H}}(-,-)$ be the Hausdorff metric,, which, on two subsets $\mathcal{Z}, \mathcal{Z}' \subset \mathbf{D}^2$, is

$$d_{\mathsf{H}}(\mathcal{Z},\mathcal{Z}') := \max\left\{\sup_{\zeta\in\mathcal{Z}}\inf_{\zeta'\in\mathcal{Z}'} d_{\mathbf{D}^2}(\zeta,\zeta'), \sup_{\zeta'\in\mathcal{Z}'}\inf_{\zeta\in\mathcal{Z}} d_{\mathbf{D}^2}(\zeta',\zeta)\right\}.$$

Then, the subspace topology on C_n is equal to the one defined by the Hausdorff metric on the elements of C_n as they lie as finite subsets in $\mathbf{D}^2$.

Proof First, it will be shown that the image under the quotient map $\pi\colon (\mathbf{D}^2)^n \twoheadrightarrow \mathrm{SP}^n$ of basis elements of the product topology on $(\mathbf{D}^2)^n$ are basis elements the quotient topology on SP^n. This will be done by showing that the image of the basis elements are open, and that they cover all open sets.

Because the Euclidean metric on $\mathbf{D}^2$ yields its usual topology, it is possible to write a basis element of $(\mathbf{D}^2)^n$ as a product of ε-neighborhoods $N_\varepsilon(\zeta)$ of $\zeta \in \mathbf{D}^2$, i.e. containing some $(\zeta_j)_{j=1}^n \in (\mathbf{D}^2)^n$, they are of the form $N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)$. By definition of the quotient topology, a subset $U \subseteq \mathrm{SP}^n$ is called open iff $\pi^{-1}(U) \subseteq (\mathbf{D}^2)^n$ is open, so it needs to be shown that $\pi^{-1} \circ \pi\left(N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)\right)$ is open. Since the equivalence relation in SP^n is determined by the action of the permutation group $\mathfrak{S}_n$ on the coordinates, it follows that that preimage is of the form

$$\pi^{-1} \circ \pi\left(N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)\right) = \bigcup_{\sigma\in\mathfrak{S}_n} \left(N_{\varepsilon_{\sigma(1)}}(\zeta_{\sigma(1)}) \times \cdots \times N_{\varepsilon_{\sigma(n)}}(\zeta_{\sigma(n)})\right),$$

which is, of course, a union of basis elements of the product topology, making it open. Now, let $U \subseteq \mathrm{SP}^n$ be open. Then, because $\pi^{-1}(U)$ is necessarily open, it can be written as a union $\bigcup_{k\in K} B_k$ of basis elements. From what was just shown, $\pi(\bigcup_{k\in K} B_k) = \bigcup_{k\in K} \pi(B_k)$ is open in SP^n. Thus, $U = \bigcup_{k\in K}\pi(B_k)$, and by its arbitrariness, this shows that the collection of images under π of basis elements of the product $(\mathbf{D}^2)^n$ are basis elements of the quotient SP^n.

Let $\mathcal{Z} = \{\zeta_j\}_{j=1}^n \in \mathrm{C}_n \subset \mathrm{SP}^n$. Using what was just shown, a basis element of C_n containing $\mathcal{Z}$ is then the intersection of C_n with the projection of a basis element $N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)$ of $(\mathbf{D}^2)^n$ containing a point $\widetilde{\mathcal{Z}} = (\zeta_j)_{j=1}^n$ of the preimage $\pi^{-1}(\mathcal{Z}) \subset (\mathbf{D}^2)^n$:

$$\mathcal{Z} \in N_{(\varepsilon_j)_j}(\mathcal{Z}) := \mathrm{C}_n \cap \pi\left(N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)\right),$$

where $(\varepsilon_j)_j$ is an ordered n-tuple of positive real numbers, in bijective correspondence with elements of $\mathcal{Z}$.

Deciphering this more closely: the image $\pi\left(N_{\varepsilon_1}(\zeta_1) \times \cdots \times N_{\varepsilon_n}(\zeta_n)\right)$ in SP^n of that basis element can be looked at as all unordered n-tuples[1] $[p_j]_{j=1}^n$ (i.e. with possible multiplicities) of points contained in $\bigcup_{k=1}^n N_{\varepsilon_k}(\zeta_k) \subseteq \mathbf{D}^2$, such that there exists a bijective correspondence $f\colon \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ between the indices j and $k = f(j)$, satisfying[2] $p_j \in N_{\varepsilon_k}(\zeta_k)$. In C_n, the basis element $N_{(\varepsilon_j)_j}(\mathcal{Z})$ can then be thought of as all subsets $\{p_j\}_{j=1}^n$ (i.e. without multiplicities) of n points in $\bigcup_{k=1}^n N_{\varepsilon_k}(\zeta_k)$, again having that correspondence of indices. In particular, note that if $\mathcal{P} = \{p_j\}_j \in N_{(\varepsilon_k)_k}(\mathcal{Z})$, then, by the discussion, there is a bijective correspondence $j \leftrightsquigarrow k = f(j)$ of indices such that $p_j \in N_{\varepsilon_k}(\zeta_k)$ for each j; this is exactly saying that $\zeta_k \in N_{\varepsilon_k}(p_j)$ for each j, which is iff $\mathcal{Z} \in N_{(\varepsilon_k)_{k=f^{-1}(j)}}(\mathcal{P})$. Hence, $\mathcal{P} \in N_{(\varepsilon_k)_k}(\mathcal{Z})$ iff $\mathcal{Z} \in N_{(\varepsilon_k)_{k=f^{-1}(j)}}(\mathcal{P})$, which is the statement

$$\mathcal{P} \subset \bigcup_{k=1}^n N_{\varepsilon_k}(\zeta_k) \quad \text{iff} \quad \mathcal{Z} \subset \bigcup_{j=1}^n N_{\varepsilon_{f(j)}}(p_j), \text{ given such an } f.$$

Let $d_{\mathbf{D}^2}(-,-)$ by the Euclidean metric on $\mathbf{D}^2$; then, an ε-neighborhood of $\mathcal{Z} \in \mathrm{C}_n$ under the Hausdorff metric $d_{\mathsf{H}}(-,-)$ is given by

$$N_\varepsilon^{\mathsf{H}}(\mathcal{Z}) := \left\{\mathcal{Z}' \in \mathrm{C}_n \,\middle|\, \varepsilon > d_{\mathsf{H}}(\mathcal{Z},\mathcal{Z}') := \max\left\{\sup_{\zeta\in\mathcal{Z}}\inf_{\zeta'\in\mathcal{Z}'} d_{\mathbf{D}^2}(\zeta,\zeta'), \sup_{\zeta'\in\mathcal{Z}'}\inf_{\zeta\in\mathcal{Z}} d_{\mathbf{D}^2}(\zeta',\zeta)\right\}\right\},$$

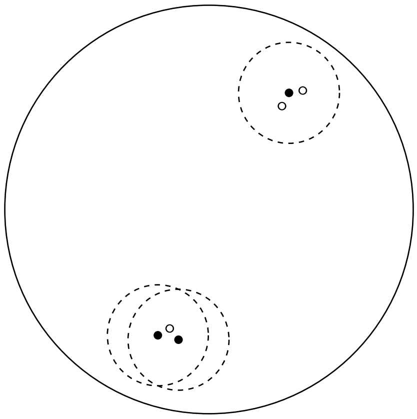
[1] Here, there indices j are not to suggest an ordering, rather they are just a convenient way of labelling, and referring to, the n points.

[2] Please note that it is possible for some j to have multiple k satisfying $p_j \in N_{\varepsilon_k}(\zeta_k)$ — and, likewise, vice versa. The key is that there exists some k for each j satisfying that relation, and that this assignment of k is to only one j, via a possible choice — or vice versa, for j to a single k.

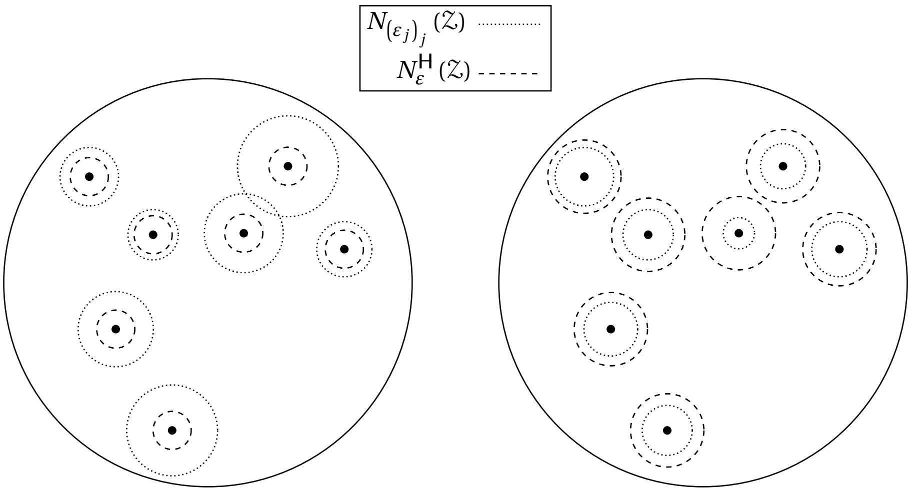
which amounts to another formulation of this neighborhood that is very similar to the one obtained earlier for the basis elements of the quotient topology, but with a uniform ε across the unions,

$$N_\varepsilon^{\mathsf{H}}(\mathcal{Z}) = \left\{ \mathcal{Z}' \in \mathrm{C}_n \;\middle|\; \mathcal{Z}' \subset \bigcup_{\zeta \in \mathcal{Z}} N_\varepsilon(\zeta) \text{ and } \mathcal{Z} \subset \bigcup_{\zeta' \in \mathcal{Z}'} N_\varepsilon(\zeta') \right\}.$$

The only possible discrepancy is that, for $\varepsilon > 0$ not small enough: let $\varepsilon > \min_{j,k} \frac{d_{\mathbf{D}^2}(\zeta_j,\zeta_k)}{2}$, where each $\zeta_j, \zeta_k \in \mathcal{Z}$; then, in the neighborhood $N_\varepsilon^{\mathsf{H}}(\mathcal{Z})$, there will be configurations $\mathcal{Z}'$ of the form, for example, the following depiction, where $\bullet$ represent $\mathcal{Z}$, the dashed lines represent the ε-neighborhoods, and $\circ$ represent $\mathcal{Z}'$.



Thus, these neighborhoods can be reconciled in the following way for any $\mathcal{Z} \in \mathrm{C}_n$:



$$N_\varepsilon^{\mathsf{H}}(\mathcal{Z}) \subseteq N_{(\varepsilon_j)_j}(\mathcal{Z}), \text{ when } \varepsilon \le \min_{j,k} \frac{\varepsilon_j + \varepsilon_k}{2}, \quad \text{and} \quad N_{(\varepsilon_j)_j}(\mathcal{Z}) \subseteq N_\varepsilon^{\mathsf{H}}(\mathcal{Z}), \text{ when } \varepsilon_j \le \varepsilon \;\; \forall j. \qquad \blacksquare$$

1. The Space of Green's Functions Gr_n and the First Map

This chapter is devoted to Green's functions with n sinks, ranging from their construction and properties, to defining a topology on the set of them to form the space $\mathrm{Gr}_n\,\mathbf{D}^2$. And then, there is a section about the first map, which is from the configuration space $\mathrm{C}_n\,\mathbf{D}^2$ of n points of the 2-disk $\mathbf{D}^2$ to the space of Green's functions with n sinks:

$$\mathrm{C}_n\,\mathbf{D}^2 \longrightarrow \mathrm{Gr}_n\,\mathbf{D}^2.$$

1.1 The Space of Green's Functions Gr_n

This section will construct a Green's function, and introduce Green's functions with multiple sinks. To better understand their construction, first, a few concepts, and results, from complex Analysis, and Geometry, will be discussed. For a derivation of the Green's function from the realm of distribution theory, see the chapter A.

1.1.1 Remark

Recall the disk automorphism $\alpha_\zeta\colon \mathbf{D}^2 \longrightarrow \mathbf{D}^2$ for each $\zeta \in \mathbf{D}^2$, the open 2-disk: [SS03, Ch. 8, §2.1]

$$\alpha_\zeta(z) := \frac{\zeta - z}{1 - \overline{\zeta} z}.$$

This α_ζ set-wise fixes $\partial\mathbf{D}^2$, and is a biholomorphism that has the following inherent properties:

$$\alpha_\zeta(\zeta) = 0, \quad \alpha_\zeta(0) = \zeta, \quad \text{and} \quad \alpha_\zeta \circ \alpha_\zeta(z) = \frac{\zeta - \frac{\zeta - z}{1-\overline{\zeta}z}}{1 - \overline{\zeta}\frac{\zeta - z}{1-\overline{\zeta}z}} = \frac{\zeta\left(1-\overline{\zeta}z\right) - \zeta + z}{1 - \overline{\zeta}z - \overline{\zeta}(\zeta - z)} = \frac{z - |\zeta|^2 z}{1 - |\zeta|^2} = z.$$

Also, note that all automorphisms of the 2-disk $\mathbf{D}^2$ have the following form: [SS03, Ch. 8, Theorem 2.2]

$$\mathrm{e}^{\mathrm{i}\theta}\alpha_\zeta(z)$$

for some $\theta \in \mathbf{R}$ and $\zeta \in \mathbf{D}^2$. ◀

1.1.2 Lemma

The disk automorphism α_ζ is an isometry of the Poincaré metric on the 2-disk,

$$\frac{4\,\mathrm{d}z \otimes \mathrm{d}\overline{z}}{\left(|z|^2 - 1\right)^2}.$$

Proof From complex realm, in terms of the usual partial-differential operators $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$, there are the following complex differential operators

$$\partial_z := \frac{\partial}{\partial z} := \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right) \quad \text{and} \quad \partial_{\overline{z}} := \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right),$$

wherein $z = x + \mathrm{i}y$, of course. For example,

$$\partial_z \alpha_\zeta(z) = \frac{-1}{1 - \overline{\zeta}z} - \frac{-\overline{\zeta}(\zeta - z)}{\left(1 - \overline{\zeta}z\right)^2} = \frac{|\zeta|^2 - 1}{\left(1 - \overline{\zeta}z\right)^2}.$$

Conveniently using that, the computation of the pullback of the Poincaré metric via α_ζ unfolds as:

$$\alpha_\zeta^*\left(\frac{4\,\mathrm{d}z \otimes \mathrm{d}\overline{z}}{\left(|z|^2 - 1\right)^2}\right) = \frac{4}{\left(\left|\alpha_\zeta(z)\right|^2 - 1\right)^2}\left(\partial_z \alpha_\zeta(z)\ \mathrm{d}z\right) \otimes \left(\partial_{\overline{z}}\overline{\alpha_\zeta(z)}\ \mathrm{d}\overline{z}\right)$$

$$\begin{aligned}
&= \frac{4\left|1-\overline{\zeta}z\right|^4}{\left(|\zeta-z|^2-\left|1-\overline{\zeta}z\right|^2\right)^2}\frac{\left(|\zeta|^2-1\right)^2}{\left|1-\overline{\zeta}z\right|^4}\,\mathrm{d}z\otimes\mathrm{d}\overline{z}\\
&= \frac{4\left(|\zeta|^2-1\right)^2}{\left(|\zeta-z|^2-\left|1-\overline{\zeta}z\right|^2\right)^2}\,\mathrm{d}z\otimes\mathrm{d}\overline{z}\\
&= \frac{4\left(|\zeta|^2-1\right)^2}{\left(\left(|\zeta|^2-\zeta\overline{z}-\overline{\zeta}z+|z|^2\right)-\left(1-\zeta\overline{z}-\overline{\zeta}z+|\zeta|^2|z|^2\right)\right)^2}\,\mathrm{d}z\otimes\mathrm{d}\overline{z}\\
&= \frac{4}{\left(1-|z|^2\right)^2}\,\mathrm{d}z\otimes\mathrm{d}\overline{z},
\end{aligned}$$

showing that the metric is preserved under such a pullback. Thus, by definition, α_ζ is an isometry of the Poincaré metric. ■

That disk automorphism α_ζ lies at the heart of a Green's function, and enables its definition, as follows.

1.1.3 Definition

The *Green's function with a sink at* $\zeta\in\mathbf{D}^2$ is a function $g_\zeta\colon \mathbf{D}^2\smallsetminus\{\zeta\}\longrightarrow\mathbf{R}$ defined as

$$g_\zeta(z) := \ln\left|\alpha_\zeta(z)\right|.$$

In order to take its complex derivative, there is the following lemma.

1.1.4 Lemma

Let $f\colon\mathbf{C}\longrightarrow\mathbf{C}$ be a holomorphic function on $\mathbf{D}^2$. Then,

$$\partial_z\ln|f| = \frac{\partial_z\operatorname{Re}f}{f} = \frac{\partial_z f}{2f}.$$

Proof Without loss of generality, let $f=u+\mathrm{i}v$ for some real functions u, v, and note that the Cauchy-Riemann equations take the form: [SS03, Ch. 1, Proposition 2.3]

$$0=\partial_{\overline{z}}f=\frac{1}{2}\left(\left(\partial_x u-\partial_y v\right)+\mathrm{i}\left(\partial_y u+\partial_x v\right)\right).$$

Then,

$$\begin{aligned}
\partial_z\ln|f| = \frac{1}{2}\partial_z\ln|f|^2 = \frac{1}{2}\partial_z\ln\left(u^2+v^2\right) &= \frac{1}{2|f|^2}\left(u\partial_x u+v\partial_x v-\mathrm{i}\left(u\partial_y u+v\partial_y v\right)\right)\\
&= \frac{1}{2|f|^2}\left(u\partial_x u-v\partial_y u-\mathrm{i}\left(u\partial_y u+v\partial_x u\right)\right) && \text{by the Cauchy-Riemann equations}\\
&= \frac{1}{2|f|^2}\left((u-\mathrm{i}v)\,\partial_x u-\mathrm{i}\,(u-\mathrm{i}v)\,\partial_y u\right)\\
&= \frac{\partial_z u}{f} && \overline{f}=u-\mathrm{i}v \text{ and } \partial_z u=\tfrac{1}{2}\left(\partial_x u-\mathrm{i}\partial_y u\right),
\end{aligned}$$

and

$$\partial_z f=\frac{1}{2}\left(\partial_x u+\mathrm{i}\partial_x v-\mathrm{i}\partial_y u+\partial_y v\right)=\partial_x u-\mathrm{i}\partial_y u=2\partial_z u.$$

Putting those two results together, obtains the desired results. ■

1.1.5 Lemma

The complex derivative of the Green's function $g_\zeta\colon\mathbf{D}^2\smallsetminus\{\zeta\}\longrightarrow\mathbf{R}$ is

$$\partial_z g_\zeta(z)=\frac{|\zeta|^2-1}{2\,(\zeta-z)\left(1-\overline{\zeta}z\right)}.$$

Moreover, g_ζ solves the Dirichlet problem on $\mathbf{D}^2\smallsetminus\{\zeta\}\subset\mathbf{D}^2$ with the usual Euclidean metric:

$$\begin{cases}\Delta g_\zeta(z)=0 & \forall z\in\mathbf{D}^2\smallsetminus\{\zeta\}\\ g_\zeta(z)=0 & \forall z\in\partial\mathbf{D}^2.\end{cases}$$

Proof Since α_ζ is holomorphic on $\mathbf{D}^2 \smallsetminus \{\zeta\}$, it follows from lemma ¶1.1.4 that, on $\mathbf{D}^2 \smallsetminus \{\zeta\}$,

$$\partial_z g_\zeta(z) = \partial_z \ln\left|\alpha_\zeta(z)\right| = \frac{\partial_z \alpha_\zeta(z)}{2\alpha_\zeta(z)} = \frac{|\zeta|^2 - 1}{2(\zeta - z)\left(1 - \overline{\zeta} z\right)}.$$

Next, using the operators defined in the proof of lemma ¶1.1.2, the Laplacian can be written as

$$\Delta = \partial_x^2 + \partial_y^2 = 4\partial_{\overline{z}}\partial_z = 4\partial_z\partial_{\overline{z}}.$$

Then, outside the singularity at ζ,

$$\Delta\, g_\zeta\big|_{\mathbf{D}^2 \smallsetminus \{\zeta\}}(z) = \partial_{\overline{z}} \frac{\partial_z \alpha_\zeta(z)}{2\alpha_\zeta(z)}.$$

Because $\alpha_\zeta(z)$ is holomorphic and vanishes only at ζ, it follows that

$$\frac{\partial_z\, \alpha_\zeta(z)\big|_{\mathbf{D}^2 \smallsetminus \{\zeta\}}}{2\, \alpha_\zeta(z)\big|_{\mathbf{D}^2 \smallsetminus \{\zeta\}}}$$

is holomorphic. [SS03, Ch. 1, Proposition 2.2 (iii)] Thus, it is in the kernel of the operator $\partial_{\overline{z}}$, making

$$\Delta\, g_\zeta\big|_{\mathbf{D}^2 \smallsetminus \{\zeta\}}(z) = 0.$$

Lastly, on the boundary $\partial\mathbf{D}^2 = \mathbf{S}^1$,

$$g_\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \ln\left|\alpha_\zeta\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \ln 1 = 0$$

since $\alpha_\zeta(z)$ set-wise fixes $\partial\mathbf{D}^2 \ni \mathrm{e}^{\mathrm{i}\theta}$. ■

And, finally, the Green's function with multiple sinks is defined as the following.

