Operations Research Letters 44 (2016) 129–135

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A note on the never-early-exercise region of American power exchange options

Daniel Wei-Chung Miao [a,b,*], Xenos Chang-Shuo Lin [c], Steve Hsin-Ting Yu [d]

[a] National Taiwan University of Science and Technology, Taipei 106, Taiwan
[b] Imperial College Business School, London SW7 2AZ, UK
[c] Aletheia University, No.32, Zhenli St., Danshui Dist., New Taipei City, 25103, Taiwan
[d] Taiwan Futures Exchange, 14th Fl., 100 Roosevelt Road, Sec. 2, Taipei, 100, Taiwan

ARTICLE INFO

Article history:
Received 21 May 2014
Received in revised form
13 December 2015
Accepted 13 December 2015
Available online 19 December 2015

Keywords:
American option
Power exchange option
Early exercise
Never-early-exercise region
Analytical upper bound
Geske–Johnson method

ABSTRACT

This note discusses how the never-early-exercise region of American power exchange options is influenced by the nonlinearity from its power coefficients. We consider a class of models which satisfy the *power invariant property* and show that early exercise depends crucially on the quantities termed *effective dividend yields*. Our mathematical analysis extends an existing model-free result and indicates how early exercise should depend on parameters. A numerical analysis is conducted to complement the analytical results and provide further observations.

1. Introduction

An exchange option gives its holder the right to exchange one asset for another. Earlier studies can be dated back to Fisher [3] and Margrabe [9] who derived the closed-form pricing formula for a European exchange option under the classical Black–Scholes model. Lindset [7] extended the pricing formula to Merton's jump–diffusion model [10] and proposed to use the Geske–Johnson method [4] to price its American version. One extension from the plain-vanilla exchange option is the power exchange option where nonlinear dependence is introduced by its power coefficients. As seen in Johnson and Tian [6] and Blenman and Clark [1], power exchange options provide more flexibility in the design of indexed executive stock options. Under the Black–Scholes model, the price of its European version was derived in closed form in [1]. By using the martingale property of the underlying stock prices, [1] also gave a sufficient condition under which its American version should never be exercised early.

The merit of this sufficient condition for the never-early-exercise (NEE) property is that it is model-free. However, the fact that it applies to a wide range of models also makes it a conservative condition. In fact, there are plenty of option parameters which actually lead to NEE but cannot be identified by this condition. In this note we consider a specific class of stock price models and show that the sufficient condition for NEE can be considerably weakened such that much more option parameters can be identified as never-early-exercise. In the model class of interest, we assume the *power invariant property* holds, meaning that the powered process of stock price remains in the same family as the original stock price process. This property enables us to introduce the *power martingale condition* and define the *effective dividend yields* which play important roles in the analysis of early exercise. A number of popular stock price models belong to this model class, including the Black–Scholes models, jump–diffusion models [10], and variance gamma models [8]. In fact, it contains all the exponential Levy models. A commonly used model not satisfying the power invariant property is the Heston stochastic volatility model [5].

When the sufficient condition for NEE is met, the American power exchange option price must be equal to the price of its European version. Contrarily, if the condition is not satisfied, early exercise may be possible and it is of interest to discuss the value contributed by early exercise. Taking the perspective of effective dividend yields, the pricing problem can be reduced to its plain-vanilla version except that the dividend yields are no longer

* Corresponding author at: National Taiwan University of Science and Technology, Taipei 106, Taiwan.
E-mail address: miao@mail.ntust.edu.tw (D.W.-C. Miao).

http://dx.doi.org/10.1016/j.orl.2015.12.011
0167-6377/

nonnegative. This gives rise to some new results which cannot be seen in the plain-vanilla setting. We derive upper bounds on the American power exchange option price and show that they take different forms according to the signs of effective dividend yields. These results enable us to see in what way the real-valued effective dividend yields affect the value and likelihood of early exercise. Moreover, they provide some computational implications for the pricing of American power exchange options. When an option's parameters are in the NEE region (condition for NEE is met), it can be valued by standard European option pricing methods. When the parameters are outside the NEE region, the upper bounds (which may be evaluated in the same way as European options) provide an indication of whether it is worth incurring the computational cost of accurately pricing the American version.

To investigate the contribution of early exercise in a more accurate way, we apply the Geske–Johnson method [4,7] to conduct a numerical analysis which provides further observations. We find that the numerical NEE region generally covers an even wider range of option parameters than the theoretical condition would suggest. Through our numerical examples, we provide in-depth discussions on how early exercise is influenced by the two power coefficients.

2. Mathematical analysis

Consider the power exchange option which gives the payoff $(S_{1t}^{n_1} - S_{2t}^{n_2})^+$ where S_{1t}, S_{2t} are stock price processes and the power coefficients n_1, n_2 are positive real numbers. The vanilla exchange option corresponds to the special case $n_1 = n_2 = 1$. Denote the option maturity time as T and let $r, q_1, q_2 \geq 0$ respectively stand for the nonnegative interest rate and the dividend yields of the two stock price processes. For the market to be free of arbitrage, the stock prices must satisfy the martingale condition

$$\mathsf{E}_t[S_{iT}] = S_{it}e^{(r-q_i)(T-t)}, \quad i = 1, 2, \tag{1}$$

for $t \leq T$ under the risk-neutral measure. It simply means that $e^{-(r-q_i)t}S_{it}$ is a martingale and this should hold for any stock price model.

2.1. The never-early-exercise conditions

Based on the fundamental relation (1), [1] gave a sufficient condition (Theorem 4, p.104) for the American power exchange option to be never-early-exercise. The result is rephrased in our notation as below.

Theorem 2.1. *Consider the American power exchange option with power coefficients n_1, n_2. If $n_1 \geq 1$ (such that x^{n_1} is a convex function of x), $0 < n_2 \leq 1$ (such that x^{n_2} is a concave function of x), and $n_1(r - q_1) \geq r$, $n_2(r - q_2) \leq r$, then early exercise is not optimal, and its value is the same as that of the European version.*

The main idea behind this theorem is that a convex function of a martingale (say $X_t = e^{-(r-q_i)t}S_{it}$) is a submartingale ($\mathsf{E}_t[X_T^{n_1}] \geq X_t^{n_1}$) while a concave function of a martingale is a supermartingale ($\mathsf{E}_t[X_T^{n_2}] \leq X_t^{n_2}$). As no specific assumption is made on the stock price model (except that r, q_1, q_2 are constant), this sufficient condition for NEE is model-free. We intend to show that for specific models, this condition can be considerably weakened such that more option parameters may lead to NEE. Before we proceed, it is worth noting that the volatilities (in a wide sense, and may include contributions from the diffusion and jump parts) of both stock price processes are absent in Theorem 2.1. This is natural since the condition is model-free whereas volatilities are model specific parameters. As will be seen later, they become present in our weakened condition for a certain class of models.

