

Lecture 7 - Cost Minimization

ECON 3070 - Intermediate Microeconomic Theory

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Overview

In the previous lecture,

- We looked at production functions, and how they can be used to represent the relationship between quantity of inputs used, and quantity of output produced.
- We also considered how the level of output varies when we vary the quantity of one or all of the inputs.

Overview

In this chapter, we will discuss how a firm might choose the mix of inputs that it uses to produce a given level of output, given the prices of those inputs.

- We will also analyze how changes in the prices of those inputs might impact the firm's decision.
- Finally, we will consider how the firm's problem varies between the short run and the long run.

Cost Minimization

Let's assume for the moment that a firm has decided to produce Q units of output. (We will consider in future lectures how they might choose Q .)

Then a firm which seeks to generate as much profits as possible will be concerned with producing that output at the lowest cost.

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Then a firm which seeks to generate as much profits as possible will be concerned with producing that output at the lowest cost.

- The problem of finding the lowest-cost mix of inputs that produces a given level of output is the **cost-minimization problem**.

Cost Minimization

For now, let's assume that the firm uses only labor (L), and capital (K), in production (just like consumers only choosing between two goods)

- Each input has a price. The price of a unit of labor is called the wage rate (w). The price of a unit of capital services is the rental rate of capital (r).
- These prices will affect the amount of labor and capital that a firm chooses to use in production.

Cost Minimization

Note that costs may be implicit or explicit, or both.

Explicit costs - Costs that involve outlays of money.

- Examples: Wages, rental fees for capital

Implicit costs - Costs that *do not* involve outlays of money

- Examples: Forgone earnings by owner, forgone rental fees on office building

Opportunity Cost

Opportunity cost is the value of a sacrificed opportunity.

- Example: You quit your job as a corporate lawyer and start a plumbing business.
- If your salary before you quit was \$150K per year, then the opportunity cost of the plumbing business is \$150K.

Economic & Accounting Costs

Economic costs are all relevant explicit and implicit costs.

- If you use your office building instead of renting it, the foregone rent is part of your economic cost.
- 'Hidden' (implicit) costs are often why people make decisions, so look out for these everywhere!

Accounting costs are simply your explicit costs (wages, rent payments, etc...)

Isocost Lines

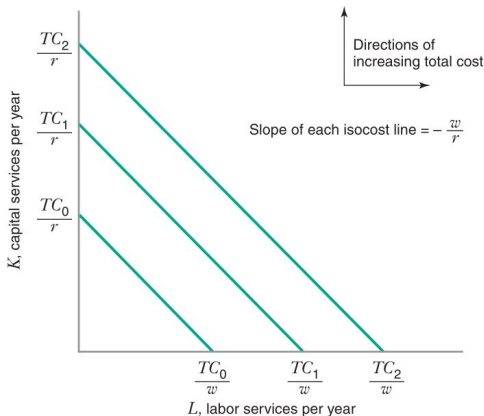
Before we look at the cost minimization problem mathematically, let's think about it graphically.

For this, we will use isocost lines. **Isocost lines** represent combinations of labor and capital that all have the same *total cost* (TC).

- Total cost is given by $TC = wL + rK$ (similar to the budget line for a consumer).

Isocost Lines

Below is a set of isocost lines for a firm, where each line represents a different total cost (TC). Note that as we move northeast in the graph, the isocost lines represent higher and higher total cost levels.



Solving the Cost Minimization Problem

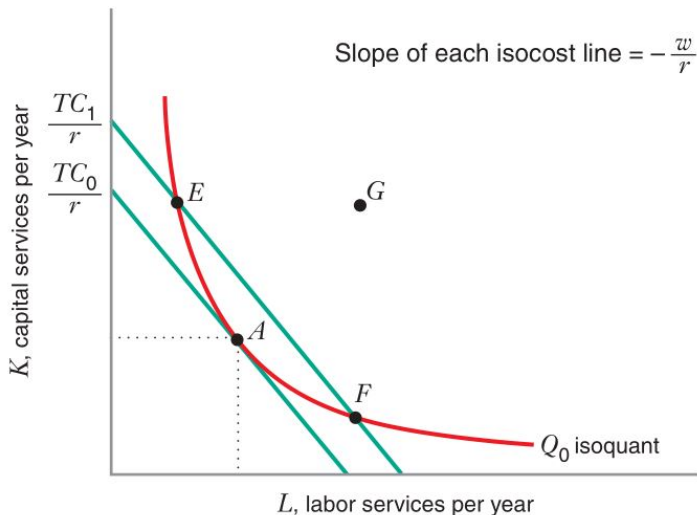
In general, the cost-minimizing choice of inputs will be the point where the isoquant is just tangent to an isocost line. Consider why that is...