1.1.6 Definition

The *Green's function with sinks at* $\mathcal{Z} = \{\zeta_j\}_{j=1}^n \subset \mathbf{D}^2$ is

$$\begin{aligned} g_{\mathcal{Z}} : \mathbf{D}^2 \smallsetminus \mathcal{Z} &\longrightarrow \mathbf{R} \\ z &\longmapsto \sum_{\zeta \in \mathcal{Z}} g_\zeta(z) = \sum_{j=1}^n g_{\zeta_j}(z); \end{aligned}$$

this is likewise harmonic outside $\mathbf{D}^2 \smallsetminus \mathcal{Z}$ and zero on $\partial\mathbf{D}^2$.

Green's functions with multiple sinks have the following symmetry property.

1.1.7 Lemma

Let $\mathcal{Z}$ be as before, and let $\mathcal{W} = \{\omega_j\}_{j=1}^m \subset \mathbf{D}^2 \smallsetminus \mathcal{Z}$. Define the following evaluation of $g_{\mathcal{Z}}$ on $\mathcal{W}$:

$$g_{\mathcal{Z}}[\mathcal{W}] := \sum_{\omega \in \mathcal{W}} g_{\mathcal{Z}}(\omega) = \sum_{j=1}^m g_{\mathcal{Z}}(\omega_j).$$

Then,

$$g_{\mathcal{Z}}[\mathcal{W}] = g_{\mathcal{W}}[\mathcal{Z}].$$

Proof Note that, since conjugation does not change magnitude,

$$g_\zeta(\omega) = \ln\left|\frac{\zeta - \omega}{1 - \overline{\zeta}\omega}\right| = \ln\left|\frac{\omega - \zeta}{1 - \zeta\overline{\omega}}\right| = g_\omega(\zeta).$$

Using that to carry out the calculation:

$$g_{\mathcal{Z}}[\mathcal{W}] = \sum_{j=1}^m g_{\mathcal{Z}}(\omega_j) = \sum_{j=1}^m \sum_{k=1}^n g_{\zeta_k}(\omega_j) = \sum_{j=1}^m \sum_{k=1}^n g_{\omega_j}(\zeta_k) = \sum_{k=1}^n \sum_{j=1}^m g_{\omega_j}(\zeta_k) = g_{\mathcal{W}}[\mathcal{Z}].$$ ■

Finally, the set of Green's functions with n sinks will be defined, which will become a space after its topology is constructed right afterwards.

1.1.8 DEFINITION

Let $\mathrm{Gr}_n = \mathrm{Gr}_n\,\mathbf{D}^2$ be the *set of Green's functions on* $\mathbf{D}^2$ *with* n *sinks*, which, via definition ¶1.1.6, is to say

$$\mathrm{Gr}_n := \left\{ g_{\mathcal{Z}} \colon \mathbf{D}^2 \smallsetminus \mathcal{Z} \longrightarrow \mathbf{R},\ z \longmapsto \sum_{\zeta \in \mathcal{Z}} g_\zeta(z) \,\middle|\, \mathcal{Z} \in \mathrm{C}_n \right\}.$$

1.1.9 CONSTRUCTION

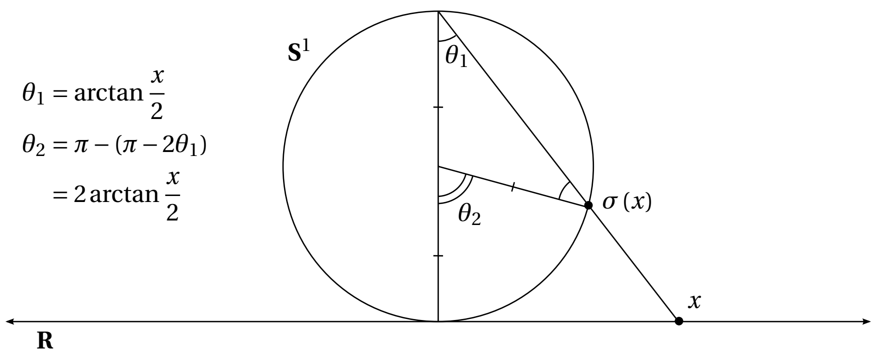
The *topology on* Gr_n will be the compact-open topology, obtained after some work of defining an appropriate metric on the codomain $\mathbf{R}$ of the Green's functions $(g_{\mathcal{Z}} \colon \mathbf{D}^2 \smallsetminus \mathcal{Z} \longrightarrow \mathbf{R}) \in \mathrm{Gr}_n$.

Since the functions in Gr_n are not defined everywhere continuously, they will first be uniquely extended to the compactification of $\mathbf{R}$ as:

$$(g_{\mathcal{Z}} \colon \mathbf{D}^2 \smallsetminus \mathcal{Z} \longrightarrow \mathbf{R}) \in \mathrm{Gr}_n \quad \text{is extended to} \quad \widehat{g_{\mathcal{Z}}} \colon \mathbf{D}^2 \longrightarrow \widehat{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$$
$$z \longmapsto \begin{cases} g_{\mathcal{Z}}(z) & z \in \mathbf{D}^2 \smallsetminus \mathcal{Z} \\ \infty & z \in \mathcal{Z}. \end{cases}$$

Now, that the function is made to be defined everywhere, it will be projected into a familiar compact space, so that differences of its values can be taken. To this end, the extension $\widehat{g_{\mathcal{Z}}}$ will be precomposed with a stereographic projection of $\widehat{\mathbf{R}}$ to the 1-sphere $\mathbf{S}^1$, which is given by

$$\Pi \colon \widehat{\mathbf{R}} \longrightarrow \mathbf{S}^1$$
$$x \longmapsto \exp \mathrm{i}\left(2 \arctan \frac{x}{2}\right).$$



In this way, for each $g_{\mathcal{Z}} \in \mathrm{Gr}_n$, the map $\Pi \circ \widehat{g_{\mathcal{Z}}} \colon \mathbf{D}^2 \longrightarrow \mathbf{S}^1$ is now continuous, and its image lies in the lower-half $\mathbf{S}^1_-$ — from $\mathrm{e}^{-\mathrm{i}\pi}$ to e^0 — of $\mathbf{S}^1$, because $g_{\mathcal{Z}}$ only takes non-positive values. As such, there is an unambiguous distance between points within the image, defined by the metric

$$d_{\mathbf{S}^1_-}\left(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta'}\right) := |\theta - \theta'| \leq \pi,$$

which is consistent with the usual topology on $\mathbf{S}^1_- \subset \mathbf{S}^1$. Because $\mathbf{D}^2$ is compact and $\mathbf{S}^1_-$ has a metric, the compact-open topology on Gr_n can be defined by the metric

$$d_{\mathrm{Gr}_n}(g_{\mathcal{Z}}, g_{\mathcal{Z}'}) := \sup_{z \in \mathbf{D}^2} d_{\mathbf{S}^1_-}\left(\Pi \circ \widehat{g_{\mathcal{Z}}}(z), \Pi \circ \widehat{g_{\mathcal{Z}'}}(z)\right) = 2 \sup_{z \in \mathbf{D}^2} \left| \arctan \frac{\widehat{g_{\mathcal{Z}}}(z)}{2} - \arctan \frac{\widehat{g_{\mathcal{Z}'}}(z)}{2} \right|.$$

Thusly, the *space of Green's functions* Gr_n is constructed.

1.2 THE FIRST MAP

Here, the proof that the first map $\mathrm{C}_n \longrightarrow \mathrm{Gr}_n$ is a homeomorphism, will be presented.

1.2.1 Theorem
The map

$$\begin{aligned} g_\bullet \colon \mathrm{C}_n &\longrightarrow \mathrm{Gr}_n \\ \mathcal{Z} &\longmapsto g_{\mathcal{Z}}. \end{aligned}$$

is a homeomorphism given the topologies defined in definition ¶0.1.1 for C_n and in construction ¶1.1.9 for Gr_n.

Proof It will be shown that the map is bijective, and then, that it is continuous with a continuous inverse.

The surjectivity of $g_\bullet$ comes from the definition of Gr_n. For injectivity, first note that the finite number of points in $\mathcal{Z} \in \mathrm{C}_n$ are exactly where $g_{\mathcal{Z}} \in \mathrm{Gr}_n$ has logarithmic singularities, by construction:

$$g_{\mathcal{Z}}(z) := \sum_{\zeta \in \mathcal{Z}} \ln \left|\alpha_\zeta(z)\right|.$$

Thus, it must be that, if two elements of Gr_n are equal, then their singularities lie at the same points, forcing their preimages in C_n to be the same.

Continuity will be shown via uniform continuity after a series of steps. First, consider the function G of two variables, taking values in $\mathbf{R}$:

$$\begin{aligned} G \colon \overline{\left(\mathbf{D}^2\right)^n} \times \overline{\mathbf{D}^2} &\longrightarrow \mathbf{R} \\ \left(\mathcal{P} = \left(p_j\right)_{j=1}^n,\ z\right) &\longmapsto \arctan \frac{\widehat{g_{\mathcal{P}}}(z)}{2}, \end{aligned}$$

where, just as the extension $\widehat{g_{\mathcal{Z}}}$ was defined for $\mathcal{Z} \in \mathrm{C}_n$ in construction ¶1.1.9,

$$\begin{aligned} \widehat{g_{\mathcal{P}}} \colon \overline{\left(\mathbf{D}^2\right)^n} &\longrightarrow \widehat{\mathbf{R}} \\ \mathcal{P} &\longmapsto \begin{cases} \sum_{j=1}^n \ln\left|\frac{p_j - z}{1 - \overline{p_j} z}\right| & z \neq p_j,\ \forall j \\ \infty & z = p_j,\ \forall j; \end{cases} \end{aligned}$$

this function G is continuous by construction. Since G is continuous on the compact $\overline{\left(\mathbf{D}^2\right)^n} \times \overline{\mathbf{D}^2}$, it follows that it is uniformly continuous there. [Rud76, Theorem 4.19] Now, note that G factors through the map $\left(\pi, \mathrm{id}_{\overline{\mathbf{D}^2}}\right) \colon \overline{\left(\mathbf{D}^2\right)^n} \times \overline{\mathbf{D}^2} \longrightarrow \left(\mathrm{SP}^n \overline{\mathbf{D}^2}\right) \times \overline{\mathbf{D}^2}$, where $\pi \colon \overline{\left(\mathbf{D}^2\right)^n} \twoheadrightarrow \mathrm{SP}^n \overline{\mathbf{D}^2}$ is the quotient map, because $\widehat{g_{\mathcal{P}}}$ is invariant under the action of $\mathfrak{S}_n$ on any $\mathcal{P} \in \overline{\left(\mathbf{D}^2\right)^n}$; thus, there is the continuous map

$$\begin{aligned} \check{G} \colon \left(\mathrm{SP}^n \overline{\mathbf{D}^2}\right) \times \overline{\mathbf{D}^2} &\longrightarrow \mathbf{R} \\ \left(\pi(\mathcal{P}), z\right) &\longmapsto \arctan \frac{\widehat{g_{\mathcal{P}}}(z)}{2}. \end{aligned}$$

Next, note that $\mathrm{SP}^n \overline{\mathbf{D}^2}$ is compact because it is defined as a quotient of the compact product $\overline{\left(\mathbf{D}^2\right)^n}$; thus, $\check{G}$ is also uniformly continuous. Upon restriction, this yields that

$$\check{G}\big|_{\mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2} \colon \mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2 \hookrightarrow \left(\mathrm{SP}^n \mathbf{D}^2\right) \times \mathbf{D}^2 \hookrightarrow \left(\mathrm{SP}^n \overline{\mathbf{D}^2}\right) \times \overline{\mathbf{D}^2} \longrightarrow \mathbf{R}$$

is uniformly continuous and agrees with

$$\begin{aligned} \mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2 &\longrightarrow \mathbf{R} \\ (\mathcal{Z}, z) &\longmapsto \arctan \frac{\widehat{g_{\mathcal{Z}}}(z)}{2}. \end{aligned}$$

Lastly, since the Hausdorff metric $d_{\mathrm{H}}(-,-)$ on $\mathrm{C}_n \mathbf{D}^2$ was shown in lemma ¶0.1.2 to yield the natural topology on $\mathrm{C}_n \mathbf{D}^2$, it follows from uniform continuity that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, for any $(\mathcal{Z}, z), \left(\mathcal{Z}', z'\right) \in \mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2$, if[1]

$$d_{\mathrm{H}}\left(\mathcal{Z}, \mathcal{Z}'\right) + d_{\mathbf{D}^2}\left(z, z'\right) =: d_{\mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2}\left((\mathcal{Z}, z), \left(\mathcal{Z}', z'\right)\right) < \delta, \quad \text{then} \quad \left|\arctan \frac{\widehat{g_{\mathcal{Z}}}(z)}{2} - \arctan \frac{\widehat{g_{\mathcal{Z}'}}(z')}{2}\right| < \varepsilon.$$

[1]It is also possible to define the metric as $d_{\mathrm{C}_n \mathbf{D}^2 \times \mathbf{D}^2}\left((\mathcal{Z}, z), (\mathcal{Z}', z')\right) := \sqrt{\left(d_{\mathrm{H}}(\mathcal{Z}, \mathcal{Z}')\right)^2 + \left(d_{\mathbf{D}^2}(z, z')\right)^2}$, but topologically, it is equal.

Hence, for any $\varepsilon > 0$, there is a $\delta > 0$ such that for any $\mathcal{Z}, \mathcal{Z}' \in \mathrm{C}_n\mathbf{D}^2$ and any $z \in \mathbf{D}^2$, if

$$d_{\mathsf{H}}\left(\mathcal{Z},\mathcal{Z}'\right) + d_{\mathbf{D}^2}(z,z) = d_{\mathsf{H}}\left(\mathcal{Z},\mathcal{Z}'\right) < \delta, \quad \text{then} \quad \underbrace{\sup_{z\in\mathbf{D}^2}\left|\arctan\frac{\widehat{g_{\mathcal{Z}}}(z)}{2} - \arctan\frac{\widehat{g_{\mathcal{Z}}}(z)}{2}\right|}_{=:\frac{1}{2}d_{\mathrm{Gr}_n\mathbf{D}^2}\left(g_{\mathcal{Z}},g_{\mathcal{Z}'}\right)} < \varepsilon,$$

which is to say that $g_\bullet \colon \mathrm{C}_n\mathbf{D}^2 \longrightarrow \mathrm{Gr}_n\mathbf{D}^2$ is continuous.

For the inverse, continuity will be shown by showing that, for each $\mathcal{Z} \in \mathrm{C}_n$ and any $\varepsilon > 0$, there is a $\delta > 0$ such that, for any $\mathcal{Z}' \in \mathrm{C}_n$, if

$$d_{\mathsf{H}}\left(\mathcal{Z},\mathcal{Z}'\right) \geq \varepsilon, \quad \text{then} \quad d_{\mathrm{Gr}_n}\left(g_{\mathcal{Z}},g_{\mathcal{Z}'}\right) \geq \delta.$$

Pick some $\{\zeta_j\}_{j=1}^n = \mathcal{Z} \in \mathrm{C}_n$, and assume that $\left\{\zeta'_j\right\}_{j=1}^n = \mathcal{Z}' \in \mathrm{C}_n$ is such that $d_{\mathsf{H}}\left(\mathcal{Z},\mathcal{Z}'\right) \geq \varepsilon$. Then, it follows from the definition of the metric $d_{\mathsf{H}}(-,-)$ that there is[2] a j such that $d_{\mathbf{D}^2}\left(\zeta_j,\zeta'_j\right) := \left|\zeta_j - \zeta'_j\right| \geq \varepsilon$. Therewith,

$$\begin{aligned}
d_{\mathrm{Gr}_n}\left(g_{\mathcal{Z}},g_{\mathcal{Z}'}\right) &:= 2\sup_{z\in\mathbf{D}^2}\left|\arctan\frac{\widehat{g_{\mathcal{Z}}}(z)}{2} - \arctan\frac{\widehat{g_{\mathcal{Z}'}}(z)}{2}\right| \\
&\geq 2\left|\arctan\frac{\widehat{g_{\mathcal{Z}}}(\zeta_j)}{2} - \arctan\frac{\widehat{g_{\mathcal{Z}'}}(\zeta_j)}{2}\right| && \text{since it is the supremum} \\
&\geq 2\left|\frac{\pi}{2} - \arctan\frac{\widehat{g_{\mathcal{Z}'}}(\zeta_j)}{2}\right| && \text{by construction of } \widehat{g_{\mathcal{Z}}} \\
&\geq \pi - 2\left|\arctan\frac{\widehat{g_{\mathcal{Z}'}}(\zeta_j)}{2}\right| \\
&= \pi - 2\arctan\frac{\left|\widehat{g_{\mathcal{Z}'}}(\zeta_j)\right|}{2} && \text{because arctan is odd}
\end{aligned}$$

wherein

$$\begin{aligned}
\left|\widehat{g_{\mathcal{Z}'}}(\zeta_j)\right| := \left|g_{\mathcal{Z}'}(\zeta_j)\right| &:= \left|\sum_{k=1}^n \ln\left|\alpha_{\zeta'_k}(\zeta_j)\right|\right| \\
&= \sum_{k=1}^n \left|\ln\left|\alpha_{\zeta'_k}(\zeta_j)\right|\right| && \text{since, for each } k, \ln\left|\alpha_{\zeta'_k}(z)\right| \text{ is a non-positive function} \\
&\geq \left|\ln\left|\alpha_{\zeta'_j}(\zeta_j)\right|\right| \\
&:= \left|\ln\left|\frac{\zeta'_j - \zeta_j}{1 - \overline{\zeta'_j}\zeta_j}\right|\right| \\
&\geq \left|\ln\frac{\varepsilon}{1 + \left|\overline{\zeta'_j}\zeta_j\right|}\right| && \text{given } \left|\zeta_j - \zeta'_j\right| \geq \varepsilon \text{ and since } \left|1 - \overline{\zeta'_j}\zeta_j\right| \leq 1 + \left|\overline{\zeta'_j}\zeta_j\right| \\
&\geq \left|\ln\frac{\varepsilon}{2}\right| && \text{because } \zeta_j, \zeta'_j \in \mathbf{D}^2 \text{ implies } \left|\overline{\zeta'_j}\right|, \left|\zeta_j\right| < 1.
\end{aligned}$$

So,

$$d_{\mathrm{Gr}_n}\left(g_{\mathcal{Z}},g_{\mathcal{Z}'}\right) \geq \pi - 2\arctan\frac{1}{2}\left|\ln\frac{\varepsilon}{2}\right| > 0,$$

where the last inequality is because $\varepsilon > 0$. Hence, setting $0 < \delta \leq \pi - 2\arctan\frac{1}{2}\left|\ln\frac{\varepsilon}{2}\right|$ suffices, and by the arbitrariness of the above argument, it follows that $g_\bullet$ has a continuous inverse.

Thereupon, it has been shown that $g_\bullet \colon \mathrm{C}_n \longrightarrow \mathrm{Gr}_n$ is a homeomorphism, as desired. ■

[2] Since the labelling indices are arbitrary, they can be chosen such that there is such a correspondence, as discussed in the proof of lemma ¶0.1.2.

