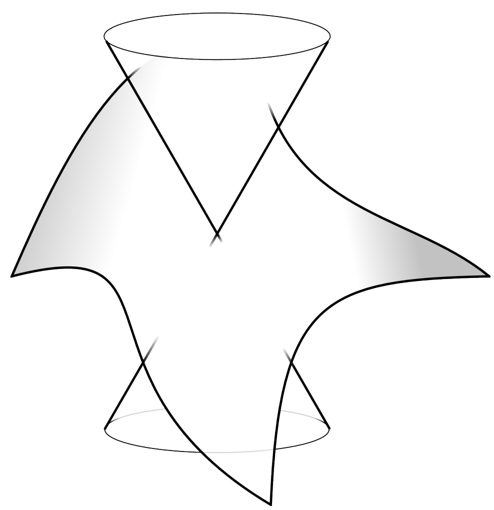
(In)completeness in Riemannian, and Lorentzian, Geometries
via the Calculus of Variations



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I pledge my honor, as a gentleman, that I have not violated the Honor Code during the utterly Princetonian process that culminated in this work before you, my thesis.

Abstract

The primary of motivation of this work is to develop notions from the Calculus of variations in the context of semi-Riemannian Geometry, using a basic background in manifold theory and Analysis. Starting off, chapter 0 outlines some of the foundational notations, and concepts, that are used throughout the following three chapters. Chapter 1 thoroughly develops notions of the Calculus of variations in the semi-Riemannian context, and is a large majority of this work. With that inspiration, chapter 2, and chapter 3, go respectively over the Riemannian, and Lorentzian, cases with a large impetus on procuring results that are elucidated by the concepts from the Calculus of variations. At the end of each of these two chapters, a main result is obtained regarding the (in)completeness of such manifolds.

This edition is a revision of the original submission. It includes the correction of quite a few typos (with much help of my adviser). Also, the proof of proposition [1.7.2] has been corrected to actually cover the generality in its statement. Finally, there is some ironing-out of details in the Lorentzian Geometry chapter, particularly in the proofs of theorem [3.0.23] and theorem [3.0.28].

Acknowledgements

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“Interesting; have fun!”

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0 | Preliminaries

Throughout later chapters of these notes, the notion of a semi-Riemannian manifold will be used, along with the associated notions of a metric, its compatible connection, and geodesics. As such, these concepts will be outlined in this chapter with some associated basic results. [O'N83, Ch. 3][Pet10, §2.3][Car92, §2.3]

0.1 The Connection

As to establish a way of differentiating on manifolds, the connection is constructed.

0.1.1 Definition A *metric tensor* is a symmetric, non-degenerate $(0,2)$-tensor field on the tangent spaces of a smooth manifold M such that the dimension of the space, on which it is negative definite, is constant. This dimension is also called the *index* of the metric. It is written as either $g(\cdot,\cdot)$ or $\langle \cdot,\cdot \rangle$.

0.1.2 Definition A *semi-Riemannian manifold* is a pair (M,g), where g is a metric tensor on M. In particular, when the index of g is zero, (M,g) is called a *Riemannian manifold*; when it is one, (M,g) is called a *Lorentzian manifold*. When it is unambiguous, a manifold is referred to simply as M.

Associated with such a metric, the tangent space to a semi-Riemannian manifold M has the following trichotomy on the *magnitude* of its elements, called *causal character*: $v \in T_pM$ is

— *timelike* if $g(v,v) < 0$
— *null* if $g(v,v) = 0$ and $v \neq 0$
— *spacelike* if $g(v,v) > 0$ or $v = 0$.

Accompanying this, $v \in T_pM$ is called *causal* if it is timelike or null. It is also handy to define the *absolute value of* v as $|v| := \sqrt{\varepsilon g(v,v)}$, with $\varepsilon = \operatorname{sgn} g(v,v)$.

Accompanying a semi-Riemannian manifold, there is a unique function on the tangent bundle TM of M:

0.1.3 Theorem The unique *Levi-Civita connection* associated with a semi-Riemannian manifold (M,g),

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)\,, \text{ where } \mathfrak{X}(M) \text{ denotes the smooth vector fields on } M,$$

is characterized by having the following properties: $X, Y, Z \in \mathfrak{X}(M)$,

(1) $\nabla_X Y$ is linear in X over all smooth functions on M, $\mathfrak{F}(M)$

(2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y

(3) for $f \in \mathfrak{F}(M)$, $\nabla_X(fY) = (Xf)\,Y + f\nabla_X Y$

(4) torsion-free, $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

(5) compatibility with the metric[1], $\nabla_X g = 0$, making

$$X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$

[1]Using these properties, the notion of a connection can be extended to an arbitrary tensor; since this will not be used, its action on the metric tensor can be regarded as a definition.

where existence follows from properties (1), (2), and (3), and uniqueness follows from properties (4) and (5).

In a local coordinate neighborhood of $p \in M^m$ (where m denotes the dimension of M), there is an induced basis $\{\partial_i\}_1^m$ on T_pM, so that vector fields[2] can be written as $X^i\partial_i, Y^i\partial_i \in \mathfrak{X}(M)$, and there is an induced dual basis $\{dx^i\}_1^m$ on T_p^*M, so that $g = g_{ij}dx^i \otimes dx^j$. Continuing with this, the connection[3] can be written locally as:

$$\begin{aligned}
\nabla_X Y &= X^i \nabla_{\partial_i}\left(Y^j\partial_j\right) && \text{using property (1)}\\
&= X^i\left(\partial_i Y^j\right)\partial_j + X^iY^j\nabla_{\partial_i}\partial_j && \text{property (3)}\\
&= X^i\left(\partial_i Y^k\right)\partial_k + X^iY^j\Gamma_{ij}^k\partial_k,
\end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols. These are gotten from the Koszul formula, which characterizes the connection: for $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
2g\left(\nabla_X Y, Z\right) = X\left(g\left(Y,Z\right)\right) + Y\left(g\left(Z,X\right)\right) - Z\left(g\left(X,Y\right)\right)\\
- g\left(X,[Y,Z]\right) + g\left(Y,[Z,X]\right) + g\left(Z,[X,Y]\right),
\end{aligned} \qquad [0.1.4]$$

where $[\cdot,\cdot]$ is the Lie bracket. The symbols are expressed as

$$\Gamma_{ij}^k = \frac{1}{2}g^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right),$$

with $\left(g^{ij}\right)$ being the matrix inverse of (g_{ij}); note that this shows that Christoffel symbols are symmetric in the lower two indices, $\Gamma_{ij}^k = \Gamma_{ji}^k$, since the metric is symmetric.

An intrinsic concept of semi-Riemannian manifolds follows from the introduction of these notions: curvature. [Pet10, §2.3][BEE96, §2.2]

0.1.5 Definition The *Hessian* is defined in terms the connection on M as the $(1,3)$-tensor

$$\nabla^2_{X,Y}Z = \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

0.1.6 Definition Using the Hessian, *Riemann curvature tensor* is a $(1,3)$-tensor on M: for $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
R(X,Y)Z &= \left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)Z\\
&= \left(\nabla_X\nabla_Y - \nabla_{\nabla_X Y} - \nabla_Y\nabla_X + \nabla_{\nabla_Y X}\right)Z\\
&= \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - X^i\nabla_{\nabla_i Y} + Y^i\nabla_{\nabla_i X}\right)Z && \nabla_i := \nabla_{\partial_i}\\
&= \left([\nabla_X,\nabla_Y] - X^i\left(\left(\partial_i Y^k\right) + \Gamma_{ij}^k X^iY^j\right)\nabla_k + Y^i\left(\left(\partial_i X^k\right) + \Gamma_{ij}^k Y^iX^j\right)\nabla_k\right)Z\\
&= \left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\right)Z && \text{since } \Gamma_{ij}^k = \Gamma_{ji}^k.
\end{aligned}$$

In local coordinates, this is

$$\begin{aligned}
R\left(\partial_i,\partial_j\right)\partial_k &= \left(\nabla^2_{\partial_i,\partial_j} - \nabla^2_{\partial_j,\partial_i}\right)\partial_k\\
&= \left(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{[\partial_i,\partial_j]}\right)\partial_k\\
&= \nabla_{\partial_i}\left(\Gamma_{jk}^n\partial_n\right) - \nabla_{\partial_j}\left(\Gamma_{ik}^n\partial_n\right) && [\partial_i,\partial_j] = 0\\
&= \left(\partial_i\Gamma_{jk}^m + \Gamma_{jk}^n\Gamma_{in}^m - \partial_j\Gamma_{ik}^m - \Gamma_{ik}^n\Gamma_{jn}^m\right)\partial_m.
\end{aligned}$$

[2]Henceforth, unless otherwise mentioned, vector fields considered will be smooth.

[3]From now onward, unless otherwise noted, when a connection is referred to in relation to a semi-Riemannian manifold, it is taken to be the Levi-Civita connection given by its inherit metric, the compatible connection.

Also, it is convenient to write this as a $(0,4)$-tensor:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

In particular, a manifold M is called *flat* if its Riemann curvature tensor vanishes everywhere; this vanishing entails that its metric, everywhere, can be written as $g_{ij} = \varepsilon_i \delta_{ij}$, where ε_i is a sign depending on the index, with its identically-zero Christoffel symbols.

This tensor has the following properties:

(1) antisymmetric in the first, and last, two entries,

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z)$$

(2) symmetric in the first two, and last two, entries,

$$R(X,Y,Z,W) = R(Z,W,X,Y)$$

(3) Bianchi's first identity,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

(4) Bianchi's second identity[4],

$$(\nabla_Z R)(X,Y)W + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W = 0.$$

From this tensor, comes a few more definitions.

0.1.7 Definition The *directional curvature operator*, or *tidal force operator*, is the map, for any $v \in T_pM$,

$$R(\cdot,v)v = R_v : T_pM \longrightarrow T_pM.$$

0.1.8 Definition The *sectional curvature* of the plane in T_pM spanned by $\{v,w\}$ is given by

$$\sec(v,w) = \frac{g(R_v(w),w)}{g(v,v)\,g(w,w) - g(v,w)^2}.$$

0.1.9 Definition If $\{e_i\}_1^n$ is an orthonormal basis on T_pM^n, i.e. $g(e_i,e_i) = \pm 1$, then the *Ricci curvature tensor* is given, either as a (symmetric) $(0,2)$-tensor by the contraction

$$\mathrm{Ric}(v,w) = \sum_{i=1}^n \varepsilon_i g(R(e_i,v)w,e_i), \text{ where } \varepsilon_i = \operatorname{sgn} g(e_i,e_i),$$

or the $(1,1)$-tensor

$$\mathrm{Ric}(v) = \sum_{i=1}^n R(v,e_i)e_i.$$

0.1.10 Definition The *scalar curvature* of T_pM is gotten by (yet another) contraction with the metric:

$$\mathrm{scal} = \sum_{i=1}^n g(\mathrm{Ric}(e_i),e_i).$$

[4] Again, as with the connection acting on the metric tensor, the action of the connection can be generalized to higher-order tensors in a natural way via its properties.

Naturally following the introduction of the connection is the question as to how it acts when restricted to vector fields on submanifolds. [O'N83, Ch. 4] Let M be a semi-Riemannian submanifold of $\overline{M}$, where $\overline{M}$ has the connection $\overline{\nabla}$; in this case, the metric on M is the one gotten from the ambient $\overline{M}$. Also, let $\overline{\mathfrak{X}}(M)$ be the set of smooth vector fields on M that are gotten by restricting each element of $\mathfrak{X}\left(\overline{M}\right)$ to M; namely, it is the set of smooth vector fields on M which assign to each $p \in M$ a member of $T_p\overline{M}$. On each coordinate neighborhood U of $\overline{M}$ intersecting M, let $\overline{X}, \overline{Y}, \in \mathfrak{X}\left(\overline{M}\right)$ be local extensions of $X \in \mathfrak{X}(M)$, and $Y \in \overline{\mathfrak{X}}(M)$, on U, and define $\overline{\nabla}_X Y = \overline{\nabla}_{\overline{X}}\overline{Y}\big|_{U\cap M}$.

0.1.11 Lemma For $X \in \mathfrak{X}(M)$ and $Y \in \overline{\mathfrak{X}}(M)$, $\overline{\nabla}_X Y$ is a well-defined member of $\overline{\mathfrak{X}}(M)$.

Proof Since $\overline{\nabla}_X Y$ is just a restriction of the already smooth $\overline{\nabla}_{\overline{X}}\overline{Y}$, it follows that it is also smooth. As such, now it is just needed to show that $\overline{\nabla}_X Y$ is independent of the extensions of X, Y. In a coordinate neighborhood U of $\overline{M}$ intersecting M, the extensions can be written as $\overline{X} = \overline{X}^i\partial_i$ and $\overline{Y} = \overline{Y}^i\partial_i$, making $\overline{\nabla}_{\overline{X}}\overline{Y} = \left(\overline{X}^i\partial_i\overline{Y}^k + \overline{X}^i\overline{Y}^j\Gamma^k_{ij}\right)\partial_k$. Evaluating this at any $p \in U\cap M$ yields

$$\begin{aligned}
\left(\overline{\nabla}_X Y\right)\big|_p &= \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)\big|_p && \text{by definition}\\
&= \left(\overline{X}^i\partial_i\overline{Y}^k\right)\Big|_p \partial_k\big|_p + \left(\overline{X}^i\overline{Y}^j\Gamma^k_{ij}\right)\Big|_p \partial_k\big|_p\\
&= X^i\big|_p\left(\partial_i\overline{Y}^k\right)\Big|_p \partial_k\big|_p + \left(X^iY^j\Gamma^k_{ij}\right)\big|_p \partial_k\big|_p && \text{since } \overline{X}, \overline{Y} \text{ are extensions of } X, Y,
\end{aligned}$$

where, in the first term,

$$\begin{aligned}
X^i\big|_p\left(\partial_i\overline{Y}^k\right)\Big|_p &= X^i\big|_p\,\partial_i\big|_p\left(\overline{Y}^k\right) && \text{by definition}\\
&= X^i\big|_p\,\partial_i\big|_p\left(Y^k\right) && \text{since } X^i\big|_p\,\partial_i\big|_p \in T_pM\\
&= \left(X^i\partial_iY^k\right)\big|_p && \text{by definition, again.}
\end{aligned}$$

Thus, $\overline{\nabla}_X Y$ is well-defined and in $\overline{\mathfrak{X}}(M)$. ∎

With this result comes a corollary that follows from the fact that $\overline{\nabla}$ is the connection on $\overline{M}$.

0.1.12 Corollary The restriction of the connection $\overline{\nabla}$ to $M \subset \overline{M}$ has the properties of a Levi-Civita connection, as mentioned in theorem `[0.1.3]`.

Restricting this induced connection to acting on $\mathfrak{X}(M) \ni X, Y$, the object $\overline{\nabla}_X Y$ need not lie in $\mathfrak{X}(M)$; in fact, it is some element of $\overline{\mathfrak{X}}(M)$. As such, it is possible to write $\overline{\nabla}_X Y$ as the sum of the component tangent to M and the component normal to M.

0.1.13 Lemma For $X, Y \in \mathfrak{X}(M)$,

$$\tan\overline{\nabla}_X Y = \nabla_X Y,$$

where ∇ is the Levi-Civita connection on M induced from the metric on $\overline{M}$, and $\tan(\cdot)$ denotes the component tangent to M

Proof The proof of this lemma hinges on the fact that $M \subset \overline{M}$ inherits its metric from $\left(\overline{M}, g\right)$, and the Koszul formula.

Consider some coordinate neighborhood U of $\overline{M}$ intersecting M; for $p \in U\cap M$, let $\{\partial_i\}$ be a basis for T_pM, and choose any $X, Y \in \mathfrak{X}(M)$. By the Koszul formula, `[0.1.4]`, it follows that, for any i,

$$g\left(\overline{\nabla}_X Y, \partial_i\right) = g\left(\nabla_X Y, \partial_i\right).$$

Since, by definition, each ∂_i is tangent to M, this can be written

$$g\left(\tan \overline{\nabla}_X Y, \partial_i\right) = g\left(\nabla_X Y, \partial_i\right).$$

Because $\nabla_X Y \in \mathfrak{X}(M)$ by definition, this implies that $\tan \overline{\nabla}_X Y = \nabla_X Y$ on any coordinate neighborhood, i.e. everywhere. ■

0.1.14 Definition The *shape tensor* is

$$\begin{aligned} \Pi : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M)^{\perp} \subset \overline{\mathfrak{X}}(M) \\ (X, Y) &\longmapsto \operatorname{nor} \overline{\nabla}_X Y, \end{aligned}$$

where $\operatorname{nor}(\cdot)$ denotes the component normal to M.

(The tensor qualification is shown in the following lemma.)

0.1.15 Lemma The shape tensor is a symmetric $(0,2)$-tensor on $\mathfrak{X}(M)$ mapping into $\mathfrak{X}(M)^{\perp}$.

Proof First, note that using the properties of the connection (guaranteed by corollary `[0.1.12]`) it follows, for any $X, Y \in \mathfrak{X}(M)$, $f \in \mathfrak{F}(M)$,

$$\overline{\nabla}_{fX} Y = f \overline{\nabla}_X Y$$

and

$$\overline{\nabla}_X (fY) = (Xf) Y + f \overline{\nabla}_X Y.$$

Now, it follows from the linearity of $\operatorname{nor}(\cdot)$ that, respectively,

$$\Pi(fX, Y) = \operatorname{nor} \overline{\nabla}_{fX} Y = f \operatorname{nor} \overline{\nabla}_X Y = f\, \Pi(X, Y)$$

and

$$\Pi(X, fY) = \operatorname{nor} \overline{\nabla}_X (fY) = \underbrace{\operatorname{nor}((Xf)Y)}_{\substack{0 \\ \text{since } Y \text{ tangent to } M}} + f \operatorname{nor} \overline{\nabla}_X Y = f\, \Pi(X, Y),$$

showing that $\Pi(\cdot,\cdot)$ is $\mathfrak{F}(M)$-bilinear since X, Y were arbitrary. Thus, given that it maps into $\mathfrak{X}(M)^{\perp}$, $\underline{\Pi(\cdot,\cdot) \text{ is a } (0,2)\text{-tensor on } \mathfrak{X}(M)}$.

Next, to show that it is symmetric: in any coordinate neighborhood U of $\overline{M}$ intersecting M, so that, for $p \in U \cap M$, $T_p\overline{M}$ has the basis $\{\partial_i\}$,

$$\begin{aligned} \left.\left(\Pi(X,Y) - \Pi(Y,X)\right)\right|_p &= \left.\left(\operatorname{nor}\left(\overline{\nabla}_X Y - \overline{\nabla}_Y X\right)\right)\right|_p && \text{using linearity of nor}(\cdot) \\ &= \operatorname{nor}\left(X^i \partial_i Y^k \partial_k + X^i Y^j \Gamma^k_{ij} \partial_k - Y^i \partial_i X^k \partial_k - Y^i X^j \Gamma^k_{ij} \partial_k\right) \\ &= \left.\left(\operatorname{nor}[X, Y]\right)\right|_p && \text{since } \Gamma^k_{ij} = \Gamma^k_{ji} \\ &= 0 && \text{since } [X, Y] \in \mathfrak{X}(M). \end{aligned}$$

Thus, since the choice of X, Y, U, and p, were arbitrary, $\underline{\Pi(\cdot,\cdot) \text{ is symmetric}}$ by definition. ■

Paired with this tensor is the following, which maps into $\mathfrak{X}(M)$.

0.1.16 Definition The *associated shape tensor* $\widetilde{\Pi}$, associated with Π, is the map

$$\begin{aligned} \widetilde{\Pi} : \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \tan \overline{\nabla}_X Y. \end{aligned}$$

Associated is meant in the following sense: since, for $Z \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)^{\perp}$, $g(Z,Y) = 0$, making, for $X \in \mathfrak{X}(M)$,

$$\begin{aligned}
0 = X\left(g\left(Z,Y\right)\right) &= g\left(\overline{\nabla}_X Z, Y\right) + g\left(Z, \overline{\nabla}_X Y\right) && \text{by compatibility of the connection with the metric} \\
\iff \; g\left(\overline{\nabla}_X Z, Y\right) &= -g\left(Z, \overline{\nabla}_X Y\right) \\
g\left(\operatorname{nor} \overline{\nabla}_X Z, Y\right) &= -g\left(Z, \tan \overline{\nabla}_X Y\right) && \text{by choice of } Y \text{ and } Z \\
g\left(\Pi\left(X,Z\right), Y\right) &= -g\left(Z, \widetilde{\Pi}\left(X,Y\right)\right) && \text{by the definitions of } \Pi, \widetilde{\Pi}\text{: respectively, by being normal, and tangent.}
\end{aligned}$$

(The proof of the tensor qualification is just as in lemma [0.1.15] but with "normal", and "tangent", switched; the proof that it is not symmetric follows similarly.)

These definitions, and results, will be used in the later sections; in the next section, the connection will be restricted to vectors fields along curves.

0.2 Geodesics

In this section, we will take the notion of a connection and restrict it to vector fields along curves; in doing this, the notion of a geodesic will follow naturally. Let $\gamma : \mathbb{R} \supset I \longrightarrow M$ be a smooth curve on a semi-Riemannian manifold M, and let the set of smooth vector fields along γ be denoted $\mathfrak{X}(\gamma)$, while the set of smooth functions along γ is denoted $\mathfrak{F}(\gamma)$. Along γ, there is an inherit vector field composed of its tangent vectors; it is denoted $\dot{\gamma}$ or γ'. Call $\gamma(0) = p$; in some coordinate neighborhood of p, with coordinate maps $\left\{x^i : M \longrightarrow \mathbb{R}\right\}$, it is possible to express this vector field in the associated basis $\{\partial_i\}$ of T_pM as

$$\dot{\gamma}(0) = \underbrace{\left.\frac{d\left(x^i \circ \gamma\right)}{dt}\right|_0}_{\left.\frac{d\gamma^i}{dt}\right|_0 = \dot{\gamma}^i(0)} \left.\partial_i\right|_p .$$

Now, recall from the definition of the connection, theorem [0.1.3], that $\nabla_X Y$ is tensorial in X, which, in particular, means that the value of $\nabla_X Y$ only depends on X at the point which it is being evaluated – not a neighborhood of that point. With this in mind [Gal07, §1.7], it is possible to restrict the connection on M to $\mathfrak{X}(\gamma)$ as such:

0.2.1 Definition The *covariant derivative along a curve*, $\gamma(t)$, is

$$\begin{aligned}
\frac{D}{dt} : \mathfrak{X}(\gamma) &\longrightarrow \mathfrak{X}(\gamma) \\
X &\longmapsto \frac{DX}{dt} = \nabla_{\dot{\gamma}} X.
\end{aligned}$$

In a coordinate neighborhood at $\gamma(0)$,

$$\begin{aligned}
\left.\frac{DX}{dt}\right|_0 &= \left.\left(\dot{\gamma}^i \partial_i X^k + \dot{\gamma}^i X^j \Gamma^k_{ij}\right)\right|_0 \left.\partial_k\right|_0 \\
&= \left.\left(\frac{dX^k}{dt} + \dot{\gamma}^i X^j \Gamma^k_{ij}\right)\right|_0 \left.\partial_k\right|_0 && \text{via the chain rule.}
\end{aligned}$$

When $\frac{DX}{dt} = 0$, it is said that X is *parallel transported* along γ.

Using this, the notion of a geodesic follows as a curve which is parallel transported along itself:

0.2.2 Definition A *geodesic* is a (non-constant) curve γ such that, along γ,

$$\frac{D\dot{\gamma}}{dt} = 0;$$

this is called the *geodesic equation.* In local coordinates, this implies, for each k,[5]

$$\begin{aligned} \dot{\gamma}^i \partial_i \dot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} &= 0 \\ \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} &= \qquad \text{via the chain rule.} \end{aligned}$$

This means that $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ, which implies that its causal character does not change. From this, it is possible to define the *causal character of* γ as simply the causal character of $\dot{\gamma}$.

The last expression in the previous definition is simply a system of second-order ODEs. Note that the constant curve trivially satisfies the geodesic equation; even so, it is not considered when talking about geodesics.

Following from the local existence-and-uniqueness theorem of ODEs, comes the next lemma.

0.2.3 Lemma Given a $p \in M$, and $v \in T_pM$, there exists an $\delta > 0$ such that $\gamma : (-\delta, \delta) \longrightarrow M$ is a unique solution to the geodesic equation with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

In this spirit, let $\gamma_{p,v}$ be the geodesic starting at p, $p = \gamma(0)$, and $\dot{\gamma} = v \in T_pM$. Going along with this uniqueness result, comes the next definition, which is a commonly-used map of geodesics from a subset of the tangent bundle into M.

0.2.4 Definition Given $p \in M$, let $U \ni v$ be the subset of T_pM such that the geodesic $\gamma_{p,v}$ is defined on $[0, 1]$. With this, the *exponential map* at p is

$$\begin{aligned} \exp_p : T_pM &\longrightarrow M \\ v &\longmapsto \gamma_{p,v}(1), \end{aligned}$$

or, as viewed as acting on the tangent bundle TM of M,

$$\begin{aligned} \exp : TM &\longrightarrow M \\ (p, v) &\longmapsto \exp_p v = \gamma_{p,v}(1), \end{aligned}$$

which is smooth, where defined.

Also, not all reparametrizations of a geodesic are geodesics themselves; because of this, the only reparametrizations of a geodesic that are considered, are those as in the following.

0.2.5 Lemma Let $\gamma(t)$ be a (non-trivial) geodesic, then $\widetilde{\gamma}(s) = \gamma(t(s))$ is a geodesics iff $t = cs + d$ for constants $c \neq 0$, d.

