# Chaos in Mechanical Systems: The Double Pendulum

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## Abstract

This paper demonstrates the numerical results of analyzing the chaotic motion of an idealized double pendulum. Double pendulums were found to be extremely sensitive to initial conditions. In other words, infinitesimally close initial data would result in arbitrarily large divergences as time evolves. As a result, the motion of such simple system is exceptionally difficult to predict. In this discussion, we are modeling the motion of a double pendulum with a set of second-order ordinary differential equations using Newtonian/Lagrangian mechanics. By obtaining a dataset to the solutions of the differential equations, we can gain some insights about the chaotic behavior in the seemingly excessively deterministic mechanical system. That is, if someone knows the precise initial conditions of the system, the future values for any given time are entailed.

## Problem description

Two double pendulums are being observed in this experiment (Figure 1). The pendulums are identical in materials, length, width, and mass. In addition, they move through the same medium (air). The experimenter used a rod to level the two pendulums so that they would have the same initial conditions. The experimenter then releases the rod so that the pendulums would fall freely due to gravity.

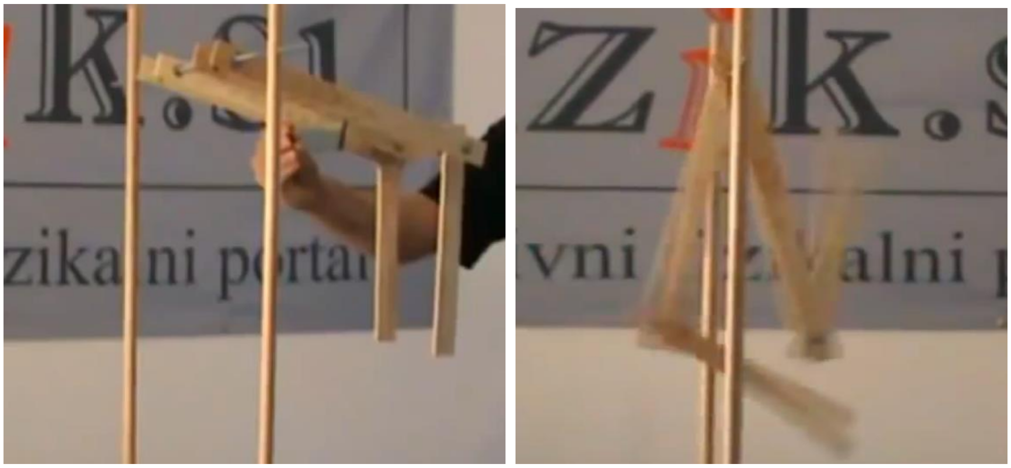


*Figure 1* - Demonstration of a system of double pendulum - snapshot from www.Fizik.si’s double pendulum video (2011)

Initially, the two rods exhibit synchronize movements. As time goes by, it is observed that the two pendulums exhibit unpredictable chaotic behaviors as shown.

## Simplifications

The double pendulum that will be analyzed consists of two simple pendulums (Figure 2). The upper simple pendulum is suspended at a fixed point *O*. The lower simple pendulum is suspended at the lower end of the upper pendulum, called $P_1$. The lower end of the lower pendulum is called $P_2$. The plane in which the pendulums swing will have *O* as the origin, the horizontal line through *O* as the x-axis, and the vertical line through *O* as the y-axis.

Let $l_1$ and $l_2$ be the lengths of the upper and lower pendulums respectively, $m_1$ and $m_2$ be the mass of $P_1$ and $P_2$. The angles created by the pendulums and the y-axis are 1and 2. The positions of $P_1$ and $P_2$ are ($x_1$, $y_1$) and ($x_2$, $y_2$) respectively.

1923).

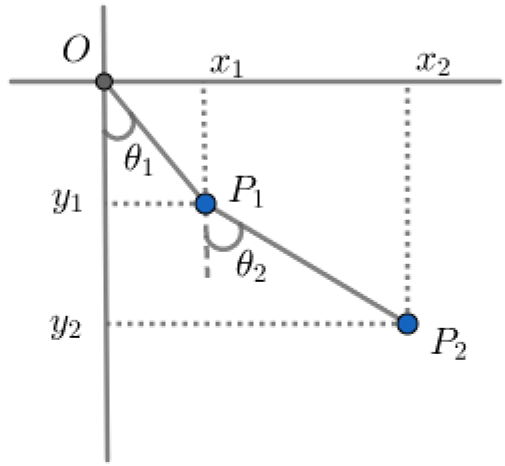



*Figure 2* - A simple double pendulum system - drawn by Geogebra

There are multiple ways to solve for the motion of the pendulums using ordinary differential equations (ODE). This paper will discuss both methods using Newton's laws of motion and Lagrangian mechanics.