Solving the Cost Minimization Problem

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- If the firm chooses a point where the isocost and isoquant lines intersect, but are not tangent, then the firm could, by using more of one input, and less of the other, reduce their cost while still being on that isoquant.
- On the other hand, if the firm chooses a point where the isocost and isoquant lines do not touch, then the firm can't feasibly produce the required number of units.

Solving the Cost Minimization Problem



The Optimality Condition

Just like in the consumer choice setting, the above graph gives us an optimality condition.

$$\frac{MP_K}{r} = \frac{MP_L}{w}$$

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$$\frac{MP_K}{r} = \frac{MP_L}{w}$$

- The **optimality condition** tells us that the produce minimizes cost when the marginal product per dollar on both inputs is equal.
- Think about what would happen if it weren't equal for the two inputs.

The Optimality Condition

If you were producing the necessary quantity (your constraint!) and

$$\frac{MP_K}{r} > \frac{MP_L}{w},$$

how could you produce the same quantity and lower costs?

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The Optimality Condition

Let's double-check that the optimality condition is correct using the Method of Lagrange. Our cost minimization problem can be formally stated as

$$\min_{(K,L)} rK + wL \quad \text{subject to} \quad \bar{Q} = Q(K, L)$$

Our Lagrangian is

$$\mathcal{L}(L, K, \lambda) = rK + wL + \lambda(\bar{Q} - Q(K, L))$$

The Optimality Condition

The first order conditions for an optimum are:

$$\frac{\delta \mathcal{L}}{\delta K} = 0 \Rightarrow r = \lambda(MP_K(K, L))$$

$$\frac{\delta \mathcal{L}}{\delta L} = 0 \Rightarrow w = \lambda(MP_L(K, L))$$

$$\frac{\delta \mathcal{L}}{\delta \lambda} = 0 \Rightarrow \bar{Q} - Q(K, L) = 0$$

If we solve the first two equations for λ , and set them equal to each other, we find

$$\frac{MP_K}{r} = \frac{MP_L}{w}$$

Solving for the Optimal Input Combination

Now we have a system of two equations:

$$\frac{MP_K}{r} = \frac{MP_L}{w} \quad \text{and} \quad \bar{Q} = Q(K, L)$$

We can solve this system to find the cost-minimizing input combination of L and K .

Solving for the Optimal Input Combination

Let's look at an example. Suppose a firm wishes to produce 1,000 units of output, and that the prices of labor and capital are \$5 and \$20, respectively. If the firm has the production function

$$Q = 50\sqrt{LK}$$

Then,

$$MP_L = 25\sqrt{\frac{K}{L}} \quad \text{and} \quad MP_K = 25\sqrt{\frac{L}{K}}$$

Solving for the Optimal Input Combination

Our optimality condition is:

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$$\frac{25\sqrt{K/L}}{5} = \frac{25\sqrt{L/K}}{20} \implies 4K = L$$

Our constraint of producing 1,000 units is:

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Our constraint of producing 1,000 units is:

$$1000 = 50\sqrt{LK}$$

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$$1000 = 50\sqrt{LK} \implies 400 = LK$$

If we plug in $L = 4K$, then do some algebra

$$K^* = 10 \quad \text{and} \quad L^* = 40$$

Solving for the Optimal Input Combination

Suppose instead we just wanted to find K^* and L^* as functions of \bar{Q} , w and r . Then our optimality condition is:

$$\frac{25\sqrt{K/L}}{25\sqrt{L/K}} = \frac{w}{r} \quad \text{or} \quad \frac{K}{L} = \frac{w}{r}$$

If we rearrange this equation, then:

$$K = \frac{w}{r}L$$

Solving for the Optimal Input Combination

Combining this with our output quota:

$$\begin{aligned}\bar{Q} &= 50\sqrt{LK} \\ &= 50\sqrt{\frac{w}{r}L^2} \\ &= 50\sqrt{\frac{w}{r}}L\end{aligned}$$

Solving for the Optimal Input Combination

If we solve the previous equation for L:

$$L^*(w, r, \bar{Q}) = \frac{\bar{Q}}{50} \sqrt{\frac{r}{w}}$$

Solving for the Optimal Input Combination

If we solve the previous equation for L:

$$L^*(w, r, \bar{Q}) = \frac{\bar{Q}}{50} \sqrt{\frac{r}{w}}$$

Then

$$K^*(w, r, \bar{Q}) = \frac{\bar{Q}}{50} \sqrt{\frac{w}{r}}$$

These are the **conditional demand equations** for labor and capital, respectively.

