APS 1022

Financial Engineering II

Course Project

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I. Introduction

Option is a financial derivative that represents a contract sold by one party to another. It is widely used for multiple purposes in the financial market. A call option is a derivative that gives the owner the right but not obligation to buy a certain asset by a maturity date for the strike price. Similarly, a put option is a derivative that gives the owner the right but not obligation to sell a certain asset by maturity date for strike price.

There are three option style that will be discuss in this project:

1. Asian option: a type of option whose payoff is determined by the average underlying price over some present time period.
2. American option: a type of option that may be exercised on any trading day on or before maturity.
3. Exotic option: a type of option that includes complicated financial structures. In our report, we are going to focus on lookback and floating lookback options.

Within our project, there are two approaches we will apply to simulate the behaviours of underlying assets, given the necessary conditions. These two methods are 'Monte Carlo Simulation' and 'Lattice Approach'.

Monte Carlo Simulation is employed to calculate the value of an option with multiple sources of uncertainty or complicated features. The principle of Monte Carlo Simulation is to generate a large amount of random and possible paths of underlying asset throughout simulation. Given these simulated prices, we can compute the payoff of each type of option accordingly.

Lattice approach, which is also known as binomial options pricing model (BOPM), is another approach to determine the valuation of an option. The principle of BOPM is to create a binomial price tree. According to different types of options, we will be able to calculate their payoffs by following the paths on the lattice.

In this project, Monte Carlo Simulation and lattice approach will be implemented to determine the price of seven different types of option including Asian call/put, lookback call/put, floating lookback call/put and American put.

II. Methodology

The objective of this project is to compute the price of the following options using Monte Carlo Simulation and lattice approach. For Monte Carlo and Lattice, the sample size is 100,000 and 4096 respectively. The options we are going to focus on are:

1. Asian call
2. Asian put
3. lookback call
4. lookback put
5. floating lookback call
6. floating lookback put
7. American put

The payoffs of above option types are summarized in Table 1. $\bar{S}$, S_{max}, S_{min} and S_T represent the average, maximum, minimum and maturity values of a particular stock price path over a period of time.

Option type	Payoff
Asian call	$(\overline{S} - K)^+$
Asian put	$(K - \overline{S})^+$
Lookback call	$(S_{max} - K)^+$
Lookback put	$(K - S_{min})^+$
Floating lookback call	$(S_T - S_{min})^+$
Floating lookback put	$(S_{max} - S_T)^+$

Table 1 Payoff from path-dependent options

The parameters that will be used in the Monte Carlo Simulation and lattice approach are listed in Table 2.

Parameters	
Risk-free rate (r)	2%
Current price (S)	$100
Volatility (σ)	25%
Strike price (K)	$105
Maturity	3 months
Simulation period	1 week

Table 2 Parameters

In the following section, we will discuss the detail about how the algorithm is applied during the simulation.

i. Monte Carlo Simulation

In order to simulate a path of the underlying asset, we shall the price of an asset on a finite set of m + 1 evenly-spaced dates, from $t_0 = 0$ to $t_m = T$, where $t_j = j\Delta t = j\frac{T}{m}$ is the time of the jth observation. The equation used for generating asset price by applying stochastic process is:

$$\boldsymbol{S_{t_j} = S_{t_{j-1}} e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_j}} \qquad \boldsymbol{Eq\ 1}$$

Where

$$r: \text{ the risk neutral rate of interest}$$

$$\sigma: \text{volatility of underlying asset}$$

$$Z_j: \text{random variable with distribution} \sim N(0,1)$$

$$j = 0, \dots, m$$

$$\boldsymbol{\tilde{h}_i = e^{-rT} * h_i} \qquad \textbf{\textit{Eq 2}}$$

Where

h_i: the maturity payoff of the selected option in path i

$\tilde{h}_i$: the present value of the payoff

1. Algorithm for Asian/Exotic Options

(1) For each simulated scenario, represented by $i = 1, \dots, n$

(a) In each simulated scenario i, for each simulated time step, represented by $j = 1, \dots, m$

(i) Compute the stock price at time t_j by using equation 1

(b) After simulating the price of the underlying asset at each time step till maturity for this scenario, we can easily compute $\bar{S}$, S_{max}, S_{min} and S_T . S_T is the asset price at the end and the average value $\bar{S} = \frac{1}{m+1}\sum_{j=0}^{m} S_{t_j}(\omega_i)$.

