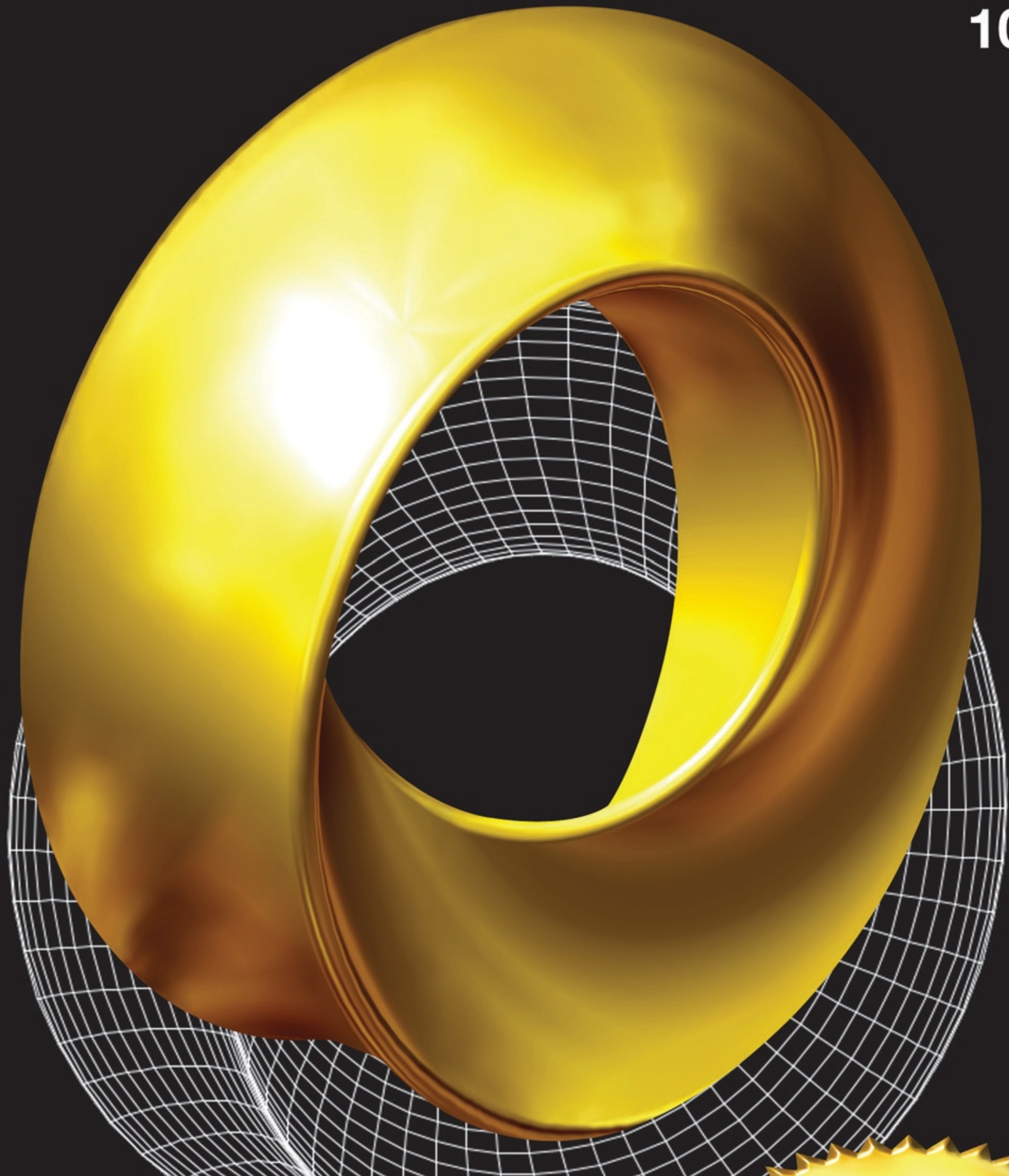
AP* EDITION

CALCULUS OF A SINGLE VARIABLE

10e

$$x = (\sin u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right] \quad y = (\cos u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right] \quad z = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right)$$



Ron Larson
Bruce Edwards

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1 Limits and Their Properties

Inventory Management *(Exercise 110, p. 81)*

Average Speed *(Exercise 62, p. 89)*

Free-Falling Object *(Exercises 101 and 102, p. 69)*

Sports *(Exercise 62, p. 57)*

Bicyclist *(Exercise 3, p. 47)*

Clockwise from top left, Christian Delbert/Shutterstock.com; WendellandCarolyn/iStockphoto.com;
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1.1 A Preview of Calculus

- **Understand what calculus is and how it compares with precalculus.**
- **Understand that the tangent line problem is basic to calculus.**
- **Understand that the area problem is also basic to calculus.**

REMARK As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as, “The area of the region is 4.6 square meters.”
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

What Is Calculus?

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process. So, one way to answer the question “What is calculus?” is to say that calculus is a “limit machine” that involves three stages. The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle. The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

Precalculus mathematics → Limit process → Calculus

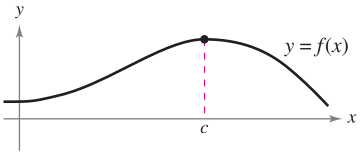
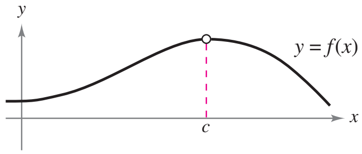
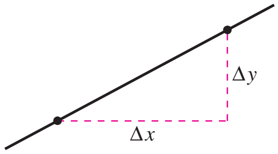
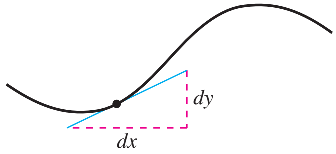
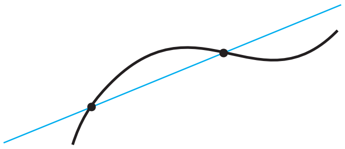
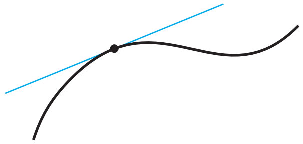
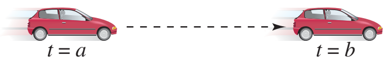
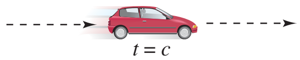
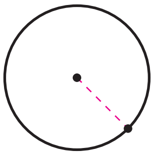
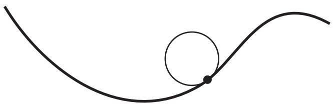
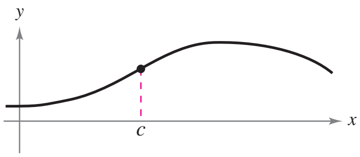
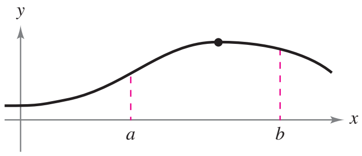
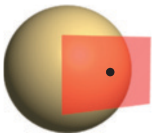
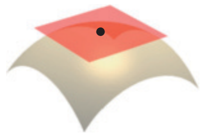
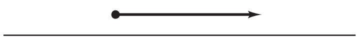
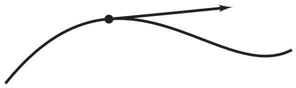
Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

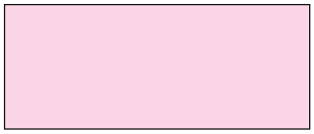
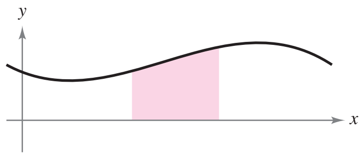
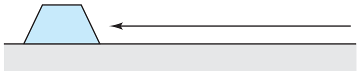
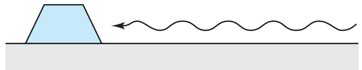
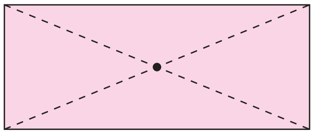
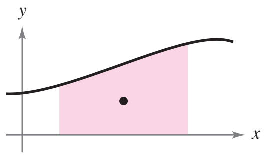
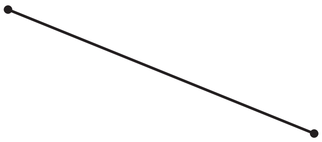
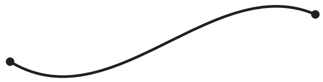
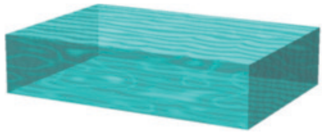
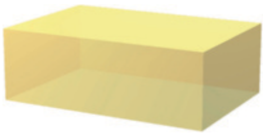
On the next two pages are listed some familiar precalculus concepts coupled with their calculus counterparts. Throughout the text, your goal should be to learn how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don’t worry if you are unfamiliar with some of the “old formulas” listed on the next two pages—you will be reviewing all of them.

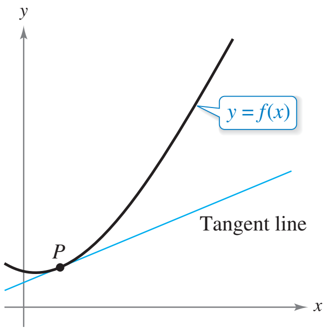
As you proceed through this text, come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For instance, note how these chapters relate to the three stages.

Chapter P: Preparation for Calculus	Precalculus
Chapter 1: Limits and Their Properties	Limit process
Chapter 2: Differentiation	Calculus

This cycle is repeated many times on a smaller scale throughout the text.

Without Calculus		**With Differential Calculus**	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	

Without Calculus		**With Integral Calculus**	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$



The tangent line to the graph of f at P
Figure 1.1

The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

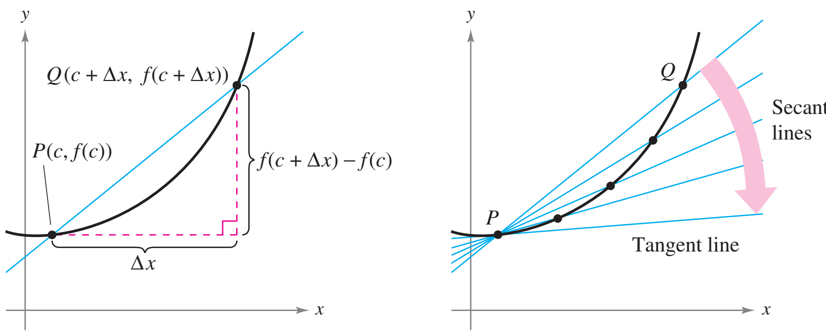
In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 1.1.

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point P is equivalent to finding the *slope* of the tangent line at P. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, then the slope of the secant line through these two points can be found using precalculus and is

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slopes of the secant lines. (Much more will be said about this important calculus concept in Chapter 2.)

GRACE CHISHOLM YOUNG (1868–1944)

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

Exploration

The following points lie on the graph of $f(x) = x^2$.

$$Q_1(1.5, f(1.5)),\quad Q_2(1.1, f(1.1)),\quad Q_3(1.01, f(1.01)),$$
$$Q_4(1.001, f(1.001)),\quad Q_5(1.0001, f(1.0001))$$

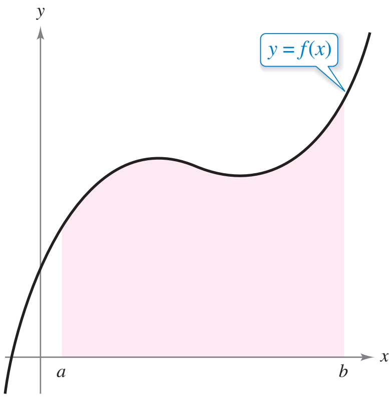
Each successive point gets closer to the point $P(1, 1)$. Find the slopes of the secant lines through Q_1 and P, Q_2 and P, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of f at the point P.

Girton College

The Area Problem

In the tangent line problem, you saw how the limit process can be applied to the slope of a line to find the slope of a general curve. A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process. In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

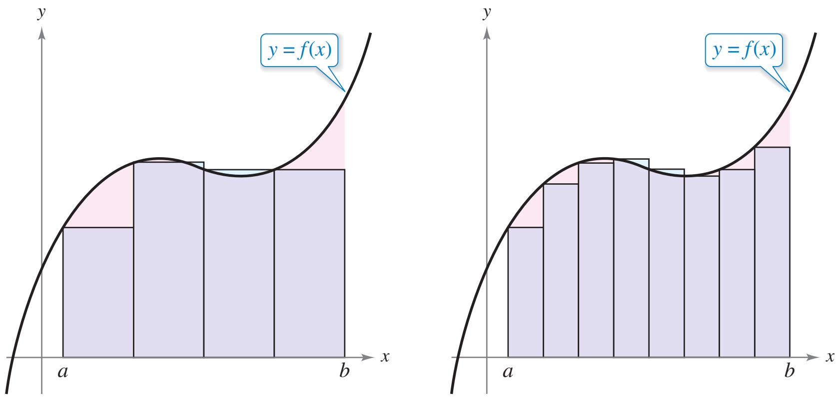
As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1.3. You can approximate the area of the region with several rectangular regions, as shown in Figure 1.4. As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases. Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.



Area under a curve
Figure 1.3

HISTORICAL NOTE

In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.



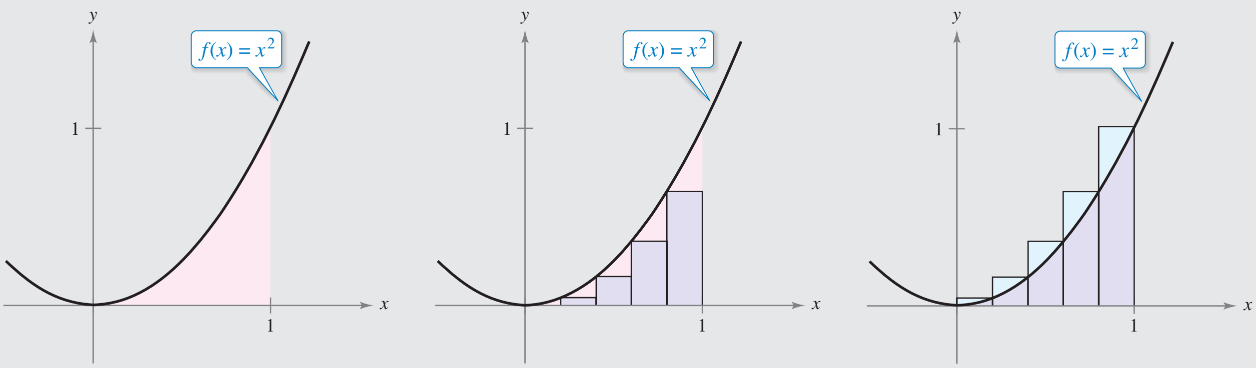
Approximation using four rectangles   Approximation using eight rectangles
Figure 1.4

Exploration

Consider the region bounded by the graphs of

$$f(x) = x^2, \quad y = 0, \quad \text{and} \quad x = 1$$

as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region   **(b)** Inscribed rectangles   **(c)** Circumscribed rectangles

1.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

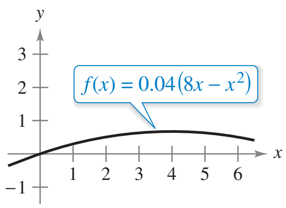
Precalculus or Calculus **In Exercises 1–5, decide whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.**

1. Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

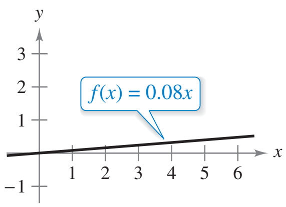
2. Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7\cos t$ feet per second.

3. Rate of Change

A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles (see figure). Find the rate of change of elevation at $x = 2$.

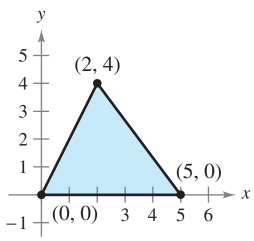


4. A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles (see figure). Find the rate of change of elevation at $x = 2$.

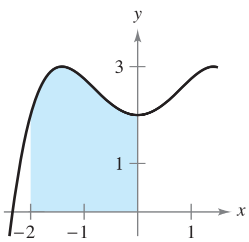


5. Find the area of the shaded region.

(a)



(b)



6. Secant Lines Consider the function

$$f(x) = \sqrt{x}$$

and the point $P(4, 2)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for x-values of 1, 3, and 5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(4, 2)$. Describe how to improve your approximation of the slope.

7. Secant Lines Consider the function $f(x) = 6x - x^2$ and the point $P(2, 8)$ on the graph of f.

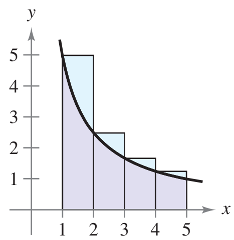
(a) Graph f and the secant lines passing through $P(2, 8)$ and $Q(x, f(x))$ for x-values of 3, 2.5, and 1.5.

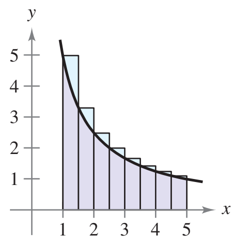
(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(2, 8)$. Describe how to improve your approximation of the slope.

8. HOW DO YOU SEE IT? How would you describe the instantaneous rate of change of an automobile's position on a highway?

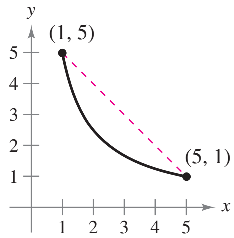
9. Approximating Area Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$. Describe how you could continue this process to obtain a more accurate approximation of the area.

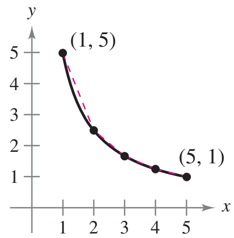




WRITING ABOUT CONCEPTS

10. Approximating the Length of a Curve Consider the length of the graph of $f(x) = 5/x$ from (1, 5) to (5, 1).





(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

1.2 Finding Limits Graphically and Numerically

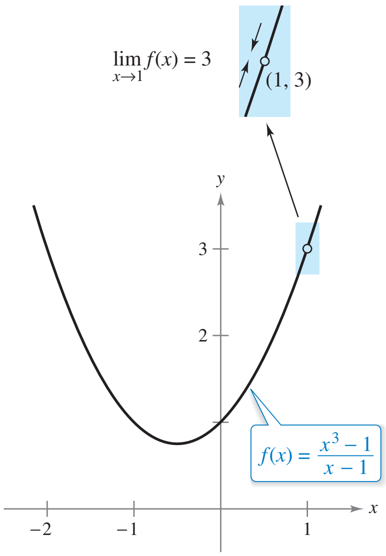
- **Estimate a limit using a numerical or graphical approach.**
- **Learn different ways that a limit can fail to exist.**
- **Study and use a formal definition of limit.**

An Introduction to Limits

To sketch the graph of the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

for values other than $x = 1$, you can use standard curve-sketching techniques. At $x = 1$, however, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.

The graph of f is a parabola that has a gap at the point $(1, 3)$, as shown in Figure 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x\to 1} f(x) = 3.$$ This is read as "the limit of $f(x)$ as x approaches 1 is 3."

AP* Tips

Limits are tested primarily on the multiple choice section of the AP Exam.

This discussion leads to an informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the **limit** of $f(x)$, as x approaches c, is L. This limit is written as

$$\lim_{x\to c} f(x) = L.$$

Exploration

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$$

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several x-values near 0 and use the results to estimate the limit

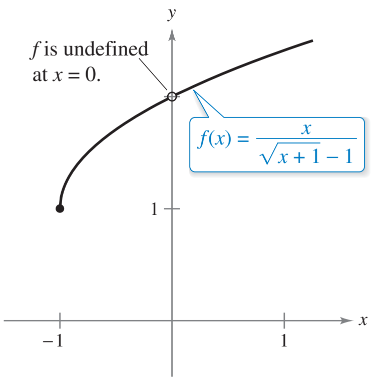
$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. → ← x approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.



The limit of $f(x)$ as x approaches 0 is 2.
Figure 1.6

From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6). ■

In Example 1, note that the function is undefined at $x = 0$, and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of* $f(x)$ *at* $x = c$ *has no bearing on the existence of the limit of* $f(x)$ *as* x *approaches* c.

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

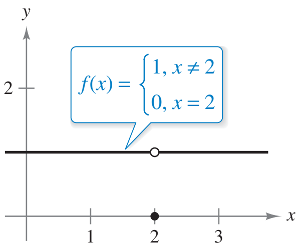
Solution Because $f(x) = 1$ for all x other than $x = 2$, you can estimate that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x \to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, as x approaches 2, the function

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

has the same limit as f. ■



The limit of $f(x)$ as x approaches 2 is 1.
Figure 1.7

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

1. **Numerical approach** Construct a table of values.
2. **Graphical approach** Draw a graph by hand or using technology.
3. **Analytic approach** Use algebra or calculus.

Limits That Fail to Exist

In the next three examples, you will examine some limits that fail to exist.

EXAMPLE 3 Different Right and Left Behavior

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution Consider the graph of the function

$$f(x) = \frac{|x|}{x}.$$

In Figure 1.8 and from the definition of absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$ Definition of absolute value

you can see that

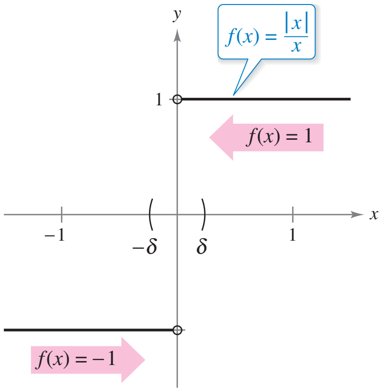
$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

So, no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$. Specifically, if δ (the lowercase Greek letter *delta*) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as

$$(-\delta, 0) \qquad \text{or} \qquad (0, \delta).$$

Negative x-values yield $|x|/x = -1$.

Positive x-values yield $|x|/x = 1$.



$\lim_{x \to 0} f(x)$ does not exist.
Figure 1.8

Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x \to 0} (|x|/x)$ does not exist.

AP* Tips

The AP Exam, especially the free response section, stresses the major applications of calculus, rather than the foundational limit concept.

EXAMPLE 4 Unbounded Behavior

Discuss the existence of the limit $\lim_{x \to 0} \frac{1}{x^2}$.

Solution Consider the graph of the function

$$f(x) = \frac{1}{x^2}.$$

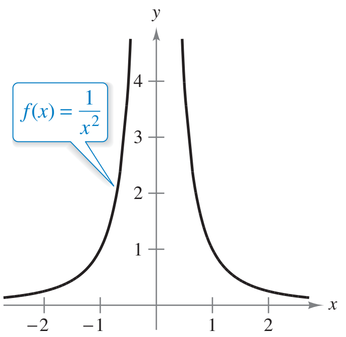
In Figure 1.9, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be greater than 100 when you choose x within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be greater than 1,000,000, as shown.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ does not become arbitrarily close to a single number L as x approaches 0, you can conclude that the limit does not exist.



$\lim_{x \to 0} f(x)$ does not exist.
Figure 1.9

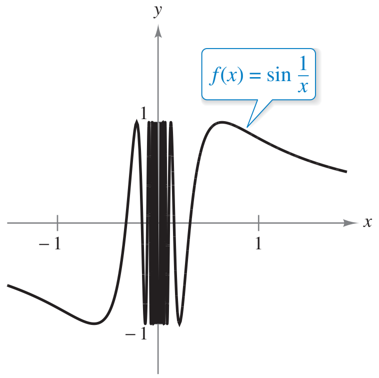
EXAMPLE 5 Oscillating Behavior

See LarsonCalculus.com for an interactive version of this type of example.

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.



$\lim_{x \to 0} f(x)$ does not exist.
Figure 1.10

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}.$$

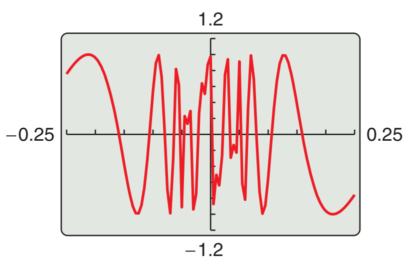
Because this function has *no limit* at any real number c, it is *not continuous* at any real number c. You will study continuity more closely in Section 1.4.

PETER GUSTAV DIRICHLET (1805–1859)

In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of function is attributed to the German mathematician Peter Gustav Dirichlet.

See LarsonCalculus.com to read more of this biography.

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. When you use a graphing utility to graph the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$
Figure 1.11

FOR FURTHER INFORMATION For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

A Formal Definition of Limit

Consider again the informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x\to c} f(x) = L.$$

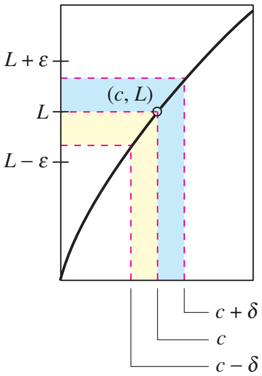
At first glance, this definition looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"$f(x)$ becomes arbitrarily close to L"

and

"x approaches c."

The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His **ε-δ definition of limit** is the standard used today.



The ε-δ definition of the limit of $f(x)$ as x approaches c
Figure 1.12

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

The first inequality

$0 < |x - c|$ The distance between x and c is more than 0.

expresses the fact that $x \neq c$. The second inequality

$|x - c| < \delta$ x is within δ units of c.

says that x is within a distance δ of c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x\to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$

REMARK Throughout this text, the expression

$$\lim_{x\to c} f(x) = L$$

implies two statements—the limit exists *and* the limit is L.

Some functions do not have limits as x approaches c, but those that do cannot have two different limits as x approaches c. That is, *if the limit of a function exists, then the limit is unique* (see Exercise 75).

The next three examples should help you develop a better understanding of the ε-δ definition of limit.

EXAMPLE 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that

$$|(2x - 5) - 1| < 0.01$$

whenever

$$0 < |x - 3| < \delta.$$

REMARK In Example 6, note that 0.005 is the *largest* value of δ that will guarantee

$$|(2x - 5) - 1| < 0.01$$

whenever

$$0 < |x - 3| < \delta.$$

Any *smaller* positive value of δ would also work.

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, try to establish a connection between the absolute values

$$|(2x - 5) - 1| \quad \text{and} \quad |x - 3|.$$

Notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose

$$\delta = \tfrac{1}{2}(0.01) = 0.005.$$

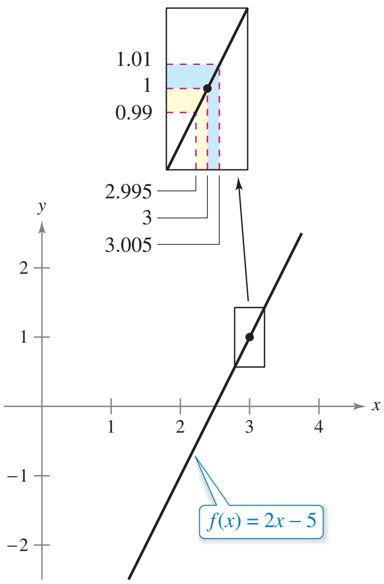
This choice works because

$$0 < |x - 3| < 0.005$$

implies that

$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01.$$

As you can see in Figure 1.13, for x-values within 0.005 of 3 $(x \neq 3)$, the values of $f(x)$ are within 0.01 of 1.



The limit of $f(x)$ as x approaches 3 is 1.

Figure 1.13

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for *any* ε, as shown in the next example.