2. The Space of Critical-Point Polynomials P_{n-1} and the Second Map

This chapter will defined the "critical-point polynomial" corresponding to a given Green's function, and then form a space of them called $\mathrm{P}_{n-1}\mathbf{D}^2$. After that in the first section, various properties about these polynomials will be shown, ending with a hypothesis for further classification of the forms of such polynomials for certain Green's functions. The second section will briefly discuss the second map, which is from the space of Green's functions $\mathrm{Gr}_n\mathbf{D}^2$ to the space of "critical-point polynomials" $\mathrm{P}_{n-1}\mathbf{D}^2$,

$$\mathrm{Gr}_n\mathbf{D}^2 \longrightarrow \mathrm{P}_{n-1}\mathbf{D}^2.$$

2.1 The Set of Critical-Point Polynomials P_{n-1}

In this section, the set of "critical-point polynomials" will be defined and made into a space, and afterwards, the remainder of the section will be devoted to discerning properties about those polynomials. To start off, there are the following definitions.

2.1.1 Definitions

Let $\mathcal{Z}\in\mathrm{C}_n$, the n-point configuration space of the 2-disk be $\mathcal{Z}=\{\zeta_j\}_{j=1}^n$, and let $g_{\mathcal{Z}}\in\mathrm{Gr}_n$ be the corresponding Green's function, as defined in definition ¶1.1.6. Then, the critical points of $g_{\mathcal{Z}}(z)$ are those $z\in\mathbf{D}^2\smallsetminus\mathcal{Z}$ which satisfy

$$0=\partial_z g_{\mathcal{Z}}(z)=\sum_{j=1}^n\partial_z g_{\zeta_j}(z)=\sum_{j=1}^n\frac{|\zeta_j|^2-1}{2(\zeta_j-z)\left(1-\overline{\zeta_j}z\right)}=\frac{\sum_{j=1}^n\left(|\zeta_j|^2-1\right)\prod_{k\neq j}(\zeta_k-z)\left(1-\overline{\zeta_k}z\right)}{2\prod_{j=1}^n(\zeta_j-z)\left(1-\overline{\zeta_j}z\right)}.$$

Clearing the denominator, those $z\in\mathbf{D}^2\smallsetminus\mathcal{Z}$ satisfy the following polynomial relation on $\mathbf{D}^2$

$$h_{\mathcal{Z}}(z):=\sum_{j=1}^n\left(|\zeta_j|^2-1\right)\prod_{k\neq j}(\zeta_k-z)\left(1-\overline{\zeta_k}z\right)=0,$$

which has degree $2n-2$, and is defined to be the *critical-point polynomial* corresponding to $\mathcal{Z}$ — so called because the critical points of $g_{\mathcal{Z}}$ are roots of it by construction. Later on in this section, the polynomial will be considered over all of $\mathbf{C}\supset\mathbf{D}^2$, so those roots will be referred as those lying in $\mathbf{D}^2$.

Let the *set of critical-point polynomials* be denoted

$$\mathrm{P}_{n-1}=\mathrm{P}_{n-1}\mathbf{D}^2:=\{h_{\mathcal{Z}}\mid\mathcal{Z}\in\mathrm{C}_n\}.$$

The natural way of topologizing this set is by considering the following embedding of these polynomials into $\mathbf{C}^{2n-1}$ via their coefficients:

$$\begin{aligned}\iota\colon\ \mathrm{P}_{n-1}&\hookrightarrow\mathbf{C}^{2n-1}\\ h_{\mathcal{Z}}&\longmapsto\left(\mathrm{coeff}\left(z^j,h_{\mathcal{Z}}\right)\right)_{j=0}^{2n-2}.\end{aligned}$$

In this way, P_{n-1} obtains the subspace topology and is thusly called the *space of critical-point polynomials*.

2.1.2 Remark

It is also possible to obtain a polynomial of degree $n-1$, whose roots correspond exactly to the critical points in $\mathbf{D}^2$ of $g_{\mathcal{Z}}$ for some $\mathcal{Z}\in\mathrm{C}_n$. However, the formula for such a polynomial, as derived here, is convoluted and cumbersome.

To determine the coefficients of a polynomial with those roots, a generalized version of the so-called argument principle [SS03, Ch. 3, §4, Theorem 4.1] [Ahl78, Ch. 4, eq. 46] will be used: let $\mathcal{U}$ be the finitely-many roots of a polynomial $h\colon\mathbf{C}\longrightarrow\mathbf{C}$; then,

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbf{D}^2}z^k\frac{h'}{h}\,\mathrm{d}z=\sum_{\rho\in\mathcal{U}\cap\mathbf{D}^2}\rho^k.$$

The coefficients [Lan02, IV.6] of the normalized polynomial with those roots are the elementary symmetric polynomials e_j in those roots up to a sign: in the case of $h = h_{\mathcal{Z}}$ as above, let $\mathcal{R} \in \mathrm{SP}^{n-1}$ be $\{\rho_j\}_{j=1}^{n-1} = \mathcal{U} \cap \mathbf{D}^2$; then,

$$\prod_{j=1}^{n-1} \left(z - \rho_j\right) = z^{n-1} + \sum_{j=1}^{n-1} (-1)^j \, e_j\,(\mathcal{R})\, z^{n-1-j},$$

where $e_j\,(\mathcal{R})$ is unambiguous by nature of e_j being a *symmetric* polynomial. Using Newton's identities [Mac95, I.2 eq. (2.11')], those coefficients can be written in terms of a sum of the power-sum polynomials (in those roots) gotten via the argument principle mentioned above: for $j > 0$,

$$j e_j\,(\mathcal{R}) = \sum_{k=1}^{j} (-1)^{k-1} \, e_{j-k}\,(\mathcal{R}) \underbrace{\sum_{s=1}^{n-1} \rho_s^k}_{\substack{k\text{-th power-sum} \\ \text{polynomial in } \mathcal{R}}} = \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{j} (-1)^{k-1} \, e_{j-k}\,(\mathcal{R}) \int_{\partial \mathbf{D}^2} \omega^k \frac{h'_{\mathcal{Z}}\,(\omega)}{h_{\mathcal{Z}}\,(\omega)} \, \mathrm{d}\omega.$$

Therefore, the sought-after polynomial of reduced degree is

$$z^{n-1} + \sum_{j=1}^{n-1} \frac{(-1)^j}{2\pi \mathrm{i} j} \left(\sum_{k=1}^{j} (-1)^{k-1} \, e_{j-k}\,(\mathcal{R}) \int_{\partial \mathbf{D}^2} \omega^k \frac{h'_{\mathcal{Z}}\,(\omega)}{h_{\mathcal{Z}}\,(\omega)} \, \mathrm{d}\omega \right) z^{n-1-j}. \quad \blacktriangleleft$$

With that settled, properties about the roots and the coefficients these of polynomials will be shown in the next two lemmas.

2.1.3 Lemma

The roots of $h_{\mathcal{Z}}$ are paired in the following way: for every root $\rho \neq 0$, there is a root also at $1/\overline{\rho}$, and the consequence for $\rho = 0$ being a root is that the leading coefficient of $h_{\mathcal{Z}}$ is zero. It follows that, there are $n-1$ roots of $h_{\mathcal{Z}}$ which lie in $\mathbf{D}^2$, and they determine all the roots.

Proof Let ρ be a root of $h_{\mathcal{Z}}$. Then, for $\rho \neq 0$,

$$\begin{aligned} 0 = h_{\mathcal{Z}}\,(\rho) &= \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \left(\zeta_k - \rho\right) \left(1 - \overline{\zeta_k} \rho\right) \\ &= \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \rho^2 \left(\frac{\zeta_k}{\rho} - 1 \right) \left(\frac{1}{\rho} - \overline{\zeta_k} \right) \\ &= \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \overline{\rho}^2 \overline{\left(\frac{\zeta_k}{\overline{\rho}} - 1 \right)} \overline{\left(\frac{1}{\overline{\rho}} - \zeta_k \right)} \\ &= \overline{\rho}^{2(n-1)} \overline{h_{\mathcal{Z}} \left(\frac{1}{\overline{\rho}} \right)} \\ \Longleftrightarrow h_{\mathcal{Z}} \left(\frac{1}{\overline{\rho}} \right) &= 0. \end{aligned}$$

For $\rho = 0$, it must be that the constant term of $h_{\mathcal{Z}}$ is zero:

$$0 = \operatorname{coeff}(1, h_{\mathcal{Z}}\,(z)) = \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \operatorname{coeff}\left(1, \prod_{k \neq j} (\zeta_k - z) \left(1 - \overline{\zeta_k} z\right) \right) = \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \zeta_k.$$

The leading coefficient is

$$\begin{aligned} \operatorname{coeff}\left(z^{2n-2}, h_{\mathcal{Z}}\,(z)\right) = \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \operatorname{coeff}\left(z^2, (\zeta_k - z) \left(1 - \overline{\zeta_k} z\right)\right) &= \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \prod_{k \neq j} \overline{\zeta_k} \\ &= \overline{\operatorname{coeff}(1, h_{\mathcal{Z}}\,(z))} = 0. \end{aligned} \quad \blacksquare$$

Expanding on a result used in the proof of the previous lemma, there is the following.

2.1.4 Lemma

In the polynomial $h_{\mathcal{Z}}\,(z)$, there is the following relation of coefficients

$$\operatorname{coeff}\left(z^t, h_{\mathcal{Z}}\,(z)\right) = \overline{\operatorname{coeff}\left(z^{2n-2-t}, h_{\mathcal{Z}}\,(z)\right)}.$$

Thus, $h_{\mathcal{Z}}$ always has a real coefficient for the term of degree $n-1$.

Proof Write

$$h_{\mathcal{Z}}(z) = \sum_{j=1}^{n} \underbrace{\left(|\zeta_j|^2 - 1\right)}_{D_j} \prod_{k \neq j} \Big(\zeta_k - \underbrace{\left(|\zeta_k|^2 + 1\right)}_{D'_k} z + \overline{\zeta_k} z^2\Big),$$

and focus on the product within the summation,

$$\prod_{j \neq k} \left(\zeta_k - D'_k z + \overline{\zeta_k} z^2\right),$$

since that is where z appears, contributing to the degree of the overall polynomial expansion. Notice that the polynomial $\zeta_k - D'_k z + \overline{\zeta_k} z^2$ satisfies the wanted property on the coefficients, so it is seems likely that the property also holds for their product.

Explicitly, a generic term of the expansion of that product is of the form

$$\left(\prod_{\alpha \in A_a} \zeta_\alpha\right)\left(\prod_{\beta \in B_b} -D'_\beta z\right)\left(\prod_{\gamma \in C_c} \overline{\zeta_\gamma} z^2\right),$$

where $A_a = \{\alpha_k\}_{k=1}^{a}$, $B_b = \{\beta_k\}_{k=1}^{b}$, and $C_c = \{\gamma_k\}_{k=1}^{c}$, are partitioning subsets of $I_j = \{1, \ldots, n\} \smallsetminus \{j\}$, in other words,

$$A_a, B_b, C_c \subseteq I_j \text{ such that } A_a \cap B_b, A_a \cap C_c, B_b \cap C_c = \varnothing \text{ and } A_a \cup B_b \cup C_c = I_j,$$

forcing

$$a + b + c = n - 1$$

and showing that the generic term written before is of degree $b + 2c$. Putting this together for each j, the conditions Ξ_j, which yield the terms in the coefficient of the degree-t term of $h_{\mathcal{Z}}(z)$, are

$$\Xi_j = \left\{\begin{array}{c} a, b, c \geq 0 \\ a + b + c = n - 1 \\ b + 2c = t \\ A_a \sqcup B_b \sqcup C_c = I_j \end{array}\right\},$$

so

$$\operatorname{coeff}\left(z^t, h_{\mathcal{Z}}(z)\right) = \sum_{j=1}^{n} D_j \sum_{\Xi_j} \zeta_{\alpha_1} \cdots \zeta_{\alpha_a} (-1)^b D'_{\beta_1} \cdots D'_{\beta_b} \overline{\zeta_{\gamma_1}} \cdots \overline{\zeta_{\gamma_c}},$$

whence

$$\overline{\operatorname{coeff}\left(z^t, h_{\mathcal{Z}}(z)\right)} = \sum_{j=1}^{n} D_j \sum_{\Xi_j} \overline{\zeta_{\alpha_1}} \cdots \overline{\zeta_{\alpha_a}} (-1)^b D'_{\beta_1} \cdots D'_{\beta_b} \zeta_{\gamma_1} \cdots \zeta_{\gamma_c}.$$

From that, it is easy to see that this last expression corresponds to the coefficient of $h_{\mathcal{Z}}(z)$ of degree

$$\begin{aligned} 2a + b &= 2(n - 1 - b - c) + b && \text{since } a + b + c = n - 1 \\ &= 2n - 2 - b - 2c \\ &= 2n - 2 - t && \text{since } b + 2c = t. \end{aligned}$$

■

2.1.5 Remark

With the previous lemma, the coefficients of terms degree of $0 \leq j \leq n - 1$ in $h_{\mathcal{Z}}$ determine all of its coefficients, and the coefficient of the term of degree $n - 1$ is real. Thusly, it is quick to notice that the natural inclusion $\iota \colon \mathrm{P}_{n-1} \hookrightarrow \mathbf{C}^{2n-1}$ factors as

$$\begin{tikzcd} \mathrm{P}_{n-1} \arrow[rr, hook, "\iota"] \arrow[dr, hook, "\iota_1"'] & & \mathbf{C}^{2n-1} \\ & \mathbf{C}^{n-1} \times \mathbf{R}, \arrow[ur, hook, "\iota_2"'] & \end{tikzcd}$$

where

$$\begin{aligned} \iota_1 &\colon h_{\mathcal{Z}} \longmapsto \operatorname{coeff}\left(z^j, h_{\mathcal{Z}}\right)_{j=0}^{n-1} \\ \iota_2 &\colon (z_1, \ldots, z_{n-1}, x) \longmapsto \left(z_1, \ldots, z_{n-1}, x, \overline{z_{n-1}}, \ldots, \overline{z_1}\right). \end{aligned}$$

◀

Also using the previous lemma, the next proposition discerns the form of $h_{\mathcal{Z}}$ for a special sets $\mathcal{Z}$.

2.1.6 PROPOSITION

For some $0 < r < 1$, let $\mathcal{Z} = \left\{ r e^{i\theta_j} \,\middle|\, 0 \le j \le n-1,\ \theta_j = \frac{2\pi j}{n} \right\}$. Then, $h_{\mathcal{Z}}$ only has a term of degree $n-1$. Furthermore, if $\{e^{i\theta}\zeta \mid \zeta \in \mathcal{Z}\} = \mathcal{Z}' \in \mathrm{C}_n$ for some $\theta \in \mathbf{R}$, which is to say that $\mathcal{Z}'$ is related to $\mathcal{Z}$ by a uniform rotation by an angle θ, then $h_{\mathcal{Z}} = h_{\mathcal{Z}'}$.

In general, it will follow that, if a $\mathcal{Z} \in \mathrm{C}_n$ is related to another $\mathcal{Z}' \in \mathrm{C}_n$ by a rotation, then

$$\operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}}(z)\right) = \operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}'}(z)\right).$$

PROOF By the lemma ¶2.1.4, it suffices to show that all coefficients of terms of degree less than $n-1$ are trivial. Proceeding as such, let Ξ_j be the j-dependent conditions on the inner summation in the previous proof, and then, the coefficient of the degree t term is

$$\operatorname{coeff}\left(z^t, h_{\mathcal{Z}}(z)\right) = \sum_{j=1}^{n} D \sum_{\Xi_j} r^a e^{i\theta_{\alpha_1}} \cdots e^{i\theta_{\alpha_a}} \left(-D'\right)^b r^c e^{-i\theta_{\gamma_1}} \cdots e^{-i\theta_{\gamma_c}},$$

since $D_j = D = r^2 - 1$ for all j and $D'_k = D' = r^2 + 1$ for all k. Now, rotating sinks in the inner summation so that "missing" j-th sink becomes real

$$\begin{aligned}
\operatorname{coeff}\left(z^t, h_{\mathcal{Z}}(z)\right) &= \sum_{j=1}^{n} D \sum_{\Xi_j} \left(e^{i\theta_j}\right)^{a-c} r^a e^{i(\theta_{\alpha_1}-\theta_j)} \cdots e^{i(\theta_{\alpha_a}-\theta_j)} \left(-D'\right)^b r^c e^{-i(\theta_{\gamma_1}-\theta_j)} \cdots e^{-i(\theta_{\gamma_c}-\theta_j)} \\
&= \sum_{j=1}^{n} D \left(e^{i\theta_j}\right)^{n-1-t} \sum_{\Xi_j} r^a e^{i(\theta_{\alpha_1}-\theta_j)} \cdots e^{i(\theta_{\alpha_a}-\theta_j)} \left(-D'\right)^b r^c e^{-i(\theta_{\gamma_1}-\theta_j)} \cdots e^{-i(\theta_{\gamma_c}-\theta_j)} \\
&\qquad \Xi_j \supset \left\{ \begin{array}{c} a+b+c = n-1 \\ b+2c = t \end{array} \right\} \implies a-c = n-1-t \\
&= \left(\sum_{j=1}^{n} D \left(e^{i\theta_j}\right)^{n-1-t} \right) \sum_{\Xi_0} r^a e^{i\theta_{\alpha_1}} \cdots e^{i\theta_{\alpha_a}} \left(-D'\right)^b r^c e^{-i\theta_{\gamma_1}} \cdots e^{-i\theta_{\gamma_c}},
\end{aligned}$$

where the rotational dependence on j of the inner summation was factored out. Thus, it suffices to show that

$$\sum_{j=1}^{n} \left(e^{i\theta_j}\right)^{n-1-t} = \sum_{j=1}^{n} e^{i\frac{2\pi j}{n}(n-1-t)} \stackrel{!}{=} 0$$

for $0 \le t < n-1$. This conclusion is reached after noting the facts that multiplication by $n-1-t$ has the following consequence

$$(n-1-t)\,(\mathbf{Z}/n\mathbf{Z}) \cong \mathbf{Z}/\gcd(n, n-1-t)\mathbf{Z}$$

along with its correspondence to an action on the roots of unity, and the fact that the sum of the m-th roots of unity for any $m > 1$ is trivial

$$\sum_{j=0}^{m-1} e^{i\frac{2\pi j}{m}} = \frac{1 - e^{2\pi i}}{1 - e^{i\frac{2\pi}{m}}} = 0.$$

Lastly, let $\mathcal{Z}'$ be as in the lemma's statement. Then,

$$\begin{aligned}
h_{\mathcal{Z}'}(z) &= \sum_{j=1}^{n} \left(\left| e^{i\theta}\zeta_j \right|^2 - 1 \right) \prod_{k \ne j} \left(e^{i\theta}\zeta_k - \left(\left| e^{i\theta}\zeta_k \right|^2 + 1 \right) z + \overline{e^{i\theta}\zeta_k} z^2 \right) \\
&= \sum_{j=1}^{n} \left(|\zeta_j|^2 - 1 \right) \left(e^{i\theta}\right)^{n-1} \prod_{k \ne j} \left(\zeta_k - (|\zeta_k|^2 + 1) \frac{z}{e^{i\theta}} + \overline{\zeta_k} \frac{z^2}{e^{2i\theta}} \right),
\end{aligned}$$

from which one can notice that

$$\operatorname{coeff}\left(\left(\frac{z}{e^{i\theta}} \right)^{n-1}, h_{\mathcal{Z}'}(z) \right) = \left(e^{i\theta}\right)^{n-1} \operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}}(z)\right).$$

Using the previous argument, $h_{\mathcal{Z}'}$ also only has a term of degree $n-1$, so it follows

$$h_{\mathcal{Z}'}(z) = \left(\frac{z}{e^{i\theta}} \right)^{n-1} \operatorname{coeff}\left(\left(\frac{z}{e^{i\theta}} \right)^{n-1}, h_{\mathcal{Z}'}(z) \right) = z^{n-1} \operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}}(z)\right) = h_{\mathcal{Z}}(z).$$

Moreover, that can be generalized to a statement about all $\mathcal{Z}$, $\mathcal{Z}'$ related by a rotation, as:

$$\operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}'}(z)\right) = \left(e^{i\theta}\right)^{-(n-1)} \operatorname{coeff}\left(\left(\frac{z}{e^{i\theta}} \right)^{n-1}, h_{\mathcal{Z}'}(z) \right) = \operatorname{coeff}\left(z^{n-1}, h_{\mathcal{Z}}(z)\right).$$

■

Appealing to intuition, the next theorem shows that, if $\mathcal{Z} \in \mathrm{C}^n$ is moved by a disk automorphism, then the roots move correspondingly so. This shows that results about critical-point polynomials can be thought of up to a disk automorphism.