(c) Let $\tilde{h}_i$ be the present value of the payoff from the selected option when path i is followed. For Asian options, the value will depend on the value of K and $\bar{S}$. The value of the lookback options depends on the value of K, S_{max} and S_{min}. The payoff of Floating lookback options is based on the value of S_T, S_{max} and S_{min}.

(d)Eq 2 will be used to discount the payoff at maturity back to present.

(2) Obtain the sample mean $\bar{h} = \frac{1}{n}\sum_{i=1}^{n} \tilde{h}_i$, which will be consider as the premium.

2. Algorithm for American put Option

$$\boldsymbol{E[S_T|S_t] = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1) \quad for\ a\ put\ option \quad Eq3}$$

where

$$\boldsymbol{d_1 = \frac{log\left(\frac{S}{k}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad Eq4}$$

$$\boldsymbol{d_2 = d_1 - \sigma\sqrt{T-t}}$$

(1) For each scenario $i = 1, \ldots, n$

(a) Set j = 0, where j represents the NO. of the time step and $t = t_j$ is the current time step.

(b) For j $\subseteq \{0, 1, \ldots, m\}$ and while an optimal stopping time for this path hasn't been found

(i) Compute d_1 and d_2 using equation 4.

(ii) Compute E[S_T | S_t] using equation 3.

(iii) Compute the payoff, Z_t from exercise at time t.

(iv) If $Z_t \geq E[S_T \mid S_t]$, the boundary has been crossed and the current time will be set as the exit time and exit the loop.

(v) Otherwise, proceed to next time step and repeat the same calculation

(c) If an optimal stopping time wasn't found for this path, set the optimal exiting time at maturity T.

(d) Calculate the payoff according to the optimal exiting time for the path and then sue equation 2 to compute the present payoff.

(2) Obtain the average optimal stopping time $\tilde{\tau} = \frac{1}{n}\sum_{i=1}^{n} \tau_i$

(3) Obtain the sample mean $\bar{h} = \frac{1}{n}\sum_{i=1}^{n} \tilde{h}_i$, which will be used as the premium.

ii. Lattice Approach

Figure 1 displays the first three level of nodes in lattice approach. There are four possible paths and three outcomes of the change of underlying asset, including uuS, udS, and ddS. In this report, the total days of simulation is 12. S is the initial price of the underlying asset. One of the most important differences between Lattice and Monte Carlo is that, we assume the price of the underlying asset can either go up or down every day. The rising and dropping ratio are represented by letters 'u' and 'd' respectively in the figure. Rising and dropping ratios are calculated as $e^{r\Delta t}$ and $e^{-r\Delta t}$. Given this assumption, we can grow the lattice from day 0 to day 12.

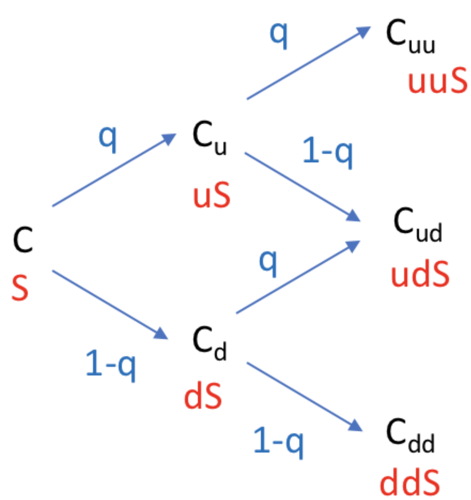

Figure 1 Three-period Option Diagram

1. Algorithm for Asian/Exotic Options

The process of calculating all the options' premium is almost the same as the one shown in Monte Carlo. The only difference here is that, instead of using Equation 1 to simulate the possible asset price at each day, we use the paths that are generated in the lattice figure. In the project, the total number of simulated paths is 4096. The following table shows the first 5 paths of the simulated prices:

Except for the construction of the simulated paths, the rest of the calculation is the same as it in Monte Carlo. For example, for Asian call options, we can take the 13 prices in scenario one and compute the average price. We can then get the payoff of the option for this particular path. We repeat this for all the paths, discount the values to present and take the average as our Asian call option premium.