EXAMPLE 7 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} (3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|(3x - 2) - 4| < \varepsilon$$

whenever

$$0 < |x - 2| < \delta.$$

Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

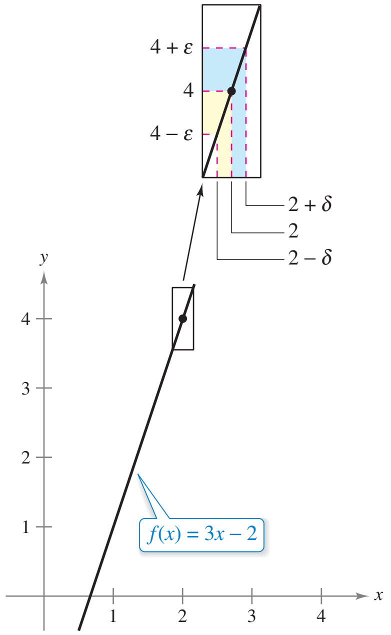
So, for a given $\varepsilon > 0$, you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

As you can see in Figure 1.14, for x-values within δ of 2 $(x \neq 2)$, the values of $f(x)$ are within ε of 4.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.14

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon$$

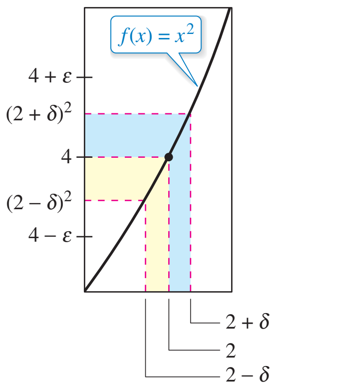
whenever

$$0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, $x + 2 < 5$ and thus $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon.$$

As you can see in Figure 1.15, for x-values within δ of 2 $(x \neq 2)$, the values of $f(x)$ are within ε of 4.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Throughout this chapter, you will use the ε-δ definition of limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of limit.

1.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating a Limit Numerically **In Exercises 1–6, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

1. $\displaystyle\lim_{x\to 4} \frac{x-4}{x^2-3x-4}$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

2. $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2-9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\displaystyle\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

5. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

6. $\displaystyle\lim_{x\to 0} \frac{\cos x-1}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

Estimating a Limit Numerically **In Exercises 7–14, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

7. $\displaystyle\lim_{x\to 1} \frac{x-2}{x^2+x-6}$

8. $\displaystyle\lim_{x\to -4} \frac{x+4}{x^2+9x+20}$

9. $\displaystyle\lim_{x\to 1} \frac{x^4-1}{x^6-1}$

10. $\displaystyle\lim_{x\to -3} \frac{x^3+27}{x+3}$

11. $\displaystyle\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6}$

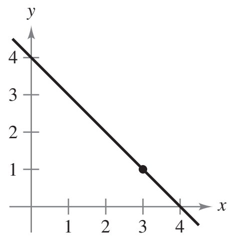
12. $\displaystyle\lim_{x\to 2} \frac{[x/(x+1)]-(2/3)}{x-2}$

13. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x}$

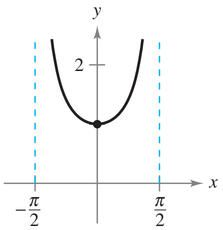
14. $\displaystyle\lim_{x\to 0} \frac{\tan x}{\tan 2x}$

Finding a Limit Graphically **In Exercises 15–22, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

15. $\displaystyle\lim_{x\to 3} (4-x)$

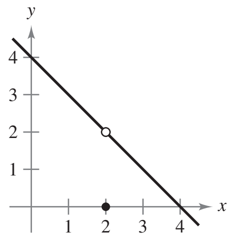


16. $\displaystyle\lim_{x\to 0} \sec x$



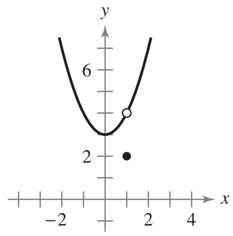
17. $\displaystyle\lim_{x\to 2} f(x)$

$$f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

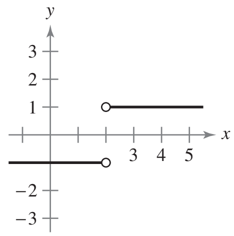


18. $\displaystyle\lim_{x\to 1} f(x)$

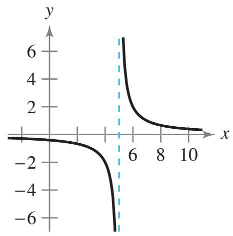
$$f(x) = \begin{cases} x^2+3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



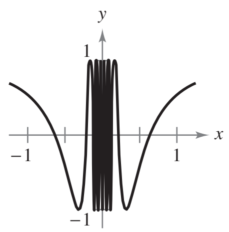
19. $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x-2}$



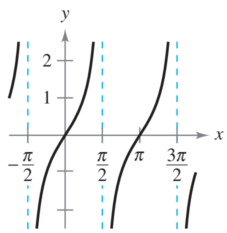
20. $\displaystyle\lim_{x\to 5} \frac{2}{x-5}$



21. $\displaystyle\lim_{x\to 0} \cos \frac{1}{x}$



22. $\displaystyle\lim_{x\to \pi/2} \tan x$



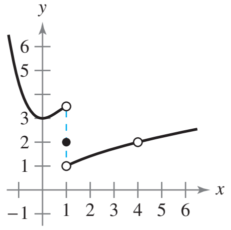
Graphical Reasoning **In Exercises 23 and 24, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.**

23. (a) $f(1)$

(b) $\lim_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to 4} f(x)$



24. (a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(0)$

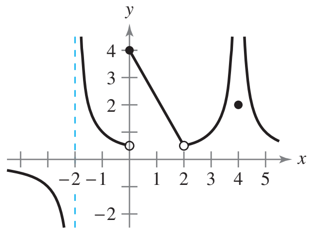
(d) $\lim_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to 2} f(x)$

(g) $f(4)$

(h) $\lim_{x\to 4} f(x)$



Limits of a Piecewise Function **In Exercises 25 and 26, sketch the graph of f. Then identify the values of c for which $\lim_{x\to c} f(x)$ exists.**

25. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$

26. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

Sketching a Graph **In Exercises 27 and 28, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)**

27. $f(0)$ is undefined.

$\lim_{x\to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3$

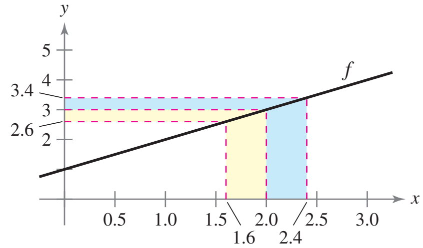
28. $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist.

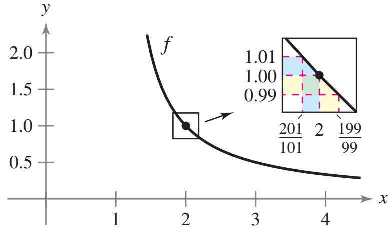
29. Finding a δ for a Given ε The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.4$.



30. Finding a δ for a Given ε The graph of

$$f(x) = \frac{1}{x - 1}$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 1| < 0.01$.



31. Finding a δ for a Given ε The graph of

$$f(x) = 2 - \frac{1}{x}$$

is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$, then $|f(x) - 1| < 0.1$.

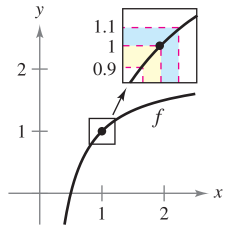


Figure for 31

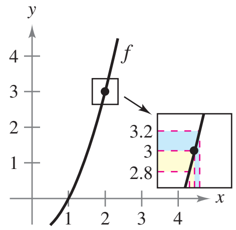


Figure for 32

32. Finding a δ for a Given ε The graph of

$$f(x) = x^2 - 1$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.2$.

Finding a δ for a Given ε **In Exercises 33–36, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.**

33. $\lim_{x\to 2} (3x + 2)$

34. $\lim_{x\to 6} \left(6 - \frac{x}{3}\right)$

35. $\lim_{x\to 2} (x^2 - 3)$

36. $\lim_{x\to 4} (x^2 + 6)$

Using the ε-δ Definition of Limit **In Exercises 37–48, find the limit L. Then use the ε-δ definition to prove that the limit is L.**

37. $\lim_{x\to 4} (x + 2)$

38. $\lim_{x\to -2} (4x + 5)$

39. $\lim_{x\to -4} \left(\frac{1}{2}x - 1\right)$

40. $\lim_{x\to 3} \left(\frac{3}{4}x + 1\right)$

41. $\lim_{x\to 6} 3$

42. $\lim_{x\to 2} (-1)$

43. $\lim_{x\to 0} \sqrt[3]{x}$

44. $\lim_{x\to 4} \sqrt{x}$

45. $\lim_{x\to -5} |x - 5|$

46. $\lim_{x\to 3} |x - 3|$

47. $\lim_{x\to 1} (x^2 + 1)$

48. $\lim_{x\to -4} (x^2 + 4x)$

49. Finding a Limit What is the limit of $f(x) = 4$ as x approaches π?

50. Finding a Limit What is the limit of $g(x) = x$ as x approaches π?

Writing **In Exercises 51–54, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

51. $f(x) = \dfrac{\sqrt{x+5}-3}{x-4}$

$\lim\limits_{x\to 4} f(x)$

52. $f(x) = \dfrac{x-3}{x^2-4x+3}$

$\lim\limits_{x\to 3} f(x)$

53. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

$\lim\limits_{x\to 9} f(x)$

54. $f(x) = \dfrac{x-3}{x^2-9}$

$\lim\limits_{x\to 3} f(x)$

55. Modeling Data For a long distance phone call, a hotel charges \$9.99 for the first minute and \$0.79 for each additional minute or fraction thereof. A formula for the cost is given by

$$C(t) = 9.99 - 0.79[\![-(t-1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 6$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find $\lim\limits_{t\to 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

56. Repeat Exercise 55 for

$$C(t) = 5.79 - 0.99[\![-(t-1)]\!].$$

WRITING ABOUT CONCEPTS

57. Describing Notation Write a brief description of the meaning of the notation

$$\lim_{x\to 8} f(x) = 25.$$

58. Using the Definition of Limit The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

59. Limits That Fail to Exist Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

60. Comparing Functions and Limits

(a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

(b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

61. Jewelry A jeweler resizes a ring so that its inner circumference is 6 centimeters.

(a) What is the radius of the ring?

(b) The inner circumference of the ring varies between 5.5 centimeters and 6.5 centimeters. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

62. Sports

A sporting goods manufacturer designs a golf ball having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) The volume of the golf ball varies between 2.45 cubic inches and 2.51 cubic inches. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

63. Estimating a Limit Consider the function

$$f(x) = (1+x)^{1/x}.$$

Estimate

$$\lim_{x\to 0} (1+x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

64. Estimating a Limit Consider the function

$$f(x) = \frac{|x+1| - |x-1|}{x}.$$

Estimate

$$\lim_{x\to 0} \frac{|x+1| - |x-1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

65. Graphical Analysis The statement

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x-2| < \delta$, then

$$\left|\frac{x^2-4}{x-2} - 4\right| < \varepsilon.$$

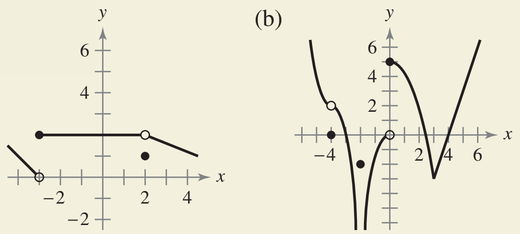
If $\varepsilon = 0.001$, then

$$\left|\frac{x^2-4}{x-2} - 4\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2-\delta, 2+\delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

66. HOW DO YOU SEE IT? Use the graph of f to identify the values of c for which $\lim_{x\to c} f(x)$ exists.

(a) (b)



True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

68. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

69. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

70. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

Determining a Limit **In Exercises 71 and 72, consider the function $f(x) = \sqrt{x}$.**

71. Is $\lim_{x\to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

72. Is $\lim_{x\to 0} \sqrt{x} = 0$ a true statement? Explain.

73. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x\to 0} \frac{\sin nx}{x}$$

for several values of n. What do you notice?

74. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x\to 0} \frac{\tan nx}{x}$$

for several values of n. What do you notice?

75. Proof Prove that if the limit of $f(x)$ as x approaches c exists, then the limit must be unique. [*Hint:* Let $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$ and prove that $L_1 = L_2$.]

76. Proof Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x\to c} f(x) = mc + b$.

77. Proof Prove that

$$\lim_{x\to c} f(x) = L$$

is equivalent to

$$\lim_{x\to c} [f(x) - L] = 0.$$

78. Proof

(a) Given that

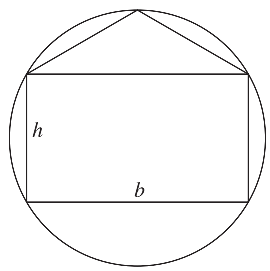
$$\lim_{x\to 0} (3x+1)(3x-1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x+1)(3x-1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x\to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

PUTNAM EXAM CHALLENGE

79. Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in a region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?



80. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

These problems were composed by the Committee on the Putnam Prize Competition.

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x\to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c*.** You will examine this concept more closely in Section 1.4.

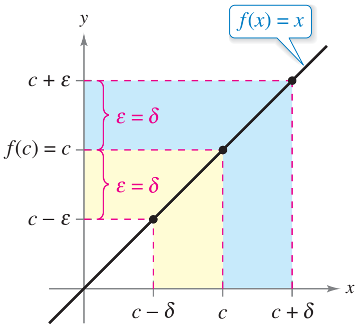


Figure 1.16

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$

Proof The proofs of Properties 1 and 3 of Theorem 1.1 are left as exercises (see Exercises 107 and 108). To prove Property 2, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. To do this, choose $\delta = \varepsilon$. The second inequality then implies the first, as shown in Figure 1.16.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

REMARK When encountering new notations or symbols in mathematics, be sure you know how the notations are read. For instance, the limit in Example 1(c) is read as "the limit of x^2 as x approaches 2 is 4."

EXAMPLE 1 Evaluating Basic Limits

a. $\lim_{x\to 2} 3 = 3$ **b.** $\lim_{x\to -4} x = -4$ **c.** $\lim_{x\to 2} x^2 = 2^2 = 4$

REMARK The proof of Property 1 is left as an exercise (see Exercise 109).

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K.$$

1. Scalar multiple: $\lim_{x\to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 2 The Limit of a Polynomial

Find the limit: $\lim_{x\to 2} (4x^2 + 3)$.

Solution

$$\lim_{x\to 2} (4x^2 + 3) = \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 \qquad \text{Property 2, Theorem 1.2}$$
$$= 4\left(\lim_{x\to 2} x^2\right) + \lim_{x\to 2} 3 \qquad \text{Property 1, Theorem 1.2}$$
$$= 4(2^2) + 3 \qquad \text{Properties 1 and 3, Theorem 1.1}$$
$$= 19 \qquad \text{Simplify.}$$

In Example 2, note that the limit (as x approaches 2) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

AP* Tips

Algebraic limit evaluation methods are not explicitly tested on the free response section of the AP Exam. They may be helpful on some multiple choice questions.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\lim_{x\to 1} \dfrac{x^2 + x + 2}{x + 1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1} \frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2.$$

Polynomial functions and rational functions are two of the three basic types of algebraic functions. The next theorem deals with the limit of the third type of algebraic function—one that involves a radical.

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase *r*, to stand for the Latin word *radix*, meaning root.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The next theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 The Limit of a Composite Function

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit.

a. $\lim_{x \to 0} \sqrt{x^2 + 4}$ **b.** $\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$

Solution

a. Because

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

you can conclude that

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

you can conclude that

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$ **2.** $\lim_{x \to c} \cos x = \cos c$ **3.** $\lim_{x \to c} \tan x = \tan c$

4. $\lim_{x \to c} \cot x = \cot c$ **5.** $\lim_{x \to c} \sec x = \sec c$ **6.** $\lim_{x \to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

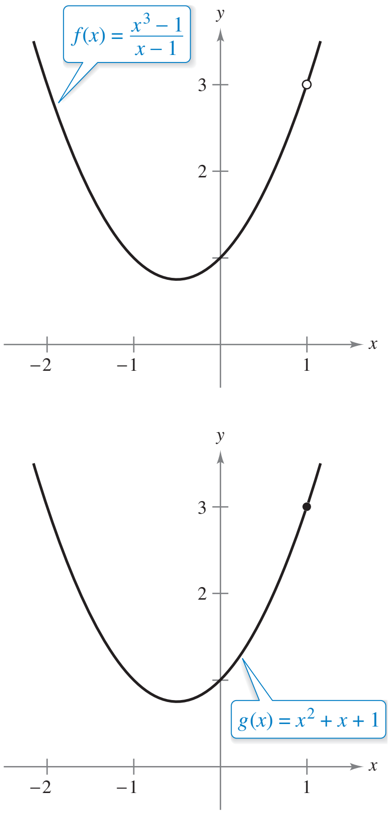
b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the next theorem, can be used to develop a strategy for finding limits.

> **THEOREM 1.7 Functions That Agree at All but One Point**
>
> Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$
>
> A proof of this theorem is given in Appendix A.
>
> *See LarsonCalculus.com for Bruce Edwards's video of this proof.*



f and *g* agree at all but one point.
Figure 1.17

EXAMPLE 6 Finding the Limit of a Function

Find the limit.

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$$

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 1.17. Because $\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} && \text{Factor.} \\
&= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}} && \text{Divide out like factors.} \\
&= \lim_{x\to 1} (x^2 + x + 1) && \text{Apply Theorem 1.7.} \\
&= 1^2 + 1 + 1 && \text{Use direct substitution.} \\
&= 3 && \text{Simplify.}
\end{aligned}$$

REMARK When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the limit below does not exist.

$$\lim_{x\to 1} \frac{x^3 + 1}{x - 1}$$

> **A Strategy for Finding Limits**
>
> 1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
> 2. When the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.] Then apply Theorem 1.7 to conclude *analytically* that
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$
> 3. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out Technique

One procedure for finding a limit analytically is the **dividing out technique.** This technique involves dividing out common factors, as shown in Example 7.

EXAMPLE 7 Dividing Out Technique

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit: $\lim\limits_{x\to -3} \dfrac{x^2 + x - 6}{x + 3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} \qquad \lim_{x\to -3}(x^2 + x - 6) = 0, \quad \lim_{x\to -3}(x + 3) = 0$$

Direct substitution fails.

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x + 3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{\cancel{(x + 3)}(x - 2)}{\cancel{x + 3}} = x - 2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x\to -3}(x - 2) \qquad \text{Apply Theorem 1.7.}$$

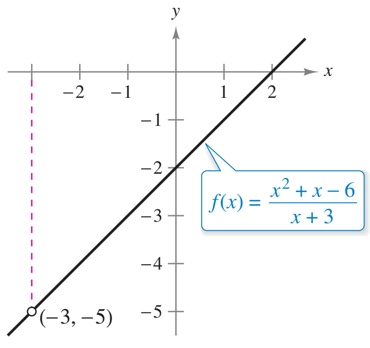
$$= -5. \qquad \text{Use direct substitution.}$$

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$.

REMARK In the solution to Example 7, be sure you see the usefulness of the Factor Theorem of Algebra. This theorem states that if c is a zero of a polynomial function, then $(x - c)$ is a factor of the polynomial. So, when you apply direct substitution to a rational function and obtain

$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0}$$

you can conclude that $(x - c)$ must be a common factor of both $p(x)$ and $q(x)$.



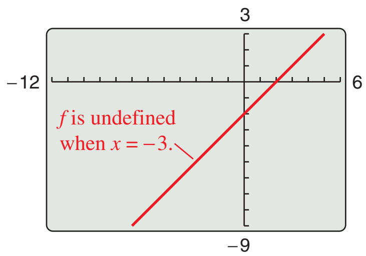
f is undefined when $x = -3$.
Figure 1.18

In Example 7, direct substitution produced the meaningless fractional form $0/0$. An expression such as $0/0$ is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*. Another way is to use the *rationalizing technique* shown on the next page.

TECHNOLOGY PITFALL A graphing utility can give misleading information about the graph of a function. For instance, try graphing the function from Example 7

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

on a standard viewing window (see Figure 1.19). On most graphing utilities, the graph appears to be defined at every real number. However, because f is undefined when $x = -3$, you know that the graph of f has a hole at $x = -3$. You can verify this on a graphing utility using the *trace* or *table* feature.



Misleading graph of f
Figure 1.19

Rationalizing Technique

Another way to find a limit analytically is the **rationalizing technique,** which involves rationalizing the numerator of a fractional expression. Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, to rationalize the numerator of

$$\frac{\sqrt{x}+4}{x}$$

multiply the numerator and denominator by the conjugate of $\sqrt{x}+4$, which is

$$\sqrt{x}-4.$$

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \quad \begin{cases} \lim_{x\to 0} \left(\sqrt{x+1}-1\right) = 0 \\ \lim_{x\to 0} x = 0 \end{cases}$$

Direct substitution fails.

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\ &= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} \\ &= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} \\ &= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0 \end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\begin{aligned} \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.20.)

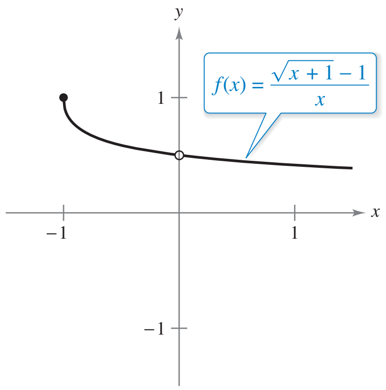
x approaches 0 from the left. → ← *x* approaches 0 from the right.

x	-0.25	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

REMARK The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 8, the convenient form is

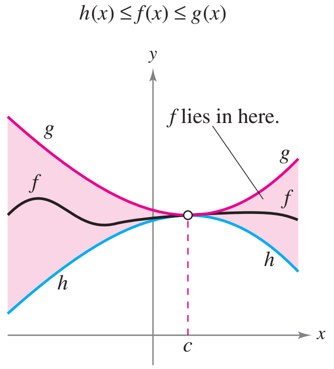
$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21.



The Squeeze Theorem
Figure 1.21

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

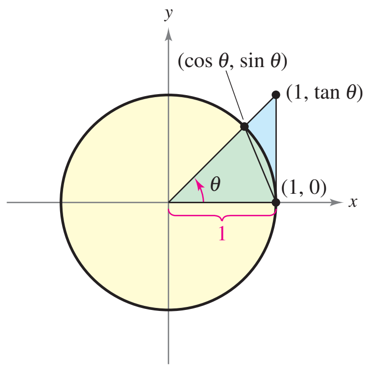
A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

You can see the usefulness of the Squeeze Theorem (also called the Sandwich Theorem or the Pinching Theorem) in the proof of Theorem 1.9.

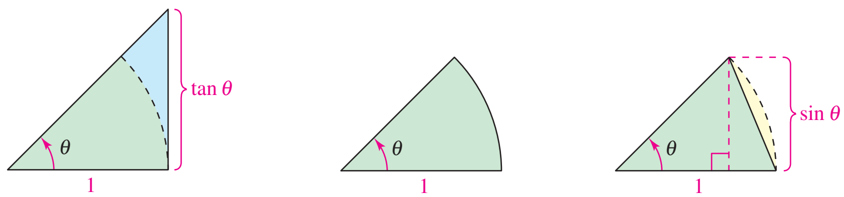
THEOREM 1.9 Two Special Trigonometric Limits

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$



A circular sector is used to prove Theorem 1.9.
Figure 1.22

Proof The proof of the second limit is left as an exercise (see Exercise 121). To avoid the confusion of two different uses of x, the proof of the first limit is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



Area of triangle $\ge$ Area of sector $\ge$ Area of triangle

$$\frac{\tan\theta}{2} \ge \frac{\theta}{2} \ge \frac{\sin\theta}{2}$$

Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta\to 0} \cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$. *See LarsonCalculus.com for Bruce Edwards's video of this proof.* ■

EXAMPLE 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

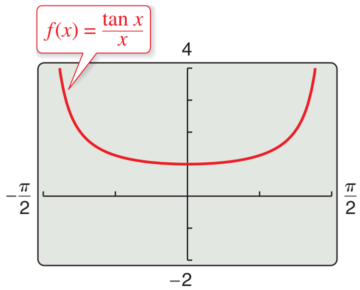
and

$$\lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

REMARK Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10, $\sin 4x$ means $\sin(4x)$.

EXAMPLE 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim\limits_{x\to 0} \dfrac{\sin 4x}{x}$.

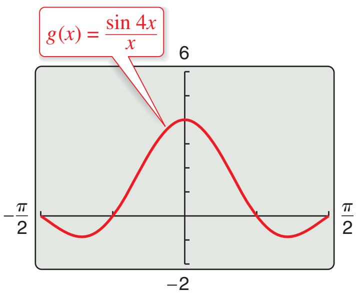
Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right). \quad \text{Multiply and divide by 4.}$$

Now, by letting $y = 4x$ and observing that x approaches 0 if and only if y approaches 0, you can write

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) && \text{Let } y = 4x. \\ &= 4(1) && \text{Apply Theorem 1.9(1).} \\ &= 4. \end{aligned}$$

(See Figure 1.24.)



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

TECHNOLOGY Use a graphing utility to confirm the limits in the examples and in the exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \quad \text{and} \quad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second graph appears to contain the point (0, 4), which lends support to the conclusions obtained in Examples 9 and 10.