In order to obtain an intuitive model, there are numbers of simplifications to the system. We assume the two rods are massless and independent of motion. As a result, $m_1$ and $m_2$ are point masses positioned at $P_1$ and $P_2$ with the entire masses of $m_1$ and $m_2$ concentrated at their centers of gravity. In addition, gravitation is the only source of external force interacts with the system. In other words, the system will not considered air resistance or any kind of damping.

## Mathematical model

Physically analyzed, there are four fixed points. In other words, there are four possible initial conditions where the pendulum would not move regardless of gravity. After achieving the set of differential equations, we will simulate the model and test the four fixed points (Cederberg 1923).

- *Fixed point 1*: $\theta_1 = \theta_2 = 0$
- *Fixed point 2:* $\theta_1 = \theta_2 =$

At this point, the system is in a stable state. If there is any small perturbation in the angles, the system would lose its stability.

- *Fixed point 3:* $\theta_1 = 0$; $\theta_2 =$

The upper pendulum in this case is in stable state and the lower pendulum is in unstable state. Upper pendulum's stability is in center direction and the lower pendulum's stability is in saddle direction. As a result, the system would be unstable and would approach the state of fixed point 1.

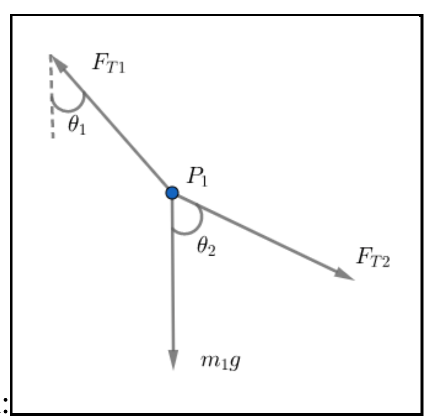
- *Fixed point 4:* $\theta_1 =$ ; $\theta_2 = 0$

The lower pendulum in this case is in stable state and the lower pendulum is in unstable state. Similarly to fixed point 3, the lower pendulum and upper pendulum's stabilities are in saddle direction and center direction respectively. Therefore, the system would also be unstable and would approach the state of fixed point 1.

As mentioned above, there will be two models discussed in this paper to demonstrate the chaotic behaviors of the double pendulum system.

- *Model 1*: Analyzing the problem using Newtonian Mechanics (Cederberg 1923)

Consider the mechanics of a simple pendulum:



$F_{T1}$: the force of tension of the upper pendulum on $P_1$

$F_{T2}$: the force of tension of the lower pendulum on $P_2$

$m_1g$: gravitational force acting on point mass $P_1$

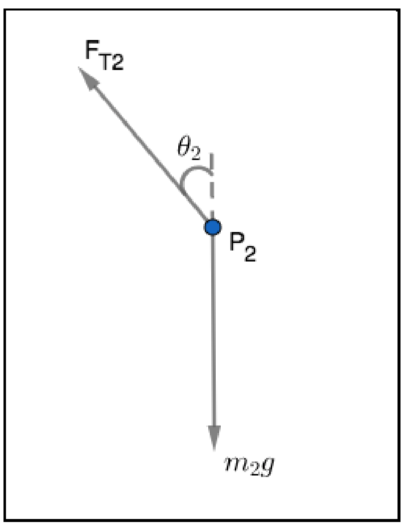
*Figure 3* - Free body diagram of point mass $P_1$ - drawn by Geogebra

The net force acting on $P_1$ in the x-direction obtained by Newton's Second law, where $a_x$ is the acceleration in the x-direction:

$$F_{x1} = ma_x = m_1\ddot{x}_1 = - F_{T1} \sin \theta_1 + F_{T2} \sin \theta_2 \qquad (1)$$

The net force acting on $P_1$ in the y-direction obtained by Newton's Second law, where $a_y$ is the acceleration in the y-direction:

$$F_{y1} = ma_y = m_1\ddot{y}_1 = F_{T1} \cos \theta_1 - F_{T2} \cos \theta_2 - m_1 g \qquad (2)$$