2. Algorithm for American put Option

The strategy used in Lattice approach for American put option, is to compare the payoff between exercising right away to waiting at each node. Then, calculate option value at each preceding node until the root node.

For example, in Figure 1, the price of American put option at maturity is calculated first using the following equations:

$$C_{uu} = \max\{u^2S - K, 0\}$$

$$C_{ud} = \max\{udS - K, 0\}$$

$$C_{dd} = \max\{d^2S - K, 0\}$$

The payoffs at maturity are calculated. The next step is to calculate the option payoff for the nodes one step to the left. The following equations are applied:

$$q = e^{r\Delta t}$$

$$C_d = \max\{ds - K, e^{-r\Delta t}(qC_{ud} + (1-q)C_{dd})\}$$

$$C_u = \max\{us - K, e^{-r\Delta t}(qC_{uu} + (1-q)C_{ud})\}$$

$$C = \max\{S - K, e^{-r\Delta t}(qC_u + (1-q)C_d)\}$$

For example, '$ds - K$' means the option payoff if we exercise right now. '$e^{-r\Delta t}(qC_{ud} + (1-q)C_{dd}$' gives the value of the option if we wait to next time step. We then compare which value is larger and use it as the value of the option at the time step.

These approaches will be generalized and used for all the nodes until we reach the root node.

