A PROOF OF A THEOREM

1. Introduction

I claimed at the end of the Tuesday talk that one can reconstruct a compact metric space X up to homeomorphism from the $\mathbb{R}$-algebra $C(X)$ of continuous functions $X \to \mathbb{R}$. Let me explain how this can be done.

Probably this is best phrased as saying that $X \mapsto C(X)$ is a 'fully faithful functor,' but I will try to avoid the language of categories and functors.

2. Step 1: Recovering points

Let us show how to recover the underlying point-set of X from the ring $C(X)$. We do this by identifying points with $\mathbb{R}$-linear functionals on X. Crucial to this is Urysohn's lemma, and a few related results.

Theorem 2.1 (Urysohn's lemma)**.** *Let X be a compact metric space. Then for any disjoint closed sets A, B, there is a continuous function $f : X \to [0,1]$ so that $f(A) = 0$ and $f(B) = 1$.*

Proof. For metric spaces this is actually not so hard to prove, but for general compact Hausdorff spaces it can be a mess (see papa Rudin for a proof there). Define

$$d_A : X \to \mathbb{R}$$

as

$$d_A(x) = \inf_{a \in A} d(a, x),$$

for d the metric on X.

Note that $d_A(x) = 0$ if and only if $x \in A$: indeed, if $d_A(x) = 0$ then there is a sequence $a_1, a_2, ... \in A$ so that $d(a_n, x) < 1/n$. As X is compact, this sequence has a convergent subsequence; but the limit of any convergent subsequence is clearly x, and so (since A is closed) we deduce that $x \in A$.

Define

$$f(x) = \frac{d_B(x)}{d_A(x) + d_B(x)}.$$

The denominator is never 0, since $A \cap B = \emptyset$, and so this is a continuous function. If $x \in A$, then $f(x) = 0/1 = 0$, and if $x \in B$, then $f(x) = 1/(0+1) = 1$. We conclude. $\square$

Corollary 2.2 (Partitions of unity)**.** *Let X be a compact metric space, and $\{U_i\}_{i \in I}$ an open cover of X. Then there are smooth functions $\varphi_i : X \to [0,1]$ so that*

(1) all but finitely many φ_i are identically 0,
(2) $\operatorname{supp} \varphi_i \subseteq U_i$,
(3) and for each $x \in X$, $\sum_i \varphi_i(x) = 1$.

Date: June 27, 2024.

Proof. This is an exercise to the reader, but you can find a solution in most differential geometry textbooks. ☐

Now, we show that $C(X)$ remembers the points of X; precisely, points of X are the $\mathbb{R}$-algebra homomorphisms $C(X) \to \mathbb{R}$. (To me, a ring homomorphism – and in particular an $\mathbb{R}$-algebra homomorphism – has to send 1 to 1.)

Proposition 2.3. *Every $\mathbb{R}$-linear homomorphism $\varphi : C(X) \to \mathbb{R}$ is of the form*

$$\varphi(f) = f(x)$$

for some fixed point $x \in X$. In other words, the only maps $C(X) \to \mathbb{R}$ are of the form "evaluation at a point."

Proof. First, I claim that there is some point $x \in X$ so that if $f \in C(X)$ has $f(x) = 0$, then $\varphi(f) = 0$.

Assume this is not the case. Then for every $x \in X$, we can find some $f_x \in C(X)$ with $f_x(x) \neq 0$ but $\varphi(f_x) = 0$. Set

$$U_x = \{y \in X \mid f_x(y) > 1/2\}.$$

Then each U_x is open and $x \in U_x$, so that these U_x form an open cover of X. By Corollary 2.2, we can find a partition of unity $\varphi_x : X \to [0, 1]$ subordinate to this open cover.

Define

$$f = \sum_{x \in X} \varphi_x \cdot f_x. \tag{2.4}$$

Then $f > 1/2$ everywhere, and in particular f is a unit in the ring $C(X)$, since $1/f$ is a continuous function. But also, since all but finitely many φ_x are identically 0, (2.4) is really a *finite* linear combination of the f_x. As $\varphi(f_x) = 0$ for each x, we deduce (from $\mathbb{R}$-linearity of φ) that $\varphi(f) = 0$.