$F_{T2}$: the force of tension of the upper pendulum on $P_2$

$m_2g$: gravitational force acting on point mass $P_2$

*Figure 4* - Free body diagram of point mass $P_2$ - drawn by Geogebra

Also using Newton's Second Law, the net force on $P_2$ in x and y direction are, respectively:

$$F_{x2} = ma_x = m_2\ddot{x}_2 = - F_{T2} \sin \theta_2 \qquad (3)$$

$$F_{y2} = ma_y = m_2\ddot{y}_2 = F_{T2} \cos \theta_2 - m_2 g \qquad (4)$$

By combining (1) and (3), then (2) and (4):

$$F_{x1} = - F_{T1} \sin\theta_1 - F_{x2} \qquad (5)$$

$$F_{y1} = F_{T1} \cos \theta 1 - F_{y2} - m_1 g - m_2 g \qquad (6)$$

Through algebraic manipulations to (5) and (6):

$$F_{T1} \sin\theta_1 = - F_{x1} - F_{x2} \qquad (7)$$

$$F_{T1} \cos\theta_1 = F_{y1} + F_{y2} + g( m_1 + m_2 ) \qquad (8)$$

When multiply the left hand side (LHS) of (7) and (8) with $\cos\theta_1$ and $\sin\theta_1$ respectively, LHS of both equations can be observed to be the same ($F_{T1}\sin\theta_1\cos\theta_1$). Therefore:

$$\sin\theta_1 [F_{y1} + F_{y2} + g ( m_1 + m_2 ) ] = \cos\theta_1 ( - F_{x1} - F_{x2} ) \qquad (9)$$

Next, multiplying (3) by $\cos\theta_2$ on both sides:

$$\cos\theta_2 ( F_{x2} ) = \cos\theta_2 ( - F_{T2} \sin\theta_2 ) \qquad (10)$$

As a result, (4) can be rewritten as:

$$F_{y2} + m_2 g = F_{T2} \cos\theta_2 \qquad (11)$$

After multiplying (11) by $- \sin\theta_2$ on both sides, the right hand side (RHS) of (10) and the altered (11) are the same ( $- F_{T2} \cos\theta_2 \sin\theta_2$). Therefore:

$$\cos\theta_2 ( F_{x2} ) = - \sin\theta_2 ( F_{y2} + m_2 g ) \qquad (12)$$

Recall that the positions of $P_1$ and $P_2$ are $(x_1, y_1)$ and $(x_2, y_2)$ respectively:

$$( x_1, y_1 ) = ( l_1 \sin\theta_1, - l_1 \cos\theta_1 ) \qquad (13)$$

$$( x_2, y_2 ) = ( l_1\sin\theta_1 + l_2\sin\theta_2, - l_1 \cos\theta_1 - l_2\cos\theta_2 ) \qquad (14)$$

The velocities of the point masses in x and y directions would then be:

$$dx_1/dt = \dot{x}_1 = l_1\cos\theta_1(d\theta_1/dt)$$

$$dy_1/dt = \dot{y}_1 = l_1\sin\theta_1(d\theta_1/dt)$$

$$dx_2/dt = \dot{x}_2 = l_1\cos\theta_1(d\theta_1/dt) + l_2\cos\theta_2(d\theta_2/dt)$$

$$dy_2/dt = \dot{y}_2 = l_1\sin\theta_1(d\theta_1/dt) + l_2\sin\theta_2(d\theta_2/dt)$$

The accelerations of the point masses in x and y directions would then be achieved by deriving the velocity functions:

$$\ddot{x}_1 = - (d\theta_1/dt)^2\, l_1 \sin\theta_1 + \ddot{\theta}_1\, l_1 \cos\theta_1$$

$$\ddot{y}_1 = (d\theta_1/dt)^2\, l_1 \cos\theta_1 + \ddot{\theta}_1\, l_1 \sin\theta_1$$

$$\ddot{x}_2 = \ddot{x}_1 - (d\theta_2/dt)^2\, l_2 \sin\theta_2 + \ddot{\theta}_2\, l_2 \cos\theta_1$$

$$\ddot{y}_2 = \ddot{y}_1 + (d\theta_2/dt)^2\, l_2 \cos\theta_2 + \ddot{\theta}_2\, l_2 \sin\theta_2$$

In order to solve for the coordinates and then analyze the motion of the double pendulum system, plug the acceleration functions into (9) and (12). By doing so, we achieve another equation with many variables. Consequently, it makes the solution inefficient and overcomplicated. The reason for this complication is that Newton’s Second law operates in terms of forces vectors. As a result, we have to use the Cartesian coordinate system and deal with quantities and directions of different forces.

By contrary, Lagrangian mechanics operate in terms of energies. Lagrangian mechanics starts from a deeper principle and Newton’s laws can be derived using Lagrangian mechanics. For this problem, using the Lagrangian formulation would make the calculations easier.