Proof By definition, the curve $\widetilde{\gamma}$ is a geodesic iff

$$\begin{aligned} \frac{D\widetilde{\gamma}'}{ds} &= 0 \\ \frac{D\left(\frac{dt}{ds}\dot{\gamma}\right)}{ds} &= \qquad \widetilde{\gamma}'(s) = \frac{d\gamma(t(s))}{ds} = \frac{dt}{ds}\dot{\gamma} \end{aligned}$$

[5] The evaluation of this expression at some point in the coordinate neighborhood is implicit.

$$\begin{aligned}
\frac{d^2t}{ds^2}\dot{\gamma} + \frac{dt}{ds}\underbrace{\frac{D\dot{\gamma}}{ds}}_{0} &= && \frac{D\dot{\gamma}}{ds} = \nabla_{\widetilde{\gamma}'}\dot{\gamma} = \frac{dt}{ds}\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \text{ by assumption on } \gamma \\
\iff \frac{d^2t}{ds^2} &= 0 && \text{since } \dot{\gamma} \neq 0 \\
\iff t &= cs + d && c \neq 0 \text{ so that } \widetilde{\gamma} \text{ is not trivial.}
\end{aligned}$$

■

Another interesting note coming from the parallel-transport quality of a geodesic is the following lemma.

0.2.6 Lemma For a non-null geodesic γ (i.e. $|\dot{\gamma}| > 0$), let $X = X^\perp + X^\top$ be a vector field along γ, where the perpendicular component with respect to γ is $X^\perp$, and the tangential component is $X^\top$. Then,

$$\frac{DX^\perp}{dt} = \left(\frac{DX}{dt}\right)^\perp \quad \text{and} \quad \frac{DX^\top}{dt} = \left(\frac{DX}{dt}\right)^\top.$$

Proof First, the restriction to non-null γ is to ensure that the perpendicular, and tangential, components of X are defined. Using linearity of the covariant derivative, it follows that

$$\frac{DX}{dt} = \frac{DX^\perp}{dt} + \frac{DX^\top}{dt}.$$

With that, now each term on the right-hand side will be considered.

For a vector field Y along γ:

$$\begin{aligned}
\frac{d}{dt}\langle Y, \dot{\gamma}\rangle &= \left\langle \frac{DY}{dt}, \dot{\gamma}\right\rangle + \left\langle Y, \frac{D\dot{\gamma}}{dt}\right\rangle && \text{using the fact that the connection is compatible with the metric} \\
&= \left\langle \frac{DY}{dt}, \dot{\gamma}\right\rangle && \tfrac{D\dot{\gamma}}{dt} = 0;
\end{aligned}$$

in particular, if $\langle Y, \dot{\gamma}\rangle = 0$ along γ, then $\left\langle \frac{DY}{dt}, \dot{\gamma}\right\rangle = 0$. As such, the fact that $X^\perp$ is perpendicular to γ is preserved after a covariant derivative along γ:

$$\langle X^\perp, \dot{\gamma}\rangle = 0 \implies \left\langle \frac{DX^\perp}{dt}, \dot{\gamma}\right\rangle = 0.$$

Also, since $X^\top$ is tangent to γ, it can be written as $f\dot{\gamma}$, where f is a smooth function along γ; it follows that

$$\frac{DX^\top}{dt} = \dot{f}\dot{\gamma} + \underbrace{f\frac{D\dot{\gamma}}{dt}}_{\substack{0 \\ \text{since } \gamma \text{ is a geodesic}}} = \dot{f}\dot{\gamma},$$

i.e. the fact that $X^\top$ is tangent is preserved after a covariant derivative along γ. Thus, there is no mixing of the components of X after taking a covariant derivative along γ,

$$\frac{DX^\perp}{dt} = \left(\frac{DX}{dt}\right)^\perp \quad \text{and} \quad \frac{DX^\top}{dt} = \left(\frac{DX}{dt}\right)^\top, \text{ respectively.}$$

■

1 | Calculus of Variations

A major part of the proofs of the Bonnet-Myers theorem of Riemannian Geometry, and the Hawking incompleteness theorem of Lorentzian Geometry, is the use of results of the Calculus of variations applied in each geometric context. In particular, this tool helps in understanding how the length functional behaves, which is essential to each theorem. As such, this chapter will be dedicated to establishing these concepts and their results. [O'N83, Ch. 8 & 10][Pet10, Ch. 5]

The motivating definition is, of course, that of length

1.0.1 Definition The *length* of a piecewise-smooth curve $\gamma : [a, b] \longrightarrow M$, a semi-Riemannian manifold, is

$$\int_a^b \underbrace{\sqrt{\varepsilon g(\dot{\gamma}, \dot{\gamma})}}_{|\dot{\gamma}|},$$

where $\varepsilon = \operatorname{sgn} g(\dot{\gamma}, \dot{\gamma})$.

And, in order to talk about changes in the length functional, the notion of a variation needs to be solidified.

1.0.2 Definition Given a smooth curve $\gamma : [a, b] \longrightarrow M$, a *variation* of γ is a smooth mapping

$$\chi : [a, b] \times (-\delta, \delta) \longrightarrow M,$$

where $\delta > 0$, such that $\chi(t, 0) = \gamma(t)$, $t \in [a, b]$. Specific to a variation: the curve γ is called the *base curve*; the curves $\chi(t_0, s)$, for fixed t_0, are the *transverse curves*; the curves $\chi(t, s_0)$, for fixed s_0, are the *longitudinal curves*. The variation is called *proper*, if $\chi(a, s) = \gamma(a)$, and $\chi(b, s) = \gamma(b)$, for any $s \in (-\delta, \delta)$.

With such a mapping, comes its associated *variational field*, which is the vector field along γ defined locally[1] as

$$\chi_s(t, 0) := \frac{\partial \chi^i}{\partial s}(t, 0)\, \partial_i|_{\gamma(t)}.$$

This "subscript" derivative is also used to mean $\chi_t(t, s) := \frac{\partial \chi^i}{\partial t}(t, s)\, \partial_i|_{\gamma(t)}$. The action of these subscript derivatives[2] on an $X \in \mathfrak{X}(M)$ can be defined via definition `[0.2.1]`: given the transverse curve $\sigma(s) = \chi(s, t_0)$, for fixed t_0, in M, then,

$$X_s(t, s) = \frac{DX}{ds}(t, s);$$

similarly, working with longitudinal curves, obtains $X_t = \frac{DX}{dt}(t, s)$. Higher-order subscript derivatives are just higher-order covariant derivatives along the appropriate curves in M.

Also to be considered are variations of piecewise-smooth curves, or with piecewise-smooth variational fields X. Let the smooth segments of γ, or X, be on (finitely-many) $[t_i, t_{i+1}]$. On each smooth segment, the definition of a variation remains intact; thus, in order to extend the definition, it is only needed to consider what happens at the break points t_i. First off, the *piecewise variation*[3] should be continuous. This ensures that each curve $\chi(t_0, s)$, for fixed

[1] With a slight abuse of notation, χ^i is the composition of the i-th component of a local coordinate chart with χ.
[2] Of course, this definition is not restricted to use with curves in variations, but this best illustrates its current use.
[3] Depending on context, "variation", and "piecewise variation", will be used interchangeably.

t_0, is piecewise-smooth. For simplicity of defining the variational field, each curve $\chi(t_i, s)$ is to be piecewise-smooth[4]. Also, the ambiguity of $\chi_t(t_i, s)$ can be resolved in the following manner: using one-sided derivatives from the left, and right, respectively,

$$\begin{aligned}
(\chi_t(t_i, s))|_{[t_{i-1}, t_i]} &:= \chi_{t^-}(t_i, s) \\
(\chi_t(t_i, s))|_{[t_i, t_{i+1}]} &:= \chi_{t^+}(t_i, s).
\end{aligned}$$

As is to be expected, higher-order derivatives of the two-parameter map χ will be taken; to this end, results pertaining to them will be shown here for reference in later sections of this chapter. [O'N83, Ch. 4]

1.0.3 Proposition For a smooth two-parameter map $\chi(t, s)$ on M,

(1) $\chi_{ts} = \chi_{st}$

(2) if X is a vector field on M restricted to the image of χ, $X_{ts} = X_{st} - R(\chi_t, \chi_s) X$

Proof The main idea here is to just write the expressions locally.

(1) As mentioned in definition [1.0.2], $\chi_s := \frac{\partial \chi^i}{\partial s}\partial_i$. Using this,

$$\begin{aligned}
\chi_{ts} &= \left(\frac{\partial \chi^i}{\partial t}\partial_i\right)_s \\
&= \frac{\partial \chi^i}{\partial s}\partial_i \frac{\partial \chi^k}{\partial t}\partial_k + \frac{\partial \chi^i}{\partial s}\frac{\partial \chi^j}{\partial t}\Gamma^k_{ij}\partial_k && \text{using definition [1.0.2]} \\
&= \frac{\partial}{\partial s}\frac{\partial \chi^k}{\partial t}\partial_k + \frac{\partial \chi^i}{\partial s}\frac{\partial \chi^j}{\partial t}\Gamma^k_{ij}\partial_k && \text{via chain rule} \\
&= \frac{\partial}{\partial t}\frac{\partial \chi^k}{\partial s}\partial_k + \frac{\partial \chi^i}{\partial s}\frac{\partial \chi^j}{\partial t}\Gamma^k_{ji}\partial_k && \text{since } \chi \text{ is smooth, and } \Gamma^k_{ij} = \Gamma^k_{ji} \\
&= \frac{\partial \chi^i}{\partial t}\partial_i \frac{\partial \chi^k}{\partial s}\partial_k + \frac{\partial \chi^j}{\partial t}\frac{\partial \chi^i}{\partial s}\Gamma^k_{ji}\partial_k && \text{via chain rule} \\
&= \left(\frac{\partial \chi^i}{\partial s}\partial_i\right)_t && \text{using definition [1.0.2]} \\
&= \chi_{st}.
\end{aligned}$$

(2) First, using definition [1.0.2],

$$\begin{aligned}
X_{ts} &= \left(\frac{\partial X^k}{\partial t}\partial_k + \frac{\partial \chi^i}{\partial t}X^j\Gamma^k_{ij}\partial_k\right)_s \\
&= \frac{\partial}{\partial s}\left(\frac{\partial X^k}{\partial t} + \frac{\partial \chi^i}{\partial t}X^j\Gamma^k_{ij}\right)\partial_k + \frac{\partial \chi^m}{\partial s}\left(\frac{\partial X^n}{\partial t} + \frac{\partial \chi^i}{\partial t}X^j\Gamma^n_{ij}\right)\Gamma^k_{mn}\partial_k,
\end{aligned}$$

and, using definition [0.1.6],

$$R(\chi_t, \chi_s) X = \frac{\partial \chi^i}{\partial t}\frac{\partial \chi^j}{\partial s}X^k R^l_{ijk}\partial_l \qquad \text{using the fact that it is a tensor,}$$

where

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} + \Gamma^n_{jk}\Gamma^l_{in} - \partial_j\Gamma^l_{ik} - \Gamma^n_{ik}\Gamma^l_{jn}.$$

[4] At least in some neighborhood of zero, so that the later considerations are well-defined.

With that,

$$\begin{aligned}
(X_{st} - X_{ts})^l &= \frac{\partial}{\partial t}\left(\frac{\partial X^l}{\partial s} + \frac{\partial \chi^i}{\partial s} X^j \Gamma^l_{ij}\right) - \frac{\partial}{\partial s}\left(\frac{\partial X^l}{\partial t} - \frac{\partial \chi^i}{\partial t} X^j \Gamma^l_{ij}\right) \\
&\qquad + \frac{\partial \chi^m}{\partial t}\left(\frac{\partial X^n}{\partial s} + \frac{\partial \chi^i}{\partial s} X^j \Gamma^n_{ij}\right)\Gamma^l_{mn} - \frac{\partial \chi^m}{\partial s}\left(\frac{\partial X^n}{\partial t} - \frac{\partial \chi^i}{\partial t} X^j \Gamma^n_{ij}\right)\Gamma^l_{mn} \\
&= \underbrace{\frac{\partial \chi^i}{\partial s}\frac{\partial X^j}{\partial t}\Gamma^l_{ij}}_{(*)} + \frac{\partial \chi^i}{\partial s} X^j \frac{\partial \Gamma^l_{ij}}{\partial t} \underbrace{- \frac{\partial \chi^i}{\partial t}\frac{\partial X^j}{\partial s}\Gamma^l_{ij}}_{(**)} - \frac{\partial \chi^i}{\partial t} X^j \frac{\partial \Gamma^l_{ij}}{\partial s} \\
&\qquad + \underbrace{\frac{\partial \chi^m}{\partial t}\frac{\partial X^n}{\partial s}\Gamma^l_{mn}}_{-(**)} + \frac{\partial \chi^m}{\partial t}\frac{\partial \chi^i}{\partial s} X^j \Gamma^n_{ij}\Gamma^l_{mn} \\
&\qquad \underbrace{- \frac{\partial \chi^m}{\partial s}\frac{\partial X^n}{\partial t}\Gamma^l_{mn}}_{-(*)} - \frac{\partial \chi^m}{\partial s}\frac{\partial \chi^i}{\partial t} X^j \Gamma^n_{ij}\Gamma^l_{mn} && \text{using the smoothness of } \chi \\
&= \frac{\partial \chi^i}{\partial s} X^j \frac{\partial \chi^n}{\partial t}\partial_n \Gamma^l_{ij} - \frac{\partial \chi^i}{\partial t} X^j \frac{\partial \chi^n}{\partial s}\partial_n \Gamma^l_{ij} + \frac{\partial \chi^m}{\partial t}\frac{\partial \chi^i}{\partial s} X^j \Gamma^n_{ij}\Gamma^l_{mn} - \frac{\partial \chi^m}{\partial s}\frac{\partial \chi^i}{\partial t} X^j \Gamma^n_{ij}\Gamma^l_{mn} \\
&= \frac{\partial \chi^i}{\partial t}\frac{\partial \chi^j}{\partial s} X^k \left(\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^n_{jk}\Gamma^l_{in} - \Gamma^n_{ik}\Gamma^l_{jn}\right) \\
&= \left(R\left(\chi_t, \chi_s\right) X\right)^l.
\end{aligned}$$

■

Now, given a (smooth) vector field along a (smooth) curve, it is possible to construct, in a simple way, a variation whose variational field is that vector field.

1.0.4 Lemma For a piecewise-smooth vector field V along γ, a variation of the piecewise-smooth $\gamma : [0, l] \longrightarrow M$ with a variational field of V is

$$\chi(t, s) = \exp_{\gamma(t)}(sV(t)),$$

where $s \in (-\delta, \delta)$, for small enough $\delta > 0$.

Proof At any point $\gamma(t)$, the existence of an $\delta_{\gamma(t)} > 0$ follows from the local existence-and-uniqueness theorems of differential equations; because the image $\gamma([0, l])$ is compact, it follows that

$$\inf_{t \in [0,l]} \delta_{\gamma(t)} > 0.$$

As such, choose $0 < \delta \leq \inf_{t \in [0,l]} \delta_{\gamma(t)}$, so that $\exp_{\gamma(t)}(sV(t))$ is defined for each $t \in [0, l]$. Now, since $\chi(t, 0) = \exp_{\gamma(t)}(0V(t)) = \gamma(t)$, it is needed to check that χ has the variational field V:

$$\begin{aligned}
\chi_s(t, 0) &= \left.\frac{\partial\left(\exp_{\gamma(t)}(sV(t))\right)^i}{\partial s}\right|_{(t,0)} \left.\partial_i\right|_{\chi(t,0)} \\
&= \left(dx^i \circ d\left(\exp_{\gamma(t)}\right)_0 (V(t))\right) \left.\partial_i\right|_{\gamma(t)} && \text{noting that the derivative } \left.\tfrac{d(sV)}{ds}\right|_0 \text{ is a map } T_{\gamma(t)}M \longrightarrow T_0 T_{\gamma(t)}M \\
&= \left(dx^i(V(t))\right) \left.\partial_i\right|_{\gamma(t)} \\
&= V^i(t) \left.\partial_i\right|_{\gamma(t)}
\end{aligned}$$

$$= V(t).$$

Finally, as to make this actually a variation of γ with variational field V, it is just needed to note that each longitudinal curve $\chi(t, s_0)$, for a fixed $s_0 \in (-\delta, \delta)$, is piecewise-smooth by the use of the smooth exponential map with the piecewise-smooth V and γ. ■

The majority of variations considered will be of geodesics, since such variations allow for a more regular approach to determining the intrinsic "size" of a manifold, a key part of the theorems. Such variations will be the concentration of the following sections.

1.1 First Variation

As will be shown in this section, the study of first variation is connected with geodesics. [O'N83, Ch. 10]

1.1.1 Definition For a given variation χ of a (possibly piecewise) smooth curve $\gamma : [a, b] \longrightarrow M$, the *length functional with respect to* χ is

$$L_\chi(s) = \int_a^b |\chi_t(t, s)| \; dt$$

Now, to derive the expression called the first variation:

1.1.2 Proposition Given a variation χ of a piecewise-smooth curve $\gamma : [a, b] \longrightarrow M$ with p breaks $\{t_i\} \subset (a, b)$ such that $|\dot\gamma| = c$ (a non-zero constant)[5] and $\operatorname{sgn}\dot\gamma = \varepsilon$, which is assumed to be constant[6], the *first variation* is given by[7]

$$L'_\chi(0) = \frac{\varepsilon}{c}\left(\left[\langle \chi_s(t, 0), \dot\gamma \rangle\right]_a^b + \sum_{i=1}^{p} \langle \chi_s, \chi_{t^-} - \chi_{t^+} \rangle (t_i, 0) - \int_a^b \left\langle \chi_s(t, 0), \frac{D\dot\gamma}{dt} \right\rangle \; dt \right).$$

Proof The main aim of this proof is to derive an expression for the smooth segments of γ, which will then be summed to yield the desired expression.

Since $|\chi_t(t, 0)| = |\dot\gamma| > 0$, it follows from continuity that there is a neighborhood of zero $(-\delta, \delta) \ni s$ on which $|\chi_t(t, s)| > 0$, and, hence differentiable with respect to s. Thus, on a segment $[t_i, t_{i+1}]$,

$$\begin{aligned}
L'_\chi(0) &= \left.\frac{d}{ds}\right|_0 \int_{t_i}^{t_{i+1}} |\chi_t| \; dt \\
&= \int_{t_i}^{t_{i+1}} \left.\frac{d}{ds}\right|_0 |\chi_t| \; dt && \text{using what was just mentioned with the dominated convergence theorem} \\
&= \frac{\varepsilon}{c} \int_{t_i}^{t_{i+1}} \langle \chi_{ts}, \chi_t \rangle (t, 0) \; dt && |\chi_t(t, 0)| = c. \\
&= \frac{\varepsilon}{c} \int_{t_i}^{t_{i+1}} \langle \chi_{st}, \chi_t \rangle (t, 0) \; dt && \text{using proposition [1.0.3]}
\end{aligned}$$

[5] The definition of a geodesic makes this evident, so the comment is written parenthetically.

[6] Because smooth curves are made so that their tangent vector is no-where zero, this means that the causal character of piecewise-smooth γ cannot change. From this, by the smooth of variations χ, it follows that ε of its longitudinal curves is the same as γ in some neighborhood of γ. In light of later considerations only being of smooth curves, restricting to curves with one causal character is sufficient for our needs.

[7] Here, the metric is written as $\langle \cdot, \cdot \rangle$ to simplify notation.

$$= \frac{\varepsilon}{c}\left(\left[\langle \chi_s, \chi_t\rangle (t,0)\right]_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \langle \chi_s, \chi_{tt}\rangle (t,0)\ dt\right) \qquad \text{using integration by parts on } t.$$

Finally, summing over i and including the endpoints, while noting the remark in definition [1.0.2] about using one-sided derivatives to handle $\chi_t(t_i,0)$ for each segment,

$$\begin{aligned} L'_\chi(0) = \frac{\varepsilon}{c}\Bigg(& \left[\langle \chi_s, \chi_t\rangle (t,0)\right]_a^b + \sum_{i=1}^p \langle \chi_s, \chi_{t^-}\rangle (t_i,0) - \sum_{i=1}^p \langle \chi_s, \chi_{t^+}\rangle (t_i,0) \\ & - \int_a^{t_1} \langle \chi_s, \chi_{tt}\rangle (t,0)\ dt - \sum_{i=2}^{p-1} \int_{t_i}^{t_{i+1}} \langle \chi_s, \chi_{tt}\rangle (t,0)\ dt - \int_{t_p}^b \langle \chi_s, \chi_{tt}\rangle (t,0)\ dt\Bigg) \\ = \frac{\varepsilon}{c}\Bigg(& \left[\langle \chi_s(t,0), \dot\gamma\rangle\right]_a^b + \sum_{i=1}^p \langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0) - \int_a^b \left\langle \chi_s(t,0), \frac{D\dot\gamma}{dt}\right\rangle dt\Bigg) \qquad \chi_t(t,0) = \dot\gamma(t). \end{aligned}$$

■

Geodesics can now be brought back into the discussion with a corollary to the previous proposition.

1.1.3 Corollary A piecewise-smooth curve γ as in the previous proposition is a geodesic iff, for every proper variation, the first variation is zero.

Proof First, it is worth noting that, if the variation χ is proper, then the first variation simplifies to

$$L'_\chi(0) = \frac{\varepsilon}{c}\left(\sum_{i=1}^p \langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0) - \int_a^b \left\langle \chi_s(t,0), \frac{D\dot\gamma}{dt}\right\rangle dt\right).$$

If γ is a geodesic, then, as stated in definition [0.2.2], $\frac{D\dot\gamma}{dt} = 0$, which is written as $\frac{D\dot\gamma}{dt} = 0$ in the notation of the proposition; also, γ is smooth, making all breaks trivial. As such, for any proper variation χ,

$$L'_\chi(0) = \frac{\varepsilon}{c}\left(\sum_{i=1}^p \underbrace{\langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0)}_{0} - \int_a^b \underbrace{\left\langle \chi_s(t,0), \frac{D\dot\gamma}{dt}\right\rangle}_{0} dt\right) = 0.$$

As for the converse, the first objective will be to show that each smooth segment of γ must be a geodesic. Let the smooth segments be on the intervals $\{[t_i, t_{i+1}]\}$ (including the endpoints a, b, not just the breaks). Pick $v \in T_{\gamma(t_i)}M$, and parallel transport it along $\gamma|_{[t_i,t_{i+1}]}$ as to construct a vector field field W along $\gamma|_{[t_i,t_{i+1}]}$; from this, create a proper variational field along $\gamma|_{[t_i,t_{i+1}]}$, written as $V = fW$, where f is a bump function along γ such that $\operatorname{supp} f \subset [t_i, t_{i+1}]$. This vector field V is then considered along the entirety of γ by having it act trivially outside of $[t_i, t_{i+1}]$. For conciseness, the accompanying variation can be written as $\chi(t,s) = \exp_{\gamma(t)}(sV(t))$, as shown in lemma [1.0.4]. Now, by the assumption that the first variation vanishes for any variation of γ,

$$0 = L'_\chi(0) = -\frac{\varepsilon}{c}\int_{t_i}^{t_{i+1}} \left\langle V, \frac{D\dot\gamma}{dt}\right\rangle dt = -\frac{\varepsilon}{c}\int_{\operatorname{supp} f} \left\langle fW, \frac{D\dot\gamma}{dt}\right\rangle dt;$$

by the arbitrariness of W, this implies that $\left\langle fW, \frac{D\dot\gamma}{dt}\Big|_{[t_i,t_{i+1}]}\right\rangle = 0 \iff \frac{D\dot\gamma}{dt}\Big|_{[t_i,t_{i+1}]} = 0$, making it a geodesic by definition. To take care of the breaks, consider another proper variation but so that a break point $t_i \in \operatorname{supp} f \subset [t_{i-1}, t_{i+1}]$. Given what was just shown, $\frac{D\dot\gamma}{dt}\Big|_{[t_{i-1},t_i]} = 0 = \frac{D\dot\gamma}{dt}\Big|_{[t_i,t_{i+1}]}$, making

$$0 = L'_\chi(0) = \frac{\varepsilon}{c}\langle fW, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0).$$

Again by the arbitrariness of W, this is iff $\chi_{t^-} - \chi_{t^+} = 0$, i.e. the break is trivial. Thus, summing up, each segment is a geodesic, and all the breaks are trivial; this makes γ a geodesic. ■

1.2 Second Variation

In order to learn more about the behavior of the length function of geodesics, another derivative needs to be taken. [O'N83, Ch. 10]

1.2.1 Proposition Let $\gamma : [a,b] \longrightarrow M$ be a geodesic such that $|\dot{\gamma}| = c$ and $\varepsilon = \operatorname{sgn} \dot{\gamma}$. Given a variation χ of γ, then the (Synge's formula for the) *second variation* is

$$L''_{\chi}(0) = \frac{\varepsilon}{c}\left(\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle (t,0) - \left\langle R\left(\dot{\gamma}, \chi_s(t,0)\right)\chi_s(t,0), \dot{\gamma}\right\rangle\right)\, dt + \left[\left\langle \chi_{ss}(t,0), \dot{\gamma}\right\rangle\right]_a^b\right),$$

where $\chi_{st}^{\perp}(t,0)$ is the component of $\chi_{st}(t,0)$ perpendicular to $\dot{\gamma}(t)$.