To proceed, let us consider a class of stock price processes which satisfy the following *power invariant property*.

Definition 2.2. The stock price processes $S_{it}, i = 1, 2$ are said to be power invariant if for a pair of positive $n_i, i = 1, 2$, the powered processes $S_{it}^{n_i}, i = 1, 2$ remain in the same family as the original processes $S_{it}, i = 1, 2$.

Because (1) must hold for the stock prices S_{it} (under the risk-neutral measure), one natural consequence of the above definition is that the following *power martingale condition* must hold for their powered processes

$$\mathsf{E}_t[S_{iT}^{n_i}] = S_{it}^{n_i}e^{(r-Q_i)(T-t)}, \quad i = 1, 2, \tag{2}$$

where $Q_i \in \mathbb{R}$ is called the *effective dividend yield* of $S_{it}^{n_i}$. Unlike dividend yield q_i which is nonnegative, here Q_i is an artificial quantity and can take real values. This condition was first introduced in [11] for the discussion of single asset power options. As it turns out, this condition is essential in the analysis of power exchange options.

The explicit formulas of effective dividend yields under these specific models can be derived without difficulty. Take the bivariate Black–Scholes model for example:

$$dS_{it} = (r - q_i)S_{it}dt + \sigma_i S_{it}dW_{it}, \quad i = 1, 2, \tag{3}$$

where $W_{it}, i = 1, 2$ are two standard Brownian motions with correlation ρ(i.e. $dW_{1t}dW_{2t} = \rho dt$). Using the result $\mathsf{E}[S_{iT}^{n_i}] = S_{it}^{n_i}e^{[n_i(r-q_i)+\frac{n_i(n_i-1)}{2}\sigma_i^2](T-t)}$ and matching it to (2), one obtains

$$Q_i = (1 - n_i)r + n_i q_i - \frac{n_i(n_i - 1)}{2}\sigma_i^2. \tag{4}$$

From (4), we see the role of volatility σ_i in Q_i. Specifically, under proper conditions (e.g. $n_i > 1$, r large and q_i small), a greater σ_i may lead to quite negative value of Q_i. Similar observation can be made under some other models (with Q_i formulas given in [11]) in that more volatile processes tend to make Q_i more negative (when $n_i > 1$).

Based on the notion of effective dividend yields, our weakened version of the NEE condition for this class of models is presented as follows.

Theorem 2.3. *Consider an American power exchange option with power coefficients n_1 and n_2. Suppose that the stock price processes S_{1t} and S_{2t} are power invariant with the effective dividend yields Q_1 and Q_2 well defined by (2) (i.e. $Q_i = r + \frac{1}{T-t}\ln(S_{it}^{n_i}/\mathsf{E}[S_{iT}^{n_i}])$, $i = 1, 2$, which do not depend on S_{it} or $T - t$). If $Q_1 \leq 0$ and $Q_2 \geq 0$, then early exercise is never optimal, and its value is the same as that of its European version.*

Proof. We prove the claim by showing that the price of European power exchange (EPE) option is always higher than the exercise value of its American counterpart, regardless of how high S_{1t} is or how low S_{2t} is. This is seen from

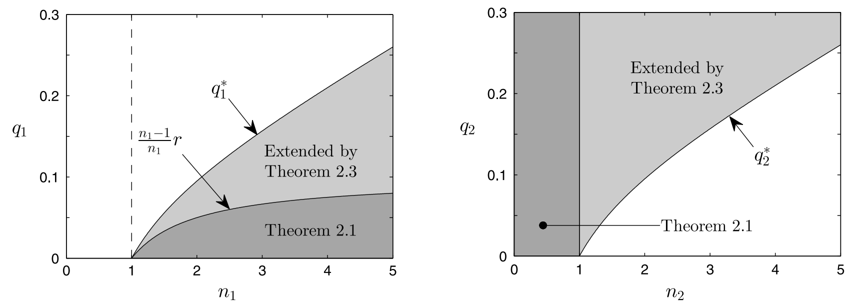
$$\begin{aligned}
\text{EPE price} &= e^{-r(T-t)}\mathsf{E}_t[(S_{1T}^{n_1} - S_{2T}^{n_2})^+] \\
&\geq e^{-r(T-t)}(\mathsf{E}_t[S_{1T}^{n_1}] - \mathsf{E}_t[S_{2T}^{n_2}])^+ \quad (\because \text{Jensen's inequality}) \\
&= e^{-r(T-t)}(S_{1t}^{n_1}e^{(r-Q_1)(T-t)} - S_{2t}^{n_2}e^{(r-Q_2)(T-t)})^+ \\
&= (S_{1t}^{n_1}e^{-Q_1(T-t)} - S_{2t}^{n_2}e^{-Q_2(T-t)})^+ \geq (S_{1t}^{n_1} - S_{2t}^{n_2})^+,
\end{aligned}$$

where the conditions $Q_1 \leq 0$ and $Q_2 \geq 0$ are used to ensure the last inequality holds. □

To show how the NEE condition is weakened, we take the Black–Scholes model for example. Let q_i^* stand for the solution of $Q_i = 0$ with Q_i defined in (4), i.e.

$$q_i^* = \frac{n_i - 1}{n_i}r + \frac{n_i - 1}{2}\sigma_i^2, \quad i = 1, 2. \tag{5}$$

Table 1 provides a comparison between the sufficient conditions in Theorems 2.1 and 2.3 which are equivalently expressed in terms



(a) NEE condition on q_1. (b) NEE condition on q_2.

Fig. 1. The two NEE conditions on q_1 and q_2 as seen in Table 1 are sketched for different n_1 and n_2 in (a) and (b) respectively. The darker areas correspond to Theorem 2.1 and the whole shaded (darker and lighter) areas correspond to Theorem 2.3. Other parameters are $r = 0.1$, $\sigma_1 = \sigma_2 = 0.3$.

Table 1
A comparison of the sufficient conditions for NEE in Theorems 2.1 and 2.3.