1.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Limits **In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.**

1. $h(x) = -x^2 + 4x$
   (a) $\lim_{x\to 4} h(x)$
   (b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
   (a) $\lim_{x\to 4} g(x)$
   (b) $\lim_{x\to 9} g(x)$

3. $f(x) = x \cos x$
   (a) $\lim_{x\to 0} f(x)$
   (b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
   (a) $\lim_{t\to 4} f(t)$
   (b) $\lim_{t\to -1} f(t)$

Finding a Limit **In Exercises 5–22, find the limit.**

5. $\lim_{x\to 2} x^3$
6. $\lim_{x\to -3} x^4$
7. $\lim_{x\to 0} (2x - 1)$
8. $\lim_{x\to -4} (2x + 3)$
9. $\lim_{x\to -3} (x^2 + 3x)$
10. $\lim_{x\to 2} (-x^3 + 1)$
11. $\lim_{x\to -3} (2x^2 + 4x + 1)$
12. $\lim_{x\to 1} (2x^3 - 6x + 5)$
13. $\lim_{x\to 3} \sqrt{x + 1}$
14. $\lim_{x\to 2} \sqrt[3]{12x + 3}$
15. $\lim_{x\to -4} (x + 3)^2$
16. $\lim_{x\to 0} (3x - 2)^4$
17. $\lim_{x\to 2} \dfrac{1}{x}$
18. $\lim_{x\to -5} \dfrac{5}{x + 3}$
19. $\lim_{x\to 1} \dfrac{x}{x^2 + 4}$
20. $\lim_{x\to 1} \dfrac{3x + 5}{x + 1}$
21. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x + 2}}$
22. $\lim_{x\to 3} \dfrac{\sqrt{x + 6}}{x + 2}$

Finding Limits **In Exercises 23–26, find the limits.**

23. $f(x) = 5 - x,\ g(x) = x^3$
    (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
    (a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
    (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
    (a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

Finding a Limit of a Trigonometric Function **In Exercises 27–36, find the limit of the trigonometric function.**

27. $\lim_{x\to \pi/2} \sin x$
28. $\lim_{x\to \pi} \tan x$
29. $\lim_{x\to 1} \cos \dfrac{\pi x}{3}$
30. $\lim_{x\to 2} \sin \dfrac{\pi x}{2}$
31. $\lim_{x\to 0} \sec 2x$
32. $\lim_{x\to \pi} \cos 3x$
33. $\lim_{x\to 5\pi/6} \sin x$
34. $\lim_{x\to 5\pi/3} \cos x$
35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

Evaluating Limits **In Exercises 37–40, use the information to evaluate the limits.**

37. $\lim_{x\to c} f(x) = 3$
    $\lim_{x\to c} g(x) = 2$
    (a) $\lim_{x\to c} [5g(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = 2$
    $\lim_{x\to c} g(x) = \frac{3}{4}$
    (a) $\lim_{x\to c} [4f(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
    (a) $\lim_{x\to c} [f(x)]^3$
    (b) $\lim_{x\to c} \sqrt{f(x)}$
    (c) $\lim_{x\to c} [3f(x)]$
    (d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
    (a) $\lim_{x\to c} \sqrt[3]{f(x)}$
    (b) $\lim_{x\to c} \dfrac{f(x)}{18}$
    (c) $\lim_{x\to c} [f(x)]^2$
    (d) $\lim_{x\to c} [f(x)]^{2/3}$

Finding a Limit **In Exercises 41–46, write a simpler function that agrees with the given function at all but one point. Then find the limit of the function. Use a graphing utility to confirm your result.**

41. $\lim_{x\to 0} \dfrac{x^2 + 3x}{x}$
42. $\lim_{x\to 0} \dfrac{x^4 - 5x^2}{x^2}$
43. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$
44. $\lim_{x\to -2} \dfrac{3x^2 + 5x - 2}{x + 2}$
45. $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$
46. $\lim_{x\to -1} \dfrac{x^3 + 1}{x + 1}$

Finding a Limit **In Exercises 47–62, find the limit.**

47. $\lim_{x\to 0} \dfrac{x}{x^2 - x}$
48. $\lim_{x\to 0} \dfrac{2x}{x^2 + 4x}$
49. $\lim_{x\to 4} \dfrac{x - 4}{x^2 - 16}$
50. $\lim_{x\to 5} \dfrac{5 - x}{x^2 - 25}$
51. $\lim_{x\to -3} \dfrac{x^2 + x - 6}{x^2 - 9}$
52. $\lim_{x\to 2} \dfrac{x^2 + 2x - 8}{x^2 - x - 2}$
53. $\lim_{x\to 4} \dfrac{\sqrt{x + 5} - 3}{x - 4}$
54. $\lim_{x\to 3} \dfrac{\sqrt{x + 1} - 2}{x - 3}$
55. $\lim_{x\to 0} \dfrac{\sqrt{x + 5} - \sqrt{5}}{x}$
56. $\lim_{x\to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$
57. $\lim_{x\to 0} \dfrac{[1/(3 + x)] - (1/3)}{x}$
58. $\lim_{x\to 0} \dfrac{[1/(x + 4)] - (1/4)}{x}$

59. $\lim_{\Delta x\to 0} \dfrac{2(x + \Delta x) - 2x}{\Delta x}$

60. $\lim_{\Delta x\to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$

61. $\lim_{\Delta x\to 0} \dfrac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}$

62. $\lim_{\Delta x\to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x}$

Finding a Limit of a Trigonometric Function **In Exercises 63–74, find the limit of the trigonometric function.**

63. $\lim_{x\to 0} \dfrac{\sin x}{5x}$

64. $\lim_{x\to 0} \dfrac{3(1 - \cos x)}{x}$

65. $\lim_{x\to 0} \dfrac{\sin x(1 - \cos x)}{x^2}$

66. $\lim_{\theta\to 0} \dfrac{\cos \theta \tan \theta}{\theta}$

67. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

68. $\lim_{x\to 0} \dfrac{\tan^2 x}{x}$

69. $\lim_{h\to 0} \dfrac{(1 - \cos h)^2}{h}$

70. $\lim_{\phi\to \pi} \phi \sec \phi$

71. $\lim_{x\to \pi/2} \dfrac{\cos x}{\cot x}$

72. $\lim_{x\to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x}$

73. $\lim_{t\to 0} \dfrac{\sin 3t}{2t}$

74. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x\to 0}\left(\dfrac{2 \sin 2x}{2x}\right)\left(\dfrac{3x}{3 \sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 75–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

75. $\lim_{x\to 0} \dfrac{\sqrt{x + 2} - \sqrt{2}}{x}$

76. $\lim_{x\to 16} \dfrac{4 - \sqrt{x}}{x - 16}$

77. $\lim_{x\to 0} \dfrac{[1/(2 + x)] - (1/2)}{x}$

78. $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2}$

79. $\lim_{t\to 0} \dfrac{\sin 3t}{t}$

80. $\lim_{x\to 0} \dfrac{\cos x - 1}{2x^2}$

81. $\lim_{x\to 0} \dfrac{\sin x^2}{x}$

82. $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}}$

Finding a Limit **In Exercises 83–88, find**

$$\lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

83. $f(x) = 3x - 2$

84. $f(x) = -6x + 3$

85. $f(x) = x^2 - 4x$

86. $f(x) = \sqrt{x}$

87. $f(x) = \dfrac{1}{x + 3}$

88. $f(x) = \dfrac{1}{x^2}$

Using the Squeeze Theorem **In Exercises 89 and 90, use the Squeeze Theorem to find $\lim_{x\to c} f(x)$.**

89. $c = 0$

$4 - x^2 \le f(x) \le 4 + x^2$

90. $c = a$

$b - |x - a| \le f(x) \le b + |x - a|$

Using the Squeeze Theorem **In Exercises 91–94, use a graphing utility to graph the given function and the equations $y = |x|$ and $y = -|x|$ in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x\to 0} f(x)$.**

91. $f(x) = |x| \sin x$

92. $f(x) = |x| \cos x$

93. $f(x) = x \sin \dfrac{1}{x}$

94. $h(x) = x \cos \dfrac{1}{x}$

WRITING ABOUT CONCEPTS

95. Functions That Agree at All but One Point

(a) In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

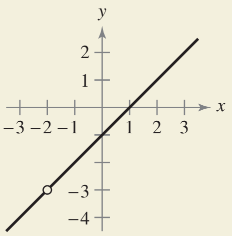
(b) Give an example of two functions that agree at all but one point.

96. Indeterminate Form What is meant by an indeterminate form?

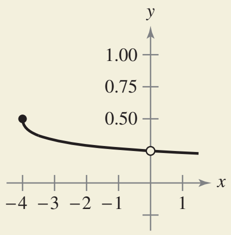
97. Squeeze Theorem In your own words, explain the Squeeze Theorem.

98. HOW DO YOU SEE IT? Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim_{x\to -2} \dfrac{x^2 + x - 2}{x + 2}$



(b) $\lim_{x\to 0} \dfrac{\sqrt{x + 4} - 2}{x}$



99. Writing Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x) = 1.$$

100. Writing Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin^2 x, \quad \text{and} \quad h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x) = 0.$$

Free-Falling Object

In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

101. A construction worker drops a full paint can from a height of 500 feet. How fast will the paint can be falling after 2 seconds?

102. A construction worker drops a full paint can from a height of 500 feet. When will the paint can hit the ground? At what velocity will the paint can impact the ground?

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Finding Functions Find two functions f and g such that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but

$$\lim_{x\to 0} [f(x) + g(x)]$$

does exist.

106. Proof Prove that if $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} [f(x) + g(x)]$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

107. Proof Prove Property 1 of Theorem 1.1.

108. Proof Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Proof Prove Property 1 of Theorem 1.2.

110. Proof Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

111. Proof Prove that if $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x\to c} f(x)g(x) = 0$.

112. Proof

(a) Prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\big||f(x)| - |L|\big| \le |f(x) - L|$.]

113. Think About It Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x\to c} |f(x)| = |L|$ but $\lim_{x\to c} f(x)$ does not exist.]

114. Think About It When using a graphing utility to generate a table to approximate

$$\lim_{x\to 0} \frac{\sin x}{x}$$

a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

True or False? **In Exercises 115–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

115. $\lim_{x\to 0} \frac{|x|}{x} = 1$

116. $\lim_{x\to \pi} \frac{\sin x}{x} = 1$

117. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} g(x) = L$.

118. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

119. $\lim_{x\to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

120. If $f(x) < g(x)$ for all $x \ne a$, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

121. Proof Prove the second part of Theorem 1.9.

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

122. Piecewise Functions Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}.$$

Find (if possible) $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.

123. Graphical Reasoning Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x\to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

124. Approximation

(a) Find $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

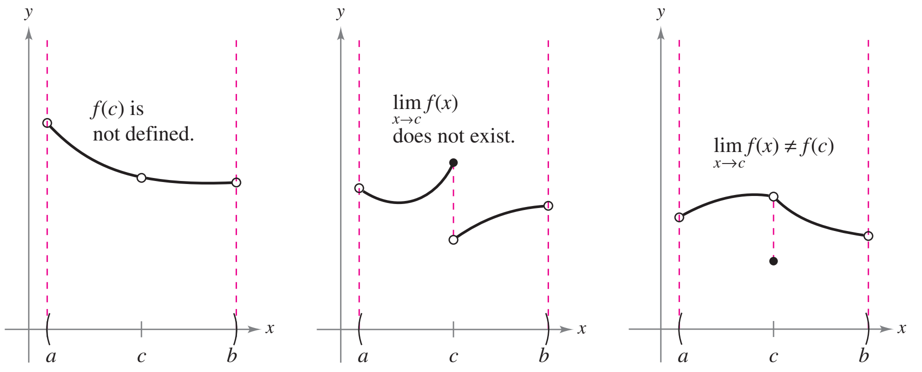
(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

1.4 Continuity and One-Sided Limits

- **Determine continuity at a point and continuity on an open interval.**
- **Determine one-sided limits and continuity on a closed interval.**
- **Use properties of continuity.**
- **Understand and use the Intermediate Value Theorem.**

Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c, and there are no holes, jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is *not* continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and **continuous.**



Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of three conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the three conditions is true, then the function f is called **continuous at c,** as indicated in the important definition below.

Definition of Continuity

Continuity at a Point
A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

Continuity on an Open Interval
A function is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous.**

AP* Tips

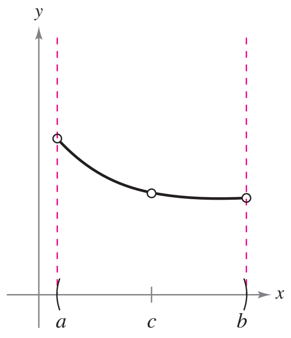
You must know the precise definition of continuity for the multiple choice section (rarely the free response section) of the AP Exam, and be able to understand which condition of the definition is not met if a function is not continuous at a particular point.

Exploration

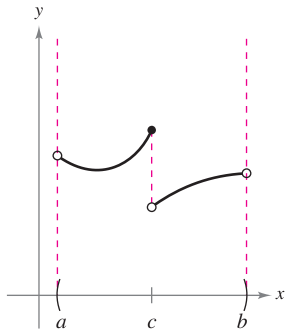
Informally, you might say that a function is *continuous* on an open interval when its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing utility to graph each function on the given interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

	Function	Interval
a.	$y = x^2 + 1$	$(-3, 3)$
b.	$y = \dfrac{1}{x - 2}$	$(-3, 3)$
c.	$y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d.	$y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$

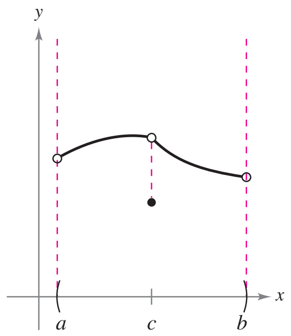
FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.



(a) Removable discontinuity



(b) Nonremovable discontinuity



(c) Removable discontinuity

Figure 1.26

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figures 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.

EXAMPLE 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

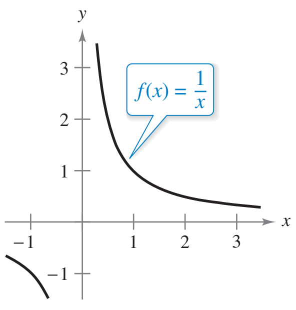
b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). By defining $g(1)$ as 2, the "redefined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because

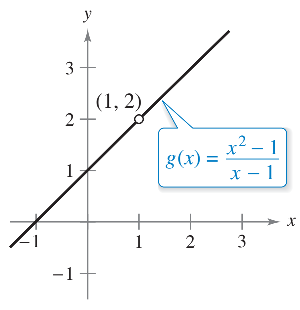
$$\lim_{x \to 0} h(x) = 1$$

h is continuous on the entire real number line, as shown in Figure 1.27(c).

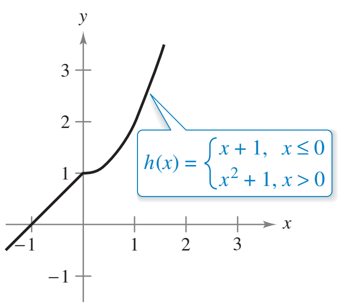
d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



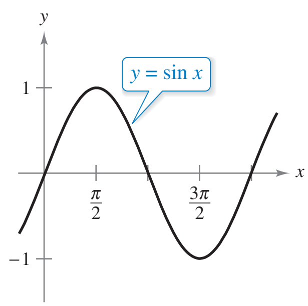
(a) Nonremovable discontinuity at $x = 0$



(b) Removable discontinuity at $x = 1$



(c) Continuous on entire real number line



(d) Continuous on entire real number line

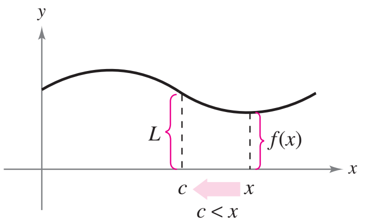
Figure 1.27

REMARK Some people may refer to the function in Example 1(a) as "discontinuous." We have found that this terminology can be confusing. Rather than saying that the function is discontinuous, we prefer to say that it has a discontinuity at $x = 0$.

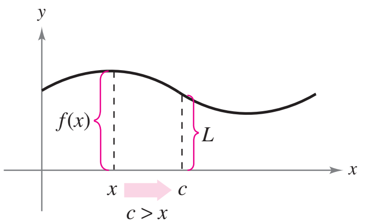
One-Sided Limits and Continuity on a Closed Interval

To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.** For instance, the **limit from the right** (or right-hand limit) means that x approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted as

$$\lim_{x \to c^+} f(x) = L. \qquad \text{Limit from the right}$$



(a) Limit as x approaches c from the right.



(b) Limit as x approaches c from the left.

Figure 1.28

Similarly, the **limit from the left** (or left-hand limit) means that x approaches c from values less than c [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x \to c^-} f(x) = L. \qquad \text{Limit from the left}$$

One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer, then

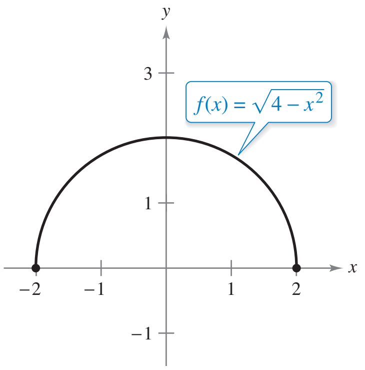
$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

EXAMPLE 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution As shown in Figure 1.29, the limit as x approaches -2 from the right is

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.

Figure 1.29

One-sided limits can be used to investigate the behavior of **step functions.** One common type of step function is the **greatest integer function** $[\![x]\!]$, defined as

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \leq x. \qquad \text{Greatest integer function}$$

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.

EXAMPLE 3 The Greatest Integer Function

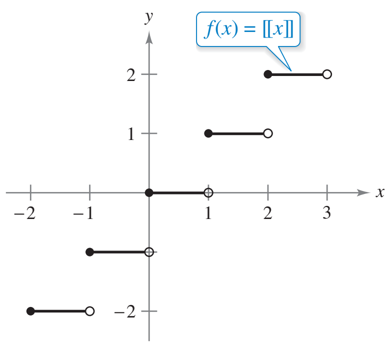
Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from the left and from the right.

Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is

$$\lim_{x \to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is

$$\lim_{x \to 0^+} [\![x]\!] = 0.$$



Greatest integer function

Figure 1.30

The greatest integer function has a discontinuity at zero because the left- and right-hand limits at zero are different. By similar reasoning, you can see that the greatest integer function has a discontinuity at any integer n.

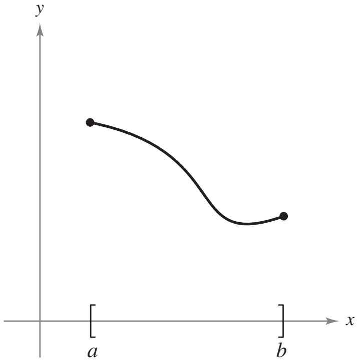
When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval when it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally in the next definition.



Continuous function on a closed interval
Figure 1.31

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval $[a, b]$** when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example,

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}.$$

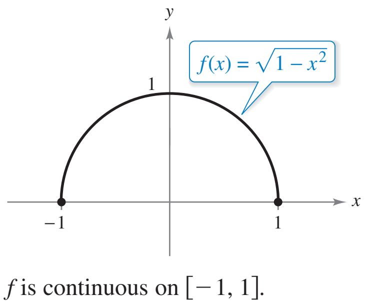
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32.



f is continuous on $[-1, 1]$.
Figure 1.32

The next example shows how a one-sided limit can be used to determine the value of absolute zero on the Kelvin scale.

REMARK Charles's Law for gases (assuming constant pressure) can be stated as

$$V = kT$$

where V is volume, k is a constant, and T is temperature.

EXAMPLE 5 Charles's Law and Absolute Zero

On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperatures very close to 0 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero *cannot* be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?

Solution The determination of absolute zero stems from the work of the French physicist Jacques Charles (1746–1823). Charles discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. The table illustrates this relationship between volume and temperature. To generate the values in the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is approximated and is measured in liters, and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

The points represented by the table are shown in Figure 1.33. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

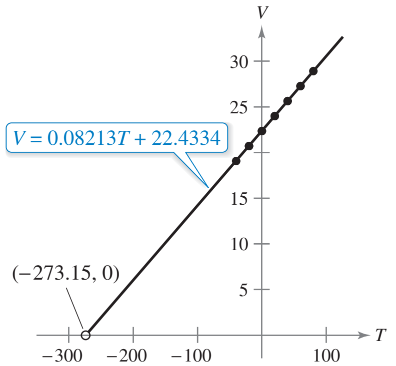
$$V = 0.08213T + 22.4334.$$

Solving for T, you get an equation for the temperature of the gas.

$$T = \frac{V - 22.4334}{0.08213}$$

By reasoning that the volume of the gas can approach 0 (but can never equal or go below 0), you can determine that the "least possible temperature" is

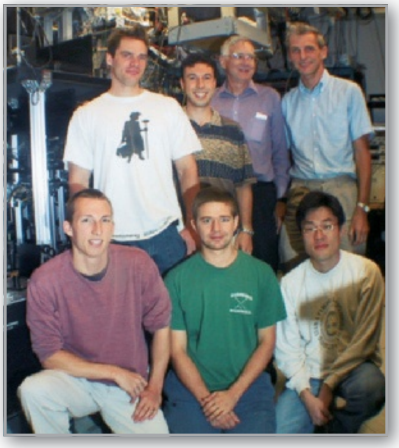
$$\begin{aligned}\lim_{V\to 0^+} T &= \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213}\\ &= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.}\\ &\approx -273.15.\end{aligned}$$



The volume of hydrogen gas depends on its temperature.

Figure 1.33

So, absolute zero on the Kelvin scale (0 K) is approximately $-273.15°$ on the Celsius scale.



In 2003, researchers at the Massachusetts Institute of Technology used lasers and evaporation to produce a supercold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. They measured a temperature of about 450 pK (picokelvin), or approximately $-273.14999999955°$C. *(Source: Science magazine, September 12, 2003)*

The table below shows the temperatures in Example 5 converted to the Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperatures and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale.

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

AUGUSTIN-LOUIS CAUCHY (1789–1857)

The concept of a continuous function was first introduced by Augustin-Louis Cauchy in 1821. The definition given in his text *Cours d'Analyse* stated that indefinite small changes in y were the result of indefinite small changes in x. "… $f(x)$ will be called a *continuous* function if … the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α …"

See LarsonCalculus.com to read more of this biography.

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2.

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf **2.** Sum or difference: $f \pm g$

3. Product: fg **4.** Quotient: $\dfrac{f}{g}$, $g(c) \neq 0$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

It is important for you to be able to recognize functions that are continuous at every point in their domains. The list below summarizes the functions you have studied so far that are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this list, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

EXAMPLE 6 Applying Properties of Continuity

See LarsonCalculus.com for an interactive version of this type of example.

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3 \tan x, \quad f(x) = \frac{x^2 + 1}{\cos x}$$

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \quad f(x) = \sqrt{x^2 + 1}, \quad \text{and} \quad f(x) = \tan \frac{1}{x}.$$

REMARK One consequence of Theorem 1.12 is that when f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Proof By the definition of continuity, $\lim\limits_{x \to c} g(x) = g(c)$ and $\lim\limits_{x \to g(c)} f(x) = f(g(c))$. Apply Theorem 1.5 with $L = g(c)$ to obtain $\lim\limits_{x \to c} f(g(x)) = f\left(\lim\limits_{x \to c} g(x)\right) = f(g(c))$. So, $(f \circ g)(x) = f(g(x))$ is continuous at c.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

At all other points, f is continuous. So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows from Theorem 1.12 that

$$y = \sin \frac{1}{x}$$

is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Section 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).

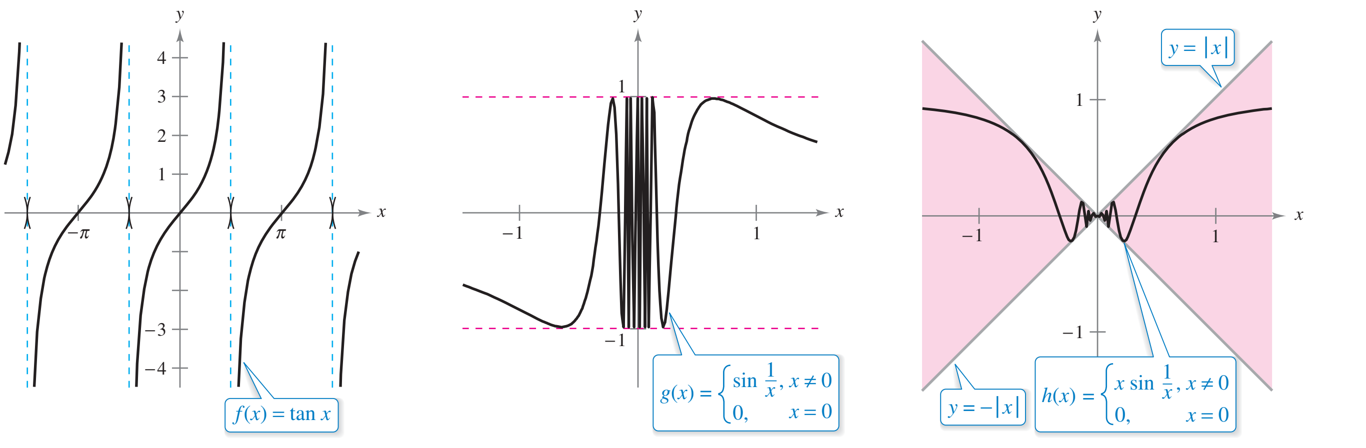
c. This function is similar to the function in part (b) except that the oscillations are damped by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, h is continuous on the entire real number line, as shown in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

(c) h is continuous on the entire real number line.

Figure 1.34

AP* Tips
You must understand the Intermediate Value Theorem and be able to apply it for both the multiple choice and free response sections of the AP Exam.

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

> **THEOREM 1.13 Intermediate Value Theorem**
>
> If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that
>
> $$f(c) = k.$$

REMARK The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.** By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, then $f(x)$ must take on all values between $f(a)$ and $f(b)$.

AP* Tips
For the free response section of the AP Exam, you should be able to apply the Intermediate Value Theorem, whether the function is presented as an equation or by a table.

As an example of the application of the Intermediate Value Theorem, consider a person's height. A girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

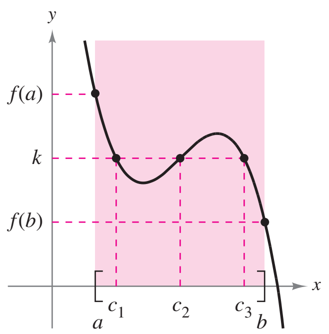
The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that

$$f(c) = k$$

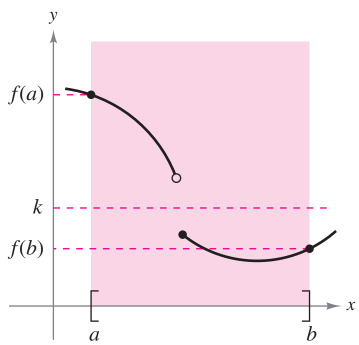
as shown in Figure 1.35. A function that is not continuous does not necessarily exhibit the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line

$$y = k$$

and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.



f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]
Figure 1.35



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]
Figure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

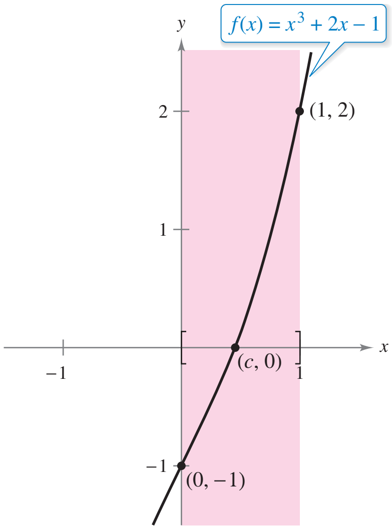
Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

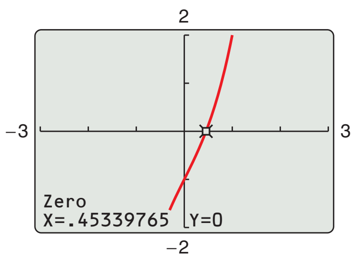
as shown in Figure 1.37.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.
Figure 1.37

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, then the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can use the *root* or *zero* feature of a graphing utility to approximate the real zeros of a continuous function. Using this feature, the zero of the function in Example 8, $f(x) = x^3 + 2x - 1$, is approximately 0.453, as shown in Figure 1.38.



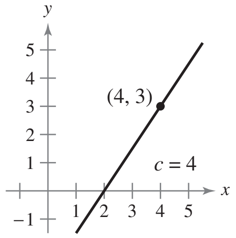
Zero of $f(x) = x^3 + 2x - 1$
Figure 1.38

1.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Limits and Continuity In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim\limits_{x\to c^+} f(x)$ (b) $\lim\limits_{x\to c^-} f(x)$ (c) $\lim\limits_{x\to c} f(x)$

1.


