Available online at www.sciencedirect.com

ScienceDirect

The Journal of Finance and Data Science 2 (2016) 1–25
http://www.keaipublishing.com/en/journals/jfds/

Pricing and hedging of arithmetic Asian options via the Edgeworth series expansion approach

Weiping Li [a,b,d], Su Chen [c,*]

[a] Southwest Jiaotong University, Chengdu, Sichuan Province 610031, PR China
[b] Oklahoma State University, Stillwater, OK 74078-0613, USA
[c] Mathematical Sciences Dept., The University of Memphis, Memphis, TN 38152, USA

Received 15 January 2016; accepted 20 January 2016
Available online 6 June 2016

Abstract

In this paper, we derive a pricing formula for arithmetic Asian options by using the Edgeworth series expansion. Our pricing formula consists of a Black-Scholes-Merton type formula and a finite sum with the estimation of the remainder term. Moreover, we present explicitly a method to compute each term in our pricing formula. The hedging formulas (greek letters) for the arithmetic Asian options are obtained as well. Our formulas for the long lasting question on pricing and hedging arithmetic Asian options are easy to implement with enough accuracy. Our numerical illustration shows that the arithmetic Asian options worths less than the European options under the standard Black-Scholes assumptions, verifies theoretically that the volatility of the arithmetic average is less than the one of the underlying assets, and also discovers an interesting phenomena that the arithmetic Asian option for large fixed strikes such as stocks has higher volatility (elasticity) than the plain European option. However, the elasticity of the arithmetic Asian options for small fixed strikes as trading in currencies and commodity products is much less than the elasticity of the plain European option. These findings are consistent with the ones from the hedgings with respect to the time to expiration, the strike, the present underlying asset price, the interest rate and the volatility.

JEL classification: G10; G12; G13

Keywords: Arithmetic Asian option; Edgworth series expansion; Cumulant; Elasticity; Hedge

1. Introduction

Banker Trust's Tokyo office first issued the arithmetic Asian option for pricing average options on crude oil contracts in1987.[e] The payoff of arithmetic Asian options depends on the arithmetic average price of the underlying

* Corresponding author. Tel.: +1 901 678 1145.
E-mail addresses: w.li@okstate.edu (W. Li), su.chen@memphis.edu (S. Chen).
Peer review under responsibility of China Science Publishing & Media Ltd.
[d] Tel.: +1 405 744 5852.
[e] Boyle and Emanuel[11] first introduce the Asian options. Ingersoll[22] gives the first formulation of a PDE with the T-forward measure. Rogers and Shi[23] formulated this problem using a stock as a numeraire.

http://dx.doi.org/10.1016/j.jfds.2016.01.001

asset, commonly traded as currencies and commodity products. The averaging procedure reduces the significance of the closing price and reduces the effects of abnormal price changes at the maturity of the option. Turnbull and Wakeman[1] point out that the arithmetic Asian options provide a way to ameliorate any possible price distortions that might arise because of a lack of depth in the market of the underlying asset.

Pricing arithmetic Asian options is a difficult issue in finance for several decades. In general, there are two different approaches to find the option pricing formula by (1) solving the corresponding partial differential equation with boundary conditions, and (2) evaluating the option by the known distribution for the underlying price. See Boyle and Boyle[2] for the history and evolution of Asian options. There are different numerical approaches to price the arithmetic Asian options, Linetsky[3] provides a pesudo closed form solution with numerical finding on certain eigenvalues, Hoogland and Neumann[4] and Vecer[5] price the Asian options with simple integral representations of Laplace transformations and with the use of Whittaker functions, see also Alziary, Décamps and Koehl,[6] Chen and Lyuu,[7] Elshegmani, Ahmed and Jaaman[8] and Elshegmani, Ahmed and Jaaman, Zakaria (2011), Sun et al[9] and references therein.[f]

Shreve[10] shows that there is a single PDE with boundary conditions to price the Asian call option (Theorem 7.5.1 and Theorem 7.5.3 of Shreve[10] provide PDEs for the Asian options.). There is no analytic solution for the partial differential equation (PDE) of the arithmetic Asian option. Solving the PDE of the arithmetic Asian option has been an equivalently outstanding issue in mathematical finance for a long time since the PDE is a degenerate partial differential equation in three dimension, and the numerical solution of this PDE is not very accurate due to the low volatility level of the arithmetic Asian option. Elshegmani, Ahmed and Jaaman, Zakaria (2011) transform the PDE of arithmetic Asian option to a parabolic equation with constant coefficients and obtain the analytic solution of the arithmetic Asian option PDE. Their closed form analytical solution for the arithmetic Asian option PDE is lack of tractability (in terms of solutions for a heat equation *without initial condition*), and does not satisfy the necessary boundary conditions for the Asian option. Their solution is only for the PDE of the arithmetic Asian option, *not for the boundary conditions required from the arithmetic Asian options*. The solution form in their Formula (2.2) of Elshegmani, Ahmed and Jaaman, Zakaria (2011) does not satisfy any boundary condition (2.5)−(2.7) for the arithmetic Asian option. Vecer[5] derives the Black-Scholes representation of the value of the Asian option in terms of the corresponding probabilities that the option will end up in the money, and both the price and hedge are computed numerically from the partial differential equations, where the PDEs are in their simplest form under the respective numeraire measures.

A closed-form formula for pricing and hedging of arithmetic Asian options does not exist since this exotic Asian option is introduced by Boyle and Emanuel.[11] This is due to the difficult fact that the average of lognormal random variables is no longer log-normally distributed.[g] The arithmetic Asian options are path-dependent options over the averaging period. This makes binomial tree approach infeasible. Various numerical methods to price arithmetic Asian options have been studied extensively in the literature to compete with the efficiency and accuracy. Kemna and Vorst[12] show that the combined process of the underlying asset and its average is not Gaussian in character which implies that it is impossible to obtain an explicit formula for the arithmetic Asian option pricing.[h] Kemna and Vorst[12] obtain an explicit pricing formula for geometric Asian options and apply the Monte Carlo simulation with variance reduction to price geometric Asian options. Geman and Yor[13] develop some efficient analytic tools to get quasi-explicit pricing formulas in terms of Laplace transformations, and evaluate the Laplace transform of an arithmetic Asian option by a number of standard methods to invert numerically. Milevsky and Posner[14] use moment-matching methods to fit the distribution function of the arithmetic average of the underlying asset with a reciprocal gamma distribution function. Ju[15] makes use of the Taylor expansion of the ratio of the characteristic function of the average to that of the approximating lognormal variable near zero volatility. Dufresne[16] utilizes Laguerre series expansions to regulate the density of the integral of a geometric Brownian motion and uses the results to get another expression for the price of an arithmetic Asian option.

Pricing and hedging of arithmetic Asian options in a tractable manner is not only useful in the financial practice, but also beneficent in mathematical finance. The objective of this paper is to provide a reasonable answer to this challenge

[f] Vecer[24] does have the standard approach used in practice to cover discrete monitoring. Many formulas in literature are less efficient with various approximations or Monte Carlo simulations.

[g] A random variable is lognormal if its logarithm is normally distributed. Standard option pricing methods that rely on the lognormal assumption, cannot be applied in the arithmetic Asian option.

[h] This no-Gaussian process indicates the conditional expectation with respect to the underlying asset price, its arithmetic average and the present time cannot be evaluated explicitly. Arnold (1974) in his book *Stochastic differential equations, theory and applications* gives an account of discussion on this point.

question for several decades with simplest computations. Although it is almost impossible to know the exact probability distribution for the average of the underlying asset prices during the maturity period, one can carefully choose a lognormal probability distribution. With this choice, the genuine distribution can be obtained from the Edgeworth series approximation. Turnbull and Wakeman[1] use this method to approximate the true distribution by first four terms and numerically compare the accuracy of this approximation. Unfortunately, Turnbull and Wakeman[1] neither specify the lognormal probability distribution used in the approximation nor estimate the remaining terms from the truncated Edgeworth series. We start with the choice of this lognormal probability distribution by computing its mean and variance from the arithmetic average of the underlying asset prices (*This has not been addressed in Turnbull and Wakeman*[1] *and in the literature*). We provide the explicit formulas to determine the mean and variance for the arithmetic average of underlying asset prices. Then we derive precise relations for all cumulants in terms of the moments. By the inverse Fourier transformation, we have the distribution identity for the arithmetic average of the underlying asset prices with exact formulas for each coefficients in the Edgeworth series expansion. In this way our closed form formula can be easily implemented, and the hedging formulas follow as well.[i] Very few results have been discussed on the arithmetic Asian option hedging problems in the literature. Our formulas for the arithmetic Asian option hedging fulfill this financial instruments for the financial industry.

As we derive the Edgeworth series of the distribution for the arithmetic average of underlying asset prices, we obtain formulas of pricing and hedging of arithmetic Asian options with this distribution of the arithmetic average of underlying asset prices. Then we further simplify the evaluation formula of the pricing arithmetic Asian options by expressing it into a finite sum of the Edgeworth series and a remainder term. The remainder term is estimated in terms of the fixed strike price among other parameters. This important estimate for the remainder term shows that the pricing formula as a series in terms of Edgeworth series is convergent. In practice, the most common contracts are based on the arithmetic average of the underlying asset prices. Hence, the Edgeworth series expansion formula of the arithmetic Asian options we find in this paper, is indispensable in practice. With the Edgeworth series expansion formula and relations among those parameters in determining the distribution of the arithmetic average of underlying asset prices, we extract the delta, the gamma, the rho and the theta of a fixed strike arithmetic Asian option. This is the first time to have explicit hedging formulas for arithmetic Asian options. It allows us to compare the elasticities of the European options and arithmetic Asian options.

Our findings from the numerical study for the pricing and hedging are (1) arithmetic Asian options are worth less than European options for the same underlying asset and the same expiration time, this confirms the intuitive fact on the arithmetic Asian options; (2) the volatility of the arithmetic average of underlying asset prices is less than the one of the underlying asset prices, and we provide precise formulas for these volatilities and prove this fact (this is the first time to have a concrete proof of this intuitive fact on volatility. Hence, the pricing of the arithmetic Asian option is less than the pricing of the plain vanilla option); (3) arithmetic Asian options are less sensitive to the change of the present underlying asset price than the plain vanilla options only if the underlying asset price is very low, usually less than 1. However, if the present underlying asset price is high, the arithmetic Asian options are way more sensitive to the change of the present underlying asset price than the plain vanilla options. This result explains the reason that arithmetic Asian options are frequently applied on currencies and commodity products, instead of stocks or indexes.

The remainder of the paper is organized as follows. Section 2 presents the basic Asian option and its partial differential equation with boundary conditions. In Section 3, we present our distribution identities, find the distribution of the arithmetic average of underlying asset prices in terms of the chosen lognormal distribution and then derive our closed form formula for the arithmetic Asian option price and the hedging in terms of the initial lognormal distribution. Section 4 illustrates some numerical results and how to determine those parameters in order to use the closed form formulas for the pricing and hedging. Summary is given in Section 5 and some technical proofs are given in the Appendix.