2.1.7 Theorem

Let $\mathcal{Z} \in \mathrm{C}_n$ and let $\mathcal{R}$ be the roots of $h_{\mathcal{Z}}$ in $\mathbf{D}^2$. Then, for any automorphism $\Phi\colon \mathbf{D}^2 \longrightarrow \mathbf{D}^2$, the roots of $h_{\Phi(\mathcal{Z})}$ in $\mathbf{D}^2$ are $\Phi(\mathcal{R})$.

Moreover, via lemma ¶2.1.3, it follows those roots $1/\overline{\rho}$ of $h_{\mathcal{Z}}$ outside of $\mathbf{D}^2$ correspond with those roots $1\big/\overline{\Phi(\rho)}$ of $h_{\Phi(\mathcal{Z})}$, again outside of $\mathbf{D}^2$.

Proof Let $\mathcal{Z} = \{\zeta_j\}_{j=1}^n$, as per usual. And, recall from remark ¶1.1.1 the properties of the maps $\alpha_\omega(z) = \frac{\omega - z}{1-\overline{\omega}z}$, and that the form of Φ is necessarily

$$\Phi(z) = \mathrm{e}^{\mathrm{i}\theta}\alpha_\omega(z)$$

for some $\theta \in \mathbf{R}$ and $\omega \in \mathbf{D}^2$.

With that, an easier case will first be shown: (I.) if $\rho \in \mathcal{R} \subset \mathbf{D}^2$ is a root of $h_{\mathcal{Z}}$, then, for any $\theta \in \mathbf{R}$, zero is a root of $h_{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\mathcal{Z})}$, and vice versa, (II.) if zero is a root of $h_{\mathcal{Z}}$, then, for any $\theta \in \mathbf{R}$ and any $\rho \in \mathbf{D}^2$, ρ is a root of $h_{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\mathcal{Z})}$.

I. It suffices to show that

$$\operatorname{coeff}\left(1, h_{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\mathcal{Z})}(z)\right) \overset{!}{=} 0.$$

As per the definitions,

$$\begin{aligned} h_{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\mathcal{Z})}(z) &= \sum_{j=1}^n \left(\left|\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_j)\right|^2 - 1\right) \prod_{k\neq j}\left(\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_k) - z\right)\left(1 - \overline{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_k)}z\right) \\ &= \sum_{j=1}^n \left(\left|\alpha_\rho(\zeta_j)\right|^2 - 1\right) \prod_{k\neq j}\left(\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_k) - z\right)\left(1 - \overline{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_k)}z\right) \end{aligned}$$

so

$$\begin{aligned} \operatorname{coeff}\left(1, h_{\mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\mathcal{Z})}(z)\right) &= \sum_{j=1}^n \left(\left|\alpha_\rho(\zeta_j)\right|^2 - 1\right) \prod_{k\neq j} \mathrm{e}^{\mathrm{i}\theta}\alpha_\rho(\zeta_k) \\ &= \mathrm{e}^{(n-1)\mathrm{i}\theta} \sum_{j=1}^n \left(\left|\frac{\rho - \zeta_j}{1-\overline{\rho}\zeta_j}\right|^2 - 1\right) \prod_{k\neq j} \frac{\rho-\zeta_k}{1-\overline{\rho}\zeta_k} \\ &= \mathrm{e}^{(n-1)\mathrm{i}\theta}\left(\prod_{l=1}^n \left|1-\overline{\rho}\zeta_l\right|^{-2}\right)\left(\sum_{j=1}^n\left(\left|\rho-\zeta_j\right|^2 - \left|1-\overline{\rho}\zeta_j\right|^2\right)\prod_{k\neq j}(\rho-\zeta_k)\overline{(1-\overline{\rho}\zeta_k)}\right), \end{aligned}$$

wherein the factor $\prod_{l=1}^n \left|1-\overline{\rho}\zeta_l\right|^{-2} \neq 0$ because ρ, and each $\zeta_l \in \mathcal{Z}$, are in the disk while

$$\text{if} \quad 1-\overline{\rho}\zeta_l \overset{!}{=} 0, \quad \text{then} \quad \left(\zeta_l \overset{!}{=} 1/\overline{\rho} \in \mathbf{C}\smallsetminus\mathbf{D}^2 \quad \text{or} \quad \overline{\rho} \overset{!}{=} 1/\zeta_l \in \mathbf{C}\smallsetminus\mathbf{D}^2\right).$$

Simplifying parts further yields

$$\begin{aligned} \left|\rho-\zeta_j\right|^2 - \left|1-\overline{\rho}\zeta_j\right|^2 &= (\rho-\zeta_j)\left(\overline{\rho}-\overline{\zeta_j}\right) - (1-\overline{\rho}\zeta_j)\left(1-\rho\overline{\zeta_j}\right) \\ &= |\rho|^2 - \rho\overline{\zeta_j} - \zeta_j\overline{\rho} + |\zeta_j|^2 - \left(1-\rho\overline{\zeta_j} - \overline{\rho}\zeta_j + |\rho|^2|\zeta_j|^2\right) \\ &= |\rho|^2 + |\zeta_j|^2 - 1 - |\rho|^2|\zeta_j|^2 \\ &= \left(1-|\rho|^2\right)\left(|\zeta_j|^2 - 1\right) \end{aligned}$$

and

$$\begin{aligned} (\rho-\zeta_k)\overline{(1-\overline{\rho}\zeta_k)} &= \rho - \rho^2\overline{\zeta_k} - \zeta_k + \rho|\zeta_k|^2 \\ &= -\left(\zeta_k - (|\zeta_k|^2+1)\rho + \overline{\zeta_k}\rho^2\right) \\ &= -(\zeta_k-\rho)\left(1-\overline{\zeta_k}\rho\right). \end{aligned}$$

Thus,

$$\operatorname{coeff}\left(1, h_{e^{i\theta}\alpha_\rho(\mathcal{Z})}(z)\right) = (-1)^{n-1} e^{(n-1)i\theta}\left(1-|\rho|^2\right)\left(\prod_{l=1}^{n}\left|1-\overline{\rho}\zeta_l\right|^{-2}\right)\underbrace{\left(\sum_{j=1}^{n}\left(\left|\zeta_j\right|^2-1\right)\prod_{k\neq j}\left(\zeta_k-\rho\right)\left(1-\overline{\zeta_k}\rho\right)\right)}_{=h_{\mathcal{Z}}(\rho)=0}$$

$$= 0,$$

as was to be shown.

II. Notice that the equation

$$\operatorname{coeff}\left(1, h_{e^{i\theta}\alpha_\rho(\mathcal{Z})}(z)\right) = (-1)^{n-1} e^{(n-1)i\theta}\underbrace{\left(1-|\rho|^2\right)}_{\substack{\neq 0\\ \text{because}\\ \rho\in\mathbf{D}^2}}\left(\prod_{\zeta\in\mathcal{Z}}\left|1-\overline{\rho}\zeta\right|^{-2}\right)h_{\mathcal{Z}}(\rho),$$

obtained in (I.), is independent of the hypotheses of (I.), so it is applicable here. For simplicity, call the non-zero constant of the right-hand side of that equation

$$D(\theta,\rho,\mathcal{Z}) := (-1)^{n-1} e^{(n-1)i\theta}\left(1-|\rho|^2\right)\left(\prod_{\zeta\in\mathcal{Z}}\left|1-\overline{\rho}\zeta\right|^{-2}\right).$$

By the involutive property of α_ρ from remark ¶1.1.1, it is easy to see that the inverse of $e^{i\theta}\alpha_\rho$ is $\alpha_\rho\circ\left(e^{-i\theta}\cdot-\right)$, where $\left(e^{-i\theta}\cdot-\right)$ is multiplication by $e^{-i\theta}$; the inverse can also be written

$$\left(\alpha_\rho\circ\left(e^{-i\theta}\cdot-\right)\right)(z) = \frac{\rho-e^{-i\theta}z}{1-\overline{\rho}e^{-i\theta}z} = e^{-i\theta}\frac{e^{i\theta}\rho-z}{1-\overline{e^{i\theta}\rho}z} = e^{-i\theta}\alpha_{e^{i\theta}\rho}(z).$$

Therewith, writing the identity as $\left(e^{-i\theta}\alpha_{e^{i\theta}\rho}\right)\circ\left(e^{i\theta}\alpha_\rho\right)$, the equation from (I.) yields

$$0 = \operatorname{coeff}(1, h_{\mathcal{Z}}(z)) = D\left(-\theta,\ e^{i\theta}\rho,\ e^{i\theta}\alpha_\rho(\mathcal{Z})\right)h_{e^{i\theta}\alpha_\rho(\mathcal{Z})}(\rho),$$

whereby the zero-product property implies

$$h_{e^{i\theta}\alpha_\rho(\mathcal{Z})}(\rho) = 0$$

as desired.

Lastly, the results (I.), and (II.), will be used to show the overall claim. To do this, the note about disk automorphisms in remark ¶1.1.1 will be used again: for a disk automorphism Φ, once the images of $0\in\mathbf{D}^2$, and any another point $\rho\in\mathbf{D}^2$, are determined, then the entire map Φ is determined and is of the form

$$\Phi(z) = \exp\left(i\arg(\Phi(\rho)) - i\arg\left(\alpha_{\Phi(0)}(\Phi(\rho))\right)\right)\alpha_{\Phi(0)}(z).$$

In particular, it will be shown that $\Phi(z) = e^{i\theta}\alpha_\omega(z)$ can be written as a composition of two disk automorphisms, one taking any one root ρ of $h_{\mathcal{Z}}$ to zero with a to-be-determined rotation by θ', and then another taking zero to $\Phi(\rho)$, namely,

$$\Phi = e^{i\theta}\alpha_\omega \overset{!}{=} \alpha_{\Phi(\rho)}\circ\left(e^{i\theta'}\alpha_\rho\right),$$

from which it will follow by (I.), and (II.), that $\Phi(\rho)$ is a root of $h_{\Phi(\mathcal{Z})}$. This decomposition of Φ can be then be done for all roots $\mathcal{R}$ in $\mathbf{D}^2$ of $h_{\mathcal{Z}}$, which, by the bijectivity of Φ, forces $h_{\Phi(\mathcal{Z})}$ to have exactly the roots $\Phi(\mathcal{R})$ in $\mathbf{D}^2$.

Determining θ': there are the following images

$$0 \longmapsto e^{i\theta}\omega \overset{!}{=} \alpha_{\Phi(\rho)}\left(e^{i\theta'}\rho\right)$$

$$\rho \longmapsto e^{i\theta}\alpha_\omega(\rho) \overset{!}{=} \alpha_{\Phi(\rho)}(0) = \Phi(\rho),$$

where the second line is already guaranteed, so it is just needed to show the first line. Using the fact $\alpha_{\Phi(\rho)}$ is an automorphism and an involution, the first line is equivalent to showing,

$$\alpha_{\Phi(\rho)}\left(e^{i\theta}\omega\right) \stackrel{!}{=} e^{i\theta'}\rho.$$

Working from the left-hand side,

$$\begin{aligned}
\alpha_{\Phi(\rho)}\left(e^{i\theta}\omega\right) &= \frac{e^{i\theta}\alpha_\omega(\rho) - e^{i\theta}\omega}{1-\overline{e^{i\theta}\alpha_\omega(\rho)}e^{i\theta}\omega} && \text{by definition, with } \Phi = e^{i\theta}\alpha_\omega \\
&= e^{i\theta}\frac{\alpha_\omega(\rho)-\omega}{1-\overline{\alpha_\omega(\rho)}\omega} \\
&= e^{i\theta}\frac{\frac{\omega-\rho}{1-\overline{\omega}\rho}-\omega}{1-\frac{\overline{\omega}-\overline{\rho}}{1-\omega\overline{\rho}}\omega} \\
&= e^{i\theta}\frac{(\omega-\rho)(1-\omega\overline{\rho})-\omega(1-\overline{\omega}\rho)(1-\omega\overline{\rho})}{(1-\overline{\omega}\rho)(1-\omega\overline{\rho})-(\overline{\omega}-\overline{\rho})\omega(1-\overline{\omega}\rho)} \\
&= e^{i\theta}\frac{\left(\omega-\omega^2\overline{\rho}-\rho+\omega|\rho|^2\right)-\omega\left(1-\omega\overline{\rho}-\overline{\omega}\rho+|\omega|^2|\rho|^2\right)}{\left(1-\omega\overline{\rho}-\overline{\omega}\rho+|\omega|^2|\rho|^2\right)-\omega\left(\overline{\omega}-\overline{\omega}^2\rho-\overline{\rho}+\overline{\omega}|\rho|^2\right)} \\
&= e^{i\theta}\frac{-\rho+\omega|\rho|^2+|\omega|^2\rho-\omega|\omega|^2|\rho|^2}{1-\overline{\omega}\rho-|\omega|^2+\overline{\omega}|\omega|^2\rho} \\
&= -\rho e^{i\theta}\underbrace{\frac{1-\omega\overline{\rho}-|\omega|^2+\omega|\omega|^2\overline{\rho}}{1-\overline{\omega}\rho-|\omega|^2+\overline{\omega}|\omega|^2\rho}}_{\in \mathbf{S}^1} \\
&= -\rho e^{i\theta}e^{i\varphi(\omega,\rho)} \\
&= \rho e^{i(\theta+\varphi(\omega,\rho)+\pi)}.
\end{aligned}$$

Therefore, for any root $\rho\in\mathbf{R}\subset\mathbf{D}^2$ of $h_{\mathcal{Z}}$, it suffices to set

$$\theta' = \theta+\varphi(\omega,\rho)+\pi,$$

which, by the earlier argument, finishes the proof. ∎

2.1.8 HYPOTHESIS

It may be the case that, up to disk automorphism, the only elements $\mathcal{Z}\in\mathrm{C}_n$, which have a critical-point polynomial $h_{\mathcal{Z}}$ with only a single root $\rho\in\mathbf{D}^2$, are those $\mathcal{Z}$ which are a rescaled set of the n-th roots of unity. Formally, this is to say that there exists an $\omega\in\mathbf{D}^2$, $\theta\in\mathbf{R}$, and an $r\in(0,1)$, such that

$$e^{i\theta}\alpha_\omega(\mathcal{Z}) = \left\{re^{i\frac{2\pi(j-1)}{n}}\right\}_{j=1}^{n}.$$

This hypothesis comes from the intuition about the correlation between critical points of $h_{\mathcal{Z}}$ and symmetries of the elements of $\mathcal{Z}\in\mathrm{C}_n$, as well the inspiration from the combination of proposition ¶2.1.6 and theorem ¶2.1.7. The former shows that rescaled n-th roots of unity yield a critical-point polynomial with only one root in $\mathbf{D}^2$, and it lies at the origin. And, the latter reduces the scope of the question to considering $\mathcal{Z}\in\mathrm{C}_n$ such that $h_{\mathcal{Z}}$ has only one root in $\mathbf{D}^2$, and it lies at the origin — which is to say that setting $\omega=\rho$ should suffice for the aforementioned expression.

A possible way of approaching this is to consider the $n-1$ equations in the n variables $\{\zeta_j\}_{j=1}^n=\mathcal{Z}$ obtained from the conditions on the coefficients of $h_{\mathcal{Z}}$ so that it has only one root, and it lies at the origin,

$$\begin{cases}
\operatorname{coeff}(1,h_{\mathcal{Z}}(z)) \stackrel{!}{=} 0 \\
\operatorname{coeff}(z,h_{\mathcal{Z}}(z)) \stackrel{!}{=} 0 \\
\qquad\vdots \\
\operatorname{coeff}\left(z^{n-2},h_{\mathcal{Z}}(z)\right) \stackrel{!}{=} 0,
\end{cases}$$

and then conclude that either $\mathcal{Z}$ must of that special form, or that, if a solution exists, as shown in proposition ¶2.1.6, then it must be unique.

2.2 The Second Map

The qualifications of the second map $\mathrm{Gr}_n \longrightarrow \mathrm{P}_n$ will be discussed in this section, relying on the properties and comments made in the previous section about the space of critical-point polynomials P_{n-1}.

2.2.1 Hypothesis

With the topologies defined in construction ¶1.1.9 for Gr_n and in the definitions ¶2.1.1 for P_{n-1}, the map

$$\begin{aligned} \chi\colon \mathrm{Gr}_n &\longrightarrow \mathrm{P}_{n-1} \\ g_{\mathcal{Z}} &\longmapsto h_{\mathcal{Z}} \end{aligned}$$

is, at most, homotopy equivalence.

The continuity of χ follows from the fact from theorem ¶1.2.1 that $\mathrm{Gr}_n \cong \mathrm{C}_n$ and the coefficients of

$$h_{\mathcal{Z}}(z) := \sum_{j=1}^{n} \left(\left|\zeta_j\right|^2 - 1\right) \prod_{k \neq j} (\zeta_k - z)\left(1 - \overline{\zeta_k} z\right)$$

vary continuously with the elements of $\mathcal{Z} = \{\zeta_j\}_{j=1}^n \in \mathrm{C}_n$. The failure for χ to be a homeomorphism follows from the factoring of the inclusion $\mathrm{P}_{n-1} \hookrightarrow \mathbf{C}^{2n-1}$ through $\mathbf{C}^{n-1} \times \mathbf{R}$, as mentioned in remark ¶2.1.5, and the fact that C_n has n complex dimensions.

In finding the homotopy inverse of this map, it might be helpful to reduce the problem to considering the polynomials, and Green's functions, up to disk automorphism. Furthermore, using lemma ¶1.1.2 might then also be helpful, since it implies that the hyperbolic distances are preserved between the points which are the sinks $\mathcal{Z} \in \mathrm{C}_n$ and are the roots in $\mathbf{D}^2$ of the critical-point polynomial $h_{\mathcal{Z}}$.

Lastly, note that, because of the homeomorphism $\mathrm{Gr}_n \cong \mathrm{C}_n$, finding a map backward from P_{n-1} to Gr_n is inherently connected to (dis)proving hypothesis ¶2.1.8.

3. The Chord Space Ch_{n-1} and the Third Map

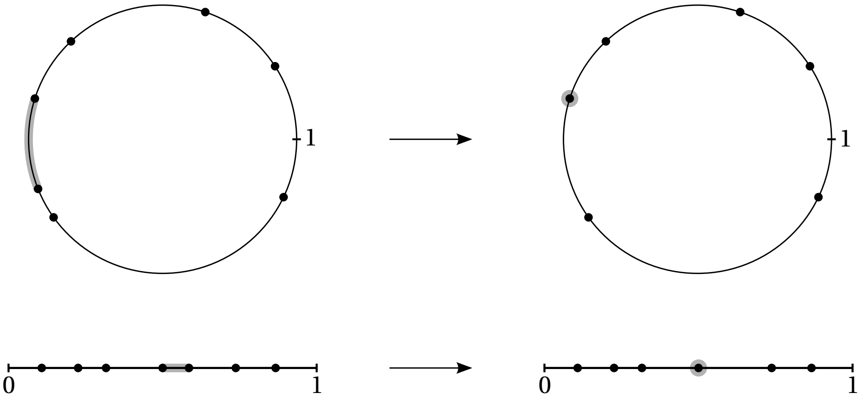
The first purpose of this chapter is to define the "chord space" $\mathrm{Ch}_{n-1}\,\mathbf{S}^1$ on the boundary of the 2-disk $\partial\mathbf{D}^2 = \mathbf{S}^1$, which will be a subspace of the larger "chord space" $\mathrm{Ch}\,\mathbf{S}^1$. After that, this chapter will describe the third map from the space of Green's functions on the 2-disk with n sinks $\mathrm{Gr}_n\,\mathbf{D}^2$ to $\mathrm{Ch}_{n-1}\,\mathbf{S}^1$,

$$\mathrm{Gr}_n\,\mathbf{D}^2 \longrightarrow \mathrm{Ch}_{n-1}\,\mathbf{S}^1 \subset \mathrm{Ch}\,\mathbf{S}^1.$$

3.1 The Chord Space Ch

The "chord space" will be constructed as the geometric realization of a semi-simplicial set. [GM03, ¶I.2.2] In order to form this semi-simplicial set, a graded set S and face maps d^j_k for $0 \le k \le j$ will need to be constructed. [nCa] [May92, Ch. 1] [GM03, §I.1.1] As to better understand the third map $\mathrm{Gr}_n\,\mathbf{D}^2 \longrightarrow \mathrm{Ch}_{n-1}\,\mathbf{S}^1$ as described in §3.2, the construction of this semi-simplicial set will be obtained via a geometric setup, which will be elucidated more in remark ¶3.1.5.

