

# Math Review

ECON 3070 - Intermediate Microeconomic Theory

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# Math Review Topics

- Limits
- Derivative Definition
- Derivative Rules
- Partial Derivatives

# Limits

The limit of  $f(x)$  as  $x$  approaches  $a$  is written as:

$$\lim_{x \rightarrow a} f(x)$$

- The behavior of the function as its input approaches  $a$
- In certain cases, can be evaluated by plugging in  $x = a$
- $\lim_{x \rightarrow a} c = c$  for a constant  $c$
- $\lim_{x \rightarrow a} x = a$  for a constant  $c$

# Limits

The limit from the left and right are

→  $\lim_{x \rightarrow a^-} f(x) = A$  "as  $x$  approaches  $a$  from the left ( $-\infty$ )"

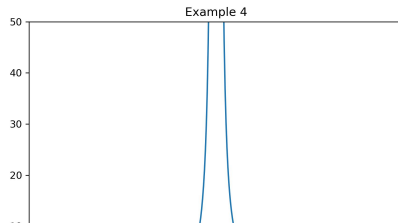
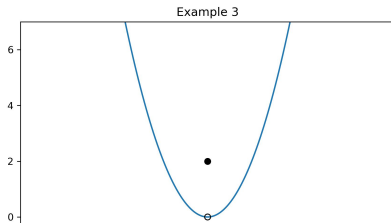
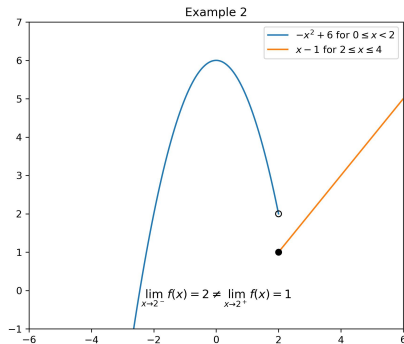
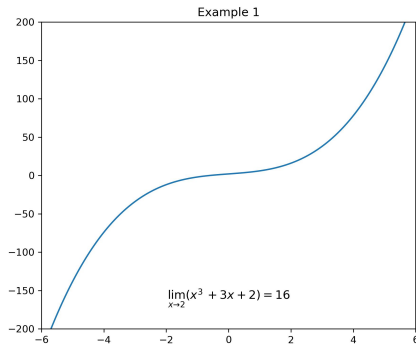
→  $\lim_{x \rightarrow a^+} f(x) = A$  "as  $x$  approaches  $a$  from the right ( $\infty$ )"

If  $\lim_{x \rightarrow a^-} f(x) = A = \lim_{x \rightarrow a^+} f(x)$  then the limit exists and we write

$$\lim_{x \rightarrow a} f(x) = A$$

- Note: the limit does not exist if  $A = \pm\infty$ , even if the left and right limit both equal  $\pm\infty$

# Limits



# Limit Rules

$$\lim_{x \rightarrow a} f(x) = L \quad \lim_{x \rightarrow a} g(x) = M$$

- **Constant Multiple Rule:**  $\lim_{x \rightarrow a} [af(x)] = aL$
- **Sum/Difference Rule:**  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M$
- **Product Rule:**  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L \cdot M$
- **Quotient Rule:**  $\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{L}{M}, \quad M \neq 0$
- **Power Rule:**  $\lim_{x \rightarrow a} [(f(x))^n] = L^n, \quad n > 0$

# Definition of a Derivative

The derivative of  $f$  is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists, then we say  $f$  is differentiable at  $x$ .

# Derivative Notation

All of these mean “take the derivative of  $f$  with respect to  $x$ ”:

→  $f'(x)$

→  $\frac{df(x)}{dx}$

→  $\frac{d}{dx}f(x)$

→  $\frac{dy}{dx}$  if  $y = f(x)$

→  $y'$  (sometimes)

→  $\dot{y}$  (particularly in Macro)



# Derivative Rules

Let  $f(x)$  and  $g(x)$  be differentiable functions:

- **Derivative of a Constant:**  $f(x) = c, f'(x) = 0$
- **“Power Rule”:**  $f(x) = x^n, f'(x) = nx^{n-1}$
- **Sum/Difference Rule:**  $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- **Product Rule:**  $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- **Quotient Rule:**  $\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- **Chain Rule:**  $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

# Derivatives Practice Problems

$$f(x) = 2x$$

# Derivatives Practice Problems

$$f(x) = 3x^2$$

# Derivative Rules for Exponents and Natural Log

$$\rightarrow f(x) = e^x, f'(x) = e^x$$

$$\rightarrow f(x) = \ln(x), f'(x) = \frac{1}{x}$$

Useful review of the properties of **exponents** and **logs**.