2. Arithmetic Asian options

An Asian option is a type of option whose payoff includes a time average of the underlying asset price. There are two classes of Asian options: (i) the fixed strike (average rate) and (ii) the floating strike (average strike) Asian options. The average may be over the entire time period between the starting time and the expired time or may be over some

[i] As indicated earlier in Kemna and Vorst,[12] it is impossible to get an explicit formula for the arithmetic Asian option price, the series solution for the arithmetic Asian option price is best we can hope for.

period of time that begins later than the starting time of the option and ends with the expired time. Asian options are quite popular in the financial market. Most Asian options are traded in a discretely sample data, but the discrete sample case can be approximated by the continuous model.

Asian options are of European style since an Asian option with the American early exercise feature may be redeemed as early as the start of the averaging period and lose the intent of protection from averaging. We only focus on the European style of the arithmetic Asian option in this paper.

We define the asset price $S(t)$ as a geometric Brownian motion given by

$$\frac{dS(t)}{S(t)} = rdt + \sigma_0 dB(t),$$

where $B(t)$ is a Brownian motion under the risk-neutral probability Q, σ_0 is the volatility of the underlying asset. The fixed-strike Asian call option with payoff at time T is

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+, \tag{2.1}$$

where the strike price K is a nonnegative constant. The price at time $t(0 \le t \le T)$ of this call option is given by

$$V(t) = E^Q\left[e^{-r(T-t)}V(T)\middle|\mathcal{F}_t\right], \tag{2.2}$$

where r is the riskless rate and $\mathcal{F}_t$, $t \geq 0$, is the filtration for the Brownian motion $B(t)$.

By defining another stochastic process $Y(t)$ as $Y(t) = \int_0^t S(u)du$, we have the Asian call option price given by

$$v(t, S(t), Y(t)) = E^Q\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T) - K\right)^+\middle|\mathcal{F}_t\right]. \tag{2.3}$$

The function $v(t,x,y)$ satisfies a partial differential equation as following. For $0 \le t \le T, x \ge 0, y \in \mathbb{R}$, $v(t,x,y)$ is the solution of the following boundary value problem.

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{\sigma^2 x^2}{2}v_{xx}(t,x,y) = rv(t,x,y), \tag{2.4}$$

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+, \tag{2.5}$$

$$\lim_{y\to-\infty} v(t,x,y) = 0, \tag{2.6}$$

$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+. \tag{2.7}$$

See Shreve[10] for more details.

When $tY_t - KT > 0$, there is a pricing formula for the Asian call option with the fixed strike K (see Section 4.3.3 of Kwok[17]; there is no analytically evaluational solution when $tY_t - KT < 0$. Bouaziz, Briys and Crouhy[18] and Chung, Shackleton and Wojakowski[19] find the linear and quadratic approximation of the joint lognormal distribution to evaluate the Asian call option with the fixed strike respectively. Several other analytic approximation methods for pricing Asian option can be found in the references of Kwok.[17] Under the Black-Scholes assumption in Black and Scholes,[20] there is no closed analytic solution for the arithmetic Asian option price with the fixed strike. The difficulty is the source of the technical issue that the average of lognormal random variables is no longer lognormally distributed. There are various numerical methods to price the Asian options in more efficient and accurate manner. For instance, Chen and Lyuu[7] extend the Rogers-Shi formula to get more accuracy in the pricing of Asian option.

Alziary, Décamps and Koehl[6] take a PDE to characterize the price of an Asian option and give a detailed comparison between Asian and European options. They also provide the hedging of Asian option implicitly in terms of the measure defined by using the underlying asset price as a numéraire. Recently Elshegmani, Ahmed, Jaaman and Zakaria[8] transform the PDE of the arithmetic Asian option to a simple parabolic equation with constant coefficients by using some general transformation techniques. They only give a solution to the PDE of the arithmetic Asian option without boundary constrains for this option. In another words, they only solve the pure PDE, not the pricing formula for the arithmetic Asian option. Vecer[5] obtains the numerical values in the geometric Brownian motion model by solving a partial differential equation or by computing the Laplace transform numerically. Vecer's[5] computations of probabilities for the Asian options do not admit a simple closed form solution.

3. An evaluation of arithmetic Asian option prices by Edgeworth series expansion

In order to evaluate the Asian call option (2.3), it is necessary to have the probability density function for the process $1/TY(T)$. The difficulty of evaluation of arithmetic Asian option is that the distribution of the sum of lognormal is not exactly lognormal distribution any more. So the averaged asset price $1/TY(T)$ does not follow lognormal distribution and thus it is not easy to obtain the accessible formula for the valuation of arithmetic Asian option. In our study, we exploit the Edgeworth series expansion to obtain the probability density function of $1/TY(T)$, in which way we achieve the pricing evaluation for arithmetic Asian option with the fixed strike.

Let $F(s)$ be the probability distribution of the random variable $1/TY(T)$, and $A(s)$ be the lognormal probability distribution such that both $dF(s)/ds = f(s)$ and $dA(s)/ds = a(s)$ exist. We will discuss how to choose the lognormal density $a(s)$ in the numerical study section. This restricted class of distributions on the continuous density function, can be easily generalized from the measure theory. For simplicity of presentation in this paper, we assume that the density functions are continuous.

The following quantities will be employed:

$$\alpha_j(F) = \int_{\mathbb{R}} s^j f(s) ds, \tag{3.1}$$

$$\mu_j(F) = \int_{\mathbb{R}} (s - \alpha_1)^j f(s) ds, \tag{3.2}$$

$$\phi(F, t) = \int_{\mathbb{R}} e^{its} f(s) ds, \tag{3.3}$$

where $i^2 = -1$, $\alpha_j(F)$ is the j-th moment of distribution $F(s)$, $\mu_j(F)$ is the j-th central moment of distribution $F(s)$, and $\phi(F,t)$ is the characteristic function of $F(s)$.

Assume that the moments $\alpha_j(F)$ exist for $j \geq 1$. The cumulants $k_j(F)$ also exists and defined by

$$\log\phi(F, t) = \sum_{j=1}^{\infty} k_j(F) \frac{(it)^j}{j!}. \tag{3.4}$$

See Kendall and Stuart.[21] Analogous notions will be employed for the moments, cumulants and characteristic function of $A(s)$. We have the distribution identity from the Edgeworth series expansion (see Appendix for details)

$$f(s) = a(s) + \sum_{j=1}^{\infty} E_j(F, A) \frac{(-1)^j}{j!} \frac{d^j a(s)}{ds^j},$$

The estimated lognormal distribution for the stock price $S(t)$ depends on two parameters: the first and second cumulants of the random variable $\log S(t)$. In the Edgeworth series expansion, we have to choose $\alpha_1(A) = \alpha_1(F) = S(0)e^{rt}$ due to the risk-neutralality argument. I.e., $k_1(A) = k_1(F)$. But the second cumulant of $\log S(t)$ is not fixed. We need to choose an efficiently closest lognormal distribution $A(s)$:

(i) $k_2(A) = k_2(F)$, or

(ii) $a(S(t)) = (S(t)\sigma\sqrt{2\pi t})^{-1}\exp[-\{\log S(t) - (\log\alpha_1(A) - \sigma^2 t/2)\}^2/(2\sigma^2 t)]$, where the parameters $\alpha_1(A) = S(0)e^{rt}$ and $\sigma^2 t = \int_{\mathbb{R}} (\log S(t))^2 dF(S(t)) - \{\int_{\mathbb{R}} \log S(t) dF(S(t))\}^2$.

For simplicity we are going to choose (i) for the presentation of this paper. One can also choose (ii) or other starting lognormal distribution $A(s)$. For our choice, we have

$$f(s) = a(s) + \sum_{j=3}^{\infty} E_j(F,A)\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j}. \tag{3.5}$$

Theorem 3.1. *The price of the Asian option exercised at timunder the risk neutral measure Q is given by*

$$V(t) = e^{-r(T-t)+\mu+\sigma^2/2}N(d_1) - Ke^{-r(T-t)}N(d_2) + e^{-r(T-t)}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-2}a(K)}{d^{j-2}s},$$

where $d_1 = \frac{\mu+\frac{\sigma^2}{2}-\ln K}{\sigma}$, $d_2 = d_1 - \sigma$, $N(x) = \int_{-\infty}^{x}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt$, and μ and σ *are the two parameters in the lognormal probability density function* $a(s)$:

$$a(s) = \frac{1}{s\sigma\sqrt{2\pi}}e^{-\frac{(lns-\mu)^2}{2\sigma^2}}.$$

Theorem 3.1 provides an explicit formula for pricing the arithmetic average Asian options, and settles the difficulty question for several decades with the drawback on dealing with infinite series.[j] Both parameters μ and σ for the lognormal distribution are given in (6.13) and (6.14) for our choice of $A(s)$. The first two terms are the Black-Scholes-Merton type formula for the European option with the lognormal distribution, and differs from the Black-Scholes formula in an essential way. In the risk neutral measure, the expect rate of return for the underlying asset is the risk-free rate of interest. But the expect rate of return for the arithmetic average of the underlying asset prices, under the risk neutral measure, is not the risk-free rate of interest. We give the explicit relation (6.13) in the next section due to the lack of information in the literature. The third term gives the Edgeworth series correction term due to the fact that the arithmetic average of lognormal random variables will not be log-normally distributed. Once the $A(s)$ is determined, the terms $E_j(F,A)$ is determined explicitly in this section. Thus there is an feasible way to implement our closed form formula for pricing, as we will explain further details in the next section.

The leading term $e^{-r(T-t)+\mu+\sigma^2/2}N(d_1) - Ke^{-r(T-t)}N(d_2)$ is similar to the expression for Black-Scholes-Merton European call option, but the parameters are essentially different. Both μ and σ are parameters for a lognormal distribution given in (4.4) which are not the expected rate of return and volatility of the underlying asset price. These parameters μ and σ depends on the initial stock price. See the next section for more details on how to determine μ and σ.

Hedging arithmetic Asian options would be indispensable for market practitioners. We derive analytic results for various hedgings in terms of parameters in Theorem 3.1.