	Range of n_i	Conditions on q_1 and q_2	
		Theorem 2.1	Theorem 2.3
n_1	$n_1 \geq 1$	$q_1 \leq \frac{n_1-1}{n_1}r (< q_1^*)$	$q_1 \leq \frac{n_1-1}{n_1}r + \frac{n_1-1}{2}\sigma_1^2 (= q_1^*)$
	$0 < n_1 < 1$	$q_1 \in \varnothing$	$q_1 \in \varnothing$
n_2	$n_2 > 1$	$q_2 \in \varnothing$	$q_2 \geq \frac{n_2-1}{n_2}r + \frac{n_2-1}{2}\sigma_2^2 (= q_2^*)$
	$0 < n_2 \leq 1$	$q_2 \geq 0$	$q_2 \geq 0$

of the conditions on q_i. We see that the differences between the two NEE conditions mainly come from the volatility related terms $\frac{n_i-1}{2}\sigma_i^2$, $i = 1, 2$. With their presence, the conditions on q_1 and q_2 are made weaker, and there will be more option parameters leading to NEE. Another notable difference is that Theorem 2.1 applies only to $n_1 \geq 1$ (x^{n_1} is convex) and $0 < n_2 \leq 1$ (x^{n_2} is concave) but Theorem 2.3 applies to all positive values of n_1 and n_2. This is attributed to the fact that Theorem 2.1 is based on the martingale property of the underlying processes whereas Theorem 2.3 is based on the martingale property of both the original and the powered processes. A graphical presentation of Table 1 is given in Fig. 1 to observe the differences between the two NEE conditions. The regions extended by Theorem 2.3 cover significantly large areas on the q_i–n_i planes, and the areas get even larger when σ_1 is high and σ_2 is low. This shows that ignoring the effects from volatilities may induce considerable misidentification.

2.2. The upper bounds of American power exchange option prices

When the option parameters do not satisfy the NEE condition, early exercise cannot be ruled out and American options may have higher values than European options. To examine the contribution from early exercise, we derive upper bounds on the American option prices where the roles of the two effective dividend yields can be clearly observed. The analysis relies on Jensen's inequality (as in Theorem 2.3) and a result in [2] (Theorem 2, p.212), which we restate below for our specific models of interest.

Theorem 2.4. *Consider an American option with maturity T and intrinsic value $X(t)$ at time $t \leq T$. Let r be the constant risk-free interest rate. If the random variable $Y(t,T)$ defined at time t satisfies the following three conditions: 1. $Y(T,T) \geq X(T)$, 2. $\mathsf{E}_t[Y(t,T)] \geq e^{-r\triangle t}\mathsf{E}_t[Y(t+\triangle t, T)]$ for any $t \in [t, T-\triangle t]$, and 3. $\mathsf{E}_t[Y(t,T)] \geq X(t)$ for all $t \in [0,T]$, then $\mathsf{E}_t[Y(t,T)]$ is an upper bound of the American option price.*

Note that the above theorem does not specify option type or stock price model, and therefore it is applicable to the American power exchange option under our chosen model class. The following theorem generalizes Theorem 2.3 to give the upper bounds of American option price according to the signs of Q_1 and Q_2.

Theorem 2.5. *Suppose that the two stock price processes S_{1t} and S_{2t} are power invariant with the effective dividend yields Q_1 and Q_2 well defined by (2). The upper bounds of the American power exchange option price APE, denoted as $\overline{\text{APE}}$, are given respectively for the four quadrants of (Q_1, Q_2) as follows:*

1. *If* $Q_1 \geq 0,\ Q_2 \geq 0$ (quadrant (I))
$$\text{APE} \leq \overline{\text{APE}} = e^{-r(T-t)}\mathsf{E}_t[(e^{Q_1(T-t)}S_{1T}^{n_1} - e^{\min(Q_1,Q_2)(T-t)}S_{2T}^{n_2})^+]. \quad (6)$$
2. *If* $Q_1 \leq 0,\ Q_2 \geq 0$ (quadrant (II))
$$\text{APE} = \overline{\text{APE}} = e^{-r(T-t)}\mathsf{E}_t[(S_{1T}^{n_1} - S_{2T}^{n_2})^+]. \quad (7)$$
3. *If* $Q_1 \leq 0,\ Q_2 \leq 0$ (quadrant (III))
$$\text{APE} \leq \overline{\text{APE}} = e^{-r(T-t)}\mathsf{E}_t[(S_{1T}^{n_1} - e^{Q_2(T-t)}S_{2T}^{n_2})^+]. \quad (8)$$
4. *If* $Q_1 \geq 0,\ Q_2 \leq 0$ (quadrant (IV))
$$\text{APE} \leq \overline{\text{APE}} = e^{-r(T-t)}\mathsf{E}_t[(e^{Q_1(T-t)}S_{1T}^{n_1} - e^{Q_2(T-t)}S_{2T}^{n_2})^+]. \quad (9)$$

Proof. Since case 2 is equivalent to Theorem 2.3, we only need to consider the other three cases. To apply Theorem 2.4, denote the intrinsic value as $X(t) = (S_{1t}^{n_1} - S_{2t}^{n_2})^+$. The proof for each case is based on a properly defined $Y(t,T)$ which is shown to satisfy the three conditions specified in Theorem 2.4. Below we present the proof for case 4 in full. For cases 1 and 3, we show how $Y(t,T)$ should be defined such that the three conditions hold but omit the remaining details as they can be checked easily.

In case 4, we define $Y(t,T) = (e^{(Q_1-r)(T-t)}S_{1T}^{n_1} - e^{(Q_2-r)(T-t)}S_{2T}^{n_2})^+$. The first condition $Y(T,T) = X(T)$ is obviously satisfied. The second condition holds because

$$\begin{aligned}\mathsf{E}_t[Y(t,T)] &= \mathsf{E}_t[(e^{(Q_1-r)(T-t)}S_{1T}^{n_1} - e^{(Q_2-r)(T-t)}S_{2T}^{n_2})^+] \\ &\geq e^{-r\triangle t}\mathsf{E}_t[Y(t+\triangle t, T)].\end{aligned}$$