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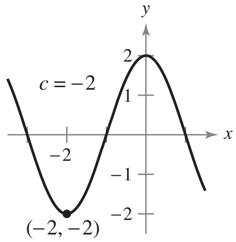

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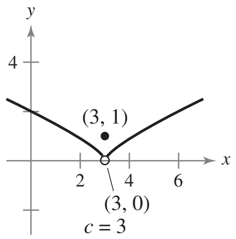

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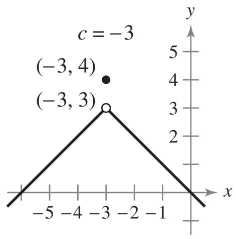

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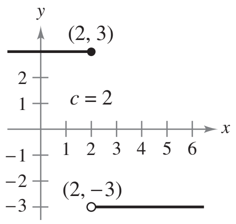

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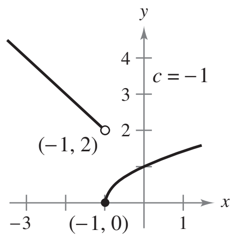


Finding a Limit In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim\limits_{x\to 8^+} \dfrac{1}{x+8}$

8. $\lim\limits_{x\to 2^-} \dfrac{2}{x+2}$

9. $\lim\limits_{x\to 5^+} \dfrac{x-5}{x^2-25}$

10. $\lim\limits_{x\to 4^+} \dfrac{4-x}{x^2-16}$

11. $\lim\limits_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

12. $\lim\limits_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$

13. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$

14. $\lim\limits_{x\to 10^+} \dfrac{|x-10|}{x-10}$

15. $\lim\limits_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}$

16. $\lim\limits_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+x+\Delta x-(x^2+x)}{\Delta x}$

17. $\lim\limits_{x\to 3^-} f(x)$, where $f(x)=\begin{cases}\dfrac{x+2}{2}, & x\le 3\\[2mm] \dfrac{12-2x}{3}, & x>3\end{cases}$

18. $\lim\limits_{x\to 3} f(x)$, where $f(x)=\begin{cases}x^2-4x+6, & x<3\\ -x^2+4x-2, & x\ge 3\end{cases}$

19. $\lim\limits_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^3+1, & x<1\\ x+1, & x\ge 1\end{cases}$

20. $\lim\limits_{x\to 1^+} f(x)$, where $f(x)=\begin{cases}x, & x\le 1\\ 1-x, & x>1\end{cases}$

21. $\lim\limits_{x\to \pi} \cot x$

22. $\lim\limits_{x\to \pi/2} \sec x$

23. $\lim\limits_{x\to 4^-} (5[\![x]\!]-7)$

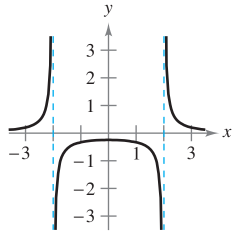
24. $\lim\limits_{x\to 2^+} (2x-[\![x]\!])$

25. $\lim\limits_{x\to 3} (2-[\![-x]\!])$

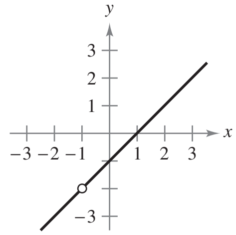
26. $\lim\limits_{x\to 1} \left(1-\left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

Continuity of a Function In Exercises 27–30, discuss the continuity of each function.

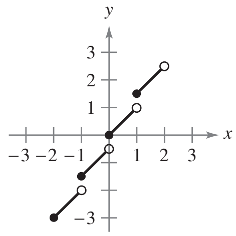
27. $f(x)=\dfrac{1}{x^2-4}$



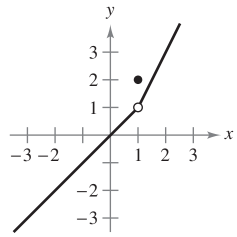
28. $f(x)=\dfrac{x^2-1}{x+1}$



29. $f(x)=\frac{1}{2}[\![x]\!]+x$



30. $f(x)=\begin{cases}x, & x<1\\ 2, & x=1\\ 2x-1, & x>1\end{cases}$



Continuity on a Closed Interval In Exercises 31–34, discuss the continuity of the function on the closed interval.

Function	Interval
31. $g(x)=\sqrt{49-x^2}$	$[-7, 7]$
32. $f(t)=3-\sqrt{9-t^2}$	$[-3, 3]$
33. $f(x)=\begin{cases}3-x, & x\le 0\\ 3+\frac{1}{2}x, & x>0\end{cases}$	$[-1, 4]$
34. $g(x)=\dfrac{1}{x^2-4}$	$[-1, 2]$

Removable and Nonremovable Discontinuities In Exercises 35–60, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

35. $f(x)=\dfrac{6}{x}$

36. $f(x)=\dfrac{4}{x-6}$

37. $f(x)=x^2-9$

38. $f(x)=x^2-4x+4$

39. $f(x) = \dfrac{1}{4 - x^2}$

40. $f(x) = \dfrac{1}{x^2 + 1}$

41. $f(x) = 3x - \cos x$

42. $f(x) = \cos \dfrac{\pi x}{2}$

43. $f(x) = \dfrac{x}{x^2 - x}$

44. $f(x) = \dfrac{x}{x^2 - 4}$

45. $f(x) = \dfrac{x}{x^2 + 1}$

46. $f(x) = \dfrac{x - 5}{x^2 - 25}$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6}$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

50. $f(x) = \dfrac{|x - 5|}{x - 5}$

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$

58. $f(x) = \tan \dfrac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$

60. $f(x) = 5 - [\![x]\!]$

Making a Function Continuous In Exercises 61–66, find the constant a, or the constants a and b, such that the function is continuous on the entire real number line.

61. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

62. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

63. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

64. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

65. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

Continuity of a Composite Function In Exercises 67–72, discuss the continuity of the composite function $h(x) = f(g(x))$.

67. $f(x) = x^2$
$g(x) = x - 1$

68. $f(x) = 5x + 1$
$g(x) = x^3$

69. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

70. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

71. $f(x) = \tan x$
$g(x) = \dfrac{x}{2}$

72. $f(x) = \sin x$
$g(x) = x^2$

Finding Discontinuities In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

73. $f(x) = [\![x]\!] - x$

74. $h(x) = \dfrac{1}{x^2 + 2x - 15}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

Testing for Continuity In Exercises 77–84, describe the interval(s) on which the function is continuous.

77. $f(x) = \dfrac{x}{x^2 + x + 2}$

78. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

79. $f(x) = 3 - \sqrt{x}$

80. $f(x) = x\sqrt{x + 3}$

81. $f(x) = \sec \dfrac{\pi x}{4}$

82. $f(x) = \cos \dfrac{1}{x}$

83. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

84. $f(x) = \begin{cases} 2x - 4, & x \ne 3 \\ 1, & x = 3 \end{cases}$

Writing In Exercises 85 and 86, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.

85. $f(x) = \dfrac{\sin x}{x}$

86. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing In Exercises 87–90, explain why the function has a zero in the given interval.

Function	Interval
87. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
88. $f(x) = x^3 + 5x - 3$	$[0, 1]$
89. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
90. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

Using the Intermediate Value Theorem In Exercises 91–94, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval $[0, 1]$. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

91. $f(x) = x^3 + x - 1$

92. $f(x) = x^4 - x^2 + 3x - 1$

93. $g(t) = 2 \cos t - 3t$

94. $h(\theta) = \tan \theta + 3\theta - 4$

Using the Intermediate Value Theorem **In Exercises 95–98, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.**

95. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

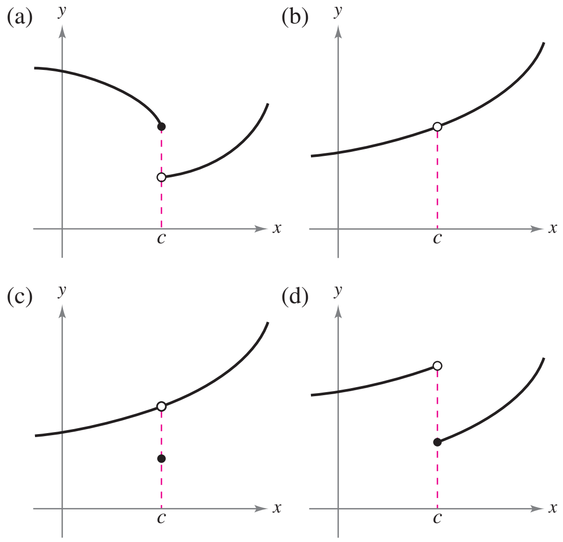
96. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

97. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

98. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

WRITING ABOUT CONCEPTS

99. Using the Definition of Continuity State how continuity is destroyed at $x = c$ for each of the following graphs.



100. Sketching a Graph Sketch the graph of any function f such that

$$\lim_{x \to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

101. Continuity of Combinations of Functions If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

102. Removable and Nonremovable Discontinuities Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 103–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. If $\lim_{x \to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

104. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

105. A rational function can have infinitely many x-values at which it is not continuous.

106. The function

$$f(x) = \frac{|x - 1|}{x - 1}$$

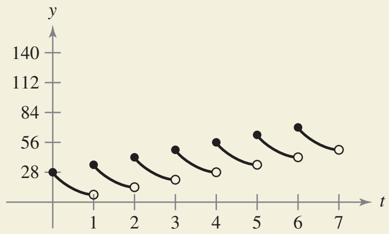
is continuous on $(-\infty, \infty)$.

107. Think About It Describe how the functions

$$f(x) = 3 + [\![x]\!] \quad \text{and} \quad g(x) = 3 - [\![-x]\!]$$

differ.

108. HOW DO YOU SEE IT? Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days. Estimate and interpret $\lim_{t \to 4^-} f(t)$ and $\lim_{t \to 4^+} f(t)$.



109. Telephone Charges A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

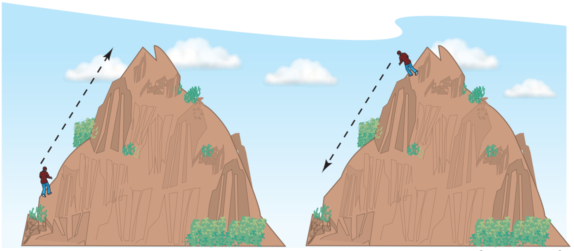
110. Inventory Management

The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\!\left[\frac{t + 2}{2}\right]\!\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

111. Déjà Vu At 8:00 A.M. on Saturday, a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M., he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Saturday 8:00 A.M. Sunday 8:00 A.M.

112. Volume Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

113. Proof Prove that if f is continuous and has no zeros on $[a, b]$, then either

$$f(x) > 0 \text{ for all } x \text{ in } [a, b] \quad \text{or} \quad f(x) < 0 \text{ for all } x \text{ in } [a, b].$$

114. Dirichlet Function Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

115. Continuity of a Function Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

116. Signum Function The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0. \\ 1, & x > 0 \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim\limits_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim\limits_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim\limits_{x\to 0} \operatorname{sgn}(x)$

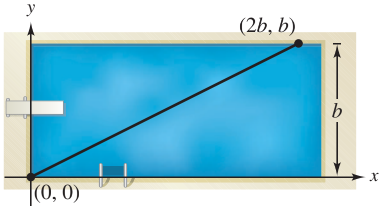
117. Modeling Data The table lists the speeds S (in feet per second) of a falling object at various times t (in seconds).

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

118. Creating Models A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.

119. Making a Function Continuous Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

120. Proof Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

121. Making a Function Continuous Let

$$f(x) = \frac{\sqrt{x + c^2} - c}{x}, \quad c > 0.$$

What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

122. Proof Prove that if

$$\lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$$

then f is continuous at c.

123. Continuity of a Function Discuss the continuity of the function $h(x) = x[\![x]\!]$.

124. Proof

(a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $[0, \pi/2]$ such that $\cos x = x$. Use a graphing utility to approximate c to three decimal places.

PUTNAM EXAM CHALLENGE

125. Prove or disprove: If x and y are real numbers with $y \ge 0$ and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

126. Determine all polynomials $P(x)$ such that

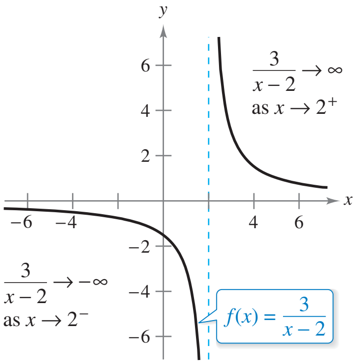
$$P(x^2 + 1) = (P(x))^2 + 1 \text{ and } P(0) = 0.$$

1.5 Infinite Limits

- **Determine infinite limits from the left and from the right.**
- **Find and sketch the vertical asymptotes of the graph of a function.**

Infinite Limits

Consider the function $f(x) = 3/(x - 2)$. From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right.



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

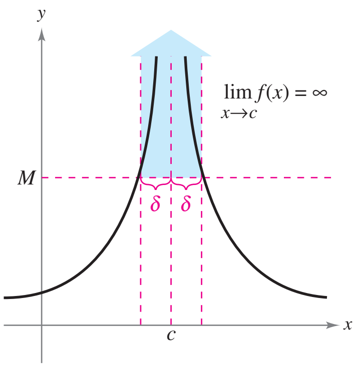
This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty \qquad f(x) \text{ decreases without bound as } x \text{ approaches 2 from the left.}$$

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty. \qquad f(x) \text{ increases without bound as } x \text{ approaches 2 from the right.}$$

The symbols ∞ and $-\infty$ refer to positive infinity and negative infinity, respectively. These symbols do not represent real numbers. They are convenient symbols used to describe unbounded conditions more concisely. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**



Infinite limits
Figure 1.40

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever

$$0 < |x - c| < \delta.$$

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit **fails to exist** by denoting the unbounded behavior of $f(x)$ as x approaches c.

Exploration

Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit (if it exists) of $f(x)$ as x approaches c from the left and from the right.

a. $f(x) = \dfrac{3}{x - 4}$

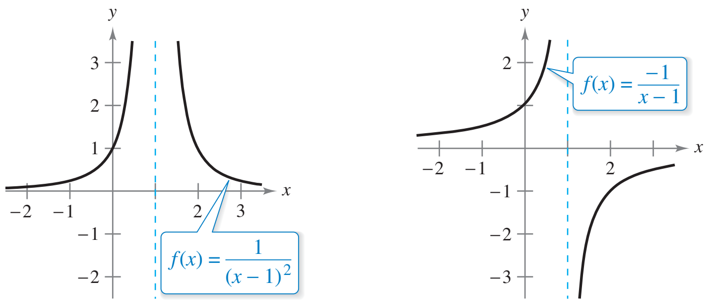
b. $f(x) = \dfrac{1}{2 - x}$

c. $f(x) = \dfrac{2}{(x - 3)^2}$

d. $f(x) = \dfrac{-3}{(x + 2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Determine the limit of each function shown in Figure 1.41 as x approaches 1 from the left and from the right.



(a) **(b)**

Each graph has an asymptote at $x = 1$.

Figure 1.41

Solution

a. When x approaches 1 from the left or the right, $(x - 1)^2$ is a small positive number. Thus, the quotient $1/(x - 1)^2$ is a large positive number, and $f(x)$ approaches infinity from each side of $x = 1$. So, you can conclude that

$$\lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty. \qquad \text{Limit from each side is infinity.}$$

Figure 1.41(a) confirms this analysis.

b. When x approaches 1 from the left, $x - 1$ is a small negative number. Thus, the quotient $-1/(x - 1)$ is a large positive number, and $f(x)$ approaches infinity from the left of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{-1}{x - 1} = \infty. \qquad \text{Limit from the left side is infinity.}$$

When x approaches 1 from the right, $x - 1$ is a small positive number. Thus, the quotient $-1/(x - 1)$ is a large negative number, and $f(x)$ approaches negative infinity from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^+} \frac{-1}{x - 1} = -\infty. \qquad \text{Limit from the right side is negative infinity.}$$

Figure 1.41(b) confirms this analysis.

TECHNOLOGY Remember that you can use a numerical approach to analyze a limit. For instance, you can use a graphing utility to create a table of values to analyze the limit in Example 1(a), as shown in Figure 1.42.

Enter x-values using *ask* mode.

X	Y1
.9	100
.99	10000
.999	1E6
1	ERROR
1.001	1E6
1.01	10000
1.1	100

X=1

As x approaches 1 from the left, $f(x)$ increases without bound.

As x approaches 1 from the right, $f(x)$ increases without bound.

Figure 1.42

Use a graphing utility to make a table of values to analyze the limit in Example 1(b).

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

REMARK If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number at which the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation.

THEOREM 1.14 Vertical Asymptotes

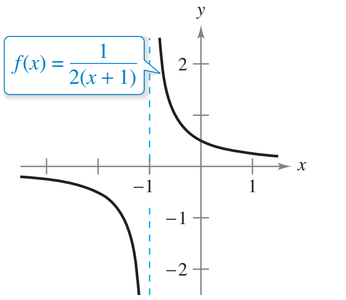
Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

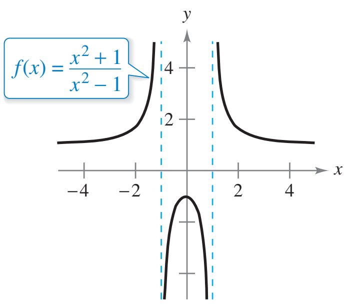
has a vertical asymptote at $x = c$.

A proof of this theorem is given in Appendix A.

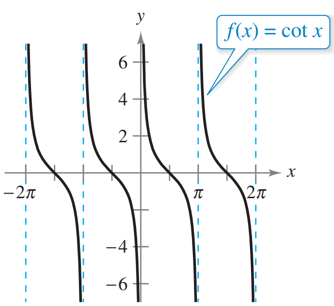
See LarsonCalculus.com for Bruce Edwards's video of this proof.



(a)



(b)



(c)

Functions with vertical asymptotes
Figure 1.43

EXAMPLE 2 Finding Vertical Asymptotes

See LarsonCalculus.com for an interactive version of this type of example.

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x + 1)}$$

is 0 and the numerator is not 0. So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.43(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2 + 1}{x^2 - 1} = \frac{x^2 + 1}{(x - 1)(x + 1)}$$

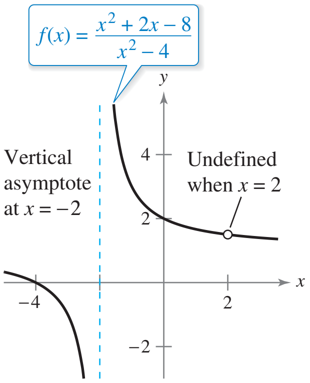
you can see that the denominator is 0 at $x = -1$ and $x = 1$. Also, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.43(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.43(c). So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer.

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. When both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.44

EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Solution Begin by simplifying the expression, as shown.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.44. From the graph, you can see that

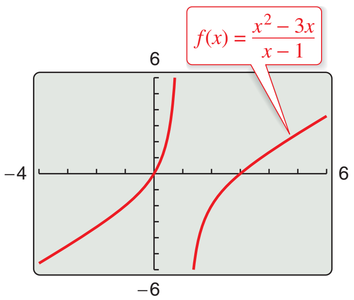
$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$



f has a vertical asymptote at $x = 1$.
Figure 1.45

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

$$f(x) = \frac{x^2 - 3x}{x - 1}$$

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. You can determine the result by analyzing f at values of x close to 1, or by using a graphing utility. From the graph of f shown in Figure 1.45, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty \qquad \text{The limit from the left is infinity.}$$

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty. \qquad \text{The limit from the right is negative infinity.}$$

TECHNOLOGY PITFALL When using a graphing utility, be careful to interpret correctly the graph of a function with a vertical asymptote—some graphing utilities have difficulty drawing this type of graph.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

1. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$

$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$ [see Example 5(d)].

Proof Here is a proof of the sum property. (The proofs of the remaining properties are left as an exercise [see Exercise 70].) To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that $[f(x) + g(x)] > M$ whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive. Let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x \to c} [f(x) + g(x)] = \infty.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 5 Determining Limits

a. Because $\lim_{x \to 0} 1 = 1$ and $\lim_{x \to 0} \frac{1}{x^2} = \infty$, you can write

$$\lim_{x \to 0} \left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x \to 1^-} (x^2 + 1) = 2$ and $\lim_{x \to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x \to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0. \qquad \text{Property 3, Theorem 1.15}$$

c. Because $\lim_{x \to 0^+} 3 = 3$ and $\lim_{x \to 0^+} \cot x = \infty$, you can write

$$\lim_{x \to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

d. Because $\lim_{x \to 0^-} x^2 = 0$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, you can write

$$\lim_{x \to 0^-} \left(x^2 + \frac{1}{x}\right) = -\infty. \qquad \text{Property 1, Theorem 1.15}$$

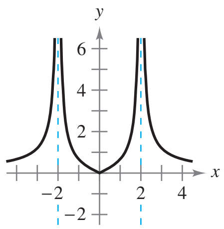
REMARK Note that the solution to Example 5(d) uses Property 1 from Theorem 1.15 for which the limit of $f(x)$ as x approaches c is $-\infty$.

1.5 Exercises

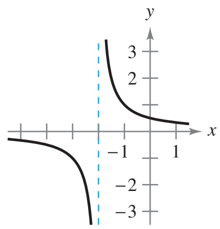
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Determining Infinite Limits from a Graph In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

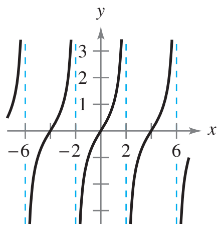
1. $f(x) = 2\left|\dfrac{x}{x^2 - 4}\right|$



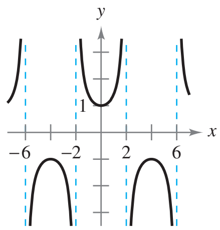
2. $f(x) = \dfrac{1}{x + 2}$



3. $f(x) = \tan \dfrac{\pi x}{4}$



4. $f(x) = \sec \dfrac{\pi x}{4}$



Determining Infinite Limits In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

5. $f(x) = \dfrac{1}{x - 4}$

6. $f(x) = \dfrac{-1}{x - 4}$

7. $f(x) = \dfrac{1}{(x - 4)^2}$

8. $f(x) = \dfrac{-1}{(x - 4)^2}$

Numerical and Graphical Analysis In Exercises 9–12, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function to confirm your answer.

x	-3.5	-3.1	-3.01	-3.001	-3
$f(x)$					?

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

9. $f(x) = \dfrac{1}{x^2 - 9}$

10. $f(x) = \dfrac{x}{x^2 - 9}$

11. $f(x) = \dfrac{x^2}{x^2 - 9}$

12. $f(x) = \cot \dfrac{\pi x}{3}$

Finding Vertical Asymptotes In Exercises 13–28, find the vertical asymptotes (if any) of the graph of the function.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{2}{(x - 3)^3}$

15. $f(x) = \dfrac{x^2}{x^2 - 4}$

16. $f(x) = \dfrac{3x}{x^2 + 9}$

17. $g(t) = \dfrac{t - 1}{t^2 + 1}$

18. $h(s) = \dfrac{3s + 4}{s^2 - 16}$

19. $f(x) = \dfrac{3}{x^2 + x - 2}$

20. $g(x) = \dfrac{x^3 - 8}{x - 2}$

21. $f(x) = \dfrac{4x^2 + 4x - 24}{x^4 - 2x^3 - 9x^2 + 18x}$

22. $h(x) = \dfrac{x^2 - 9}{x^3 + 3x^2 - x - 3}$

23. $f(x) = \dfrac{x^2 - 2x - 15}{x^3 - 5x^2 + x - 5}$

24. $h(t) = \dfrac{t^2 - 2t}{t^4 - 16}$

25. $f(x) = \csc \pi x$

26. $f(x) = \tan \pi x$

27. $s(t) = \dfrac{t}{\sin t}$

28. $g(\theta) = \dfrac{\tan \theta}{\theta}$

Vertical Asymptote or Removable Discontinuity In Exercises 29–32, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

29. $f(x) = \dfrac{x^2 - 1}{x + 1}$

30. $f(x) = \dfrac{x^2 - 2x - 8}{x + 1}$

31. $f(x) = \dfrac{x^2 + 1}{x + 1}$

32. $f(x) = \dfrac{\sin(x + 1)}{x + 1}$

Finding a One-Sided Limit In Exercises 33–48, find the one-sided limit (if it exists).

33. $\displaystyle\lim_{x \to -1^+} \frac{1}{x + 1}$

34. $\displaystyle\lim_{x \to 1^-} \frac{-1}{(x - 1)^2}$

35. $\displaystyle\lim_{x \to 2^+} \frac{x}{x - 2}$

36. $\displaystyle\lim_{x \to 2^-} \frac{x^2}{x^2 + 4}$

37. $\displaystyle\lim_{x \to -3^-} \frac{x + 3}{x^2 + x - 6}$

38. $\displaystyle\lim_{x \to (-1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3}$

39. $\displaystyle\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right)$

40. $\displaystyle\lim_{x \to 0^+} \left(6 - \frac{1}{x^3}\right)$

41. $\displaystyle\lim_{x \to -4^-} \left(x^2 + \frac{2}{x + 4}\right)$

42. $\displaystyle\lim_{x \to 3^+} \left(\frac{x}{3} + \cot \frac{\pi x}{2}\right)$

43. $\displaystyle\lim_{x \to 0^+} \frac{2}{\sin x}$

44. $\displaystyle\lim_{x \to (\pi/2)^+} \frac{-2}{\cos x}$

45. $\displaystyle\lim_{x \to \pi^+} \frac{\sqrt{x}}{\csc x}$

46. $\displaystyle\lim_{x \to 0^-} \frac{x + 2}{\cot x}$

47. $\lim_{x\to(1/2)^-} x \sec \pi x$

48. $\lim_{x\to(1/2)^+} x^2 \tan \pi x$

One-Sided Limit **In Exercises 49–52, use a graphing utility to graph the function and determine the one-sided limit.**

49. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x\to 1^+} f(x)$

50. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\lim_{x\to 1^-} f(x)$

51. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim_{x\to 5^-} f(x)$

52. $f(x) = \sec \dfrac{\pi x}{8}$

$\lim_{x\to 4^+} f(x)$

WRITING ABOUT CONCEPTS

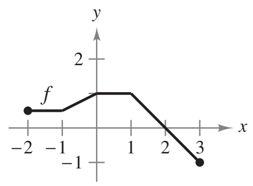
53. Infinite Limit In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

54. Asymptote In your own words, describe what is meant by an asymptote of a graph.

55. Writing a Rational Function Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Rational Function Does the graph of every rational function have a vertical asymptote? Explain.

57. Sketching a Graph Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



58. Relativity According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c from the left.

59. Numerical and Graphical Analysis Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

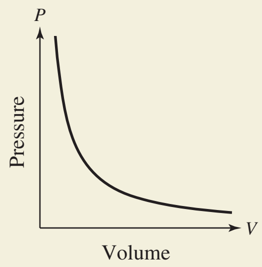
(a) $\lim_{x\to 0^+} \dfrac{x - \sin x}{x}$

(b) $\lim_{x\to 0^+} \dfrac{x - \sin x}{x^2}$

(c) $\lim_{x\to 0^+} \dfrac{x - \sin x}{x^3}$

(d) $\lim_{x\to 0^+} \dfrac{x - \sin x}{x^4}$

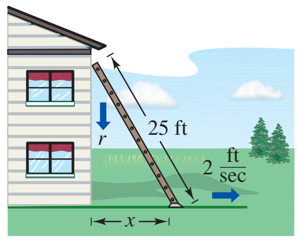
60. HOW DO YOU SEE IT? For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. What is the limit of P as V approaches 0 from the right? Explain what this means in the context of the problem.