Notice that there is no fundamental difference between the two methods. Energy and forces are closely related to each other. Newtonian mechanics is incorporated in Lagrangian mechanics.

- *Model 2*: Analyzing the problem using Lagrangian mechanics (Herrath 2000)

Lagrangian equation is defined by the difference of kinetic energy and potential energy:

$$L\ K - U \qquad (15)$$

where $K$ is the kinetic energy and $U$ is the potential energy.

Recall the positions and velocities functions from the Newtonian mechanics model:

Positions:

$(x_1, y_1) = (l_1 \sin\theta_1, -l_1 \cos\theta_1)$

$(x_2, y_2) = (l_1 \sin\theta_1 + l_2 \sin\theta_2, -l_1 \cos\theta_1 - l_2 \cos\theta_2)$

Velocities:

$dx_1/dt = \dot{x}_1 = l_1 \cos\theta_1 (d\theta_1/dt)$

$dy_1/dt = \dot{y}_1 = l_1 \sin\theta_1 (d\theta_1/dt)$

$dx_2/dt = \dot{x}_2 = l_1 \cos\theta_1 (d\theta_1/dt) + l_2 \cos\theta_2 (d\theta_2/dt)$

$dy_2/dt = \dot{y}_2 = l_1 \sin\theta_1 (d\theta_1/dt) + l_2 \sin\theta_2 (d\theta_2/dt)$

Using these values to obtain the kinetic and potential energy:

- *Kinetic Energy*:

$K = 12mv^2 = 12( m_1 v12 + m_2 v22 )$

$$= 12[(m_1 + m_2) l12 (d\theta_1/dt)^2 + m_2 l22 (d\theta_2/dt)^2 + 2 m_2 l_1 l_2 \cos(\theta_2 - \theta_1)(d\theta_1/dt)(d\theta_1/dt)] \quad (16)$$

- *Potential Energy*:

$U = mgh = g ( m_1 y_1 + m_2 y_2 )$

$$= - m_1 g l_1 \cos\theta_1 - m_2 g ( l_1 \cos\theta_1 + l_2 \cos\theta_2 ) \qquad (17)$$

where g is the gravitational force ( 9.8 m/s²)

By substituting (16) and (17) into Eq. 15, we obtain the Lagrangian of the double pendulum:

$L$ = 12[($m_1$+ $m_2$)$l$12($d\theta_1$/dt)$^2$ + $m_2$$l$22($d\theta_2$/dt)$^2$ + 2$m_2$$l_1$$l_2$ cos($\theta_2$ – $\theta_1$)($d\theta_1$/dt) ($d\theta_1$/dt)] -

[ - $m_1$g$l_1$cos$\theta_1$ – $m_2$g( $l_1$cos$\theta_1$ + $l_2$cos$\theta_2$ ) ]

Through algebraic manipulation,

$$
\begin{aligned}
L = \; & \frac{1}{2}m_1\left[(l_1\dot{\theta}_1\cos\theta_1)^2+(-l_1\dot{\theta}_1\sin\theta_1)^2\right] \\
& +\frac{1}{2}m_2\left[(l_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_2\cos\theta_2)^2+(-l_1\dot{\theta}_1\sin\theta_1-l_2\dot{\theta}_2\sin\theta_2)^2\right] \\
& +m_1gl_1\cos\theta_1+m_2g(l_1\cos\theta_1+l_2\cos\theta_2).
\end{aligned}
$$

The Lagrange equations in the generalized coordinates $\theta_1$ and $\theta_2$:

$$
\frac{\partial L}{\partial \theta_1}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)=0;
$$

$$
\frac{\partial L}{\partial \theta_2}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)=0
$$

Derive the above set of equations to get:

$l_1^2$($m_1$ + $m_2$) $\ddot{\theta}_1$ + $l_1l_2$ $m_2$cos($\theta_2$ – $\theta_1$) $\ddot{\theta}_2^2$ – $l_1l_2$ $m_2$sin ($\theta_2$ – $\theta_1$)($d\theta_2$/dt)$^2$ = - ( $m_1$ + $m_2$)g$l_1$sin$\theta_1$ (18)

$l_1l_2$ $m_2$cos ($\theta_2$ – $\theta_1$)$\ddot{\theta}_1$+ $l_2^2$$m_2$ $\ddot{\theta}_2$+ $l_1l_2$ $m_2$sin ($\theta_2$ – $\theta_1$) ($d\theta_1$/dt)$^2$ = - $m_2$g$l_2$sin$\theta_2$ (19)