Proof As mentioned in the proof of proposition `[1.1.2]`, it is possible to take the derivative with respect to s under the integral. Using that, if χ is smooth,

$$\begin{aligned}
L''_{\chi}(0) &= \int_a^b \left.\frac{d^2}{ds^2}\right|_0 |\chi_t|\, dt \\
&= \int_a^b \left.\frac{d}{ds}\right|_0 \left(\frac{\varepsilon}{|\chi_t|}\langle \chi_{ts}, \chi_t\rangle\right) dt \\
&= \int_a^b \left(-\frac{1}{|\chi_t(t,0)|^3}\left(\langle \chi_{ts}, \chi_t\rangle (t,0)\right)^2 + \frac{\varepsilon}{|\chi_t(t,0)|}\langle \chi_{tss}, \chi_t\rangle (t,0) + \frac{\varepsilon}{|\chi_t(t,0)|}\langle \chi_{ts}, \chi_{ts}\rangle (t,0)\right) dt && \varepsilon^2 = 1 \\
&= \frac{\varepsilon}{c}\int_a^b \left(-\frac{\varepsilon}{c^2}\left(\langle \chi_{st}, \chi_t\rangle (t,0)\right)^2 + \langle \chi_{sts}, \chi_t\rangle (t,0) + \langle \chi_{st}, \chi_{st}\rangle (t,0)\right) dt && \text{by definition of } c = |\dot{\gamma}| = \chi_t(t,0) \text{, and using proposition [1.0.3] to get } \chi_{ts} = \chi_{st} \\
&= \frac{\varepsilon}{c}\int_a^b \left(-\frac{\varepsilon}{c^2}\left(\langle \chi_{st}, \chi_t\rangle (t,0)\right)^2 + \langle \chi_{sst} - R(\chi_t, \chi_s)\chi_s, \chi_t\rangle (t,0) + \langle \chi_{st}, \chi_{st}\rangle (t,0)\right) dt && \text{by proposition [1.0.3] } \chi_{sts} = \chi_{sst} - R(\chi_t, \chi_s)\chi_s.
\end{aligned}$$

If $\chi_s(t,0)$ is only piecewise-smooth, then the above can be can be done along each of the smooth segments, and it sums to the same integral expression.

Now, it is just needed to simplify the expression to obtain the one in the statement. Motivated to get a term involving $\chi_{st}^{\perp}$, note that χ_{st} can be broken up into components as such

$$\chi_{st}(t,0) = \chi_{st}^{\perp}(t,0) + \underbrace{\left\langle \chi_{st}(t,0), \frac{\varepsilon\dot{\gamma}(t)}{c}\right\rangle \frac{\dot{\gamma}(t)}{c}}_{\text{tangent to } \dot{\gamma}(t)},$$

where ε it taken into consideration to ensure that the component lies in the correct direction, and $\chi_{st}^{\perp} = \left(\chi_s^{\perp}\right)_t = (\chi_{st})^{\perp}$ is unambiguous via lemma `[0.2.6]`. Recalling that $\dot{\gamma}(t) = \chi_t(t,0)$, this makes

$$\langle \chi_{st}, \chi_{st}\rangle (t,0) = \left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle (t,0) + \frac{\varepsilon}{c^2}\left(\langle \chi_{st}, \chi_t\rangle (t,0)\right)^2,$$

using the fact that $\frac{\dot{\gamma}}{c}$ is a unit vector. Also, since $\gamma(t) = \chi(t,0)$ is a geodesic,

$$\left.\frac{\partial}{\partial t}\right|_{(t,0)} \langle \chi_t, \chi_{ss}\rangle = \underbrace{\langle \chi_{tt}, \chi_{ss}\rangle (t,0)}_{0} + \langle \chi_t, \chi_{sst}\rangle (t,0).$$

With these results,

$$\begin{aligned}
L''_\chi(0) &= \frac{\varepsilon}{c}\int_a^b \left(\langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) + \left.\frac{\partial}{\partial t}\right|_{(t,0)} \langle \chi_{ss}, \chi_t\rangle - \langle R(\chi_t,\chi_s)\chi_s,\chi_t\rangle(t,0)\right) dt \\
&= \frac{\varepsilon}{c}\left(\int_a^b \left(\langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) - \langle R(\chi_t,\chi_s)\chi_s,\chi_t\rangle(t,0)\right) dt + \left[\langle \chi_{ss}(t,0), \dot\gamma\rangle\right]_a^b \right) && \chi_t(t,0) = \dot\gamma(t) \\
&= \frac{\varepsilon}{c}\left(\int_a^b \left(\langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) - \langle R(\dot\gamma,\chi_s(t,0))\chi_s(t,0),\dot\gamma\rangle\right) dt + \left[\langle \chi_{ss}(t,0), \dot\gamma\rangle\right]_a^b \right) && \text{using the fact that } R \text{ is a tensor, allowing for the evaluation of its components at } (t,0).
\end{aligned}$$

■

Here, it is useful to note that, along geodesic a γ, only the component of a proper variational field perpendicular to γ contributes to the second variation.

1.2.2 Corollary If γ is non-null, and χ is proper, as in the proposition, the second variation does not depend on the tangential component of χ_s:

$$L''_\chi(0) = \frac{\varepsilon}{c}\int_a^b \left(\langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s^\perp(t,0)\right)\chi_s^\perp(t,0), \dot\gamma\right\rangle\right) dt.$$

Proof Given that $|\dot\gamma| > 0$ (i.e. it is non-null), $\chi_s(t,0)$ can be written as $\chi_s^\perp(t,0) + \chi_s^\top(t,0)$. Also, since χ is proper, it follows that $\chi_{ss}(a,0) = 0 = \chi_{ss}(b,0)$. Additionally, note that $\left(R\left(\chi_s^\top,\chi_t\right)\chi_t\right)(t,0) = 0$:

$$\begin{aligned}
\left(R\left(\chi_s^\top,\chi_t\right)\chi_t\right)(t,0) &= R\left(\chi_s^\top(t,0),\chi_t(t,0)\right)\chi_t(t,0) && \text{using the fact that it is a tensor} \\
&= f(t)\, R(\dot\gamma(t),\dot\gamma(t))\,\dot\gamma(t) && \chi_s^\top(t,0) = f(t)\,\dot\gamma(t) \text{ for some } f \in \mathfrak{F}(\gamma) \text{ because it is tangent to } \gamma(t) = \chi(t,0), \text{ and then linearity}
\end{aligned}$$

but

$$\begin{aligned}
& R(\dot\gamma(t),\dot\gamma(t))\,\dot\gamma(t) = -R(\dot\gamma(t),\dot\gamma(t))\,\dot\gamma(t) && \text{since it is antisymmetric in its first two entries} \\
\iff\; & R(\dot\gamma(t),\dot\gamma(t))\,\dot\gamma(t) = 0,
\end{aligned}$$

so

$$\left(R\left(\chi_s^\top,\chi_t\right)\chi_t\right)(t,0) = 0.$$

Substituting these facts into the second variation for the proper variation χ

$$\begin{aligned}
L''_\chi(0) &= \frac{\varepsilon}{c}\int_a^b \Bigg(\underbrace{\left\langle \left(\left(\chi_s^\perp + \chi_s^\top\right)_t\right)^\perp, \left(\left(\chi_s^\perp + \chi_s^\top\right)_t\right)^\perp \right\rangle(t,0)}_{\substack{\left\langle \left(\chi_s^\perp\right)_t, \left(\chi_s^\perp\right)_t\right\rangle(t,0) = \langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) \\ \text{using lemma [0.2.6]}}} - \left\langle R\left(\chi_t, \chi_s^\perp + \chi_s^\top\right)\chi_s^\perp + \chi_s^\top, \chi_t\right\rangle(t,0)\Bigg) dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\langle \chi_{st}^\perp, \chi_{st}^\perp\rangle(t,0) - \left\langle R\left(\chi_t,\chi_s^\perp\right)\chi_s^\perp,\chi_t\right\rangle(t,0) \underbrace{- \left\langle R\left(\chi_t,\chi_s^\perp\right)\chi_s^\top,\chi_t\right\rangle(t,0)}_{\substack{-\left\langle R\left(\chi_s^\top,\chi_t\right)\chi_t, \chi_s^\perp\right\rangle(t,0) = 0 \\ \text{left-hand side by rearranging entries of the tensor, and} \\ \text{right-hand side by what was just shown}}}\right.
\end{aligned}$$

$$\underbrace{- \left\langle R\left(\chi_t, \chi_s^{\top}\right)\chi_s^{\perp}, \chi_t\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\top}\right)\chi_s^{\top}, \chi_t\right\rangle(t,0)}_{\substack{0 \\ \text{similarly, as with the other term}}}\Big)\ dt \qquad \text{using linearity}$$

$$= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\perp}\right)\chi_s^{\perp}, \chi_t\right\rangle(t,0)\right)\ dt \qquad [*]$$

$$= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s^{\perp}(t,0)\right)\chi_s^{\perp}(t,0), \dot\gamma\right\rangle\right)\ dt \qquad \text{using the fact } R \text{ is a tensor,}$$

which was to be shown. ■

1.2.3 Remark It is worth noting that, in using the reasoning in getting to [*] in the previous proof, it is possible to have obtained

$$\begin{aligned}
L''_\chi(0) &= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\perp} + \chi_s^{\top}\right)\chi_s, \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\perp}\right)\chi_s, \chi_t\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\top}\right)\chi_s, \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s^{\perp}\right)\chi_s, \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s^{\perp}(t,0)\right)\chi_s(t,0), \dot\gamma\right\rangle\right)\ dt
\end{aligned}$$

or

$$\begin{aligned}
L''_\chi(0) &= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s\right)\left(\chi_s^{\perp} + \chi_s^{\top}\right), \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s\right)\chi_s^{\perp}, \chi_t\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s\right)\chi_s^{\top}, \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\chi_t, \chi_s\right)\chi_s^{\perp}, \chi_t\right\rangle(t,0)\right)\ dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \chi_{st}^{\perp}, \chi_{st}^{\perp}\right\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s(t,0)\right)\chi_s^{\perp}(t,0), \dot\gamma\right\rangle\right)\ dt,
\end{aligned}$$

analogous forms of $L''_\chi(0)$ related by the symmetries of the $(0,4)$-tensor R. ◀

As such, only variational fields that a perpendicular to a geodesic provide for useful consideration.

1.3 Index Form

When handling of vector fields along smooth curves, it is handy to define the space $\Omega_{p,q}$ for $p, q \in M$. [O'N83, Ch. 10][Car92, §11.2]

1.3.1 Definition For $M \ni p, q$, let $\Omega_{p,q}$ be the set of piecewise-smooth curves joining p and q.

For $\gamma \in \Omega_{p,q}$, let $\widetilde{\mathfrak{X}}_0(\gamma)$ be the set of piecewise-smooth vector fields along γ that vanish at p and q. As such, associated with $\Omega_{p,q}$, is the module of $\widetilde{\mathfrak{X}}_0$ over the ring of piecewise-smooth functions along γ; this space is denoted $V_\gamma \Omega_{p,q}$.

On this module, there is a useful symmetric bilinear form.

1.3.2 Definition The *index form* I_γ for a non-null geodesic $\gamma \in \Omega_{p,q}$ is the map

$$I_\gamma : V_\gamma\Omega_{p,q} \times V_\gamma\Omega_{p,q} \longrightarrow \mathbb{R}$$

such that

$$I_\gamma(X,Y) = \frac{\varepsilon}{c}\int_a^b \left(\left\langle \frac{DX^\perp}{dt}, \frac{DY^\perp}{dt}\right\rangle(t) - \langle R(\dot\gamma, X)Y, \dot\gamma\rangle(t)\right)\, dt,$$

where ε, and c, are as in the expression shown in proposition [1.2.1], by which this expression is inspired. It is symmetric by the symmetries of the metric and the Riemann curvature tensor; similarly, it is bilinear by the linearity of the metric, covariant derivative, and the Riemann curvature tensor.

Using what was discussed in remark [1.2.3], or corollary [1.2.2], it can also be written, respectively, as

$$\begin{aligned} I_\gamma(X,Y) &= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \frac{DX^\perp}{dt}, \frac{DY^\perp}{dt}\right\rangle(t) - \left\langle R\left(\dot\gamma, X^\perp\right)Y, \dot\gamma\right\rangle(t)\right)\, dt \\ &= \frac{\varepsilon}{c}\int_a^b \left(\left\langle \frac{DX^\perp}{dt}, \frac{DY^\perp}{dt}\right\rangle(t) - \left\langle R(\dot\gamma, X)Y^\perp, \dot\gamma\right\rangle(t)\right), \end{aligned}$$

or

$$I_\gamma(X,Y) = \frac{\varepsilon}{c}\int_a^b \left(\left\langle \frac{DX^\perp}{dt}, \frac{DY^\perp}{dt}\right\rangle(t) - \left\langle R\left(\dot\gamma, X^\perp\right)Y^\perp, \dot\gamma\right\rangle(t)\right)\, dt.$$

Also, note that it immediately follows from this definition that

$$I_\gamma(X,Y) = I_\gamma\left(X^\perp, Y^\perp\right),$$

and, if X is the variational field of a (necessarily proper) variation χ (e.g. as constructed in lemma [1.0.4]), then

$$I_\gamma(X,X) = L''_\chi(0),$$

via corollary [1.2.2], for example.

1.4 Jacobi Fields

With the general discussion of the variation of arc length in the previous sections, focus will be shifted to more specific variations – namely, variations whose longitudinal curves are geodesics. Such a field manifests naturally through the use of the exponential map. [Car92, §5.2]

Let $v \in T_pM$ be such that $\exp_p v$ is defined. Then, consider the map, for small enough $\delta > 0$ (guaranteed by smoothness),

$$\chi(t,s) = \exp_p(tf(s)), \ t \in [0,1], \ s \in (-\delta,\delta),$$

where $f : (-\delta,\delta) \longrightarrow T_pM$ is smooth such that $f(0) = v$ and $f'(0) = w \in T_vT_pM$. Now, using the fact that, for fixed s_0, $\chi(t,s_0)$ is a geodesic, it follows from the definition of a geodesic that

$$\begin{aligned} \chi_{tts} &= 0 && \text{[1.4.1]} \\ \chi_{tst} - R(\chi_t,\chi_s)\chi_t &= && \text{using proposition [1.0.3]} \\ \chi_{stt} - R(\chi_t,\chi_s)\chi_t &= && \text{using proposition [1.0.3].} \end{aligned}$$

With that, notice that $\chi_s(t,0) = J(t)$ is a vector field along the geodesic $\chi(t,0) = \exp_p(tv) = \gamma(t)$, so that the equation can be written as

$$\frac{D^2J}{dt^2} - R(\dot\gamma, J)\dot\gamma = 0, \qquad \text{[1.4.2]}$$

the *Jacobi equation*.

1.4.3 Definition A vector field along a geodesic that satisfies the Jacobi equation is said to be a *Jacobi field* along that geodesic.

Considering the Jacobi equation more closely, comes the next lemma.

1.4.4 Lemma For a geodesic $\gamma : [0,l] \longrightarrow M^n$, defining J, and $\frac{DJ}{dt}$, at a point in the domain of γ, determines J uniquely along γ.

Proof Let $|\dot{\gamma}|^2 = 1$, and let the orthonormal vectors $\{e_i\}_2^n$ be so that $\operatorname{span}\{e_i\} = \dot{\gamma}^{\perp}(0)$ with $\{E_i\}_2^n$ being the vector fields along γ gotten by parallel transport of $\{e_i\}$; also, let $E_1 = \dot{\gamma}$. In this construction, $J = J^i E_i$, where $\{J^i\}$ are smooth functions along γ, so that

$$\frac{D^2 J}{dt^2} = \frac{D}{dt}\left(\dot{J}^i E_i + J^i \underbrace{\frac{DE_i}{dt}}_{0}\right) = \ddot{J}^i E_i + J^i \underbrace{\frac{DE_i}{dt}}_{0} = \ddot{J}^i E_i,$$

and

$$R\left(\dot{\gamma}, J^i E_i\right)\dot{\gamma} = J^i R\left(E_1, E_i\right) E_1 = J^i R^j_{1i1} E_j,$$

making the Jacobi equation the n equations

$$\ddot{J}^i = J^j R^i_{1j1},$$

where $\{R^i_{1j1}\}$ are the coefficients of the Riemann curvature tensor. Written as such, it is easy[8] to see that setting the values of $\{J^i\}$, and $\{\dot{J}^i\}$, at some point along γ, determines J uniquely, i.e. defining J, and $\frac{DJ}{dt}$, at a point, determines J uniquely. ■

As to show the significance of the initial construction which yielded the Jacobi equation, it will be shown that any Jacobi field which vanishes at a point can be constructed using the exponential map.

1.4.5 Proposition Let J be a Jacobi field along a geodesic $\gamma : [0,b] \longrightarrow M$ with $\gamma(0) = p$, and $\dot{\gamma}(0) = v$, such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w \in T_v T_p M$. Also, let K be the Jacobi field gotten from the variation $\chi(t,s) = \exp_p(tf(s))$, where, for sufficiently small $\delta > 0$, $f : (-\delta, \delta) \longrightarrow T_p M$ is smooth with $f(0) = v$ and $f'(0) = w$. Then, $J = K$.

Proof By the previous lemma, it is just needed to find the initial conditions of K to show that they agree with J, making the Jacobi fields the same. Doing so,

$$K(0) = \chi_s(0,0) = \left(d\exp_p\right)_{(tf(s))|_{(0,0)}} \left.\frac{d(tf(s))}{ds}\right|_{(0,0)} = \left(d\exp_p\right)_0 0 = 0 = J(0),$$

and

$$\begin{aligned}
\frac{DK}{dt}(0) &= \chi_{st}(0,0) \\
&= \frac{D}{dt}\left(t\left(d\exp_p\right)_{tf(s)} f'(s)\right)(0,0) && \text{using linearity of the differential} \\
&= \left(\left(d\exp_p\right)_{tf(s)} f'(s) + t\frac{D}{dt}\left(d\exp_p\right)_{tf(s)} f'(s)\right)(0,0) \\
&= \left(d\exp_p\right)_0 w \\
&= w = \frac{DJ}{dt}(0).
\end{aligned}$$

■

[8] This is to say that it follows by the existence, and uniqueness, conditions of second-order ODEs.

1.4.6 Remark Of course, there is the consideration of reparametrizations of Jacobi fields, i.e. reparametrizations of the base geodesic. As shown in lemma `[0.2.5]`, only linear reparametrizations of geodesics are still geodesics. Hence, linearly reparametrizing the base geodesic to $\gamma(t) = \chi(t, 0)$ of the variation χ, which corresponds (in the light of the previous proposition) to the Jacobi field $J(t) = \chi_s(t, 0)$ along γ, maintains the fact J satisfies the Jacobi equation via `[1.4.1]`: $\chi_{tts} = 0$. It follows that the interval on which a Jacobi field is defined can be altered linearly without disturbing the Jacobi equation. ◀

Looking at these fields a bit more closely, brings up the following. [O'N83, Ch. 8]

1.4.7 Lemma For a vector field J along a geodesic γ:

(1) If J is tangent to γ, then J is a Jacobi field iff $\frac{D^2 J}{dt^2} = 0$ iff $J = f\dot{\gamma}$ where f is linear.

(2) If J is a Jacobi field, then the following are equivalent:

  (a) $J \perp \gamma$

  (b) there exist distinct p, q such that $J(p) \perp \gamma(p)$, and $J(q) \perp \gamma(q)$

  (c) there exists p such that $J(p) \perp \gamma(p)$ and $\frac{DJ}{dt}(p) \perp \gamma(p)$.

(3) If γ is non-null, then J is a Jacobi field iff both the tangent, $J^\top$, and perpendicular, $J^\perp$, components of J, with respect to γ, are Jacobi fields.

Proof (1) Since J is tangent to γ, it follows that it must be of the form $J = f\dot{\gamma}$ where f is a smooth function along γ. It is a Jacobi field iff

$$\frac{D^2 J}{dt^2} - R(\dot{\gamma}, J)\dot{\gamma} = 0$$

$$\iff \frac{D^2 J}{dt^2} = 0 \qquad \text{using linearity, and the fact that } \gamma \text{ is a geodesic, } R(\dot{\gamma}, J) = fR(\dot{\gamma}, \dot{\gamma}) = 0$$

$$\frac{D}{dt}\left(f'\dot{\gamma} + f \underbrace{\frac{D\dot{\gamma}}{dt}}_{\substack{0 \\ \text{since } \gamma \text{ is a geodesic}}}\right) =$$

$$f''\dot{\gamma} + f' \underbrace{\frac{D\dot{\gamma}}{dt}}_{0} =$$

$$\iff f = ct + d \qquad \text{where } c, d \text{ are constants.}$$

(2) First off, the statement (a) trivially implies (b). Also, it follows from lemma `[0.2.6]` that (a) implies (c).

Now, to show that either (b), or (c), imply the (a), as to complete the proof. Consider

$$\frac{d^2}{dt^2}\langle J, \dot{\gamma}\rangle = \left\langle \frac{D^2 J}{dt^2}, \dot{\gamma}\right\rangle + 2\left\langle \frac{DJ}{dt}, \frac{D\dot{\gamma}}{dt}\right\rangle + \left\langle J, \frac{D^2\dot{\gamma}}{dt^2}\right\rangle$$

$$= R(\dot{\gamma}, J, \dot{\gamma}, \dot{\gamma}) \qquad \text{using the Jacobi equation, and } \frac{D\dot{\gamma}}{dt} = 0$$

$$= -R(\dot{\gamma}, J, \dot{\gamma}, \dot{\gamma}) \qquad \text{by antisymmetric property of the Riemann curvature tensor}$$

$$\iff \frac{d^2}{dt^2}\langle J, \dot{\gamma}\rangle = 0$$

$$\iff \langle J, \dot\gamma \rangle = ct + d \qquad \text{where } c, d \text{ are constants.} \qquad \texttt{[*]}$$

$$\implies \frac{d}{dt} \langle J, \dot\gamma \rangle = c$$

$$\left\langle \frac{DJ}{dt}, \dot\gamma \right\rangle = \qquad \text{using compatibility of the metric, and the fact that } \gamma \text{ is a geodesic, } \tfrac{D\dot\gamma}{dt} = 0. \qquad \texttt{[**]}$$

As such, if there is a point p such that $\frac{DJ}{dt}(p) \perp \gamma(p)$, and $J(p) \perp \gamma(p)$, then, respectively, it follows that $c = 0$ by `[**]`, making $d = 0$ by `[*]`, i.e. $J \perp \gamma$. Similarly, if there are two distinct points p, q such that $J(p) \perp \gamma(p)$, and $J(q) \perp \gamma(q)$, then it must be the case, by `[*]`, that $c = 0$, making $d = 0$, i.e. $J \perp \gamma$.

(3) As to ensure that J can be decomposed into $J^\perp + J^\top$ with respect to γ, consideration is restricted to geodesics such that $|\dot\gamma| > 0$, i.e. non-null. With that noted, it is just needed to show that each component satisfies the Jacobi equation iff J does.