Proposition 3.2. *The delta, the gamma and the elasticity of a fixed-strike arithmetic Asian call option are given by the following.*

$$\begin{aligned}\Delta^a &= \frac{\partial V_t}{\partial S_t}\\ &= \left[e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left(N(d_1)+\frac{n(d_1)}{\sigma}\right) - Ke^{-r\tau}\frac{n(d_2)}{\sigma}\right]\frac{\partial\mu}{\partial S_t} + \left[e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left(\sigma N(d_1)+\left(1-\frac{d_2}{\sigma}\right)n(d_1)\right)\right.\\ &\quad\left.+Ke^{-r\tau}n(d_2)\frac{d_2}{\sigma}\right]\frac{\partial\sigma}{\partial S_t} + e^{-r\tau}a(K)\left[-\frac{d_2}{\sigma}\frac{\partial\mu}{\partial S_t}+\left(-\frac{1}{\sigma}+\frac{d_2^2}{\sigma}\right)\frac{\partial\sigma}{\partial S_t}\right]\cdot\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}),\end{aligned} \tag{3.6}$$

[j] Geman and Yor,[13] Linetsky[3] and Vecer[24] and (2014) also provide a similar formula for the pricing arithmetic Asian options. However, the formulas that appears in the papers of Hoogland and Neumann[4] or in Vecer[5] are simple integral representations of the prices (Laplace transforms) that can be computed in a commercial software such as Mathematica. These computations are quite intense. Vecer[5] finds the probabilistic representation of hedging and pricing at the same time and the partial differential equation for the pricing and hedging with numerical evaluations.

$$\Gamma^a = \frac{\partial^2 V_t}{\partial S_t^2} = \frac{\partial \Delta^a}{\partial S_t}, \tag{3.7}$$

$$\Omega^a = \Delta^a \frac{S_t}{V_t} = \frac{\partial V_t}{V_t} \Big/ \frac{\partial S_t}{S_t}, \tag{3.8}$$

where $\frac{\partial\mu}{\partial S_t}, \frac{\partial^2\mu}{\partial S_t^2}$ and $\frac{\partial\sigma}{\partial S_t}, \frac{\partial^2\sigma}{\partial S_t^2}$ are given in the following,

$$\frac{\partial\mu}{\partial S_t} = \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau} - 1}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)\tau} = \frac{\mu_{a(s)}}{S_t}, \frac{\partial^2\mu}{\partial S_t^2} = 0,$$

$$\frac{\partial\sigma}{\partial S_t} = \frac{\sigma_{a(s)}^2}{S_t\sigma},\ \frac{\partial^2\sigma}{\partial S_t^2} = \frac{\sigma_{a(s)}^2}{S_t^2\sigma}\left(1 - \frac{\sigma_{a(s)}^2}{\sigma}\right).$$

Proposition 3.2 shows that the delta, the gamma and the elasticity of a fixed-strike arithmetic Asian call option is more complicated than the delta, the gamma and the elasticity of the plain European call option, due to the non-lognormal distribution property for the arithmetic average of the underlying asset prices. Our formula give the delta, the gamma hedgings in explicit formulas at the first time. It is computable from the numerical approach and doable in practice.

Proposition 3.3. *The vega, the rho and the theta of a fixed-strike arithmetic Asian call option are given by the following*.

$$\begin{aligned} \upsilon^a &= \frac{\partial V_t}{\partial\sigma} \\ &= e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left[\sigma N(d_1) + \left(1 - \frac{d_2}{\sigma}\right)n(d_1)\right] + Ke^{-r\tau}n(d_2)\frac{d_2}{\sigma} + e^{-r\tau}\frac{a(K)\left(d_2^2-1\right)}{\sigma}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \rho^a = {}& e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left[N(d_1)a_1 + n(d_1)\frac{a_1}{\sigma}\left(\frac{d_2}{\sigma}\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - n(d_1)\frac{a_2 d_2}{2\sigma^2}e^{-2\left(\mu+\sigma^2\right)}\right] - \tau N(d_2) - Ke^{-r\tau}n(d_2) \\ & \left[\frac{a_1}{\sigma}\left(\left(1+\frac{d_2}{\sigma}\right)\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - \frac{a_2 e^{-2\left(\mu+\sigma^2\right)}}{2\sigma}\left(1+\frac{d_2}{\sigma}\right)\right] + e^{-r\tau}\frac{\partial a}{\partial\mu_0}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}. \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Theta^a &= -\frac{\partial V_t}{\partial\tau} \\ &= rV_t - e^{-r\tau}n(d_1)\left(\frac{S_t}{\tau\sigma} - \frac{d_2\mu_{a(s)}}{\sigma}\frac{\partial\sigma}{\partial\tau}\right) + Ke^{-r\tau}n(d_2)\left[\frac{S_t}{\tau\sigma\mu_{a(s)}} - \left(1+\frac{d_2}{\sigma}\right)\frac{\partial\sigma}{\partial\tau}\right] \\ &\quad - e^{-r\tau}a(K)\left[-\frac{d_2 S_t}{\sigma\tau\mu_{a(s)}} + \frac{d_2^2 + d_2\sigma - 1}{\sigma}\frac{\partial\sigma}{\partial\tau}\right]\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}, \end{aligned} \tag{3.11}$$

where the rho of the arithmetic Asian call option is same as the hedging of the expected return of the underlying asset due to the relation $\mu_0 = r - \sigma_0^2$, *and*

$$a_1 = \frac{S_t e^{-\left(\mu + \frac{\sigma^2}{2}\right)}}{\mu_0 + \frac{\sigma_0^2}{2}} = \frac{S_t}{\mu_{a(s)}\left(\mu_0 + \frac{\sigma_0^2}{2}\right)},$$

$$a_2 = -\frac{\sigma_{a(s)}^2}{\mu_0 + \frac{3\sigma_0^2}{2}} - \frac{S_t^2\sigma_0^2}{\left(\mu_0 + \frac{3\sigma_0^2}{2}\right)\left(\mu_0 + \sigma_0^2\right)\left(\mu_0 + \frac{\sigma_0^2}{2}\right)\tau}$$
$$-\frac{S_t^2\left(e^{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)\tau} - 1\right)}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)^2\tau}\left(\frac{e^{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)\tau} - 1}{\left(\mu_0 + \frac{3\sigma_0^2}{2}\right)\tau} - 2\right),$$

$$\frac{\partial a}{\partial \mu_0} = a(K)\left[\frac{e^{-2\left(\mu + \sigma^2\right)}}{\sigma^2}\left(a_1\sigma_{a(s)}^2 - \frac{a_2}{2}\right)\left(1 - d_2\sigma - d_2^2\right) - \frac{a_1 d_2}{\sigma}\right],$$

$$\frac{\partial \sigma}{\partial \tau} = \frac{e^{-2\left(\mu + \sigma^2\right)}}{2\tau\sigma\mu_{a(s)}}\left(2S_t\left(\mu_{a(s)}^2 - \sigma_{a(s)}^2\right) - \mu_{a(s)}\left(\mu_{a(s)}^2 + \sigma_{a(s)}^2\right)\right).$$

Proposition 3.3 shows that the vega, the rho and the theta of a fixed-strike arithmetic Asian call option is implicitly intertwining with almost every parameter involved. The principal part of the vega ($e^{-r\tau + \mu + \frac{\sigma^2}{2}}\left[\sigma N(d_1) + \left(1 - \frac{d_2}{\sigma}\right)n(d_1)\right] + Ke^{-r\tau}n(d_2)\frac{d_2}{\sigma}$) shows that the hedging against the parameter σ in the lognormal distribution always positive. The rho hedging is way complicated than the plain vanilla European call option case due to all parameters related to the r in an implicit manner.

Proposition 3.4. *The evaluation of the price of the arithmetic Asian call option in Theorem 3.1 with first N-terms,* $N \geq 3$,

$$V_N(t) = e^{-r(T-t) + \mu + \sigma^2/2}N(d_1) - Ke^{-r(T-t)}N(d_2) + e^{-r(T-t)}\sum_{j=3}^{N}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-2}a(K)}{d^{j-2}s},$$

has the following properties.

(1) $V(t) = V_N(t) + R_N(t)$, *where the remainder term* $R_N(t)$ *is given by*

$$R_N(t) = e^{-r(T-t)}r_N(F,A)\frac{(-1)^{N+1}}{(N+1)!}\frac{d^{N-1}a(K)}{ds^{N-1}},$$

for the remainder tern $r_N(F,A)$ *in the comparing ratio of characteristic functions.*

(2) The remainder term $R_N(t)$ *can be estimated by*

Table 1
The comparison of Asian option value using Theorem 3.1 and European option value using Black-Scholes: I.

Strike price	Volatility								
	$\sigma_0 = 0.2$			$\sigma_0 = 0.3$			$\sigma_0 = 0.4$		
Option type	Asian	Euro	Diff	Asian	Euro	Diff	Asian	Euro	Diff
$K = 90$	12.56	13.30	−0.74	13.3	14.67	−1.37	14.45	16.37	−1.92
$K = 95$	8.03	9.38	−1.35	9.18	11.25	−2.07	10.63	13.28	−2.65
$K = 100$	4.27	6.17	−1.90	5.79	8.38	−2.59	7.45	10.61	−3.16
$K = 105$	1.80	3.76	−1.96	3.31	6.06	−2.75	4.95	8.36	−3.41
$K = 110$	0.58	2.13	−1.55	1.70	4.26	−2.56	3.12	6.50	−3.38

$$\left|R_N(t)\right| = \left|V(t) - V_N(t)\right| \le C\frac{e^{-r(T-t)}a(K)}{(N+1)!K^{N-1}} \cdot \frac{\left|\left(1+\frac{\ln K-\mu}{\sigma^2}\right)^N - 1\right|}{|\ln K - \mu|},$$

where C is a finite positive constant depending on μ and σ^2.

Proposition 3.4 gives the remainder term and an estimate for the numerical computation of the arithmetic average Asian option with first N terms in the series of Theorem 3.1. The estimate in Proposition 3.4 (2) is different from the usual estimate since the parameter K can be large or small. We have to carefully estimate the remainder term for both large fixed strike and the small fixed strike so that terms involving with K cannot be bounded by a simple term. The remainder estimate in Proposition 3.4 (2) works for both the large and small strike parameter K. However, the estimate shows that $\lim_{N\to\infty}|R_N(t)| = 0$ for any fixed positive K. Therefore the series evaluation in Theorem 3.1 is convergent. See Table 3, we fix the other parameters and compare the remainder estimates for one large $K = 90$ and one small $K = 0.45$. For the large K, R_N is getting smaller when $N \ge 3$ and bigger; for the smaller K, the remainder term R_N is already infinitesimally small for $N = 3$. Therefore the bound given in Proposition 3.4 (2) serves the purpose for any numerical implementation.

4. Numerical illustrations

In this section, we illustrate the evaluation of the arithmetic average Asian option explicitly by using Theorem 3.1 step by step. A concrete example is also given to compare prices of the arithmetic average Asian call options and European call options in Table 2. Meanwhile, the properties of the arithmetic average Asian option with respect to the strike price, interest rate, volatility, asset return and time to maturity are illustrated in this section.

Table 2
The Greeks and Value of Asian option.

Initial setup	$r = 0.09$, $T = 1/3$									
	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
	$\mu_0 = 0.15$ $\sigma_0 = 0.2$ $S_0 = 100$ K = 90		$\mu_0 = 0.1$ $\sigma_0 = 0.15$ $S_0 = 50$ K = 45		$\mu_0 = 0.035$ $\sigma_0 = 0.03$ $S_0 = 5$ K = 4.5		$\mu_0 = 0.015$ $\sigma_0 = 0.02$ $S_0 = 0.5$ K = 0.45		$\mu_0 = 0.015$ $\sigma_0 = 0.02$ $S_0 = 0.1$ K = 0.09	
Option type	Euro	Asian	Euro	Asian	Euro	Asian	Euro	Asian	Euro	Asian
Option price	13.2999	12.5612	6.4325	5.7683	0.6330	0.5140	0.0633	0.0498	0.0127	0.0100
Delta	0.8906	183.7933	0.9459	58.7927	1	4.9124	1	0.4877	1	0.0975
Omega	6.6967	1463.2	7.3525	509.6173	7.8990	47.7859	7.8990	4.9011	7.8990	0.9802
Vega	10.8102	11.6821	3.1698	3.3519	5.88E-14	0.0489	1.56E-31	0.0032	3.22E-32	6.49E-04
Rho	25.2550	−71.7104	13.6208	121.0292	1.4557	136.5419	0.1456	31.5892	0.0291	6.0512
Theta	−10.0619	−634.3797	−4.3908	−163.5583	−0.3930	0.0463	−0.0393	0.0045	−0.0079	8.96E-04

The Greeks of the arithmetic average Asian option for large initial underlying asset value and large fixed strike price behave differently for those of smaller initial underlying asset value and small fixed strike price (see Table 2 on 4 scenarios). We compute the Greeks for these two different scenarios by using Proposition 3.2 and Proposition 3.3. The comparisons from these two scenarios implies an important information about the arithmetic average Asian option used in the market. Finally, the residual terms of arithmetic average Asian option in Proposition 3.4 are evaluated numerically in Table 3.