Then solve (18) and (19) for $\ddot{\theta}_1$ and $\ddot{\theta}_2$:

$$
\ddot{\theta}_1=\frac{-g(2m_1+m_2)\sin\theta_1-m_2g\sin(\theta_1-2\theta_2)-2\sin(\theta_1-\theta_2)m_2\left[\dot{\theta}_2^2L_2+\dot{\theta}_1^2L_1\cos(\theta_1-\theta_2)\right]}{L_1\left[2m_1+m_2-m_2\cos(2\theta_1-2\theta_2)\right]}
$$

$$
\ddot{\theta}_2=\frac{2\sin(\theta_1-\theta_2)\left[\dot{\theta}_1^2L_1(m_1+m_2)+g(m_1+m_2)\cos\theta_1+\dot{\theta}_2^2L_2m_2\cos(\theta_1-\theta_2)\right]}{L_2\left[2m_1+m_2-m_2\cos(2\theta_1-2\theta_2)\right]}
$$

After achieving differential equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$, the next step is to use a computational system (MATLAB) to solve for numerical values of the generalized coordinates.

## Solution of the mathematical problem

In this section, we will use MATLAB to do two simulations to further observe the chaotic behaviors that the system exhibit. Before we could do all that, one of the most important steps is to plug the differential equations we obtained from Lagrangian mechanics:

```
pendulumfunction.m

function [yprime] = pend(t, y)

l1=1; l2=2 ; m1=2 ; m2=1; g=9.80665;
y_prime=zeros(4,1);

a = (m1+m2)*l1 ;
b = m2*l2*cos(y(1)-y(3)) ;
c = m2*l1*cos(y(1)-y(3)) ;
d = m2*l2 ;
e = -m2*l2*y(4)* y(4)*sin(y(1)-y(3))-g*(m1+m2)*sin(y(1)) ;
f = m2*l1*y(2)*y(2)*sin(y(1)-y(3))-m2*g*sin(y(3)) ;

yprime(1) = y(2);
yprime(3)= y(4) ;
yprime(2)= (e*d-b*f)/(a*d-c*b) ;
yprime(4)= (a*f-c*e)/(a*d-c*b) ;

yprime=yprime';

end
```

*__Note:__* Codes modified from Sathyanarayan Rao's double pendulum simulation code (2014).

Now, we will setup the initial conditions:

```
%Modeling a double pendulum system
%Mainly adopted from Sathyanarayan Rao
%The ordinary differetial equation is solved using Matlab ODE45
clc
close all
clear all
%Initial Conditions
l1=1; %length of the upper rod
l2=2; %length of the lower rod
m1=2; %mass of the first rod
m2=1; %mass of the second rod
g=9.80665; %gravitational acceleration
theta1 = 2.5; %the angle y-axis and the upper rod
theta1_prime=0;% initial angular velocity for the upper rod
theta2= 1 ; %the angle y-axis and the lower rod
theta2_prime=0;% initial angular velocity for the lower rod
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%
y0=[theta1 theta1_prime theta2 theta2_prime];
tspan=50; %simulation time
[t,y]=ode45(@pend, [0 ,tspan],y0);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%
%x-y coordinates of p1 and p2
x1=l1*sin(y(:,1));
y1=-l1*cos(y(:,1));
x2=l1*sin(y(:,1))+l2*sin(y(:,3));
y2=-l1*cos(y(:,1))-l2*cos(y(:,3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%
```

The next snippet of codes will exhibit the trajectory of point masses $m_1$ and $m_2$ on the x-y plane (Figure 5) determined above:

```
%Positions of point masses m1 and m2 as time progresses
  figure(1)
  plot(x1,y1,'linewidth',2)
  hold on
  plot(x2,y2,'r','linewidth',2)
  h=gca;
  get(h,'fontSize')
  set(h,'fontSize',14)
  xlabel('X','fontSize',14);
  ylabel('Y','fontSize',14);
  title('Trajectory of upper and lower pendulums','fontsize',14)
  fh = figure(1);
  set(fh, 'color', 'white');
```