Finally, the third condition holds because

$$\begin{aligned}\mathsf{E}_t[Y(t,T)] &= \mathsf{E}_t[(e^{(Q_1-r)(T-t)}S_{1T}^{n_1} - e^{(Q_2-r)(T-t)}S_{2T}^{n_2})^+] \\ &\geq (e^{(Q_1-r)(T-t)}\mathsf{E}_t[S_{1T}^{n_1}] - e^{(Q_2-r)(T-t)}\mathsf{E}_t[S_{2T}^{n_2}])^+ \\ &\geq (S_{1t}^{n_1} - S_{2t}^{n_2})^+ = X(t), \quad \forall t \leq T,\end{aligned}$$

where the first inequality uses Jensen's inequality. As for cases 1 and 3, by respectively defining $Y(t,T) = (e^{(Q_1-r)(T-t)}S_{1T}^{n_1} - e^{(\min(Q_1,Q_2)-r)(T-t)}S_{2T}^{n_2})^+$ and $Y(t,T) = (e^{-r(T-t)}S_{1T}^{n_1} - e^{(Q_2-r)(T-t)}S_{2T}^{n_2})^+$, we are able to check in a similar way that all the three conditions are satisfied. Therefore, in each case, $\mathsf{E}_t[Y(t,T)]$ is the upper bound of the American power exchange option and this completes the proof. □

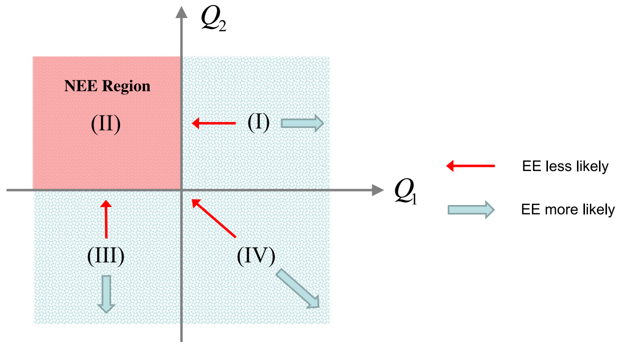


Fig. 2. The likelihood of early exercise suggested by Theorem 2.5 is viewed on the Q_1–Q_2 plane. The quadrant (II) (including the two half-axes) is the NEE region. The upper bounds for the other three quadrants suggest that early exercise tends to be less (more) likely if (Q_1, Q_2) approaches (moves away from) quadrant (II) as indicated by the thin (thick) arrows.

Note that the bounds given in Theorem 2.5 are asymptotically correct. For quadrant (I), as $Q_1 \to 0$, (6) will converge to the correct formula (7). The same observations can be made in quadrant (III) ((8) tends to (7) as $Q_2 \to 0$) and quadrant (IV) ((9) tends to (7) as both Q_1 and Q_2 tend to 0). As shown in Fig. 2, when the location of (Q_1, Q_2) is nearer to quadrant (II), the bounds get closer to EPE option price and early exercise becomes less likely. On the contrary, as (Q_1, Q_2) moves away from quadrant (II) (i.e. either Q_1 or $-Q_2$ gets large), the greater difference between the upper bound and the European price suggests that early exercise tends to contribute more value.

Also note that the upper bounds in Theorem 2.5 are given in the form of expected values under the risk-neutral measure. In fact, they can be further expressed in terms of the European exchange (EE) option pricing formula. This provides computational advantages for the exact calculation of EPE price and upper bounds of APE price, as long as there exist closed-form or easy-to-compute expressions for the EE price. Below we show how the EPE price and upper bounds of APE price are related to the EE price.

Proposition 2.6. *Consider the bivariate stock price model (S_{1t}, S_{2t}). Suppose that S_{it} follows the risk-neutral dynamics*

$$S_{it} = S_{i0}e^{(r-q_i)t+X_{it}}, \quad i = 1, 2,$$

where X_{it} is a stochastic process with $\mathsf{E}[e^{X_{it}}] = 1$ such that (1) holds. Let $\boldsymbol{\theta}$ denote the parameter set of (X_{1t}, X_{2t}) and $F(S_{1t}, S_{2t}, T, q_1, q_2, \boldsymbol{\theta})$ denote the time t price of the European exchange option maturing at time T. Suppose that the processes (S_{1t}, S_{2t}) are power invariant for a pair of positive (n_1, n_2) with $\mathsf{E}[e^{n_iX_{it}}] = e^{\alpha_i t}$ where $t > 0$ and α_i does not depend on t, $i = 1, 2$. Then the powered processes $(S_{1t}^{n_1}, S_{2t}^{n_2})$ satisfy the power martingale condition with (Q_1, Q_2) well defined as

$$Q_i = (1 - n_i)r + n_iq_i - \alpha_i, \quad i = 1, 2. \tag{10}$$

Let $\boldsymbol{\Theta}$ (corresponding to $\boldsymbol{\theta}$) represent the parameter set of their stochastic parts. Then the European power exchange option price has the expression $F(S_{1t}^{n_1}, S_{2t}^{n_2}, T, Q_1, Q_2, \boldsymbol{\Theta})$. The upper bounds of the American power exchange option price as given by (6), (8), (9) can be expressed as $F(e^{Q_1(T-t)}S_{1t}^{n_1}, e^{\min(Q_1,Q_2)(T-t)}S_{2t}^{n_2}, T, Q_1, Q_2, \boldsymbol{\Theta})$, $F(S_{1t}^{n_1}, e^{Q_2(T-t)}S_{2t}^{n_2}, T, Q_1, Q_2, \boldsymbol{\Theta})$, and $F(e^{Q_1(T-t)}S_{1t}^{n_1}, e^{Q_2(T-t)}S_{2t}^{n_2}, T, Q_1, Q_2, \boldsymbol{\Theta})$, respectively.

