Math Review

ECON 3070 - Intermediate Microeconomic Theory

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January 15, 2025

Math Review Topics

→ Limits
→ Derivative Definition
→ Derivative Rules
→ Partial Derivatives

Limits

The limit of $f(x)$ as x approaches a is written as:

$$\lim_{x \to a} f(x)$$

- $\rightarrow$ The behavior of the function as its input approaches a
- $\rightarrow$ In certain cases, can be evaluated by plugging in $x = a$
- $\rightarrow$ $\lim_{x \to a} c = c$ for a constant c

Limits

The limit from the left and right are

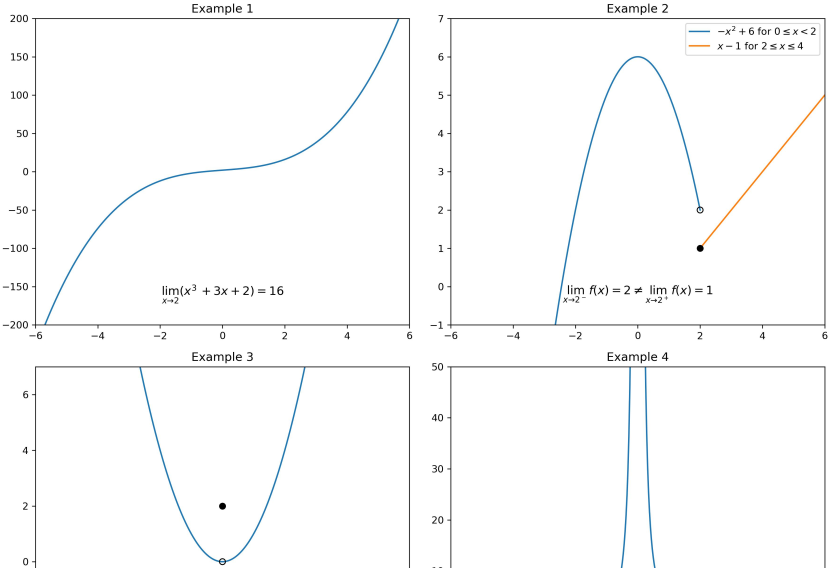
- $\lim_{x \to a^-} f(x) = A$ "as x approaches a from the left ($-\infty$)"
- $\lim_{x \to a^+} f(x) = A$ "as x approaches a from the right (∞)"

If $\lim_{x \to a^-} f(x) = A = \lim_{x \to a^+} f(x)$ then the limit exists and we write

$$\lim_{x \to a} f(x) = A$$

- Note: the limit does not exist if $A = \pm\infty$, even if the left and right limit both equal $\pm\infty$

Limits


Example 1
$\lim_{x\to 2}(x^3 + 3x + 2) = 16$
Example 2
$-x^2 + 6$ for $0 \le x < 2$
$x - 1$ for $2 \le x \le 4$
$\lim_{x\to 2^-} f(x) = 2 \neq \lim_{x\to 2^+} f(x) = 1$
Example 3
Example 4

Limit Rules

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$$

- **Constant Multiple Rule:** $\lim_{x \to a}[af(x)] = aL$
- **Sum/Difference Rule:** $\lim_{x \to a}[f(x) \pm g(x)] = L \pm M$
- **Product Rule:** $\lim_{x \to a}[f(x) \cdot g(x)] = L \cdot M$
- **Quotient Rule:** $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}, \quad M \neq 0$
- **Power Rule:** $\lim_{x \to a}[(f(x))^n] = L^n, \quad n > 0$

Definition of a Derivative

The derivative of f is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists, then we say f is differentiable at x.

