1. On the notion of a linear ODE

We will write $\mathbb{C}(z)$ for the field of *rational* functions: this means ratios of polynomials; functions like

$$\frac{1}{z^2+1}, \frac{\sqrt{3}-\pi i z}{178z^3}, z^8-4.$$

Definition 1.1. A *rank n, first order linear ODE* is an equation of the form

$$u'(z) = A(z)u(z),$$

$A(z)$ be an $n \times n$ matrix with entries in $\mathbb{C}(z)$.

For an open set $U \subseteq \mathbb{C}$ over which no entry of $\mathbb{C}(z)$ has a pole (for example, if $1/(z^2+1)$ and $1/z$ were entries of $A(z)$, then this would mean U does not contain $i, -i$, or 0), we say a function

$$u : U \to \mathbb{C}^n$$

is a *solution* to our ODE if it's derivative obeys $u'(z) = A(z)u(z)$. Observe that $u(z)$ is vector valued!

Problem 1.2. *I mentioned in the first lecture that every linear ODE can be rewritten as a system of first order linear ODEs. To understand this, rewrite*

$$f''(z) + \frac{1}{z}f'(z) - zf(z) = 0$$

as a rank 2 first order linear ODE. As a hint, think about $u(z) = (f(z), f'(z))$.

Problem 1.3. *Solve the linear ODEs*

$$u'(z) = u(z),$$

$$u'(z) = \frac{\alpha}{z}u(z),$$

and

$$u'(z) = \frac{1}{z^2}u(z).$$

Note the first ODE will make sense at zero, but the last two ODEs will not. Can you see why in their solutions?

Theorem 1.4 (ODEs admit power series solutions)**.** *If*

$$u'(z) = A(z)u(z)$$

is an ODE and $A(z)$ has entries lying in the ring $\mathbb{C}[[z]]$ of formal power series, then there always exists a vector of formal power series $u(z)$ solving this ODE.

Problem 1.5. *Prove Theorem 1.4.*

2. On p-curvature

We now explain p-curvature in a bit more computational of a way. Given a rank n linear ODE

$$u'(z) = A(z)u(z)$$

where the entires lie in $\mathbb{Q}(z)$ (meaning they have rational coefficients), for almost every p we can make sense of the matrix $A(z)$ as a matrix with entries in $\mathbb{F}_p(z)$, by reducing the coefficients mod p.

The *p-curvature* of this ODE is the linear operator

$$\psi_p = \left(\frac{d}{dz} + A(z)\right)^p,$$

by which we mean the operator on $\mathbb{F}_p(z)^n$ obtained by composing $d/dz + A(z)$ with itself p times. Here, $A(z)$ is a matrix, which is how we view it as a linear operator on $\mathbb{F}_p(z)^n$. It is a great miracle that ψ_p is linear!

Remark 2.1. The derivative of a formal power series

$$\sum_n a_n z^n$$

is defined as

$$\sum_n n a_n z^{n-1}.$$

In other words, we just take derivatives termwise using the usual formula; don't be scared by the $\mathbb{F}_p$.

Problem 2.2. *Suppose $a(z)$ is a function. We can view $a(z)$ as a linear operator on $\mathbb{F}_p(z)$, by the rule*

$$a(z) \cdot f(z) = (a(z)f(z)).$$

In other words, $a(z)$ the operator means "multiplication by $a(z)$." Prove that

$$\frac{d}{dz} \circ a(z) = a(z) \circ \frac{d}{dz} + a'(z),$$

where the $\circ$ is function composition, and this equality means an equality of linear operators. (The difficulty here is entirely in understanding the notation, and not really at all in proving the identity once you understand the notation.)

Problem 2.3. *Prove that, modulo p,*

$$\frac{d^p}{dz^p} = 0.$$

This is the operator which sends a function $f(z)$ to its p^{th} derivative.

Now we will give an extended discussion of 2 curvatures.

We fix a rank 1 ODE

$$u'(z) = a(z)u(z)$$

for the rest of these questions. Write

$$u(z) = u_0 + u_1 z + u_2 z^2 + \cdots$$

as a power series, and

$$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

as a formal power series.

Problem 2.4. *Compute*

$$\psi_2 = a'(z) + a(z)^2.$$

Remember you are working modulo 2!

Now let's try to solve our ODE mod 2. This should be possible if and only if the 2-curvature vanishes; aka if and only if

$$a'(z) + a(z)^2 = 0.$$

Let's see why. Suppose you are given an initial condition u_0.

Problem 2.5. *Show that*

$$u'(z) = u_1 + u_3 z^2 + u_5 z^4 + \cdots$$

modulo 2, and

$$a(z)u(z) = a_0 u_0 + (a_1 u_0 + a_0 u_1)z + (a_2 u_0 + a_1 u_1 + a_0 u_2)z^2 + \cdots .$$

Thus we need to choose $u_1 := a_0 u_0$, and we need

$$a_1 u_0 + a_0 u_1 = 0$$

(since $u'(z)$ has no z^1 term, but $a(z)u(z)$ has a coefficient of $a_1 u_0 + a_0 u_1$ on its z^1 term). Rewriting $u_1 = a_0 u_0$, we need

$$(a_1 + a_0^2) \cdot u_0 = 0.$$

This is true for every initial condition u_0 if and only if $a_1 + a_0^2 = 0$.

