COMP0127

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1. Part 1

1.1. since the rotation matrices in all axes only include the following entries: $\cos\theta$ $\sin\theta$ and 1. the limits of the entries are bounded from -1 to 1 which is ≤ 1

1.2. Since the dot product of two vectors is commutative, we see that $\mathrm{R}_0^1 = (\mathrm{R}_0^1)^T$ the orientation of frame $o_o x_o y_o$ with the respect to frame $o_1 x_1 y_1$ is the inverse of the orientation $o_1 x_1 y_1$ in respect to $o_o x_o y_o$. Algebraically using the fact that coordinates frames are mutually orthogonal it can be seen that $(R_0^1)^T = (R_1^0)^{-1}$

for example in a simple 2D rotation:

$$R = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \tag{1.1}$$

the inverse and the transpose are equal: $R^{-1} == R^T$

1.3. the notation ${}^aR_b$ refers to the rotational transform from one coordinate system to another coordinate system b. The rows of the rotational matrix are the orthogonal unit vectors. Each row in a rotational matrix refers to the operations required to go from axis_o to axis_1. For example in the example above of the rotation matrix $\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ relates to the 2D transform:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{1.2}$$

Therefore the transformation to x_1 is described as: $\mathrm{x}_1 = \cos\theta * \mathrm{x}_0 + \sin\theta * \mathrm{y}_0$.

1.4. It can be shown that for a 3x3 rotation matrix the eigenvalues can be shown to be:

$$\lambda = 1, \lambda = e^{\pm i\theta} \tag{1.3}$$

which means that the eigenvectors can be are related to the angle via the Euler's identity and angle θ.

2. Part 2

2.1. Gimbal lock refers to the loss of degrees of freedom when two axis are aligned with each other. in order to avoid Gimball lock we alternative rotation notation such as quaterninons which prevent the the occurrence of a gimball lock situation.
example of an extrinsic Euler angle YZY matrix.

$$\begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix} \begin{bmatrix} cos\beta & -sin\beta & 0 \\ sin\beta & cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\gamma & 0 & sin\gamma \\ 0 & 1 & 0 \\ -sin\gamma & 0 & cos\gamma \end{bmatrix} \tag{2.1}$$

The matrix multiplies to:

$$\begin{bmatrix} cos(\gamma)cos(\alpha)cos(\beta) - sin(\gamma)sin(\alpha) & -cos(\gamma)sin(\beta)cos(\beta) + sin(\gamma)cos(\alpha) & cos(\alpha)cos(\beta)sin(\gamma) + sin(\alpha)sin(\gamma) \\ cos(\alpha)sin(\beta) & cos(\beta) & sin(\alpha)sin(\beta) \\ -sin(\gamma)cos(\alpha)cos(\beta) - cos(\alpha)sin(\gamma) & sin(\alpha)sin(\beta) & sin(\alpha)cos(\gamma) + sin(\gamma)sin(\alpha)cos(\beta) \end{bmatrix} \tag{2.2}$$

Gimbal lock occurs when $\beta = 0$, if we set $\theta = \alpha + \beta$. The resulting matrix can be simplified to:

$$\begin{bmatrix} cos(\gamma+\alpha) & 0 & sin(\gamma+\alpha) \\ 0 & 1 & 0 \\ -sin(\gamma+\alpha) & 0 & cos(\gamma+\alpha) \end{bmatrix} \tag{2.3}$$

Any set of values which satisfy the criterion θ will give a gimbal lock. For example, $\alpha = 30$' and $\gamma = 60$' results in the matrix looking like:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.4}$$