Proof. For $i = 1, 2$, we have $S_{it}^{n_i} = S_{i0}^{n_i}e^{n_i(r-q_i)t+n_iX_{it}}$ and $\mathsf{E}[S_{it}^{n_i}] = S_{i0}^{n_i}e^{n_i(r-q_i)t+\alpha_i t}$. Since α_i does not depend on t, the power martingale condition (2), i.e. $\mathsf{E}[S_{it}^{n_i}] = S_{i0}^{n_i}e^{(r-Q_i)t}$ holds with Q_i as defined in (10). Denote the powered processes by $Z_{it} = S_{it}^{n_i}$, $i = 1, 2$, then their risk-neutral dynamics follow

$$Z_{it} = Z_{i0}e^{n_i(r-q_i)t+n_iX_{it}} = Z_{i0}e^{(r-Q_i)t+Y_{it}}, \quad i = 1, 2, \tag{11}$$

where $Y_{it} = n_iX_{it} - \alpha_i t$. Since $\mathsf{E}[e^{Y_{it}}] = 1$, $i = 1, 2$, (Z_{1t}, Z_{2t}) have the same specification as (S_{1t}, S_{2t}) with the original parameters $q_1, q_2, \boldsymbol{\theta}$ changed to $Q_1, Q_2, \boldsymbol{\Theta}$. Because the EE option price is given by $e^{-r(T-t)}\mathsf{E}[(S_{1T} - S_{2T})^+] = F(S_{1t}, S_{2t}, T, q_1, q_2, \boldsymbol{\theta})$, the EPE option price is obtained by an analogy as $e^{-r(T-t)}\mathsf{E}[(Z_{1T} - Z_{2T})^+] = F(Z_{1t}, Z_{2t}, T, Q_1, Q_2, \boldsymbol{\Theta})$. The expressions for the three upper bounds may be obtained by redefining Z_{it}, $i = 1, 2$. Below we derive the expression for (9) with similar treatments for (6), (8) omitted. Taking t and T as constants and letting $Z_{iu} = e^{Q_i(T-t)}S_{iu}^{n_i}$ where $u \in [0, T]$, we have

$$\begin{aligned} Z_{iu} &= e^{Q_i(T-t)}S_{iu}^{n_i} = e^{Q_i(T-t)}S_{i0}^{n_i}e^{(r-Q_i)u+Y_{iu}} \\ &= Z_{i0}e^{(r-Q_i)u+Y_{iu}}, \quad i = 1, 2, \end{aligned}$$

which has the same form as (11). Consequently, the upper bound in (9) is obtained by $e^{-r(T-t)}\mathsf{E}_t[(Z_{1T} - Z_{2T})^+] = F(Z_{1t}, Z_{2t}, T, Q_1, Q_2, \boldsymbol{\Theta})$ as claimed. □

Table 2 summarizes the parameter sets $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ for the two important examples, the Black–Scholes model and Merton's jump–diffusion model. In the former case where (3) is expressed in the form $S_{it} = S_{i0}e^{(r-q_i-\frac{\sigma_i^2}{2})t+\sigma_iW_{it}}$, $i = 1, 2$, we have $\boldsymbol{\theta} = \{\sigma_1, \sigma_2, \rho\}$ and it is easy to check $\boldsymbol{\Theta} = \{n_1\sigma_1, n_2\sigma_2, \rho\}$ for a given pair of (n_1, n_2). In the latter case where the stock price dynamics follow

$$\frac{dS_{it}}{S_{it}} = (r - q_i - \lambda_ik_i)dt + \sigma_idW_{it} + dJ_{it}, \quad i = 1, 2, \tag{12}$$

the jumps are governed by the compound Poisson process $J_{it} = \sum_{j=1}^{N_i(t)}(e^{Y_{ij}} - 1)$. Here $N_i(t)$ is the Poisson process (with intensity λ_i) generating jumps and Y_{ij} is the jump size following a normal distribution $Y_{ij} \sim \mathrm{N}(\gamma_i, \delta_i^2)$. The mean jump size is $k_i = \mathsf{E}[e^{Y_{ij}}] - 1 = e^{\gamma_i+\frac{\delta_i^2}{2}} - 1$. When (12) is expressed in this form $S_{it} = S_{i0}e^{(r-q_i-\lambda_ik_i-\frac{\sigma_i^2}{2})t+\sigma_iW_{it}+\sum_{j=1}^{N_i(t)}Y_{ij}}$, we see the parameter set for the random part is given by $\boldsymbol{\theta} = \{\sigma_1, \sigma_2, \rho, \lambda_1, \gamma_1, \delta_1, \lambda_2, \gamma_2, \delta_2\}$. For a given pair of (n_1, n_2), (α_1, α_2) can be found as

$$\begin{aligned} \alpha_i &= \frac{n_i(n_i-1)}{2}\sigma_i^2 \\ &\quad - \lambda_i\left[n_i\left(e^{\gamma_i+\frac{\delta_i^2}{2}} - 1\right) - \left(e^{n_i\gamma_i+\frac{n_i^2\delta_i^2}{2}} - 1\right)\right], \quad i = 1, 2, \end{aligned}$$

which determine (Q_1, Q_2) through (10). The parameter set $\boldsymbol{\Theta}$ for the random part of the powered processes is shown in Table 2. The pricing formulas $F(S_{1t}, S_{2t}, T, q_1, q_2, \boldsymbol{\theta})$ under these two models were given in closed-form in Margrabe (1978) [9] (p.179, Eq. (7)) and in Lindset (2007) [7] (p.264, Proposition 1), respectively.

2.3. A generalization to high-dimensional American power exchange options

The power exchange option discussed so far involves two underlying stocks. In fact, our main results can be generalized to its high-dimensional version involving more stocks. Consider two groups of stock price processes S_{it}, $i = 1, \ldots, k$ and S_{jt}, $j = k+1, \ldots, k+\ell$, and define a generalized power exchange option giving payoff $(\sum_{i=1}^k S_{iT}^{n_i} - \sum_{j=k+1}^{k+\ell} S_{jT}^{n_j})^+$. Using the same argument as in Theorem 2.3, the NEE condition for the American version of this option is $Q_i \le 0$, $\forall i$, $Q_j \ge 0$, $\forall j$, under which the prices of the APE and EPE options are equal and given by $e^{-r(T-t)}\mathsf{E}_t[(\sum_{i=1}^k S_{iT}^{n_i} - \sum_{j=k+1}^{k+\ell} S_{jT}^{n_j})^+]$. One can see that the NEE region takes about $2^{-(k+\ell)}$ of all parameter sets over the space of $(Q_1, \ldots, Q_{k+\ell})$ if Q_i, Q_j are equally likely to be positive or negative (recall that in Fig. 2, the NEE region, i.e. quadrant (II) and half axes, roughly takes 1/4

Table 2
The parameter sets $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ in the BS and JD models.