Problem 2.6. *Show that*

$$a'(z) + a(z)^2 = 0$$

implies $a_1 + a_0^2 = 0$.

Woohoo! So $\psi_2 = 0$ gives us that we can solve for u_1 consistently. Comparing the z^2 terms of $u'(z)$ and $a(z)u(z)$, we need

$$u_3 = a_2 u_0 + a_1 u_1 + a_0 u_2.$$

The trouble is this has two unknowns: $a(z)$ is known to us (so a_2, a_1, a_0 are OK); u_0 is our initial condition so we know it; and u_1 we already solved for... but u_2, u_3 are both unknowns!

Problem 2.7. *Prove that any formal power series in* $\mathbb{F}_2[[x^2]]$ *has derivative 0.*

Explain why this means solutions to an ODE mod 2 are not uniquely determined by their initial condition.

Problem 2.8. *Show that if* $a'(z) + a(z)^2 = 0$, *then the ODE*

$$u'(z) = a(z)u(z)$$

has a solution for every choice of initial condition; moreover, prove that if $u_1(z), u_2(z) \in \mathbb{F}_2[[z]]$ *are two formal power series both solving this ODE, and if* $u_1(0) = u_2(0)$, *then* $u_1(z) - u_2(z) \in \mathbb{F}_2[[z^2]]$.

Problem 2.9. *Conjecture a positive characteristic analogue to Theorem 1.4.*

3. Convergent solutions

Definition 3.1. We write

$$\mathbb{C}\{z\}$$

for the ring of power series which have *some* nonzero radius of convergence. So, for example,

$$1 - z + z^2 - z^3 + z^4 - \cdots$$

(which has radius of convergence one) lies in $\mathbb{C}\{z\}$, as does

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots ,$$

but $1 + z + 2z + 6z^2 + 24z^3 + \cdots$ does not (as this is a formal power series of radius of convergence zero).

Problem 3.2. *Prove $\mathbb{C}\{z\}$ is actually a ring (closed under addition, multiplication, etc.).*

Theorem 3.3. *Let $A(z)$ be an $n \times n$ matrix with entries in $\mathbb{C}\{z\}$. Then any formal power series solution $u(z)$ to $u'(z) = A(z)u(z)$ is actually a convergent power series. (Here I mean $u(z)$ is a vector whose n components are formal power series.)*

Problem 3.4. *Prove Theorem 3.3, by carefully writing down what the coefficients of a formal power series solution to an ODE are, and then bounding them to show the radius of convergence is nonzero. As a hint, one strategy is to use the following technique: if you have two power series $\sum_n a_n z^n$ and $\sum_n b_n z^n$ and $|a_n| \leq |b_n|$ for all but finitely many n, then the first power series converges wherever the second does.*

Corollary 3.5. *Let $A(z), u(z)$ be as in Theorem 3.3. If each entry of $A(z)$ converges in the open disk of radius R centered at 0, then the components of the vector $u(z)$ also converge inside this radius R disk.*

Problem 3.6. *Prove the corollary Corollary 3.5 using Theorem 3.3. We did this in lecture, so this is mostly a test of filling in details.*

4. Monodromy of square root

We now explore monodromy of $\sqrt{z}$. Take the ODE

$$u'(z) = \frac{1}{2z}u(z). \tag{4.1}$$

We will be working with tihs ODE throughout the rest of the section.

Problem 4.2. *Find the unique formal power series solution to (4.1) centered at $z = 1$ which obeys $u(1) = 1$. This should be a power series of the form $\sum_n a_n(z-1)^n$.*

The radius of convergence of the power series you found in Problem 4.2 is at least 1, by Corollary 3.5. (In fact it is exactly one but we will not need this.) In particular, if we call that power series $u(z)$, then it makes sense to ask what $u(e^{2\pi i/8}) = u(e^{\pi i/4})$ is, since $|1 - e^{\pi i/4}| < 1$.

Problem 4.3. *Compute $u(e^{2\pi i/8})$ using your first formal power series (hint: the answer is just $e^{\pi i/8}$, but do you see why?). Then, find a formal power series solution to (4.1) centered at $e^{2\pi i/8}$ obeying $u(e^{2\pi i/8}) = e^{\pi i/8}$. This is an* analytic continuation *of our first power series.*

Problem 4.4. *Repeat this procedure, going all the way around the circle: get power series solutions at*

(1) 1,
(2) $e^{2\pi i/8}$,
(3) $e^{4\pi i/8} = i$,
(4) $e^{6\pi i/8}$,
(5) $e^{8\pi i/8} = -1$,
(6) $e^{10\pi i/8}$,
(7) $e^{12\pi i/8} = -i$,
(8) $e^{14\pi i/8}$,
(9) $e^{16\pi i/8} = 1$.

Show that when you go all the way around the circle, using the previous power series' value at the new point as your new initial condition, the final power series you get at 1 is the negative *of the original power series you started with.*

This is the $\mathbb{Z}/2$ monodromy of $\sqrt{z}$.