CU Denver Math Camp - Limits & Derivatives

Day 2

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Day 2 Topics

- → Practice Problems
- → Increasing/Decreasing Functions
- → Concave/Convex Functions
- → Implicit Differentiation
- → Partial Derivatives
- → Taylor Series

Day 1 Practice Problems

$$f(x) = \frac{2x}{x^2 + 2}$$

Day 1 Practice Problems

$$f(x) = e^{x^2 + ln(x)}$$

Day 1 Practice Problems

$$f(x) = x(x^2 + 1)$$

Find $f'(x)$ and $f''(x)$

Day 1 Practice Problems

$$f(x) = ln(x^2 + 1)$$

Find $f'(x)$ and $f''(x)$

Day 1 Practice Problems

Prove that $\frac{d}{dx}e^x = e^x$

Increasing/Decreasing Functions

Let f be a differentiable function defined on an interval $[a, b]$

- f is increasing on $[a, b]$ if, for every $a \leq x \leq b$, $f'(x) > 0$
- f is decreasing on $[a, b]$ if, for every $a \leq x \leq b$, $f'(x) \leq 0$

Could replace $\leq$ with $<$ and say "strictly increasing" (no flat parts of f on $[a, b]$), likewise with decreasing. When $f'(x) = 0$ the function is not changing, an "optimal" point.

Increasing/Decreasing Functions Practice Problems

$$f(x) = x^2$$

Find which intervals which $f(x)$ is increasing and which intervals which $f(x)$ is decreasing.

Increasing/Decreasing Functions Practice Problems

$$f(x) = ln(x)$$

Find which intervals which $f(x)$ is increasing and which intervals which $f(x)$ is decreasing.

Increasing/Decreasing Functions Practice Problems

$$f(x) = e^x$$

Find which intervals which $f(x)$ is increasing and which intervals which $f(x)$ is decreasing.

Concave/Convex Functions

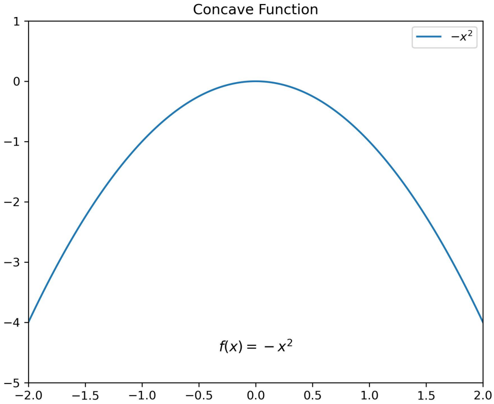
Often it is not enough to know if a function is increasing/decreasing. We need to know its shape. Let f and I be as before. Then,

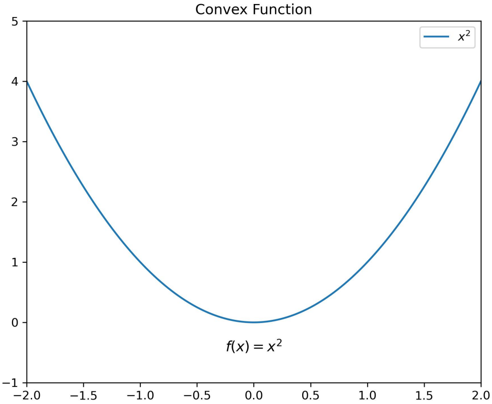
- f is convex on $[a, b]$ if, for every $a \leq x \leq b$, $f''(x) > 0$
- f is concave on $[a, b]$ if, for every $a \leq x \leq b$, $f''(x) < 0$

Convex: function is increasing at an increasing rate to the bottom of a "valley." Concave: function is increasing at a decreasing rate to the peak of a "mountain."

- If $f''(x) = 0$, f is at an inflection point – the acceleration switches (e.g. skiing)

Concave/Convex Functions





Concave/Convex Functions Practice Problems

$$f(x) = x - 2ln(x+1)$$

Implicit Differentiation

Sometimes we can't solve for $y = f(x)$ (or it would be very difficult) but still need to differentiate. Implicit differentiation allows us to find $f'(x)$ without a "closed-form" solution for $y = f(x)$.

- $\rightarrow$ Differentiate both sides of the equation using all the typical rules of derivatives, treating x and y as variables that are both changing
- $\rightarrow$ If you differentiate something involving y, multiply the answer by y' (the derivative of $y = f(x)$)
- $\rightarrow$ If you differentiate something involving x, do not multiply by anything new
- $\rightarrow$ After everything is differentiated, solve for y' using algebra
- $\rightarrow$ The answer for y' should be a function of x (and potentially y)

Implicit Differentiation

Suppose $x + y + xy = 5$.

$$1 + 1 \cdot y' + \underbrace{1 \cdot y}_{f'(x)g(x)} + \underbrace{x \cdot 1 \cdot y'}_{f(x)g'(x)} = 0$$

Using product rule:

$$1 + y' + y + xy' = 0$$

$$y'(1 + x) = -1 - y$$

$$y' = \frac{-1 - y}{1 + x}$$

The derivative is a function of both x and y.