61. Rate of Change A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, then the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house, and r is the rate in feet per second.



(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as x approaches 25 from the left.

62. Average Speed

On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip, the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that

$$y = \frac{25x}{x - 25}.$$

What is the domain?

(b) Complete the table.

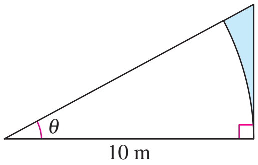
x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as x approaches 25 from the right and interpret its meaning.

WendellandCarolyn/iStockphoto.com

63. Numerical and Graphical Analysis Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).



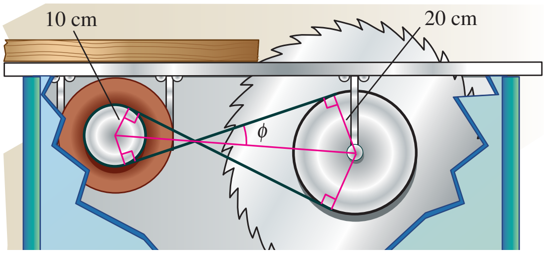
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as θ approaches $\pi/2$ from the left.

64. Numerical and Graphical Reasoning A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\lim_{\phi \to (\pi/2)^-} L$. Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.

True or False? **In Exercises 65–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. The graph of a rational function has at least one vertical asymptote.

66. The graphs of polynomial functions have no vertical asymptotes.

67. The graphs of trigonometric functions have no vertical asymptotes.

68. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

69. Finding Functions Find functions f and g such that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = \infty$, but $\lim_{x \to c} [f(x) - g(x)] \neq 0$.

70. Proof Prove the difference, product, and quotient properties in Theorem 1.15.

71. Proof Prove that if $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} \dfrac{1}{f(x)} = 0$.

72. Proof Prove that if

$$\lim_{x \to c} \frac{1}{f(x)} = 0$$

then $\lim_{x \to c} f(x)$ does not exist.

Infinite Limits **In Exercises 73 and 74, use the ε-δ definition of infinite limits to prove the statement.**

73. $\lim_{x \to 3^+} \dfrac{1}{x - 3} = \infty$ **74.** $\lim_{x \to 5^-} \dfrac{1}{x - 5} = -\infty$

SECTION PROJECT

Graphs and Limits of Trigonometric Functions

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \le x \le \pi$. Explain how the graph helps confirm this theorem.

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula at $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Precalculus or Calculus **In Exercises 1 and 2, determine whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.**

1. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the curve $y = x^2$.

2. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the line $y = 4x - 3$.

Estimating a Limit Numerically **In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

3. $\displaystyle\lim_{x \to 3} \frac{x - 3}{x^2 - 7x + 12}$

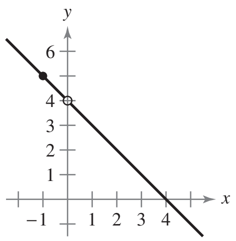
x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

4. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

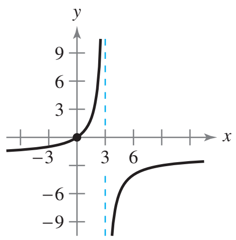
Finding a Limit Graphically **In Exercises 5 and 6, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

5. $h(x) = \dfrac{4x - x^2}{x}$



(a) $\displaystyle\lim_{x \to 0} h(x)$ (b) $\displaystyle\lim_{x \to -1} h(x)$

6. $g(x) = \dfrac{-2x}{x - 3}$



(a) $\displaystyle\lim_{x \to 3} g(x)$ (b) $\displaystyle\lim_{x \to 0} g(x)$

Using the ϵ-δ Definition of a Limit **In Exercises 7–10, find the limit L. Then use the ϵ-δ definition to prove that the limit is L.**

7. $\displaystyle\lim_{x \to 1} (x + 4)$

8. $\displaystyle\lim_{x \to 9} \sqrt{x}$

9. $\displaystyle\lim_{x \to 2} (1 - x^2)$

10. $\displaystyle\lim_{x \to 5} 9$

Finding a Limit **In Exercises 11–28, find the limit.**

11. $\displaystyle\lim_{x \to -6} x^2$

12. $\displaystyle\lim_{x \to 0} (5x - 3)$

13. $\displaystyle\lim_{t \to 4} \sqrt{t + 2}$

14. $\displaystyle\lim_{x \to -5} \sqrt[3]{x - 3}$

15. $\displaystyle\lim_{x \to 6} (x - 2)^2$

16. $\displaystyle\lim_{x \to 7} (x - 4)^3$

17. $\displaystyle\lim_{x \to 4} \frac{4}{x - 1}$

18. $\displaystyle\lim_{x \to 2} \frac{x}{x^2 + 1}$

19. $\displaystyle\lim_{x \to -2} \frac{t + 2}{t^2 - 4}$

20. $\displaystyle\lim_{x \to 4} \frac{t^2 - 16}{t - 4}$

21. $\displaystyle\lim_{x \to 4} \frac{\sqrt{x - 3} - 1}{x - 4}$

22. $\displaystyle\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$

23. $\displaystyle\lim_{x \to 0} \frac{[1/(x + 1)] - 1}{x}$

24. $\displaystyle\lim_{s \to 0} \frac{\left(1/\sqrt{1 + s}\right) - 1}{s}$

25. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

26. $\displaystyle\lim_{x \to \pi/4} \frac{4x}{\tan x}$

27. $\displaystyle\lim_{\Delta x \to 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x}$

[*Hint:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$]

28. $\displaystyle\lim_{\Delta x \to 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x}$

[*Hint:* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$]

Evaluating a Limit **In Exercises 29–32, evaluate the limit given $\lim_{x \to c} f(x) = -6$ and $\lim_{x \to c} g(x) = \frac{1}{2}$.**

29. $\displaystyle\lim_{x \to c} [f(x)g(x)]$

30. $\displaystyle\lim_{x \to c} \frac{f(x)}{g(x)}$

31. $\displaystyle\lim_{x \to c} [f(x) + 2g(x)]$

32. $\displaystyle\lim_{x \to c} [f(x)]^2$

Graphical, Numerical, and Analytic Analysis **In Exercises 33–36, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

33. $\displaystyle\lim_{x \to 0} \frac{\sqrt{2x + 9} - 3}{x}$

34. $\displaystyle\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$

35. $\displaystyle\lim_{x \to -5} \frac{x^3 + 125}{x + 5}$

36. $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x}$

Free-Falling Object **In Exercises 37 and 38, use the position function $s(t) = -4.9t^2 + 250$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 250 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

37. Find the velocity of the object when $t = 4$.

38. At what velocity will the object impact the ground?

Finding a Limit **In Exercises 39–48, find the limit (if it exists). If it does not exist, explain why.**

39. $\displaystyle\lim_{x \to 3^+} \frac{1}{x + 3}$

40. $\displaystyle\lim_{x \to 6^-} \frac{x - 6}{x^2 - 36}$

41. $\displaystyle\lim_{x \to 4^-} \frac{\sqrt{x} - 2}{x - 4}$

42. $\displaystyle\lim_{x \to 3^-} \frac{|x - 3|}{x - 3}$

43. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

44. $\lim_{x\to 1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

45. $\lim_{t\to 1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

46. $\lim_{s\to -2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

47. $\lim_{x\to 2^-} (2[\![x]\!]+1)$

48. $\lim_{x\to 4} [\![x-1]\!]$

Removable and Nonremovable Discontinuities **In Exercises 49–54, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?**

49. $f(x) = x^2 - 4$

50. $f(x) = x^2 - x + 20$

51. $f(x) = \dfrac{4}{x-5}$

52. $f(x) = \dfrac{1}{x^2-9}$

53. $f(x) = \dfrac{x}{x^3-x}$

54. $f(x) = \dfrac{x+3}{x^2-3x-18}$

55. Making a Function Continuous Determine the value of c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

56. Making a Function Continuous Determine the values of b and c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

Testing for Continuity **In Exercises 57–62, describe the intervals on which the function is continuous.**

57. $f(x) = -3x^2 + 7$

58. $f(x) = \dfrac{4x^2+7x-2}{x+2}$

59. $f(x) = \sqrt{x-4}$

60. $f(x) = [\![x+3]\!]$

61. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

62. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

63. Using the Intermediate Value Theorem Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

64. Delivery Charges The cost of sending an overnight package from New York to Atlanta is \$12.80 for the first pound and \$2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Sketch the graph of this function and discuss its continuity.

65. Finding Limits Let

$$f(x) = \frac{x^2-4}{|x-2|}.$$

Find each limit (if it exists).

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

66. Finding Limits Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\lim_{x\to 0^-} f(x)$.

(c) Find $\lim_{x\to 1^+} f(x)$.

Finding Vertical Asymptotes **In Exercises 67–72, find the vertical asymptotes (if any) of the graph of the function.**

67. $f(x) = \dfrac{3}{x}$

68. $f(x) = \dfrac{5}{(x-2)^4}$

69. $f(x) = \dfrac{x^3}{x^2-9}$

70. $h(x) = \dfrac{6x}{36-x^2}$

71. $g(x) = \dfrac{2x+1}{x^2-64}$

72. $f(x) = \csc \pi x$

Finding a One-Sided Limit **In Exercises 73–82, find the one-sided limit (if it exists).**

73. $\lim_{x\to 1^-} \dfrac{x^2+2x+1}{x-1}$

74. $\lim_{x\to (1/2)^+} \dfrac{x}{2x-1}$

75. $\lim_{x\to -1^+} \dfrac{x+1}{x^3+1}$

76. $\lim_{x\to -1^-} \dfrac{x+1}{x^4-1}$

77. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right)$

78. $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2-4}}$

79. $\lim_{x\to 0^+} \dfrac{\sin 4x}{5x}$

80. $\lim_{x\to 0^+} \dfrac{\sec x}{x}$

81. $\lim_{x\to 0^+} \dfrac{\csc 2x}{x}$

82. $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x}$

83. Environment A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 15% of the pollutants.

(b) Find the cost of removing 50% of the pollutants.

(c) Find the cost of removing 90% of the pollutants.

(d) Find the limit of C as p approaches 100 from the left and interpret its meaning.

84. Limits and Continuity The function f is defined as shown.

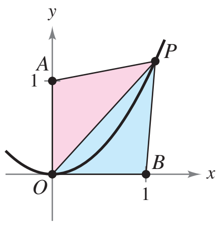
$$f(x) = \frac{\tan 2x}{x}, \quad x \ne 0$$

(a) Find $\lim_{x\to 0} \dfrac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

P.S. Problem Solving

1. **Perimeter** Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the perimeter of each triangle in terms of x.

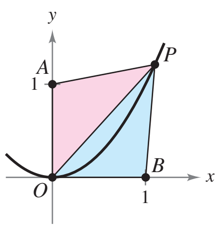
(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table. Calculate $\lim_{x \to 0^+} r(x)$.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

2. **Area** Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

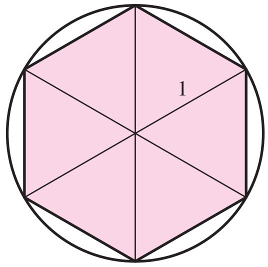
$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table. Calculate $\lim_{x \to 0^+} a(x)$.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

3. **Area of a Circle**

(a) Find the area of a regular hexagon inscribed in a circle of radius 1. How close is this area to that of the circle?



(b) Find the area A_n of an n-sided regular polygon inscribed in a circle of radius 1. Write your answer as a function of n.

(c) Complete the table. What number does A_n approach as n gets larger and larger?

n	6	12	24	48	96
A_n					

4. **Tangent Line** Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 3} m_x$. How does this number relate to your answer in part (b)?

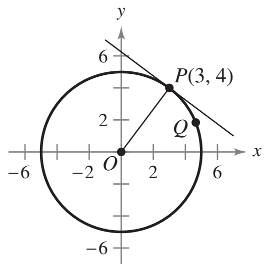


Figure for 4

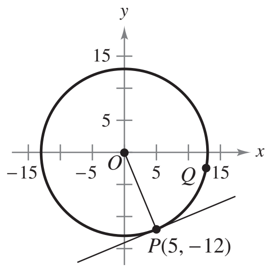


Figure for 5

5. **Tangent Line** Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.

(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 5} m_x$. How does this number relate to your answer in part (b)?

6. Finding Values Find the values of the constants a and b such that

$$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}.$$

7. Finding Limits Consider the function

$$f(x) = \frac{\sqrt{3+x^{1/3}}-2}{x-1}.$$

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

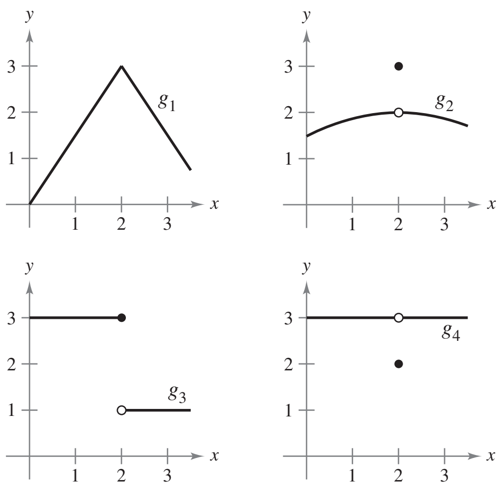
(c) Calculate $\lim\limits_{x\to -27^+} f(x)$.

(d) Calculate $\lim\limits_{x\to 1} f(x)$.

8. Making a Function Continuous Determine all values of the constant a such that the following function is continuous for all real numbers.

$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \ge 0 \\ a^2-2, & x < 0 \end{cases}$$

9. Choosing Graphs Consider the graphs of the four functions g_1, g_2, g_3, and g_4.



For each given condition of the function f, which of the graphs could be the graph of f?

(a) $\lim\limits_{x\to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\lim\limits_{x\to 2^-} f(x) = 3$

10. Limits and Continuity Sketch the graph of the function

$$f(x) = \left[\!\left[\frac{1}{x}\right]\!\right].$$

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0^+} f(x)$.

(c) Discuss the continuity of the function.

11. Limits and Continuity Sketch the graph of the function $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, and $\lim\limits_{x\to 1/2} f(x)$.

(c) Discuss the continuity of the function.

12. Escape Velocity To escape Earth's gravitational field, a rocket must be launched with an initial velocity called the **escape velocity.** A rocket launched from the surface of Earth has velocity v (in miles per second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket to the center of Earth, G is the gravitational constant, M is the mass of Earth, and R is the radius of Earth (approximately 4000 miles).

(a) Find the value of v_0 for which you obtain an infinite limit for r as v approaches zero. This value of v_0 is the escape velocity for Earth.

(b) A rocket launched from the surface of the moon has velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a planet has velocity v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of this planet larger or smaller than that of Earth? (Assume that the mean density of this planet is the same as that of Earth.)

13. Pulse Function For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x-a) - H(x-b) = \begin{cases} 0, & x < a \\ 1, & a \le x < b \\ 0, & x \ge b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\lim\limits_{x\to a^+} P_{a,b}(x)$ (ii) $\lim\limits_{x\to a^-} P_{a,b}(x)$

(iii) $\lim\limits_{x\to b^+} P_{a,b}(x)$ (iv) $\lim\limits_{x\to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is $U(x) = \dfrac{1}{b-a} P_{a,b}(x)$ called the **unit pulse function?**

14. Proof Let a be a nonzero constant. Prove that if $\lim\limits_{x\to 0} f(x) = L$, then $\lim\limits_{x\to 0} f(ax) = L$. Show by means of an example that a must be nonzero.

AP* Review Questions for Chapter 1

1. (no calculator)

The function f is defined as follows: $f(x) = \dfrac{x^2 + 5x + 6}{2x^2 + 7x + 3}$.

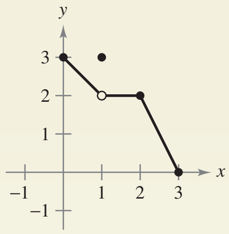
(a) State the value(s) of x for which f is not continuous.

(b) Evaluate $\lim_{x\to -3} f(x)$. Show the work that leads to your answer.

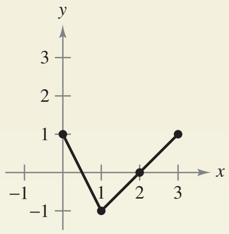
(c) State the equation(s) for the vertical asymptote(s) for the graph of $y = f(x)$.

(d) State the equation(s) for the horizontal asymptote(s) for the graph of $y = f(x)$. Show the work that leads to your answer.

2.



Graph of f



Graph of g

The graphs of functions f and g are shown above. Evaluate each limit using the graphs provided. Show the computations that lead to your answer.

(a) $\lim_{x\to 1} (f(x) + 4)$

(b) $\lim_{x\to 3} \dfrac{5}{g(x)}$

(c) $\lim_{x\to 2} (f(x) \cdot g(x))$

(d) $\lim_{x\to 3^-} \dfrac{f(x)}{g(x) - 1}$ (Assume that f and g are linear on the interval $[2, 3]$.)

3. A hot cup of tea is placed on a counter and left to cool. The temperature of the tea, in degrees Fahrenheit (correct to the nearest degree), x minutes after the cup is placed on the counter is modeled by a continuous function $T(x)$ for $0 \le x < 10$. Values of $T(x)$ at various times x are shown in the table below.

x	0	3	4	6	8	9
$T(x)$	180°	174°	172°	168°	164°	162°

(a) Evaluate: $\lim_{x\to 4} T(x)$. Justify your answer.

(b) Using the data in the table, find the average rate of change in the temperature of the tea for $3 \le x \le 8$. Include units in your final answer.

(c) Identify, using the times listed in the table, the shortest interval during which there must exist a time x for which the temperature of the tea is 166.5°. Justify your answer.

(d) Use the data in the table to find the best estimate of the slope of the line tangent to the graph of T at $x = 8$.

4. (no calculator)

Find the value of each of these limits, or else explain why the limit does not exist. Show the computations which lead to your answers.

(a) $\lim_{x\to\infty} \dfrac{3x^4 - 6x^3 + x^2 - x - 1}{2x^3 - 9x^4 - 5}$

(b) $\lim_{x\to 0} \dfrac{\sqrt{x+5} - \sqrt{5}}{5x}$

(c) $\lim_{x\to 0} \dfrac{2 - 2\cos^2 x}{x \sin x}$

(d) $\lim_{x\to -2} \dfrac{\frac{5}{x} + \frac{5}{2}}{x + 2}$

5. (no calculator)

The position function $s(t) = -4.9t^2 + 396.9$ gives the height (in meters) of an object that has fallen from a height of 396.9 meters after t seconds.

(a) Explain why there must exist a time t, $1 < t < 2$, at which the height of the object must be 382 meters above the ground.

(b) Find the time at which the object hits the ground.

(c) Find the average rate of change in s over the interval $t \in [8, 9]$. Include units of measure. Explain why this is a good estimate of the velocity at which the object hits the ground. How can this estimate be improved?

(d) Evaluate $\lim_{t\to 3} \dfrac{s(t) - s(3)}{t - 3}$. Show the work that leads to your answer. Include units.

6. Let a and b represent real numbers. Define

$$f(x) = \begin{cases} ax^2 + x - b & \text{if } x \le 2 \\ ax + b & \text{if } 2 < x < 5. \\ 2ax - 7 & \text{if } x \ge 5 \end{cases}$$

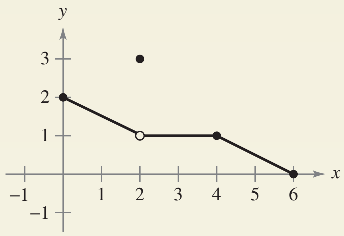
(a) Find the values of a and b such that f is continuous everywhere.

(b) Evaluate $\lim_{x\to 3} f(x)$.

(c) Let $g(x) = \dfrac{f(x)}{x - 1}$.

Evaluate $\lim_{x\to 1} g(x)$.

7.



The graph of function g is shown above. Which of the following is true?

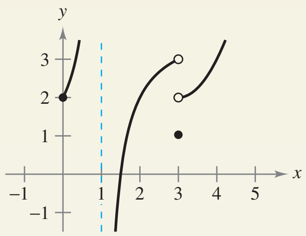
I. $\lim_{x \to 2} g(x) = 1$

II. $\lim_{x \to 2} g(x) = g(2)$

III. g is continuous at $x = 3$.

(A) I only
(B) I and II only
(C) I and III only
(D) III only
(E) I, II and III

8.



The graph of the function f is shown above. The line $x = 1$ is a vertical asymptote. Which of the following statements about f is true?

(A) $\lim_{x \to 1} = \infty$

(B) $\lim_{x \to 3} = 1$

(C) $\lim_{x \to 3^-} f(x) < \lim_{x \to 3^+} f(x)$

(D) $\lim_{x \to 4} f(x)$ does not exist

(E) $\lim_{x \to 0^+} f(x) = \lim_{x \to 3^+} f(x)$

9. Define $f(x) = \begin{cases} \dfrac{x^2 + 4x - 32}{x^2 - 2x - 8} & \text{if } x \neq -2, 4 \\ 8 & \text{if } x = 4 \end{cases}$

Which of the following statements about f are true?

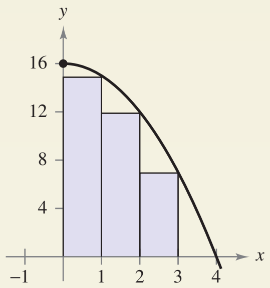
I. f is not continuous at $x = 4$.

II. $\lim_{x \to \infty} f(x) = 4$

III. $x = 4$ is a vertical asymptote of the graph of $y = f(x)$.

(A) None
(B) I only
(C) I and II only
(D) I and III only
(E) I, II and III

10.



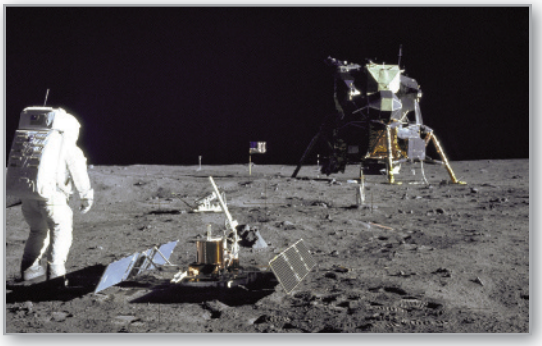
The figure above shows three rectangles each with a vertex on the graph of $y = 16 - x^2$. Find the sum of the areas of these rectangles

(A) 42 sq. units
(B) 40 sq. units
(C) 34 sq. units
(D) 33 sq. units
(E) 29 sq. units

2 Differentiation

Rate of Change (*Example 2, p. 149*)

Bacteria (*Exercise 111, p. 139*)



Acceleration Due to Gravity (*Example 10, p. 124*)

Velocity of a Falling Object (*Example 9, p. 112*)

Stopping Distance (*Exercise 107, p. 117*)

2.1 The Derivative and the Tangent Line Problem

- **Find the slope of the tangent line to a curve at a point.**
- **Use the limit definition to find the derivative of a function.**
- **Understand the relationship between differentiability and continuity.**

ISAAC NEWTON (1642–1727)

In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Law of Universal Gravitation and his three laws of motion.

See LarsonCalculus.com to read more of this biography.

The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

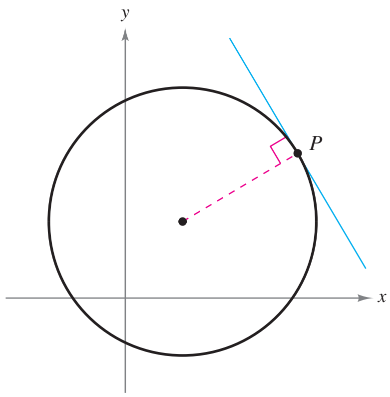
1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

A brief introduction to the tangent line problem is given in Section 1.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at point P, as shown in Figure 2.1.

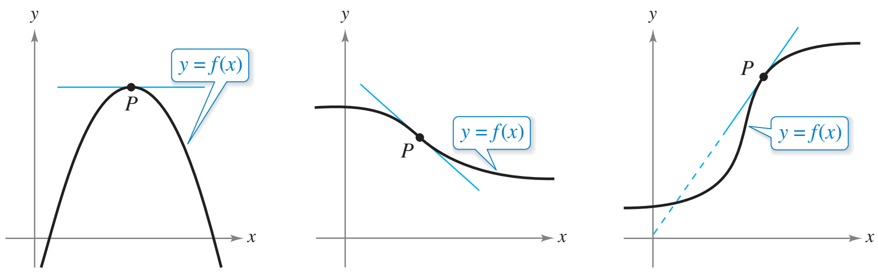
For a general curve, however, the problem is more difficult. For instance, how would you define the tangent lines shown in Figure 2.2? You might say that a line is tangent to a curve at a point P when it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in Figure 2.2, but not for the second. *Or* you might say that a line is tangent to a curve when the line touches or intersects the curve at exactly one point. This definition would work for a circle, but not for more general curves, as the third curve in Figure 2.2 shows.



Tangent line to a circle
Figure 2.1

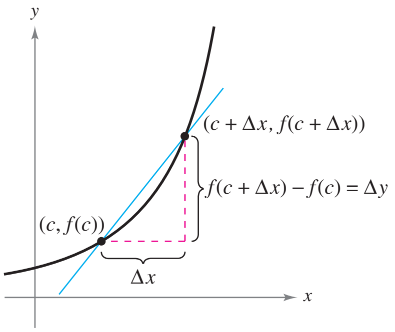
Exploration

Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, graph $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of f at the point $(0, -5)$? Explain your reasoning.



Tangent line to a curve at a point
Figure 2.2

Mary Evans Picture Library/Alamy



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$
Figure 2.3

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the *slope* of the tangent line at point P. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 2.3. If $(c, f(c))$ is the point of tangency and

$$(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, then the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$\boxed{m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.} \qquad \text{Slope of secant line}$$

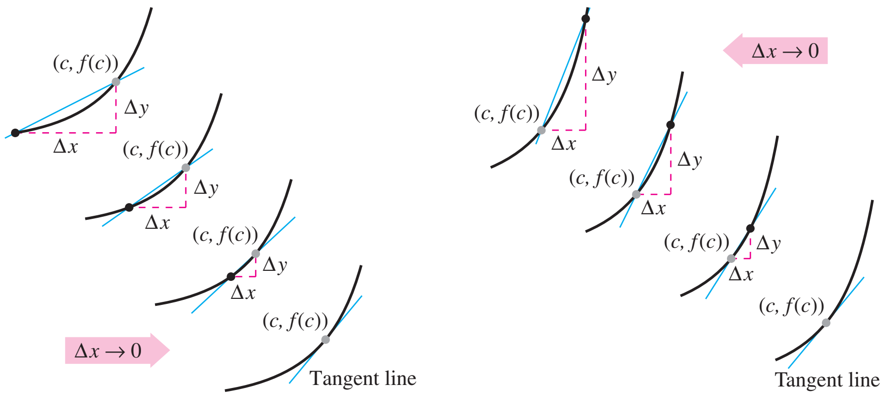
The right-hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x,** and the numerator

$$\Delta y = f(c + \Delta x) - f(c)$$

is the **change in y.**

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.

> **THE TANGENT LINE PROBLEM**
>
> In 1637, mathematician René Descartes stated this about the tangent line problem:
>
> "And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."