Derivative Notation

All of these mean “take the derivative of f with respect to x”:

- $\rightarrow$ $f'(x)$
- $\rightarrow$ $\frac{df(x)}{dx}$
- $\rightarrow$ $\frac{d}{dx}f(x)$
- $\rightarrow$ $\frac{dy}{dx}$ if $y = f(x)$
- $\rightarrow$ y' (sometimes)
- $\rightarrow$ $\dot{y}$ (particularly in Macro)

Derivative Rules

Let $f(x)$ and $g(x)$ be differentiable functions:

- **Derivative of a Constant:** $f(x) = c$, $f'(x) = 0$
- **"Power Rule":** $f(x) = x^n$, $f'(x) = nx^{n-1}$
- **Sum/Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- **Chain Rule:** $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Derivatives Practice Problems

$$f(x) = 2x$$

Derivatives Practice Problems

$$f(x) = 3x^2$$

Derivative Rules for Exponents and Natural Log

- $\rightarrow$ $f(x) = e^x$, $f'(x) = e^x$
- $\rightarrow$ $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$

Useful review of the properties of exponents and logs.

Derivatives Practice Problems

$$f(x) = ln(e^x)$$

Derivatives Practice Problems

$$f(x) = x^2 ln(x)$$

Derivatives Practice Problems

$$f(x) = e^{2x}$$

Higher-order Derivatives

The second-order derivative is

$$f''(x) = \frac{d}{dx} f'(x)$$

We could keep differentiating as long as the last derivative is differentiable. Notation:

- $f''(x)$
- $\frac{d^2 f(x)}{dx^2}$
- $\frac{d^2}{dx^2} f(x)$
- $\frac{d^2 y}{dx^2}$ if $y = f(x)$
- y'' (sometimes)

Partial Derivatives

The function $z = f(x, y)$ takes two inputs and outputs a third. We can evaluate what happens to that output when we change x or y (but not both).

- $\frac{\partial z}{\partial x}$ = "the derivative of z with respect to x"
  - $\rightarrow$ The slope of f in the cardinal direction of x (e.g. "north-south")
  - $\rightarrow$ The tangent of f in the x direction at a point (x_0, y_0)
- $\frac{\partial z}{\partial y}$ = "the derivative of z with respect to y"
  - $\rightarrow$ The slope of f in the cardinal direction of y (e.g. "east-west")
  - $\rightarrow$ The tangent of f in the y direction at a point (x_0, y_0)

Partial Derivatives

All of these mean "take the partial derivative of f with respect to x":

- $f_x(x, y)$
- $\frac{\partial f(x,y)}{\partial x}$
- $\frac{\partial}{\partial x} f(x, y)$
- $\frac{\partial z}{\partial x}$ if $z = f(x, y)$

Partial Derivatives

Interpret $\frac{\partial z}{\partial x}$ as “what happens to z if I change x, holding y constant.” Likewise for $\frac{\partial z}{\partial y}$.

Suppose $z = x + y + xy$, find $\frac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x + y + xy] = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y + \frac{\partial}{\partial x}xy = 1 + 0 + y \cdot 1 = 1 + y$$

Partial Derivatives Practice Problems

$$f(x, y) = x^2 + y^2 + x^2 y^2$$

Partial Derivatives Practice Problems

$$f(x, y) = x^2y + ln(x)y^3$$

Properties of Exponents

Let a and b be real numbers and m and n be integers. Then the following properties of exponents hold, provided that all of the expressions appearing in a particular equation are defined:

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m b^m$
- $\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$
- $a^{-m} = \frac{1}{a^m}$, $a \neq 0$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^0 = 1$, $a \neq 0$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Properties of Natural Log

Note that logs are only defined for positive values of x:

- $\rightarrow$ $\ln(xy) = \ln(x) + \ln(y)$
- $\rightarrow$ $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\rightarrow$ $\ln(x^y) = y \cdot \ln(x)$
- $\rightarrow$ $\ln(e^x) = x$
- $\rightarrow$ $e^{\ln(x)} = x$
- $\rightarrow$ $\ln(e) = 1$
- $\rightarrow$ $\ln(1) = 0$
- $\rightarrow$ $\ln(0) = undefined$