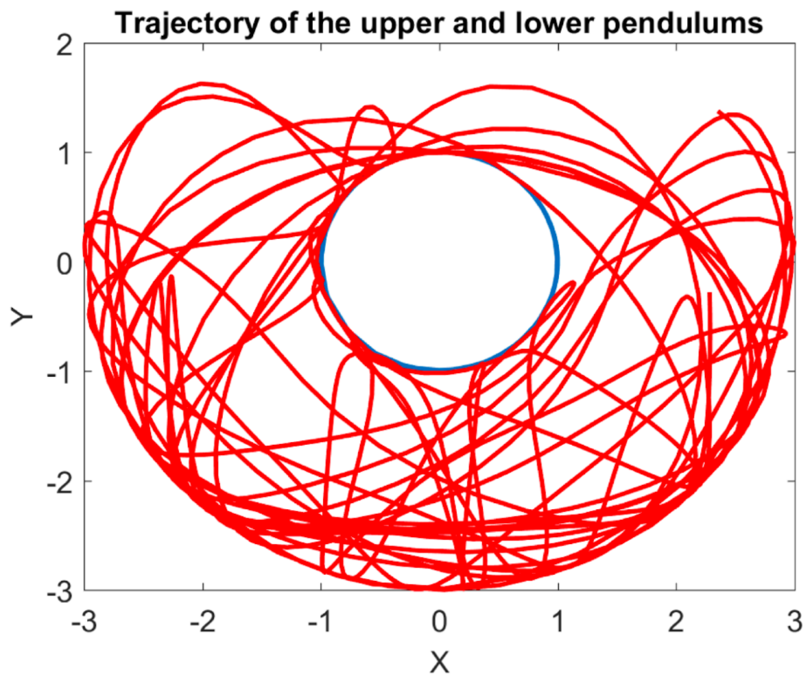



*Figure 5* - Positions of point masses $m_1$ and $m_2$ obtained using MATLAB codes

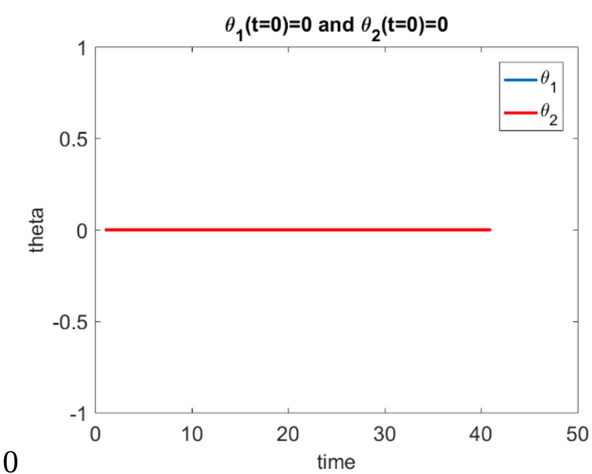
By just intuitively looking at Figure 5, it can be observed that only the upper pendulum moves in a circular pattern, while the lower pendulum exhibits no pattern at all. However, the motion of $m_1$ is not always the same. In other words, if only look at Figure 5, $m_1$ seems to move in a circular loop but in fact, it swings back and forth with no constant angle velocity.

To fully observe the chaotic behaviors, we add codes to create graphs of generalized positions of $m_1$ and $m_2$ - $\theta_1$ and $\theta_2$ on the y-axis and time on the x-axis. We will use these graphs to discuss the steady states and chaotic behaviors of the system in Results and Discussion section.

```
figure(2)
plot(y(:,1),'linewidth',2)
hold on
plot(y(:,3),'r','linewidth',2)
h=gca;
get(h,'fontSize')
set(h,'fontSize',14)
legend('\theta_1','\theta_2')
xlabel('time','fontSize',14);
ylabel('theta','fontSize',14);
title('\theta_1(t=0)=0 and \theta_2(t=0)=0','fontsize',14)
fh = figure(2);
set(fh, 'color', 'white');
```

## Results and Discussion

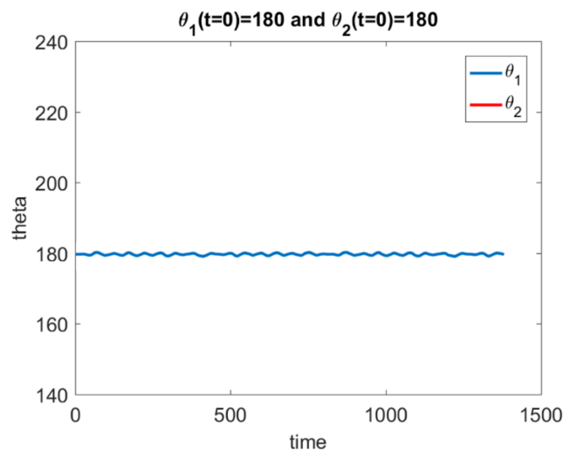
To verify the fixed points mentioned, as mentioned in Mathematical model section, we set the initial conditions accordingly to obtain the graphs then observe and analyze.