III. Result

Monte Carlo Simulation

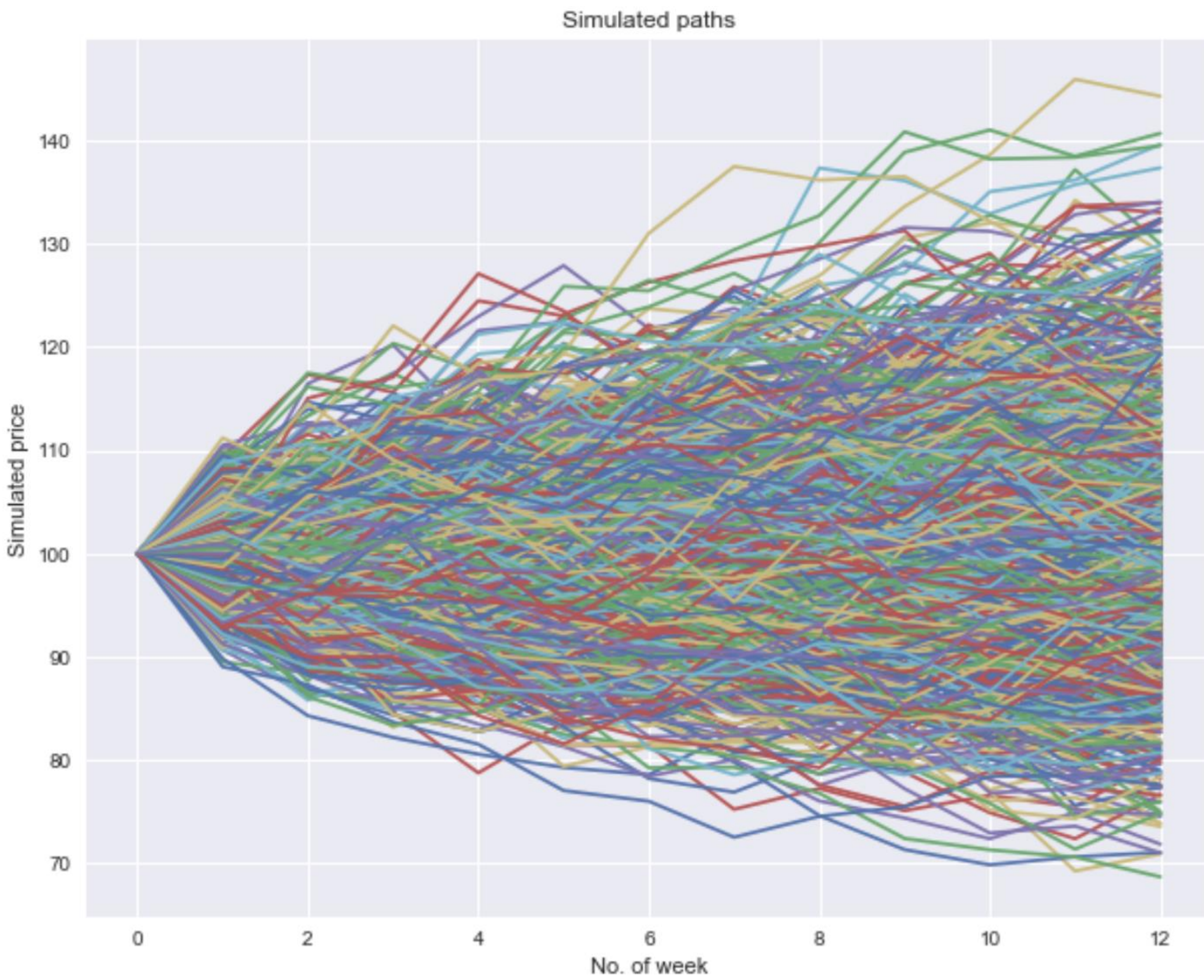

Figure 2 Simulation Path

Premium of Different Option Types

	Average Price	95% confident interval	
Asian call	$1.1413	$1.1238	$1.1587
Asian put	$5.8630	$5.8295	$5.8965
Lookback call	$5.0316	$4.9872	$5.0760
Lookback put	$12.5908	$12.5493	$12.6324
Floating lookback call	$8.0987	$8.0482	$8.1491
Floating lookback put	$8.0613	$8.0183	$8.1043

Table 3 Premium of Different Option Types

	Average Price	95% confident interval		Avg Estimated exiting week	95% confident interval	
American put option	$7.6772	$7.6353	$7.7191	9.5808	9.5616	9.5952

Table 4 Premium of Different Option Types (American Option)

Lattice Approach

	Average Price
Asian call	$1.1818
Asian put	$5.7672
Lookback call	$5.4003
Lookback put	$12.7146
Floating lookback call	$8.5197
Floating lookback put	$8.2178
American put	$7.7836

Table 5 Premium of Different Option Types

# of weeks

# of scenario	0	1	2	3	4	5	6	7	8	9	10	11	12
1	100	103.674	107.483	111.43	115.527	119.772	124.173	128.735	133.465	138.369	143.454	148.7249	154.1896
2	100	103.674	107.483	111.43	115.527	119.772	124.173	128.735	133.465	138.369	143.454	148.7249	143.454
3	100	103.674	107.483	111.43	115.527	119.772	124.173	128.735	133.465	138.369	143.454	138.3698	143.454
4	100	103.674	107.483	111.43	115.527	119.772	124.173	128.735	133.465	138.369	143.454	138.3698	133.4658
5	100	103.674	107.483	111.43	115.527	119.772	124.173	128.735	133.465	138.369	133.465	138.3698	143.454

Table 6 Lattice Approach Example

IV. Discussion

No matter in which simulation approach, a call option for Asian and Lookback always has a much lower price than a put option. The reason is we have an initial price, which is smaller than the strike price. As a result, it is harder to get a price higher than the strike price along each path. As a result, it is much easier to get 0 payoff for the call options.

The option values for floating are quite similar. It indicates that, no matter by which approach, the price will not fluctuate too dramatically.

The Monte Carlo approach incorporates the stochastic process in simulation. It takes use of the random walk and be able to take a wider range of possibilities into consideration, which can reduce our uncertainty. It is also really flexible and widely used. However, the assumptions need to be fairly accurate, so that the paths are reliable. Besides, it tends to underestimate the probability of extreme cases. This may affect the floating options extensively because their values depend on the maximum and minimum value along each simulated path.

Lattice approach is flexible as we can change the values at each node. It can also predict the values of the option precisely if enough time steps are given. It is also able to provide a more detailed view about the simulation. However, the complexity of the Lattice structure grows exponentially, which makes it really hard to check if you have number of time steps larger 4.

V. Conclusion

In conclusion, both Monte Carlo and Lattice approach are really good to use for simulations. They both have their own advantages and disadvantages. When we want to simulate the prices of an underlying asset, we can choose one of these methods based on our requirements. Thus, the objective of this project has been fulfilled.

References

[1] N. MCWILLIAMS, "PRICING AMERICAN OPTIONS USING MONTE CARLOSIMULATION," 15 July 2005.