The Total Cost Function

Suppose that, now that we have the conditional demand equations, and we want to find the total cost function.

The Total Cost Function

Suppose that, now that we have the conditional demand equations, and we want to find the total cost function.

Remember that the total cost function is $TC = wL + rK$. We can then plug in our functions for L^* and K^* .

$$\begin{aligned} TC &= w * \frac{\bar{Q}}{50} \sqrt{\frac{r}{w}} + r * \frac{\bar{Q}}{50} \sqrt{\frac{w}{r}} \\ &= \frac{\bar{Q}}{50} \sqrt{w^2 * \frac{r}{w}} + \frac{\bar{Q}}{50} \sqrt{r^2 * \frac{w}{r}} \\ &= \frac{\bar{Q}}{50} \sqrt{rw} + \frac{\bar{Q}}{50} \sqrt{rw} \\ &= \frac{\bar{Q}}{25} \sqrt{rw} \end{aligned}$$

Cost Minimization with more than 2 Inputs

As was the case with the consumer's optimality condition, the cost-minimizing producer's optimality condition can be generalized to more than 2 inputs:

$$\frac{MP_{x_1}}{P_{x_1}} = \frac{MP_{x_2}}{P_{x_2}} = \dots = \frac{MP_{x_N}}{P_{x_N}}$$

That is, the marginal product per dollar of each output must be equal at the optimal input bundle.

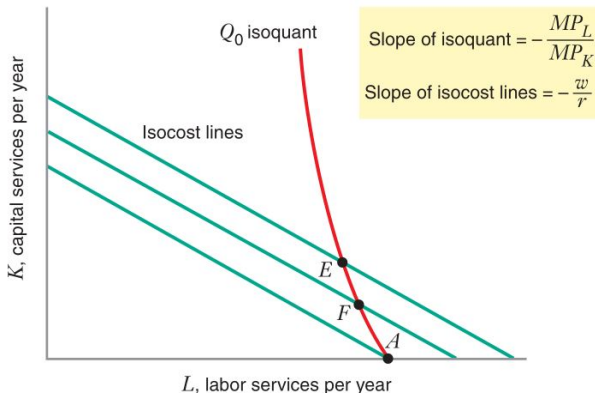
Corner Solutions

Previously, we considered only cases where there existed a tangency between the isoquant and isocost curves.

- Sometimes, there is no tangency. So how do we find the optimal bundle?
- On the next slide, we'll see a graph where the isocost lines are always flatter than the isoquant.

Corner Solutions

Here, the cost minimizing input bundle is one in which the firm uses only labor, and no capital, to produce the output good.



Corner Solutions

In these cases, the optimality condition won't hold

- Marginal product per dollar for one of the inputs will be strictly greater than the marginal product per dollar for the other.
- And cost is minimized by using all of that input.

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- Marginal product per dollar for one of the inputs will be strictly greater than the marginal product per dollar for the other.
- And cost is minimized by using all of that input.

This is just like perfect substitutes in consumer choice!

Try It Yourself

Suppose that for a firm that produces \bar{Q} t-shirts, the cost of an hour of labor is \$5, and the cost of an hour of machine time is \$10. If at the current mix of labor and capital, a worker's output per hour is 15 t-shirts, and a machine's output is 25 t-shirts, which of the following is true?

- A) The firm should use more labor and less capital in order to reduce the cost of producing \bar{Q} t-shirts.
- B) The firm should use more capital and less labor in order to reduce the cost of producing \bar{Q} t-shirts.
- C) The firm is already producing \bar{Q} t-shirts at the lowest cost. It's impossible to tell without knowing what \bar{Q} is.

Comparative Statics Analysis

Now that we have found the optimality condition for this problem, let's consider how the solution changes as our input prices and output quantity change.

Suppose that w increases, with r held constant. Then the optimal bundle becomes more expensive, and our optimality condition no longer holds. Now,

$$\frac{MP_L}{w} < \frac{MP_K}{r}$$

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- This means the firm gets more output per dollar from additional capital than from additional labor.
- And should use more capital and less labor.