Model	Processes	Parameters	
		Common	Model specific
BS	(S_{1t}, S_{2t})	r, q_1, q_2	$\boldsymbol{\theta} = \{\sigma_1, \sigma_2, \rho\}$
	$(S_{1t}^{n_1}, S_{2t}^{n_2})$	r, Q_1, Q_2	$\boldsymbol{\Theta} = \{n_1\sigma_1, n_2\sigma_2, \rho\}$
JD	(S_{1t}, S_{2t})	r, q_1, q_2	$\boldsymbol{\theta} = \{\sigma_1, \sigma_2, \rho, \lambda_1, \gamma_1, \delta_1, \lambda_2, \gamma_2, \delta_2\}$
	$(S_{1t}^{n_1}, S_{2t}^{n_2})$	r, Q_1, Q_2	$\boldsymbol{\Theta} = \{n_1\sigma_1, n_2\sigma_2, \rho, \lambda_1, n_1\gamma_1, n_1\delta_1, \lambda_2, n_2\gamma_2, n_2\delta_2\}$

of the Q_1–Q_2 plane). Other results in Theorem 2.5 can also be generalized. For example, the generalized result of case 4 is $\mathrm{APE} \le \overline{\mathrm{APE}} = e^{-r(T-t)}\mathrm{E}_t[(\sum_{i=1}^{k} e^{Q_i(T-t)}S_{iT}^{n_i} - \sum_{j=k+1}^{k+\ell} e^{Q_j(T-t)}S_{jT}^{n_j})^+]$ when $Q_i \ge 0, \forall i, Q_j \le 0, \forall j$. Cases 1 and 3 can be generalized in a similar manner.

3. Numerical analysis

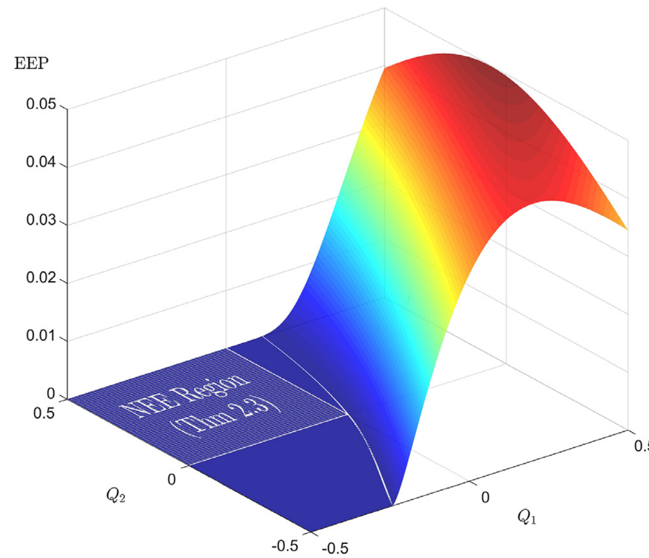
The main insight of the preceding mathematical analysis is that early exercise is more (less) likely to happen if (Q_1, Q_2) moves away from (closer to) quadrant (II) (the NEE region) on the Q_1–Q_2 plane. To further quantify the value of early exercise and its dependence on (Q_1, Q_2), this section provides a numerical analysis based on the Geske–Johnson (GJ) method [4]. Let BPE_k denote the price of a Bermudan power exchange option which is exercisable at k equally spaced time points over option life. In the two-step GJ method, the APE option price (seen as BPE_k with $k \to \infty$) is obtained from Richardson extrapolation by $\mathrm{APE}_2 = 2\mathrm{BPE}_2 - \mathrm{EPE}$, which usually provides satisfactory accuracy. Its three-step version (i.e. $\mathrm{APE}_3 = 4.5\mathrm{BPE}_3 - 4\mathrm{BPE}_2 + \mathrm{EPE}$) provides limited improvements but induces much higher computational costs. In the next subsections, we follow [7] and use the two-step GJ method to conduct our analysis.

3.1. The EE premium and NEE region viewed on the Q_1–Q_2 plane

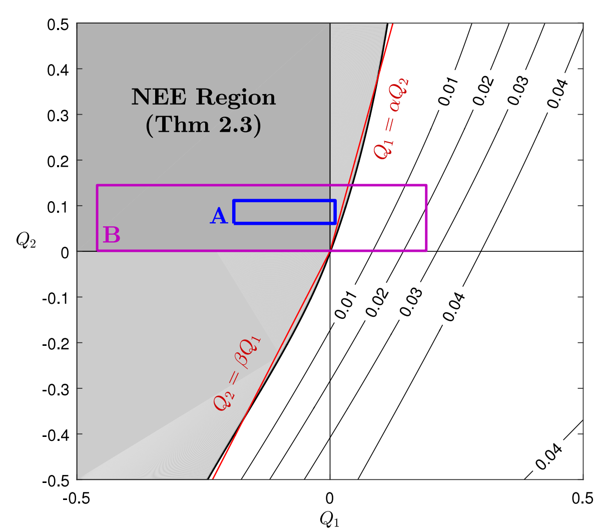
In the first example, we consider an American power exchange option with $(n_1, n_2) = (2, 0.5)$ under the bivariate Black–Scholes model with the following parameters $S_{1t} = S_{2t} = 1, T - t = 1, r = 0.1, \sigma_1 = \sigma_2 = 0.3, \rho = 0$. The dividend yields q_1, q_2 are left free (possibly negative) so that (Q_1, Q_2) may cover the designated range of $[-0.5, 0.5] \times [-0.5, 0.5]$. The value of early exercise is measured by the early exercise premium (EEP) which is the difference between the prices of APE (2-step Geske–Johnson method) and EPE (using Margrabe's formula in [9] with altered parameter set) options. We refer to the region of (Q_1, Q_2) with $\mathrm{EEP} = \mathrm{APE} - \mathrm{EPE} < 10^{-6}$ as the *numerical* NEE region, in contrast with the *theoretical* NEE region which is quadrant (II).

Fig. 3 shows the EEP on the Q_1–Q_2 plane. From the 3D plot in (a) we see that EEP increases as either Q_1 or $-Q_2$ gets large. It reaches a peak at around $(Q_1, Q_2) = (0.5, 0.065)$ after which EEP starts to decline. This is because when Q_1 and $-Q_2$ are large, $S_{1T}^{n_1}$ tends to be small and $S_{2T}^{n_2}$ tends to be large. This will significantly lower the option value as well as the EEP (even though early exercise appears to be more likely). On the other hand, as either Q_1 or $-Q_2$ gets smaller and (Q_1, Q_2) gets closer to quadrant (II), EEP decreases towards zero. It is interesting to note that the numerical NEE region (determined by $\mathrm{EEP} < 10^{-6}$ which is considered to be negligibly low) covers a significantly greater area than the theoretical NEE region (determined by Theorem 2.3). This indicates that EEPs are generally fairly small in reality. In addition, the numerical NEE boundary passes through the origin and behaves like a linear function in quadrants (I) and (III) respectively.