Let us first outline the procedure for evaluation of the arithmetic average Asian call option, and then evaluate the necessary parameters we need. The arithmetic average Asian call option price can be implemented by the following steps:

(1) Find the mean and variance of the arithmetic average of the underlying asset prices, denoted by $E[1/TY(T)]$ and $Var(1/TY(T))$, with respect to the asset return rate μ_0 and underlying asset volatility σ_0;
(2) Obtain two parameters μ and σ in the lognormal density function $a(s)$ given in Theorem (3.1) by solving the equations $\mu_{a(s)} = E\left[\frac{1}{T}Y(T)\right]$ and $\sigma^2_{a(s)} = Var\left(\frac{1}{T}Y(T)\right)$, where $\mu_{a(s)}$ and $\sigma^2_{a(s)}$ are the mean and variance of lognormal distribution with density function $a(s)$ respectively;
(3) Follow equations in Section 3 and estimate $E_i(F,A)$ by simulation;
(4) Caculate the arithmetic average Asian option price $V(t)$ by plugging all the parameters in Theorem 3.1.

The most challenge part of calculating the arithmetic average Asian option price numerically is to determine the two parameters μ and σ in Theorem 3.1. They are neither the underlying asset return μ_0 and the volatility σ_0, nor the mean $\mu_{a(s)}$ and the variance $\sigma^2_{a(s)}$ of the lognormal distribution with the density function $a(s)$. Moreover, previous works, such as Turnbull and Wakeman,[1] did not explain their method to determine μ and σ in their numerical computation and comparisons. In fact, it is nontrivial to give the explicit formulas for μ and σ since the probability distribution of the random variable $1/TY(T)$ is unknown. Unlike the geometric average Asian option, the arithmetic average Asian option involves the sum of lognormal distribution, which is obviously not a lognormal anymore. Fortunately, we can compute the expectation and variance of this unknown random variable $1/TY(T)$ and take them as the expectation and variance for the lognormal probability distribution $a(s)$.

Our method of solving μ and σ is motivated by how to choose the $a(s)$ so that it becomes a better approximation to $f(s)$ as we discussed in Section 3. We choose the lognormal probability distribution $a(s)$ such that the distributions with density function $a(s)$ and $f(s)$ share the same mean and variance, i.e., the same first two moments. The efficiently "closest" lognormal distribution $A(s)$ is chosen when the following two equations hold:

Table 3
The residual term of the Asian option value.

Initial setup	$r = 0.09, T = 1/3$				
	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
	$\mu_0 = 0.15$	$\mu_0 = 0.1$	$\mu_0 = 0.035$	$\mu_0 = 0.015$	$\mu_0 = 0.015$
	$\sigma_0 = 0.2$	$\sigma_0 = 0.15$	$\sigma_0 = 0.03$	$\sigma_0 = 0.02$	$\sigma_0 = 0.02$
	$S_0 = 100$	$S_0 = 50$	$S_0 = 5$	$S_0 = 0.5$	$S_0 = 0.1$
	$K = 90$	$K = 45$	$K = 4.5$	$K = 0.45$	$K = 0.09$
Residual			R_N		
$N = 1$	0.1755	0.2166	1.77E-24	4.01E-53	2.00E-52
$N = 2$	0.0177	0.0748	1.45E-22	7.19E-50	1.80E-48
$N = 3$	0.0014	0.0198	8.94E-21	9.68E-47	1.21E-44
$N = 4$	8.65E-05	4.20E-03	4.40E-19	1.04E-43	6.52E-41
$N = 5$	4.51E-06	7.38E-04	1.81E-17	9.36E-41	2.92E-37
$N = 6$	2.02E-07	1.11E-04	6.35E-16	7.20E-38	1.12E-33
$N = 7$	7.88E-09	1.47E-05	1.96E-14	4.85E-35	3.79E-30
$N = 8$	2.74E-10	1.73E-06	5.35E-13	2.90E-32	1.13E-26
$N = 9$	8.75E-12	1.8318E-07	1.32E-11	1.56E-29	3.05E-23
$N = 10$	2.44E-13	1.76E-08	2.95E-10	7.64E-27	7.46E-20

$$\mu_{a(s)} = E\left[\frac{1}{T}Y(T)\right] = S(0)\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T} - 1}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)T}, \tag{4.1}$$

$$\sigma_{a(s)}^2 = Var\left[\frac{1}{T}Y(T)\right] \tag{4.2}$$

$$= \frac{S(0)^2}{T^2}\left[\frac{2}{\mu_0 + \frac{3\sigma_0^2}{2}}\left(\frac{e^{2\left(\mu_0+\sigma_0^2\right)T} - 1}{2\left(\mu_0 + \sigma_0^2\right)} - \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T} - 1}{\mu_0 + \frac{\sigma_0^2}{2}}\right) - \frac{\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T} - 1\right)^2}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)^2}\right].$$

I.e., $k_1(A) = k_1(F)$ and $k_2(A) = k_2(F)$. To solve these two equations for parameters μ and σ, we first need to derive $E[1/TY(T)]$ and $Var(1/TY(T))$ with respect to the underlying asset return rate μ_0 and underlying asset volatility σ_0 in Step 1, and then write the $\mu_{a(s)}$ and $\sigma_{a(s)}^2$ in terms of μ and σ and solve the equation system (4.1) for μ and σ in Step 2.

Formulas (4.1) and (4.2) are derived in Appendix. This finishes Step 1.

Because of the average feature, it is intuitive that the arithmetic Asian options enjoys lower volatilities than their underlying asset. With the Formula (6.14), we finally can prove this explicitly:

$$Var(S(T)) = S_t^2\left(e^{2\left(\mu_0+\sigma_0^2\right)\tau} - e^{\left(2\mu_0+\sigma_0^2\right)\tau}\right) \geq Var\left(\frac{1}{T-t}Y(T-t)\right), \tag{4.3}$$

where the inequality is proved through basic calculus verifications. This gives first clear verification on the comparison of volatilities between the underlying asset and its arithmetic average since we know the formula for $Var(1/T - tY(T-t))$. Therefore the arithmetic Asian call option is cheaper than the European call option.

It is known that the mean and variance for lognormal distribution with two parameters μ and σ are:

$$\mu_{a(s)} = e^{\mu+\sigma^2/2}, \tag{4.4}$$

$$\sigma_{a(s)} = e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right).$$

Plugging Equations (6.13), (6.14) and (4.4) into the equation system (4.1), we can solve for μ and σ. This is a nonlinear equation system so it is more convenient and practical to solve them numerically. Thus we solve the system of nonlinear equations by numerical iteration method. First of all, we solve the equation for μ with respect to σ and then solve the second equation for σ with respect to μ as follows:

$$e^{\mu} = S(0)e^{-\sigma^2/2}\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T} - 1}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)T}, \tag{4.5}$$

and

$$\sigma^2 = ln\frac{1 + \sqrt{1 + 4b}}{2}, \tag{4.6}$$

where $b = \sigma^2_{a(s)}e^{-2\mu}$. Finally, assign an initial value to σ to be zero, and then plug in Equations (4.5) and (4.6) iteratively until the solutions stabilize to the approximated fixed values. In our simulation this process converges very fast, both parameters μ and σ converge to fixed values in less than 10 iterations. This finishes Step 2.

By choosing the same parameters as in Turnbull and Wakeman,[1] we compare our numerical values from formulas in Section 3 by implementing Step 3 and Step 4 with their approximations. Choose the spot price $S(0) = 100$, the interest rate $r = 9\%$, the time to maturity $T = 120$ days, the averaging period 120 days and asset return rate $\mu_0 = 15\%$. *Note that Turnbull and Wakeman*[1] *did not specify the expected return rate for the underlying asset they chose*. Therefore, it is inappropriate to compare our results with values in Turnbull and Wakeman.[1] Hence, we compare the arithmetic average Asian option price with European option price under same parameters, just as in Turnbull and Wakeman.[1] We compute both the valuations of the arithmetic average Asian call option and the vanilla European call option when the volatility varies from 0.2,0.3 to 0.4 and the strike price $K = 90$, 95, 100, 105 or 110. The results are listed in Table 1.

Table 1 indicates that European call options cost more than the arithmetic average Asian call options for all five different strike prices and three different volatilities, which is consistent with the common sense. The European call option price is based on the underlying asset price at the time of expiration, while the arithmetic average Asian option price is based on the arithmetic averaged asset price over the average period. It is known that the arithmetic averaged asset price has less variability than the asset price at the expiration. Thus, the arithmetic average Asian call option is worth less than the European call option. Table 1 also shows that the arithmetic average Asian call option price is decreasing with respect to the strike price, and increasing with respect to the volatility of the underlying asset. The deeper the arithmetic average Asian call option is in the money, the more it is worth. The higher risk the underlying asset price goes, the more the arithmetic average Asian call option costs. Moreover, the difference between the European call option price and the arithmetic average Asian call option price decreases faster as the strike price increases. When the strike price increase to 105 (a little bit greater than the spot price), the difference decreases slower. The price of the European call option increases faster than the price of the arithmetic average Asian call option when the underlying volatility risk increases. These results are further confirmed by Fig. 1a, c and the Greeks of both options in Table 2.

We present a three-dimensional graph of the price of the arithmetic average Asian call option with respect to two parameters among the time to expiration, the strike price, the interest rate, the underlying asset volatility and the rate of underlying asset return in Fig. 1. In all of Fig. 1a–d, the initial underlying asset price is set to be 100 and the strike price, the interest rate, the volatility, and the rate of underlying asset return are fixed to be 90, 0.09, 0.2, 0.15 respectively if they are not drawn on the axis of subfigures. All the four subfigures show that the value of the arithmetic average Asian call option decreases as expiration draws nearer. Time decay is a well known phenomena in the option trading. However, this is not always the case for the arithmetic average Asian call option when it applies to currency or commodity price which involve very small values of the strike price, the volatility and the rate of asset return. This special phenomena is illustrated in Table 3. Fig. 1a indicates that the value of the arithmetic average Asian call option is close to 0 when the strike price is above a certain amount, near 105. When the strike price is below 105, the value of the arithmetic average Asian call option increases dramatically as the strike price decreases. This conclusion is consistent with Table 1. Fig. 1b, c, d also show that the value of the arithmetic average Asian call option rises as the interest rate *decreases*, the volatility increases or the rate of underlying asset return increases if other factors keep constant.