Comparative Statics Analysis

Let's think about it another way. If the wage rate increases, then the slope of the isocost lines, $-\frac{w}{r}$, gets steeper.

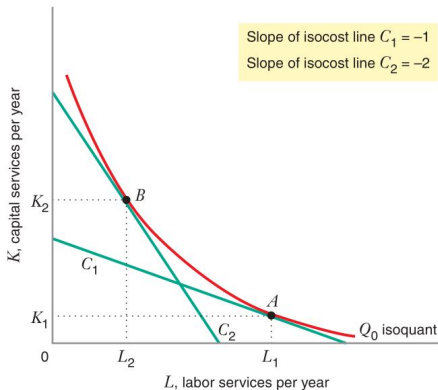
Comparative Statics Analysis

Let's think about it another way. If the wage rate increases, then the slope of the isocost lines, $-\frac{w}{r}$, gets steeper.

- The new point of tangency between the isocost and isoquant lines must be higher up on the isoquant.
- This implies that the firm will use more capital and less labor.

Comparative Statics Analysis

This diagram illustrates how, as the price of labor increases, the firm's optimal input combination is one with relatively more capital and less labor.



Comparative Statics Analysis

Now suppose the output quantity, Q , increases.

- Clearly, for the isoquant and isocost lines to remain tangent, the firm must move to a higher isocost (it costs more money to produce more output!)
- If we find the optimal bundle for a variety of output levels, we can trace what is called an **expansion path**

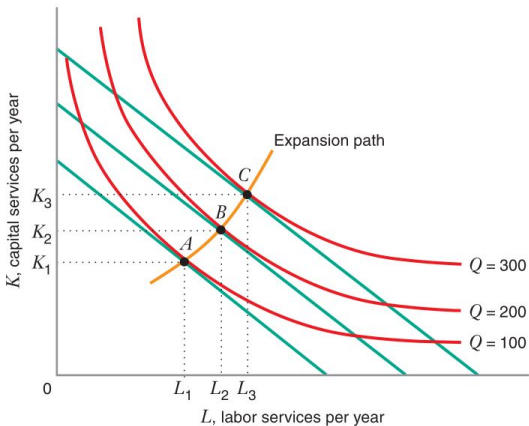
Normal and Inferior inputs

1. If as the output quantity increases, demand for the input increases we say that the input is a **normal input**
2. If as the output quantity increases, demand for the input decreases we say that the input is an **inferior input**

Note the similarities with quantity and income

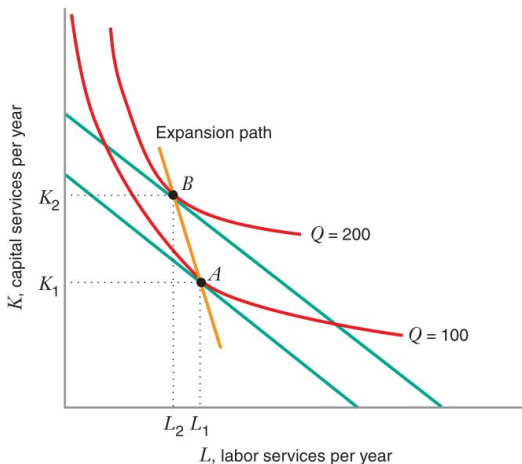
Comparative Statics Analysis

Below is an expansion path where both inputs are normal inputs (as output increases, demand for both inputs increases)



Comparative Statics Analysis

In this diagram, demand for labor decreases as quantity of output increases, implying that labor is an inferior input.



Comparative Statics

Suppose we want to do comparative statics numerically. For example, your boss asks you if they should buy more machines or hire more workers for their expanding business

- We can easily just solve for our conditional input demand functions, and plug in our old and new w , r , and \bar{Q} .
- Or, we can see how the functions vary as we change an input using partial derivatives.

Comparative Statics and Partial Derivatives

From above, we had

$$L^*(w, r, \bar{Q}) = \frac{\bar{Q}}{50} \sqrt{\frac{r}{w}}$$

Then

$$\frac{\partial L^*(w, r, \bar{Q})}{\partial w} = -\frac{\bar{Q}}{25} \sqrt{\frac{r}{w^3}}$$

Try It Yourself

Which of the following conditional factor demand functions represents demand for an *inferior* input?

