

2025 CU Denver Math Camp - Limits & Derivatives

Day 1

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About Me

Michael Karas, PhD Candidate at CU Boulder

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→ Entering my 5th year, will be on job market Fall 2025

- Environmental Economics, Economic History, Industrial Organization, Applied Micro

→ Development of the Gas Utility Industry

- The initial adoption of public utility commissions
- Role of incumbent firms in the transition from manufactured gas to natural gas
- The substitutability and complementarity between gas and electricity utilities
- The spread of block pricing tariffs in the gas industry

- Teaching: Intermediate Microeconomics, Principles of Macroeconomics

- Hobbies: Cycling, Skiing

Getting to Know Each Other

- Name
- Hometown
- Interests in Economics
- Hobbies or something about yourself

- It will be challenging
 - Everyone struggles, even the “top” students
 - It may take a while to master the material, probably won’t grasp everything on the first attempt
 - What matters is determination, discipline, and consistency: put the time in and the results will come
 - Treat it like a job
- Study Tips
 - Practice makes perfect: get old homework/exam from upper students
 - Study with each other
 - Recognize when diminishing marginal returns start to set in and take a break
- Don’t neglect your mental health. It’s okay to take breaks.
 - Have fun! CO is a great place, develop a life outside of school

Day 1 Topics

- Limits
- Limit Rules
- Derivative Definition
- Derivative Rules
- Natural Log and Exponent Rules
- Higher-order Derivatives

Limits

The limit of $f(x)$ as x approaches a is written as:

$$\lim_{x \rightarrow a} f(x)$$

- The behavior of the function as its input approaches a
- In certain cases, can be evaluated by plugging in $x = a$
- $\lim_{x \rightarrow a} c = c$ for a constant c

Limits

The limit from the left and right are

→ $\lim_{x \rightarrow a^-} f(x) = A$ "as x approaches a from the left ($-\infty$)"

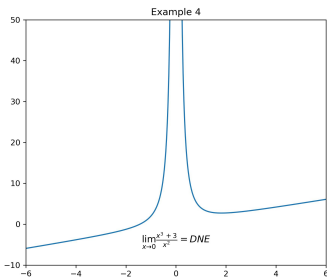
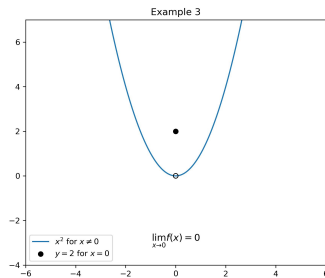
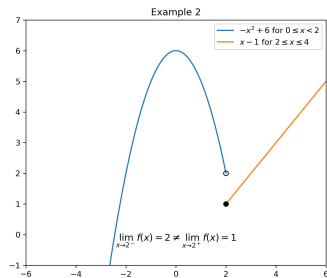
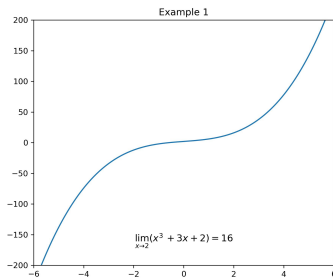
→ $\lim_{x \rightarrow a^+} f(x) = A$ "as x approaches a from the right (∞)"

If $\lim_{x \rightarrow a^-} f(x) = A = \lim_{x \rightarrow a^+} f(x)$ then the limit exists and we write

$$\lim_{x \rightarrow a} f(x) = A$$

- Note: the limit does not exist if $A = \pm\infty$, even if the left and right limit both equal $\pm\infty$

Limits



Limit Rules

$$\lim_{x \rightarrow a} f(x) = L \quad \lim_{x \rightarrow a} g(x) = M$$

- **Constant Multiple Rule:** $\lim_{x \rightarrow a} [af(x)] = aL$
- **Sum/Difference Rule:** $\lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M$
- **Product Rule:** $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L \cdot M$
- **Quotient Rule:** $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}, \quad M \neq 0$
- **Power Rule:** $\lim_{x \rightarrow a} [(f(x))^n] = L^n, \quad n > 0$

Limit Practice Problems

$$\lim_{x \rightarrow 0} (3 + 2x^2)$$

Limit Practice Problems

$$\lim_{x \rightarrow 1} \frac{x^2 + 7x - 8}{x - 1}$$

Definition of a Derivative

The derivative of f is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists, then we say f is differentiable at x .

