2025 CU Denver Math Camp - Limits & Derivatives

Day 1

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About Me

Michael Karas, PhD Candidate at CU Boulder

- → michael.karas@colorado.edu
- → Entering my 5th year, will be on job market Fall 2025

- Environmental Economics, Economic History, Industrial Organization, Applied Micro
  - → Development of the Gas Utility Industry
    - The initial adoption of public utility commissions
    - Role of incumbent firms in the transition from manufactured gas to natural gas
    - The substitutability and complementarity between gas and electricity utilities
    - The spread of block pricing tariffs in the gas industry
- Teaching: Intermediate Microeconomics, Principles of Macroeconomics
- Hobbies: Cycling, Skiing

Getting to Know Each Other

- Name
- Hometown
- Interests in Economics
- Hobbies or something about yourself

- It will be challenging
  - → Everyone struggles, even the “top” students
  - → It may take a while to master the material, probably won’t grasp everything on the first attempt
  - → What matters is determination, discipline, and consistency: put the time in and the results will come
  - → Treat it like a job
- Study Tips
  - → Practice makes perfect: get old homework/exam from upper students
  - → Study with each other
  - → Recognize when diminishing marginal returns start to set in and take a break
- Don’t neglect your mental health. It’s okay to take breaks.
  - → Have fun! CO is a great place, develop a life outside of school

Day 1 Topics

- $\rightarrow$ Limits
- $\rightarrow$ Limit Rules
- $\rightarrow$ Derivative Definition
- $\rightarrow$ Derivative Rules
- $\rightarrow$ Natural Log and Exponent Rules
- $\rightarrow$ Higher-order Derivatives

Functions

A function is a rule which assigns a number in R^1 to each number in R^1

- $\rightarrow$ R^1 is the set of all real numbers
- $\rightarrow$ ex. $f(x) = 2x$ assigns $x = 2$ to $f(2) = 2(2) = 4$
- $\rightarrow$ The domain of a function is the set of all possible input values
- $\rightarrow$ The input variable x is called the independent variable or exogenous variable
- $\rightarrow$ The output variable y is called the dependent variable or endogenous variable
- $\rightarrow$ Linear vs nonlinear functions
- $\rightarrow$ Functions can be of multiple input variables, $f(x, z) = 2xz^2$

Limits

The limit of $f(x)$ as x approaches a is written as:

$$\lim_{x \to a} f(x)$$

- $\rightarrow$ The behavior of the function as its input approaches a
- $\rightarrow$ In certain cases, can be evaluated by plugging in $x = a$
- $\rightarrow$ $\lim_{x \to a} c = c$ for a constant c

Limits

The limit from the left and right are

→ $\lim_{x \to a^-} f(x) = A$ "as x approaches a from the left ($-\infty$)"

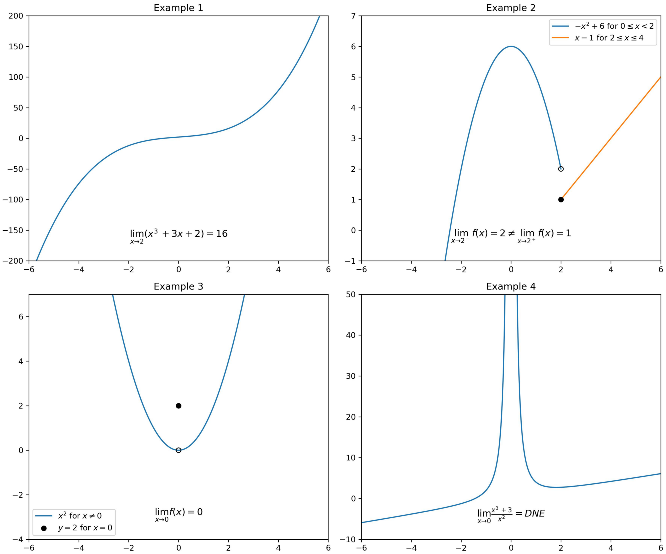
→ $\lim_{x \to a^+} f(x) = A$ "as x approaches a from the right (∞)"