The setup is the following: take some 1-dimensional manifold M — up to homeomorphism, the interval $[0,1]$ or the 1-sphere $\mathbf{S}^1 \cong [0,1]/(0\sim 1)$ — and associate to each subset $\mathcal{B} \subset M$ of j points, a depiction of the action of a permutation of j objects $\sigma \in \mathfrak{S}_j$, the j-th symmetric group. With that, the idea of "chords" is to capture an effect on σ by identifying two elements of $\mathcal{B} \cup \{0,1\} \subset [0,1]$, or $\mathcal{B} \cup \{1\} \subset \mathbf{S}^1 \subset \mathbf{C}$, which is done by collapsing one of the closed intervals in between those elements. On the level of M, for example, this looks like:

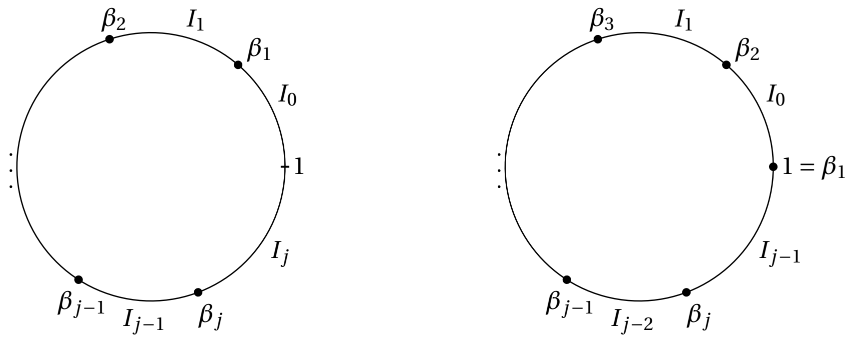


with the symbol • representing the elements of $\mathcal{B}$, the shaded intervals being those that are collapsed, and the shaded points being the points of the interval after the identifications from collapsing.

Here, as to first obtain $\mathrm{Ch}\,\mathbf{S}^1 \supset \mathrm{Ch}_{n-1}\,\mathbf{S}^1$, the focus will be on this construct for $M = \mathbf{S}^1$, which is delineated formally in the following.

3.1.1 Construction

Fix an orientation on $\mathbf{S}^1$, and let $\mathcal{B} = \{\beta_k\}_{k=1}^{j} \subset \mathbf{S}^1$, wherein an ordering of the indices can be fixed by taking the element $1 \in \mathbf{S}^1 \subset \mathbf{C}$ as the starting point, and labelling the elements of $\mathcal{B}$ going counter-clockwise. Likewise, the intermediating closed intervals between two adjacent elements of $\mathcal{B}$ can be labelled, but this will be done in two cases.



If $1 \notin \mathcal{B}$, then the intervals are $\{I_k\}_{k=0}^{j}$, where I_k goes from β_k to β_{k+1}, for $0 \le k < j$, and I_j goes from β_j to 1. If $1 \in \mathcal{B}$, then the intervals are $\{I_k\}_{k=0}^{j-1}$, there I_k goes from β_{k+1} to β_{k+2}, for $0 \le k < j-1$, and I_{j-1} goes from β_j to β_1.

The effect *collapsing* of intervals I_k is broken up into the following cases:

When $1 \notin \mathcal{B}$: For $1 \le k \le j-1$, collapsing I_k results in a set $\mathcal{B}$ obtained from $\mathcal{B}$ by

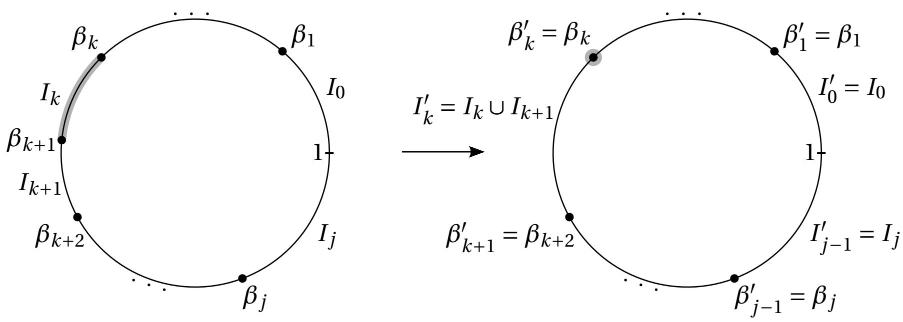
$$\mathcal{B}' = \{\beta'_\ell\}_{\ell=1}^{j-1} = \mathcal{B} \smallsetminus \{\beta_{k+1}\},$$

where the labelling of the elements of $\mathcal{B}'$ is

$$\beta'_\ell = \beta_\ell \text{ for } 1 \le \ell \le k \quad \text{and} \quad \beta'_\ell = \beta_{\ell+1} \text{ for } k+1 \le \ell \le j-1,$$

and the corresponding intervals $\{I'_\ell\}_{\ell=0}^{j-1}$ are

$$I'_\ell = I_\ell \text{ for } 0 \le \ell \le k-1, \quad I'_k = I_k \cup I_{k+1}, \quad \text{and} \quad I'_\ell = I_{\ell+1} \text{ for } k+1 \le \ell \le j-1.$$



After collapsing I_0, the resulting set is

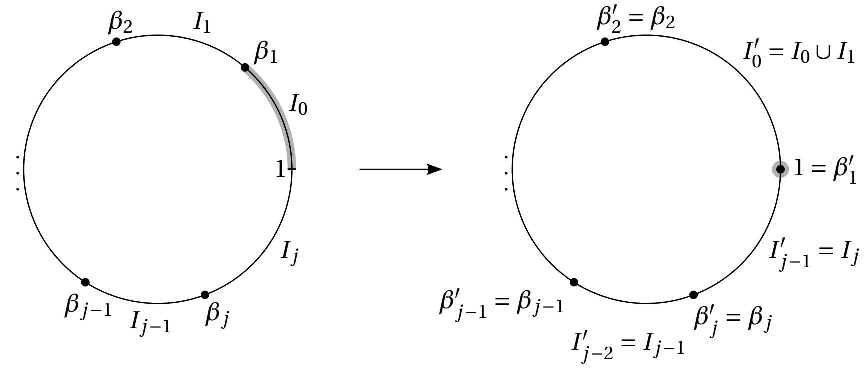
$$\mathcal{B}' = \{\beta'_\ell\}_{\ell=1}^{j} = (\mathcal{B} \smallsetminus \beta_1) \cup \{1\},$$

where

$$\beta'_1 = 1 \quad \text{and} \quad \beta'_\ell = \beta_\ell \text{ for } 2 \le \ell \le j,$$

and the corresponding intervals $\{I'_\ell\}_{\ell=0}^{j-1}$ are

$$I'_0 = I_0 \cup I_1 \quad \text{and} \quad I'_\ell = I_{\ell+1} \text{ for } 2 \le \ell \le j-1.$$



And, after collapsing I_j, the resulting set is

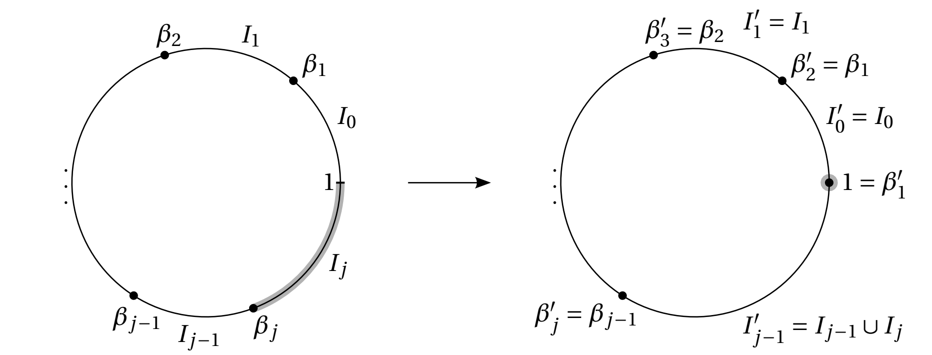
$$\mathcal{B}' = \{\beta'_\ell\}_{\ell=1}^{j} = (\mathcal{B} \smallsetminus \beta_j) \cup \{1\},$$

where

$$\beta'_1 = 1, \quad \text{and} \quad \beta'_\ell = \beta_{\ell-1} \text{ for } 2 \le \ell \le j,$$

and the corresponding intervals $\{I'_\ell\}_{\ell=0}^{j-1}$ are

$$I'_\ell = I_\ell \text{ for } 0 \le \ell \le j-2 \quad \text{and} \quad I'_{j-1} = I_{j-1} \cup I_j.$$



When $1 \in \mathcal{B}$: For $0 \le k \le j-2$, collapsing I_k yields a subset

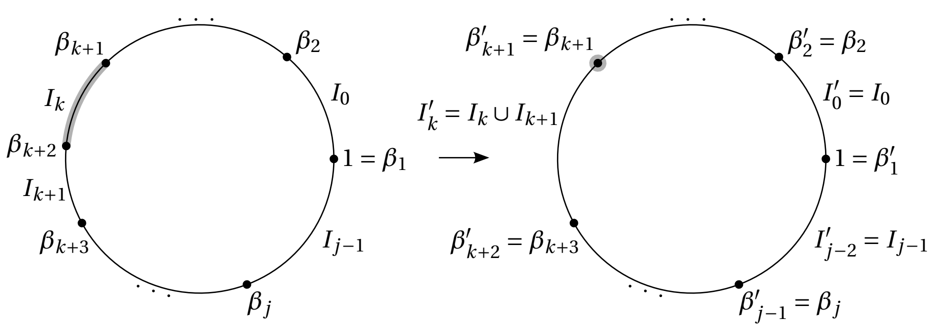
$$\mathcal{B}' = \{\beta'_\ell\}_{\ell=1}^{j-1} = \mathcal{B} \smallsetminus \{\beta_{k+2}\},$$

where

$$\beta'_\ell = \beta_\ell \text{ for } 0 \le \ell \le k+1 \quad \text{and} \quad \beta'_\ell = \beta_{\ell+1} \text{ for } k+2 \le \ell \le j-1,$$

and the corresponding intervals $\{I'_\ell\}_{\ell=0}^{j-2}$ are

$$I'_\ell = I_\ell \text{ for } 0 \le \ell \le k-1, \quad I'_k = I_k \cup I_{k+1}, \quad \text{and} \quad I'_\ell = I_{\ell+1} \text{ for } k+1 \le \ell \le j-2.$$



After collapsing I_{j-1}, the yielded set is

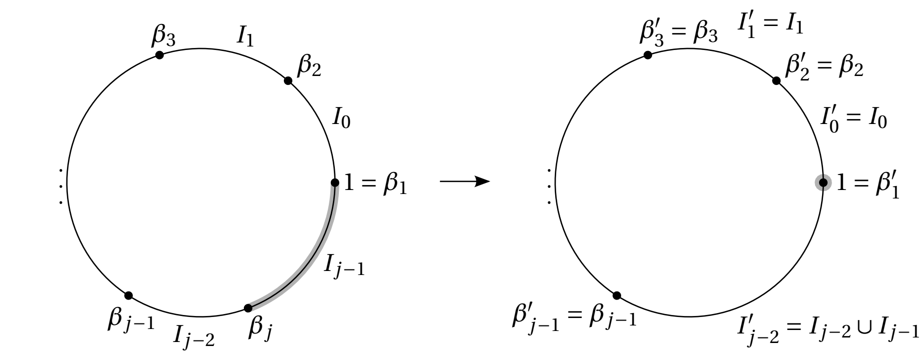
$$\mathcal{B}' = \{\beta_\ell\}_{\ell=1}^{j-1} = \mathcal{B} \smallsetminus \{\beta_j\},$$

where

$$\beta'_\ell = \beta_\ell \text{ for } 1 \le \ell \le j-1,$$

and the corresponding intervals $\{I'_\ell\}_{\ell=0}^{j-2}$ are

$$I'_\ell = I_\ell \text{ for } 0 \le \ell \le j-3 \quad \text{and} \quad I'_{j-2} = I_{j-2} \cup I_{j-1}.$$



The basis of the idea for obtaining "chords" on $\mathbf{S}^1$ is to associate elements σ of the j-th symmetric group $\mathfrak{S}_j$ to a configuration $\mathcal{B} \in \mathrm{C}_j\,\mathbf{S}^1$ of j points of $\mathbf{S}^1$ in the following way: with $\sigma \in \mathfrak{S}_j$ written as cycles, it is then possible to unambiguously make a correspondence between those integers $k \in \{1, \ldots, j\}$, which form those cycles, and elements β_k of $\mathcal{B}$.

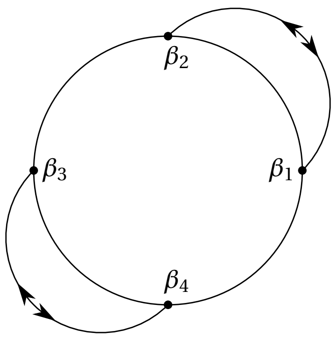
3.1.2 Definition

As before, let $\mathcal{B} = \{\beta_j\}_{j=1}^{m} \subset \mathbf{S}^1$, and let $\sigma \in \mathfrak{S}_m$, the m-th symmetric group. A *chord* is then an oriented edge connecting β_j with $\beta_{\sigma(j)}$. In the case when $\sigma^2(j) = j$, then the edge between β_j and $\beta_{\sigma(j)}$ is sometimes depicted as a bidirectional edge, to simplify pictures. Thusly, for each pair $(\mathcal{B}, \sigma)$, there is a collection of chords.

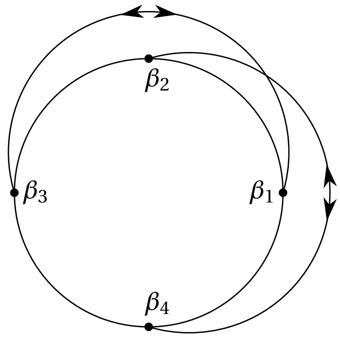
3.1.3 Examples

Let $\mathcal{B} = \left\{\beta_k = \mathrm{e}^{\frac{\mathrm{i}\pi(j-1)}{2}}\right\}_{k=1}^{4} \subset \mathbf{S}^1$. Then, the chords corresponding to the action of elements of $\mathfrak{S}_4$ on the elements of $\mathcal{B}$ can be depicted as in the following figures.

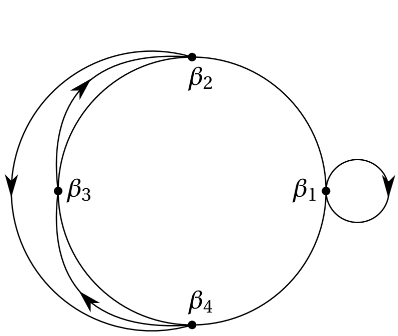
$(12)(34) \in \mathfrak{S}_4$



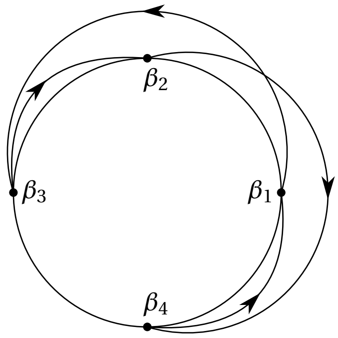
$(13)(24) \in \mathfrak{S}_4$



$(1)(324) \in \mathfrak{S}_4$



$(1324) \in \mathfrak{S}_4$



Note that the chords cross in the second and fourth figures, while in the first and third, they do not. Also, note that the fact they cross is a characteristic that cannot be avoided, regardless of how the chords are drawn outside of $\mathbf{S}^1$. ⧓

With the a-posteriori motivating background in place, the graded set $\mathsf{S} = \bigsqcup_{j=0}^{\infty} \mathsf{S}_j$ will be defined, accompanied with face maps $\left\{\mathsf{d}_k^j\right\}_{k=0}^{j}$. [nCa] [May92, Ch. 1] [GM03, §I.1.1] Together with the lemma afterwards stating the needed simplicial identity, the wanted semi-simplicial set will then be formed, from which the "chord space" $\mathrm{Ch}\,\mathbf{S}^1$ will be formed as the geometric realization. [GM03, ¶I.2.2]

3.1.4 Definition

Let $\mathfrak{S}_m$ be the m-th symmetric group, and then define the *graded set of permutations* S as

$$\mathsf{S} := \bigsqcup_{j=0}^{\infty} \mathsf{S}_j,$$

where $\mathsf{S}_j := \mathfrak{S}_j \sqcup \mathfrak{S}_{j+1}$.

Before defining the face maps, intermediary maps c_k^j for each $j > 0$, and for $0 \leq k \leq j$, will first be defined. These maps which take a permutation of that set $\{\ell\}_{\ell=1}^{j+1}$ and output a permutation of the set $\{1, \ldots, k, k+2, \ldots j+1\}$ if $k \neq j+1$, or $\{2, \ldots, j+1\}$ if $k = j+1$. When $k \neq j+1$, then it is defined on any $\sigma \in \mathfrak{S}_{j+1}$ as

$$\mathsf{c}_k^j(\sigma) : \{1, \ldots, k, k+2, \ldots j+1\} \longrightarrow \{1, \ldots, k, k+2, \ldots j+1\}$$

$$\ell \longmapsto \begin{cases} \sigma(\ell) & \ell \neq k, \sigma^{-1}(k+1) \\ \sigma(k+1) & \ell = k \\ \sigma(k) & \ell = \sigma^{-1}(k+1). \end{cases}$$

In this case, the definition for $\mathsf{c}_k^j(\sigma)$, for some $j > 0$, can be depicted on cycles of σ as the following: if k and $k+1$ lie in the same cycle of σ, the other cycles remain the same, while the cycle where they cohabitate breaks into two as

$$\big(\cdots \quad \sigma^{-1}(k) \quad k \quad \sigma(k) \quad \cdots \quad \sigma^{-1}(k+1) \quad k+1 \quad \sigma(k+1) \quad \cdots\big),$$

and, if k and $k+1$ lie is different cycles, the other cycles remain the same, while the aforementioned cycles combine as[1]

$$\big(\sigma(k) \quad \cdots \quad \sigma^{-1}(k) \quad k\big)\ \big(k+1 \quad \sigma(k+1) \quad \cdots \quad \sigma^{-1}(k+1)\big).$$

When $k = j+1$, then, on any $\sigma \in \mathfrak{S}_{j+1}$, it is defined as

$$\begin{aligned} \mathsf{c}_k^j(\sigma) : \{2, \ldots, j+1\} &\longrightarrow \{2, \ldots, j+1\} \\ \ell &\longmapsto \begin{cases} \sigma(\ell) & \ell \neq k, \sigma^{-1}(1) \\ \sigma(1) & \ell = k \\ \sigma(k) & \ell = \sigma^{-1}(1), \end{cases} \end{aligned}$$

which corresponds with the other maps when things are considered $\mathrm{mod}\ j+1$. On the cycles of σ, the depiction of the definition for $\mathsf{c}_k^j(\sigma)$, when $k = j+1$, looks just as those displayed for the case when $k \neq j+1$, except the instances of $k+1$ are replaced with 1, which is $k+1 \bmod j+1$.