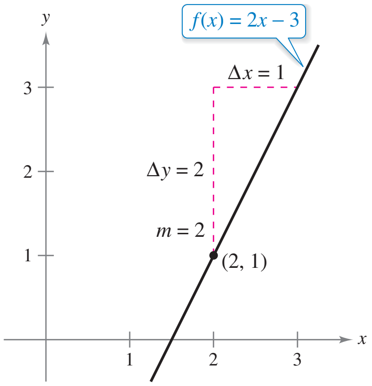


Tangent line approximations
Figure 2.4

> **Definition of Tangent Line with Slope m**
>
> If f is defined on an open interval containing c, and if the limit
>
> $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$
>
> exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

* This use of the word *secant* comes from the Latin *secare*, meaning to cut, and is not a reference to the trigonometric function of the same name.



The slope of f at $(2, 1)$ is $m = 2$.
Figure 2.5

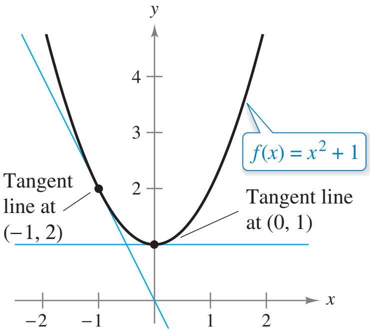
EXAMPLE 1 The Slope of the Graph of a Linear Function

To find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2.5. Notice that the limit definition of the slope of f agrees with the definition of the slope of a line as discussed in Section P.2.

The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the next example.



The slope of f at any point $(c, f(c))$ is $m = 2c$.
Figure 2.6

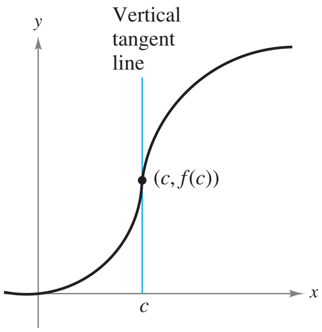
EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure 2.6.

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ can be found as shown below. [Note in the limit process that c is held constant (as Δx approaches 0).]

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2c + \Delta x) \\
&= 2c
\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$.



The graph of f has a vertical tangent line at $(c, f(c))$.
Figure 2.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. When the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

AP* Tips

The definition of derivative is tested primarily on the multiple choice section of the AP Exam. You should master both the standard and alternative definitions and their geometric interpretations.

The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

Definition of the Derivative of a Function

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

REMARK The notation $f'(x)$ is read as "f prime of x."

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point. The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval (a, b)** when it is differentiable at every point in the interval.

In addition to $f'(x)$, other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

FOR FURTHER INFORMATION For more information on the crediting of mathematical discoveries to the first "discoverers," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to *MathArticles.com*.

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "dy, dx." Using limit notation, you can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

EXAMPLE 3 Finding the Derivative by the Limit Process

See LarsonCalculus.com for an interactive version of this type of example.

To find the derivative of $f(x) = x^3 + 2x$, use the definition of the derivative as shown.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2] \\
&= 3x^2 + 2
\end{aligned}$$

REMARK When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

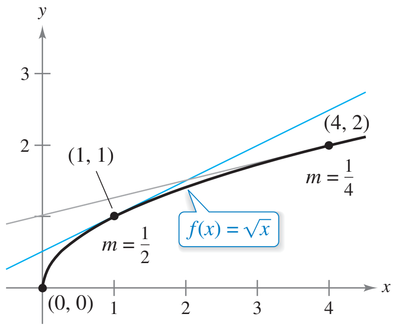
REMARK Remember that the derivative of a function f is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$. Discuss the behavior of f at $(0, 0)$.

Solution Use the procedure for rationalizing numerators, as discussed in Section 1.3.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right) \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}, \quad x > 0
\end{aligned}$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/(2\sqrt{x})$.
Figure 2.8

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$. See Figure 2.8. At the point $(0, 0)$, the slope is undefined. Moreover, the graph of f has a vertical tangent line at $(0, 0)$.

REMARK In many applications, it is convenient to use a variable other than x as the independent variable, as shown in Example 5.

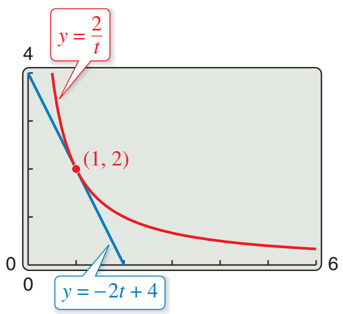
EXAMPLE 5 Finding the Derivative of a Function

See LarsonCalculus.com for an interactive version of this type of example.

Find the derivative with respect to t for the function $y = 2/t$.

Solution Considering $y = f(t)$, you obtain

$$\begin{aligned}
\frac{dy}{dt} &= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} && \text{Definition of derivative} \\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} && f(t + \Delta t) = \frac{2}{t + \Delta t} \text{ and } f(t) = \frac{2}{t} \\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} && \text{Combine fractions in numerator.} \\
&= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{\cancel{\Delta t}(t)(t + \Delta t)} && \text{Divide out common factor of } \Delta t. \\
&= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} && \text{Simplify.} \\
&= -\frac{2}{t^2}. && \text{Evaluate limit as } \Delta t \to 0.
\end{aligned}$$



At the point $(1, 2)$, the line $y = -2t + 4$ is tangent to the graph of $y = 2/t$.
Figure 2.9

TECHNOLOGY A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. Using the point-slope form, you can find that the equation of the tangent line to the graph at $(1, 2)$ is

$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 2.9.

Differentiability and Continuity

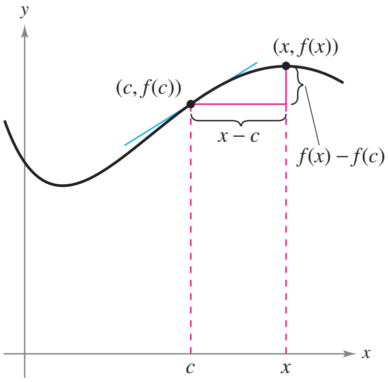
The alternative limit form of the derivative shown below is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$ Alternative form of derivative

provided this limit exists (see Figure 2.10).

REMARK A proof of the equivalence of the alternative form of the derivative is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.



As x approaches c, the secant line approaches the tangent line.
Figure 2.10

Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

and

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval** $[a, b]$ when it is differentiable on (a, b) and when the derivative from the right at a and the derivative from the left at b both exist.

When a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function

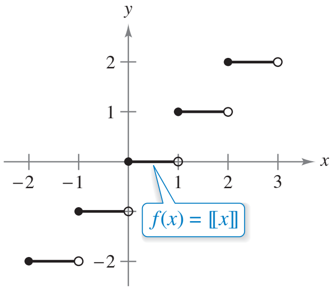
$$f(x) = [\![x]\!]$$

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 2.11). You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty$$ Derivative from the left

and

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0.$$ Derivative from the right



The greatest integer function is not differentiable at $x = 0$ because it is not continuous at $x = 0$.
Figure 2.11

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate this possibility.

EXAMPLE 6 A Graph with a Sharp Turn

See LarsonCalculus.com for an interactive version of this type of example.

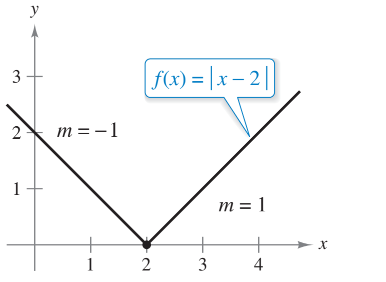
The function $f(x) = |x - 2|$, shown in Figure 2.12, is continuous at $x = 2$. The one-sided limits, however,

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \qquad \text{Derivative from the left}$$

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \qquad \text{Derivative from the right}$$

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point $(2, 0)$.



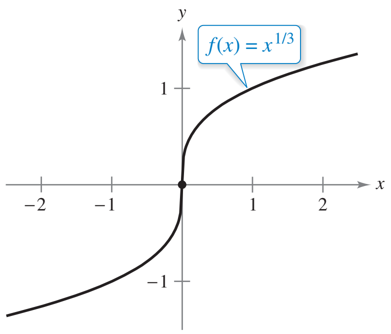
f is not differentiable at $x = 2$ because the derivatives from the left and from the right are not equal.
Figure 2.12

EXAMPLE 7 A Graph with a Vertical Tangent Line

The function $f(x) = x^{1/3}$ is continuous at $x = 0$, as shown in Figure 2.13. However, because the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - 0}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$.



f is not differentiable at $x = 0$ because f has a vertical tangent line at $x = 0$.
Figure 2.13

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x \to c} [f(x) - f(c)] &= \lim_{x \to c} \left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\ &= \left[\lim_{x \to c} (x - c)\right]\left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right] \\ &= (0)[f'(c)] \\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x \to c} f(x) = f(c)$. So, f is continuous at $x = c$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The relationship between continuity and differentiability is summarized below.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability (see Example 6).

TECHNOLOGY Some graphing utilities, such as *Maple*, *Mathematica*, and the *TI-nspire*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

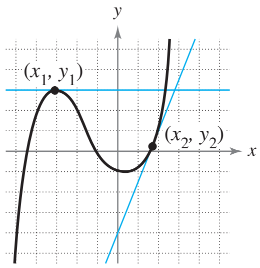
where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

2.1 Exercises

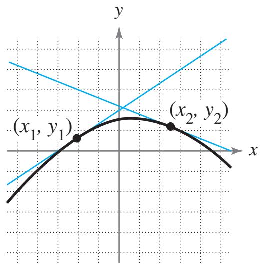
See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Slope **In Exercises 1 and 2, estimate the slope of the graph at the points (x_1, y_1) and (x_2, y_2).**

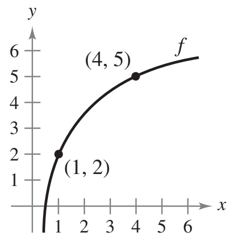
1.



2.



Slopes of Secant Lines **In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to *MathGraphs.com*.**



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol (< or >) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1}$ ▭ $\dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1}$ ▭ $f'(1)$

Finding the Slope of a Tangent Line **In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.**

5. $f(x) = 3 - 5x, \quad (-1, 8)$ **6.** $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

7. $g(x) = x^2 - 9, \quad (2, -5)$ **8.** $f(x) = 5 - x^2, \quad (3, -4)$

9. $f(t) = 3t - t^2, \quad (0, 0)$ **10.** $h(t) = t^2 + 4t, \quad (1, 5)$

Finding the Derivative by the Limit Process **In Exercises 11–24, find the derivative of the function by the limit process.**

11. $f(x) = 7$ **12.** $g(x) = -3$

13. $f(x) = -10x$ **14.** $f(x) = 7x - 3$

15. $h(s) = 3 + \frac{2}{3}s$ **16.** $f(x) = 5 - \frac{2}{3}x$

17. $f(x) = x^2 + x - 3$ **18.** $f(x) = x^2 - 5$

19. $f(x) = x^3 - 12x$ **20.** $f(x) = x^3 + x^2$

21. $f(x) = \dfrac{1}{x - 1}$ **22.** $f(x) = \dfrac{1}{x^2}$

23. $f(x) = \sqrt{x + 4}$ **24.** $f(x) = \dfrac{4}{\sqrt{x}}$

Finding an Equation of a Tangent Line **In Exercises 25–32, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.**

25. $f(x) = x^2 + 3, \quad (-1, 4)$ **26.** $f(x) = x^2 + 2x - 1, \quad (1, 2)$

27. $f(x) = x^3, \quad (2, 8)$ **28.** $f(x) = x^3 + 1, \quad (-1, 0)$

29. $f(x) = \sqrt{x}, \quad (1, 1)$ **30.** $f(x) = \sqrt{x - 1}, \quad (5, 2)$

31. $f(x) = x + \dfrac{4}{x}, \quad (-4, -5)$ **32.** $f(x) = \dfrac{6}{x + 2}, \quad (0, 3)$

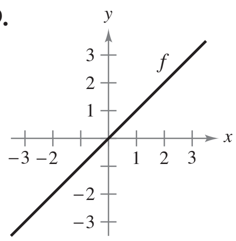
Finding an Equation of a Tangent Line **In Exercises 33–38, find an equation of the line that is tangent to the graph of f *and* parallel to the given line.**

	Function	**Line**
33.	$f(x) = x^2$	$2x - y + 1 = 0$
34.	$f(x) = 2x^2$	$4x + y + 3 = 0$
35.	$f(x) = x^3$	$3x - y + 1 = 0$
36.	$f(x) = x^3 + 2$	$3x - y - 4 = 0$
37.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
38.	$f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

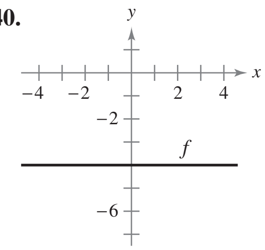
WRITING ABOUT CONCEPTS

Sketching a Derivative **In Exercises 39–44, sketch the graph of f'. Explain how you found your answer.**

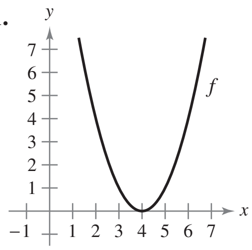
39.



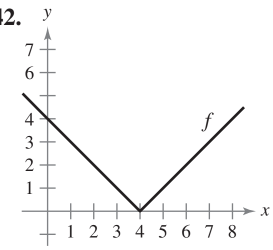
40.



41.

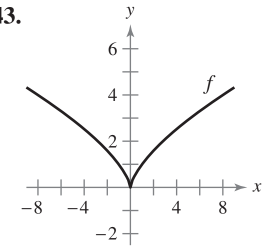


42.

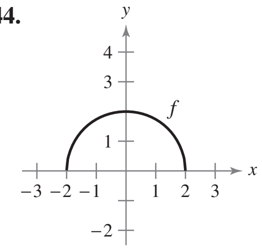


WRITING ABOUT CONCEPTS (continued)

43.



44.



45. Sketching a Graph Sketch a graph of a function whose derivative is always negative. Explain how you found the answer.

46. Sketching a Graph Sketch a graph of a function whose derivative is always positive. Explain how you found the answer.

47. Using a Tangent Line The tangent line to the graph of $y = g(x)$ at the point $(4, 5)$ passes through the point $(7, 0)$. Find $g(4)$ and $g'(4)$.

48. Using a Tangent Line The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

Working Backwards **In Exercises 49–52, the limit represents $f'(c)$ for a function f and a number c. Find f and c.**

49. $\displaystyle\lim_{\Delta x \to 0} \frac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$ **50.** $\displaystyle\lim_{\Delta x \to 0} \frac{(-2 + \Delta x)^3 + 8}{\Delta x}$

51. $\displaystyle\lim_{x \to 6} \frac{-x^2 + 36}{x - 6}$ **52.** $\displaystyle\lim_{x \to 9} \frac{2\sqrt{x} - 6}{x - 9}$

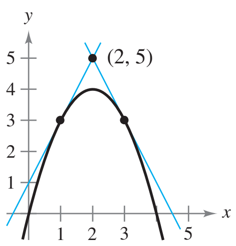
Writing a Function Using Derivatives **In Exercises 53 and 54, identify a function f that has the given characteristics. Then sketch the function.**

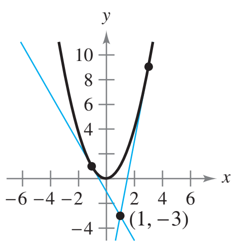
53. $f(0) = 2;\ f'(x) = -3$ for $-\infty < x < \infty$

54. $f(0) = 4;\ f'(0) = 0;\ f'(x) < 0$ for $x < 0;\ f'(x) > 0$ for $x > 0$

Finding an Equation of a Tangent Line **In Exercises 55 and 56, find equations of the two tangent lines to the graph of f that pass through the indicated point.**

55. $f(x) = 4x - x^2$ **56.** $f(x) = x^2$

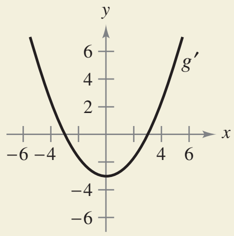




57. Graphical Reasoning Use a graphing utility to graph each function and its tangent lines at $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slopes of tangent lines to the graph of a function at different values of x are always distinct.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

58. HOW DO YOU SEE IT? The figure shows the graph of g'.



(a) $g'(0) = $ ▭ (b) $g'(3) = $ ▭

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

59. Graphical Reasoning Consider the function $f(x) = \frac{1}{2}x^2$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, and $f'(2)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, and $f'(-2)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

60. Graphical Reasoning Consider the function $f(x) = \frac{1}{3}x^3$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

Graphical Reasoning **In Exercises 61 and 62, use a graphing utility to graph the functions f and g in the same viewing window, where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

61. $f(x) = 2x - x^2$ **62.** $f(x) = 3\sqrt{x}$

Approximating a Derivative **In Exercises 63 and 64, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.**

63. $f(x) = x(4 - x)$ **64.** $f(x) = \frac{1}{4}x^3$

Using the Alternative Form of the Derivative **In Exercises 65–74, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).**

65. $f(x) = x^2 - 5,\quad c = 3$ **66.** $g(x) = x^2 - x,\quad c = 1$

67. $f(x) = x^3 + 2x^2 + 1,\quad c = -2$

68. $f(x) = x^3 + 6x, \quad c = 2$

69. $g(x) = \sqrt{|x|}, \quad c = 0$

70. $f(x) = 3/x, \quad c = 4$

71. $f(x) = (x - 6)^{2/3}, \quad c = 6$

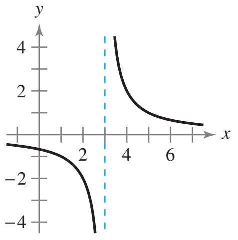
72. $g(x) = (x + 3)^{1/3}, \quad c = -3$

73. $h(x) = |x + 7|, \quad c = -7$

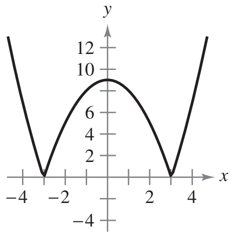
74. $f(x) = |x - 6|, \quad c = 6$

Determining Differentiability **In Exercises 75–80, describe the x-values at which f is differentiable.**

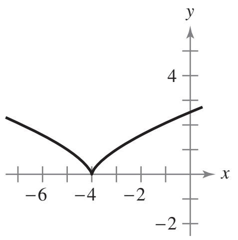
75. $f(x) = \dfrac{2}{x - 3}$



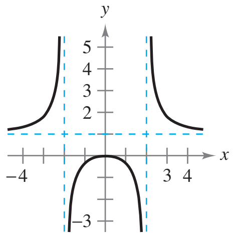
76. $f(x) = |x^2 - 9|$



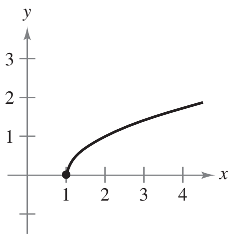
77. $f(x) = (x + 4)^{2/3}$



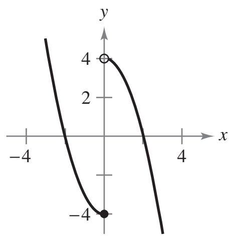
78. $f(x) = \dfrac{x^2}{x^2 - 4}$



79. $f(x) = \sqrt{x - 1}$



80. $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$



Graphical Reasoning **In Exercises 81–84, use a graphing utility to graph the function and find the x-values at which f is differentiable.**

81. $f(x) = |x - 5|$

82. $f(x) = \dfrac{4x}{x - 3}$

83. $f(x) = x^{2/5}$

84. $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \le 1 \\ x^2 - 2x, & x > 1 \end{cases}$

Determining Differentiability **In Exercises 85–88, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?**

85. $f(x) = |x - 1|$

86. $f(x) = \sqrt{1 - x^2}$

87. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

88. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

Determining Differentiability **In Exercises 89 and 90, determine whether the function is differentiable at $x = 2$.**

89. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

90. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

91. Graphical Reasoning A line with slope m passes through the point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the function d in part (a). Based on the graph, is the function differentiable at every value of m? If not, where is it not differentiable?

92. Conjecture Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify a pattern between f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

True or False? **In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

93. The slope of the tangent line to the differentiable function f at the point $(2, f(2))$ is

$$\frac{f(2 + \Delta x) - f(2)}{\Delta x}.$$

94. If a function is continuous at a point, then it is differentiable at that point.

95. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

96. If a function is differentiable at a point, then it is continuous at that point.

97. Differentiability and Continuity Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}.$$

Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

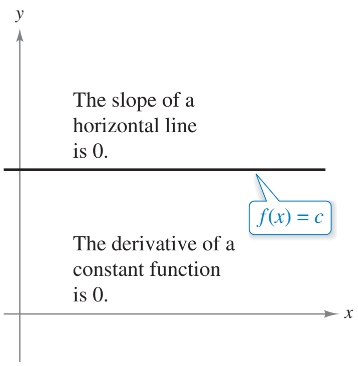
98. Writing Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Use derivatives to find rates of change.

The Constant Rule

In Section 2.1, you used the limit definition to find derivatives. In this and the next two sections, you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.
Figure 2.14

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$ (See Figure 2.14.)

Proof Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Using the Constant Rule

	Function	Derivative
a.	$y = 7$	$dy/dx = 0$
b.	$f(x) = 0$	$f'(x) = 0$
c.	$s(t) = -3$	$s'(t) = 0$
d.	$y = k\pi^2$, k is constant	$y' = 0$

Exploration

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each function. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$
$$(x + \Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$
$$(x + \Delta x)^5 = x^5 + 5x^4\Delta x + 10x^3(\Delta x)^2 + 10x^2(\Delta x)^3 + 5x(\Delta x)^4 + (\Delta x)^5$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

REMARK From Example 7 in Section 2.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

Proof If n is a positive integer greater than 1, then the binomial expansion produces

$$\begin{aligned}
\frac{d}{dx}[x^n] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right] \\
&= nx^{n-1} + 0 + \cdots + 0 \\
&= nx^{n-1}.
\end{aligned}$$

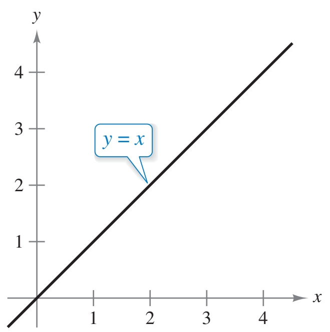
This proves the case for which n is a positive integer greater than 1. It is left to you to prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which n is a negative integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which n is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of n.)

See LarsonCalculus.com for Bruce Edwards's video of this proof.

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\boxed{\frac{d}{dx}[x] = 1.}$$

Power Rule when $n = 1$



The slope of the line $y = x$ is 1.
Figure 2.15

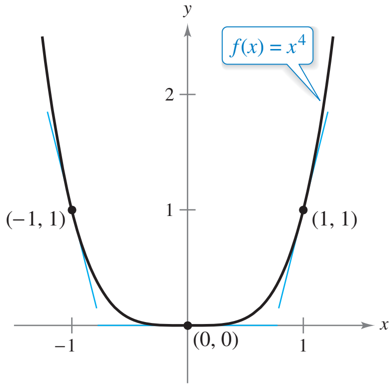
This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.

EXAMPLE 2 Using the Power Rule

Function	**Derivative**
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$
c. $y = \dfrac{1}{x^2}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given:		Rewrite:		Differentiate:		Simplify:
$y = \dfrac{1}{x^2}$	⇨	$y = x^{-2}$	⇨	$\dfrac{dy}{dx} = (-2)x^{-3}$	⇨	$\dfrac{dy}{dx} = -\dfrac{2}{x^3}$



Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.
Figure 2.16

EXAMPLE 3 Finding the Slope of a Graph

See LarsonCalculus.com for an interactive version of this type of example.

Find the slope of the graph of

$$f(x) = x^4$$

for each value of x.

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$

Solution The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.
b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.
c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

See Figure 2.16.

EXAMPLE 4 Finding an Equation of a Tangent Line

See LarsonCalculus.com for an interactive version of this type of example.

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

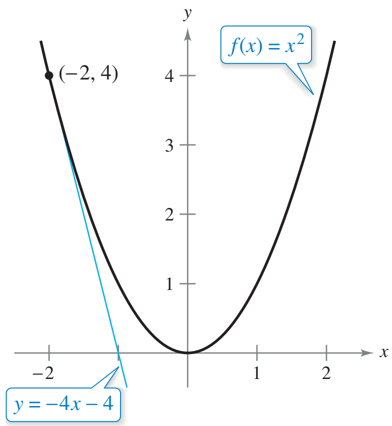
To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 4 = -4[x - (-2)] \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = -4x - 4. \qquad \text{Simplify.}$$

See Figure 2.17.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 2.17

The Constant Multiple Rule

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

Proof

$$
\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} c\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] \\
&= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right] && \text{Apply Theorem 1.2.} \\
&= cf'(x)
\end{aligned}
$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even when the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[\ f(x)] = cf'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right] = \left(\frac{1}{c}\right)\frac{d}{dx}[\ f(x)] = \left(\frac{1}{c}\right)f'(x)$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	**Derivative**
a. $y = 5x^3$	$\frac{dy}{dx} = \frac{d}{dx}[5x^3] = 5\frac{d}{dx}[x^3] = 5(3)x^2 = 15x^2$
b. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
c. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
d. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
e. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
f. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

REMARK Before differentiating functions involving radicals, rewrite the function with rational exponents.

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 **Using Parentheses When Differentiating**

	Original Function	Rewrite	Differentiate	Simplify
a.	$y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b.	$y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c.	$y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d.	$y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

The Sum and Difference Rules

THEOREM 2.5 The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Proof A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

REMARK In Example 7(c), note that before differentiating,

$$\frac{3x^2 - x + 1}{x}$$

was rewritten as

$$3x - 1 + \frac{1}{x}.$$

EXAMPLE 7 **Using the Sum and Difference Rules**

	Function	Derivative
a.	$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b.	$g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$
c.	$y = \dfrac{3x^2 - x + 1}{x} = 3x - 1 + \dfrac{1}{x}$	$y' = 3 - \dfrac{1}{x^2} = \dfrac{3x^2 - 1}{x^2}$

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

Derivatives of the Sine and Cosine Functions

In Section 1.3, you studied the limits

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0.$$

These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

THEOREM 2.6 Derivatives of Sine and Cosine Functions

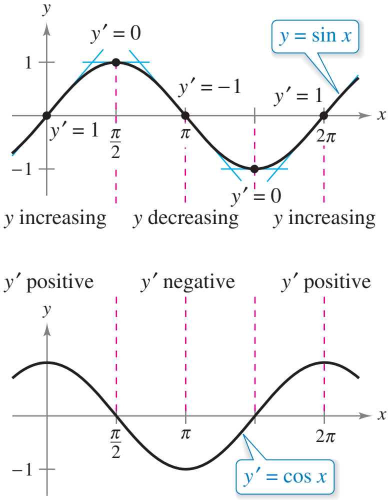
$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Proof Here is a proof of the first rule. (The proof of the second rule is left as an exercise [see Exercise 118].)

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

This differentiation rule is shown graphically in Figure 2.18. Note that for each x, the *slope* of the sine curve is equal to the value of the cosine.

See LarsonCalculus.com for Bruce Edwards's video of this proof.



The derivative of the sine function is the cosine function.

Figure 2.18

EXAMPLE 8 Derivatives Involving Sines and Cosines

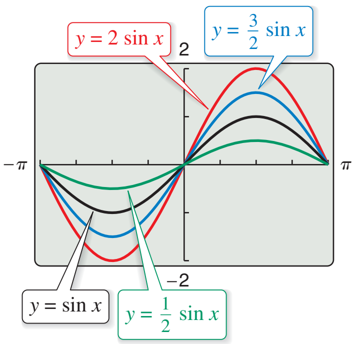
See LarsonCalculus.com for an interactive version of this type of example.