If $\lim_{x \to a^-} f(x) = A = \lim_{x \to a^+} f(x)$ then the limit exists and we write

$$\lim_{x \to a} f(x) = A$$

- Note: the limit does not exist if $A = \pm\infty$, even if the left and right limit both equal $\pm\infty$

Limits



Example 1
$\lim_{x\to 2}(x^3 + 3x + 2) = 16$
Example 2
$-x^2 + 6$ for $0 \leq x < 2$
$x - 1$ for $2 \leq x \leq 4$
$\lim_{x\to 2^-} f(x) = 2 \neq \lim_{x\to 2^+} f(x) = 1$
Example 3
x^2 for $x \neq 0$
$y = 2$ for $x = 0$
$\lim_{x\to 0} f(x) = 0$
Example 4
$\lim_{x\to 0}\frac{x^3+3}{x^2} = DNE$

Limit Rules

$$\lim_{x \to a} f(x) = L \quad \lim_{x \to a} g(x) = M$$

- **Constant Multiple Rule:** $\lim_{x\to a}[af(x)] = aL$
- **Sum/Difference Rule:** $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$
- **Product Rule:** $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$
- **Quotient Rule:** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}, \quad M \neq 0$
- **Power Rule:** $\lim_{x\to a}[(f(x))^n] = L^n, \quad n > 0$

Limit Practice Problems

$$\lim_{x \to 0} (3 + 2x^2)$$

Limit Practice Problems

$$\lim_{x \to 1} \frac{x^2 + 7x - 8}{x - 1}$$

Definition of a Derivative

The derivative of f is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists, then we say f is differentiable at x.

$\rightarrow$ Find $f(x+h)$. Ex: $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$\rightarrow$ Find $f(x+h) - f(x)$

$\rightarrow$ Find $\frac{f(x+h)-f(x)}{h}$ and simplify until $h = 0$ doesn't divide by 0

$\rightarrow$ Plug-in $h = 0$ to determine the limit

Definition of a Derivative Practice Problems

$$f(x) = x^2$$

Definition of a Derivative Practice Problems

$$f(x) = 3x + 1$$

Derivative Notation

All of these mean “take the derivative of f with respect to x”:

- $\rightarrow$ $f'(x)$
- $\rightarrow$ $\frac{df(x)}{dx}$
- $\rightarrow$ $\frac{d}{dx}f(x)$
- $\rightarrow$ $\frac{dy}{dx}$ if $y = f(x)$
- $\rightarrow$ y' (sometimes)
- $\rightarrow$ $\dot{y}$ (particularly in Macro)

Derivative Rules

Let $f(x)$ and $g(x)$ be differentiable functions:

- **Derivative of a Constant:** $f(x) = c$, $f'(x) = 0$
- **“Power Rule”:** $f(x) = x^n$, $f'(x) = nx^{n-1}$
- **Sum/Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- **Chain Rule:** $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Example: Derivative of a Constant

$$f(x) = 5$$

Example: Power Rule

$$f(x) = x^3$$

Example: Sum/Difference Rule

$$f(x) = x^2 + 4x$$

Example: Product Rule

$$f(x) = x \cdot (x + 2)$$

Example: Quotient Rule

$$f(x) = \frac{x}{x+1}$$

Example: Chain Rule

$$f(x) = (3x + 1)^2$$

Derivative Practice Problems

$$f(x) = x^{\frac{1}{2}}$$

Derivative Practice Problems

$$f(x) = 8x^4 + 2\sqrt{x}$$

Derivative Practice Problems

$$f(x) = (x^2 + 1)(\sqrt{x})$$

Derivative Practice Problems

$$f(x) = \frac{x+1}{x-1}$$

Derivative Practice Problems

$$f(x) = 5u^4; u = 1 + x^2$$

Properties of Exponents

Let a and b be real numbers and m and n be integers. Then the following properties of exponents hold, provided that all of the expressions appearing in a particular equation are defined:

- $\rightarrow$ $a^m a^n = a^{m+n}$
- $\rightarrow$ $(a^m)^n = a^{mn}$
- $\rightarrow$ $(ab)^m = a^m b^m$
- $\rightarrow$ $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
- $\rightarrow$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
- $\rightarrow$ $a^{-m} = \frac{1}{a^m}, a \neq 0$
- $\rightarrow$ $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $\rightarrow$ $a^0 = 1, a \neq 0$

Exponent Rules: Product and Power of a Power

- **Product Rule:** $a^m a^n = a^{m+n}$

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$$

- **Power of a Power Rule:** $(a^m)^n = a^{mn}$

$$(3^2)^3 = 3^{2\cdot 3} = 3^6 = 729$$

Exponent Rules: Distributing Powers and Quotients

- **Power of a Product:** $a^m b^m = (ab)^m$

$$2^3 \cdot 5^3 = (2 \cdot 5)^3 = (10)^3 = 1000$$

- **Quotient Rule:** $\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$

$$\frac{7^5}{7^2} = 7^{5-2} = 7^3 = 343$$

Exponent Rules: Power of a Quotient and Negative Exponents

- **Power of a Quotient:** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$

$$\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$$

- **Negative Exponent:** $a^{-m} = \frac{1}{a^m}$, $a \neq 0$

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

Exponent Rules: Rational and Zero Exponents

- **Fractional Exponent:** $a^{\frac{1}{n}} = \sqrt[n]{a}$

$$16^{1/2} = \sqrt{16} = 4$$

- **Zero Exponent:** $a^0 = 1,\ a \neq 0$

$$12^0 = 1$$

Exponent Rule: General Rational Exponents

- **General Rule:** $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Properties of Natural Log

Note that logs are only defined for positive values of x:

- $\rightarrow \ln(xy) = \ln(x) + \ln(y)$
- $\rightarrow \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\rightarrow \ln(x^y) = y \cdot \ln(x)$
- $\rightarrow \ln(e^x) = x$
- $\rightarrow e^{\ln(x)} = x$
- $\rightarrow \ln(e) = 1$
- $\rightarrow \ln(1) = 0$
- $\rightarrow \ln(0) = undefined$

Log and Exponent Practice Problems

$$2e^{6x} = 18$$

Log and Exponent Practice Problems

$$e^{x^2} = 1$$

Derivative Rules for Exponents and Natural Log

- $\rightarrow$ $f(x) = e^{h(x)}$, $f'(x) = e^{h(x)}h'(x)$
  - $f(x) = e^x$, $f'(x) = e^x$
- $\rightarrow$ $f(x) = \ln(h(x))$, $f'(x) = \frac{h'(x)}{h(x)}$
  - $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = 3x^2 * ln(x)$$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = x^4 * e^x$$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = e^x * (1 - x)^4$$

Higher-order Derivatives

The second-order derivative is

$$f''(x) = \frac{d}{dx} f'(x)$$

We could keep differentiating as long as the last derivative is differentiable. Notation:

- → $f''(x)$
- → $\frac{d^2 f(x)}{dx^2}$
- → $\frac{d^2}{dx^2} f(x)$
- → $\frac{d^2 y}{dx^2}$ if $y = f(x)$
- → y'' (sometimes)

Higher Order Derivatives Practice Problem

$$f(x) = 2x^3$$

Higher Order Derivatives Practice Problem

$$f(x) = x^{\frac{5}{2}}$$

Topics for Tomorrow

- $\rightarrow$ Increasing/Decreasing Functions
- $\rightarrow$ Concave/Convex Functions
- $\rightarrow$ Implicit Differentiation
- $\rightarrow$ Partial Derivatives
- $\rightarrow$ Taylor Series