Now, with that, in order to obtain a permutation that is once again on the usual set $\{1, \ldots, j\}$ from a permutation on the set $\{1, \ldots, k, k+2, \ldots, j+1\}$ if $k \neq j+1$, or $\{2, \ldots, j+1\}$ if $k = j+1$, there are the natural maps for each $j > 0$, and $1 \leq k \leq j+1$:

$$\text{skipping } k+1, \text{ for } k \neq j+1, \quad \begin{aligned} \delta_k^j \colon \{1, \ldots, j\} &\longrightarrow \{1, \ldots, k, k+2, \ldots, j+1\} \\ \ell &\longmapsto \begin{cases} \ell & \ell \leq k \\ \ell+1 & \ell > k \end{cases} \end{aligned}$$

$$\text{skipping } 1, \text{ for } k = j+1, \quad \begin{aligned} \delta_k^j \colon \{1, \ldots, j\} &\longrightarrow \{2, \ldots, j+1\} \\ \ell &\longmapsto \ell+1 \end{aligned}$$

$$\text{hitting } k \text{ twice, for } k \neq j+1, \quad \begin{aligned} \varepsilon_k^j \colon \{1, \ldots, j+1\} &\longrightarrow \{1, \ldots, j\} \\ \ell &\longmapsto \begin{cases} \ell & \ell \leq k \\ \ell-1 & \ell > k \end{cases} \end{aligned}$$

$$\text{hitting } 1 \text{ twice, for } k = j+1, \quad \begin{aligned} \varepsilon_k^j \colon \{1, \ldots, j+1\} &\longrightarrow \{1, \ldots, j\} \\ \ell &\longmapsto \begin{cases} \ell & \ell = 1 \\ \ell-1 & \ell \geq 1, \end{cases} \end{aligned}$$

which have the property that

$$\varepsilon_k^j \circ \delta_k^j = \mathrm{id}_{\{1,\ldots,j\}}.$$

These maps can be utilized as follows: when $k \neq j+1$,

$$\{1, \ldots, j\} \xrightarrow{\varepsilon_k^j} \{1, \ldots, k, k+2, \ldots, j+1\} \xrightarrow{\mathsf{c}_k^j(\sigma)} \{1, \ldots, k, k+2, \ldots, j+1\} \xrightarrow{\delta_k^j} \{1, \ldots, j\},$$

[1]Without loss of generality, the two cycles are of the form $(\cdots k)\,(k+1\cdots)$, by the property that cycles are written only up to a cyclic permutation, giving them their name.

and when $k = j+1$,

$$\{1,\ldots,j\} \xrightarrow{\varepsilon^j_k} \{2,\ldots,j+1\} \xrightarrow{\mathsf{c}^j_k(\sigma)} \{2,\ldots,j+1\} \xrightarrow{\delta^j_k} \{1,\ldots,j\}.$$

Therewith, the *face maps* on level $j>0$ of S are, for $0 \le k \le j$, can be written cohesively as

$$\begin{aligned} \mathsf{d}^j_k \colon \mathsf{S}_j &\longrightarrow \mathsf{S}_{j-1} \\ \sigma &\longmapsto \begin{cases} \sigma & \sigma \in \mathfrak{S}_j, \ \ k = 0, j \\ \varepsilon^{j-1}_k \circ \left(\mathsf{c}^{j-1}_k(\sigma)\right) \circ \delta^{j-1}_k & \sigma \in \mathfrak{S}_j, \ \ 1 \le k \le j-1 \\ \varepsilon^j_{k+1} \circ \left(\mathsf{c}^j_{k+1}(\sigma)\right) \circ \delta^j_{k+1} & \sigma \in \mathfrak{S}_{j+1}, \end{cases} \end{aligned}$$

so, if $\sigma \in \mathfrak{S}_j \subset \mathfrak{S}_j \sqcup \mathfrak{S}_{j+1} =: \mathsf{S}_j$,

$$\begin{aligned} \mathsf{d}^j_0(\sigma), \ \ \mathsf{d}^j_j(\sigma) &\in \mathfrak{S}_j \subset \mathfrak{S}_{j-1} \sqcup \mathfrak{S}_j =: \mathsf{S}_{j-1} \\ \mathsf{d}^j_k(\sigma) &\in \mathfrak{S}_{j-1} \subset \mathsf{S}_{j-1} \text{ for } 1 \le k \le j-1, \end{aligned}$$

and, if $\sigma \in \mathfrak{S}_{j+1} \subset \mathsf{S}_j$, for $0 \le k \le j$,

$$\mathsf{d}^j_k(\sigma) \in \mathfrak{S}_j \subset \mathsf{S}_{j-1}.$$

3.1.5 Remark

Notice that in the previous definition, the maps were defined on cases modulo j or modulo $j+1$ on level $j>0$. This is directly inspired from looking at the chords on $\mathbf{S}^1$ given a pair $(\mathcal{B},\sigma)$, where $\mathcal{B} \subset \mathbf{S}^1$ is a subset of j or $j+1$ points, and $\sigma \in \mathfrak{S}_j \sqcup \mathfrak{S}_{j+1} =: \mathsf{S}_j$ is a permutation of j or $j+1$ objects. In this, the maps c^j_k capture a notion of what the effect of collapsing the interval I_k has on σ, namely, resulting in a permutation that is the same or of one less object, which corresponds to an element of $\mathfrak{S}_{j-1} \sqcup \mathfrak{S}_j =: \mathsf{S}_{j-1}$. Also, notice that utilizing the maps δ^j_k, and ε^j_k, as $\varepsilon^j_k \circ \left(\mathsf{c}^j_k(\sigma)\right) \circ \delta^j_k$ mimics the relabellings done in construction ¶3.1.1 after collapsing an interval. ◀

3.1.6 Lemma

For each $j>1$, and $1 \le k < \ell \le j+1$, it is the case that the face maps satisfy the simplicial identity

$$\mathsf{d}^{j-1}_k \mathsf{d}^j_\ell = \mathsf{d}^{j-1}_{\ell-1} \mathsf{d}^j_k,$$

as maps from $\mathsf{S}_j \longrightarrow \mathsf{S}_{j-1} \longrightarrow \mathsf{S}_{j-2}$.

Sketch First, note that the ε^j_k, and δ^j_k, maps do not effect permutation $\mathsf{c}^j_k(\sigma)$ that they surround by composition; recall that δ^j_k is used just to shift everything above $k \bmod j+1$ by one, while ε^j_k undoes that shift:

$$\varepsilon^j_k \circ \delta^j_k = \mathrm{id}_{\{1,\ldots,j\}}.$$

For example, when looking at a $\sigma \in \mathfrak{S}_{j+1} \subset \mathsf{S}_j$, with that in mind, it is possible to look more directly at the maps $\mathsf{c}^j_{\ell+1}$, and c^{j-1}_{k+1}, from the left-hand side, and the maps c^j_{k+1}, and c^{j-1}_ℓ, from the right-hand side. On the left-hand side: $\ell+2$ is first deleted from the permutation σ with $\mathsf{c}^j_{\ell+1}$, and then $k+2$ is deleted from the result with c^{j-1}_{k+1}; this generically looks like:

$$\begin{tikzcd}[column sep=small] 1 \arrow[d, "\mathsf{d}^j_\ell"'] & \cdots & k+1 \arrow[d] & k+2 \arrow[d] & k+3 \arrow[d] & \cdots & \ell+1 \arrow[d] & \ell+2 & \ell+3 \arrow[dl] & \cdots & j+1 \arrow[dl] \\ 1 \arrow[d, "\mathsf{d}^{j-1}_k"'] & \cdots & k+1 \arrow[d] & k+2 & k+3 \arrow[dl] & \cdots & \ell+1 \arrow[dl] & \ell+2 \arrow[dl] & \cdots & j \arrow[dl] & \\ 1 & \cdots & k+1 & k+2 & \cdots & \ell & \ell+1 & \cdots & j-1. & & \end{tikzcd}$$

On the right-hand side: $k+2$ is first deleted with c^j_{k+1}, and then $\ell+1$ is deleted from the result with c^{j-1}_ℓ; this generically looks like:

$$\begin{tikzcd}[column sep=small]
1 \arrow[d, "\mathsf{d}^j_k"'] & \cdots & k+1 \arrow[d] & k+2 & k+3 \arrow[dl] & \cdots & \ell+1 \arrow[dl] & \ell+2 \arrow[dl] & \ell+3 \arrow[dl] & \cdots & j+1 \arrow[dl] \\
1 \arrow[d, "\mathsf{d}^{j-1}_{\ell-1}"'] & \cdots & k+1 \arrow[d] & k+2 \arrow[d] & \cdots & \ell \arrow[d] & \ell+1 & \ell+2 \arrow[dl] & \cdots & j \arrow[dl] & \\
1 & \cdots & k+1 & k+2 & \cdots & \ell & \ell+1 & \cdots & j-1. & &
\end{tikzcd}$$

Notice that, in both pictures, the only elements from the top row, which do not get mapped into the bottom row, are $k+2$ and $\ell+2$; in this way, $\mathsf{d}^{j-1}_k \mathsf{d}^{\ell}_k$ and $\mathsf{d}^{j}_{\ell-1}\mathsf{d}^{j}_k$ match up. In the case when $\sigma \in \mathfrak{S}_j \subset \mathsf{S}_j$, there is a similar result, except that for d^j_0 and d^j_j, which act as an identity on σ. Therewith, it is not hard to convince oneself that the identity holds:

$$\mathsf{d}^{j-1}_k \mathsf{d}^{\ell}_k = \mathsf{d}^{j}_{\ell-1}\mathsf{d}^{j}_k. \qquad ◪$$

The last piece needed in the formation of the geometric realization of the semi-simplicial set S are the building-block spaces called simplices.

3.1.7 Definition

Let Δ^j be the *j-dimensional simplex*, or simply, *j-simplex*, namely, [GM03, §I.1.1]

$$\Delta^j := \left\{ (x_k)_{k=0}^{n} \in \mathbf{R}^{n+1} \;\middle|\; \sum_{k=0}^{j} x_k = 1 \text{ and } x_k \geq 0,\ \forall k \right\}.$$

The *k-th face* of the simplex Δ^j is defined as the set

$$\left\{ (x_\ell)_{\ell=0}^{n} \in \mathbf{R}^{n+1} \;\middle|\; \sum_{\ell=0}^{j} x_\ell = 1 \text{ and } x_\ell \geq 0,\ \forall \ell \neq k,\ x_k = 0 \right\}.$$

Finally, with all the requisites settled, it is possible to form the "chord space" $\mathrm{Ch}\,\mathbf{S}^1$ as the geometric realization of the semi-simplicial set formed by S with the face maps d^j_k, for $j > 0$.

3.1.8 Definition

The *chord space* $\mathrm{Ch} = \mathrm{Ch}\,\mathbf{S}^1$ is defined as the *geometric realization* of S with the face maps $\left\{\mathsf{d}^j_k\right\}_{k=0}^{j}$, for $j > 0$, is the quotient

$$\mathrm{Ch} := |\mathsf{S}| := \left(\coprod_{j=0}^{\infty} \left(\Delta^j \times \mathsf{S}_j \right) \right) \Big/ \sim\, ,$$

where the equivalence $\sim$ is defined in the following way: for each $j > 0$, and any $\sigma \in \mathsf{S}_j$, the k-th face of the j-simplex $\Delta^j \times \{\sigma\}$ is identified with the $j-1$-simplex $\Delta^{j-1} \times \left\{\mathsf{d}^j_k(\sigma)\right\}$.

Now, with the chord space Ch established, the last definition of this section will be used in the the next section to restrict to a particular subspace of Ch, which will contain the image of the map $\mathrm{Gr}_n \longrightarrow \mathrm{Ch}$ from the space of Green's functions with n sinks to the chord space Ch. Specifically, this subspace will be the faces of Ch corresponding to certain, wanted permutations. Such wanted permutations will be called "non-crossing", and the next definition captures the conditions that elements of $\mathfrak{S}_j$ need to satisfy in order for them to meet that qualification.

3.1.9 Definition

Let $\sigma \in \mathfrak{S}_j$, the j-th symmetric group, and represent σ using cycles. This permutation σ is called *non-crossing* if the following conditions are met for its cycles:

1. every cycle is at least length $c > 1$, which is to say that there are not any points fixed by σ.
2. for every cycle $(a_1 \ldots a_c)$ of length $c > 3$, every subset $\{a_k, a_{k+1}, a_\ell, a_{\ell+1}\} \subset \{a_j\}_{j=1}^{c} \subset \mathbf{Z}$, where the indices are considered $\bmod c$ and are pairwise distinct residues, is such that one of the following hold with respect to the usual ordering on $\mathbf{Z}$, denoted here as $<$,

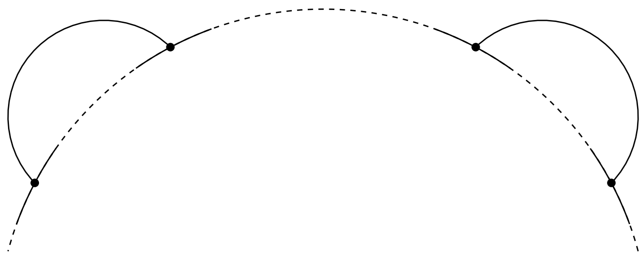
$$a_k, a_{k+1} < \min\{a_\ell, a_{\ell+1}\} \quad \text{or} \quad a_\ell, a_{\ell+1} < \min\{a_k, a_{k+1}\}$$

$$\text{or} \quad \left\{ \begin{array}{c} \min\{a_k, a_{k+1}\} < a_\ell < \max\{a_k, a_{k+1}\} \\ \text{and} \\ \min\{a_k, a_{k+1}\} < a_{\ell+1} < \max\{a_k, a_{k+1}\} \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{c} \min\{a_\ell, a_{\ell+1}\} < a_k < \max\{a_\ell, a_{\ell+1}\} \\ \text{and} \\ \min\{a_\ell, a_{\ell+1}\} < a_{k+1} < \max\{a_\ell, a_{\ell+1}\} \end{array} \right\}.$$

3. for any two cycles $(a_1 \ldots a_c)$ and $(b_1 \ldots b_d)$, every two subsets $\{a_k, a_{k+1}\} \subset \{a_j\}_{j=1}^c$, and $\{b_\ell, b_{\ell+1}\} \subset \{b_j\}_{j=1}^d$, where indices are respectively considered $\bmod c$ and $\bmod d$, are such that one of the following hold with respect to the usual ordering $<$ on $\mathbf{Z}$

$$a_k, a_{k+1} < \min\{b_\ell, b_{\ell+1}\} \quad \text{or} \quad b_\ell, b_{\ell+1} < \min\{a_k, a_{k+1}\}$$
$$\text{or} \left\{\begin{array}{c} \min\{a_k, a_{k+1}\} < b_\ell < \max\{a_k, a_{k+1}\} \\ \text{and} \\ \min\{a_k, a_{k+1}\} < b_{\ell+1} < \max\{a_k, a_{k+1}\} \end{array}\right\} \text{or} \left\{\begin{array}{c} \min\{b_\ell, b_{\ell+1}\} < a_k < \max\{b_\ell, b_{\ell+1}\} \\ \text{and} \\ \min\{b_\ell, b_{\ell+1}\} < a_{k+1} < \max\{b_\ell, b_{\ell+1}\} \end{array}\right\}.$$

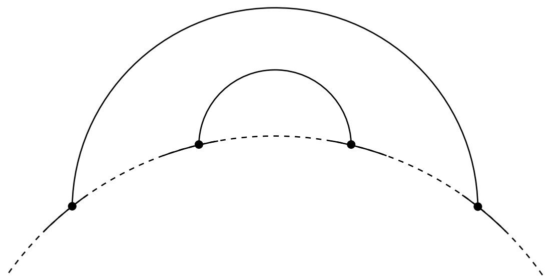
Let $\mathcal{B} \subset \mathbf{S}^1$ be any subset of m points. Then, the cumbersome inequality conditions in (2.), and (3.), can be represented pictorially using the collection of chords gotten from the pair $(\mathcal{B}, \sigma)$ in the following bare schematics, which must be satisfied for any two chords in the collection in order for σ to be considered non-crossing:



for

$$\text{(2.)} \quad a_k, a_{k+1} < \min\{a_\ell, a_{\ell+1}\} \quad \text{or} \quad a_\ell, a_{\ell+1} < \min\{a_k, a_{k+1}\}$$
$$\text{(3.)} \quad a_k, a_{k+1} < \min\{b_\ell, b_{\ell+1}\} \quad \text{or} \quad b_\ell, b_{\ell+1} < \min\{a_k, a_{k+1}\}$$

and



$$\text{(2.)} \left\{\begin{array}{c} \min\{a_k, a_{k+1}\} < a_\ell < \max\{a_k, a_{k+1}\} \\ \text{and} \\ \min\{a_k, a_{k+1}\} < a_{\ell+1} < \max\{a_k, a_{k+1}\} \end{array}\right\} \text{or} \left\{\begin{array}{c} \min\{a_\ell, a_{\ell+1}\} < a_k < \max\{a_\ell, a_{\ell+1}\} \\ \text{and} \\ \min\{a_\ell, a_{\ell+1}\} < a_{k+1} < \max\{a_\ell, a_{\ell+1}\} \end{array}\right\}$$
$$\text{(3.)} \left\{\begin{array}{c} \min\{a_k, a_{k+1}\} < b_\ell < \max\{a_k, a_{k+1}\} \\ \text{and} \\ \min\{a_k, a_{k+1}\} < b_{\ell+1} < \max\{a_k, a_{k+1}\} \end{array}\right\} \text{or} \left\{\begin{array}{c} \min\{b_\ell, b_{\ell+1}\} < a_k < \max\{b_\ell, b_{\ell+1}\} \\ \text{and} \\ \min\{b_\ell, b_{\ell+1}\} < a_{k+1} < \max\{b_\ell, b_{\ell+1}\} \end{array}\right\}.$$

3.1.10 EXAMPLES

Coming from the examples ¶3.1.3, there are examples of permutations which do not satisfy the corresponding conditions in the previous definition:

1. the figure for (1) (324), because it fixes β_1: there is the cycle (1) of length one.
2. the figure for (1324), because it includes crossing chords coming from a single cycle, regardless of how the chords are drawn:

$$3 \not< \min\{2,4\} \quad \text{and} \quad 2,4 \not< \min\{1,3\}$$
$$\text{and} \left\{\begin{array}{c} \min\{1,3\} < 2 < \max\{1,3\} \\ \text{and} \\ \min\{1,3\} \not< 4 \not< \max\{1,3\} \end{array}\right\} \text{and} \left\{\begin{array}{c} \min\{2,4\} \not< 1 \not< \max\{2,4\} \\ \text{and} \\ \min\{2,4\} < 3 < \max\{2,4\} \end{array}\right\}.$$

3. the figure for (13) (24), because it includes crossing chords coming from two different cycles, again, regardless of how the chords are drawn. It fails the conditions exactly as example (2.) does. ⋈

3.2 The Third Map

Obtaining the third map, involves the construction of the "critical-point graph" and the association to each such graph a permutation. This will be done first in this section, and then there will be a discussion about the third map.

3.2.1 Construction

For any $g_{\mathcal{Z}} \in \mathrm{Gr}_n$ with $n \geq 2$, as defined in definition ¶1.1.8, its $n-1$ critical points are the roots contained in $\mathbf{D}^2$ of the polynomial $h_{\mathcal{Z}} \in \mathrm{P}_{n-1}$, as defined in definitions ¶2.1.1. Denote the set of roots of $h_{\mathcal{Z}}$ inside $\mathbf{D}^2$ by $\mathcal{R}$, and consider the flow of

$$-\nabla\left(g_{\mathcal{Z}}(x+\mathrm{i}y)\right) := -\left(\partial_x g_{\mathcal{Z}}(x+\mathrm{i}y),\ \partial_y g_{\mathcal{Z}}(x+\mathrm{i}y)\right) = -2\left(\mathrm{Re}\,\partial_z g_{\mathcal{Z}}(z),\ -\mathrm{Im}\,\partial_z g_{\mathcal{Z}}(z)\right),$$

where $\partial_z := \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right)$, as mentioned in the proof of lemma ¶1.1.2. As such, each $\rho \in \mathcal{R}$ is a singular point of the vector field $-\nabla g_{\mathcal{Z}}$ by nature of it being a critical point of $g_{\mathcal{Z}}$. Looking near each ρ, if the multiplicity of ρ as a root of $h_{\mathcal{Z}}$ is m, then the number of flow lines approaching ρ in a limit is $m+1$; this is related to the index, or winding number, at the singular point ρ. [Nee97, §10.II.1]

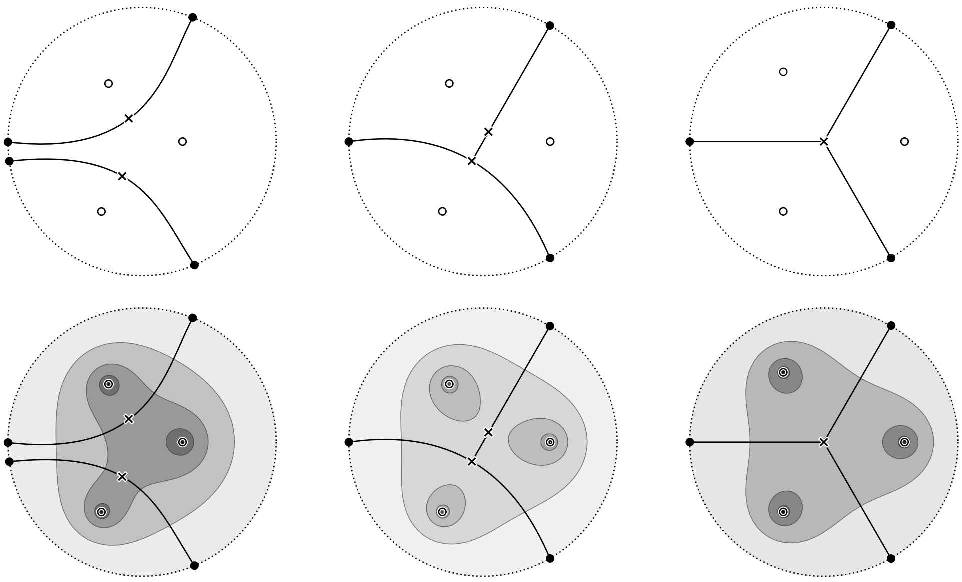
With that setup, the *critical-point graph* of $g_{\mathcal{Z}}$ is a graph embedded into $\overline{\mathbf{D}^2}$ with the following specifications:

a. The vertices are the elements of $\mathcal{R}$, along with any points $\beta \in \partial\mathbf{D}^2$ which flow under $-\nabla g_{\mathcal{Z}}$ approaching some $\rho \in \mathcal{R}$. The set of boundary points is denoted by $\mathcal{B}$.

b. The edges are determined by those distinguished flow lines of $-\nabla g_{\mathcal{Z}}$ which approach, in a limit, some $\rho \in \mathcal{R}$.