	Function	**Derivative**
a.	$y = 2 \sin x$	$y' = 2 \cos x$
b.	$y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c.	$y = x + \cos x$	$y' = 1 - \sin x$
d.	$\cos x - \dfrac{\pi}{3} \sin x$	$-\sin x - \dfrac{\pi}{3} \cos x$

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.



$$\frac{d}{dx}[a \sin x] = a \cos x$$

Figure 2.19

AP* Tips

The AP Exam requires that you have equal facility with using a function's equation, table of values, or graph in finding the average velocity or average rate of change.

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change, sometimes referred to as instantaneous rates of change, occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount

$$\Delta s = s(t + \Delta t) - s(t)$$

then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}.$$

Average velocity

EXAMPLE 9 Finding Average Velocity of a Falling Object

Time-lapse photograph of a free-falling billiard ball

A billiard ball is dropped from a height of 100 feet. The ball's height s at time t is the position function

$$s = -16t^2 + 100$$

Position function

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

Solution

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

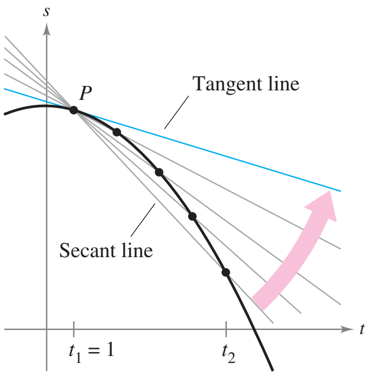
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of $s(1.5) = -16(1.5)^2 + 100 = 64$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of $s(1.1) = -16(1.1)^2 + 100 = 80.64$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 2.20

Suppose that in Example 9, you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the **velocity** of the object at time t is

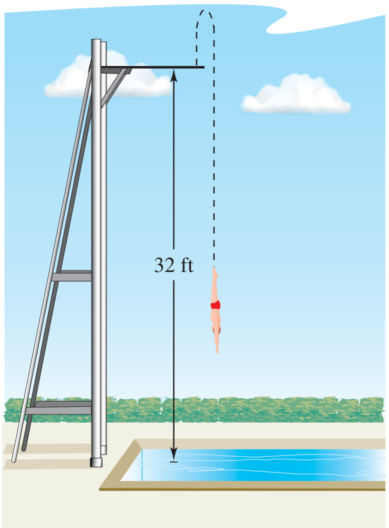
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.
Figure 2.21

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). Because the initial velocity of the diver is 16 feet per second, the position of the diver is

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$ Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$ Factor.

$$t = -1 \text{ or } 2$$ Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative

$$s'(t) = -32t + 16.$$ Velocity function

So, the velocity at time $t = 2$ is

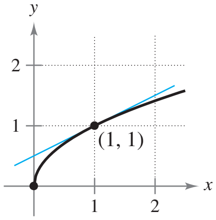
$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

2.2 Exercises

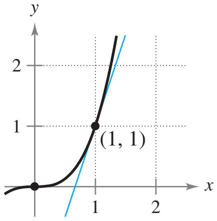
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Slope In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point $(1, 1)$. Verify your answer analytically. To print an enlarged copy of the graph, go to *MathGraphs.com*.

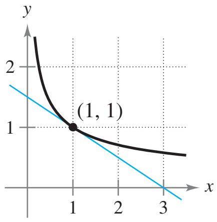
1. (a) $y = x^{1/2}$



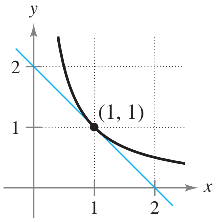
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



Finding a Derivative In Exercises 3–24, use the rules of differentiation to find the derivative of the function.

3. $y = 12$

4. $f(x) = -9$

5. $y = x^7$

6. $y = x^{12}$

7. $y = \dfrac{1}{x^5}$

8. $y = \dfrac{3}{x^7}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[4]{x}$

11. $f(x) = x + 11$

12. $g(x) = 6x + 3$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 - 3t + 1$

15. $g(x) = x^2 + 4x^3$

16. $y = 4x - 3x^3$

17. $s(t) = t^3 + 5t^2 - 3t + 8$

18. $y = 2x^3 + 6x^2 - 1$

19. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

20. $g(t) = \pi\cos t$

21. $y = x^2 - \frac{1}{2}\cos x$

22. $y = 7 + \sin x$

23. $y = \dfrac{1}{x} - 3\sin x$

24. $y = \dfrac{5}{(2x)^3} + 2\cos x$

Rewriting a Function Before Differentiating In Exercises 25–30, complete the table to find the derivative of the function.

Original Function	Rewrite	Differentiate	Simplify
25. $y = \dfrac{5}{2x^2}$			
26. $y = \dfrac{3}{2x^4}$			
27. $y = \dfrac{6}{(5x)^3}$			
28. $y = \dfrac{\pi}{(3x)^2}$			
29. $y = \dfrac{\sqrt{x}}{x}$			
30. $y = \dfrac{4}{x^{-3}}$			

Finding the Slope of a Graph In Exercises 31–38, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

Function	Point
31. $f(x) = \dfrac{8}{x^2}$	$(2, 2)$
32. $f(t) = 2 - \dfrac{4}{t}$	$(4, 1)$
33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34. $y = 2x^4 - 3$	$(1, -1)$
35. $y = (4x + 1)^2$	$(0, 1)$
36. $f(x) = 2(x - 4)^2$	$(2, 8)$
37. $f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38. $g(t) = -2\cos t + 5$	$(\pi, 7)$

Finding a Derivative In Exercises 39–52, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$

40. $f(x) = x^3 - 2x + 3x^{-3}$

41. $g(t) = t^2 - \dfrac{4}{t^3}$

42. $f(x) = 8x + \dfrac{3}{x^2}$

43. $f(x) = \dfrac{4x^3 + 3x^2}{x}$

44. $f(x) = \dfrac{2x^4 - x}{x^3}$

45. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

46. $h(x) = \dfrac{4x^3 + 2x + 5}{x}$

47. $y = x(x^2 + 1)$

48. $y = x^2(2x^2 - 3x)$

49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

50. $f(t) = t^{2/3} - t^{1/3} + 4$

51. $f(x) = 6\sqrt{x} + 5\cos x$

52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

Finding an Equation of a Tangent Line In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	Point
53. $y = x^4 - 3x^2 + 2$	$(1, 0)$
54. $y = x^3 - 3x$	$(2, 2)$
55. $f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
56. $y = (x - 2)(x^2 + 3x)$	$(1, -4)$

Horizontal Tangent Line **In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.**

57. $y = x^4 - 2x^2 + 3$

58. $y = x^3 + x$

59. $y = \dfrac{1}{x^2}$

60. $y = x^2 + 9$

61. $y = x + \sin x, \quad 0 \le x < 2\pi$

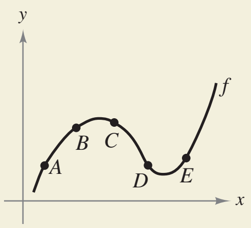
62. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

Finding a Value **In Exercises 63–68, find k such that the line is tangent to the graph of the function.**

	Function	Line
63.	$f(x) = k - x^2$	$y = -6x + 1$
64.	$f(x) = kx^2$	$y = -2x + 3$
65.	$f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
66.	$f(x) = k\sqrt{x}$	$y = x + 4$
67.	$f(x) = kx^3$	$y = x + 1$
68.	$f(x) = kx^4$	$y = 4x - 1$

69. Sketching a Graph Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

70. HOW DO YOU SEE IT? Use the graph of f to answer each question. To print an enlarged copy of the graph, go to *MathGraphs.com*.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

WRITING ABOUT CONCEPTS

Exploring a Relationship **In Exercises 71–74, the relationship between f and g is given. Explain the relationship between f' and g'.**

71. $g(x) = f(x) + 6$

72. $g(x) = 2f(x)$

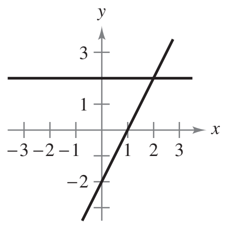
73. $g(x) = -5f(x)$

74. $g(x) = 3f(x) - 1$

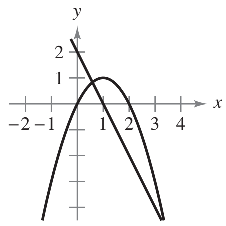
WRITING ABOUT CONCEPTS (continued)

A Function and Its Derivative **In Exercises 75 and 76, the graphs of a function f and its derivative f' are shown in the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

75.



76.



77. Finding Equations of Tangent Lines Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

78. Tangent Lines Show that the graphs of the two equations

$$y = x \quad \text{and} \quad y = \frac{1}{x}$$

have tangent lines that are perpendicular to each other at their point of intersection.

79. Tangent Line Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

80. Tangent Line Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

Finding an Equation of a Tangent Line **In Exercises 81 and 82, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation**

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

81. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

82. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

83. Linear Approximation Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4 - \tfrac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.

84. Linear Approximation Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.

85. Linear Approximation Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to graph f. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T in the same set of coordinate axes. Note that T is a good approximation of f when x is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

86. Linear Approximation Repeat Exercise 85 for the function $f(x) = x^3$, where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 85.

True or False? **In Exercises 87–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

88. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

89. If $y = \pi^2$, then $dy/dx = 2\pi$.

90. If $y = x/\pi$, then $dy/dx = 1/\pi$.

91. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

92. If $f(x) = \dfrac{1}{x^n}$, then $f'(x) = \dfrac{1}{nx^{n-1}}$.

Finding Rates of Change **In Exercises 93–96, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.**

93. $f(t) = 4t + 5,\quad [1, 2]$

94. $f(t) = t^2 - 7,\quad [3, 3.1]$

95. $f(x) = \dfrac{-1}{x},\quad [1, 2]$

96. $f(x) = \sin x,\quad \left[0, \dfrac{\pi}{6}\right]$

Vertical Motion **In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

97. A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

98. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

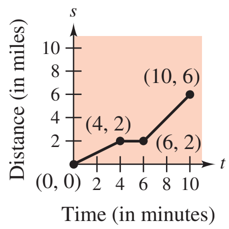
Vertical Motion **In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

99. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

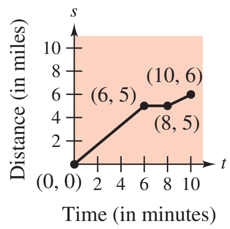
100. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. The splash is seen 5.6 seconds after the stone is dropped. What is the height of the building?

Think About It **In Exercises 101 and 102, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

101.

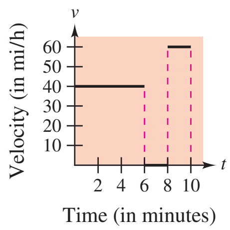


102.

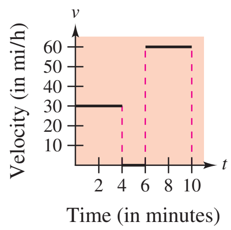


Think About It **In Exercises 103 and 104, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute trip to work. Make a sketch of the corresponding position function.**

103.



104.

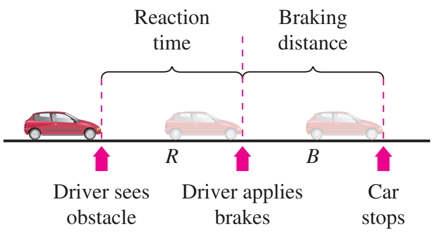


105. Volume The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 6$ centimeters.

106. Area The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 6$ meters.

107. Modeling Data

The stopping distance of an automobile, on dry, level pavement, traveling at a speed v (in kilometers per hour) is the distance R (in meters) the car travels during the reaction time of the driver plus the distance B (in meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



Speed, v	20	40	60	80	100
Reaction Time Distance, R	8.3	16.7	25.0	33.3	41.7
Braking Time Distance, B	2.3	9.0	20.2	35.8	55.9

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time distance R.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking time distance B.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

108. Fuel Cost A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$3.48 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
dC/dx							

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon, or the driver of a car that gets 35 miles per gallon? Explain.

109. Velocity Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\frac{1}{2}at^2 + c.$$

110. Inventory Management The annual inventory cost C for a manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

111. Finding an Equation of a Parabola Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

112. Proof Let (a, b) be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

113. Finding Equation(s) of Tangent Line(s) Find the equation(s) of the tangent line(s) to the graph of the curve $y = x^3 - 9x$ through the point $(1, -9)$ not on the graph.

114. Finding Equation(s) of Tangent Line(s) Find the equation(s) of the tangent line(s) to the graph of the parabola $y = x^2$ through the given point not on the graph.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

Making a Function Differentiable **In Exercises 115 and 116, find a and b such that f is differentiable everywhere.**

115. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

116. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

117. Determining Differentiability Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

118. Proof Prove that $\frac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to *MathArticles.com*.

PUTNAM EXAM CHALLENGE

119. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x + n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

2.3 Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 2.2, you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

REMARK A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0} \left[g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} f(x+\Delta x) \cdot \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x) \cdot \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

REMARK The proof of the Product Rule for products of more than two factors is left as an exercise (see Exercise 137).

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

So, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x(-\sin x) \\
&= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. *(Source: The History of Mathematics by David M. Burton)*

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of

$$f(x) = 3x - 2x^2$$

and

$$g(x) = 5 + 4x$$

with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned} h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}}\overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.} \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= (12x - 8x^2) + (15 - 8x - 16x^2) \\ &= -24x^2 + 4x + 15 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned} D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\ &= -24x^2 + 4x + 15. \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\begin{aligned} \frac{d}{dx}[3x^2 \sin x] &= 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] && \text{Apply Product Rule.} \\ &= 3x^2 \cos x + (\sin x)(6x) \\ &= 3x^2 \cos x + 6x \sin x \\ &= 3x(x \cos x + 2 \sin x) \end{aligned}$$

REMARK In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \overbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}^{\text{Product Rule}} - \overbrace{2\frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}} \\ &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\ &= -2x \sin x \end{aligned}$$

The Quotient Rule

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

REMARK From the Quotient Rule, you can see that the derivative of a quotient is not (in general) the quotient of the derivatives.

AP* Tips

Extremely complex examples of the product and quotient rules do not appear on the AP Exam. The harder problems here will help you to master and remember the concept.

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}$$

$$= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

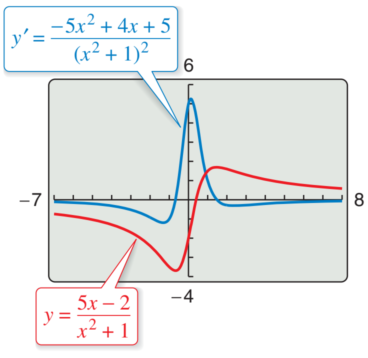
$$= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Note that $\lim_{\Delta x \to 0} g(x + \Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 2.22

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

Solution

$$\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] = \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} \qquad \text{Apply Quotient Rule.}$$

$$= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}$$

$$= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2}$$

$$= \frac{-5x^2+4x+5}{(x^2+1)^2}$$

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$f(x) = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by } x.$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

Next, apply the Quotient Rule.

$$f'(x) = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Quotient Rule}$$

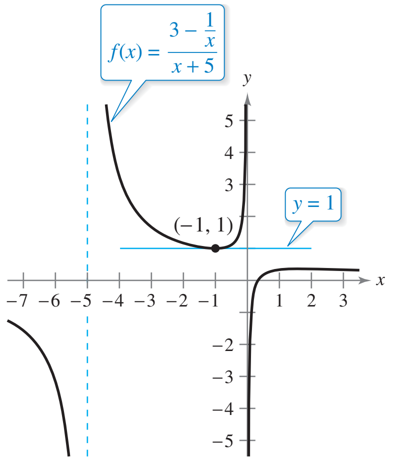
$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 2.23

Not every quotient needs to be differentiated by the Quotient Rule. For instance, each quotient in the next example can be considered as the product of a constant times a function of x. In such cases, it is more convenient to use the Constant Multiple Rule.

REMARK To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

EXAMPLE 6 Using the Constant Multiple Rule

	Original Function	Rewrite	Differentiate	Simplify
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

In Section 2.2, the Power Rule was proved only for the case in which the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Power Rule: Negative Integer Exponents

If n is a negative integer, then there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right] \\ &= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule} \\ &= \frac{0 - kx^{k-1}}{x^{2k}} \\ &= -kx^{-k-1} \\ &= nx^{n-1}. && n = -k\end{aligned}$$

So, the Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which n is any rational number.

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

REMARK In the proof of Theorem 2.9, note the use of the trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

and

$$\sec x = \frac{1}{\cos x}.$$

These trigonometric identities and others are listed in Appendix C and on the formula cards for this text.

Proof Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 87).

EXAMPLE 8 Differentiating Trigonometric Functions

See LarsonCalculus.com for an interactive version of this type of example.

Function	Derivative
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$
	$= (\sec x)(1 + x \tan x)$

EXAMPLE 9 Different Forms of a Derivative

REMARK Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

Differentiate both forms of

$$y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x.$$

Solution

First form:

$$y = \frac{1 - \cos x}{\sin x}$$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x} \qquad \sin^2 x + \cos^2 x = 1$$

Second form:

$$y = \csc x - \cot x$$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= \csc^2 x - \csc x \cot x.$$

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ After Differentiating	$f'(x)$ After Simplifying
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 6	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

The function $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as shown below.

First derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

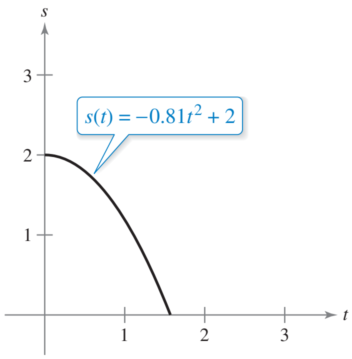
REMARK The second derivative of a function is the derivative of the first derivative of the function.

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds, as shown in the figure at the right. What is the ratio of Earth's gravitational force to the moon's?



Solution To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \qquad \text{Position function}$$
$$s'(t) = -1.62t \qquad \text{Velocity function}$$
$$s''(t) = -1.62 \qquad \text{Acceleration function}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$$

NASA

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.0.$$

2.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using the Product Rule In Exercises 1–6, use the Product Rule to find the derivative of the function.

1. $g(x) = (x^2 + 3)(x^2 - 4x)$
2. $y = (3x - 4)(x^3 + 5)$
3. $h(t) = \sqrt{t}(1 - t^2)$
4. $g(s) = \sqrt{s}(s^2 + 8)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

Using the Quotient Rule In Exercises 7–12, use the Quotient Rule to find the derivative of the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{3t^2 - 1}{2t + 5}$
9. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$
10. $f(x) = \dfrac{x^2}{2\sqrt{x} + 1}$
11. $g(x) = \dfrac{\sin x}{x^2}$
12. $f(t) = \dfrac{\cos t}{t^3}$

Finding and Evaluating a Derivative In Exercises 13–18, find $f'(x)$ and $f'(c)$.

Function	Value of c
13. $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$	$c = 0$
14. $y = (x^2 - 3x + 2)(x^3 + 1)$	$c = 2$
15. $f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16. $f(x) = \dfrac{x - 4}{x + 4}$	$c = 3$
17. $f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18. $f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

Using the Constant Multiple Rule In Exercises 19–24, complete the table to find the derivative of the function without using the Quotient Rule.

Function	Rewrite	Differentiate	Simplify
19. $y = \dfrac{x^2 + 3x}{7}$			
20. $y = \dfrac{5x^2 - 3}{4}$			
21. $y = \dfrac{6}{7x^2}$			
22. $y = \dfrac{10}{3x^3}$			
23. $y = \dfrac{4x^{3/2}}{x}$			
24. $y = \dfrac{2x}{x^{1/3}}$			

Finding a Derivative In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 + 3)^3$
33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$
36. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

Finding a Derivative of a Trigonometric Function In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x^3}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 6 \csc t$
46. $h(x) = \dfrac{1}{x} - 12 \sec x$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

Finding a Derivative Using Technology In Exercises 55–58, use a computer algebra system to find the derivative of the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

Evaluating a Derivative **In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.**

Function	Point
59. $y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62. $f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

Finding an Equation of a Tangent Line **In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.**

63. $f(x) = (x^3 + 4x - 1)(x - 2),\quad (1, -4)$

64. $f(x) = (x - 2)(x^2 + 4),\quad (1, -5)$

65. $f(x) = \dfrac{x}{x + 4},\quad (-5, 5)$

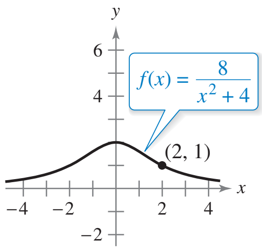
66. $f(x) = \dfrac{x + 3}{x - 3},\quad (4, 7)$

67. $f(x) = \tan x,\quad \left(\dfrac{\pi}{4}, 1\right)$

68. $f(x) = \sec x,\quad \left(\dfrac{\pi}{3}, 2\right)$

Famous Curves **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

69.



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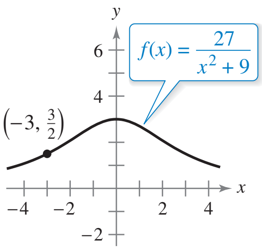

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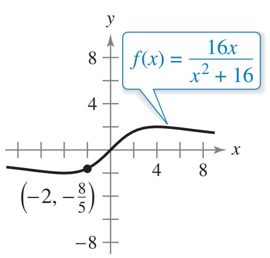

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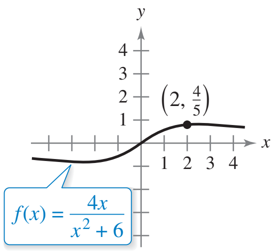


Horizontal Tangent Line **In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.**

73. $f(x) = \dfrac{2x - 1}{x^2}$

74. $f(x) = \dfrac{x^2}{x^2 + 1}$

75. $f(x) = \dfrac{x^2}{x - 1}$

76. $f(x) = \dfrac{x - 4}{x^2 - 7}$

77. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

78. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

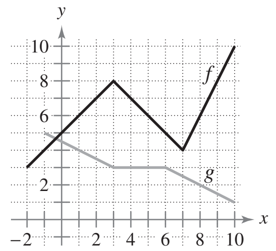
Exploring a Relationship **In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explain the relationship between f and g.**

79. $f(x) = \dfrac{3x}{x + 2},\quad g(x) = \dfrac{5x + 4}{x + 2}$

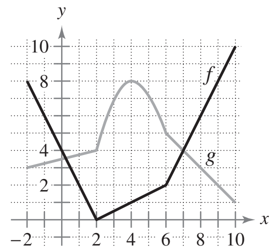
80. $f(x) = \dfrac{\sin x - 3x}{x},\quad g(x) = \dfrac{\sin x + 2x}{x}$

Evaluating Derivatives **In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.**

81. (a) Find $p'(1)$.
(b) Find $q'(4)$.



82. (a) Find $p'(4)$.
(b) Find $q'(7)$.



83. Area The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

84. Volume The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

85. Inventory Replenishment The ordering and transportation cost C for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right),\quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

86. Population Growth A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

87. Proof Prove the following differentiation rules.

(a) $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$

88. Rate of Change Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

89. Modeling Data The table shows the health care expenditures h (in billions of dollars) in the United States and the population p (in millions) of the United States for the years 2004 through 2009. The year is represented by t, with $t = 4$ corresponding to 2004. *(Source: U.S. Centers for Medicare & Medicaid Services and U.S. Census Bureau)*

Year, t	4	5	6	7	8	9
h	1773	1890	2017	2135	2234	2330
p	293	296	299	302	305	307

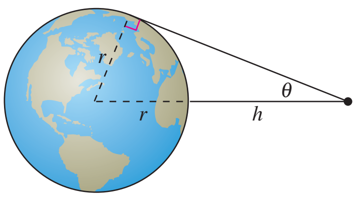
(a) Use a graphing utility to find linear models for the health care expenditures $h(t)$ and the population $p(t)$.

(b) Use a graphing utility to graph each model found in part (a).

(c) Find $A = h(t)/p(t)$, then graph A using a graphing utility. What does this function represent?

(d) Find and interpret $A'(t)$ in the context of these data.

90. Satellites When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface, and let r represent Earth's radius.



(a) Show that $h = r(\csc \theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30°$. (Assume $r = 3960$ miles.)

Finding a Second Derivative **In Exercises 91–98, find the second derivative of the function.**

91. $f(x) = x^4 + 2x^3 - 3x^2 - x$

92. $f(x) = 4x^5 - 2x^3 + 5x^2$

93. $f(x) = 4x^{3/2}$

94. $f(x) = x^2 + 3x^{-3}$

95. $f(x) = \dfrac{x}{x-1}$

96. $f(x) = \dfrac{x^2 + 3x}{x - 4}$

97. $f(x) = x \sin x$

98. $f(x) = \sec x$

Finding a Higher-Order Derivative **In Exercises 99–102, find the given higher-order derivative.**

99. $f'(x) = x^2, \quad f''(x)$

100. $f''(x) = 2 - \dfrac{2}{x}, \quad f'''(x)$

101. $f'''(x) = 2\sqrt{x}, \quad f^{(4)}(x)$

102. $f^{(4)}(x) = 2x + 1, \quad f^{(6)}(x)$

Using Relationships **In Exercises 103–106, use the given information to find $f'(2)$.**

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

103. $f(x) = 2g(x) + h(x)$

104. $f(x) = 4 - h(x)$

105. $f(x) = \dfrac{g(x)}{h(x)}$

106. $f(x) = g(x)h(x)$

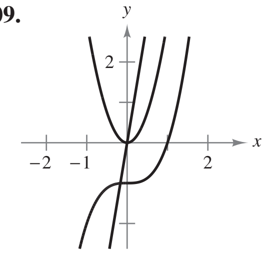
WRITING ABOUT CONCEPTS

107. Sketching a Graph Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$. Explain how you found your answer.

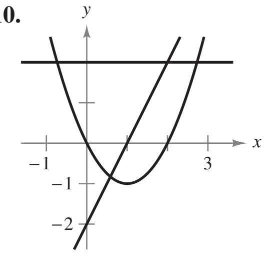
108. Sketching a Graph Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x. Explain how you found your answer.

Identifying Graphs **In Exercises 109 and 110, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

109.

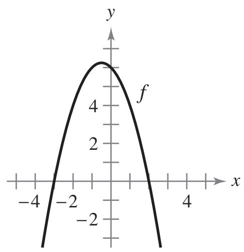


110.

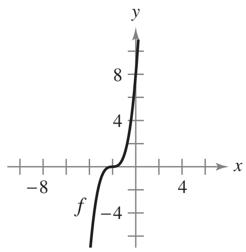


Sketching Graphs **In Exercises 111–114, the graph of f is shown. Sketch the graphs of f' and f''. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

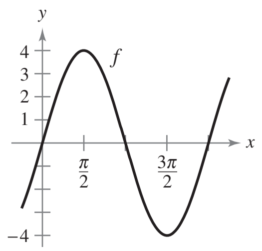
111.



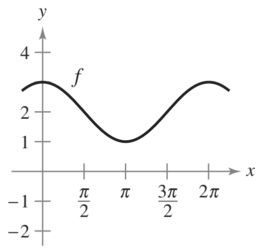
112.



113.



114.