# Derivatives Practice Problems

$$f(x) = \ln(e^x)$$

# Derivatives Practice Problems

$$f(x) = x^2 \ln(x)$$

# Derivatives Practice Problems

$$f(x) = e^{2x}$$

# Higher-order Derivatives

The second-order derivative is

$$f''(x) = \frac{d}{dx} f'(x)$$

We could keep differentiating as long as the last derivative is differentiable. Notation:

- $f''(x)$
- $\frac{d^2 f(x)}{dx^2}$
- $\frac{d^2}{dx^2} f(x)$
- $\frac{d^2 y}{dx^2}$  if  $y = f(x)$
- $y''$  (sometimes)



# Partial Derivatives

The function  $z = f(x, y)$  takes two inputs and outputs a third. We can evaluate what happens to that output when we change  $x$  or  $y$  (but not both).

- $\frac{\partial z}{\partial x} =$  “the derivative of  $z$  with respect to  $x$ ”
  - The slope of  $f$  in the cardinal direction of  $x$  (e.g. “north-south”)
  - The tangent of  $f$  in the  $x$  direction at a point  $(x_0, y_0)$
- $\frac{\partial z}{\partial y} =$  “the derivative of  $z$  with respect to  $y$ ”
  - The slope of  $f$  in the cardinal direction of  $y$  (e.g. “east-west”)
  - The tangent of  $f$  in the  $y$  direction at a point  $(x_0, y_0)$

# Partial Derivatives

All of these mean “take the partial derivative of  $f$  with respect to  $x$ ”:

- $f_x(x, y)$
- $\frac{\partial f(x, y)}{\partial x}$
- $\frac{\partial}{\partial x} f(x, y)$
- $\frac{\partial z}{\partial x}$  if  $z = f(x, y)$

# Partial Derivatives

Interpret  $\frac{\partial z}{\partial x}$  as “what happens to  $z$  if I change  $x$ , holding  $y$  constant.” Likewise for  $\frac{\partial z}{\partial y}$ .

Suppose  $z = x + y + xy$ , find  $\frac{\partial z}{\partial x}$ .

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x + y + xy] = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y + \frac{\partial}{\partial x}xy = 1 + 0 + y \cdot 1 = 1 + y$$

# Partial Derivatives Practice Problems

$$f(x, y) = x^2 + y^2 + x^2y^2$$

# Partial Derivatives Practice Problems

$$f(x, y) = x^2y + \ln(x)y^3$$

# Properties of Exponents

Let  $a$  and  $b$  be real numbers and  $m$  and  $n$  be integers. Then the following properties of exponents hold, provided that all of the expressions appearing in a particular equation are defined:

$$\rightarrow a^m a^n = a^{m+n}$$

$$\rightarrow (a^m)^n = a^{mn}$$

$$\rightarrow (ab)^m = a^m b^m$$

$$\rightarrow \frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

$$\rightarrow \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

$$\rightarrow a^{-m} = \frac{1}{a^m}, a \neq 0$$

$$\rightarrow a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$\rightarrow a^0 = 1, a \neq 0$$

$$\rightarrow a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

# Properties of Natural Log

Note that logs are only defined for positive values of  $x$ :

$$\rightarrow \ln(xy) = \ln(x) + \ln(y)$$

$$\rightarrow \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\rightarrow \ln(x^y) = y \cdot \ln(x)$$

$$\rightarrow \ln(e^x) = x$$

$$\rightarrow e^{\ln(x)} = x$$

$$\rightarrow \ln(e) = 1$$

$$\rightarrow \ln(1) = 0$$

$$\rightarrow \ln(1) = \textit{undefined}$$