But $\varphi(1/f) \cdot \varphi(f) = \varphi(1) = 1$ (since φ is a ring homomorphism). Thus we cannot have $\varphi(f) = 0$, and so by contradiction we deduce that there is some point $x \in X$ so that if $f \in C(X)$ has $f(x) = 0$, then $\varphi(f) = 0$.

We claim now that $\varphi(f) = f(x)$. Indeed, for any function $f : X \to \mathbb{R}$, we can define

$$g = f - f(x),$$

which is a function obeying $g(x) = 0$, and so $\varphi(g) = 0$. As $f(x) \in \mathbb{R}$ is a constant function and φ is $\mathbb{R}$-linear, we find that $\varphi(g) = \varphi(f) - \varphi(f(x)) = \varphi(f) - f(x)$, so that (since $\varphi(g) = 0$) we have

$$\varphi(f) = f(x),$$

as desired. ☐

3. Recovering the topology

Now, given an $\mathbb{R}$-algebra A, we can define a set

$$|A| = \{\varphi \mid \varphi : A \to \mathbb{R} \text{ is an } \mathbb{R}\text{-algebra homomorphism}\}.$$

We endow $|A|$ with a certain topology; the point of this construction is that if $A = C(X)$, then $|C(X)| = X$ (as sets), and the topology we endow on $|C(X)|$ will be the same as the topology on X.

Definition 3.1. The *Zariski topology* on $|A|$ is the topology whose closed sets are precisely the sets of the form

$$V(I) = \{\varphi \in |A| \mid \varphi(f) = 0 \text{ for every} f \in I\},$$

for $I \subseteq A$ some ideal of the ring A.

Proposition 3.2. *Let X be a compact metric space. The Zariski topology on $|C(X)|$ coincides with the topology on X.*

Proof. First we prove that every Zariski closed set is actually closed in X. This is because $V(I)$ is just the set of all points $x \in X$ so that $x \in f^{-1}(0)$ for all $f \in I$; in other words,

$$V(I) = \bigcap_{f \in I} f^{-1}(0).$$

As each $f^{-1}(0)$ is closed (by continuity) and an arbitrary intersection of closed sets is closed, we deduce that each $V(I)$ is closed.

Next, we prove that any closed set $C \subseteq X$ can be written as $V(I)$. In fact, define

$$I = \{f \in C(X) \mid f(x) = 0 \text{ for all } x \in C\}.$$

We prove now that $C = V(I)$. It is clear that $C \subseteq V(I)$. The harder part is the reverse inclusion; we use Urysohn's lemma for this.

Take a point $x \notin C$. Then, as $\{x\}$ and C are two disjoint closed subsets of X, by Urysohn's lemma there exists some continuous $f : X \to \mathbb{R}$ so that $f(C) = 0$ and $f(x) = 1$. As $f(C) = 0$ we find that $f \in I$, and so $x \notin V(I)$ since $f(x) = 1 \neq 0$. We conclude that $C = V(I)$. □

4. Exercises

Thus, a compact metric space can be reconstructed from its $\mathbb{R}$-algebra of continuous functions. In fact, one can upgrade this result: continuous functions $f : X \to Y$ of comapct metric spaces correspond to $\mathbb{R}$-algebra homomorphisms $C(Y) \to C(X)$. Can you see how?

Where do I use compactness? Can you remove the compactness assumption by slightly tweaking the reconstruction procedure?

I said $\mathbb{R}$-algebra $C(X)$. Prove that $\mathbb{R}$ has no automorphisms as a field, and then ask yourself if I need to remember the $\mathbb{R}$-algebra $C(X)$ or if I can make the above argument work by remembering just the ring $C(X)$.

If instead of compact metric space, I said compact manifold, and instead of continuous functions I used smooth functions, could you recover the manifold up to diffeomorphism? The trickiest part is probably making a smooth version of Urysohn's lemma; this can be accomplished via the analytic tool of "mollifying" a continuous function to closely approximate it by a smooth function.