- *Fixed point 1*: $\theta_1 = \theta_2 = 0$

From Figure 6, the angle does not change through time. As a result, the physical hypothesis is correct.

*Figure 6* - $\theta_1$ and $\theta_2$ through time for fixed point 1's initial conditions

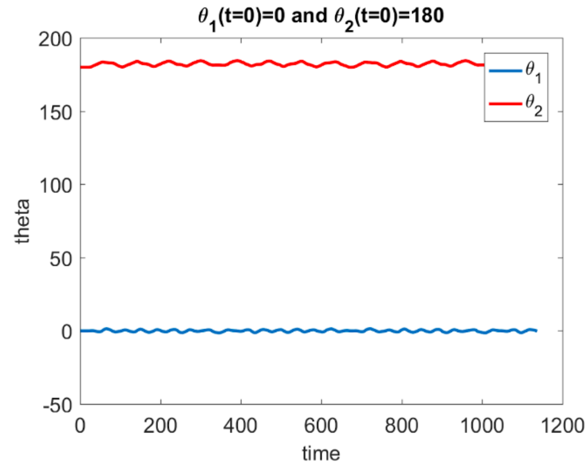



- *<u>Fixed point 2:</u>* $\theta_1 = \theta_2 =$

From Figure 7, observed that the lines represent $\theta_1$ and $\theta_2$ through time are straight and lie on top of each other.

This means the angle does not change during the motion. Therefore, the physical hypothesis is correct.

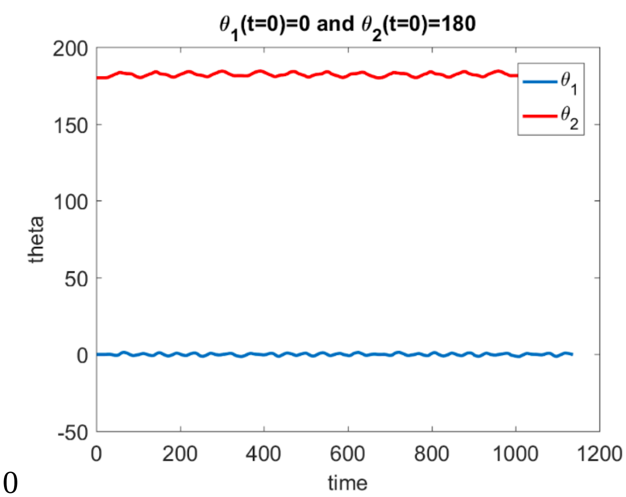
*Figure 7* - $\theta_1$ and $\theta_2$ through time for fixed point 2's initial conditions




- *<u>Fixed point 3:</u>* $\theta_1 = 0$;  $\theta_2 =$

From Figure 8, the angle does not change through time. As a result, the physical hypothesis is correct.

*Figure 8* - $\theta_1$ and $\theta_2$ through time for fixed point 3's initial conditions




- *<u>Fixed point 4:</u>* $\theta_1 =$ ;  $\theta_2 = 0$

From Figure 9, the angle does not change through time. As a result, the physical hypothesis is correct.

*Figure 9* - $\theta_1$ and $\theta_2$ through time for fixed point 4's initial conditions

Now, we will exhibit the chaotic behaviors through the graphs.

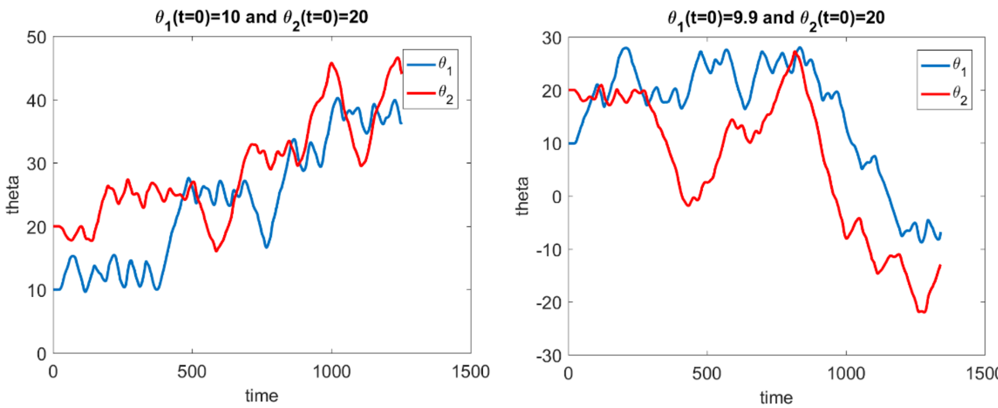



*Figure 10 -* $\theta_1$ and $\theta_2$ through time for $\theta_1$=10; $\theta_2$=20 initially (left) and with small perturbation on $\theta_1$ for $\theta_1$=10; $\theta_2$=20 (right)

By looking at Figure 10, we observe that the pendulums are extremely sensitive to small perturbations, namedly the one on $\theta_1$ (-0.1 degree). As a result, we can conclude that the double pendulum system does indeed exhibit chaotic behaviors.