A) $L(w, r, Q) = 10 \frac{w}{rQ}$

B) $L(w, r, Q) = 10(w + rQ)$

C) $L(w, r, Q) = 10 \frac{wQ}{r}$

D) $L(w, r, Q) = 10 \frac{Q}{wr}$

Price Elasticity of Input Demand

In chapter 2, we learned how to describe the sensitivity of demand for a good to a change in the price. The same principle can be applied with input demand.

The **price elasticity of demand for labor** is given by

$$\epsilon_{L,w} = \frac{\frac{\Delta L}{L} \times 100\%}{\frac{\Delta w}{w} \times 100\%} = \frac{\Delta L}{\Delta w} \frac{w}{L}$$

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The **price elasticity of demand for capital** is given by

$$\epsilon_{K,r} = \frac{\frac{\Delta K}{K} \times 100\%}{\frac{\Delta r}{r} \times 100\%} = \frac{\Delta K}{\Delta r} \frac{r}{K}$$

Price Elasticity of Input Demand

Remember that an elasticity measures the percentage change in demand for an input that results from a 1% change in some price.

- That means that if labor has a price elasticity of 3, then if the wage increases by 1%, demand for labor will decrease by 3%.

Price Elasticity of Input Demand

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- That means that if labor has a price elasticity of 3, then if the wage increases by 1%, demand for labor will decrease by 3%.

If we have input demand functions, then our price elasticity can be calculated using the derivative:

$$\epsilon_{L,w} = \frac{\Delta L}{\Delta w} \frac{w}{L} = \frac{\partial L}{\partial w} \frac{w}{L}$$
$$\epsilon_{K,r} = \frac{\Delta K}{\Delta r} \frac{r}{K} = \frac{\partial K}{\partial r} \frac{r}{K}$$

Price Elasticity of Input Demand

Price elasticity of input demand depends, in part, on the elasticity of substitution.

Suppose that a firm can more easily substitute capital for labor.

- Then if the price of labor increases slightly, they will use more capital and less labor.

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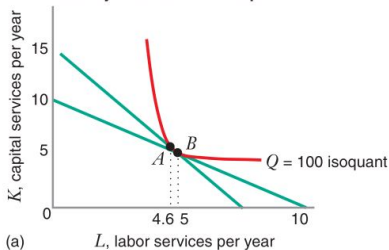
- Then if the price of labor increases slightly, they will use more capital and less labor.

On the other hand, if capital and labor are complements for a firm...

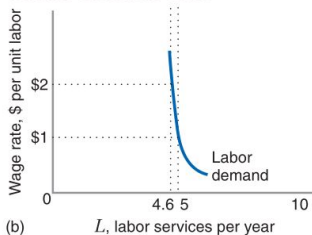
- Their mix of capital and labor may not change much in response to a change in the wage.

Price Elasticity of Input Demand

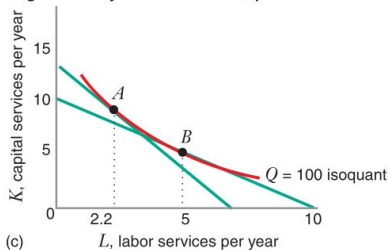
Low elasticity of substitution implies ...



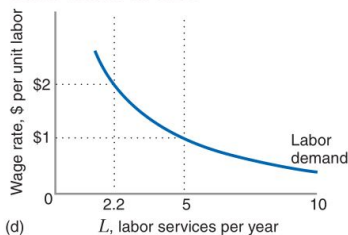
inelastic demand for labor.



High elasticity of substitution implies ...



elastic demand for labor.



Price Elasticity of Input Demand

These elasticities are important for policy decisions. For example, what will the impact on labor demand be if the government increases the minimum wage?

- If the wage elasticity of labor demand is low, then firms will still hire almost as many workers as before.
- But if the wage elasticity of labor demand is high, they may substitute to capital instead (like self-checkout machines).

Try It Yourself

Suppose that the price of low-skill labor increases as a result of an increase in the federal minimum wage. Assume that firms cannot adjust their level of output. For which type of firm will total cost be impacted the most?

- A) A retail store such as Target, for whom cashiers and self-checkout stations are substitutes.
- B) A graphic design firm, for whom designers and computers are complements.

Fixed and Variable Costs

A firm's costs can be divided into two types:

1. **Fixed costs** refer to costs that do not vary with the level of output.
 - For example, a nail salon might have to pay rent every month, regardless of whether they have no customers, or 1,000 customers.
2. **Variable costs** refer to costs that depend on the level of output.
 - For example, a bakery's quantity of output depends on how many hours their bakers work.