Table 2 compares the five Greek letters-Delta, Omega, Vega, Rho and Theta, of the arithmetic average Asian call option with those of the European call option in five distinctive scenarios. Scenario 1 and Scenario 2 demonstrate examples of the arithmetic average Asian call option when the initial asset price, the strike price, the volatility and the rate of underlying asset return of the underlying assets are in a relative large scale. A typical underlying asset in the market is a stock or an index. Thus, we set the rate of underlying asset return, the volatility, the initial asset price and the strike price to be 0.15, 0.2, 100 and 90 respectively in Scenario 1 and 0.1, 0.15, 50,45 in Scenario 2. Scenario 4 and Scenario 5 demonstrate examples of the arithmetic average Asian call option when the initial asset price, the strike price, the volatility and the rate of asset return of the underlying assets are in a relatively small scale. A typical asset in the market is a currency or a commodity. We set the rate of asset return, the asset volatility, the initial asset price and strike price to be 0.015, 0.02, 0.5 and 0.45 respectively in Scenario 4 and 0.015, 0.02, 0.1, 0.09 in Scenario 5. The initial asset price, the strike price, the volatility and the rate of underlying asset return rate in Scenario 3 are between Scenario 2 and Scenario 4. Interestingly, Scenario 1 and Scenario 2 show that the Delta of the arithmetic average Asian

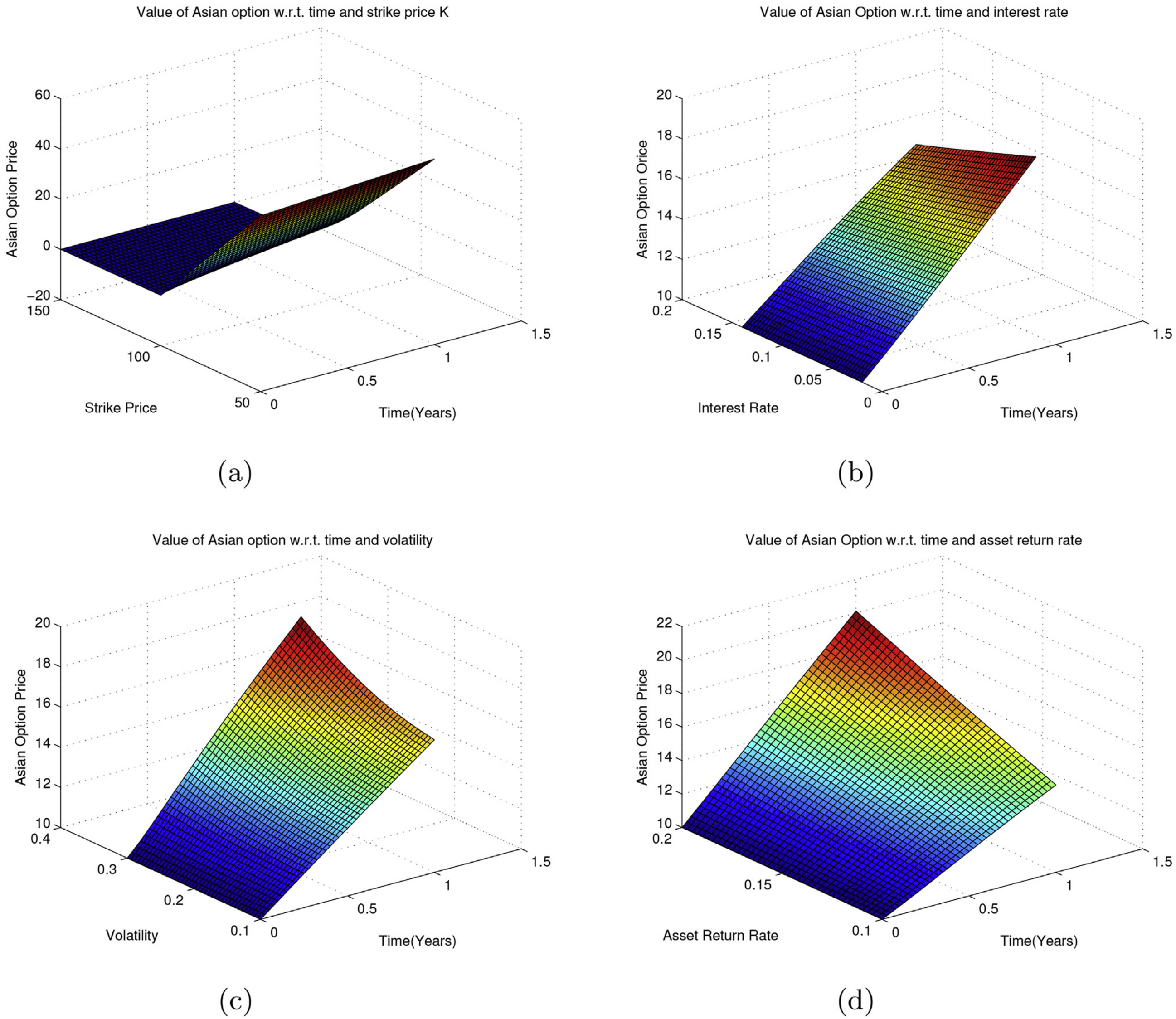

Fig. 1. The Asian option value with respect to time to expiration, strike price, interest rate, volatility and asset return rate.

call option is extremely higher than the one of the European call option. The Delta of the arithmetic average Asian call option is about 184 in Scenario 1, which indicates that the arithmetic average Asian call option price is much more sensitive to the underlying asset price. However, in Scenario 4 and Scenario 5, the Delta of the arithmetic average Asian call option is quite reasonable, and lies in between 0.09 and 0.5. The arithmetic average Asian call option in Scenario 4 and Scenario 5 become much less sensitive to the initial underlying asset price. The Omega (the elasticity of the arithmetic average Asian call option) leads to the same conclusion as the Delta does with respect to the scales of the initial underlying asset price.

The elasticity is the volatility of the option. The findings reflect the volatility of the arithmetic average Asian call option is higher when the underlying asset price is larger. This may explain the fact that there are more and more arithmetic average Asian call option tradings in currency and commodity market rather than stock market. By comparing the Vega of Scenario 1 and Scenario 2 with the one of Scenario 4 and Scenario 5, we find that in Scenario 4 and Scenario 5, the arithmetic average Asian call option price is not sensitive to the underlying asset volatility at all, almost close to 0. We also discover that the Rho value of the arithmetic average Asian call option is negative when the underlying asset price is relatively large as in Scenario 1. The Rho value of the arithmetic average Asian call option is positive when the underlying asset price is relatively small as in Scenario 2–Scenario 5. The negative effect is mainly from the fourth term in the rho of Proposition 3.3 where the strike price is relatively large to dominate the sign in Scenario 1 of Table 2.

The arithmetic Asian option is sensitive with respect to the initial stock price in an interesting way. We have an example on $K = 0.8S0$ that the delta of the arithmetic Asian option is less sensitive with small initial stock price S_0 (the delta is less than 1 when $S_0 < 1$) and the delta of the arithmetic Asian option is more sensitive with large initial stock price (the delta is greater than 1 when $S_0 > 1$). *From numerical examples, we obtain the delta is less than 1 when the initial stock price is less than a critical value and is greater than 1 when the initial stock price is greater than the critical value.*[k] The delta is proportional with the initial stock price. We believe that this is the reason behind the practical implements of arithmetic Asian options on currencies and exchange rates, rather than on stocks.

Finally, we estimate the residual error term given by Proposition 3.4 in Table 3. Table 3 lists the first 10 residual terms excluding the constant C factor, i.e., the term $\frac{e^{-r(T-t)}a(K)}{(N+1)!K^{N-1}} \cdot \frac{\left|\left(1+\frac{\ln K-\mu}{\sigma^2}\right)^N - 1\right|}{|\ln K-\mu|}$ in two different scenarios as described in the previous paragraph. The results show that errors in Scenario 1 & 2 become smaller as the number of terms N increases, while errors in Scenario 3, 4 & 5, become extremely small in first few N, and gets relatively a little bigger as the number of terms N increases (still keeps the error in the order of 10^{-27} for $N = 10$ in Scenario 4). All errors are relatively quite small. If the underlying asset prices are higher (as in Scenario 1 & 2), then choosing $N = 3$ or 4 is enough to achieve the required accuracy. If the underlying asset prices are smaller (as in Scenario 3, 4 & 5), then choosing $N = 1$ is enough to achieve the required accuracy. Our evaluations and errors are straightforward computable, and easier to control the accuracy than those approaches in literature to deal with integrals and Laplace transformations.

5. Conclusion

Bankers Trust's Tokyo office used the arithmetic Asian options for pricing average options on crude oil contracts since 1987. The arithmetic Asian options are useful for hedging future transactions whose risk is related to the average price of the underlying asset. The arithmetic Asian options are popular in financial markets where the average feature leads a smoothing effect, since the prices of arithmetic Asian options are less fluctuating to price manipulation. Easy accessible formulas for pricing arithmetic Asian options under the standard Black-Scholes assumptions are extremely important in real world.

In this paper, we obtain an evaluation formula for pricing arithmetic Asian options under the standard Black-Scholes assumptions by using the Edgeworth series expansion. This resolves the long-time puzzle to find an analytic pricing formula for the arithmetic Asian options in terms of fast convergent series. The easy accessible formula of the arithmetic Asian options is indispensable in practice. With the analytic formula of pricing arithmetic Asian options, the sensitivities of the pricing of the arithmetic Asian option with respect to the stock price, the fixed strike price,

[k] We refer the critical value as the initial stock price S_0^* such that its corresponding delta equals 1.

the volatility, the risk-free interest rate and the time to maturity are derived in explicit formulas for risk management in this paper. This is the first time, to the authors' knowledge, to have such explicit formulas for hedgings of arithmetic Asian options. For implementations, we also express the price of the arithmetic Asian option by the first N-finite sum of the series and the remainder term. The residual error estimates are obtained in terms of the fixed strike price. The fixed strike price can be any positive number, this makes the estimation of the remainder term interesting in numerical analysis.

We clarify the evaluation of the arithmetic Asian option price in terms of the underlying stock price and an initial lognormal distribution. The first and second moments of the arithmetic average of underlying stock prices over the maturity are obtained, under the standard assumption that the stock price is modeled as a geometric Brownian motion. This allows us to confirm the intuitive fact that the arithmetic average of underlying asset prices has smaller volatility than the underlying asset price. Hence, the arithmetic Asian option is cheaper than the European option. We choose the lognormal probability distribution with the same first and second moments of the arithmetic average of stock prices over the maturity. The parameters in the lognormal distribution are determined in a nonlinear system. These are important parameters in the pricing formulas for the arithmetic Asian options.

A numerical method to price arithmetic Asian options is presented in this paper. We find the error control estimates for both large fixed strike prices and small fixed strike prices. This shows an interesting phenomenon that the arithmetic Asian option for stocks has higher volatility (elasticity) than the plain European option. The elasticity of the arithmetic Asian options for small fixed strikes as trading in currencies and commodity products are much less than the elasticity of the European option. With the evaluation formula of the arithmetic Asian option in Theorem 3.1, we find that it is more practical to adapt the arithmetic Asian option with lower strike prices rather than larger strike prices.