3.2.2 Examples

In the case of $\mathcal{Z} \in \mathrm{C}_3$, the following are depictions of possible critical-point graphs: the edges are the solid lines, while the dotted lines represent $\partial\mathbf{D}^2$; the vertices are represented by the symbol $\times$ for elements of the set of critical points $\mathcal{R} \subset \mathbf{D}^2$, and $\bullet$ for the elements of the set $\mathcal{B} \subset \partial\mathbf{D}^2$; the symbol $\circ$ is not part of the graph, but it represents the locations of the sinks $\mathcal{Z} \subset \mathbf{D}^2$ and is shown for completeness. The second row is the depiction overlaid on the corresponding contour plots[2]

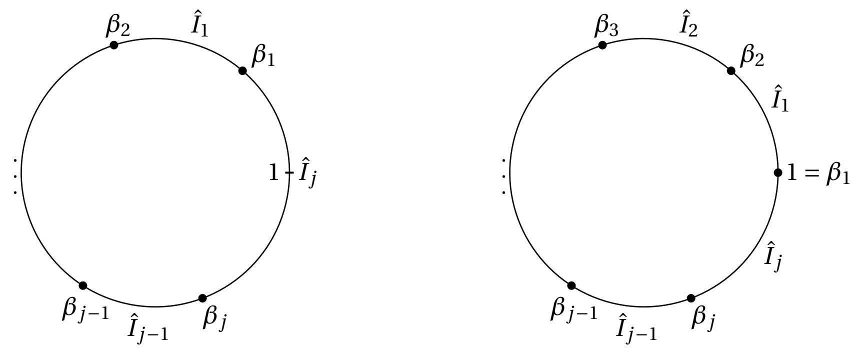


The first two columns of depictions have two critical points, each of multiplicity one, while the third column has only one but of multiplicity two. ⧓

3.2.3 Remark

Let $\mathcal{Z} \in \mathrm{C}_n$, for $n \geq 2$, be the sinks for a Green's function $g_{\mathcal{Z}}$, and let $\mathcal{B} \subset \partial\mathbf{D}^2 = \mathbf{S}^1$ be the j boundary points of the critical-point graph of $g_{\mathcal{Z}}$. Similar to what was done in construction ¶3.1.1, let $\mathcal{B} = \{\beta_k\}_{k=1}^{j}$, where the labelling is going counterclockwise starting at $1 \in \mathbf{S}^1 \subset \mathbf{C}$, and let the intervals $\{\hat{I}_k\}_{k=1}^{j}$ corresponding to $\mathcal{B}$ be defined as: $\hat{I}_k$ being the interval starting at β_k going counterclockwise to β_{k+1} for $1 \leq k \leq j-1$, and $\hat{I}_j$ being the interval starting at β_j going counterclockwise to β_1.

[2] The contour plots were made using Wolfram Mathematica 10.0, Student Edition, and exported to SVG, then saved into the current PDF image.



It is the nature of these critical-point graphs that the interior of the interval $\hat{I}_k$, for $1 \le k \le j$, flows to only one of the sinks $\zeta \in \mathcal{Z}$, as can be seen in the examples ¶3.2.2

Now, recall from lemma ¶2.1.3 that the number of critical points of $g_{\mathcal{Z}}$ is $n-1$, including multiplicities. With that, it follows that the minimum number of critical points is one with multiplicity $n-1$, and the maximum is $n-1$ each with multiplicity one. Also, recall from the construction ¶3.2.1 that it was mentioned that there are $m+1$ flow lines approaching a critical point of multiplicity m. Thus, there are the following bounds on the number $\#\mathcal{B}$ of vertices $\mathcal{B} \subset \partial \mathbf{D}^2$ of the critical-point graph for $g_{\mathcal{Z}}$:

$$n \le \#\mathcal{B} \le 2\,(n-1)\,,$$

where the lower bound comes from the case when there is only one critical point, which has $(n-1)+1$ flow lines approaching it from the boundary $\partial \mathbf{D}^1 = \mathbf{S}^1$, and the upper bound comes from the case when there is $n-1$ critical points, which each have at most $1+1$ flow lines approaching it from the boundary. Also, note that $n \ge 2$, because if $g_{\mathcal{Z}}$ has one sink then it has no critical points in $\mathbf{D}^2$, again coming from lemma ¶2.1.3. ◀

Next is the description of how to associate to each critical-point graph a permutation σ using the previous remark. This σ will then be used to form the chords for the pair $(\mathcal{B}, \sigma)$, as done in definition ¶3.1.2, where $\mathcal{B}$ is the set of vertices on the boundary $\partial \mathbf{D}^2$ from the critical-points graph.

3.2.4 Construction

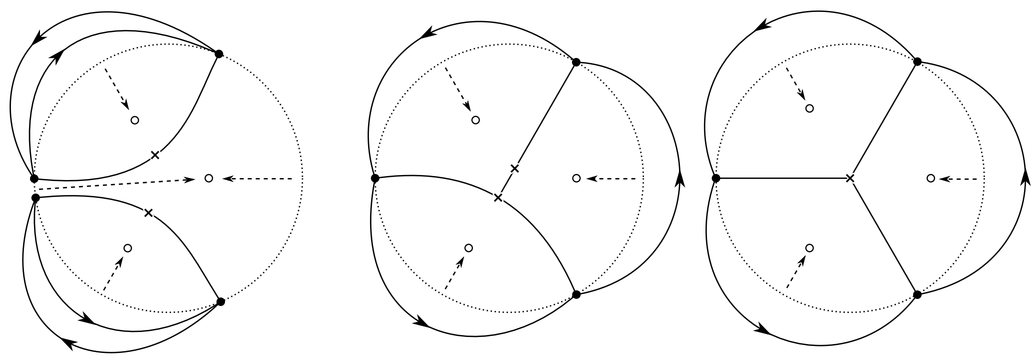
Given some critical-point graph, let its boundary points be $\mathcal{B} = \{\beta_k\}_{k=1}^{j} \subset \mathbf{S}^1 = \partial \mathbf{D}^2$, where the labelling is just as in construction ¶3.1.1, with the corresponding intermediating intervals $\hat{I}_k$, for $1 \le k \le j$, as in defined in remark ¶3.2.3. Then, a permutation $\sigma \in \mathfrak{S}_j$ can be formed in the following way: if the interior of interval $\hat{I}_k$ flows to the same sink at the interior of the interval $\hat{I}_\ell$, then

$$\sigma\,(k) = \begin{cases} \ell + 1 & 1 \le \ell \le j-1 \\ 1 & \ell = j. \end{cases}$$

By the previous remark, it follows that this map is well-defined.

3.2.5 Examples

Using the previous examples ¶3.2.2, it is possible to depict the chords for the pair $(\mathcal{B}, \sigma)$ as in definition ¶3.1.2, where $\mathcal{B} \subset \mathbf{S}^1$ is the set of boundary points of a critical point graph, and σ is the permutation obtained from construction ¶3.2.4.



The directed edges outside of $\mathbf{S}^1$ are, of course, the chords, while the dashed lines added in the interior point from the interval interiors to the sink which they flow. ⧓

3.2.6 Remark

The permutation associated to a critical-point graph of a Green's function has the property that it is non-crossing, a qualification defined in definition ¶3.1.9. ◀

To finish off, there will be a discussion about the map from the space Gr_n of Green's function with $n \geq 2$ sinks to Ch the chord space, which factors through the subspace $\mathrm{Ch}_{n-1} \subset \mathrm{Ch}$, and possible also the space of critical-point polynomials P_{n-1}.

3.2.7 Discussion

First, a map is needed to place the elements $\mathcal{B} = \{\beta_k\}_{k=1}^{j} \in \mathrm{C}_j\left(\mathbf{S}^1 \smallsetminus \{1\}\right)$, and $\mathcal{B}' = \{\beta_1 = 1\} \cup \{\beta'_k\}_{k=2}^{j+1} \in \mathrm{C}_{j+1}\mathbf{S}^1$, into a j-simplex. As done in construction ¶3.1.1, there are sets of intervals $\{I_k\}_{k=0}^{j}$ and $\{I'_k\}_{k=0}^{j}$ corresponding to $\mathcal{B}$ and $\mathcal{B}'$, respectively. Let $|K|$, for K is an interval in either $\{I_k\}_{k=0}^{j}$ or $\{I'_k\}_{k=0}^{j}$, denote the length of the interval K as part of the circumference of the unit 1-sphere $\mathbf{S}^1$. Note that it follows that

$$\sum_{k=0}^{j} |I_k| = 2\pi = \sum_{k=0}^{j} |I'_k|.$$

With that, there is a continuous map from such $\mathcal{B}$ and $\mathcal{B}'$ to Δ^j defined as

$$\begin{aligned} \mathsf{B}^j : \mathcal{B} &\longmapsto \left(|I_k|/2\pi\right)_{k=0}^{j} \\ \mathcal{B}' &\longmapsto \left(|I'_k|/2\pi\right)_{k=0}^{j}. \end{aligned}$$

It is also possible to view $\mathcal{B}'$ as lying on the face of a $(j+1)$-simplex as

$$\left(|I'_0|/2\pi, \ldots, \left|I'_j\right|/2\pi, 0\right).$$

Now, using construction ¶3.2.1, and construction ¶3.2.4, the map from the space of Green's functions Gr_n to the chord space Ch is defined by the following steps:

1. Take some $g_{\mathcal{Z}}$, and obtain its critical-point graph.
2. This graph has a subset $\mathcal{B}$ of j vertices which lie in the boundary $\mathbf{S}^1$ of the 2-disk, as well as an associated permutation $\sigma \in \mathfrak{S}_j$.
3. The pair $(\mathcal{B}, \sigma)$ can be thought of as a point of the chord space after using the map B^j on $\mathcal{B}$:

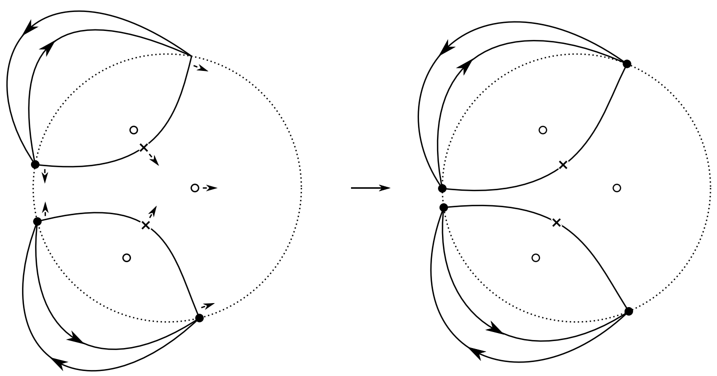
$$\left(\mathsf{B}^j(\mathcal{B}), \sigma\right) \in \mathrm{Ch} := \coprod_{k=0}^{\infty} \left(\Delta^k \times \mathsf{S}_k\right) \Big/ \sim .$$

Now, note that from remark ¶3.2.3 it follows that such a pair $(\mathcal{B}, \sigma)$ lies between level n and level $2(n-1)$, inclusively, because $n \leq \#\mathcal{B} \leq 2(n-1)$ for each $n \geq 2$, and $\#\mathcal{B} = j$ determines that $\mathcal{B}$ gets mapped into the simplex Δ^{j+1} via the map B^j. Moreover, as mentioned in remark ¶3.2.6, $\sigma \in \mathfrak{S}_j$ must be non-crossing, as described in definition ¶3.1.9, so the image of $(\mathcal{B}, \sigma)$ is again restriction in a possible smaller subset of Ch.

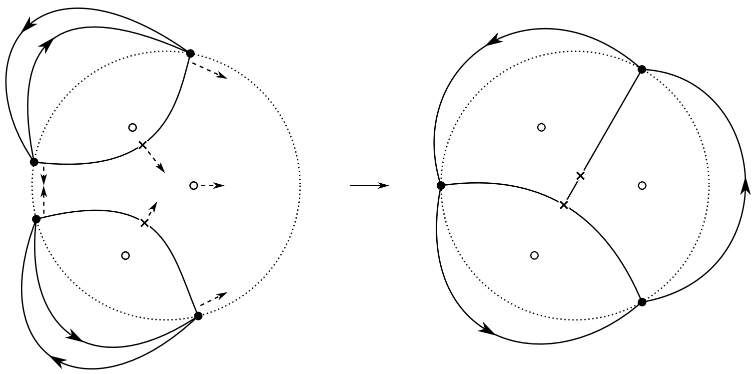
With that, let $\mathrm{Ch}_{n-1} = \mathrm{Ch}_{n-1}\mathbf{S}^1 \subset \mathrm{Ch}$ be the subspace going from level n to level $2(n-1)$, inclusively, deleting all the faces Δ^j corresponding to crossing (which is to say, not non-crossing) permutations from S_j on each level $n \leq j \leq 2(n-1)$.

Looking more closely, there is the following geometric picture:

> Given some $\mathcal{Z} \in \mathrm{C}_n$ for $n \geq 2$, there is a corresponding $\mathcal{B} \in \mathrm{C}_j\mathbf{S}^1$, which are the boundary points of the critical-point graph for $g_{\mathcal{Z}} \in \mathrm{Gr}_n$, along with a non-crossing permutation $\sigma \in \mathfrak{S}_j$. As the element $\mathcal{Z}$ moves along a short path in C_n, this corresponds to a path of $\mathcal{B}$ in $\mathrm{C}_j\mathbf{S}^1$, leaving σ unchanged.



However, for a long enough path, two points of $\mathcal{B}$ will combine because some interval k collapsing, resulting in the path going to $\mathrm{C}_{j-1}\mathbf{S}^1$ and σ turning into the permutation $\mathsf{d}_k^j(\sigma)$, as defined in definition ¶3.1.4, which must still be non-crossing by the constraints mentioned in remark ¶3.2.6.



As such, though it leaves $\mathrm{C}_j\mathbf{S}^1$ and gets to a permutation $\mathsf{d}_k^j(\sigma)$, this path remains in Ch_{n-1}, because $\mathrm{C}_{j-1}\mathbf{S}^1$ is just some face of Δ^{j+1} as discussed earlier, and $\mathsf{d}_k^j(\sigma)$ corresponds to that face in Ch_{n-1}, since it contains faces corresponding to non-crossing permutations, by its construction.

This picture suggests that the map

$$\mathrm{Gr}_n \longrightarrow \mathrm{Ch}_{n-1} \subset \mathrm{Ch}$$

is continuous and, likely, a homotopy equivalence. The map might factor through the space of critical-point polynomials P_{n-1}, as worked with in chapter 2; particularly, if the second map $\mathrm{Gr}_n \longrightarrow \mathrm{P}_{n-1}$ is a homotopy equivalence as mentioned in hypothesis ¶2.2.1, then one excepts that there is also a homotopy equivalence between P_{n-1} and Ch_{n-1}: there should the factoring

$$\begin{tikzcd}
\mathrm{Gr}_n \arrow[rr, "(3)"] \arrow[dr, "(2)"'] & & \mathrm{Ch}_{n-1} \subset \mathrm{Ch}, \\
& \mathrm{P}_{n-1} \arrow[ur, dashed] &
\end{tikzcd}$$

which also might have some invariance with respect to disk automorphisms, like what was discussed in hypothesis ¶2.2.1 for the other two maps (2) and (3).

A. The Derivation of the Green's Function

In this chapter, prerequisite notions will be discussed from distribution theory, and then a derivation of a Green's function will be presented.

A.1 Distribution Theory

Here, some requisite notions from distribution theory will be discussed. [App07, ch. 7] [DK10, ch. 3] From this, the notion of a Green's function will be introduced to segue into the following section.

A.1.1 Definition

A *test function* is a smooth (i.e. $\mathscr{C}^\infty$) function $\varphi : \mathbf{R}^n \longrightarrow \mathbf{C}$ with bounded support. These naturally form a $\mathbf{C}$-vector space, which will be denoted $\mathscr{D}(\mathbf{R}^n)$.

Acting on the test functions, will be a certain set of $\mathbf{C}$-linear functionals $T : \mathscr{D}(\mathbf{R}^n) \longrightarrow \mathbf{C}$; their action will be said to be continuous if it respects the completeness of the usual topology of $\mathbf{C}$.

A.1.2 Definition

Let a *distribution* be a $\mathbf{C}$-linear functional

$$\begin{aligned} T : \mathscr{D}\left(\mathbf{R}^n\right) &\longrightarrow \mathbf{C} \\ \varphi &\longmapsto T \centerdot \varphi, \end{aligned}$$

which is continuous in the following sense:

if, for any sequence $(\varphi_i)_i \subseteq \mathscr{D}(\mathbf{R}^n)$ such that $\bigcup_i \operatorname{supp} \varphi_i \subseteq \mathbf{R}^n$ is bounded and $\varphi_i^{(p)} \longrightarrow \varphi^{(p)} \in \mathscr{D}(\mathbf{R}^n)$ uniformly for all orders $p \geq 0$, it is the case that

$$T \centerdot \varphi_i \xrightarrow{i \to \infty} T \centerdot \varphi \in \mathbf{C},$$

then T is said to be continuous.

In this way, distributions form a $\mathbf{C}$-vector space dual to $\mathscr{D}(\mathbf{R}^n)$, and as such, will be denoted $\mathscr{D}'(\mathbf{R}^n)$.

The notion equality in this space is readily stipulated as follows: for $S, T \in \mathscr{D}'(\mathbf{R}^n)$, it is said that $S = T$ in $\mathscr{D}'(\mathbf{R}^n)$ iff $S \centerdot \varphi = T \centerdot \varphi$ for all $\varphi \in \mathscr{D}(\mathbf{R}^n)$. Or, with a smaller scope: $S = T$ on $U \subseteq \mathbf{R}^n$ iff $S \centerdot \varphi = T \centerdot \varphi$ for all $\varphi \in \mathscr{D}(\mathbf{R}^n)$ which have support contained in U. Therewith, the notion of the *support* of a distribution T is defined to be the complement of the set $U \subseteq \mathbf{R}^n$ where $T = 0$.

A.1.3 Example

A prolific exmaple is the *Dirac delta* (*distribution*) δ, which is defined as follows

$$\begin{aligned} \delta : \mathscr{D}\left(\mathbf{R}^n\right) &\longrightarrow \mathbf{C} \\ \varphi &\longmapsto \delta \centerdot \varphi := \varphi(0). \end{aligned}$$

⧓

As to certain properties that these so-called "generalized functions" should have: locally-integrable functions will serve as examples.

A.1.4 Definition

Let $f \in L^1_{\text{loc}}(\mathbf{R}^n)$, i.e. f is locally integrable on $\mathbf{R}^n$,

$$\int_U |f| < \infty \quad \text{for all compact } U \subseteq \mathbf{R}^n.$$

Then, the *regular distribution* associated to f is defined to have the following action for a $\varphi \in \mathscr{D}(\mathbf{R}^n)$

$$f \centerdot \varphi := \int_{\mathbf{R}^n} f \cdot \varphi,$$

the usual integral of the product; when the variable is noted explicitly, this is written

$$f(x) \centerdot \varphi(x) := \int_{\mathbf{R}^n} f(x) \cdot \varphi(x) \; \mathrm{d}x.$$

A.1.5 Example

The *Heaviside distribution* H is a regular distribution defined by the locally integrable function on $\mathbf{R}$ as

$$\begin{aligned} H : \mathbf{R} &\longrightarrow \mathbf{C} \\ x &\longmapsto \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases} \end{aligned}$$

⧓

With that, the convolution of two regular distributions $f, g \in \mathscr{D}'(\mathbf{R}^n)$ is defined as acting on any $\varphi \in \mathscr{D}(\mathbf{R}^n)$

$$\begin{aligned} (f * g) \mathbin{.} \varphi &:= \int_{\mathbf{R}^n} (f * g) \cdot \varphi \\ &= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} f(x) \cdot g(y - x) \, \mathrm{d}x \right) \cdot \varphi(y) \, \mathrm{d}y \\ &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x) \cdot g(y) \cdot \varphi(y + x) \, \mathrm{d}x \mathrm{d}y \\ &= f(y) \mathbin{.} \left(g(x) \mathbin{.} \varphi(y + x) \right). \end{aligned}$$

Using that that property as inspiration, there is the following definition of the convolution of two distributions.