## Improvement

There is another way to analyze the system if the assumption of no dissipation and no damping still applies. If there is no dissipation, the angular momentum is conserved. As a result, we can use Hamiltonian formalism and generate four first-order differential equations with two in terms of the generalized coordinates and two in terms of the momentums.

Hamiltonian mechanics was formulated from the Lagrangian mechanics with slight difference. While Lagrangian mechanics describes the difference in kinetic and potential energy, Hamiltonian mechanics describes the total energy (Herrath 2000):

$$H = K + U$$

To describe Hamiltonian mechanics in terms of momentums:

$$H = \theta_i\, p_i - L \qquad (20)$$

where $\theta_i$ is the generalized coordinates, $p_i$ is the generalized momentum, and L is the Lagrangian of the system.

Even though Lagrangian mechanics works well to deal with this particular double pendulum system, it is worth it to learn about the Hamiltonian mechanics. Since Hamiltonian mechanics is a powerful tool when dealing with systems containing large numbers of particles, it can be useful to analyze the system when it gets more complicated (Morin 2012).

Diving into the calculations under Hamiltonian method for the double pendulum system, we find the generalized momentums by taking the derivative of the Lagrangian of the system with respect to the angular velocity.

$$\begin{aligned} p_{\theta_1} &= \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ p_{\theta_2} &= \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2). \end{aligned}$$

By substituting the values of generalized momentums into (20), we obtain:

$$H = \frac{l_2^2 m_2 p_{\theta_1}^2 + l_1^2 (m_1 + m_2) p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2 \left[m_1 + \sin^2(\theta_1 - \theta_2) m_2\right]}$$

$$- m_2 g l_2 \cos\theta_2 - (m_1 + m_2) g l_1 \cos\theta_1.$$

By deriving Hamiltonian by appropriate values, we obtain:

$$
\begin{aligned}
\dot{\theta}_1 &= \frac{\partial H}{\partial p_{\theta_1}} = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2}\cos(\theta_1-\theta_2)}{l_1^2 l_2[m_1 + m_2\sin^2(\theta_1-\theta_2)]} \\
\dot{\theta}_2 &= \frac{\partial H}{\partial p_{\theta_2}} = \frac{l_1(m_1+m_2)p_{\theta_2} - l_2 m_2 p_{\theta_1}\cos(\theta_1-\theta_2)}{l_1 l_2^2 m_2[m_1 + m_2\sin^2(\theta_1-\theta_2)]} \\
\dot{p}_{\theta_1} &= -\frac{\partial H}{\partial \theta_1} = -(m_1+m_2)gl_1\sin\theta_1 - C_1 + C_2 \\
\dot{p}_{\theta_2} &= -\frac{\partial H}{\partial \theta_2} = -m_2 g l_2 \sin\theta_2 + C_1 - C_2,
\end{aligned}
$$

where

$$C_1 \equiv \frac{p_{\theta_1} p_{\theta_2}\sin(\theta_1-\theta_2)}{l_1 l_2[m_1 + m_2\sin^2(\theta_1-\theta_2)]}$$

$$C_2 \equiv \frac{l_2^2 m_2 p_1^2 + l_1^2(m_1+m_2)p_2^2 - l_1 l_2 m_2 p_1 p_2\cos(\theta_1-\theta_2)}{2l_1^2 l_2^2[m_1 + m_2\sin^2(\theta_1-\theta_2)]^2}\sin[2(\theta_1-\theta_2)].$$

The next ideal step would be plugging these equations into a computational system such as MATLAB similarly when using the Lagrangian method.

## Conclusions

The goal of this paper is to use different methods like Newtonian and Lagrangian to analyze the double pendulum. By using MATLAB, we observe its chaotic behaviors numerically. The motion of the system shows no particular patterns. In addition, the double pendulum is observed to be a system of deterministic chaos due to the fact that any remotely small change in the initial states of the system would result in a completely different result as time progresses.

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