Where none of the entries in the matrix are dependant on angle β

For the intrisic. xyz example, we can prove that a gimbal lock can occur when the condition for θ is respected and when the angle β is 90, the intrisic matrix takes the following form after the multiplication $R_z(\alpha)R_y(\beta)R_x(\gamma)$, resulting in the following matrix:

$$\begin{bmatrix} cos(\alpha)cos(\beta) & cos(\alpha)sin(\beta)sin(\gamma) - sin(\alpha)cos(\gamma) & cos(\alpha)sin(\beta)cos(\gamma) + sin(\alpha)sin(\gamma) \\ sin(\alpha)cos(\beta) & sin(\alpha)sin(\beta)sin(\gamma) + cos(\alpha)cos(\gamma) & cos(\alpha)sin(\beta)sin(\gamma) - sin(\alpha)cos(\gamma) \\ -sin(\beta) & sin(\alpha)cos(\beta) & cos(\alpha)cos(\beta) \end{bmatrix} \tag{2.5}$$

which can be simplified to the following once $\beta = 90$ is inserted:

$$\begin{bmatrix} 0 & sin(\gamma+\alpha) & cos(\gamma+\alpha) \\ 0 & -cos(\gamma+\alpha) & sin(\gamma+\alpha) \\ -1 & 0 & 0 \end{bmatrix} \tag{2.6}$$

It can be seen that all the values in this matrix can be defined by a rotation between γ and α , independent of β . Meaning that one degree of freedom was lost

2.2. suppose we have vector v and quaternion $q = q_0 + \vec{q}$. . A rotation v with respect to q is described as $w = qvq^*$

$$w = qvq^* = (q_0 + q) + (0 + v) + (q_0 - q) \tag{2.7}$$

$$w = (2q_0^2)v + 2(q \cdot v)q + 2q_0(q \times v) \tag{2.8}$$

The components can be expanded as:

$$(2q_0^2 - 1)v = \begin{bmatrix} (2q_0^2 - 1) & 0 & 0 \\ 0 & (2q_0^2 - 1) & 0 \\ 0 & 0 & (2q_0^2 - 1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.9}$$

$$2(q \cdot v)q = \begin{bmatrix} 2q_1^2 & 2q_1q_2 & 2q_1q_3 \\ 2q_1q_2 & 2q_2^2 & 2q_2q_3 \\ 2q_1q_3 & 2q_2q_3 & 2q_3^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.10}$$

$$2q_0(q \cdot v) = \begin{bmatrix} 0 & -2q_0q_3 & 2q_0q_2 \\ 2q_0q_3 & 0 & 2q_2q_3 \\ -q_13 & 2q_2q_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.11}$$

Combining the three equations we get a rotation matrix from the quaternion representation , $w = \text{q}v\text{q}^*$ $= \text{Q}v$, where Q is the rotation matrix

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.12}$$

2.3.

- Axis angle
- Rotation Matrix
- Quaternion
- Rotation Matrix

2.4.

3. Part 3

3.1. It can be seen that rotation quaterion q and -q are equivalent when we substitute the quaternion identity inside the rotation matrix

$$q = q_0 + q_1 + q_2 + q_3 \tag{3.1}$$

$$-q = -q_0 - q_1 - q_2 - q_3 \tag{3.2}$$

since all the terms in the rotation matrix are either squared or multiplied by one another the resulting quaternion rotation matrix -Q will be equivalent to Q

3.2. Rotation matrices R_a and R_b become commutative when the rotation occurs on the same plane. This is the case only in 2D matrix rotation or in a gimball lock situation for 3D matrices. Additionally, two rotation matrices are commutative when both their quaternion vector are the same. Assume there are two quaternions q_0 and q_1 with angles α and β

$q_0 = cos(\alpha) + u \times sin(\alpha)$

$q_1 = cos(\beta) + u \times sin(\beta)$

$p = q_0 * q_1 = (cos\alpha + u * sin\alpha) \times (cos(\beta) + u \times sin(\beta))$

Which can be written as $\cos(\alpha + \beta) + u \times \sin(\alpha + \beta)$. The factors inside the cosine and since are commutative.