First off, the forward statement follows from linearity:

$$\left.\begin{aligned} \frac{D^2 J^\perp}{dt^2} &= R\left(\dot\gamma, J^\perp\right)\dot\gamma \\ \frac{D^2 J^\top}{dt^2} &= R\left(\dot\gamma, J^\top\right)\dot\gamma \end{aligned}\right\} \implies \frac{D^2 J^\perp}{dt^2} + \frac{D^2 J^\top}{dt^2} = R\left(\dot\gamma, J^\perp\right)\dot\gamma + R\left(\dot\gamma, J^\top\right)\dot\gamma$$

$$\frac{D^2 J}{dt^2} = R(\dot\gamma, J)\dot\gamma \qquad \text{by linearity, } J = J^\perp + J^\top.$$

The converse relies on a bit more. Working from the assumptions,

$$\begin{aligned} \frac{D^2 J}{dt^2} &= R(\dot\gamma, J)\dot\gamma \\ \frac{D^2 J^\perp}{dt^2} + \frac{D^2 J^\top}{dt^2} &= R\left(\dot\gamma, J^\perp\right)\dot\gamma + \underbrace{R\left(\dot\gamma, J^\top\right)\dot\gamma}_{0} \\ \frac{D^2 J^\perp}{dt^2} + f''\dot\gamma &= R\left(\dot\gamma, J^\perp\right)\dot\gamma \qquad \text{using the proof of part (1).} \end{aligned}$$

Now, note that $R\left(\dot\gamma, J^\perp, \dot\gamma, \dot\gamma\right) = R\left(\dot\gamma, \dot\gamma, \dot\gamma, J^\perp\right) = 0$, making $R\left(\dot\gamma, J^\perp\right)\dot\gamma \perp \gamma$; from this, it follows that

$$\begin{cases} \dfrac{D^2 J^\perp}{dt^2} = R\left(\dot\gamma, J^\perp\right)\dot\gamma \\ \dfrac{D^2 J^\top}{dt^2} = f''\dot\gamma = 0 = R\left(\dot\gamma, J^\top\right)\dot\gamma \end{cases},$$

namely that $J^\perp, J^\top$ are Jacobi fields. ■

Also, useful results for later sections, are the following lemmas. [O'N83, Ch. 10]

1.4.8 Lemma For Jacobi fields X, Y along a geodesic γ, the following is a constant

$$\left\langle \frac{DX}{dt}, Y \right\rangle - \left\langle X, \frac{DY}{dt} \right\rangle.$$

Proof Showing that the derivative of the expression is zero, is sufficient:

$$\frac{d}{dt}\left(\left\langle \frac{DX}{dt}, Y \right\rangle - \left\langle X, \frac{DY}{dt} \right\rangle\right) = \left\langle \frac{D^2X}{dt^2}, Y \right\rangle + \left\langle \frac{DX}{dt}, \frac{DY}{dt} \right\rangle - \left\langle \frac{DX}{dt}, \frac{DY}{dt} \right\rangle - \left\langle X, \frac{D^2Y}{dt^2} \right\rangle \qquad \text{via compatibility with the metric}$$

$$= \langle R(\dot\gamma, X)\dot\gamma, Y\rangle - \langle R(\dot\gamma, Y)\dot\gamma, X\rangle \qquad \text{by the Jacobi equation [1.4.2]}$$
$$= 0 \qquad \text{by the symmetry of the Riemann curvature tensor } R.$$

■

1.4.9 Lemma Let $\{Y_i\}$ be Jacobi fields along the geodesic γ such that, for any i, j,

$$\left\langle \frac{DY_i}{dt}, Y_j \right\rangle = \left\langle Y_i, \frac{DY_j}{dt} \right\rangle.$$

Now, let $X = \sum_i f_i Y_i$ for $f_i \in \mathfrak{F}(\gamma)$; then,

$$\left\langle \frac{DX}{dt}, \frac{DX}{dt} \right\rangle = \langle R(\dot\gamma, X)X, \dot\gamma\rangle + \left\langle \sum_i f_i' Y_i, \sum_i f_i' Y_i \right\rangle + \frac{d}{dt}\left\langle X, \sum_i f_i \frac{DY_i}{dt} \right\rangle.$$

Proof Working through,

$$\begin{aligned}
\frac{d}{dt}\left\langle X, \sum_i f_i \frac{DY_i}{dt}\right\rangle &= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \left\langle X, \sum_i \frac{D}{dt}\left(f_i \frac{DY_i}{dt}\right)\right\rangle \\
&= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \left\langle X, \sum_i \left(f_i' \frac{DY_i}{dt} + f_i \frac{D^2 Y_i}{dt^2}\right)\right\rangle \\
&= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \left\langle X, \sum_i f_i' \frac{DY_i}{dt}\right\rangle + \left\langle X, \sum_i f_i R(\dot\gamma, Y_i)\dot\gamma\right\rangle && \text{by the Jacobi equation [1.4.2]} \\
&= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \left\langle X, \sum_i f_i' \frac{DY_i}{dt}\right\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle && \text{by linearity of } R \\
&= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \sum_{i,j}\left\langle f_j Y_j, f_i' \frac{DY_i}{dt}\right\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle && \text{by definition of } X \\
&= \left\langle \frac{DX}{dt}, \sum_i f_i \frac{DY_i}{dt}\right\rangle + \sum_{i,j}\left\langle f_j \frac{DY_j}{dt}, f_i' Y_i\right\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle && \text{assumed property of } \{Y_i\} \\
&= \left\langle \sum_j \left(f_j' Y_j + f_j \frac{DY_j}{dt}\right), \sum_i f_i \frac{DY_i}{dt}\right\rangle + \sum_{i,j}\left\langle f_j \frac{DY_j}{dt}, f_i' Y_i\right\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle \\
&= \sum_{i,j}\left\langle f_j \frac{DY_j}{dt}, f_i \frac{DY_i}{dt}\right\rangle + 2\sum_{i,j}\left\langle f_j \frac{DY_j}{dt}, f_i' Y_i\right\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle \\
&= \left\langle \frac{DX}{dt}, \frac{DX}{dt}\right\rangle - \sum_{i,j}\langle f_j' Y_i, f_i' Y_i\rangle + \langle X, R(\dot\gamma, X)\dot\gamma\rangle
\end{aligned}$$

$$\iff \left\langle \frac{DX}{dt}, \frac{DX}{dt} \right\rangle = \langle R(\dot\gamma, X)X, \dot\gamma\rangle + \left\langle \sum_i f_i' Y_i, \sum_i f_i' Y_i \right\rangle + \frac{d}{dt}\left\langle X, \sum_i f_i \frac{DY_i}{dt} \right\rangle \qquad \text{by properties of } R$$

■

1.5 Conjugate Points

The next definition speaks to the fact that, if Jacobi field along a geodesic vanishes at two points, neighboring radial geodesics from the one point become infinitely close at the other. [O'N83, Ch. 10][Car92, §5.3]

1.5.1 Definition Let J be a nontrivial Jacobi field along a geodesic γ which vanishes at distinct $\gamma(a)$, and $\gamma(b)$. Then, $\gamma(b)$ is *conjugate* to $\gamma(a)$ along γ. It follows from remark [1.4.6] that, after a reversing the parametrization of J, $\gamma(a)$ is also conjugate to $\gamma(b)$ along γ.

The notion of a conjugate point has a few equivalent definitions.

1.5.2 Proposition Let $\gamma : [0,l] \longrightarrow M$ be a geodesic with $\gamma(0) = p$ and $\gamma(l) = q$. Then, the following are equivalent:

(1) q is conjugate to p along γ

(2) there is a (nontrivial) variation χ of γ through geodesics starting at p such that $\chi_s(l,0) = 0$

(3) the differential of the exponential map $d\left(\exp_p\right)_{l\dot{\gamma}(0)}$ is singular.

Proof (2) $\Longrightarrow$ (1) Since χ is a nontrivial variation through geodesics, it follows that $\chi_s(t,0) \not\equiv 0$, but that $\chi_{tts}(t,0) \equiv 0$, so that it satisfies the Jacobi equation via [1.4.1]. Letting $J(t) = \chi_s(t,0)$, it follows that J is a Jacobi field which vanishes at 0 and l: respectively, $\chi_s(0,0) = 0$ by construction of χ, and $\chi_s(l,0) = 0$ by assumption on χ. Thus, by definition q is conjugate to p along γ.

(3) $\Longrightarrow$ (2) By definition of being singular, it follows that there is a $v \in T_{l\dot{\gamma}(0)}T_pM \simeq T_pM$ such that $d\left(\exp_p\right)_{l\dot{\gamma}(0)} v = 0$. Consider the variation $\chi(t,s) = \exp_p\left(t\left(\dot{\gamma}(0) + sv\right)\right)$; using the linearity of $d\left(\exp_p\right)_{t\dot{\gamma}(0)}$, this variation is such that

$$\chi_s(l,0) = \left.\frac{d}{ds}\right|_{(l,0)} \exp_p\left(t\left(\dot{\gamma}(0) + sv\right)\right) = d\left(\exp_p\right)_{t\dot{\gamma}(0)} lv = ld\left(\exp_p\right)_{t\dot{\gamma}(0)} v = 0,$$

as was to be shown.

(1) $\Longrightarrow$ (3) By definition, it follows there is a Jacobi field J along γ such that $J(0) = 0 = J(l)$; assume that $\frac{DJ}{dt}(0) = w \in T_{\dot{\gamma}(0)}T_pM \simeq T_pM$. Now, by proposition [1.4.5], it follows that the variation corresponding to J is $\chi(t,s) = \exp_p(tf(s))$, where $f(0) = \dot{\gamma}(0)$ and $f'(0) = w$. With this, and the assumption on J, it follows that

$$0 = J(l) = \chi_s(l,0) = d\left(\exp_p\right)_{lf(0)} lf'(0) = d\left(\exp_p\right)_{l\dot{\gamma}(0)} lw,$$

making the differential singular by definition. ■

It turns out that the consideration of these points yield results pertaining to local minima, and maxima, of the length functional; this will be discussed in a later section.

1.6 Endmanifold Variations

Attention now will be directed toward considering the case of variations in which one endpoint varies over a submanifold. First order of business is the accompanying definitions.

1.6.1 Definition Let $M \subset \overline{M}$ be a submanifold of $\overline{M}$. Analogous to definition [1.3.1], let $\Omega_{M,q}$ be the set of piecewise-smooth curves from a point in M to $q \in \overline{M} \setminus M$.

Let $\gamma : [0,l] \longrightarrow \overline{M}$ in $\Omega_{M,q}$ be such that $\gamma(0) = p$ and $\gamma(l) = q$. A variation χ of γ *across* M is a variation $\chi : [0,l] \times (-\delta,\delta) \longrightarrow \overline{M}$ of γ such that, for each s,

$$\chi(0,s) \in M \iff \chi_s(0,s) \in T_{\chi(0,s)}M.$$

More specifically, an *endmanifold variation* is defined to be a variation χ of γ across M such that

$$\chi_s(l,0)=0;$$

furthermore, this endmanifold variation is called *proper* if, for each s,

$$\chi(l,s)=q \iff \chi_s(l,s)=0.$$

Note that it is implicit that, for a given variation, $\gamma(l)$ is set as the point where the longitudinal curves "come together".

Also, analogous to what was presented in definition `[1.3.1]`: let $\widetilde{\mathfrak{X}}_1(\gamma)\ni V$ be the set of piecewise-smooth vector fields along γ such that $V(0)\in T_pM$ and $V(l)=0$. With that, let $V_\gamma\Omega_{M,q}$ be the module of $\widetilde{\mathfrak{X}}_1(\gamma)$ over the ring of piecewise-smooth functions along γ.

Following from this, is the matter as to how such a variation can be constructed – especially since lemma `[1.0.4]` is not (in general[9]) applicable here: it does not guarantee that $\chi(0,s)\in M$.

1.6.2 Lemma With γ as in the previous definition, let $V\in V_\gamma\Omega_{M,q}$. Then, from V, a variation χ of γ across M can be constructed, whose variational field is V.

Proof Let the set of breaks of V be $\{t_i\}_1^b\subset(0,l)$, where $t_0=0$ and $t_{b+1}=l$. The variation will be constructed in b parts: along each of the smooth segments of V.

Let $U\subset\overline{M}$ be a neighborhood of γ. Extend $V|_{[t_0,t_1]}$ to $X_0\in\mathfrak{X}(U)$ such that, for each $x\in U\cap M$, $X_0|_x\in T_xM$. Let $\varphi_0(\gamma(t),s):[t_0,t_1]\times(-\delta_0,\delta_0)\longrightarrow\overline{M}$ be the local flow[10] of X_0 restricted to $\gamma|_{[t_0,t_1]}$.

Now, for each of the remaining ($1\le i\le b$) smooth segments, extend $V|_{[t_i,t_{i+1}]}$ to a vector field $X_i\in\mathfrak{X}(U)$ so that, for each $s\in(-\delta_{i-1},\delta_{i-1})$, $X_i|_{\varphi_{i-1}(t_i,s)}=X_{i-1}|_{\varphi_{i-1}(t_i,s)}$, as to make sure[11] the extensions agree on the "seam" $\varphi_{i-1}(t_i,s)$. With that, let $\varphi_i(t,s):[t_i,t_{i+1}]\times(-\delta_i,\delta_i)\longrightarrow\overline{M}$ be a restriction of the local flow[10] of X_i to $\gamma|_{[t_i,t_{i+1}]}$.

With that, let $\delta=\min\{\delta_i\}>0$, since there are finitely many breaks, and let

$$\chi:[t_0,t_{b+1}]\times(-\delta,\delta)\longrightarrow\overline{M}$$

be defined as

$$\chi(t,s)=\begin{cases}\varphi_0(t,s) & t\in[t_0,t_1]\\ & \vdots\\ \varphi_i(t,s) & t\in[t_i,t_{i+1}]\\ & \vdots\\ \varphi_b(t,s) & t\in[t_b,t_{b+1}]\end{cases}.$$

By properties of local flows, it follows that χ is such that

$$\varphi(\gamma(t),0)=\gamma(t)\quad\text{and}\quad\varphi_s(\gamma(t),0)=V(t).$$

The construction of X_0 from $V|_{[t_0,t_1]}$ makes it so that its local flow is such that $\varphi_0(0,s)\in M$, since $X_0|_x\in T_xM$ for each $x\in U\cap M$. Also, if $V(l)=0$, then $\chi(l,s)=q$, since $X_b|_q=V|_q=0$. Thus, $\chi(t,s)$ is a (possibly endmanifold) variation of γ across M. ■

[9]Since, for lemma `[1.0.4]` to be sufficient, it would require that, from $\chi(0,0)\in M$, a geodesic of $\overline{M}$ is a geodesic of M.

[10]More rigorously, for each $\gamma(t)$, there is a $\delta_{\gamma(t)}>0$, for which the local flow is defined; however, since $[t_i,t_{i+1}]$ is compact, the infimum of those $\delta_{\gamma(t)}$ is obtained: it is $\delta_i>0$.

[11]This is possible, since V is piecewise-smooth.

As was done in prior sections, variations of geodesics will now be considered. In particular, the next result shows the spatial relationship of geodesics in $\Omega_{M,q}$ to that of M. [O'N83, Ch. 10]

1.6.3 Proposition Let $\gamma : [0,l] \longrightarrow \overline{M}$ be a piecewise-smooth curve from a point of M to q which is non-null, $|\dot{\gamma}| > 0$. Then, $L'_\chi(0) = 0$ for all endmanifold variations χ of γ across M iff γ is a geodesic perpendicular to M.

Proof The assertion that γ is such that $|\dot{\gamma}| > 0$ is ensure that the formula for the first variation is defined: the corresponding constant $c \neq 0$.

Assume that $L'_\chi(0) = 0$ for any endmanifold variation χ of γ across M. Since, by definition, each proper variation of γ is an endmanifold variation, it follows from corollary `[1.1.3]` that γ is a geodesic. As in the proof of corollary `[1.1.3]`, it follows that

$$\frac{\varepsilon}{c}\left(\sum_{i=1}^{p}\underbrace{\langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i, 0)}_{\substack{0 \\ \text{by smoothness of } \gamma}} - \int_0^l \underbrace{\left\langle \chi_s(t,0), \frac{D\dot{\gamma}}{dt}\right\rangle}_{\substack{0 \\ \text{since } \frac{D\dot{\gamma}}{dt} = 0}} dt\right) = 0.$$

Hence, for any endmanifold variation χ,

$$\begin{aligned} 0 = L'_\chi(0) &= \frac{\varepsilon}{c}\left(\left[\langle \chi_s(t,0), \dot{\gamma}\rangle\right]_0^l + \sum_{i=1}^{p}\langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0) - \int_0^l \left\langle \chi_s(t,0), \frac{D\dot{\gamma}}{dt}\right\rangle dt\right) \\ &= \frac{\varepsilon}{c}\langle \chi_s(0,0), \dot{\gamma}(0)\rangle \qquad \begin{array}{l}\text{also using the fact that} \\ \chi_s(l,0) = 0 \text{ by definition of} \\ \text{an endmanifold variation}\end{array} \\ \iff \chi_s(0,0) &\perp \dot{\gamma}(0) \qquad \text{by the arbitrariness of } \chi\,. \end{aligned}$$

Thus, since $\chi_s(0,0) \in T_{\gamma(0)}M$ for each such χ, by definition of an endmanifold variation, it must be that the geodesic γ is normal to M.

The converse simply follows from facts mentioned already: by γ being normal to M, i.e.

$$\chi_s(0,0) \perp \dot{\gamma}(0) \iff \langle \chi_s(0,0), \dot{\gamma}(0)\rangle = 0,$$

by χ being an endmanifold variation, i.e.

$$\chi_s(l,0) = 0,$$

and, by γ by a geodesic, i.e.

$$\sum_{i=1}^{p}\langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0) - \int_0^l \left\langle \chi_s(t,0), \frac{D\dot{\gamma}}{dt}\right\rangle dt = 0,$$

making, for any endmanifold variation χ of γ across M,

$$L'_\chi(0) = \frac{\varepsilon}{c}\left(\left[\langle \chi_s(t,0), \dot{\gamma}\rangle\right]_0^l + \sum_{i=1}^{p}\langle \chi_s, \chi_{t^-} - \chi_{t^+}\rangle (t_i,0) - \int_0^l \left\langle \chi_s(t,0), \frac{D\dot{\gamma}}{dt}\right\rangle dt\right) = 0,$$

since $\chi_s(l,0) = 0$ by definition of χ being an endmanifold variation. ∎

With that result establishing the extremal curves as geodesics, now attention will be set on obtaining the more-elucidating second variation of these endmanifold variations.

1.6.4 Lemma Let γ be non-null geodesic perpendicular to M at $\gamma(0)$. Then, for a proper endmanifold variation χ of γ across $M \subset \overline{M}$, its second variation is

$$L''_\chi(0) = \frac{\varepsilon}{c}\int_0^l \left(\left\langle \chi_{st}^\perp, \chi_{st}^\perp\right\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s^\perp(t,0)\right)\chi_s^\perp(t,0), \dot\gamma\right\rangle\right)\,dt - \frac{\varepsilon}{c}\left\langle \Pi(\chi_s,\chi_s)(0,0), \dot\gamma(0)\right\rangle.$$

Proof The first term on the right-hand side simply follows from that same argument used in corollary `[1.2.2]`, which relied on properties of the tensor R and not on the fact that χ was proper there.

Given that χ is proper so that

$$\chi_{ss}(l,0) = 0 \implies \langle \chi_{ss}(l,0), \dot\gamma(l)\rangle = 0,$$

the second variation can be written as shown in proposition `[1.2.1]` as

$$L''_\chi(0) = \frac{\varepsilon}{c}\int_0^l \left(\left\langle \chi_{st}^\perp, \chi_{st}^\perp\right\rangle(t,0) - \left\langle R\left(\dot\gamma, \chi_s^\perp(t,0)\right)\chi_s^\perp(t,0), \dot\gamma\right\rangle\right)\,dt - \frac{\varepsilon}{c}\left\langle \chi_{ss}(0,0), \dot\gamma(0)\right\rangle,$$

including the boundary term. Looking further at the boundary term: since $\dot\gamma(0)$ is normal to M, it follows that

$$\langle \chi_{ss}(0,0), \dot\gamma(0)\rangle = \langle \operatorname{nor}\chi_{ss}(0,0), \dot\gamma(0)\rangle,$$

where $\operatorname{nor}\chi_{ss}(0,0)$ denotes the component of $\chi_{ss}(0,0)$ normal to M; since, recalling definition `[1.0.2]`, $\chi_{ss} = \frac{D\chi_s}{ds}$ is simply a covariant derivative in $\overline{M}$ of χ_s along $\chi(t_0,s)$, for fixed t_0, it is the case that

$$\operatorname{nor}\chi_{ss}(0,0) = \left(\operatorname{nor}\overline\nabla_{\chi_s}\chi_s\right)\Big|_{(0,0)} = \Pi(\chi_s,\chi_s)(0,0),$$

by definition `[0.1.14]`. Substitution of this into the boundary term yields the expression that was to be shown. ■

Coming with this, is a generalization of the index form defined in definition `[1.3.2]` to the space $V_\gamma\Omega_{M,q}$.

1.6.5 Definition For a non-null geodesic $\gamma \in \Omega_{M,q}$, the generalization of the index form from definition `[1.3.2]` is

$$I_\gamma : V_\gamma\Omega_{M,q} \times V_\gamma\Omega_{M,q} \longrightarrow \mathbb{R}$$

such that

$$I_\gamma(X,Y) = \frac{\varepsilon}{c}\int_0^l \left(\left\langle X_t^\perp, Y_t^\perp\right\rangle(t) - \langle R(\dot\gamma, X)Y, \dot\gamma\rangle(t)\right)\,dt - \frac{\varepsilon}{c}\langle \Pi(X,Y)(0), \dot\gamma(0)\rangle.$$

Note that it follows from the discussion in definition `[1.3.2]`, and lemma `[0.1.15]` about the shape tensor Π, that this generalization is still symmetric and bilinear.

Just as in definition `[1.3.2]`, it is possible to write this strictly in terms of the perpendicular components. As used in definition `[1.3.2]`, the first term follows from corollary `[1.2.2]`, which is used in lemma `[1.6.4]`, and remark `[1.2.3]`. As for the second term: since $X(0), Y(0) \in T_{\gamma(0)}M$, and $\gamma \perp M$ at $\gamma(0)$, it follows that

$$X(0) = X^\perp(0) \quad \text{and} \quad Y(0) = Y^\perp(0) \implies \Pi(X,Y)(0) = \Pi\left(X^\perp, Y^\perp\right)(0),$$

using the fact that Π is a tensor. With that, analogous to definition `[1.3.2]`, it can be written as

$$\begin{aligned} I_\gamma(X,Y) &= \frac{\varepsilon}{c}\int_0^l \left(\left\langle X_t^\perp, Y_t^\perp\right\rangle(t) - \left\langle R\left(\dot\gamma, X^\perp\right)Y, \dot\gamma\right\rangle(t)\right)\,dt - \frac{\varepsilon}{c}\left\langle \Pi\left(X^\perp, Y\right)(0), \dot\gamma(0)\right\rangle \\ &= \frac{\varepsilon}{c}\int_0^l \left(\left\langle X_t^\perp, Y_t^\perp\right\rangle(t) - \left\langle R(\dot\gamma, X)Y^\perp, \dot\gamma\right\rangle(t)\right)\,dt - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle \end{aligned}$$

$$= \frac{\varepsilon}{c} \int_0^l \left(\left\langle X_t^\perp, Y_t^\perp \right\rangle (t) - \left\langle R\left(\dot\gamma, X^\perp\right) Y^\perp, \dot\gamma \right\rangle (t) \right) \, dt - \frac{\varepsilon}{c} \left\langle \Pi\left(X^\perp, Y^\perp\right)(0), \dot\gamma(0) \right\rangle,$$

with further expressions gotten by using $X(0) = X^\perp(0)$ and $Y(0) = Y^\perp$. From this, it immediately follows that, for any $X, Y \in V_\gamma \Omega_{M,q}$,

$$I_\gamma(X, Y) = I_\gamma\left(X^\perp, Y^\perp\right),$$

and, if X is the variational field of a proper endmanifold variation χ of γ across M (e.g. as constructed in lemma `[1.6.2]`), then

$$I_\gamma(X, X) = L''_\chi(0)$$

by lemma `[1.6.4]`.

1.7 Focal Points

As is the theme of the previous sections, attention will now be focused on variations whose longitudinal curves are geodesics; in particular, sights will be directed to Jacobi fields along geodesics normal to the endmanifold, whose variational longitudinal geodesics are also normal to the endmanifold. The aim is to generalize the notion of conjugate points; in that notion, Jacobi fields perturb geodesics in the same fashion: by keeping the longitudinal curves normal to a point. [Car92, §10.4][O'N83, Ch. 10]

First: to establish the conditions on such normal-preserving Jacobi fields.

1.7.1 Definition A *normal variation*, across some submanifold $M \subset \overline{M}$, is a variation $\chi : [0, l] \times (-\delta, \delta) \longrightarrow \overline{M}$ of a geodesic γ, which is normal[13] to M at $\gamma(0)$, across M such that, for each fixed s_0,

(1) $\chi(t, s_0)$ is a geodesic

(2) these geodesics are such that $\chi_t(0, s_0) \in \left(T_{\chi(0,s_0)} M\right)^\perp$,

it follows immediately that $J(t) = \chi_s(t, 0)$ is a Jacobi field by the Jacobi equation `[1.4.1]`.

Moreover, a *normal endmanifold variation* is defined to be a normal variation, which is also an endmanifold variation, and it is called *proper* if it is a proper endmanifold variation.

1.7.2 Proposition Let the terms be as defined in definition `[1.7.1]`. The Jacobi field J is such that

(i) $J(0) \in T_{\gamma(0)} M$

(ii) $\dfrac{DJ}{dt}(0) - \widetilde{\Pi}(J, \dot\gamma)(0) \in \left(T_{\gamma(0)} M\right)^\perp$,

where $\widetilde{\Pi}$ is the associated shape tensor from definition `[0.1.16]`.

Conversely, if J is a Jacobi field along γ, which is normal to M at $\gamma(0)$, that satisfies (i) and (ii), then there is a normal variation χ of γ across M (i.e. χ satisfies (1), and (2), from definition `[1.7.1]`), whose variational field is J.

Proof Showing that a J as defined in definition `[1.7.1]` has the properties (i) and (ii):

(i) By virtue of χ being a variation across M, it follows that $J(0) = \chi_s(0, 0) \in T_{\chi(0,0)} M$.

[13] This is to say that $\dot\gamma(0) \in \left(T_{\gamma(0)} M\right)^\perp$; it is sometimes denoted simply as $\gamma \perp M$ if there is no ambiguity.