The 2D plot in (b) provides more detailed observations. The contour lines with $\mathrm{EEP} = 0.01$–0.04 exhibit a similar pattern to the NEE boundary (interpreted as $\mathrm{EEP} \approx 0$). This boundary can be respectively approximated by a line $Q_1 = \alpha Q_2$ in quadrant (I) and by another line $Q_2 = \beta Q_1$ in quadrant (III). The theoretical NEE region corresponds to $\alpha = \beta = 0$. The fact that the actual $\alpha, \beta > 0$ (in plot (b) they are $\alpha = 0.248$ and $\beta = 2.154$) indicates that the numerical NEE region covers a significantly larger area. In quadrant (I), for a given Q_2 (say $Q_2 = 0.1$), Q_1 needs to go up to $Q_1 > \alpha Q_2$ (=0.0248 instead of 0) for EEP to grow to a noticeable

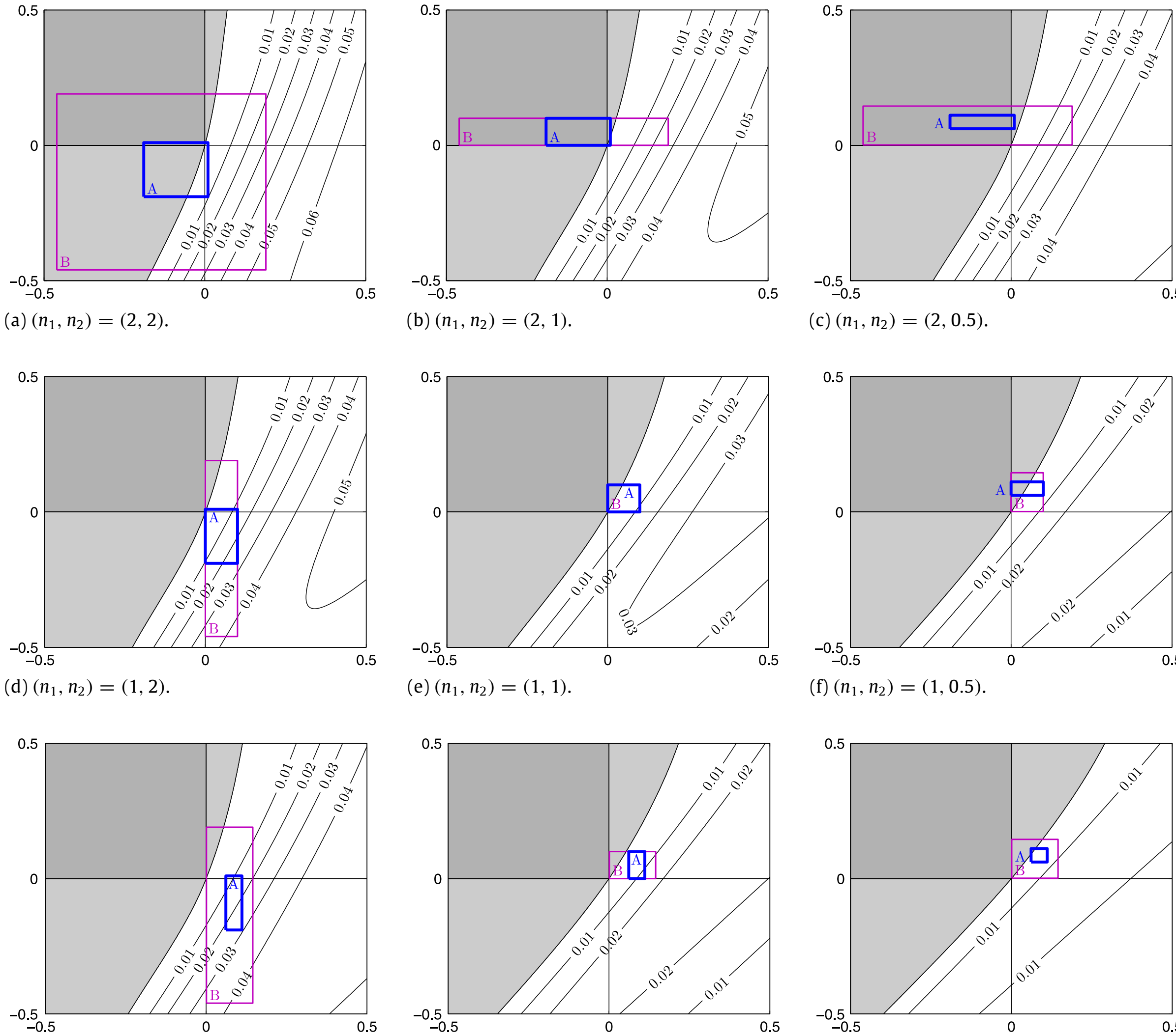


(a) 3D plot of EEP.



(b) 2D plot of EEP.

Fig. 3. EEP on the Q_1–Q_2 plane under power coefficients $(n_1, n_2) = (2, 0.5)$. (a) 3D plot with EEP growing from 0 to higher level over 0.04; (b) 2D plot with gray areas representing the theoretical (dark gray) and numerical (light gray) NEE regions. The contour lines correspond to $\mathrm{EEP} = 0.01$–0.04. Range A is formed by (Q_1, Q_2) with $q_i \in [0, 0.1]$, and range B is formed by varying both $q_i \in [0, 0.1]$ and $\sigma_i \in [0.1, 0.6]$.



(a) $(n_1, n_2) = (2, 2)$. (b) $(n_1, n_2) = (2, 1)$. (c) $(n_1, n_2) = (2, 0.5)$.

(d) $(n_1, n_2) = (1, 2)$. (e) $(n_1, n_2) = (1, 1)$. (f) $(n_1, n_2) = (1, 0.5)$.

(g) $(n_1, n_2) = (0.5, 2)$. (h) $(n_1, n_2) = (0.5, 1)$. (i) $(n_1, n_2) = (0.5, 0.5)$.