→ Find $f(x+h)$. Ex: $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

→ Find $f(x+h) - f(x)$

→ Find $\frac{f(x+h)-f(x)}{h}$ and simplify until $h = 0$ doesn't divide by 0

→ Plug-in $h = 0$ to determine the limit

Definition of a Derivative Practice Problems

$$f(x) = x^2$$

Definition of a Derivative Practice Problems

$$f(x) = 3x + 1$$

Derivative Notation

All of these mean “take the derivative of f with respect to x ”:

$$\rightarrow f'(x)$$

$$\rightarrow \frac{df(x)}{dx}$$

$$\rightarrow \frac{d}{dx} f(x)$$

$$\rightarrow \frac{dy}{dx} \text{ if } y = f(x)$$

$$\rightarrow y' \text{ (sometimes)}$$

$$\rightarrow \dot{y} \text{ (particularly in Macro)}$$

Derivative Rules

Let $f(x)$ and $g(x)$ be differentiable functions:

- **Derivative of a Constant:** $f(x) = c, f'(x) = 0$
- **“Power Rule”:** $f(x) = x^n, f'(x) = nx^{n-1}$
- **Sum/Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- **Quotient Rule:** $\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- **Chain Rule:** $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Example: Derivative of a Constant

$$f(x) = 5$$

Example: Power Rule

$$f(x) = x^3$$

Example: Sum/Difference Rule

$$f(x) = x^2 + 4x$$

Example: Product Rule

$$f(x) = x \cdot (x + 2)$$

Example: Quotient Rule

$$f(x) = \frac{x}{x+1}$$

Example: Chain Rule

$$f(x) = (3x + 1)^2$$

Derivative Practice Problems

$$f(x) = 9x^{10}$$

Derivative Practice Problems

$$f(x) = 8x^4 + 2\sqrt{x}$$

Derivative Practice Problems

$$f(x) = \sqrt{x} * 6x^4$$

Derivative Practice Problems

$$f(x) = \frac{x+1}{x-1}$$

Derivative Practice Problems

$$f(x) = 5u^4; u = 1 + x^2$$

Properties of Exponents

Let a and b be real numbers and m and n be integers. Then the following properties of exponents hold, provided that all of the expressions appearing in a particular equation are defined:

$$\rightarrow a^m a^n = a^{m+n}$$

$$\rightarrow (a^m)^n = a^{mn}$$

$$\rightarrow (ab)^m = a^m b^m$$

$$\rightarrow \frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

$$\rightarrow \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

$$\rightarrow a^{-m} = \frac{1}{a^m}, a \neq 0$$

$$\rightarrow a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$\rightarrow a^0 = 1, a \neq 0$$

Properties of Natural Log

Note that logs are only defined for positive values of x :

$$\rightarrow \ln(xy) = \ln(x) + \ln(y)$$

$$\rightarrow \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\rightarrow \ln(x^y) = y \cdot \ln(x)$$

$$\rightarrow \ln(e^x) = x$$

$$\rightarrow e^{\ln(x)} = x$$

$$\rightarrow \ln(e) = 1$$

$$\rightarrow \ln(1) = 0$$

$$\rightarrow \ln(0) = \textit{undefined}$$

Log and Exponent Practice Problems

$$2e^{6x} = 18$$

Log and Exponent Practice Problems

$$e^{x^2} = 1$$

Derivative Rules for Exponents and Natural Log

$$\rightarrow f(x) = e^{h(x)}, f'(x) = e^{h(x)}h'(x)$$

$$\blacksquare f(x) = e^x, f'(x) = e^x$$

$$\rightarrow f(x) = \ln(h(x)), f'(x) = \frac{h'(x)}{h(x)}$$

$$\blacksquare f(x) = \ln(x), f'(x) = \frac{1}{x}$$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = 3x^2 * \ln(x)$$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = x^4 * e^x$$

Natural Log/Exponent Derivative Practice Problems

$$f(x) = e^x * (1 - x)^4$$

Higher-order Derivatives

The second-order derivative is

$$f''(x) = \frac{d}{dx} f'(x)$$

We could keep differentiating as long as the last derivative is differentiable. Notation:

$$\rightarrow f''(x)$$

$$\rightarrow \frac{d^2 f(x)}{dx^2}$$

$$\rightarrow \frac{d^2}{dx^2} f(x)$$

$$\rightarrow \frac{d^2 y}{dx^2} \text{ if } y = f(x)$$

$$\rightarrow y'' \text{ (sometimes)}$$

Topics for Tomorrow

- Increasing/Decreasing Functions
- Concave/Convex Functions
- Implicit Differentiation
- Partial Derivatives
- Taylor Series