Fixed and Variable Costs

The total cost function is a sum of the two costs:

1. **Total variable cost** is made up of all of the firm's expenditures on variable inputs, such as labor and materials.
2. **Total fixed cost** is made up of all expenditures that do not vary with output.

Sunk Costs

When it comes to fixed costs, some can be avoided if the firm chooses not to produce (e.g. lighting for an office building).

But some cannot be recovered even if a firm shuts down (e.g. license fees or a very specialized piece of equipment).

- Fixed costs that cannot be recovered or avoided are known as **sunk costs**.
- Fixed costs that can be recovered, or are avoidable in the event that the firm does not produce, are known as **nonsunk costs**.

Sunk Costs

In general, sunk costs are irrelevant to a firm's decisions. If the cost cannot be recovered, then whether the firm continues producing (and if so, how much), do not depend on that cost.

- However, if the sunk cost has not yet been paid, it is relevant.
- A firm can decide whether or not to incur the cost in the first place.

Short-Run vs. Long-Run

Until this point, we've really only considered the long run.

- The **long run** refers to a time period over which all inputs can vary (capital and labor).
- In the **short run**, at least one input is fixed.

Usually we assume capital is fixed (e.g. rent on a building).

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- In the **short run**, at least one input is fixed.

Usually we assume capital is fixed (e.g. rent on a building). But it could also be labor (maybe firms hire employees on a 1-year contract).

Cost Minimization in the Short Run

In the short run, a firm's total cost will always be greater than or equal to a firm's long-run total cost.

- Because if it were lower in the short run, they would use that same input combination in the long run.

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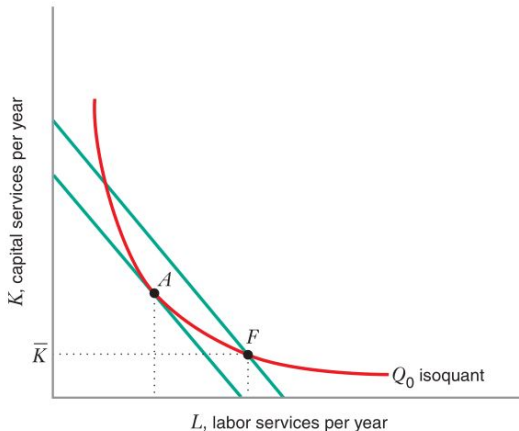
- Because if it were lower in the short run, they would use that same input combination in the long run.

Suppose for example that capital is fixed at \bar{K} units.

- Then in the short run, the firm may not be able to achieve the cost-minimizing input combination, so costs will be higher.

Cost Minimization in the Short Run

Because capital is fixed, the firm's lowest-cost input combination leaves them at a higher isocost line than the long-run cost-minimizing input combination (F in the short-run; A in the long-run)



Cost Minimization in the Short Run

What about our optimality condition?

- You can see from the previous diagram that at the short-run cost minimizing point, our isoquant and isocost lines are no longer tangent.
- Our optimality condition doesn't hold.

Cost Minimization in the Short Run

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- Our optimality condition doesn't hold.

Let's try to find the cost-minimizing point using calculus.

Cost Minimization in the Short Run

Suppose a firm's production function is given by $Q = 50\sqrt{KL}$. In the short-run, the firm's problem becomes

$$\min_L wL + r\bar{K} \quad \text{subject to} \quad \bar{Q} = 50\sqrt{\bar{K}L}$$

Cost Minimization in the Short Run

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Note that, because K is fixed, we only have 1 choice variable. In this case, we now simply have 1 equation, $\bar{Q} = 50\sqrt{\bar{K}L}$, and 1 unknown, L .

We can then simply solve for L .

$$L^*(w, r, \bar{Q}) = \frac{\bar{Q}^2}{(2500\bar{K})}$$

Try It Yourself

Suppose that a firm has the following production function:

$Q(K, L) = 15KL$. In the short run, suppose that K is fixed at $\bar{K} = 5$. If a cost-minimizing firm decides to produce $Q = 1500$ units, find the cost-minimizing level of labor.

- A) 5 units
- B) 10 units
- C) 15 units
- D) 20 units

Comparative Statics

What if the output quantity changes?

- Well, in the short run, K is still fixed, so the firm can only vary the amount of labor used.
- This results in a short-run expansion path that's horizontal.
- In the long run, however, both inputs can vary, resulting in a different long-run expansion path.

Comparative Statics

Below are the short-run and long-run expansion paths for a production function.