Appendix

Derivation of the Formula (3.5)

Note that

$$\phi(F,t) = 1 + \sum_{j=1}^{\infty} \alpha_j(F) \frac{(it)^j}{j!}.$$

By using the series expansion $log(1+x)$ within the convergent region, we have

$$\log\phi(F,t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{j=1}^{\infty} \alpha_j(F) \frac{(it)^j}{j!} \right)^k, \tag{6.1}$$

where the above identity holds in the sense of the convergence for series.

$$\left(\sum_{j=1}^{\infty} \alpha_j(F) \frac{(it)^j}{j!} \right)^k = \sum_{j=k}^{\infty} c_j^{(k)}(F) \frac{(it)^j}{j!}, \tag{6.2}$$

$$c_j^{(k)}(F) = \sum_{\substack{n_1+n_2+\cdots n_k=j \\ n_1 \geq 1, \cdots, n_k \geq 1}} j! \frac{\alpha_{n_1}(F)}{n_1!} \cdot \frac{\alpha_{n_2}(F)}{n_2!} \cdots \frac{\alpha_{n_k}(F)}{n_k!}. \tag{6.3}$$

We have $\alpha_j(F) = c_j^{(1)}(F)$. Thus we obtain the cumulants in terms of the moments in the following precise relations. For $j \geq 1$,

$$k_j(F) = \alpha_j(F) - \frac{c_j^{(2)}(F)}{2} + \frac{c_j^{(3)}(F)}{3} - \cdots + (-1)^{j-1} \frac{c_j^{(j)}(F)}{j} = \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k} c_j^{(k)}(F). \tag{6.4}$$

Analogous notions will be employed for the moments, cumulants and characteristic function of $A(s)$, i.,e., $\alpha_j(A), \mu_j(A), k_j(A), c_j^{(k)}(A)$ and $\phi(A,t)$.

$$\log\frac{\phi(F,t)}{\phi(A,t)} = \sum_{j=1}^{\infty}\left(k_j(F)-k_j(A)\right)\frac{(it)^j}{j!}, \tag{6.5}$$

$$k_j(F)-k_j(A) = \sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\left(c_j^{(k)}(F)-c_j^{(k)}(A)\right). \tag{6.6}$$

Thus we obtain the ratio of comparing characteristic functions in the following.

$$\frac{\phi(F,t)}{\phi(A,t)} = \exp\left\{\sum_{j=1}^{\infty}\left(k_j(F)-k_j(A)\right)\frac{(it)^j}{j!}\right\} = \sum_{j=0}^{\infty}E_j(F,A)\frac{(it)^j}{j!},$$

where the exponential function is an analytic function, the power series expansion converges everywhere for the value $\sum_{j=1}^{\infty}(k_j(F)-k_j(A))\frac{(it)^j}{j!}$, and $E_0(F,A)=1$ and for $j \geq 1$

$$E_j(F,A) = \sum_{n=1}^{j}\frac{k_j^{(n)}(F,A)}{n!}$$

$$k_j^{(n)}(F,A) = \sum_{\substack{m_1+m_2+\cdots+m_n=j \\ m_1\geq 1, m_2\geq 1,\cdots,m_n\geq 1}} j!\prod_{j=1}^{n}\frac{k_{m_j}(F)-k_{m_j}(A)}{m_j!}$$

where the formulas of $E_j(F,A)$ and $k_j^{(n)}(F,A)$ are obtained by the same method in (6.3) and (6.4). By taking the inverse Fourier transform of the following

$$\phi(F,t) = \left(\sum_{j=0}^{\infty}E_j(F,A)\frac{(it)^j}{j!}\right)\cdot\phi(A,t), \tag{6.7}$$

we get the distribution identity from the Edgeworth series expansion,

$$f(s) = a(s) + \sum_{j=1}^{\infty}E_j(F,A)\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j}, \tag{6.8}$$

where the following identities hold,

$$f(s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-its}\phi(F,t)dt,$$

$$a(s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-its}\phi(A,t)dt,$$

$$(-1)^j\frac{d^j a(s)}{ds^j} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-its}(it)^j\phi(A,t)dt.$$

Proof of Theorem 3.1

The price of Asian option at time $t(0 \le t \le T)$ is

$$V(t) = E^Q\left[e^{-r(T-t)}\left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+ \middle| \mathcal{F}_t\right].$$

Using the probability distribution $f(s)$ of the random variable $1/TY(T)$, we have

$$\begin{aligned} V(t) &= \int_K^\infty e^{-r(T-t)}(s-K)f(s)ds \\ &= e^{-r(T-t)}\int_K^\infty (s-K)\left(a(s) + \sum_{j=3}^\infty E_j(F,A)\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j}\right)ds \\ &= e^{-r(T-t)}\int_K^\infty (s-K)a(s)ds + e^{-r(T-t)}\sum_{j=3}^\infty \int_K^\infty (s-K)\left(E_j(F,A)\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j}\right)ds. \end{aligned}$$

The first term can be evaluated by the standard method for Black-Scholes,

$$e^{-r(T-t)}\int_K^\infty (s-K)a(s)ds = e^{-r(T-t)+\mu+\sigma^2/2}N(d_1) - Ke^{-r(T-t)}N(d_2),$$

with $d_1 = \frac{\mu+\frac{\sigma^2}{2}-\ln K}{\sigma}$, $d_2 = d_1 - \sigma$, where μ and σ are the mean and standard deviation for the lognormal $a(s)$. For $j \geq 3$, we have

$$\int_K^\infty (s-K)\frac{d^j a(s)}{d^j s} = \frac{d^{j-2}a(K)}{d^{j-2}s},$$

by using the lognormal distribution properties $\lim_{x\to\infty} x^n a(x) = 0$ for $n>0$ in Kendall and Stuart (1977, p. 180) and the integration by parts. Now our result follows.

Proof of Proposition 3.2

By Theorem 3.1, we have

$$\begin{aligned} \Delta^a &= e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left(\frac{\partial\mu}{\partial S_t} + \sigma\frac{\partial\sigma}{\partial S_t}\right)N(d_1) + e^{-r\tau+\mu+\frac{\sigma^2}{2}}n(d_1)\frac{\partial d_1}{\partial S_t} - Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial S_t} \\ &\quad + e^{-r\tau}\sum_{j=3}^\infty (-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}\left(\frac{\partial a}{\partial\mu}\frac{\partial\mu}{\partial S_t} + \frac{\partial a}{\partial\sigma}\frac{\partial\sigma}{\partial S_t}\right), \end{aligned}$$

where $n(d_i) = \frac{1}{\sqrt{2\pi}}e^{-d_i^2/2}$ with d_i in Theorem 3.1 for $i = 1,2$. By (6.13) and (6.14), we have

$$\frac{\partial\mu}{\partial S_t} = \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau} - 1}{\left(\mu_0 + \frac{\sigma_0^2}{2}\right)\tau}, \tag{6.9}$$

$$\sigma\frac{\partial\sigma}{\partial S_t}=\frac{S_t}{\tau^2}\left[\frac{2}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2\left(\mu_0+\sigma_0^2\right)\tau}-1}{2(\mu_0+\sigma_0^2)}-\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right)-\frac{\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1\right)^2}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2}\right]. \tag{6.10}$$

Hence $\frac{\partial\sigma}{\partial S_t}=\frac{\sigma_{a(s)}^2}{S_t\sigma}$.

$$\frac{\partial^2\sigma}{\partial S_t^2}=\frac{1}{\sigma\tau^2}\left[\frac{2}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2\left(\mu_0+\sigma_0^2\right)\tau}-1}{2(\mu_0+\sigma_0^2)}-\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right)-\frac{\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1\right)^2}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2}\right]-\left(\frac{\partial\sigma}{\partial S_t}\right)^2.$$
$$=\frac{\sigma_{a(s)}^2}{S_t^2\sigma}\left(1-\frac{\sigma_{a(s)}^2}{\sigma}\right).$$

$$\frac{\partial d_1}{\partial S_t}=\frac{\partial d_1}{\partial\mu}\frac{\partial\mu}{\partial S_t}+\frac{\partial d_1}{\partial\sigma}\frac{\partial\sigma}{\partial S_t}$$
$$=\frac{1}{\sigma}\frac{\partial\mu}{\partial S_t}+\left(1-\frac{d_2}{\sigma}\right)\frac{\partial\sigma}{\partial S_t},$$

where $\frac{\partial\mu}{\partial S_t}$ is given in (6.9) and $\frac{\partial\sigma}{\partial S_t}$ is given in (6.10).

$$\frac{\partial d_2}{\partial S_t}=\frac{1}{\sigma}\left(\frac{\partial\mu}{\partial S_t}-d_2\frac{\partial\sigma}{\partial S_t}\right).$$

By the lognormal probability density $a(s)$, we have

$$\frac{\partial a}{\partial\mu}\frac{\partial\mu}{\partial S_t}+\frac{\partial a}{\partial\sigma}\frac{\partial\sigma}{\partial S_t}=a(K)\left[\frac{(\ln K-\mu)}{\sigma^2}\frac{\partial\mu}{\partial S_t}+\left(-\frac{1}{\sigma}+\frac{(\ln K-\mu)^2}{\sigma^3}\right)\frac{\partial\sigma}{\partial S_t}\right],$$

where $\frac{\partial a(s)}{\partial\mu}=a(s)\frac{(\ln K-\mu)}{\sigma^2}$ and $\frac{\partial a(s)}{\partial\sigma}=a(s)\left(-\frac{1}{\sigma}+\frac{(\ln s-\mu)^2}{\sigma^3}\right)$ from the direct calculations. Thus we obtain the formula by re-grouping terms together.

Proof of Proposition 3.3

By Theorem 3.1, we have

$$v^a=e^{-r\tau+\mu+\frac{\sigma^2}{2}}N(d_1)\sigma+e^{-r\tau+\mu+\frac{\sigma^2}{2}}n(d_1)\frac{\partial d_1}{\partial\sigma}-Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial\sigma}+e^{-r\tau}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}\frac{\partial a}{\partial\sigma}.$$

$$\frac{\partial d_1}{\partial\sigma}=1-\frac{d_2}{\sigma},$$

$$\frac{\partial d_2}{\partial\sigma}=-\frac{d_2}{\sigma},$$

$$\frac{\partial a}{\partial \sigma} = a(K)\left(-\frac{1}{\sigma}+\frac{(\ln K-\mu)^2}{\sigma^3}\right)$$

Therefore we have the vega for the arithmetic Asian call option by combining terms.
From the relation $\mu_0 = r - \frac{\sigma_0^2}{2}$, $\frac{\partial V_t}{\partial r} = \frac{\partial V_t}{\partial \mu_0}$. Thus we have

$$\rho^a = e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left(-\tau+\frac{\partial \mu}{\partial \mu_0}+\sigma\frac{\partial \sigma}{\partial \mu_0}\right)N(d_1)+e^{-r\tau+\mu+\frac{\sigma^2}{2}}n(d_1)\frac{\partial d_1}{\partial \mu_0}+Ke^{-r\tau}\tau N(d_2)-Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial \mu_0}-\tau\Big(V_t - e^{-r\tau+\mu+\frac{\sigma^2}{2}}N(d_1)+Ke^{-r\tau}N(d_2)\Big)+e^{-r\tau}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}\frac{\partial a}{\partial \mu_0}.$$