(ii) It is sufficient to show that $\tan\left(\frac{DJ}{dt}(0) - \widetilde{\Pi}(J,\dot{\gamma})(0)\right) = 0$:

$$\begin{aligned}
\tan\left(\frac{DJ}{dt}(0) - \widetilde{\Pi}(J,\dot{\gamma})(0)\right) &= \tan\left(\frac{D\chi_s}{dt}(0,0) - \widetilde{\Pi}(J,\dot{\gamma})(0)\right) && \text{by definition of } J \\
&= \tan\left(\frac{D\chi_s}{dt}(0,0)\right) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{using the linearity of } \tan(\cdot) \text{ and the definition of } \widetilde{\Pi}, \widetilde{\Pi} \text{ maps into } \mathfrak{X}(M) \\
&= \tan\left(\frac{D\chi_t}{ds}(0,0)\right) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{by proposition [1.0.3]} \\
&= \tan\left(\frac{D\chi_t(0,s)}{ds}(0)\right) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{since the derivative is only in the parameter } s \\
&= \tan\left(\left(\overline{\nabla}_{\chi_s(0,s)}\chi_t(0,s)\right)(0)\right) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{by definition of covarient derivative} \\
&= \tan\left(\overline{\nabla}_{\chi_s(0,s)}\chi_t(0,s)\right)(0) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{by linearity of } \tan(\cdot) \\
&= \widetilde{\Pi}(\chi_s(0,s), \chi_t(0,s))(0) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{by definition of } \widetilde{\Pi} \\
&= \widetilde{\Pi}(J(0), \dot{\gamma}(0)) - \widetilde{\Pi}(J,\dot{\gamma})(0) && \text{using the fact that } \widetilde{\Pi} \text{ is a tensor, and } \chi_s(0,0) = J(0), \chi_t(0,0) = \dot{\gamma}(0) \text{, by construction of } \chi \\
&= 0 && \text{using the fact that } \widetilde{\Pi} \text{ is a tensor.}
\end{aligned}$$

As for the converse, it is just needed to construct a variation χ from J which satisfies (1) and (2). Pick a smooth curve $\alpha : (-\delta, \delta) \longrightarrow M$ such that $\alpha(0) = \gamma(0)$, where $\gamma \perp M$, and $\alpha'(0) = J(0)$. Along α, a smooth vector field Z normal to M will be constructed such that $Z(0) = \dot{\gamma}(0)$ and $\frac{DZ}{ds}(0) = \frac{DJ}{dt}(0)$, as to allow for an easy construction of an accompanying variation χ later. Let X be the parallel transport of $\dot{\gamma}(0)$ along α, and let Y be the parallel transport of nor $\frac{DJ}{dt}(0)$ along α. Then, let

$$Z(s) = X(s) + sY(s)$$

so that

$$Z(0) = X(0) = \dot{\gamma}(0)$$

and

$$\begin{aligned}
\frac{DZ}{ds}(0) &= \frac{DX}{ds}(0) + Y(0) \\
&= \tan\left(\overline{\nabla}_{\alpha'}X\right)(0) + \operatorname{nor}\frac{DJ}{dt}(0) && \text{by definition of the covariant derivative and construction of } X, Y \\
&= \widetilde{\Pi}(\alpha', X)(0) + \operatorname{nor}\frac{DJ}{dt}(0) && \text{by definition of } \widetilde{\Pi} \\
&= \widetilde{\Pi}(J, \dot{\gamma})(0) + \operatorname{nor}\frac{DJ}{dt}(0) && \text{by linearity of } \widetilde{\Pi} \text{ and contruction of } \alpha, X \\
&= \frac{DJ}{dt}(0) && \text{by assumption (2) on } J.
\end{aligned}$$

With that, let $\chi(t,s) = \exp_{\alpha(s)}(tZ(s))$. By this construction, it follows that: for each s,

$$\chi(0,s) = \exp_{\alpha(s)} 0 = \alpha(s) \in M,$$

showing that it varies across M, and, using the uniqueness of geodesics (from lemma [0.2.3], which is inherent to the definition of exp),

$$\chi(t,0) = \exp_{\alpha(0)}(tZ(0)) = \exp_{\gamma(0)}(t\dot{\gamma}(0)) = \gamma(t),$$

for each t, shows that γ is the base curve of χ. From this, it follows that χ is an variation of the normal geodesic γ across M. Now, it is just needed to show that χ satisfies (1), and (2), from definition [1.7.1], as to show that it is normal:

(1) With use of the exp map, this is already guaranteed: the longitudinal curves are geodesics.

(2) For each of these geodesics, it is the case that their starting tangent vector is

$$\chi_t(0,s) = \left(d\exp_{\alpha(s)}\right)_0 Z(s) = Z(s) \in \left(T_{\alpha(s)}M\right)^{\perp},$$

by construction of Z.

Finally, it is to be shown that χ has J as its variational field. To do so, it suffices, by lemma [1.4.4], to show that $\chi_s(0,0) = J(0)$ and $\chi_{st}(0,0) = \frac{DJ}{dt}(0)$. First off, by construction,

$$\chi_s(0,0) = \alpha'(0) = J(0).$$

Next,

$$\begin{aligned}
\chi_{st}(0,0) &= \chi_{ts}(0,0) && \text{by proposition [1.0.3]}\\
&= \frac{DZ}{ds}(0) && \text{as shown in } \chi \text{ satisfying property (2), } \chi_t(0,s) = Z(s)\\
&= \frac{DJ}{dt}(0) && \text{from the construction of } Z.
\end{aligned}$$

Therefore, J is the variational field of the normal variation χ. ∎

Influenced by this proposition, the next definition gives name to these Jacobi fields.

1.7.3 Definition In the context of proposition [1.7.2], if a Jacobi field J satisfies (i), and (ii), for some submanifold $M \subset \overline{M}$, to which γ is normal, then J is called an *M-Jacobi field* along γ.

With that, there is enough corresponding terminology as to generalize the case of conjugate points (cf. definition [1.5.1]).

1.7.4 Definition Let $\gamma : [0,l] \longrightarrow \overline{M}$ be a geodesic of $\overline{M} \supset M$ normal to M at $\gamma(0)$. A point $\gamma(t_0) \in \overline{M}$, $0 < t_0 \leq l$, is called a *focal point* of M along γ if there is a nonzero M-Jacobi field along γ which vanishes at t_0, $J(t_0) = 0$.

Note that all conjugate points are just focal points with M being a point.

In particular, this definition draws the parallel between variations with conjugate points, and endmanifold variations with focal points. Coming with this, is the analogous form of proposition [1.5.2].

1.7.5 Proposition Let $\gamma : [0,l] \longrightarrow \overline{M}$ be a geodesic normal to $M \subset \overline{M}$ with $\gamma(0) = p$ and $\gamma(l) = q$. Then, the following are equivalent:

(1) q is a focal point of M along γ

(2) there exists a (nontrivial) normal endmanifold variation χ of γ across M

(3) the differential of the normal exponential map $d\left(\exp^{\perp}\right)_{(p,l\dot{\gamma}(0))}$ is singular, where the *normal exponential map* is defined as of restriction of the exp map of $\overline{M}$:

$$\exp_p^{\perp} : (T_pM)^{\perp} \longrightarrow \overline{M}$$

$$v \longmapsto \exp_p v,$$

or, as viewed as acting on the normal tangent bundle $T(M)^{\perp}$ of M,

$$\begin{aligned} \exp^{\perp} : T(M)^{\perp} &\longrightarrow \overline{M} \\ (p, v) &\longmapsto \exp_p v. \end{aligned}$$

Proof (1) $\Longleftrightarrow$ (2) This immediately follows from proposition [1.7.2] via the corresponding definitions: definition [1.7.4], definition [1.7.3], and definition [1.7.1].

(2) $\Longrightarrow$ (3) By the uniqueness of geodesics (inherent in the definition of the exp map), it follows that the normal endmanifold variation χ (i.e. a variation whose longitudinal curves are geodesics normal to M) can be written as

$$\chi(t, s) = \exp^{\perp}_{\chi(0,s)}(t\chi_t(0, s)) = \exp^{\perp}(\chi(0, s), t\chi_t(0, s)),$$

after recalling that $\chi_t(0, s) \in \left(T_{\chi(0,s)}M\right)^{\perp}$ by definition of χ. Also, by definition

$$\chi_s(l, 0) = 0,$$

which makes

$$\begin{aligned} 0 &= \chi_s(l, 0) \\ &= \left(d\exp^{\perp}\right)_{(\chi(0,0), l\chi_s(0,0))}(\chi_s(0, 0), l\chi_s(0, 0)) \\ &= \left(d\exp^{\perp}\right)_{(p, l\dot\gamma(0))}(\chi_s(0, 0), l\chi_{ts}(0, 0)), \end{aligned}$$

showing that the differential at $(p, l\dot\gamma(0))$ is singular by definition.

(3) $\Longrightarrow$ (2) Let $(u, v) \in T_pM \times T_{l\dot\gamma(0)}T_pM$ be such that

$$\left(d\exp^{\perp}\right)_{(p, l\dot\gamma(0))}(u, v) = 0.$$

Pick a (smooth) curve σ in M, and a vector field $X \in \mathfrak{X}(\sigma) \cap \mathfrak{X}(M)^{\perp}$, such that

$$(\sigma(0), lX(0)) = (p, l\dot\gamma(0))$$

and

$$(\sigma'(0), lX_s(0)) = (u, v).$$

Now, define the variation χ as

$$\chi(t, s) = \exp^{\perp}(\sigma(s), tX(s));$$

it follows that χ is a variation of γ:

$$\chi(t, 0) = \exp^{\perp}(\sigma(0), tX(0)) = \exp_p(t\dot\gamma(0)) = \gamma(t).$$

It is also a normal endmanifold variation: it is an endmanifold variation since

$$\chi(0, s) = \exp^{\perp}(\sigma(s), 0) = \exp_{\sigma(s)} 0 = \sigma(s) \in M,$$

and

$$\begin{aligned} \chi_s(l, 0) &= \left(d\exp^{\perp}\right)_{(\sigma(0), lX(0))}(\sigma'(0), lX_s(0)) \\ &= \left(d\exp^{\perp}\right)_{(p, l\dot\gamma(0))}(u, v) && \text{by construction of } \sigma \text{ and } X \\ &= 0, \end{aligned}$$

and it is normal since it uses the exp map, and

$$\begin{aligned}
\chi_t(0,s) &= \left(d\exp^{\perp}\right)_{(\sigma(s),0)}(0,X(s)) \\
&= \left(d\exp^{\perp}_{\sigma(s)}\right)_0 X(s) && \text{since there is no } t\text{-dependence on the point } \sigma(s)\text{, from which the exp map comes} \\
&= X(s) \\
&\in \left(T_{\sigma(s)}M\right)^{\perp} && \text{by construction of } X \\
&\equiv \left(T_{\chi(0,s)}M\right)^{\perp} && \text{by construction of } \chi.
\end{aligned}$$

(Quickly note that, since $\dot{\gamma}(0) \neq 0$ as per usual consideration, it must be that X is nontrivial by its smoothness.) Thus, χ is a (nontrivial) normal endmanifold variation of γ across M, for which was sought. ■

To accompany this, this a nice result as to the focal points of submanifolds flat space: they lie along the normal geodesic γ at the reciprocals of the eigenvalues of $-\widetilde{\mathrm{II}}(\cdot,\dot{\gamma}(0))$, which is simply information from M at $\gamma(0)$.

1.7.6 Proposition Let $\overline{M}$ be flat, and let γ be a geodesic in normal to $M \subset \overline{M}$ at $\gamma(0)$. Then, all the focal points of M along γ, if they exist in $\overline{M}$, are exactly $\left\{\gamma\left(\frac{1}{\lambda_i}\right)\right\}$, where $\{\lambda_i\}$ are eigenvalues of $-\widetilde{\mathrm{II}}(\cdot,\dot{\gamma}(0))$.

Proof Let $v \in T_{\gamma(0)}M$ be such that

$$-\widetilde{\mathrm{II}}(v,\dot{\gamma}(0)) = \lambda v,$$

for some $\lambda > 0$. Now, construct $V \in \mathfrak{X}(\gamma)$ as the parallel transport of v along γ, and consider the vector field

$$J(t) = (1-\lambda t)\,V(t).$$

Since the ambient $\overline{M}$ is flat, it follows that J is a Jacobi field: R is identically zero, so the Jacobi equation `[1.4.2]` reduces to

$$\begin{aligned}
\frac{D^2 J}{dt^2} &= \left(\frac{d^2}{dt^2}(1-\lambda t)\right)V + 2\left(\frac{d}{dt}(1-\lambda t)\right)\underbrace{\frac{DV}{dt}}_{\substack{0\\ \text{by parallel}\\ \text{transport}}} + (1-\lambda t)\underbrace{\frac{D^2V}{dt^2}}_{\substack{0\\ \text{similarly}}} \\
&= 0.
\end{aligned}$$

Also, note that $J\left(\frac{1}{\lambda}\right) = 0$; as such, it is only needed to show that J is an M-Jacobi field along γ, as to show that $\gamma\left(\frac{1}{\lambda}\right)$ is a focal point of M along γ. Checking the for the properties (i), and (ii), given in proposition `[1.7.2]` as per definition of M-Jacobi fields:

(i) $J(0) = V(0) = v \in T_{\gamma(0)}M$

(ii) First, note

$$\frac{DJ}{dt}(0) = \left(\left.\frac{d}{dt}\right|_0 (1-\lambda t)\right)V(0) + (1-\lambda t)\underbrace{\frac{DV}{dt}}_{0}(0) = -\lambda V(0) = \lambda v = \widetilde{\mathrm{II}}(v,\dot{\gamma}(0)).$$

This makes

$$\frac{DJ}{dt}(0) - \widetilde{\mathrm{II}}(J,\dot{\gamma})(0) = 0 \in \left(T_{\gamma(0)}M\right)^{\perp},$$

since $J(0) = v$.

Therefore, J is an M-Jacobi field along γ, which vanishes at $\frac{1}{\lambda}$, making $\gamma\left(\frac{1}{\lambda}\right)$ a focal point by definition.

Now, for the other inclusion, in order to finish the proof. Let $\gamma(r)$ be a focal point of M along γ; it is needed to show that $\frac{1}{r}$ is an eigenvalue of $\widetilde{\Pi}(\cdot, \dot{\gamma}(0))$. By definition, it follows that there is an M-Jacobi field $J \in \mathfrak{X}(\gamma)$ which vanishes at r, $J(r) = 0$. Since, as mentioned in the first part of the proof, the Jacobi equation is simply $\frac{D^2 J}{dt^2} = 0$; from this, it is possible to conclude that J has the form

$$J(t) = X(t) + tY(t),$$

where $X, Y \in \mathfrak{X}(\gamma)$. In particular, this can be done using the uniqueness guaranteed by lemma [1.4.4], as follows:

$$J(0) = X(0),$$

so let X be the parallel transport of $J(0) \in T_{\gamma(0)}M$ along γ, and, with this,

$$\frac{DJ}{dt}(0) = \underbrace{\frac{DX}{dt}}_{\substack{0\\ \text{by being a parallel}\\ \text{transport}}}(0) + Y(0) + t\frac{DY}{dt}(0),$$

so let Y be the parallel transport of $\frac{DJ}{dt}(0) \in T_{\dot{\gamma}(0)}T_{\gamma(0)}\overline{M}$ along γ to make $\frac{DY}{dt} = 0$ and

$$\frac{DJ}{dt}(0) = Y(0);$$

together, it is still the case that the Jacobi equation is satisfied, again using $\frac{DX}{dt} = 0 = \frac{DY}{dt}$,

$$\frac{D^2 J}{dt^2} = \frac{D}{dt}\left(\frac{DX}{dt} + Y + t\frac{DY}{dt}\right) = 0.$$

As such, it follows

$$\begin{aligned}
0 = J(r) = X(r) + rY(r) &\iff Y(r) = -\frac{1}{r}X(r) \\
&\iff \frac{DJ}{dt}(0) = Y(0) = -\frac{1}{r}X(0) = -\frac{1}{r}J(0) && \text{by } X, Y \text{ being parallel transports of } J(0), \tfrac{DJ}{dt}(t)\text{, respectively} \quad [*] \\
&\implies \frac{DJ}{dt}(0) \in T_{\gamma(0)}M && \text{since } J(0) \in T_{\gamma(0)}M \\
&\iff \frac{DJ}{dt}(0) = \widetilde{\Pi}(J, \dot{\gamma})(0) && \text{by property (ii) of } M\text{-Jacobi fields, since } \widetilde{\Pi}(J,\dot{\gamma})(0) \in T_{\gamma(0)}M \text{ by definition} \\
&\iff -\widetilde{\Pi}(J, \dot{\gamma})(0) = -\frac{DJ}{dt}(0) = \frac{1}{r}J(0) && \text{with } [*],
\end{aligned}$$

which, therefore, makes $\frac{1}{r}$ an eigenvalue of $\widetilde{\Pi}(\cdot, \dot{\gamma}(0))$ by definition. ■

With that, the next section is regarding the goal of this chapter: the local minima, and maxima, of the length functional.

1.8 Local Minima and Maxima

Using the concepts defined in the previous sections of this chapter, this section will discuss the end-goal of this chapter: local extrema of the length functional. The primary tool in this discussion will be the index form. [O'N83, Ch. 10]

This first result determines the nullspace of the index form.

1.8.1 Lemma Let $\gamma \in \Omega_{M,q}$, $\gamma : [0,l] \longrightarrow \overline{M}$, be a non-null geodesic in $\overline{M} \supset M$ that is normal to M at $\gamma(0)$. The nullspace of the index form I_γ is exactly the space of M-Jacobi fields along γ, which vanish at l, $J(l) = 0$.

Proof This fact simply comes out of the expression for the index form in definition `[1.6.5]` and use of proposition `[1.7.2]`.

Let $X \in V_\gamma \Omega_{M,q}$ have b breaks t_i, and rewriting the expression of the index form,

$$
\begin{aligned}
I_\gamma(X,Y) &= \frac{\varepsilon}{c}\int_0^l \left(\left\langle X_t^\perp, Y_t^\perp\right\rangle(t) - \left\langle R(\dot\gamma, X)Y^\perp, \dot\gamma\right\rangle(t)\right)\, dt - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{chosen from definition [1.6.5] with hindsight}\\
&= \frac{\varepsilon}{c}\int_0^l \left(\left(\frac{d}{dt}\left\langle X_t^\perp, Y^\perp\right\rangle\right)(t) - \left\langle X_{tt}^\perp, Y^\perp\right\rangle(t) - \left\langle R(\dot\gamma, X)Y^\perp, \dot\gamma\right\rangle(t)\right)\, dt\\
&\qquad - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{chain rule with lemma [0.2.6]}\\
&= \frac{\varepsilon}{c}\int_0^l \left(-\left\langle X_{tt}^\perp, Y^\perp\right\rangle(t) - \left\langle R(\dot\gamma, X)Y^\perp, \dot\gamma\right\rangle(t)\right)\, dt + \frac{\varepsilon}{c}\underbrace{\left[\left\langle X_t^\perp, Y^\perp\right\rangle\right]_0^l}_{\substack{-\left\langle X_t^\perp, Y^\perp\right\rangle(0)\\ \text{since } Y \in V_\gamma\Omega_{M,q},\\ Y(l)=0 \implies Y^\perp(l)=0}}\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{taking into account the breaks } \{t_i\}\text{, where } X_{t^-}, X_{t^+} \text{ are one-sided derivatives}\\
&= \frac{\varepsilon}{c}\int_0^l \left(-\left\langle X_{tt}^\perp, Y^\perp\right\rangle(t) + \left\langle R(\dot\gamma, X)\dot\gamma, Y^\perp\right\rangle(t)\right)\, dt - \frac{\varepsilon}{c}\left\langle X_t^\perp, Y^\perp\right\rangle(0)\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{using symmetries of the } (0,4)\text{-tensor } R\\
&= \frac{\varepsilon}{c}\int_0^l \left(-\left\langle X_{tt}, Y^\perp\right\rangle(t) + \left\langle R(\dot\gamma, X)\dot\gamma, Y^\perp\right\rangle(t)\right)\, dt - \frac{\varepsilon}{c}\left\langle X_t, Y^\perp\right\rangle(0)\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{using linearity, since } \left\langle X_{tt}^\top, Y^\perp\right\rangle = 0 \text{ and } \left\langle X_t^\top, Y^\perp\right\rangle = 0\\
&= \frac{\varepsilon}{c}\int_0^l \left\langle R(\dot\gamma, X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\, dt - \frac{\varepsilon}{c}\left\langle X_t, Y^\perp\right\rangle(0)\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) - \frac{\varepsilon}{c}\left\langle \Pi\left(X, Y^\perp\right)(0), \dot\gamma(0)\right\rangle && \text{by linearity}\\
&= \frac{\varepsilon}{c}\int_0^l \left\langle R(\dot\gamma, X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\, dt - \frac{\varepsilon}{c}\left\langle X_t, Y^\perp\right\rangle(0)\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) + \frac{\varepsilon}{c}\left\langle \widetilde{\Pi}(X, \dot\gamma)(0), Y^\perp(0)\right\rangle && \text{from the definition of } \widetilde{\Pi}\\
&= \frac{\varepsilon}{c}\int_0^l \left\langle R(\dot\gamma, X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\, dt + \frac{\varepsilon}{c}\left\langle \widetilde{\Pi}(X, \dot\gamma) - X_t, Y^\perp\right\rangle(0) && \text{[1.8.2]}\\
&\qquad + \frac{\varepsilon}{c}\sum_{i=1}^b \left\langle X_{t^-}^\perp - X_{t^+}^\perp, Y^\perp\right\rangle(t_i) && \text{by linearity.}
\end{aligned}
$$

From this, and the definition of an M-Jacobi field J, which is smooth (i.e. no breaks) and entails that

$$J_{tt} - R(\dot{\gamma}, J)\dot{\gamma} = 0 \qquad \text{the Jacobi equation [1.4.2]} \qquad [*]$$

and

$$J_t(0) - \widetilde{\Pi}(J, \dot{\gamma})(0) \in \left(T_{\gamma(0)}M\right)^{\perp} \qquad \text{property (ii) of } M\text{-Jacobi fields,} \qquad [**]$$

it immediately follows that, for any $Y \in V_\gamma \Omega_{M,q}$,

$$I_\gamma(J, Y) = \frac{\varepsilon}{c} \int_0^l \underbrace{\left\langle R(\dot{\gamma}, J)\dot{\gamma} - J_{tt}, Y^{\perp}\right\rangle(t)}_{\substack{0 \\ \text{by } [*]}} \; dt + \frac{\varepsilon}{c} \underbrace{\left\langle \widetilde{\Pi}(J, \dot{\gamma}) - J_t, Y^{\perp}\right\rangle(0)}_{\substack{0 \\ \text{by } [**] \text{ since} \\ Y^{\perp}(0) = Y(0) \in T_{\gamma(0)}M \text{ by definition} \\ \text{of being in } V_\gamma \Omega_{M,q}}} = 0,$$

showing that each M-Jacobi field J is in the nullspace of I_γ by definition.

The reverse inclusion will be shown in the usual fashion: by contradiction, arguing via support. Let $X, Y \in V_\gamma \Omega_{M,q}$, and let X be in the nullspace of I_γ with c breaks at t_i and smooth segment $[t_i, t_{i+1}]$ for $0 \leq i \leq c$. It will be shown that X is an M-Jacobi field. By definition of being in the nullspace, this means that, for any such Y,

$$I_\gamma(X, Y) = 0.$$

As such, say that $Y = fW$ for some $f \in \mathfrak{F}(\gamma)$, and $W \in V_\gamma \Omega_{M,q}$, which is only zero at l, $W(l) = 0$, for simplicity. Let $f(0) = 0$ and $\operatorname{supp} f \subset [t_i, t_{i+1}]$, and it follows that

$$0 = I_\gamma(X, fW) = \frac{\varepsilon}{c} \int_0^l \left\langle R(\dot{\gamma}, X)\dot{\gamma} - X_{tt}, fW^{\perp}\right\rangle(t) \; dt.$$

Now, assume for the sake of contradiction, that

$$h(t) = \left(R(\dot{\gamma}, X)\dot{\gamma} - X_{tt}\right)(t) \neq 0,$$

for some $t \in [t_i, t_{i+1}]$. By continuity, it follows that there is some open neighborhood $U \subset [t_i, t_{i+1}]$ such that $U \subset \operatorname{supp} h$. To abuse this, pick f such that $\operatorname{supp} f \subset U$, and it follows that, by being an integral of a non-zero function on $\operatorname{supp} f$,

$$I_\gamma(X, fW) = \frac{\varepsilon}{c} \int_0^l \left\langle R(\dot{\gamma}, X)\dot{\gamma} - X_{tt}, fW^{\perp}\right\rangle(t) \; dt = \frac{\varepsilon}{c} \int_{\operatorname{supp} f} \left\langle R(\dot{\gamma}, X)\dot{\gamma} - X_{tt}, fW^{\perp}\right\rangle(t) \; dt \neq 0,$$

contradiction. From this, it follows that $h(t) = 0$, which is the Jacobi equation, making X a Jacobi field by definition.

Next, it is needed to show that X is smooth, that it has no breaks. The argument for this follows as it does in the proof of corollary [1.1.3]: at each break t_i pick f so that $t_i \in \operatorname{supp} f \subset [t_{i-1}, t_{i+1}]$; it follows that

$$0 = I_\gamma(X, Y) = \left\langle X_{t^-}^{\perp} - X_{t^+}^{\perp}, Y^{\perp}\right\rangle(t_i) \implies X_{t^-}^{\perp} - X_{t^+}^{\perp} = 0$$

by the arbitrariness of Y.

Now, it is needed to show that X has properties (i), and (ii), of M-Jacobi fields. Since $X \in V_\gamma \Omega_{M,q}$, it immediately follows that X has property (i). Looking at the index form again, knowing that X is Jacobi, the integral vanishes as to make it so that

$$0 = I_\gamma(X, Y) = \frac{\varepsilon}{c} \left\langle \widetilde{\Pi}(X, \dot{\gamma}) - X_t, Y^{\perp}\right\rangle(0)$$

$$= \frac{\varepsilon}{c} \left\langle \widetilde{\Pi}(X, \dot{\gamma}) - X_t, Y \right\rangle (0) \qquad \begin{array}{l} Y^{\perp}(0) = Y(0) \text{ since} \\ Y \in V_\gamma \Omega_{M,q}. \end{array}$$

Finally, since that holds for arbitrary $Y \in V_\gamma \Omega_{M,q}$, it follows that

$$\widetilde{\Pi}(X, \dot{\gamma}) - X_t \in \left(T_{\gamma(0)} M\right)^{\perp},$$

which, by linearity of $\left(T_{\gamma(0)} M\right)^{\perp}$, is iff

$$X_t - \widetilde{\Pi}(X, \dot{\gamma}) \in \left(T_{\gamma(0)} M\right)^{\perp},$$

showing that X has property (ii). Therefore, X is an M-Jacobi field. ■

Now, recall that, for a variation χ associated to a vector field $X \in V_\gamma \Omega_{M,q}$, it follows from the definition that $I_\gamma(X, X) = L''_\chi(0)$. As such, talking about the positive, or negative, definiteness of I_γ is talking about whether γ is a local minimum, or a local maximum, respectively. The next result shows what the definiteness of I_γ implies about the index of metric on M, which is particularly useful in focusing considerations of the Riemannian, and Lorentzian, cases.