Implicit Differentiation Practice Problems

$$x^2 + y^2 = 1$$

Partial Derivatives

The function $z = f(x, y)$ takes two inputs and outputs a third. We can evaluate what happens to that output when we change x or y (but not both).

- $\frac{\partial z}{\partial x}$ = "the derivative of z with respect to x"
  - $\rightarrow$ The slope of f in the cardinal direction of x (e.g. "north-south")
  - $\rightarrow$ The tangent of f in the x direction at a point (x_0, y_0)
- $\frac{\partial z}{\partial y}$ = "the derivative of z with respect to y"
  - $\rightarrow$ The slope of f in the cardinal direction of y (e.g. "east-west")
  - $\rightarrow$ The tangent of f in the y direction at a point (x_0, y_0)

Partial Derivatives

All of these mean "take the partial derivative of f with respect to x":

- $f_x(x, y)$
- $\frac{\partial f(x,y)}{\partial x}$
- $\frac{\partial}{\partial x} f(x, y)$
- $\frac{\partial z}{\partial x}$ if $z = f(x, y)$

Partial Derivatives

Interpret $\frac{\partial z}{\partial x}$ as "what happens to z if I change x, holding y constant." Likewise for $\frac{\partial z}{\partial y}$.

Suppose $z = x + y + xy$, find $\frac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x + y + xy] = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y + \frac{\partial}{\partial x}xy = 1 + 0 + y \cdot 1 = 1 + y$$

Partial Derivatives

The first partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are themselves functions of x and y. We could differentiate each of them with respect to either x or y, so there are four second-order partial derivatives.

$$f(x, y)$$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y^2}$$

Partial Derivatives

All of these mean "take the partial derivative of f with respect to x twice":

- $f_{xx}(x, y)$
- $\frac{\partial^2 f(x,y)}{\partial x^2}$
- $\frac{\partial^2}{\partial x^2} f(x, y)$
- $\frac{\partial^2 z}{\partial x^2}$ if $z = f(x, y)$

Partial Derivatives

All of these mean "take the partial derivative of f with respect to x, then with respect to y":

- $f_{xy}(x, y)$
- $\frac{\partial^2 f(x,y)}{\partial y \partial x}$
- $\frac{\partial^2}{\partial y \partial x} f(x, y)$
- $\frac{\partial^2 z}{\partial y \partial x}$ if $z = f(x, y)$

Young's Theorem

If $f(x, y)$ is a twice differentiable function and continuous at the point (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

The cross-partial derivatives are the same, regardless of the order in which you take them.

Partial Derivatives Practice Problems

$$f(x, y) = x^2 + y^2 + x^2y^2$$

Partial Derivatives Practice Problems

$$f(x, y) = x^2y + ln(x)y^3$$

Taylor Series

Sometimes it is too difficult (or not possible) to differentiate a function. We can use a Taylor Series expansion to approximate the value of a function around different points.

For a function $f(x)$ with derivatives of all orders at $x = a$, the Taylor series is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a = 0$, it is called the Maclaurin series.

Taylor Series

Taylor Series for $f(x) = \frac{1}{x}$

$$f(x) = \frac{1}{x} \quad f'(x) = -\frac{1}{x^2} \quad f''(x) = \frac{2}{x^3} \quad f^{(k)}(x) = \frac{k!}{x^{k+1}}$$

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

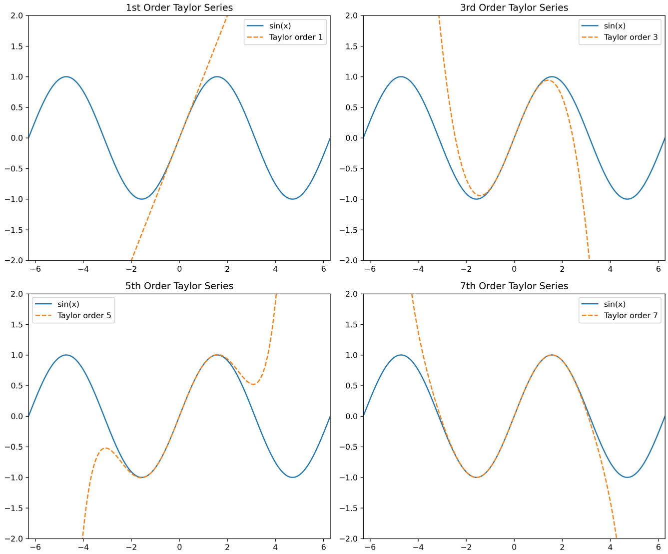
At $a = 1$ of order $k == 2$:

$$P_2(x) = 1 - (x-1) + (x-1)^2$$

At $a = 3$ of order $k = 2$:

$$P_3(x) = \frac{1}{3} - \frac{x-3}{3^2} + \frac{(x-3)^2}{3^3}$$

Taylor Series



Taylor Series Practice Problems

Write the Taylor Series Expansion of order k generated for the function $f(x) = e^x$ around the point $x = 0$