From the identities (6.13), (6.14) and (4.4), we have

$$e^{\mu+\sigma^2/2}\left(\frac{\partial \mu}{\partial \mu_0}+\sigma\frac{\partial \sigma}{\partial \mu_0}\right) = \frac{S_t}{\mu_0+\frac{\sigma_0^2}{2}},$$

$$\sigma^2_{a(s)}2\frac{\partial \mu}{\partial \mu_0}+e^{2\mu}\left(2e^{2\sigma^2}-e^{\sigma^2}\right)2\sigma\frac{\partial \sigma}{\partial \mu_0} = -\frac{\sigma^2_{a(s)}}{\mu_0+\frac{3\sigma_0^2}{2}}-\frac{S_t^2\sigma_0^2}{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)(\mu_0+\sigma_0^2)\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau} - \frac{S_t^2\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1\right)}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2\tau}\left(\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)\tau}-2\right),$$

where the chain rule and $\left(\frac{e^x-1}{x}\right)' = \frac{1}{x}$ are applied in the first identity. Then we solve for $\frac{\partial \mu}{\partial \mu_0}$ and $\sigma\frac{\partial \sigma}{\partial \mu_0}$ from the system of linear equations.

$$\frac{\partial \mu}{\partial \mu_0} = a_1\left(\sigma^2_{a(s)}e^{-2\left(\mu+\sigma^2\right)}+1\right)-\frac{a_2}{2}e^{-2\left(\mu+\sigma^2\right)},$$

$$\sigma\frac{\partial \sigma}{\partial \mu_0} = \frac{a_2}{2}e^{-2\left(\mu+\sigma^2\right)}-a_1\sigma^2_{a(s)}e^{-2\left(\mu+\sigma^2\right)}$$

where $a_1 = \frac{S_t e^{-\left(\mu+\frac{\sigma^2}{2}\right)}}{\mu_0+\frac{\sigma_0^2}{2}}$ and

$$a_2 = -\frac{\sigma^2_{a(s)}}{\mu_0+\frac{3\sigma_0^2}{2}}-\frac{S_t^2\sigma_0^2}{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)(\mu_0+\sigma_0^2)\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-\frac{S_t^2\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1\right)}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2\tau}\left(\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)\tau}-2\right).$$

By a straightforward calculation, we have

$$\frac{\partial d_1}{\partial \mu_0} = \frac{1}{\sigma}\frac{\partial \mu}{\partial \mu_0} + \left(1 - \frac{d_2}{\sigma}\right)\frac{\partial \sigma}{\partial \mu_0}$$
$$= \frac{a_1}{\sigma}\left(\frac{d_2}{\sigma}\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - \frac{a_2 d_2}{2\sigma^2}e^{-2\left(\mu+\sigma^2\right)},$$

$$\frac{\partial d_2}{\partial \mu_0} = \frac{a_1}{\sigma}\left(\left(1 + \frac{d_2}{\sigma}\right)\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - \frac{a_2 e^{-2\left(\mu+\sigma^2\right)}}{2\sigma}\left(1 + \frac{d_2}{\sigma}\right).$$

$$\frac{\partial a}{\partial \mu_0} = \frac{\partial a}{\partial \mu}\frac{\partial \mu}{\partial \mu_0} + \frac{\partial a}{\partial \sigma}\frac{\partial \sigma}{\partial \mu_0}$$
$$= a(K)\left[\frac{e^{-2\left(\mu+\sigma^2\right)}}{\sigma^2}\left(a_1\sigma_{a(s)}^2 - \frac{a_2}{2}\right)\left(\ln K - \mu + 1 - \frac{(\ln K - \mu)^2}{\sigma^2}\right) + \frac{a_1}{\sigma^2}(\ln K - \mu)\right].$$

Now we obtain the rho for the arithmetic Asian call option by adding these terms together after simplifications.

$$\rho^a = e^{-r\tau+\mu+\frac{\sigma^2}{2}}\left[N(d_1)a_1 + n(d_1)\frac{a_1}{\sigma}\left(\frac{d_2}{\sigma}\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - n(d_1)\frac{a_2 d_2}{2\sigma^2}e^{-2\left(\mu+\sigma^2\right)}\right] - \tau N(d_2) - Ke^{-r\tau}n(d_2)$$
$$\left[\frac{a_1}{\sigma}\left(\left(1 + \frac{d_2}{\sigma}\right)\sigma_{a(s)}^2 e^{-2\left(\mu+\sigma^2\right)} + 1\right) - \frac{a_2 e^{-2\left(\mu+\sigma^2\right)}}{2\sigma}\left(1 + \frac{d_2}{\sigma}\right)\right] + e^{-r\tau}\frac{\partial a}{\partial \mu_0}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}.$$

The theta of the arithmetic Asian call option can be obtained by $\frac{\partial V_t}{\partial t} = -\frac{\partial V_t}{\partial \tau}$ due to $\tau = T - t$. And we have

$$\frac{\partial V_t}{\partial \tau} = -rV_t + e^{-r\tau+\mu+\frac{\sigma^2}{2}}n(d_1)\frac{\partial d_1}{\partial \tau} - Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial \tau} + e^{-r\tau}\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}\frac{\partial a}{\partial \tau}.$$

From the identities (6.13), (6.14) and (4.4), we have

$$\frac{\partial \mu}{\partial \tau} + \sigma\frac{\partial \sigma}{\partial \tau} = \frac{S_t}{\tau\mu_{a(s)}},$$

$$\sigma_{a(s)}^2 2\frac{\partial \mu}{\partial \tau} + \left(e^{2\left(\mu+\sigma^2\right)} + \sigma_{a(s)}^2\right)2\sigma\frac{\partial \sigma}{\partial \tau} = \frac{S_t^2 - \sigma_{a(s)}^2 - \left(\mu_{a(s)} - S_t\right)^2}{\tau}.$$

Similar to the μ_0 case, we get

$$\frac{\partial d_1}{\partial \tau} = \frac{1}{\sigma}\frac{\partial \mu}{\partial \tau} + \left(1 - \frac{d_2}{\sigma}\right)\frac{\partial \sigma}{\partial \tau}$$
$$= \frac{1}{\sigma}\left(\frac{\partial \mu}{\partial \tau} + \sigma\frac{\partial \sigma}{\partial \tau}\right) - \frac{d_2}{\sigma}\frac{\partial \sigma}{\partial \tau}$$
$$= \frac{S_t}{\tau\sigma\mu_{a(s)}} - e^{-2\left(\mu+\sigma^2\right)}\frac{d_2}{2\tau\sigma^2\mu_{a(s)}}\left(2S_t\left(\mu_{a(s)}^2 - \sigma_{a(s)}^2\right) - \mu_{a(s)}\left(\mu_{a(s)}^2 + \sigma_{a(s)}^2\right)\right),$$

$$\frac{\partial d_2}{\partial \tau} = \frac{\partial d_1}{\partial \tau} - \frac{\partial \sigma}{\partial \tau}$$
$$= \frac{S_t}{\tau\sigma\mu_{a(s)}} - \left(1 + \frac{d_2}{\sigma}\right)\frac{\partial \sigma}{\partial \tau},$$

where $\frac{\partial \sigma}{\partial \tau} = \frac{e^{-2(\mu+\sigma^2)}}{2\tau\sigma\mu_{a(s)}}(2S_t(\mu_{a(s)}^2 - \sigma_{a(s)}^2) - \mu_{a(s)}(\mu_{a(s)}^2 + \sigma_{a(s)}^2))$.

$$\begin{aligned}\frac{\partial a}{\partial \tau} &= a(K)\left[-\frac{d_2}{\sigma}\frac{\partial \mu}{\partial \tau} + \frac{d_2^2-1}{\sigma}\frac{\partial \sigma}{\partial \tau}\right] \\ &= a(K)\left[-\frac{d_2}{\sigma}\left(\frac{\partial \mu}{\partial \tau} + \sigma\frac{\partial \sigma}{\partial \tau}\right) + \frac{d_2^2+d_2\sigma-1}{\sigma}\frac{\partial \sigma}{\partial \tau}\right] \\ &= a(K)\left[-\frac{d_2 S_t}{\sigma\tau\mu_{a(s)}} + \frac{d_2^2+d_2\sigma-1}{\sigma}\frac{\partial \sigma}{\partial \tau}\right].\end{aligned}$$

$$\begin{aligned}\frac{\partial V_t}{\partial \tau} = &-rV_t + e^{-r\tau}n(d_1)\left(\frac{S_t}{\tau\sigma} - \frac{d_2\mu_{a(s)}}{\sigma}\frac{\partial \sigma}{\partial \tau}\right) - Ke^{-r\tau}n(d_2)\left[\frac{S_t}{\tau\sigma\mu_{a(s)}} - \left(1+\frac{d_2}{\sigma}\right)\frac{\partial \sigma}{\partial \tau}\right] \\ &+ e^{-r\tau}a(K)\left[-\frac{d_2 S_t}{\sigma\tau\mu_{a(s)}} + \frac{d_2^2+d_2\sigma-1}{\sigma}\frac{\partial \sigma}{\partial \tau}\right]\sum_{j=3}^{\infty}(-1)^j\frac{E_j(F,A)}{j!}\frac{d^{j-1}a(K)}{d^{j-1}s}.\end{aligned}$$

Therefore the result follows.

Proof of Proposition 3.4

We have the remainder part from comparing the ratio of characteristic functions,

$$\frac{\phi(F,t)}{\phi(A,t)} = \sum_{j=0}^{N} E_j(F,A)\frac{(it)^j}{j!} + r_N(F,A)\frac{(it)^{N+1}}{(N+1)!}.$$

Thus the distribution identity from the Edgeworth series expansion is given by

$$f(s) = a(s) + \sum_{j=0}^{N} E_j(F,A)\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j} + r_N(F,A)\frac{(-1)^{N+1}}{(N+1)!}\frac{d^{N+1}a(s)}{ds^{N+1}}.$$

By the evaluation of the arithmetic Asian option, we obtain

$$V(t) = V_N(t) + R_N(t),$$

where $V_N(t)$ and $R_N(t)$ are given in Proposition 3.4.