1.8.3 Lemma Let the manifold $\overline{M}^n$ have a metric with index i, and let $\gamma : [0, l] \longrightarrow \overline{M}$ be a non-null geodesic such that $|\dot{\gamma}| = c$, where $\varepsilon = \operatorname{sgn}\langle \dot{\gamma}, \dot{\gamma}\rangle$. Then,

(1) if I_γ is positive semidefinite, then $i = 0$ or $i = n$

(2) if I_γ is negative semidefinite, then $i = 1$ and $\varepsilon = -1$, or $i = n - 1$ and $\varepsilon = 1$.

Proof The proofs of (1), and (2), follow from the same argument via contradiction; as such, (1) will be proved, and (2) will more quickly follow.

(1) Assume the I_γ is positive semidefinite, but, for the sake of contradiction, that $0 < i < n$. The objective is to show that $I_\gamma(X, X)$ is negative for some $X \in V_\gamma \Omega_{\gamma(0), \gamma(l)}$.

Given that the index has that bound, it is possible to find a unit vector $w \in T_{\gamma(0)} M$ such that $w \perp \dot{\gamma}(0)$ and w has a causal character opposite[14] that of γ, i.e. $\varepsilon \langle w, w\rangle = -1$. From w, construct a vector field $W \in \mathfrak{X}(\gamma)$ by parallel transport of w along γ.

Let $X = \delta \sin\left(\frac{t}{\delta}\right) W$, where $\delta > 0$ is chosen so that $\sin\left(\frac{t}{\delta}\right)$ vanishes at 0 and l: $\delta = \frac{l}{m\pi + \frac{\pi}{2}}$ for $m \in \{0, 1, \ldots\}$. Then,

$$I_\gamma(X, X) = \frac{\varepsilon}{c} \int_0^l \left(\left\langle X_t^{\perp}, X_t^{\perp} \right\rangle (t) - \langle R(\dot{\gamma}, X) X, \dot{\gamma}\rangle (t) \right) dt + \frac{\varepsilon}{c} \underbrace{\langle \Pi(X, X)(0), \dot{\gamma}(0)\rangle}_{\substack{0 \\ \text{since } X(0) = 0 \implies \\ \Pi(X, X)(0) = 0}}$$

$$= \frac{\varepsilon}{c} \int_0^l \left(\langle (\delta f W)_t, (\delta f W)_t \rangle (t) - \langle R(\dot{\gamma}, \delta f W) \delta f W, \dot{\gamma}\rangle (t) \right) dt \qquad \begin{array}{l} \text{where } f = \sin\left(\frac{t}{\delta}\right), \\ \text{since } W \perp \gamma \text{ by} \\ \text{construction} \end{array}$$

$$= \frac{\varepsilon}{c} \int_0^l \left(\delta^2 \left(f'(t)\right)^2 \langle W, W\rangle (t) - \delta^2 \left(f(t)\right)^2 \langle R(\dot{\gamma}, W) W, \dot{\gamma}\rangle (t) \right) dt \qquad \begin{array}{l} W_t = 0 \text{ by} \\ \text{construction, and} \\ \text{using linearity} \end{array}$$

$$= \frac{\varepsilon}{c} \int_0^l \left(\cos^2\left(\frac{t}{\delta}\right) \langle W, W\rangle (t) - \delta^2 \sin^2\left(\frac{t}{\delta}\right) \langle R(\dot{\gamma}, W) W, \dot{\gamma}\rangle (t) \right) dt$$

[14] With the detail that γ is non-null, there are only two causal characters that it can have; thus, "opposite" is to mean "the other".

$$= \frac{1}{c}\int_0^l \left(-\cos^2\left(\frac{t}{\delta}\right) - \varepsilon\delta^2\sin^2\left(\frac{t}{\delta}\right)\langle R(\dot\gamma, W)W, \dot\gamma\rangle(t)\right)\, dt$$

by being a parallel transport $\varepsilon\langle w, w\rangle = -1 \Longrightarrow \varepsilon\langle W, W\rangle = -1$.

Now, it is needed to show that the integral is negative; in particular, to ensure that the second term in the integrand is either negative, or positive and small enough. Since $\delta^2\sin^2\left(\frac{t}{\delta}\right)$ is non-negative, it is just needed to look at $\varepsilon\langle R(\dot\gamma, W)W, \dot\gamma\rangle$. Where it is positive, it leaves the second term negative. Where it is negative, it makes the second term positive, but, since it is bounded, δ can be chosen small enough (i.e. with large enough m) as to make the second term less than the first in absolute value. In each case, the integrand is made non-positive, making the integral negative; contradicting the assumption that I_γ is positive semidefinite. Thus, it must be that $i = 0$ or $i = n$.

(2) Assume that I_γ is negative semidefinite and, for the sake of contradiction, that neither $i = 1$ and $\varepsilon = -1$, nor $i = n-1$ and $\varepsilon = 1$. It follows $\dot\gamma(0)$ is not timelike in the first case, and not spacelike in the second; as such, it is possible to pick a w of the same causal character, i.e. $\varepsilon\langle w, w\rangle = 1$.

Following the argument for (1), it is possible to construct W, and X, in the same manner, and obtain

$$I_\gamma(X, X) = \frac{\varepsilon}{c}\int_0^l \left(\cos^2\left(\frac{t}{\delta}\right)\langle W, W\rangle(t) - \delta^2\sin^2\left(\frac{t}{\delta}\right)\langle R(\dot\gamma, W)W, \dot\gamma\rangle(t)\right)\, dt$$

$$= \frac{1}{c}\int_0^l \left(\cos^2\left(\frac{t}{\delta}\right) - \varepsilon\delta^2\sin^2\left(\frac{t}{\delta}\right)\langle R(\dot\gamma, W)W, \dot\gamma\rangle(t)\right)\, dt$$

by being a parallel transport $\varepsilon\langle w, w\rangle = -1 \Longrightarrow \varepsilon\langle W, W\rangle = -1$.

It is then possible to make the second term in the integrand small enough with δ, as to ensure the integral is positive given that first term is positive; this contradicts the assumption that I_γ is negative semidefinite. Thus, either $i = 1$ and $\varepsilon = -1$, or $i = n-1$ and $\varepsilon = 1$. ■

From these results, it immediately follows that, in the Riemannian case, when considering the index form I_γ for some geodesic γ, that I_γ can only be a positive definite, if it is definite; in the Lorentzian case, for a timelike geodesic γ, I_γ can only be negative definite, if it is definite. Even so, discussion about definiteness of the index form for an arbitrary semi-Riemannian manifold can still can continue via the commonality defined in the next definition.

1.8.4 Definition A geodesic $\gamma : [0, l] \longrightarrow \overline{M}$ is *cospacelike* if, for some[15] t, the subspace $(\dot\gamma(t))^\perp \subset T_{\gamma(t)}\overline{M}$ is spacelike.

(Note that γ must be non-null for, if it were, then $\dot\gamma^\perp$ would be null as well.)

1.8.5 Remark A nice consequence of this definition is that, if a geodesic γ is cospacelike in $\overline{M}$, then the causal character of γ entirely determines whether $\overline{M}$ is Riemannian or Lorentzian, i.e. it cannot have any other index. If γ is spacelike and cospacelike, then, by definition, each vector in $\dot\gamma^\perp$ must also be spacelike, forcing $\overline{M}$ to be Riemannian. And, if γ is timelike, and cospacelike, then, by definition, $\dot\gamma$ is the only timelike "direction", forcing $\overline{M}$ to be Lorentzian. ◀

With this notion, there is an eased discussion connecting the definiteness of I_γ, for some cospacelike geodesic γ, with the focal points along γ; in particular, the following establishes the relationship between the location of focal points along a geodesic and the index form.

1.8.6 Theorem Let $\gamma \in \Omega_{M,q}$, $\gamma : [0, l] \longrightarrow \overline{M}$, be a cospacelike geodesic in $\overline{M}$, which is such that $\gamma \perp M$ at $\gamma(0)$, and let $\gamma(l) = q$ and $\varepsilon = \operatorname{sgn}\langle\dot\gamma, \dot\gamma\rangle$. Then,

[15] And, hence, for any t simply because γ is a geodesic.

(1) if there are no focal points of M along γ, then εI_γ is positive definite, i.e. I_γ negative, or positive, definite, when ε is -1, or 1, respectively

(2) if q is the only focal point of M along γ, then I_γ is semidefinite, and not definite

(3) if $\gamma(r)$, $0<r<l$, is a focal point of M along γ, then I_γ is indefinite.

Proof (1) To show this, first a "basis" of M-Jacobi fields $\{J_i\}$ along γ will be constructed – as to express any $X \in V_\gamma \Omega_{M,q}$ in terms of the J_i. With that, the computation of the index form $I_\gamma(X,X)$ will follow from previous results.

Let $\{w_i\}$ be some basis of $T_{\gamma(0)}M$, and note that it follows that $w_i \perp \dot\gamma(0)$, since $\gamma \perp M$. Then, for each i, let the vector field J_i along γ be such that $J_i(0) = w_i$, and let

$$\frac{DJ_i}{dt}(0) = \widetilde{\Pi}(J_i, \dot\gamma)(0) \in T_{\gamma(0)}M;$$

in this way, each J_i is fully determined as a Jacobi field by means of lemma [1.4.4]. Moreover, constructed as such, it follows that each J_i is an M-Jacobi field by definition, satisfying properties (i) and (ii). Also, since each J_i is such that $J_i(0) \perp \dot\gamma(0)$, and $\frac{DJ_i}{dt}(0) \perp \dot\gamma(0)$, lemma [1.4.7] implies that each $J_i(t) \perp \dot\gamma(t)$ for all $t \in [0,l]$.

Furthermore, these Jacobi fields $\{J_i\}$ remain linearly independent along γ. To see this, assume otherwise; then, for some distinct j,k, there is a point $t_0 \in [0,l]$ such that $J_j(t_0) = -cJ_k(t_0)$, for some constant c. Consider $(J_j + cJ_k)(t)$; since J_j, and J_k, are M-Jacobi fields, it follows that: $(J_j + cJ_k)$ is a Jacobi field by linearity of the covariant derivatives in the Jacobi equation,

$$(J_j + cJ_k)(0) \in T_{\gamma(0)}M$$

by linearity of $T_{\gamma(0)}M$, and

$$\begin{aligned}
\frac{D}{dt}(J_j + cJ_k)(0) - \widetilde{\Pi}((J_j + cJ_k), \dot\gamma)(0) &= \frac{DJ_j}{dt}(0) + c\frac{DJ_k}{dt}(0) \\
&\quad - \widetilde{\Pi}(J_j, \dot\gamma)(0) - \widetilde{\Pi}(cJ_k, \dot\gamma)(0) && \text{by linearity of the covariant derivative and the } \widetilde{\Pi} \text{ tensor} \\
&= 0 && \text{by construction} \\
&\in \left(T_{\gamma(0)}M\right)^\perp .
\end{aligned}$$

From that, it follows that $(J_j + cJ_k)$ is an M-Jacobi field. Note, now, that $(J_j + cJ_k)(t_0) = 0$, making $\gamma(t_0)$ a focal point of M along γ by definition; this contradicts the assumption that there are no focal point along γ. Thus, $\{J_i\}$ remains linearly independent along γ.

Next, consider $X \in V_\gamma \Omega_{M,q}$. At each t, it is possible to express $X(t)$ as $\sum_i c_i J_i(t)$, for some constants c_i, since $\{J_i(t)\}$ spans $(\dot\gamma(t))^\perp$ by virtue of being linearly independent. Hence, by the continuity of X, it follows that X can be written

$$X = \sum_i f_i J_i,$$

for piecewise-smooth f_i.

Now, the idea is to use the two lemmas about Jacobi fields as to rewrite the index form integral expression. In order to use the result of lemma [1.4.9], which has the form of the index form's integrand, lemma [1.4.8] will be used to show that $\{J_i\}$ satisfy its hypothesis. Consider[16]

$$\left\langle \frac{DJ_i}{dt}, J_j \right\rangle (0) = \left\langle \tan \frac{DJ_i}{dt}, J_j \right\rangle (0) \qquad \text{since } J_j(0) \in T_{\gamma(0)}M \text{ by definition of being } M\text{-Jacobi}$$

[16]Using implicitly the fact that the metric is linear to simplify notation.

$$
\begin{aligned}
&= \left\langle \widetilde{\Pi}\left(J_i, \dot{\gamma}\right), J_j \right\rangle (0) && \text{by property (ii) of } M\text{-Jacobi fields} \\
&= -\left\langle \Pi\left(J_i, J_j\right), \dot{\gamma} \right\rangle (0) && \text{from the definition of } \widetilde{\Pi} \\
&= -\left\langle \Pi\left(J_j, J_i\right), \dot{\gamma} \right\rangle (0) && \text{by symmetry of } \Pi \\
\iff \left\langle \frac{DJ_i}{dt}, J_j \right\rangle (0) &= \left\langle \frac{DJ_j}{dt}, J_i \right\rangle (0) && \text{using the same reasoning} \\
\iff \left\langle \frac{DJ_i}{dt}, J_j \right\rangle (0) - \left\langle \frac{DJ_j}{dt}, J_i \right\rangle (0) &= 0 \\
\implies \left\langle \frac{DJ_i}{dt}, J_j \right\rangle - \left\langle \frac{DJ_j}{dt}, J_i \right\rangle &= 0 && \text{by lemma } \texttt{[1.4.8]} \\
\iff \left\langle \frac{DJ_i}{dt}, J_j \right\rangle &= \left\langle \frac{DJ_j}{dt}, J_i \right\rangle .
\end{aligned}
$$

With that, lemma `[1.4.9]` concludes that

$$
\left\langle \frac{DX}{dt}, \frac{DX}{dt} \right\rangle - \left\langle R\left(\dot{\gamma}, X\right) X, \dot{\gamma} \right\rangle = \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle + \frac{d}{dt} \left\langle X, \sum_i f_i \frac{DJ_i}{dt} \right\rangle ,
$$

which makes

$$
\begin{aligned}
\varepsilon I_\gamma\left(X, X\right) &= \frac{1}{c} \left(\int_0^l \left(\left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle (t) + \left(\frac{d}{dt} \left\langle X, \sum_i f_i \frac{DJ_i}{dt} \right\rangle \right) (t) \right) \ dt - \left\langle \Pi\left(X, X\right), \dot{\gamma} \right\rangle (0) \right) \\
&= \frac{1}{c} \left(\int_0^l \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle (t) \ dt + \left[\left\langle X, \sum_i f_i \frac{DJ_i}{dt} \right\rangle (t) \right]_0^l - \left\langle \Pi\left(X, X\right), \dot{\gamma} \right\rangle (0) \right) \\
&= \frac{1}{c} \left(\int_0^l \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle (t) \ dt - \left\langle X, \sum_i f_i \frac{DJ_i}{dt} \right\rangle (0) - \left\langle \Pi\left(X, X\right), \dot{\gamma} \right\rangle (0) \right) && \text{since } X \in V_\gamma \Omega_{M,q},\ X(l) = 0
\end{aligned}
$$

Looking at the second-to-last term further,

$$
\begin{aligned}
\left\langle X, \sum_i f_i \frac{DJ_i}{dt} \right\rangle (0) &= \left\langle X, \tan \sum_i f_i \frac{DJ_i}{dt} \right\rangle (0) && \text{since } X \in V_\gamma \Omega_{M,q},\ X(0) \in T_{\gamma(0)} M \\
&= \left\langle X, \sum_i f_i \tan \frac{DJ_i}{dt} \right\rangle (0) && \text{by linearity} \\
&= \left\langle X, \sum_i f_i \, \widetilde{\Pi}\left(J_i, \dot{\gamma}\right) \right\rangle (0) && \text{by property (ii) of } M\text{-Jacobi fields} \\
&= \sum_i f_i \left\langle X, \widetilde{\Pi}\left(J_i, \dot{\gamma}\right) \right\rangle (0) && \text{by linearity} \\
&= -\sum_i f_i \left\langle \dot{\gamma}, \Pi\left(J_i, X\right) \right\rangle (0) && \text{from the definition of } \widetilde{\Pi} \\
&= \left\langle \dot{\gamma}, \Pi\left(\sum_i f_i J_i, X \right) \right\rangle (0) && \text{by linearity and tensor properties of } \Pi \\
&= \left\langle \dot{\gamma}, \Pi\left(X, X\right) \right\rangle (0) ,
\end{aligned}
$$

making

$$
\varepsilon I_\gamma\left(X, X\right) = \frac{1}{c} \int_0^l \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle (t) \ dt.
$$

Since γ is cospacelike, and $J_i \in (\dot{\gamma})^{\perp}$, it follows that

$$\begin{aligned} \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle &\geq 0 \qquad \texttt{[*]} \\ \implies \varepsilon I_\gamma (X, X) &\geq 0, \end{aligned}$$

showing that the index form εI_γ is positive semidefinite. To show that it is definite, it is simply needed to show that it is zero only if $\sum_i f_i' J_i = 0$:

$$\begin{aligned} \varepsilon I_\gamma (X, X) &= 0 \\ \iff \int_0^l \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle (t) \; dt &= 0 \\ \iff \left\langle \sum_i f_i' J_i, \sum_i f_i' J_i \right\rangle &= 0 && \text{by } \texttt{[*]} \\ \iff \sum_i f_i' J_i &= 0 && \text{since each } J_i \perp \dot{\gamma}\text{, making them spacelike, i.e. non-null} \\ \iff f_i' &= 0 \\ \iff f_i &= c_i && \text{for each } i\text{, for some constant } c_i, \end{aligned}$$

which implies that $f_i = 0$ for each i, since $X(l) = 0$; by definition, this is iff $X = 0$. Thus, εI_γ is positive definite.

(2) By definition of there being a focal point of M along γ, it follows from lemma [1.8.1] that the nullspace of I_γ contains at least the M-Jacobi fields along γ that vanish at l. As such, it cannot be definite.

As to show that it is not indefinite, part (1) will be used. Pick an $X \in V_\gamma \Omega_{M,q}$, and consider $X_i \in V_\gamma \Omega_{M,q}$ such that X_i vanishes at some point $0 < r_i < l$ and past it, i.e. it vanishes on $[r_i, l]$. It follows from part (1) that $\varepsilon I_\gamma (X_i, X_i) \geq 0$. From these, construct[17] a sequence of X_i such that the corresponding $r_i \to l$ as $i \to \infty$ so that $\lim_{i\to\infty} X_i = X$, making

$$\lim_{i\to\infty} \varepsilon I_\gamma (X_i, X_i) = \varepsilon I_\gamma (X, X).$$

By continuity, it follows that

$$\varepsilon I_\gamma (X, X) = \lim_{i\to\infty} \varepsilon I_\gamma (X_i, X_i) \geq 0.$$

Thus, by the arbitrariness of X, it is the case that εI_γ is never negative, making it positive semidefinite, i.e. I_γ is semidefinite.

(3) From remark [1.8.5], it follows that, since γ is a cospacelike geodesic in $\overline{M}$, that $\overline{M}$ is either Riemannian or Lorentzian. In either case, it follows from lemma [1.8.3] that εI_γ is positive semidefinite. Moreover, with lemma [1.8.1], this means that there is at least some[18] $X \in V_\gamma \Omega_{M,q}$ (which is not M-Jacobi) such that $\varepsilon I_\gamma (X, X) > 0$. Hence, the objective here to find a $Z \in V_\gamma \Omega_{M,q}$ along γ such that $\varepsilon I_\gamma (Z, Z) < 0$; this will be constructed as a perturbation of an extension of an M-Jacobi field along $\gamma|_{[0,r]}$, which is guaranteed by the focal point.

Without loss of generality, assume that $\gamma(r)$ is the first focal point of M along γ. By definition, this means that there is an M-Jacobi field J along $\gamma|_{[0,r]}$ which vanishes at r. Extend J to a

[17] For example, by having $X_i = f_i X$ where $f_i \in \mathfrak{F}(\gamma)$ such that $[0, l] \setminus \operatorname{supp} f_i = [r_i, l]$.

[18] If there were not, then $I_\gamma (X, X) = 0$ for all $X \in V_\gamma \Omega_{M,q}$, which implies that $0 = I_\gamma (X + Y, X + Y) = I_\gamma (X, X) + 2 I_\gamma (X, Y) + I_\gamma (Y, Y) = 2 I_\gamma (X, Y)$ for all $X, Y \in V_\gamma \Omega_{M,q}$; this shows that each $X \in V_\gamma \Omega_{M,q}$ is in the nullspace of I_γ, which is clearly not the case by lemma [1.8.1] since there are non-M-Jacobi fields along γ.

vector field $X \in V_\gamma \Omega_{M,q}$ along γ by defining it to be J on $[0,r]$, and to also vanish on $(r,l]$. As such, X only has one break. It follows, using the expression [1.8.2] of the index form, that

$$\begin{aligned}
I_\gamma(X,X) &= \frac{\varepsilon}{c}\int_0^l \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, X^\perp\right\rangle(t)\;dt + \frac{\varepsilon}{c}\left\langle \widetilde{\Pi}(X,\dot\gamma) - X_t, X^\perp\right\rangle(0)\\
&\quad + \frac{\varepsilon}{c}\left(\left\langle X^\perp_{t^-}, X^\perp\right\rangle(r) - \left\langle X^\perp_{t^+}, X^\perp\right\rangle(r)\right)\\
&= \underbrace{\frac{\varepsilon}{c}\int_0^r \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, X^\perp\right\rangle(t)\;dt + \frac{\varepsilon}{c}\left\langle \widetilde{\Pi}(X,\dot\gamma) - X_t, X^\perp\right\rangle(0)}_{\substack{0\\ \text{by lemma [1.8.1], since } X|_{[0,r]} = J}}\\
&\quad + \underbrace{\frac{\varepsilon}{c}\int_r^l \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, X^\perp\right\rangle(t)\;dt}_{\substack{0\\ \text{since } X|_{[r,l]} = 0}}\\
&\quad + \underbrace{\frac{\varepsilon}{c}\left(\left\langle X^\perp_{t^-}, X^\perp\right\rangle(r) - \left\langle X^\perp_{t^+}, X^\perp\right\rangle(r)\right)}_{\substack{0\\ \text{since } X(r) = 0 \implies X^\perp(r) = 0}}\\
&= 0.
\end{aligned}$$

Now, perturbing X with some $Y \in V_\gamma \Omega_{M,q}$ such that

$$Y(r) = X_{t^+}(r) - X_{t^-}(r)$$

to obtain

$$Z = X + \delta Y,$$

and evaluating with the index form εI_γ,

$$\begin{aligned}
\varepsilon I_\gamma(Z,Z) &= \underbrace{\varepsilon I_\gamma(X,X)}_{\substack{0\\ \text{as just shown}}} + \varepsilon\delta I_\gamma(X,Y) + \varepsilon\delta I_\gamma(Y,X) + \varepsilon\delta^2 I_\gamma(Y,Y) && \text{using linearity}\\
&= 2\varepsilon\delta I_\gamma(X,Y) + \varepsilon\delta^2 I_\gamma(Y,Y) && \text{using symmetry.}
\end{aligned}$$

Looking at the first term,

$$\begin{aligned}
I_\gamma(X,Y) &= \frac{\varepsilon}{c}\int_0^l \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\;dt + \frac{\varepsilon}{c}\underbrace{\left\langle \widetilde{\Pi}(X,\dot\gamma) - X_t, Y^\perp\right\rangle(0)}_{\substack{0\\ \text{since } X \text{ is } M\text{-Jacobi and } Y \in V_\gamma\Omega_{M,q},\\ \widetilde{\Pi}(X,\dot\gamma)(0) - X_t(0) \in T_{\gamma(0)}M, \text{ and}\\ Y^\perp(0) \in T_{\gamma(0)}M}}\\
&\quad + \frac{\varepsilon}{c}\left(\left\langle X^\perp_{t^-}, Y^\perp\right\rangle(r) - \left\langle X^\perp_{t^+}, Y^\perp\right\rangle(r)\right)\\
&= \frac{\varepsilon}{c}\underbrace{\int_0^r \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\;dt}_{\substack{0\\ \text{since } X|_{[0,r]} = J, \text{ which is Jacobi}}} + \frac{\varepsilon}{c}\underbrace{\int_r^l \left\langle R(\dot\gamma,X)\dot\gamma - X_{tt}, Y^\perp\right\rangle(t)\;dt}_{\substack{0\\ \text{since } X|_{[r,l]} = 0}}\\
&\quad + \frac{\varepsilon}{c}\left(\left\langle X^\perp_{t^-}, Y^\perp\right\rangle(r) - \left\langle X^\perp_{t^+}, Y^\perp\right\rangle(r)\right)\\
&= \frac{\varepsilon}{c}\left(\left\langle X^\perp_{t^-}, X^\perp_{t^+} - X^\perp_{t^-}\right\rangle(r) - \left\langle X^\perp_{t^+}, X^\perp_{t^+} - X^\perp_{t^-}\right\rangle(r)\right) && \text{by construction of } Y
\end{aligned}$$

$$= -\frac{\varepsilon}{c}\left\langle X^{\perp}_{t^+} - X^{\perp}_{t^-}, X^{\perp}_{t^+} - X^{\perp}_{t^-}\right\rangle (r) \qquad \text{using linearity}$$

$$\iff \varepsilon I_\gamma (X,Y) = -\frac{1}{c}\left\langle X^{\perp}_{t^+} - X^{\perp}_{t^-}, X^{\perp}_{t^+} - X^{\perp}_{t^-}\right\rangle (r).$$

Since γ is cospacelike, and X piecewise-smooth with a break at r (i.e. $X^{\perp}_{t^+} - X^{\perp}_{t^-} \neq 0$), it follows that

$$\left\langle X^{\perp}_{t^+} - X^{\perp}_{t^-}, X^{\perp}_{t^+} - X^{\perp}_{t^-}\right\rangle (r) > 0$$
$$\implies \varepsilon I_\gamma (X,Y) < 0.$$

This means that, for some small enough $\delta > 0$,

$$\varepsilon I_\gamma (Z,Z) = 2\varepsilon\delta I_\gamma (X,Y) + \varepsilon\delta^2 I_\gamma (Y,Y) < 0.$$

Therefore, with what was mentioned at the start, it must be that I_γ is indefinite by definition. ■

With all that now established, the following results handle generalizing the idea of proposition [1.7.6], motivated by using the notion of a cospacelike geodesic: given information about a submanifold $M \subset \overline{M}$, determine the existence of focal points of M.