4. part 4

4.1. The service that coverts quaternions to Euler angles follows the following formula:

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} arctan(\frac{2(q_0q_1 + q_2q_3}{1 - 2(q_1^2 + q_2^2)}) \\ arcsin(2(q_0q_2 - q_3q_1)) \\ arctan(\frac{2(q_0q_3 + q_1q_2}{1 - 2(q_2^2 q_3^3)}) \end{bmatrix} \tag{4.1}$$

This equation implemented on computers only gives a result between π and -π, this is why in the code the eqation used uses arcsin2 , atan2. Singularities occur when the pitch angle approaches $\pm 90'$.

4.2. quaternions are represented like: q = $q_w + \vec{q}$, we know that this can be written as q= $\cos(\theta/2) + \vec{v}\sin(\theta/2)$. therefore the axis vector is given by $\vec{v} = \frac{\vec{q}}{sin(\theta/2)}$. The Rodrigues vector states that the $\vec{r} = \frac{\vec{q}}{sin(\theta)/2}\theta$. Therefore one can find the Rodrigues formulation by equating coefficients.

5. part 5

5.1. The derived DH parameters are shown in the table below:

θ	d	a	α
$\pi/2$	0.147	0	$\pi/2$
$\pi/2$	0	0.155	0
0	0	0.135	0
$\pi/2$	0	0	$\pi/2$
π	0.218	0	0

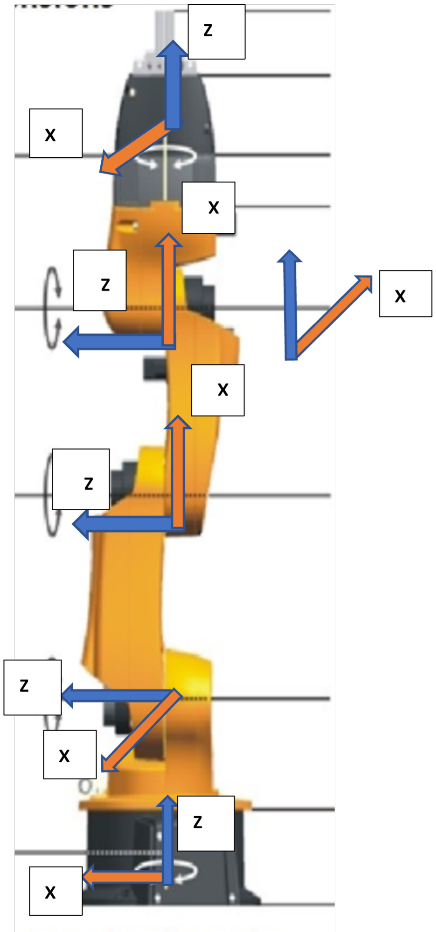
Table 1. Caption

The four parameters are a convection for connecting the reference frames to the links of a robot. The axes associated with each frame are derived by taking the cross product between the axis. The parameters

are defined in the following way:
d parameter is the offset along the previous z to the common normal.
θ is the angle about the previous z from x_{-1} to a new x
a is the length along the common normal between point z_{-1} and z
d is the distance along the z−1 axis between point x_{-1} and x

The positioning of the frames is determined by the following rules:

- The z_i is the axis of rotation for a revolute joint.
- The x_i must be perpendicular to both the current z_i and the previous z_{i-1}.
- The y_i is determined from the x_i and z_i by using the right-hand coordinate rule.
- The x_i must intersect z_{i-1}



5.2. **2.** Using the xacro file we can determine the corrected DH parameters, we can extract the position of the new axis. From the xacro file we can extract the following useful information about how one joint relates to the next:

- Base to joint0 = [-0.024 , 0 , 0.03]
- Joint0 to Joint1 = [0.024 , 0 , 0.096]
- Joint1 to Joint2 = [0.033 , 0 , 0.019]
- joint2 to joint3 = [0 , 0 , 0.155]
- joint3 to joint4 = [0 , 0 , 0.135]
- joint4 to joint5 = [-0.002 , 0 , 0.130]
- joint5 to end effector = [0 ,0 ,0.055]

The parameters for α and θ will be the same as for part 5.1