From the comparison on characteristic functions, $E_j(F,A)$ is uniformly bounded by a fixed constant and so is the remainder tern $r_N(F,A)$ from the uniform convergence estimates. Hence we have

$$\left|R_N(t)\right| \le \frac{ce^{-r(T-t)}}{(N+1)!}\left|\frac{d^{N-1}a(K)}{ds^{N-1}}\right|. \tag{6.11}$$

Note that the lognormal probability density function $a(s) = \frac{1}{s\sigma\sqrt{2\pi}}e^{-\frac{(\ln s-\mu)^2}{2\sigma^2}}$ has the following identities.

$$a'(s) = \left(-\frac{a(s)}{s}\right)\left(1+\frac{\ln s-\mu}{\sigma^2}\right)$$

$$a''(s) = \frac{a(s)}{s^2}\left[\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^2 + \left(1+\frac{\ln s-\mu}{\sigma^2}\right) - \frac{1}{\sigma^2}\right]$$

$$a'''(s) = \left(-\frac{a(s)}{s^3}\right)\left[\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^3 + 3\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^2 + \left(1-\frac{2}{\sigma^2}\right)\left(1+\frac{\ln s-\mu}{\sigma^2}\right) - \frac{3}{\sigma^2}\right]$$

$$a^{(4)}(s) = \frac{a(s)}{s^4}\left[\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^4 + 6\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^3 + \left(10-\frac{5}{\sigma^2}\right)\left(1+\frac{\ln s-\mu}{\sigma^2}\right)^2 + \left(3-\frac{15}{\sigma^2}\right)\left(1+\frac{\ln s-\mu}{\sigma^2}\right) + \left(\frac{2}{\sigma^4}-\frac{10}{\sigma^2}\right)\right].$$

By the Leibnitz rule and induction,

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}\cdot g^{(k)}.$$

$$\begin{aligned} a^{(n+1)}(s) &= (a(s)\cdot h(s))^{(n)} \\ &= \sum_{k=0}^{n}\binom{n}{k} a^{(n-k)}(s)\cdot h^{(k)}(s) \end{aligned}$$

where $h(s) = \left(-\frac{1}{s}\right)\left(1+\frac{\ln s-\mu}{\sigma^2}\right)$, we have

$$a^{(N-1)}(s) = \frac{a(s)}{s^{N-1}} p_{N-1}\left(\left(1+\frac{\ln s-\mu}{\sigma^2}\right), \sigma^2\right),$$

where $p_{N-1}(x,\sigma^2)$ is a polynomial of degree $N-1$ with leading coefficient $(-1)^{N-1}$ and other coefficients only depending on σ^2. Therefore one gets the estimate

$$\left|a^{(N-1)}(s)\right| \le \frac{a(K)}{K^{N-1}}\left|p_{N-1}\left(\left(1+\frac{\ln K-\mu}{\sigma^2}\right), \sigma^2\right)\right|,$$

where $\left|p_{N-1}\left(\left(1+\frac{\ln K-\mu}{\sigma^2}\right), \sigma^2\right)\right| \le C(\sigma^2)\frac{\left|\left(1+\frac{\ln K-\mu}{\sigma^2}\right)\right|^N - 1}{|\ln K-\mu|}$ by the geometric sequence estimate. Combining with (6.11), we have

$$\left|R_N(t)\right| \le C\frac{e^{-r(T-t)}a(K)}{(N+1)!K^{N-1}}\cdot\frac{\left|\left(1+\frac{\ln K-\mu}{\sigma^2}\right)^N - 1\right|}{|\ln K-\mu|}.$$

Proof of formulas (4.1) and (4.2)

Suppose the underlying asset price follows the geometric Brownian motion as $S_t = S(0)e^{\mu_0 t+\sigma_0 W(t)}$ with the realized volatility σ_0 and drift μ_0. Under the risk neutral measure Q, $\mu_0 = r-\sigma_0^2/2$. Hence the arithmetic average of the underlying asset prices is

$$\frac{1}{T}\int_0^T S_t dt = \frac{1}{T}\int_0^T S(0)e^{\mu_0 t+\sigma_0 W(t)}dt.$$

We compute the mean of the arithmetic average of the underlying asset prices, by using the property of the Brownian motion $W(t)$,

$$E\left[\frac{1}{T}Y(T)\right]=E\left[\frac{1}{T}\int_0^T S(0)e^{\mu_0 t+\sigma_0 W(t)}dt\right]$$
$$=\frac{S(0)}{T}\int_0^T e^{\mu_0 t+\frac{\sigma_0^2 t}{2}}dt=S(0)\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}, \tag{6.12}$$

By our choice of the lognormal distribution, we have

$$E\left[\frac{1}{T-t}Y(T-t)\right]=S_t\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau} \tag{6.13}$$

where $\tau=T-t$ for any $0\leq t\leq T$. The variance of the arithmetic average of the underlying asset prices can be evaluated in the following

$$Var\left(\frac{1}{T}Y(T)\right)=E\left[\frac{1}{T^2}Y^2(T)\right]-\mu_{a(s)}^2.$$

$$E\left[\frac{1}{T^2}Y^2(T)\right]=E\left[\frac{S(0)^2}{T^2}\left(\int_0^T e^{\mu_0 t+\sigma_0 W(t)}dt\right)^2\right]=\frac{S(0)^2}{T^2}E\left[\int_0^T\int_0^T e^{\mu_0(s+t)+\sigma_0(W(s)+W(t))}dsdt\right]$$
$$=\frac{S(0)^2}{T^2}E\left[\int\int_{R_1}+\int\int_{R_2}e^{\mu_0(s+t)+\sigma_0(W(s)+W(t))}dsdt\right]=\frac{S(0)^2}{T^2}(I_1+I_2),$$

where R_1 is the region in the square $(s,t)\in[0,T]\times[0.T]$ such that $t\leq s$ (the lower right triangle) and R_2 is the region in the square such that $t>s$ (the upper left right triangle), and $I_j=\int\int_{R_j}E[e^{\mu_0(s+t)+\sigma_0(W(s)+W(t))}dsdt]$ for $j=1,2$.

$$I_1=\int\int_{R_1}E\left[e^{\mu_0(s+t)+\sigma_0(W(s)+W(t))}dsdt\right]=\int\int_{R_1}e^{\mu_0(s+t)+\frac{\sigma_0^2}{2}(s+t)+\sigma_0^2 Cov(W(s),W(t))}dsdt$$
$$=\int\int_{R_1}e^{\mu_0(s+t)+\frac{\sigma_0^2}{2}(s+t)+\sigma_0^2 t}dsdt=\int_0^T\left(\int_0^s e^{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)t}\cdot e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)s}dt\right)ds$$
$$=\frac{1}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2\left(\mu_0+\sigma_0^2\right)T}-1}{2(\mu_0+\sigma_0^2)}-\frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right).$$

where the second identity follows from the moment function calculation for bivariate normal random variables, the third identity from $Cov(W(s),W(t))=t$ for $t\leq s$ in the first region, the fourth from the integration t-direction first then s-direction, the last one from the simple integrations. Similarly we evaluate I_2,

$$I_2 = \int\int_{R_2} E\left[e^{\mu_0(s+t)+\sigma_0(W(s)+W(t))}dsdt\right] = \int\int_{R_2} e^{\mu_0(s+t)+\frac{\sigma_0^2}{2}(s+t)+\sigma_0^2 s}dsdt$$

$$= \int_0^T \left(\int_0^t e^{\left(\mu_0+\frac{3\sigma_0^2}{2}\right)s}\cdot e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)t} ds\right) dt = \frac{1}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2(\mu_0+\sigma_0^2)T}-1}{2(\mu_0+\sigma_0^2)} - \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right).$$

where $Cov(W(s),W(t)) = t$ for $s < t$ in the second region R_2, and the third identity is integration s-direction first, the last one follows from the simple integrations. Combining I_1 and I_2, we obtain the variance for the average of the underlying asset prices

$$Var\left(\frac{1}{T}Y(T)\right) = \frac{2S(0)^2}{T^2}\cdot\frac{1}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2(\mu_0+\sigma_0^2)T}-1}{2(\mu_0+\sigma_0^2)} - \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right) - \mu_{a(s)}^2$$

$$= \frac{S(0)^2}{T^2}\left[\frac{2}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2(\mu_0+\sigma_0^2)T}-1}{2(\mu_0+\sigma_0^2)} - \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right) - \frac{\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)T}-1\right)^2}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2}\right].$$

Therefore the variance formula for the arithmetic average of the underlying asset prices is given by

$$Var\left(\frac{1}{T-t}Y(T-t)\right) = \frac{S_t^2}{\tau^2}\left[\frac{2}{\mu_0+\frac{3\sigma_0^2}{2}}\left(\frac{e^{2(\mu_0+\sigma_0^2)\tau}-1}{2(\mu_0+\sigma_0^2)} - \frac{e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1}{\mu_0+\frac{\sigma_0^2}{2}}\right) - \frac{\left(e^{\left(\mu_0+\frac{\sigma_0^2}{2}\right)\tau}-1\right)^2}{\left(\mu_0+\frac{\sigma_0^2}{2}\right)^2}\right], \tag{6.14}$$

where $\tau = T - t$ for any $0 \le t \le T$.

References

1. Turnbull S, Wakeman LM. A quick algorithm for pricing European average options. *J Financ Quant Anal*. 1991;26:377−389.
2. Boyle P, Boyle F. *Derivatives: The Tool that Changed Finance*. London: Risk Books; 2001.
3. Linetsky V. Spectral expansions for Asian (average price) options. *Operations Res*. 2004;52:856−867.
4. Hoogland J, Neumann D. *Asians and Cash Dividends: Exploiting Symmetries in Pricing Theory*. Centrum voor Wiskunde en Informatica; 2000. MAS Working Paper, (R0019).
5. Vecer J. Black-Scholes representation for Asian options. *Math Finance*. 2014;24:598−626.
6. Alziary B, Décamps J-P, Koehl P-F. A P. D. E. approach to Asian options: analytical and numerical evidence. *J Bank Finance*. 1997;21:613−640.
7. Chen K-W, Lyuu Y-D. Accurate pricing formulas for Asian options. *Appl Math Comput*. 2007;188:1711−1724.
8. Elshegmani ZA, Ahmad RR, Jaaman SH, Zakaria RH. Transforming Arithmetic Asian Option PDE to the parabolic equation with constant coefficients. *Int J Math Math Sci*. 2011;2011. http://dx.doi.org/10.1155/2011/401547. Article ID 401547.
9. Sun J, Chen L, Li S. A quasi-analytic pricing model for arithmetic Asian options. *J Future Mark*. 2012;00:1−24.
10. Shreve S. *Stochastic Calculus for Finance II: Continuous-time Models*. Springer; 2004.
11. Boyle PP, Emanuel D. Discretely adjusted option hedges. *J Finan Econ*. 1980;8:259−282.
12. Kemna AGZ, Vorst ACF. A pricing method for options based on average asset values. *J Bank Finance*. 1990;14:113−129.
13. Geman H, Yor M. Bessel processes, Asian options and perpetuities. *Math Finance*. 1993;3:349−375.
14. Milevsky M, Posner S. Asian options, the sum of lognormals, and the reciprocal gamma distribution. *J Financ Quant Anal*. 1998;33:409−422.
15. Ju N. Pricing Asian and basket options via Taylor expansion. *J Comput Finance*. 2002;5:79−103.
16. Dufresne D. The Laguerre series for Asian and other options. *Math Finance*. 2000;10:407−428.
17. Kwok Yue-Kuen. *Mathematical Models of Financial Derivatives*. 2nd ed. Springer; 2008.

18. Bouaziz L, Briys E, Crouhy M. The pricing of forward-starting Asian options. *J Bank Finance*. 1994;18:823–839.
19. Chung SL, Shackleton M, Wojakowski R. Efficient quadratic approximation of floating strike Asian option values. *Finance*. 2003;24:49–62.
20. Black F, Scholes M. The pricing of options and corporate liabilities. *J Political Econ*. 1973;81:637–659.
21. Kendall M, Stuart A. *The Advanced Theory of Statistics*. In: *Distribution Theory*. 4th ed. vol. 1. New York: Macmillan; 1977.
22. Ingersoll Jr JE. *Theory of Financial Decision Making*. Savage, MD: Rowman &Littlefield; 1987.
23. Rogers L, Shi Z. The value of an Asian option. *J Appl Probab*. 1995;32:1077–1088.
24. Vecer J. A new PDE approach for pricing arithmetic average Asian options. *J Comput Finance*. 2001;4:105–113.