1.8.7 Proposition Let M be a spacelike[19] submanifold of $\overline{M}$, and let γ be a cospacelike[20] geodesic normal to M at $\gamma(0)$. Also, suppose the following bounds

(1) for some unit vector $v \in T_{\gamma(0)}M$, $\langle \Pi(v,v), \dot\gamma(0)\rangle = k > 0$

(2) for any t along γ, and for all $w \in (\dot\gamma(t))^{\perp}$, $\langle R(\dot\gamma(t), w) w, \dot\gamma(t)\rangle \geq 0$.

Then, there is a focal point $\gamma(r)$ of P along γ with $0 < r \leq \frac{1}{k}$, given that γ is defined on that interval.

(Note that, if $k < 0$ for some unit vector v, then reversing the parametrization γ allows for use of the proposition: property (1) is then satisfied; property (2) remains unchanged since the sign-change of $\dot\gamma$ appears twice in the $(0,4)$-tensor R; the conclusion is that there is a focal point $\gamma(r)$, $-\frac{1}{k} \leq r < 0$.)

Proof In light of the results on the trichotomy[21] established by theorem [1.8.6], it is sufficient to show that I_γ, or equivalently εI_γ, is not definite on $\left[0, \frac{1}{k}\right]$. Let γ be considered on $\left[0, \frac{1}{k}\right]$, and $\gamma\left(\frac{1}{k}\right) = q$. As noted in the proof of theorem [1.8.6] (3), lemma [1.8.1] guarantees that there is some $X \in V_\gamma\Omega_{M,q}$ such that $\varepsilon I_\gamma (X,X) > 0$. With that, it is just needed to find an $X \in V_\gamma\Omega_{M,q}$, which is not identically zero, such that $\varepsilon I_\gamma (X,X) \leq 0$, as to show that I_γ is not definite.

Suggested by the bound (1): from $v \in T_{\gamma(0)}M$ construct a vector field V which is the parallel transport of v along γ. In turn, let $X = (1-kt)V$, as in proposition [1.7.6], to obtain an $X \in V_\gamma\Omega_{M,q}$. Evaluating,

$$\varepsilon I_\gamma (X,X) = \frac{1}{c}\int_0^{\frac{1}{k}} \left(\left\langle X^{\perp}_t, X^{\perp}_t\right\rangle (t) - \langle R(\dot\gamma, X) X, \dot\gamma\rangle (t)\right)\, dt - \frac{1}{c}\underbrace{\langle \Pi(X,X), \dot\gamma\rangle (0)}_{\substack{\langle \Pi(v,v), \dot\gamma\rangle (0) = k \\ \text{by construction of} \\ X}}$$

$$= \frac{1}{c}\int_0^{\frac{1}{k}} \left(\langle -kV, -kV\rangle (t) - \langle R(\dot\gamma, X) X, \dot\gamma\rangle (t)\right)\, dt - \frac{k}{c} \qquad \begin{array}{l}\text{by parallel transport: } V_t = 0 \\ \text{and } v \perp \dot\gamma(0) \iff V^{\perp} = V\end{array}$$

[19] This is not mean that, at every $p \in M$, T_pM is spacelike subspace of $T_p\overline{M}$.
[20] Recall remark [1.8.5]: this means that $\overline{M}$ is necessarily Riemannian or Lorentzian.
[21] Either the focal point is not in the domain of γ, is at the endpoint of γ, or is in the domain of γ but not on the endpoint.

$$= \frac{1}{c}\int_0^{\frac{1}{k}} \left(k^2 - \langle R(\dot\gamma, X)X, \dot\gamma\rangle (t)\right)\ dt - \frac{k}{c}$$

using linearity and, by parallel transport, $\langle v, v\rangle = 1 \iff \langle V, V\rangle = 1$

$$= -\frac{1}{c}\int_0^{\frac{1}{k}} \langle R(\dot\gamma, X)X, \dot\gamma\rangle (t)\ dt$$

$\frac{1}{c}\int_0^{\frac{1}{k}} k^2 = \frac{k}{c}$

$$\leq 0$$

by bound (2).

Therefore, from what was mentioned at the start, it follows that I_γ is not definite, and, in turn, that there is a focal point at $\gamma(r)$, for some $0 < r \leq \frac{1}{k}$, if γ is defined there. ■

The next result will seek to weaken the bounds assumed in the previous result in the case of a spacelike hypersurface of $\overline{M}$. To do this, a couple definitions will first be established.

1.8.8 Definition For a submanifold $M^n \subset \overline{M}$, the *mean curvature vector field* H of M is, at each $p \in M$,

$$H|_p = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \,\Pi(e_i, e_i),$$

where $\{e_i\}$ is any orthonormal basis of T_pM, and $\varepsilon_i = \operatorname{sgn}\langle e_i, e_i\rangle$.

1.8.9 Definition Let M^n be a submanifold of $\overline{M}$. The *convergence* of M is a function k on the normal tangent bundle $T(M)^\perp$ such that, each $p \in M$,

$$\begin{aligned} k : (T_pM)^\perp &\longrightarrow \mathbb{R} \\ v &\longmapsto \left\langle H|_p, v\right\rangle \end{aligned}$$

Note that, using that previous definition and the definition of $\widetilde{\Pi}$,

$$\left\langle H|_p, v\right\rangle = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i \langle \Pi(e_i, e_i), v\rangle = -\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i \left\langle \widetilde{\Pi}(e_i, v), e_i\right\rangle = -\frac{1}{n}\operatorname{tr}\widetilde{\Pi}(\cdot, v),$$

which is the average of the eigenvalues of $-\widetilde{\Pi}(\cdot, v)$.

Now, for the result, which uses these concepts.

1.8.10 Proposition Let M be a spacelike hypersurface of $\overline{M}$, and let γ be a cospacelike geodesic normal to M at $\gamma(0) = p$. Also, suppose that

(1) $k(\dot\gamma(0)) = \left\langle H|_p, \dot\gamma(0)\right\rangle > 0$

(2) $\operatorname{Ric}(\dot\gamma, \dot\gamma) \geq 0$.

Then, there is a focal point $\gamma(r)$ of M along γ with $0 < r \leq \frac{1}{k(\dot\gamma(0))}$, given that γ is defined there.

(Note, as in proposition [1.8.7], that reversing the orientation of γ may be useful in meeting the first bound.)

Proof Just as in the proof of proposition [1.8.7], it follows from theorem [1.8.6], that it is sufficient to show that εI_γ is not definite on $\left[0, \frac{1}{k}\right]$, where $k = k(\dot\gamma(0))$. Let γ be considered on $\left[0, \frac{1}{k}\right]$ with $\gamma(0) = p$ and $\gamma\left(\frac{1}{k}\right) = q$. Also, as mentioned in the proof of proposition [1.8.7], lemma [1.8.1] guarantees that there is some $X \in V_\gamma\Omega_{M,q}$ such that $\varepsilon I_\gamma(X, X) > 0$. Given that, it is just needed to find an $X \in V_\gamma \Omega_{M,q}$, which is not identically zero, such that $\varepsilon I_\gamma(X,X) \leq 0$, as to show that I_γ is not definite.

Once again, suggested by bound (1): let $\{e_i\}_1^{n-1}$ be an orthonormal basis that spans T_pM; from each of these, construct a corresponding E_i which is gotten by a parallel transport of e_i along γ. Let $f(t) = 1 - kt$ as to make $fE_i \in V_\gamma \Omega_{M,q}$ for each i.

Now, evaluating, for each i,

$$\begin{aligned}
\varepsilon I_\gamma(fE_i, fE_i) &= \frac{1}{c}\int_0^{\frac{1}{k}} \left(\left\langle (fE_i)_t^\perp, (fE_i)_t^\perp \right\rangle(t) - \langle R(\dot\gamma, fE_i) fE_i, \dot\gamma\rangle(t)\right)\, dt - \frac{1}{c}\langle \Pi(fE_i, fE_i), \dot\gamma\rangle(0) \\
&= \frac{1}{c}\int_0^{\frac{1}{k}} \left(\langle -kE_i, -kE_i\rangle - \langle R(\dot\gamma, fE_i) fE_i, \dot\gamma\rangle(t)\right)\, dt - \frac{1}{c}\langle \Pi(fE_i, fE_i), \dot\gamma\rangle(0) && \text{by parallel transport: } (E_i)_t = 0 \text{, and } e_i \perp \dot\gamma(0) \iff E_i^\perp = E_i \\
&= \frac{1}{c}\int_0^{\frac{1}{k}} \left(k^2 - \langle R(\dot\gamma, fE_i) fE_i, \dot\gamma\rangle(t)\right)\, dt - \frac{1}{c}\langle \Pi(fE_i, fE_i), \dot\gamma\rangle(0) && \text{using linearity and, by parallel transport, } \langle e_i, e_i\rangle = 1 \iff \langle E_i, E_i\rangle = 1 \\
&= \frac{k}{c} - \frac{1}{c}\int_0^{\frac{1}{k}} (f(t))^2 \left(\langle R(\dot\gamma, E_i) E_i, \dot\gamma\rangle(t)\right)\, dt - \frac{1}{c}\langle \Pi(E_i, E_i), \dot\gamma\rangle(0) && \text{using linearity of } R \text{ and } \Pi \text{, with } f(0) = 1.
\end{aligned}$$

Summing these, as to obtain terms which are in the bounds,

$$\sum_{i=1}^{n-1} \varepsilon I_\gamma(fE_i, fE_i) = \frac{k(n-1)}{c} - \frac{1}{c}\int_0^{\frac{1}{k}} (f(t))^2 \sum_{i=1}^{n-1} \langle R(\dot\gamma, E_i) E_i, \dot\gamma\rangle(t)\, dt - \frac{1}{c}\left\langle \sum_{i=1}^{n-1} \Pi(E_i, E_i), \dot\gamma \right\rangle(0),$$

using linearity. Looking further:

$$\begin{aligned}
\sum_{i=1}^{n-1} \langle R(\dot\gamma, E_i) E_i, \dot\gamma\rangle &= \sum_{i=1}^{n-1} \langle R(E_i, \dot\gamma)\dot\gamma, E_i\rangle && \text{using symmetries of } R \\
&= \frac{1}{c^2}\underbrace{\langle R(\dot\gamma, \dot\gamma)\dot\gamma, \dot\gamma\rangle}_{\substack{0 \\ \text{using the (anti)symmetries} \\ \text{of } R}} + \sum_{i=1}^{n-1} \langle R(E_i, \dot\gamma)\dot\gamma, E_i\rangle \\
&= \left\langle R\left(\dot\gamma, \frac{\dot\gamma}{c}\right)\frac{\dot\gamma}{c}, \dot\gamma\right\rangle + \sum_{i=1}^{n-1} \langle R(E_i, \dot\gamma)\dot\gamma, E_i\rangle && \text{using linearity} \\
&= \mathrm{Ric}(\dot\gamma, \dot\gamma) && \text{by definition, since } \tfrac{\dot\gamma}{c} \text{ is the } n\text{-th “unit-direction”}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{n-1} \Pi(E_i, E_i)(0) &= \sum_{i=1}^{n-1} \Pi(e_i, e_i) && \text{using linearity and the construction of each } E_i \\
&= (n-1)\, H|_p && \text{by definition.}
\end{aligned}$$

Compiling this, using linearity,

$$\sum_{i=1}^{n-1} \varepsilon I_\gamma(fE_i, fE_i) = \frac{k(n-1)}{c} - \frac{1}{c}\int_0^{\frac{1}{k}} (f(t))^2\, \mathrm{Ric}(\dot\gamma, \dot\gamma)(t)\, dt - \frac{n-1}{c}\underbrace{\left\langle H|_p, \dot\gamma(0)\right\rangle}_{\substack{k \\ \text{by definition}}}$$

$$= -\frac{1}{c}\int_0^{\frac{1}{k}} (f(t))^2 \operatorname{Ric}(\dot\gamma, \dot\gamma)(t)\ dt$$
$$\leq 0 \qquad \text{using bound (2).}$$

Thus, it follows from that discussion at the start of the proof that I_γ is not definite, and, in turn, that there is a focal point $\gamma(r)$ of M along γ, for $0 < r < \frac{1}{k}$, if γ is defined there. ■

The bounds are, of course, weaker ones than those in proposition `[1.8.7]`, but that comes with the restriction that M is a hypersurface; in particular, the punchline to this proof was that

$$\operatorname{Ric}(\dot\gamma, \dot\gamma) = \sum_{i=1}^{n} \langle R(e_i, \dot\gamma)\dot\gamma, e_i\rangle = \sum_{i=1}^{\dim M} \langle R(e_i, \dot\gamma)\dot\gamma, e_i\rangle,$$

as to make summing the $n - 1 = \dim M$ index forms fruitful. Also, it should be noted that the first bounds in each are just to ensure that k, or $k(\dot\gamma(0))$, is non-zero; if it were zero, then it makes their conclusions vacuous.

And, with that, all the machinery is in place for discourse about (in)completeness in the Riemanninan, and Lorentzian, cases.

2 | Riemannian Geometry

As the title suggests, the content of this chapter will focus on the Riemmanian case, and derive results about completeness using the concepts of the Calculus of variations. First on the to-do list, is to establish the idea of a metric on a manifold (as a space); this metric naturally differs from the metric tensor used prior, but is innately connected to it. [O'N83, Ch. 5][Car92, §3.3 & §7.2]

2.0.1 Definition Recall the length functional L mentioned in definition `[1.0.1]`, and definition `[1.1.1]`. For a Riemannian manifold $M \ni p, q$, let $\gamma \in \Omega_{p,q}$ be such that $\gamma : [0, l] \longrightarrow M$. The length of γ, via definition `[1.0.1]`, is

$$L(\gamma) = \int_0^l |\dot{\gamma}| \, .$$

With that, the *metric* $d : M \times M \longrightarrow \mathbb{R}$ on M is

$$d(p, q) = \inf_{\gamma \in \Omega_{p,q}} L(\gamma) \, ,$$

which can be seen as being motivated by lemma `[1.8.3]` (1).

With this, it is possible to define a neighborhood of radius $\delta > 0$ about $p \in M$ as

$$N_\delta(p) := \{q \in M : d(p, q) < \delta\} \subset M.$$

Before showing that this indeed has the usual properties of a metric, it is needed to establish the notion neighborhoods of M which are diffeomorphic with a tangent space at some point in the neighborhood via the exponential map.

2.0.2 Definition For $p \in M$, the neighborhood U of p is called a *normal neighborhood* of p iff there is some neighborhood V of the origin of T_pM such that such that $\exp_p : V \longrightarrow U$ is a diffeomorphism.

2.0.3 Lemma For any $p \in M$:

(1) there is a $\delta > 0$ such that $N_\delta(p)$ is a normal neighborhood of p.

(2) for a normal neighborhood $N_\delta(p) \ni q$, there is a unique minimizing geodesic in $N_\delta(p)$ joining p and q.

Proof (Sketches)

(1) Since $(d\exp_p)_0$ is identity, it follows, by the implicit function theorem, that there is a neighborhood U of $0 \in T_pM$ on which $\exp_p$ invertible. Thus, taking $\delta > 0$ small enough such that $\exp_p U \supset N_\delta(p)$, obtains a normal neighborhood of p.

(2) Given that $N_\delta(p)$ is a normal neighborhood of p, it follows there is only a single vector in the preimage of any $q \in N_\delta(p)$. Thus, by the uniqueness of geodesics guaranteed in lemma `[0.2.3]`, it follows that there is a unique geodesic connecting p with any such q. Also, by corollary `[1.1.3]`, this unique geodesic is minimizing; moreover, the length of this geodesic is less than δ by construction of $N_\delta(p) \in q$.

■

2.0.4 Lemma This metric d on M has the usual properties of a metric: for any $p, q, r \in M$,

(1) positivity, $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$

(2) symmetry, $d(p,q) = d(q,p)$

(3) triangle inequality, $d(p,r) \leq d(p,q) + d(q,r)$.

Proof (1) First, note that, since $L(\gamma)$ is an integral of the non-negative function $|\dot{\gamma}|$, $L(\gamma) \geq 0$ for all $\gamma \in \Omega_{p,q}$. From this, it follows that $d(p,q) \geq 0$ for all $p, q \in M$.

Now, the case when it is zero. If $p = q$, it trivially follows that $d(p,q) = 0$. As for the converse, assume for the sake of contradiction that $d(p,q) = 0$, but $p \neq q$. Since manifolds are assumed to be Hausdorff, it follows that there is a neighborhood $U$of p not containing q. Take a normal neighborhood $N_\delta(p)$ which is a subset of U; it follows that $d(p,q) \geq \delta$ as discussed in lemma `[2.0.3]` – contradiction.

(2) Given any curve $\gamma \in \Omega_{p,q}$, it is possible to reverse the orientation of $\gamma : [0,l] \longrightarrow M$ as to obtain a curve $\overline{\gamma} \in \Omega_{q,p}$, i.e. $\overline{\gamma}(t) = \gamma(l-t)$ on $[0,l]$. This reparametrization, of course, does not affect the integral for the length: since

$$\dot{\overline{\gamma}} = -\dot{\gamma},$$

it follows that

$$L(\gamma) = \int_0^l |\dot{\gamma}| = \int_0^l \left|\dot{\overline{\gamma}}\right| = L(\overline{\gamma}).$$

As such, it is the case that

$$\{L(\gamma) : \gamma \in \Omega_{p,q}\} = \{L(\gamma) : \gamma \in \Omega_{q,p}\} \implies d(p,q) = d(q,p).$$

(3) Using[1] the definition of $d(\cdot,\cdot)$, it follows that, for each $\delta > 0$, there is an $\alpha \in \Omega_{p,q}$ such that

$$L(\alpha) - \delta \leq d(p,q),$$

and there is a $\beta \in \Omega_{q,r}$ such that

$$L(\beta) - \delta \leq d(q,r).$$

Joining α, and β, a curve $\gamma \in \Omega_{p,r}$ is obtained. With that, it follows that

$$\begin{aligned} d(p,r) &\leq L(\gamma) && \text{by definition of } d(\cdot,\cdot) \\ &= L(\alpha) + L(\beta) && \text{by construction of } \gamma \\ &\leq d(p,q) + d(q,r) + 2\delta. \end{aligned}$$

Therefore, since that holds for each $\delta > 0$, it follows that

$$d(p,r) \leq d(p,q) + d(q,r).$$

■

From this result, it also follows that the topology induced by d is compatible with the topology on M.

Before the first result involving completeness, a lemma[2] is needed asserting the existence of a minimizing geodesic between two points by means of a condition on the the domain of the exponential map.

2.0.5 Lemma For a Riemannian manifold $M \ni p$, if $\exp_p$ is defined on all of T_pM, then there is a minimizing geodesic connecting p and q, for any $q \in M$.

[1] The definition of $d(p,q)$ requires that there is a sequence of $\gamma_i \in \Omega_{p,q}$ whose lengths converge to $d(p,q)$.
[2] See [O'N83, Ch. 5 lemma 24].

With that, it is possible to obtain the first result establishing the equivalent definitions of completeness; it is the theorem[3] by Hopf-Rinow.

2.0.6 Theorem Let M be a connected Riemannian manifold. Then, the following are equivalent:

(1) (as a metric space) M with the metric d is complete

(2) there is a point $p \in M$ such that $\exp_p$ is defined on the entirety of T_pM

(3) M is *geodesically complete*, which is iff $\exp_p$ is defined on the entirety of T_pM for each $p \in M$

(4) every closed, and bounded, subset of M is compact.

Finally, with those notions acknowledged, comes the main result of this chapter, regarding completeness using the results from the Calculus of variations: the Bonnet-Myers theorem.[O'N83, Ch. 5][Cha06, §II.6]

2.0.7 Theorem Let M^n be a complete, connected Riemannian manifold such that there is a constant $C > 0$ so that, for all $p \in M$, and any $v \in T_pM$, $\mathrm{Ric}(v, v) \geq (n-1)\, C \langle v, v\rangle > 0$. Then,

$$\sup_{p,q\in M} d(p,q) \leq \frac{\pi}{\sqrt{C}}.$$

Proof First off, since M is complete, it follows from theorem `[2.0.6]` that, for each $p \in M$, the domain of $\exp_p$ is the entirety of T_pM; with lemma `[2.0.5]`, it follows that for any $p, q \in M$, there is a minimizing geodesic joining them. Moreover, by being minimizing along with the fact that geodesics are the only extrema of the length functional (corollary `[1.1.3]`), it follows that γ is such that $L(\gamma) = d(p,q)$. With that, it suffices to show that, for any $p, q \in M$, any such minimizing geodesic γ joining them is such that $L(\gamma) \leq \frac{\pi}{\sqrt{C}}$.

Now, recall the fact from definition `[1.6.5]` that $I_\gamma(X, X) = L''_\chi(0)$ for $X \in V_\gamma\Omega_{p,q}$ with accompanying variation χ. As such, γ is (locally) minimizing iff I_γ is positive semidefinite. With that, it will be shown by means of the index form that, for any $p, q \in M$, if there is a geodesic γ joining p, q such that $L(\gamma) > \frac{\pi}{\sqrt{C}}$, then γ is no longer minimizing. To do this, the trichotomy established in theorem `[1.8.6]` will be used: if there is a geodesic γ joining any $p, q \in M$ such that $L(\gamma) = \frac{\pi}{\sqrt{C}}$, it will be shown that I_γ is not positive definite, showing that there is a conjugate point[4] along γ of p. Furthermore, by the theorem, this forces the index form along any extension[5] of γ past q to be indefinite, which shows that any extension of γ is not minimizing.

Suppose that, for some $p, q \in M$, there is a geodesic γ such that $L(\gamma) = \frac{\pi}{\sqrt{C}}$; without loss of generality[6], it is possible to assume that γ has unit speed, i.e. $|\dot\gamma| = 1$, making

$$\gamma : \left[0, \frac{\pi}{\sqrt{C}}\right] \longrightarrow M.$$

As to show that I_γ is not positive definite, the objective now is to find an non-trivial $X \in V_\gamma\Omega_{p,q}$ such that $I_\gamma(X, X) \leq 0$. Let $\{e_i\}_1^{n-1}$ be an orthonormal basis of $(\dot\gamma(0))^\perp$; construct vector fields E_i, for each i, as the parallel transport of e_i along γ.

[3] See [O'N83, Ch. 5 theorem 21] or [Car92, Ch. 7 theorem 2.8].

[4] Recalling the note in definition `[1.7.4]` that all conjugate points are focal points, as to allow use of theorem `[1.8.6]` in regards to conjugate points.

[5] The ability to do this follows from M being complete by theorem `[2.0.6]`.

[6] Since γ is a geodesic, $|\dot\gamma|$ is constant, allowing for a linear reparametrization to make γ unit-speed, which maintains the fact that γ is a geodesic by lemma `[0.2.5]`.