Fig. 4. The influence of the power coefficients (n_1, n_2) on the NEE region, the EEP contour lines, and the typical parameter ranges on the Q_1–Q_2 plane. The theoretical NEE regions are fixed in quadrant (II) but the numerical NEE regions (shaded areas) vary with (n_1, n_2). Ranges A and B also vary with (n_1, n_2) and this significantly influences the probability of a given (Q_1, Q_2) falling on the NEE region.

level. Likewise, in quadrant (III), for a given Q_1 (say $Q_1 = -0.1$), this happens when Q_2 drops down to $Q_2 < \beta Q_1$ ($= -0.2154$ instead of 0). The smaller α indicates that the numerical NEE region in quadrant (I) does not deviate too much from Theorem 2.3. But the relatively larger β suggests that many parameter sets in quadrant (III) actually lead to almost zero EEP, which is a greater deviation from Theorem 2.3.

It is also important to consider the range of (Q_1, Q_2) under typical parameters of (q_1, q_2) because not all the points on the Q_1–Q_2 plane are achievable. We consider two parameter ranges. Range A is formed by $q_1, q_2 \in [0, 0.1]$ (with $\sigma_1, \sigma_2 = 0.3$ fixed) which are intended to cover most market values of dividend yields. Range B is formed by further varying $\sigma_1, \sigma_2 \in [0.1, 0.6]$ on top of $q_1, q_2 \in [0, 0.1]$ which will cover a much greater area than range A. Both ranges of (Q_1, Q_2) are obtained via (4) and shown in Fig. 3(b). (Note that α, β may depend on σ_1, σ_2, and range B is only intended to show the location of (Q_1, Q_2) under these typical parameters.) We see that both ranges A and B overlap with the numerical NEE region by more than a half of their own areas. In fact, the numerical NEE region and the typical ranges of (Q_1, Q_2) depend in different ways on the power coefficients, which will be discussed in the next subsection.

3.2. The influence of power coefficients (n_1, n_2) on the shape of NEE region

To give an in-depth investigation on the effects from the power coefficients, we consider 9 combinations of (n_1, n_2) where $n_i \in \{2, 1, 0.5\}$. The 2D plots (same form as Fig. 3(b)) for these cases are shown in Fig. 4 where the numerical NEE regions are found to vary with the power coefficients in contrast with the theoretical NEE regions which are fixed. The previously defined α and β not only describe the shape of the numerical NEE regions but also reveal the patterns of other contour lines (particularly β). Through a simple linear regression model (with a number of $n_1, n_2 = 0.1, 0.2, \ldots, 2.0$), α and β can be related to the power coefficients as below:

$$\begin{cases} \alpha \approx 0.381025 - 0.171746(n_1 - 1) - 0.171776(n_2 - 1), & \text{adjusted } R^2 = 0.937155; \\ \beta \approx 1.777669 + 0.458254(n_1 - 1) + 0.458311(n_2 - 1), & \text{adjusted } R^2 = 0.986751. \end{cases}$$

We see that α is generally much smaller than β and less sensitive to n_1, n_2. Note that greater n_i tends to make the power process $S_{it}^{n_i}$ more volatile. The fact that α is negatively related to n_1, n_2 indicates that the region with positive premium expands in quadrant (I) when $S_{it}^{n_i}$ is more volatile. The fact that β is positively related to n_1, n_2 indicates that the NEE region expands in quadrant (III) when $S_{it}^{n_i}$ is more volatile.

It is more important to look at the range of (Q_1, Q_2) covered by the typical market observed values of (q_1, q_2) under different (n_1, n_2). The ranges A and B as defined earlier are shown in each plot of Fig. 4. Unlike the numerical NEE region whose shape does not vary too much under different (n_1, n_2), the shapes of ranges A and B are very sensitive to (n_1, n_2). For instance, using (4) with varying $q_i \in [0, 0.1]$ and fixed $r = 0.1, \sigma_i = 0.3$, the range of $Q_i, i = 1, 2$, is $Q_i \in [-0.19, 0.01]$ when $n_i = 2$, $Q_i \in [0, 0.1]$ when $n_i = 1$, and $Q_i \in [0.06125, 0.11125]$ when $n_i = 0.5$. These make the locations and areas of range A (as well as B) very different across the 9 combinations of (n_1, n_2). To discuss the different ways they overlap with the shaded region, these 9 cases are divided into three categories according to their locations.

- The first category contains plots (a), (b), and (c) where the overlapped areas are significantly larger. Since $n_1 = 2$, both ranges A and B in each plot span into quadrant (II). When $n_2 = 2$ (plot (a)), they further span into quadrant (III) of which a large part is shaded.
- The second category contains plots (d) and (g) where the overlapped areas become much smaller (almost no overlap for range A). Since $n_2 = 2$ but $n_1 \leq 1$, both ranges span into quadrant (IV) which is not shaded. There are small overlapped areas that can be seen in quadrant (I).
- The third category contains the rest plots (e), (f), (h), and (i), again with much smaller overlapped areas but at different locations. With $n_1, n_2 \leq 1$, ranges A and B are confined in quadrant (I) and located near to the origin. The overlapped areas are much smaller than the first category but greater than the second category.

Acknowledgment

The authors acknowledge the support from the Ministry of Science and Technology of Taiwan under the grant number 101-2410-H-011-003.

References

[1] L.P. Blenman, S.P. Clark, Power exchange options, Finance Res. Lett. 2 (2005) 97–106.
[2] S.L. Chung, H.C. Chang, Generalized analytical upper bounds for American options prices, J. Finan. Quant. Anal. 42 (2007) 209–228.
[3] S. Fischer, Call option pricing when the exercise price is uncertain, and the valuation of index bonds, J. Finance 33 (1978) 169–176.
[4] R. Geske, H.E. Johnson, The American put option valued analytically, J. Finance 39 (1984) 1511–1524.
[5] S.L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, Rev. Financ. Stud. 6 (1993) 327–343.
[6] S.A. Johnson, Y.S. Tian, Indexed executive stock options, J. Financ. Econ. 57 (2000) 35–64.
[7] S. Lindset, Pricing American exchange options in a jump-diffusion model, J. Futures Mark. 27 (2007) 257–273.
[8] D.B. Madan, P.P. Carr, E.C. Chang, The variance gamma process and option pricing, Eur. Financ. Rev. 2 (1998) 79–105.
[9] W. Margrabe, The value of an options to exchange one asset for another, J. Finance 33 (1978) 177–186.
[10] R.C. Merton, Option pricing when underlying stock returns are discontinuous, J. Financ. Econ. 3 (1976) 123–144.
[11] D.W. Miao, S.L. Chung, Y.H. Lee, The never-early-exercise condition and analytical price upper bounds of American power call options, J. Futures Options 7 (2014) 1–24.