A.1.6 Definition

Let $S, T \in \mathscr{D}'(\mathbf{R}^n)$. Then, their *convolution* $S * T$ is defined by its action on any $\varphi \in \mathscr{D}(\mathbf{R}^n)$ as

$$(S * T) \mathbin{.} \varphi := S(x) \mathbin{.} \left(T(y) \mathbin{.} \varphi(y + x) \right),$$

where the "evaluations" $S(x)$ and $T(y)$ are just formal expressions for S and T acting on the translated function $\varphi(y + x) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{C}$.

Similarly, there is a notion of the derivative of a distribution ascertained by the following: let $f \in \mathscr{D}'(\mathbf{R}^n)$ be a regular distribution (which is differentiable, of course), then for any $\varphi \in \mathscr{D}(\mathbf{R}^n)$,

$$\begin{aligned} \frac{\partial f}{\partial x^i} \mathbin{.} \varphi &:= \int_{\mathbf{R}^n} \frac{\partial f}{\partial x^i} \cdot \varphi \\ &= \underbrace{\left[f \cdot \varphi \right]_{x^i = -\infty}^{\infty}}_{\substack{0, \\ \text{since } \varphi \in \mathscr{D}(\mathbf{R}^n) \text{ has} \\ \text{bounded support}}} - \int_{\mathbf{R}^n} f \cdot \frac{\partial \varphi}{\partial x^i} \\ &= -f \mathbin{.} \frac{\partial \varphi}{\partial x^i}. \end{aligned}$$

A.1.7 Definition

For a distribution $T \in \mathscr{D}'(\mathbf{R}^n)$, its (*distributional*) *derivative* is defined to have its action on any $\varphi \in \mathscr{D}(\mathbf{R}^n)$ as

$$T' \mathbin{.} \varphi := -T \mathbin{.} \varphi',$$

where the prime denotes a derivative in some (one-dimensional) variable in $\mathbf{R}^n$.

A.1.8 Remark

Consider the Laplacian Δ on $\mathbf{R}^n$,

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial (x^i)^2}.$$

From definition ¶A.1.6, it follows that, for any distributions $T \in \mathscr{D}'(\mathbf{R}^n)$ and any $\varphi \in \mathscr{D}(\mathbf{R}^n)$,

$$\Delta T \mathbin{.} \varphi = T \mathbin{.} \Delta \varphi.$$

◀

A.1.9 Example

The derivative of the Heaviside distribution H: for any $\varphi \in \mathscr{D}(\mathbf{R})$,

$$H' \mathbin{.} \varphi = H \mathbin{.} \varphi'$$

$$
\begin{aligned}
&= \int_{\mathbf{R}} H \cdot \varphi' \\
&= \int_{\mathbf{R}_{\geq 0}} \varphi' \\
&= \varphi(0) \qquad \text{since } \varphi \text{ has bounded support} \\
\iff H' &= \delta.
\end{aligned}
$$

⋈

Using these concepts, there is the following result of taking the distributional derivative of a discontinuous function.

A.1.10 Theorem [App07, 7.47–48][Sch61, (II, 2; Proposition 4)][Sch66b, (II, 2; 8)]
Let $f : \mathbf{R} \longrightarrow \mathbf{C}$ be a piecewise smooth function, let its (finite) set of discontinuties be $\{p_k\}_{k\in K} \subset \mathbf{R}$. Then, as a distribution, it has the following n-th derivative

$$f^{(n)} + \sum_{j=1}^{n} \sum_{k\in K} \sigma_{j-1}(k)\,\delta^{(n-j)}\left(x - p_k\right),$$

where $f^{(n)}$ is the usual n-th derivative defined almost everywhere, making it a regular distribution, and $\sigma_j(k)$ is the jump in $f^{(j)}$ at p_k in the direction of increasing $x \in \mathbf{R}$.

Sketch This is done by writing $f^{(n)}$ as continuous function plus Heaviside distributions to account for the discontinuities, and then, considering it as a regular distribution and using definition ¶A.1.7. Using integration by parts n times, the derivatives are moved onto f from the test function. Because of the discontinuities of f, attention to given to the boundary terms of the continuous segments of $f^{(j)}$. The $\sigma_j(k)\,\delta^{(j)}\left(x - p_k\right)$ are exactly these boundary terms. ◪

Finally, there is a result involving these definitions.

A.1.11 Theorem [App07, theorem 8.24]
Let $\varphi \in \mathscr{D}(\mathbf{R}^n)$ be a smooth function. Then, for any $T \in \mathscr{D}'(\mathbf{R}^n)$,

$$T * \varphi(x) := T(y) \,.\, \varphi(x - y)$$

it a distribution, and there is a smooth function $f \in \mathscr{D}'(\mathbf{R}^n)$ for which this it the associated regular distribution. Also, there is the following relation

$$T' * \varphi(x) = T'(y) \,.\, \varphi(x - y) = T(y) \,.\, \varphi'(x - y).$$

■

With all of that now in place, focus narrowed to the certain case at hand, a certain set of distributions, wherein the wanted Green's function lies.

A.1.12 Theorem [App07, theorem 8.28]
Let $\mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n) \subseteq \mathscr{D}'(\mathbf{R}^n)$ be the subspace of distributions which has bounded support. Then, $\left(\mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n), *\right)$ is a $\mathbf{C}$-algebra, where $*$ denotes convolution, and wherein the Dirac delta distribution,

$$
\begin{aligned}
\delta : \mathscr{D}\left(\mathbf{R}^n\right) &\longrightarrow \mathbf{C} \\
\varphi &\longmapsto \delta \,.\, \varphi := \varphi(0),
\end{aligned}
$$

is the unit. ■

Now, note that differential operators can be written as a bounded distribution by using the Dirac delta δ and its derivatives.

A.1.13 Remark
Consider the Laplacian Δ on $\mathbf{R}^n$ upon some smooth $f : \mathbf{R}^n \longrightarrow \mathbf{C}$: this is just

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial\left(x^i\right)^2}.$$

This expression can be written as element of $\mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n)$ convolved with such an $f \in \mathscr{D}(\mathbf{R}^n)$ in the following way: note first that

$$\frac{\partial^2 \delta}{\partial\left(x^i\right)^2} * f(x) = \frac{\partial^2 \delta}{\partial\left(x^i\right)^2}(y) \,.\, f(x - y) \qquad \text{by theorem ¶A.1.11}$$

$$= \delta(y) \cdot \frac{\partial^2 f}{\partial (x^i)^2}(x-y) \qquad \text{similarly}$$

$$= \frac{\partial^2 f}{\partial (x^i)^2}(x) \qquad \text{by definition of } \delta \text{ in example ¶A.1.3,}$$

so as a distribution, Δ can be written with abuse of notation as

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2 \delta}{\partial (x^i)^2},$$

acting as

$$\Delta * f = \Delta f.$$

◀

Putting this all together, the definition of a Green's function can be expressed.

A.1.14 Definition

Considering the Laplacian as a distribution $\Delta \in \mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n)$, its *Green's function* is its inverse in $\mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n)$ with respect to the $*$ product, i.e. it is a $G \in \mathscr{D}'_{\mathrm{bdd}}(\mathbf{R}^n)$ such that

$$G * \Delta = \delta = \Delta * G.$$

A.1.15 Remark

In this way, finding the Green's function is useful in solving the Dirichlet problem on some bounded domain $U \subset \mathbf{R}^n$

$$\begin{cases} \Delta f = g \\ f|_{\partial U} = 0. \end{cases}$$

Since the sought-for f is smooth and has bounded support, the bounded U, it follows that it can be considered to be in $\mathscr{D}'(\mathbf{R}^n)$; thus, it can be written, up to a harmonic term, as

$$f = G * \Delta * f = G * g,$$

where G also satisfies the boundary conditions that f does, i.e. $G|_{\partial U} = 0$. ◀

A.2 A Green's Function for the Laplacian

With the motivating background of Green's functions in place, the task at hand is to derived one for the Laplacian on the 2-disk.

A.2.1 Proposition [App07, 7.55][Sch66b, (II, 3; 14)][Sch66a, (II, 2; Theorem 6)][Wu][Uni03]

Let the 2-disk be denoted by $\mathbf{D}^2 \subset \mathbf{R}^2 \cong \mathbf{C}$. A radially-symmetric function $G : \mathbf{R}^2 \longrightarrow \mathbf{R} \subset \mathbf{R}^2 \cong \mathbf{C}$, which satisfies

$$\begin{cases} \Delta * G = \delta \\ G|_{\partial \mathbf{D}^2} = 0, \end{cases}$$

is

$$G(x) = \frac{1}{2\pi} \ln |x|.$$

Thus, the Green's function, after taking into account boundary conditions, is regular distribution written as

$$G_p(z) = \frac{1}{2\pi}\left(\ln\left|\frac{p-z}{1-\overline{p}z}\right|\right), \qquad G_p * g := \int_{\mathbf{D}^2} G_p(z) \cdot g(p) \, \mathrm{d}p$$

for some $p \in \mathbf{D}^2$, here considered as in the complex line $\mathbf{C}$, and solves the problem mentioned in remark ¶A.1.15.

Proof Since G is to be radially-symmetric, it suffices to write $\Delta * G$ in radial coordinates and find the form of G outside of the origin $0 \in \mathbf{R}^2$. Pulling back under the map $(x^1, x^2) \longmapsto \left(\sqrt{(x^1)^2 + (x^2)^2}, \arctan \frac{x^2}{x^1}\right) = (r, \theta)$,

$$\frac{\partial^2}{\partial (x^1)^2} = \left(\frac{\partial r}{\partial x^1}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x^1}\frac{\partial}{\partial \theta}\right)^2$$

$$= \left(\frac{x^1}{r}\frac{\partial}{\partial r} - \frac{x^2}{r^2}\frac{\partial}{\partial\theta}\right)^2$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2 \qquad x^1 = r\cos\theta \text{ and } x^2 = r\sin\theta$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}$$

$$\frac{\partial^2}{\partial\left(x^2\right)^2} = \left(\frac{\partial r}{\partial x^2}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x^2}\frac{\partial}{\partial\theta}\right)^2$$

$$= \left(\frac{x^2}{r}\frac{\partial}{\partial r} + \frac{x^1}{r^2}\frac{\partial}{\partial\theta}\right)^2$$

$$= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\cos^2\theta\frac{\partial}{\partial r} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2},$$

which shows that as a differential operator on $\mathbf{R}^2 \smallsetminus \{0\}$

$$\Delta = \frac{\partial^2}{\partial\left(x^1\right)^2} + \frac{\partial^2}{\partial\left(x^1\right)^2}$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Thus, the problem can be rewritten as

$$\left.\left(\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r}\right)\right|_{\mathbf{R}^2\smallsetminus\{0\}} = (\Delta * G)|_{\mathbf{R}^2\smallsetminus\{0\}} = \delta|_{\mathbf{R}^2\smallsetminus\{0\}} = 0$$

since $\frac{\partial G}{\partial\theta} = 0$ by being radially symmetric. Solving this yields,

$$G(x) = C\ln|x| + D \quad \text{or in polar coordinates,} \quad G(r) = C\ln r + D,$$

for constants $C, D \in \mathbf{R}$; with the boundary conditions $G|_{\partial\mathbf{D}^2} = 0$, it must be that $D = 0$. To determine C, an approach via the dominated convergence theorem will be used. First, let

$$L^1_{\text{loc}}\left(\mathbf{R}^2\right) \ni G^{\text{cont}}_\varepsilon(r,\theta) := \begin{cases} 0 & r < \varepsilon \\ C\ln r - C\ln\varepsilon & r \geq \varepsilon \end{cases},$$

which is continuous, and

$$G_\varepsilon(r,\theta) := G^{\text{cont}}_\varepsilon(r,\theta) + C\ln\varepsilon\cdot H(r-\varepsilon) \in L^1_{\text{loc}}\left(\mathbf{R}^2\right).$$

Using theorem ¶A.1.10, the distributional derivatives are

$$\frac{\partial}{\partial r}G_\varepsilon = \frac{\partial G^{\text{cont}}_\varepsilon}{\partial r} + C\ln\varepsilon\cdot\delta(r-\varepsilon) \quad \text{and} \quad \frac{\partial^2}{\partial r^2}(rG_\varepsilon) = \frac{\partial^2\left(rG^{\text{cont}}_\varepsilon\right)}{\partial r^2} + C\varepsilon\ln\varepsilon\cdot\frac{\partial}{\partial r}\delta(r-\varepsilon) + (C\ln\varepsilon + C)\cdot\delta(r-\varepsilon).$$

Then, for any $\varphi \in \mathscr{D}\left(\mathbf{R}^2\right)$ via remark ¶A.1.8,

$$\Delta G_\varepsilon \,.\, \varphi = G_\varepsilon \,.\, \Delta\varphi$$

$$= \int_{\mathbf{R}^2} G_\varepsilon\cdot\Delta\varphi \; \mathrm{d}x^1\wedge\mathrm{d}x^2 \qquad \text{since } G_\varepsilon \text{ is a regular distribution}$$

$$= \int_{\mathbf{R}^2\smallsetminus\{0\}} G_\varepsilon\cdot\Delta\varphi \; \mathrm{d}x^1\wedge\mathrm{d}x^2 \qquad \text{since } \{0\} \text{ is a set of measure } 0$$

$$= \int_0^{2\pi}\int_{\mathbf{R}_{>0}} rG_\varepsilon\cdot\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi \; r\mathrm{d}r\wedge\mathrm{d}\theta \qquad \text{pulling back to polar coordinates}$$

$$= \int_0^{2\pi} (rG_\varepsilon) \,.\, \frac{\partial^2}{\partial r^2}\varphi + G_\varepsilon \,.\, \frac{\partial}{\partial r}\varphi + \left(\frac{1}{r}G_\varepsilon\right) \,.\, \frac{\partial^2}{\partial\theta^2}\varphi \qquad \text{by definition of being a regular distribution}$$

$$= \int_0^{2\pi} \frac{\partial^2}{\partial r^2}(rG_\varepsilon) \,.\, \varphi - \frac{\partial}{\partial r} G_\varepsilon \,.\, \varphi \; \mathrm{d}\theta$$

using definition ¶A.1.7 and $\frac{\partial}{\partial\theta}\left(\frac{1}{r}G_\varepsilon\right) = 0$

$$= \int_0^{2\pi} \left(\frac{\partial^2 \left(rG_\varepsilon^{\text{cont}}\right)}{\partial r^2} + C\varepsilon \ln\varepsilon \cdot \frac{\partial}{\partial r}\delta(r-\varepsilon) + (C\ln\varepsilon + C)\cdot\delta(r-\varepsilon) \right) \,.\, \varphi$$
$$- \left(\frac{\partial G_\varepsilon^{\text{cont}}}{\partial r} + C\ln\varepsilon \cdot \delta(r-\varepsilon) \right) \,.\, \varphi \; \mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(2\frac{\partial G_\varepsilon^{\text{cont}}}{\partial r} + r\frac{\partial^2 G_\varepsilon^{\text{cont}}}{\partial r^2} + C\varepsilon \ln\varepsilon \cdot \frac{\partial}{\partial r}\delta(r-\varepsilon) + (C\ln\varepsilon + C)\cdot\delta(r-\varepsilon) \right) \,.\, \varphi$$
$$- \left(\frac{\partial G_\varepsilon^{\text{cont}}}{\partial r} + C\ln\varepsilon \cdot \delta(r-\varepsilon) \right) \,.\, \varphi \; \mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(\frac{\partial G_\varepsilon^{\text{cont}}}{\partial r} + r\frac{\partial^2 G_\varepsilon^{\text{cont}}}{\partial r^2} + C\varepsilon \ln\varepsilon \cdot \frac{\partial}{\partial r}\delta(r-\varepsilon) + C\delta(r-\varepsilon) \right) \,.\, \varphi \; \mathrm{d}\theta$$

by linearity

$$= \int_0^{2\pi} \left(C\varepsilon \ln\varepsilon \cdot \frac{\partial}{\partial r}\delta(r-\varepsilon) + C\delta(r-\varepsilon) \right) \,.\, \varphi \; \mathrm{d}\theta$$

$\frac{\partial G_\varepsilon^{\text{cont}}}{\partial r} = \frac{1}{r} = -r\frac{\partial^2 G_\varepsilon^{\text{cont}}}{\partial r^2}$, or note that is also solves the above Laplace equation

$$= \int_0^{2\pi} \left(\left(-C\varepsilon \ln\varepsilon \cdot \delta(r-\varepsilon)\right) \,.\, \frac{\partial\varphi}{\partial r} + \left(C\delta(r-\varepsilon)\right) \,.\, \varphi \right) \; \mathrm{d}\theta$$

using definition ¶A.1.7

$$= 2\pi C \left(\varepsilon \ln\varepsilon \cdot \frac{\partial\varphi}{\partial r}(\varepsilon, 2\pi) + \varphi(\varepsilon, 2\pi) \right).$$

Wherefore, via the dominated convergence theorem and l'Hôpital's rule,

$$(\Delta * G) \,.\, \varphi = \Delta G \,.\, \varphi = G \,.\, \Delta\varphi = \lim_{\varepsilon\to 0} G_\varepsilon \,.\, \Delta\varphi = \lim_{\varepsilon\to 0} 2\pi C \left(\varepsilon\ln\varepsilon \cdot \frac{\partial\varphi}{\partial r}(\varepsilon, 2\pi) + \varphi(\varepsilon, 2\pi) \right) = 2\pi C\varphi(0, 2\pi),$$

which, when $C = \frac{1}{2\pi}$, is exactly

$$\Delta * G = \delta.$$

Hence,

$$G(x) = \frac{1}{2\pi}\ln|x|$$

satisfies

$$\begin{cases} \Delta * G = \delta \\ G|_{\partial\mathbf{D}^2} = 0. \end{cases}$$

Lastly, in wanting this to solve

$$\Delta * f = g$$

as mentioned in remark ¶A.1.15, it must be that, upon translation that G preserves the boundary conditions. As such, there needs to be a correcting function added to the translation $G(p-z)$ to achieve this; there are two ways of obtain this. The quickest way is to consider the automorphism of $\mathbf{D}^2$ [SS03, Ch. 8, Theorem 2.2]

$$\alpha_p(z) = \frac{p-z}{1-\overline{p}z},$$

which fixes the boundary set-wise, so the boundary conditions are preserved with

$$G_p(z) := G\left(\frac{p-z}{1-\overline{p}z}\right) = \frac{1}{2\pi}\left(\ln|p-z| - \ln|1-\overline{p}z|\right),$$

wherein the correcting function is the $-\ln|1-\overline{p}z|$. Another way to obtain the correcting function, is to use the "method of images". [Nee97, §12.VI.5] [Uni03] The "image" of point p is $p^* := 1/\overline{p}$, which is the reflection in the plane $\mathbf{C}$ with respect to the unit 1-sphere $\mathbf{S}^1 = \partial\mathbf{D}^2 \subset \mathbf{C}$. It has that property, that, for any $z \in \mathbf{S}^1$,

$$|p|^2|z-p^*| = |p|^2\left(1 - z\overline{p^*} - \overline{z}p^* + |p^*|^2\right) = |p|^2 - z\overline{p} - \overline{z}p + |z|^2 = |p-z|^2,$$

and for any $z \in \mathbf{D}^2$,

$$|p||z-p^*| = |pz - p/\overline{p}| = |p/\overline{p}||\overline{p}z - 1| = |1-\overline{p}z|.$$

Thus, the Green's function is

$$G_p(z) := G(p-z) - G(p(z-p^*)) = \frac{1}{2\pi}\left(\ln|p-z| - \ln|p||z-p^*|\right) = \frac{1}{2\pi}\ln\left|\frac{p-z}{1-\overline{p}z}\right| = G(\alpha_p(z)). \quad \blacksquare$$

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