For now, let $f \in \mathfrak{F}(\gamma)$ be chosen so that it vanishes at p and q, as to make $fE_i \in V_\gamma \Omega_{p,q}$. Next, consider, for each i,

$$\begin{aligned}
I_\gamma(fE_i, fE_i) &= \int_0^{\frac{\pi}{\sqrt{C}}} \left(\left\langle (fE_i)_t^\perp, (fE_i)_t^\perp \right\rangle (t) - \langle R(\dot\gamma, fE_i) fE_i, \dot\gamma \rangle (t) \right) dt && \text{by definition of } I_\gamma,\ \varepsilon = 1 \text{ by } M \text{ being Riemannian, } c = |\dot\gamma| = 1 \text{ by assumption} \\
&= \int_0^{\frac{\pi}{\sqrt{C}}} \left((f'(t))^2 \langle E_i, E_i \rangle (t) - (f(t))^2 \langle R(\dot\gamma, E_i) E_i, \dot\gamma \rangle (t) \right) dt && \text{by linearity of } R, \text{ and by parallel transport: } E_i^\perp = E_i \text{ and } (E_i)_t = 0 \\
&= \int_0^{\frac{\pi}{\sqrt{C}}} \left((f'(t))^2 - (f(t))^2 \langle R(\dot\gamma, E_i) E_i, \dot\gamma \rangle (t) \right) dt && \text{by parallel transport: } \langle E_i, E_i \rangle = \langle e_i, e_i \rangle = 1.
\end{aligned}$$

Before summing over $1 \leq i \leq n-1$, as done in the proof of proposition `[1.8.10]`, it should be noted that

$$\sum_{i=1}^{n-1} \langle R(\dot\gamma, E_i) E_i, \dot\gamma \rangle = \operatorname{Ric}(\dot\gamma, \dot\gamma),$$

as noted there. Summing

$$\begin{aligned}
\sum_{i=1}^{n-1} I_\gamma(fE_i, fE_i) &= \int_0^{\frac{\pi}{\sqrt{C}}} \left((n-1)(f'(t))^2 - (f(t))^2 \operatorname{Ric}(\dot\gamma, \dot\gamma)(t) \right) dt && \text{by linearity, and substituting the noted expression} \\
&\leq \int_0^{\frac{\pi}{\sqrt{C}}} \left((n-1)(f'(t))^2 - (n-1) C (f(t))^2 \right) dt && \text{using the bound on Ric, } |\dot\gamma| = 1.
\end{aligned}$$

Thus, letting $f(t) = \sin\left(\sqrt{C} t\right)$, which vanishes at p and q as needed,

$$\begin{aligned}
\sum_{i=1}^{n-1} I_\gamma(fE_i, fE_i) &\leq (n-1) \int_0^{\frac{\pi}{\sqrt{C}}} \left(C \cos^2\left(\sqrt{C} t\right) - C \sin^2\left(\sqrt{C} t\right) \right) dt \\
&= (n-1) C \int_0^{\frac{\pi}{\sqrt{C}}} \cos\left(2\sqrt{C} t\right) dt \\
&= (n-1) C \left[-\frac{\sin\left(2\sqrt{C} t\right)}{2\sqrt{C}} \right]_0^{\frac{\pi}{\sqrt{C}}} \\
&= 0.
\end{aligned}$$

From which, it follows that $I_\gamma(fE_i, fE_i) \leq 0$ for some i, making I_γ not positive definite by definition. Therefore, following from the discussion at the start, any geodesic of length greater than $\frac{\pi}{\sqrt{C}}$ is not minimizing, which forces

$$\sup_{p,q \in M} d(p,q) \leq \frac{\pi}{\sqrt{C}}.$$

■

3 | Lorentzian Geometry

As done in the previous chapter for the Riemannian case, this chapter will seek to utilize results from the Calculus of variations as to obtain results for the Lorentzian case. Though, a slightly different approach needs to be taken than in the Riemannian case. Since the metric of a Lorentzian manifold is indefinite, there is more of a structure on the tangent bundle, leading to a few more notions about a manifold's structure that are not relevant in the more-homogeneous Riemannian case.

As considered right from chapter 0 (cf. definition `[0.1.2]` and the later definition `[1.8.4]`), there is a trichotomy on the tangent space of a Lorentzian manifold gotten from its metric. Particular to a Lorentzian manifold M, there is an innate sense (or at least possible sense) of orientation coming from the metric's establishment of a distinguished timelike "direction" in the tangent space T_pM at each point of $p \in M$. To make this concrete, here are the accompanying definitions. [O'N83, Ch. 5][Hol11, §4]

3.0.1 Definition Let $M \ni p$ be a Lorentzian manifold, and let $\mathcal{T}_p \subset T_pM$ set of timelike vectors in T_pM, i.e. with definition `[0.1.2]`,

$$\mathcal{T}_p = \{x \in T_pM : \langle x, x\rangle < 0\}.$$

For a $p \in M$, define the *timecone* containing $v \in \mathcal{T}_p$ to be

$$C(x) = \{y \in \mathcal{T}_p : \langle x, y\rangle < 0\}.$$

Also, the set vectors in the *opposite timecone* of $v \in \mathcal{T}_p$ is

$$C_-(x) = C(-x) = -C(x) = \{y \in \mathcal{T}_p : \langle x, y\rangle > 0\},$$

where the usage of the negative sign follows from the definition of C.

Note that, since $x^\perp$ is spacelike for $x \in \mathcal{T}_p$, it follows that $C(x)$, and $C_-(x)$, are disjoint, but their union[1] is $\mathcal{T}_p$.

The definitions for the timecone hint at the notion that timelike vectors, which lie in the same timecone, point in the same "direction". Next is a lemma that helps in developing the idea that M can be oriented in some way.

3.0.2 Lemma For a Lorentzian manifold $M \ni p$, $x, y \in \mathcal{T}_p$ are in the same timecone iff $\langle x, y\rangle < 0$.

This shows that, for $x, y \in \mathcal{T}_p$,

$$x \in C(y) \iff y \in C(x) \iff C(x) = C(y).$$

Proof Assume that $x, y, z \in \mathcal{T}_p$ are such that $x \in C(z)$. By linearity, it is possible to express x, y as

$$x = az + v$$
$$y = bz + w,$$

where a, b are constants and $v, w \in z^\perp$. Since $x, y \in \mathcal{T}_p$, it follows that $|a| > |v|$ and $|b| > |w|$. Also, since $x \in C(z)$, $\langle x, z\rangle < 0 \iff a > 0$. With that,

$$\langle x, y\rangle = -ab + \langle v, w\rangle. \qquad [*]$$

[1] If some $y \in \mathcal{T}_p$ is not in the union $C(x) \cup C_-(x)$, then that would require that it is perpendicular to x, forcing it to be spacelike; this a contradiction.

By the Schwarz inequality and the previously mentioned bounds,

$$|\langle v, w\rangle| \leq |v|\,|w| < |a|\,|b|\,. \qquad \texttt{[**]}$$

Now,

$$\begin{aligned}\operatorname{sgn}\langle x, y\rangle &= \operatorname{sgn}\left(-ab + \langle v, w\rangle\right) && \text{by } \texttt{[*]}\\ &= \operatorname{sgn}\left(-ab\right) && \text{by } \texttt{[**]}\\ &= -\operatorname{sgn} b && \text{since } a > 0.\end{aligned}$$

Thus, $\langle x, y\rangle < 0$ iff $b > 0$, which is iff $\langle y, z\rangle = -b < 0$, i.e. $y \in C(z)$, which already contains x. ■

With this result, it follows[2] that on each T_pM there are only two timelike "directions": with respect to some $x \in \mathcal{T}_p$, each element of $\mathcal{T}_p$ is either in the same timecone as x or in the opposite timecone of x. Using this, comes the next definition, and result.

3.0.3 Definition A *time orientation* for a Lorentzian manifold M, is a smooth assignment of timecone in T_pM to each $p \in M$.

If M admits such an assignment, then it is called *time-orientable.*

This in conjunction with the previous lemma yields the following.

3.0.4 Lemma A Lorentzian manifold M time-orientable iff it admits a smooth timelike vector field.

The proof of this comes from a usual partition-of-unity argument for the forward statement, and the use of lemma `[3.0.2]` for the converse.

From now on, when a Lorentzian manifold is mentioned, it will implicitly be time-oriented or orientable, unless otherwise stated. With that, marching onward to the next notions.

3.0.5 Definition Given a time-oriented Lorentzian manifold $M \ni p$ with its timelike vector field X, a vector is called *future-pointing* if it lies in the same timecone as $X|_p$, and it is called *past-pointing* if it lies in the opposite time cone.

3.0.6 Definition For a Lorentzian manifold[3] $M \ni p, q$, define[4] the following *causality relations*:

— $p \ll q$ if there is a future-pointing timelike curve in M from p to q

— $p \gg q$ if there is a past-pointing timelike curve in M from p to q

— $p < q$ if there is a future-pointing causal curve in M from p to q

— $p > q$ if there is a past-pointing causal curve in M from p to q.

Also, $p \leq q$ means $p < q$ or $p = q$, and $p \geq q$ means $p > q$ or $p = q$.

3.0.7 Definition For a subset $U \subset M$, the *chronological future* of U is

$$I^+(U) = \{p \in M : \exists q \in U \text{ such that } q \ll p\}$$

and the *causal future* of U is

$$J^+(U) = \{p \in M : \exists q \in U \text{ such that } q \leq p\}\,.$$

[2]Recall that $\mathcal{T}_p = C(x) \cup C_-(x)$ for some timelike x.
[3]Again, implicitly time-oriented.
[4]When attributing a causal character to a curve, it is to mean that the tangent vectors of the curve have that character, as per the usual meaning.

The corresponding past versions are naturally:

$$I^{-}(U) = \{p \in M : \exists q \in U \text{ such that } q \gg p\}$$

and

$$J^{-}(U) = \{p \in M : \exists q \in U \text{ such that } q \geq p\}.$$

As done in this definition, it is the case that definitions, and results, can be shown for future/past objects and, then, be immediately obtained for past/future objects, respectively, by reversing the time-orientation.

With all that structure established, it is now possible to define a notion of "distance" on a Lorentzian manifold. [O'N83, Ch. 14][Hol11, §4 & §5][Gal07, §2.3]

3.0.8 Definition Let $p, q \in M$ for a Lorentzian manifold M, and let $\Omega_{p<q}$ be the set of future-pointing causal curves from p to q. Then, the *time separation* from p to q is

$$\tau(p,q) = \begin{cases} \sup_{\gamma\in\Omega_{p<q}} L(\gamma) & \Omega_{p<q} \neq \varnothing, \text{ i.e. } q \in J^{+}(p) \\ 0 & \Omega_{p<q} = \varnothing, \text{ i.e. } q \notin J^{+}(p) \end{cases}.$$

This can be seen as being motivated by lemma `[1.8.3]`, just as in the case of the Riemannian distance function from definition `[2.0.1]`.

Also, given $U, V \subset M$, then let

$$\tau(U,V) = \sup_{p\in U, q\in V} \tau(p,q).$$

Note that τ is not symmetric, except in trivial cases, unlike the Riemannian manifold's metric (cf. lemma `[2.0.4]`).

Before investigating this function further, a proposition[5] is needed that discusses the possibility of finding timelike curves near certain null ones.

3.0.9 Proposition For a causal curve $\gamma \in \Omega_{p,q}$ in a Lorentzian manifold M, which is not a null geodesic, there is a timelike curve in $\Omega_{p,q}$ arbitrarily close to γ via a variation.

With that, comes the following about the time-separation function τ.

3.0.10 Lemma For a Lorentzian manifold $M \ni p, q, r$:

(1) $\tau(p,q) > 0$ iff $p \ll q$

(2) if $p \leq q \leq r$, then $\tau(p,q) + \tau(q,r) \leq \tau(p,r)$.

Proof (1) If $\tau(p,q) > 0$, then, by definition, there must be a future-pointing curve $\gamma \in \Omega_{p,q}$ with $L(\gamma) > 0$; as such, γ cannot be a null geodesic. By proposition `[3.0.9]`, this means that there is a nearby future-pointing timelike curve from p to q, i.e. $p \ll q$ by definition.

If $p \ll q$, then, by definition, there is a future-pointing timelike curve γ from p to q, which naturally has positive length; it follows that $\tau(p,q) \geq L(\gamma) > 0$ by definition of τ as a supremum over such timelike curves.

[5] See [O'N83, Ch. 10 proposition 46] for a proof.

(2) Assume that p, q, r are all distinct. By definition[6] of τ as a supremum, for any $\delta > 0$, it is possible to pick future-pointing causal curves α from p to q such that

$$L(\alpha) + \delta \geq \tau(p,q),$$

and β from q to r such that

$$L(\beta) + \delta \geq \tau(q,r).$$

Letting γ be the future-pointing causal curve from p to r gotten by going along α and then β, it follows that

$$\begin{aligned} \tau(p,r) &\geq L(\gamma) \\ &= L(\alpha) + L(\beta) \\ &\geq \tau(p,q) + \tau(q,r) - 2\delta && \text{by the bounds mentioned.} \end{aligned}$$

Thus, since this is for any $\delta > 0$, it follows that

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r).$$

Now, if $p = q$ for example, then there is no causal curve between p and q, making $\tau(p,q) = 0$ and, in turn,

$$\tau(p,r) = \tau(q,r) \implies \tau(p,q) + \tau(q,r) \leq \tau p, r.$$

In the other cases, the inequality holds in a similar fashion. ■

Note that property (2) does not hold for metrics (e.g. as shown in the Riemannian lemma `[2.0.4]`); with this fact, it should not be expected for Lorentzian manifolds that there is a version of Hopf-Rinow (theorem `[2.0.6]`) since that theorem relies on a Riemannian manifold being made into a metric space.

On to more definitions, involving specific subsets of a Lorentzian manifold M.

3.0.11 Definition A subset $U \subset M$ is called *achronal* provided that, for any $p, q \in U$, there does not exist a timelike curve joining them; this means that no timelike curve meets U more than once.

Similarly, a subset $U \subset M$ is called *acausal* if, for any $p, q \in U$, there does not exist a causal curve joining them; this means that no causal curve meets U more than once.

3.0.12 Definition A *Cauchy hypersurface* a subset $S \subset M$ which is met exactly once by every inextendible timelike curve in M.

3.0.13 Definition For an achronal set $U \subset M$, the *future Cauchy development* of U is the set $D^+(U)$ containing all $p \in M$ such that every past-inextendible causal curve through p meets U. Note that this means $U \subset D^+(U)$.

Similarly, there is a *past Cauchy development* $D^-(U)$ which is determined using future-inextendible curves instead.

Also, the *Cauchy development* of U is $D(U) = D^-(U) \cup D^+(U)$.

3.0.14 Definition For an achronal set $U \subset M$, the *future Cauchy horizon* of U is

$$H^+(U) = \overline{D^+}(U) \setminus I^-\left(D^+(U)\right),$$

which is[7] the set of points $p \in \overline{D^+}(U)$ such that $I^+(p) \cap D^+(U) = \varnothing$.

[6] The definition of $\tau(p,q)$ requires that there is a sequence of $\gamma_i \in \Omega_{p,q}$ whose lengths converge to $\tau(p,q)$.

[7] Let H_1 be the set formed by first definition, and let H_2 be the second. Say there is a $p \in H_1$ but $p \notin H_2$; it follows that $I^+(p) \cap D^+(U) \neq \varnothing$, but then p lies in $I^-(q)$ for some $q \in D^+(U)$, contradiction. Now, say that there is a $p \in H_2$ but $p \notin H_1$; since $H_2 \subset \overline{D^+}(U)$, it follows that $p \in I^-\left(D^+(U)\right)$, which makes $p \in I^-(q)$ for some $q \in D^+(U)$, forcing $I^+(p) \cap D^+(U) \neq \varnothing$, contradiction.

Again, similarly, there is a *past Cauchy horizon*

$$H^- (U) = \overline{D^-} (U) \setminus I^+ \left(D^- (U)\right),$$

or the set of point $p \in \overline{D^-} (U)$ such that $I^- (p) \cap D^- (U) = \varnothing$.

3.0.15 Definition A closed achronal set $U \subset M$ is called a *future Cauchy hypersurface* if $H^+ (U) = \varnothing$. It is called a *past Cauchy hypersurface* if $H^- (U) = \varnothing$.

In the Riemannian case (cf. definition `[2.0.2]`), there was the existence of normal neighborhoods that helped in the discussion of locally minimizing geodesics. Here, the analogue is the following.

3.0.16 Definition The *strong causality condition* holds at $p \in M$ if, for any neighborhood $U \ni p$, there is a neighborhood $V \subset U$ of p such that every smooth causal curve with endpoints in V stays within U. If it holds on a subset of M, then it holds at each point of that subset.

Enough is at disposal now to approach a few more results.

3.0.17 Lemma Suppose that the strong causality condition holds on a compact $U \subset M$. Let $\{\gamma_i : [0, l] \longrightarrow U\}$ be a sequence of future-pointing smooth causal curves such that

$$\lim_{i\to\infty} \gamma_i (0) = p \quad \text{and} \qquad \lim_{i\to\infty} \gamma_i (0) = q \neq p;$$

note that $p, q \in U$ by compactness. Then, there is a future-point broken causal geodesic from p to q, and there is a subsequence $\{\gamma_{i_k}\}$ of $\{\gamma_i\}$ such that

$$\lim_{k\to\infty} L (\gamma_{i_k}) \leq L (\gamma) .$$

This lemma[8], though not proved here, helps in the proof of the following.

3.0.18 Proposition Let $p, q \in M$ such that $p < q$. If the set $J^+ (p) \cap J^- (q)$ is compact, and the strong causality condition holds on it, then there is a causal geodesic from p to q achieving the length $\tau (p, q)$.

Proof Let $\{\gamma_i\}$ be a sequence of future-pointing smooth causal curves from p to q whose lengths converge to $\tau (p, q)$. By definition of the causal future and past, it follows that each $\gamma_i \in J^+ (p) \cap J^- (q)$. As such, by lemma `[3.0.17]`, there is a broken causal geodesic γ from p to q, with $L (\gamma) = \tau (p, q)$. However, by corollary `[1.1.3]`, γ can only be an extremum of L iff it is an unbroken geodesic. Thus, since τ is length of a extremal curve, γ only has trivial breaks, making it the geodesic for which was sought. ■

Of course, this is a nice result to have on a Lorentzian manifold. Furthermore, note that the result of assuming those properties of M, it similar to the result of assuming completeness of a Riemannian manifold (cf. theorem `[2.0.6]` with lemma `[2.0.5]`). With all this significance, there is motivation for the following definition. [Hol11, §5][O'N83, Ch. 10 & 14][HE75, Ch. 8]

3.0.19 Definition A Lorentzian manifold M is called *globally hyperbolic* if the strong causality condition holds on M, and, for any $p, q \in M$, $J^+ (p) \cap J^- (q)$ is compact.

For a subset $U \subset M$, U is called globally hyperbolic if the strong causality condition holds on U, and, for any $p, q \in U$ such that $p < q$, $J^+ (p) \cap J^- (q)$ is compact and is contained in U.

[8] See [O'N83, Ch. 14 lemma 14] for proof.

This quality also has an implication on the continuity of the time-separation function that is stated in the following lemma[9].

3.0.20 Lemma Let $U \subset M$ be a globally hyperbolic open set, then the time-separation function τ of M is continuous on $U \times U$.

Given that, now sights will be a set on utilizing proposition `[1.6.3]`; to do this, the following results[10] are needed.

3.0.21 Lemma If S is a closed achronal spacelike hypersurface of M, then $D(S)$ is a globally hyperbolic open set.

3.0.22 Lemma Let $U \subset M$ be achronal. If[11] $p \in D(U) \setminus \partial D(U) \setminus I^-(U)$, then $J^-(p) \cap D^+(U)$ is compact.

Now, on to the theorem.

3.0.23 Theorem Let S be a closed achronal spacelike hypersurface of M. Then, for $p \in D^+(S)$, there is a geodesic $\gamma \in \Omega_{S,p}$ with length $\tau(S,p)$, which is normal to S and has no focal points of S before p.

Proof First, note that by the assumptions on $S \subset M$, it follows from lemma `[3.0.21]` that $D(S)$ is a globally hyperbolic open set. As such,

$$\partial D(S) = \varnothing$$

by it being open, and

$$I^-(S) \cap D^+(S) = \varnothing$$

by it being achronal. From this, for any $p \in D^+(S)$, it follows that

$$p \in D^-(S) \cup D^+(S) = D(S) = D(S) \setminus \partial D(S) \setminus I^-(S),$$

which implies that $J^-(p) \cap D^+(S)$ is compact by lemma `[3.0.22]` since S is achronal. Now, since $S \subset D^+(S)$ by definition of $D^+(S)$, it follows that

$$J^-(p) \cap D^+(S) \cap S = J^-(p) \cap S,$$

which is closed since S is closed and $J^-(p) \cap D^+(S)$ is compact.

Since $D(S)$ is globally hyperbolic and open, it follows by lemma `[3.0.20]` that the time-separation function τ is continuous on $D(S) \times D(S)$. With that, there is continuity of the map $q \longmapsto \tau(q,p)$ on the compact set $J^-(p) \cap S \subset D(S)$, meaning that it achieves a maximum; say it is at p_0. Moreover, again since $D(S)$ is globally hyperbolic, there is a causal geodesic γ that achieves this maximum by proposition `[3.0.18]`. Now, there are two cases to consider:

— if $p \in S$, then, since S is achronal, any curve joining some $q \in S$ to p is not timelike; this means that $\tau(S,p) = 0$ by definition of τ. As such, it is possible to take the the constant curve at p, which is obviously a geodesic that is normal to S and has no focal points of S.

— if $p \notin S$, then there is a causal curve connecting p_0 to p since $p \in D^+(S)$; by proposition `[3.0.9]`, it follows that there must be a timelike curve connecting them, i.e. $p_0 \ll p$. In turn, by lemma `[3.0.10]`, this means that $\tau(p_0,p) > 0$, forcing γ timelike because it is maximizing. Since γ is timelike, it is possible to use proposition `[1.6.3]` to show that γ is normal to S, and to use theorem `[1.8.6]` to show that there are no focal points of S along γ before p, since γ is maximizing.

[9] See [O'N83, Ch. 14 lemma 21] for proof.

[10] See [O'N83, Ch. 14 lemma 42 & 43] for a proof of lemma `[3.0.21]`, and [O'N83, Ch. 14 lemma 40] for a proof of lemma `[3.0.22]`.

[11] Here, ∂ denotes "the boundary of".

Therefore, γ is the geodesic for which was sought. ■

Now, here final new pieces[12] needed for the main theorem are the following.

3.0.24 Definition A subset $S \subset M$ is called a *topological hypersurface* of M if, for each $p \in U$, there is a neighborhood U of p in M such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$ with $\varphi(S \cap U) = \varphi(U) \cap \Sigma$, where Σ is some hyperplane of $\mathbb{R}^n$.

3.0.25 Proposition Let $S \subset M$ be a closed acausal topological hypersurface. Then,

$$H^+(S) = I^+(S) \cap \partial D^+(S) = \overline{D^+}(S) \setminus D^+(S).$$

Moreover, $S \cap H^+(S) = \varnothing$.

3.0.26 Lemma A future Cauchy hypersurface is a closed topological hypersurface.

Nearly there now, it is only needed to recall an earlier definition as to simplify discussion in the statement of the theorem.

3.0.27 Definition Recall definition `[1.8.9]` as to consider the following specific instance of it.

For a spacelike hypersurface $S \subset M^n$, the *future convergence* of S is simply the value of the convergence k from definition `[1.8.9]` restricted to the future-pointing timelike vector field normal to S, which is guaranteed by S being spacelike.

With that, it is possible finally moving to the main result of this chapter: the Hawking singularity theorem. It has a similar bound to the Riemannian theorem `[2.0.7]` and even uses the same results from the Calculus of variation, but it yields different conclusions. Also, note the more striking resemblance with the semi-Riemannian proposition `[1.8.10]`, which will be used in this theorem's proof.

3.0.28 Theorem Let $M \ni p$ be a Lorentzian manifold such that, for every $x \in \mathcal{T}_p$, $\operatorname{Ric}(x,x) \geq 0$. Also, let S be a spacelike future Cauchy hypersurface with future convergence $k\left(X|_p\right) \geq C > 0$ for each $p \in S$, where X is the future-pointing timelike unit-vector field on S. Then, every future-pointing timelike curve starting in S has a length of at most $\frac{1}{C}$.

Proof To start off, note that, since S is a future Cauchy hypersurface, it follows that $H^+(S) = \varnothing$ by definition.

Now, it will be shown that $I^+(S) \subset D^+(S)$. Say that this is not the case, then there is a future-pointing timelike curve from S that ventures to a point outside of $D^+(S)$; this means that the curve would go through $\partial D^+(S)$, making $\partial D^+(S) \neq \varnothing$. Moreover, by this consideration, this point would also be in $I^+(S)$. Using lemma `[3.0.26]` and, then, proposition `[3.0.25]` since S is acausal by being a spacelike Cauchy hypersurface, it follows that

$$\varnothing \neq I^+(S) \cap \partial D^+(S) = H^+(S),$$

contradiction.

With that, considerations can be done in $D^+(S)$. By the proof of theorem `[3.0.23]`, it follows that considering $p \in S \subset D^+(S)$ is trivial: the geodesic is constant, trivially meeting the bound on its length. As such, let $p \in D^+(S) \setminus S$; using theorem `[3.0.23]` again, it follows that there is a timelike geodesic from S to p such that $L(\gamma) = \tau(S,p)$; it is also normal to S and has no focal points

[12] See [O'N83, Ch. 14 proposition 53] for a proof of proposition `[3.0.25]`, and see [O'N83, Ch. 14 exercise 9] for the statement of lemma `[3.0.26]`.

of S before p. Moreover, after recalling the definition of cospacelike (cf. definition `[1.8.4]`), it follows from proposition `[1.8.10]` that there is a focal point before p if $L(\gamma) > \frac{1}{C}$.

To put these conclusions together, recall from definition `[1.6.5]` that the index form $I_\gamma(X, X) = L''_\chi(0)$ for $X \in V_\gamma \Omega_{S,p}$ with corresponding variation χ, and recall theorem `[1.8.6]`; in particular, the timelike geodesic γ from S to p cannot be maximizing if there are focal points of S along γ before p. It follows that from this, and the previous conclusions, that γ cannot be maximizing if $L(\gamma) > \frac{1}{C}$. In turn, by the definition of τ as a supremum, it must be that

$$D^+(S) \subset \left\{ q \in S : \tau(q, p) \leq \frac{1}{C} \right\}.$$

Therefore, by the inclusion $I^+(S) \subset D^+(S)$, it follows that all future-pointing timelike curves from S have length of at most $\frac{1}{C}$. ■

Looking more at this result, it reaches a significant conclusion: future-pointing timelike geodesics cannot be extended beyond a certain length. It is in this sense, that a Lorentzian manifold meeting the conditions of the theorem is timelike-incomplete. This particularly draws a contrast to the theorem proved in the Riemannian case: theorem `[2.0.7]` has similar bounds on the entirety of the Riemannian M, yet completeness was essential in reaching its conclusion. As remarked earlier (cf. definition `[3.0.8]` & lemma `[3.0.10]`), the fact that the metric on a Lorentzian manifold is indefinite can be looked at as bringing rise to such highlighted differences in these semi-Riemannian manifolds' structure and behavior.

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