115. Acceleration The velocity of an object in meters per second is

$v(t) = 36 - t^2$

for $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

116. Acceleration The velocity of an automobile starting from rest is

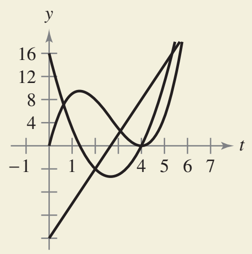
$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

117. Stopping Distance A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

118. HOW DO YOU SEE IT? The figure shows the graphs of the position, velocity, and acceleration functions of a particle.



(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

Finding a Pattern In Exercises 119 and 120, develop a general rule for $f^{(n)}(x)$ given $f(x)$.

119. $f(x) = x^n$

120. $f(x) = \dfrac{1}{x}$

121. Finding a Pattern Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results of part (a) to write a general rule for $f^{(n)}(x)$.

122. Finding a Pattern Develop a general rule for $[xf(x)]^{(n)}$, where f is a differentiable function of x.

Finding a Pattern In Exercises 123 and 124, find the derivatives of the function f for $n = 1, 2, 3,$ and 4. Use the results to write a general rule for $f'(x)$ in terms of n.

123. $f(x) = x^n \sin x$

124. $f(x) = \dfrac{\cos x}{x^n}$

Differential Equations In Exercises 125–128, verify that the function satisfies the differential equation.

	Function	Differential Equation
125.	$y = \dfrac{1}{x},\ x > 0$	$x^3 y'' + 2x^2 y' = 0$
126.	$y = 2x^3 - 6x + 10$	$-y''' - xy'' - 2y' = -24x^2$
127.	$y = 2 \sin x + 3$	$y'' + y = 3$
128.	$y = 3 \cos x + \sin x$	$y'' + y = 0$

True or False? In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

129. If $y = f(x)g(x)$, then $\dfrac{dy}{dx} = f'(x)g'(x)$.

130. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $\dfrac{d^5y}{dx^5} = 0$.

131. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

132. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

133. The second derivative represents the rate of change of the first derivative.

134. If the velocity of an object is constant, then its acceleration is zero.

135. Absolute Value Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist? (*Hint:* Rewrite the function as a piecewise function and then differentiate each part.)

136. Think About It Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$ (b) $fg'' + f''g = (fg)''$

137. Proof Use the Product Rule twice to prove that if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

2.4 The Chain Rule

- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a trigonometric function using the Chain Rule.**

The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

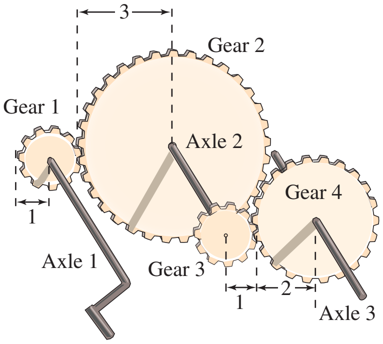
Without the Chain Rule	**With the Chain Rule**
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

EXAMPLE 1 **The Derivative of a Composite Function**

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 2.24

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\begin{aligned}\frac{dy}{dx} &= \boxed{\begin{matrix}\text{Rate of change of first axle}\\ \text{with respect to second axle}\end{matrix}} \cdot \boxed{\begin{matrix}\text{Rate of change of second axle}\\ \text{with respect to third axle}\end{matrix}}\\ &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 3 \cdot 2\\ &= 6\\ &= \boxed{\begin{matrix}\text{Rate of change of first axle}\\ \text{with respect to third axle}\end{matrix}}.\end{aligned}$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

Exploration

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x+1}$

b. $(x+2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated in the next theorem.

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Proof Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs when there are values of x, other than c, such that

$$g(x) = g(c).$$

Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x)$ approaches $g(c)$ as x approaches c.

REMARK The alternative limit form of the derivative was given at the end of Section 2.1.

$$\begin{aligned}
h'(c) &= \lim_{x\to c}\frac{f(g(x)) - f(g(c))}{x-c} && \text{Alternative form of derivative}\\
&= \lim_{x\to c}\left[\frac{f(g(x)) - f(g(c))}{x-c}\cdot\frac{g(x)-g(c)}{g(x)-g(c)}\right], && g(x)\neq g(c)\\
&= \lim_{x\to c}\left[\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\cdot\frac{g(x)-g(c)}{x-c}\right]\\
&= \left[\lim_{x\to c}\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\right]\left[\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\right]\\
&= f'(g(c))g'(c)
\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

AP* Tips

Very complex chain rule examples do not appear on the AP Exam. However, forgetting the chain rule is a frequent cause of loss of credit on a problem. The difficult problems here will help reinforce the concept.

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u)\cdot u'$$

EXAMPLE 2 **Decomposition of a Composite Function**

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

EXAMPLE 3 **Using the Chain Rule**

Find dy/dx for

$$y = (x^2 + 1)^3.$$

Solution For this function, you can consider the inside function to be $u = x^2 + 1$ and the outer function to be $y = u^3$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2+1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2+1)^2.$$

REMARK You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

Proof Because $y = [u(x)]^n = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$
$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = nu^{n-1}$, and it follows that

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

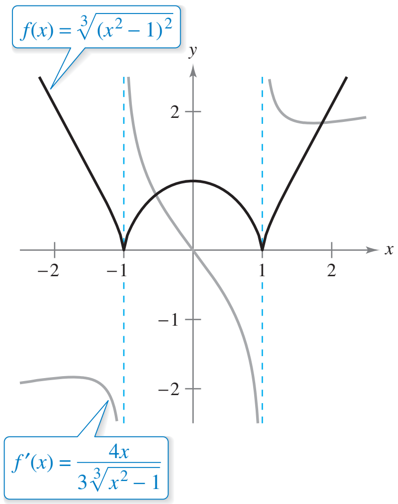
Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$\begin{aligned} f'(x) &= \overset{n}{3}\overbrace{(3x - 2x^2)^2}^{u^{n-1}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'} && \text{Apply General Power Rule.} \\ &= 3(3x - 2x^2)^2(3 - 4x). && \text{Differentiate } 3x - 2x^2. \end{aligned}$$

EXAMPLE 5 Differentiating Functions Involving Radicals



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.25

Find all points on the graph of

$$f(x) = \sqrt[3]{(x^2 - 1)^2}$$

for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$\begin{aligned} f'(x) &= \overset{n}{\frac{2}{3}}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} && \text{Apply General Power Rule.} \\ &= \frac{4x}{3\sqrt[3]{x^2 - 1}}. && \text{Write in radical form.} \end{aligned}$$

So, $f'(x) = 0$ when $x = 0$, and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 2.25.

EXAMPLE 6 Differentiating Quotients: Constant Numerators

REMARK Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Differentiate the function

$$g(t) = \frac{-7}{(2t - 3)^2}.$$

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule (with $u = 2t - 3$) produces

$$\begin{aligned} g'(t) &= \underbrace{(-7)}_{\substack{\text{Constant} \\ \text{Multiple Rule}}}\overbrace{(-2)}^{n}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} && \text{Apply General Power Rule.} \\ &= 28(2t - 3)^{-3} && \text{Simplify.} \\ &= \frac{28}{(2t - 3)^3}. && \text{Write with positive exponent.} \end{aligned}$$

Simplifying Derivatives

The next three examples demonstrate techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

Find the derivative of $f(x) = x^2\sqrt{1 - x^2}$.

Solution

$$
\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Write original function.}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}
$$

EXAMPLE 8 Simplifying the Derivative of a Quotient

▷ **TECHNOLOGY** Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given in these examples.

$$
\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}
$$

EXAMPLE 9 Simplifying the Derivative of a Power

▷ See LarsonCalculus.com for an interactive version of this type of example.

$$
\begin{aligned}
y &= \left(\frac{3x-1}{x^2+3}\right)^2 && \text{Original function}\\
y' &= \underset{n}{2}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3) - (3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}
$$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are shown below.

$$\frac{d}{dx}[\sin u] = (\cos u)u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

EXAMPLE 10 The Chain Rule and Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos\overbrace{(x - 1)}^{u}$ $\qquad y' = \overbrace{-\sin(x-1)}^{-(\sin u)}\overbrace{\frac{d}{dx}[x-1]}^{u'} = -\sin(x-1)$

c. $y = \tan \overbrace{3x}^{u}$ $\qquad y' = \overbrace{\sec^2 3x}^{(\sec^2 u)}\overbrace{\frac{d}{dx}[3x]}^{u'} = (\sec^2 3x)(3) = 3\sec^2(3x)$ ■

Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x\cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x\sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$ ■

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$$\begin{aligned}
f(t) &= \sin^3 4t && \text{Original function}\\
&= (\sin 4t)^3 && \text{Rewrite.}\\
f'(t) &= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t] && \text{Apply Chain Rule once.}\\
&= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t] && \text{Apply Chain Rule a second time.}\\
&= 3(\sin 4t)^2(\cos 4t)(4) &&\\
&= 12\sin^2 4t \cos 4t && \text{Simplify.}
\end{aligned}$$

■

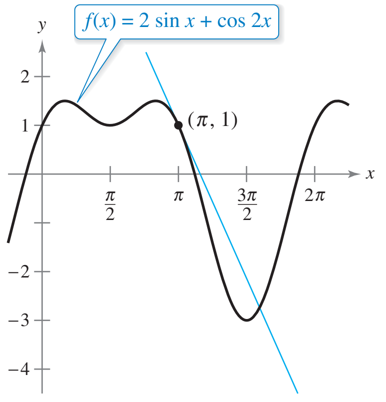


Figure 2.26

EXAMPLE 13 Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of $f(x) = 2\sin x + \cos 2x$ at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Solution Begin by finding $f'(x)$.

$$f(x) = 2\sin x + \cos 2x \qquad \text{Write original function.}$$
$$f'(x) = 2\cos x + (-\sin 2x)(2) \qquad \text{Apply Chain Rule to } \cos 2x.$$
$$= 2\cos x - 2\sin 2x \qquad \text{Simplify.}$$

To find the equation of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$f'(\pi) = 2\cos\pi - 2\sin 2\pi \qquad \text{Substitute.}$$
$$= -2 \qquad \text{Slope of graph at } (\pi, 1)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 1 = -2(x - \pi) \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = 1 - 2x + 2\pi. \qquad \text{Equation of tangent line at } (\pi, 1)$$

You can then determine that $f'(x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. So, f has horizontal tangents at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

This section concludes with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules

Let f, g, and u be differentiable functions of x.

Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$

Sum or Difference Rule: $\frac{d}{dx}[f \pm g] = f' \pm g'$

Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$

Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

Derivatives of Algebraic Functions

Constant Rule: $\frac{d}{dx}[c] = 0$

(Simple) Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$

Chain Rule

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\, u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}\, u'$

2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Decomposition of a Composite Function In Exercises 1–6, complete the table.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (5x - 8)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^3 - 7}$		
4. $y = 3\tan(\pi x^2)$		
5. $y = \csc^3 x$		
6. $y = \sin \dfrac{5x}{2}$		

Finding a Derivative In Exercises 7–34, find the derivative of the function.

7. $y = (4x - 1)^3$

8. $y = 5(2 - x^3)^4$

9. $g(x) = 3(4 - 9x)^4$

10. $f(t) = (9t + 2)^{2/3}$

11. $f(t) = \sqrt{5 - t}$

12. $g(x) = \sqrt{4 - 3x^2}$

13. $y = \sqrt[3]{6x^2 + 1}$

14. $f(x) = \sqrt{x^2 - 4x + 2}$

15. $y = 2\sqrt[4]{9 - x^2}$

16. $f(x) = \sqrt[3]{12x - 5}$

17. $y = \dfrac{1}{x - 2}$

18. $s(t) = \dfrac{1}{4 - 5t - t^2}$

19. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

20. $y = -\dfrac{3}{(t - 2)^4}$

21. $y = \dfrac{1}{\sqrt{3x + 5}}$

22. $g(t) = \dfrac{1}{\sqrt{t^2 - 2}}$

23. $f(x) = x^2(x - 2)^4$

24. $f(x) = x(2x - 5)^3$

25. $y = x\sqrt{1 - x^2}$

26. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

31. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

33. $f(x) = ((x^2 + 3)^5 + x)^2$

34. $g(x) = (2 + (x^2 + 1)^4)^3$

Finding a Derivative Using Technology In Exercises 35–40, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

35. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

36. $y = \sqrt{\dfrac{2x}{x + 1}}$

37. $y = \sqrt{\dfrac{x + 1}{x}}$

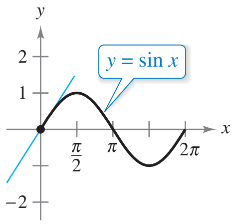
38. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

39. $y = \dfrac{\cos \pi x + 1}{x}$

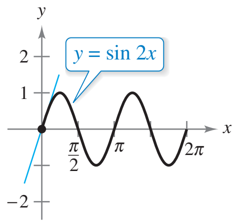
40. $y = x^2 \tan \dfrac{1}{x}$

Slope of a Tangent Line In Exercises 41 and 42, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

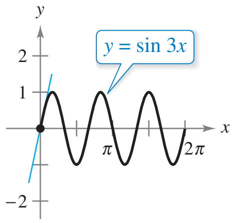
41. (a)



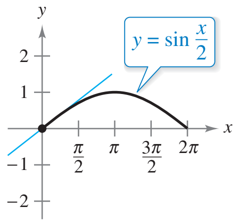
(b)



42. (a)



(b)



Finding a Derivative In Exercises 43–64, find the derivative of the function.

43. $y = \cos 4x$

44. $y = \sin \pi x$

45. $g(x) = 5\tan 3x$

46. $h(x) = \sec x^2$

47. $y = \sin(\pi x)^2$

48. $y = \cos(1 - 2x)^2$

49. $h(x) = \sin 2x \cos 2x$

50. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

51. $f(x) = \dfrac{\cot x}{\sin x}$

52. $g(v) = \dfrac{\cos v}{\csc v}$

53. $y = 4\sec^2 x$

54. $g(t) = 5\cos^2 \pi t$

55. $f(\theta) = \tan^2 5\theta$

56. $g(\theta) = \cos^2 8\theta$

57. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

58. $h(t) = 2\cot^2(\pi t + 2)$

59. $f(t) = 3\sec^2(\pi t - 1)$

60. $y = 3x - 5\cos(\pi x)^2$

61. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

62. $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

63. $y = \sin(\tan 2x)$

64. $y = \cos\sqrt{\sin(\tan \pi x)}$

Evaluating a Derivative In Exercises 65–72, find and evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

65. $y = \sqrt{x^2 + 8x}$, $(1, 3)$

66. $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$

67. $f(x) = \dfrac{5}{x^3 - 2}$, $\left(-2, -\dfrac{1}{2}\right)$

68. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$

69. $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

70. $f(x) = \dfrac{x + 4}{2x - 5}$, $(9, 1)$

71. $y = 26 - \sec^3 4x$, $(0, 25)$

72. $y = \dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

Finding an Equation of a Tangent Line **In Exercises 73–80, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.**

73. $f(x) = \sqrt{2x^2 - 7}$, $(4, 5)$

74. $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$, $(2, 2)$

75. $y = (4x^3 + 3)^2$, $(-1, 1)$

76. $f(x) = (9 - x^2)^{2/3}$, $(1, 4)$

77. $f(x) = \sin 2x$, $(\pi, 0)$

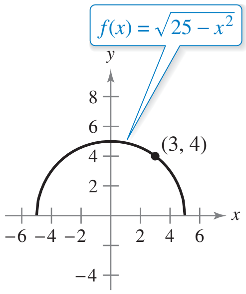
78. $y = \cos 3x$, $\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

79. $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

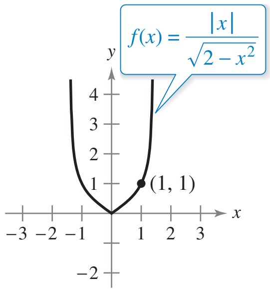
80. $y = 2\tan^3 x$, $\left(\frac{\pi}{4}, 2\right)$

Famous Curves **In Exercises 81 and 82, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

81. Top half of circle



82. Bullet-nose curve



83. Horizontal Tangent Line Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of

$$f(x) = 2\cos x + \sin 2x$$

has a horizontal tangent.

84. Horizontal Tangent Line Determine the point(s) at which the graph of

$$f(x) = \frac{x}{\sqrt{2x - 1}}$$

has a horizontal tangent.

Finding a Second Derivative **In Exercises 85–90, find the second derivative of the function.**

85. $f(x) = 5(2 - 7x)^4$

86. $f(x) = 6(x^3 + 4)^3$

87. $f(x) = \frac{1}{x - 6}$

88. $f(x) = \frac{8}{(x - 2)^2}$

89. $f(x) = \sin x^2$

90. $f(x) = \sec^2 \pi x$

Evaluating a Second Derivative **In Exercises 91–94, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.**

91. $h(x) = \frac{1}{9}(3x + 1)^3$, $\left(1, \frac{64}{9}\right)$

92. $f(x) = \frac{1}{\sqrt{x + 4}}$, $\left(0, \frac{1}{2}\right)$

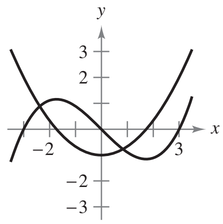
93. $f(x) = \cos x^2$, $(0, 1)$

94. $g(t) = \tan 2t$, $\left(\frac{\pi}{6}, \sqrt{3}\right)$

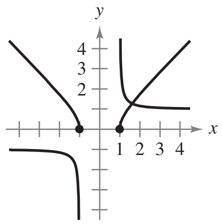
WRITING ABOUT CONCEPTS

Identifying Graphs **In Exercises 95–98, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

95.



96.



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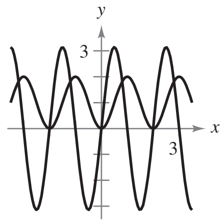
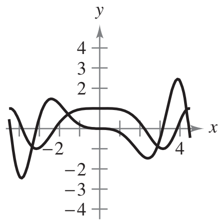


98.



Describing a Relationship **In Exercises 99 and 100, the relationship between f and g is given. Explain the relationship between f' and g'.**

99. $g(x) = f(3x)$

100. $g(x) = f(x^2)$

101. Think About It The table shows some values of the derivative of an unknown function f. Complete the table by finding the derivative of each transformation of f, if possible.

(a) $g(x) = f(x) - 2$

(b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$

(d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

102. Using Relationships Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

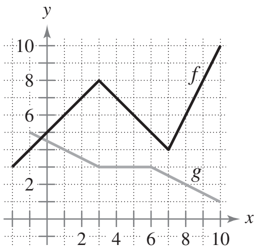
(b) $f(x) = g(h(x))$

(c) $f(x) = \frac{g(x)}{h(x)}$

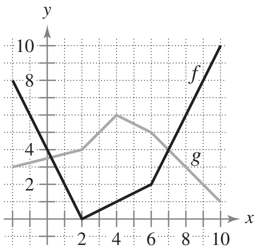
(d) $f(x) = [g(x)]^3$

Finding Derivatives **In Exercises 103 and 104, the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.**

103. (a) Find $h'(1)$.
(b) Find $s'(5)$.



104. (a) Find $h'(3)$.
(b) Find $s'(9)$.



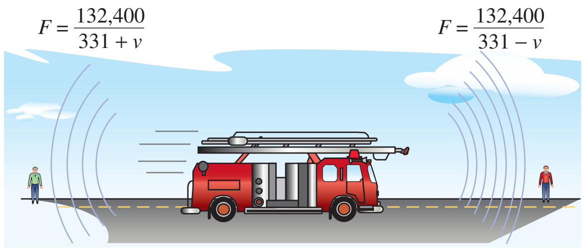
105. Doppler Effect The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



106. Harmonic Motion The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

107. Pendulum A 15-centimeter pendulum moves according to the equation $\theta = 0.2 \cos 8t$, where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

108. Wave Motion A buoy oscillates in simple harmonic motion $y = A \cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

109. Modeling Data The normal daily maximum temperatures T (in degrees Fahrenheit) for Chicago, Illinois, are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr
Temperature	29.6	34.7	46.1	58.0

Month	May	Jun	Jul	Aug
Temperature	69.9	79.2	83.5	81.2

Month	Sep	Oct	Nov	Dec
Temperature	73.9	62.1	47.1	34.4

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(ct - d)$$

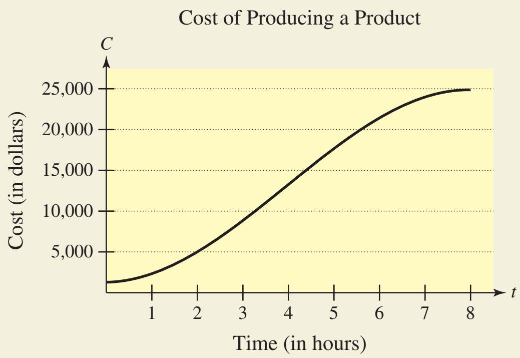
where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

110. HOW DO YOU SEE IT? The cost C (in dollars) of producing x units of a product is $C = 60x + 1350$. For one week, management determined that the number of units produced x at the end of t hours can be modeled by $x = -1.6t^3 + 19t^2 - 0.5t - 1$. The graph shows the cost C in terms of the time t.



(a) Using the graph, which is greater, the rate of change of the cost after 1 hour or the rate of change of the cost after 4 hours?

(b) Explain why the cost function is not increasing at a constant rate during the eight-hour shift.

111. Biology The number N of bacteria in a culture after t days is modeled by

$$N = 400\left[1 - \frac{3}{(t^2 + 2)^2}\right].$$

Find the rate of change of N with respect to t when (a) $t = 0$, (b) $t = 1$, (c) $t = 2$, (d) $t = 3$, and (e) $t = 4$. (f) What can you conclude?

112. Depreciation The value V of a machine t years after it is purchased is inversely proportional to the square root of $t + 1$. The initial value of the machine is \$10,000.

(a) Write V as a function of t.

(b) Find the rate of depreciation when $t = 1$.

(c) Find the rate of depreciation when $t = 3$.

113. Finding a Pattern Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

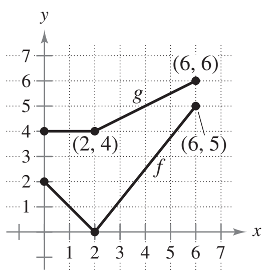
(c) Use the results of part (a) to write general rules for the even- and odd-order derivatives $f^{(2k)}(x)$ and $f^{(2k-1)}(x)$. [*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

114. Conjecture Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

115. Think About It Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



116. Using Trigonometric Functions

(a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$$f'(x) = g'(x).$$

117. Even and Odd Functions

(a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

118. Proof Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

Using Absolute Value **In Exercises 119–122, use the result of Exercise 118 to find the derivative of the function.**

119. $g(x) = |3x - 5|$

120. $f(x) = |x^2 - 9|$

121. $h(x) = |x| \cos x$

122. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \quad \text{and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 123 and 124, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

123. $f(x) = \tan x; \quad a = \dfrac{\pi}{4}$

124. $f(x) = \sec x; \quad a = \dfrac{\pi}{6}$

True or False? **In Exercises 125–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

125. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

126. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

127. If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x.

128. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

PUTNAM EXAM CHALLENGE

129. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

130. Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k - 1}$ has the form $\dfrac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

These problems were composed by the Committee on the Putnam Prize Competition.

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2.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation $y = 3x^2 - 5$, the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation

$xy = 1.$ Implicit form

To find dy/dx for this equation, you can write y explicitly as a function of x and then differentiate.

Implicit Form	Explicit Form	Derivative
$xy = 1$	$y = \dfrac{1}{x} = x^{-1}$	$\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation

$$x^2 - 2y^3 + 4y = 2?$$

For this equation, it is difficult to express y as a function of x explicitly. To find dy/dx, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

EXAMPLE 1 Differentiating with Respect to *x*

a. $\dfrac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

b. $\dfrac{d}{dx}[\overbrace{y^3}^{u^n}] = \overbrace{3y^2}^{nu^{n-1}}\,\overbrace{\dfrac{dy}{dx}}^{u'}$ Variables disagree: use Chain Rule.

Variables disagree

c. $\dfrac{d}{dx}[x + 3y] = 1 + 3\dfrac{dy}{dx}$ Chain Rule: $\dfrac{d}{dx}[3y] = 3y'$

d. $\dfrac{d}{dx}[xy^2] = x\dfrac{d}{dx}[y^2] + y^2\dfrac{d}{dx}[x]$ Product Rule

$= x\left(2y\dfrac{dy}{dx}\right) + y^2(1)$ Chain Rule

$= 2xy\dfrac{dy}{dx} + y^2$ Simplify.

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to* x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation and move all other terms to the right side of the equation.

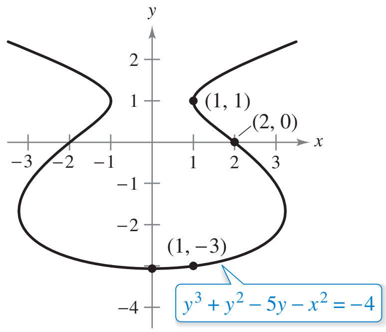
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$



Point on Graph	Slope of Graph
(2, 0)	$-\frac{4}{5}$
(1, −3)	$\frac{1}{8}$
$x = 0$	0
(1, 1)	Undefined

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

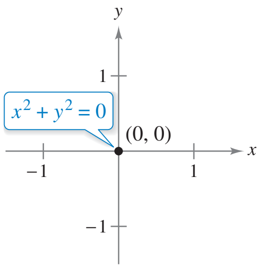
has the derivative

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

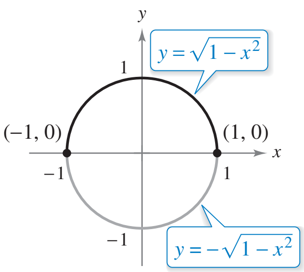
Figure 2.27

To see how you can use an *implicit derivative*, consider the graph shown in Figure 2.27. From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.

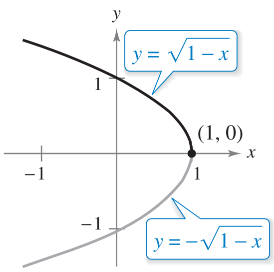
TECHNOLOGY With most graphing utilities, it is easy to graph an equation that explicitly represents y as a function of x. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 2.27?



(a)



(b)



(c)

Some graph segments can be represented by differentiable functions.

Figure 2.28

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, then dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (a) points with vertical tangents and (b) points at which the function is not continuous.

EXAMPLE 3 Graphs and Differentiable Functions

If possible, represent y as a differentiable function of x.

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. So, it does not define y as a differentiable function of x. See Figure 2.28(a).

b. The graph of this equation is the unit circle centered at $(0, 0)$. The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(b).

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(c).

EXAMPLE 4 Finding the Slope of a Graph Implicitly

See LarsonCalculus.com for an interactive version of this type of example.

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$. See Figure 2.29.

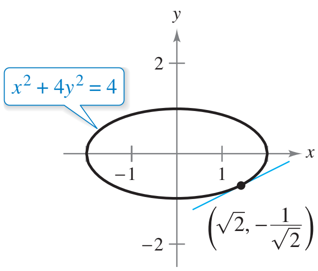


Figure 2.29

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} \qquad \text{Solve for } \frac{dy}{dx}.$$

$$= \frac{-x}{4y} \qquad \text{Simplify.}$$

So, at $\left(\sqrt{2}, -1/\sqrt{2}\right)$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

REMARK To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$.

EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of

$$3(x^2 + y^2)^2 = 100xy$$

at the point (3, 1).

Solution

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

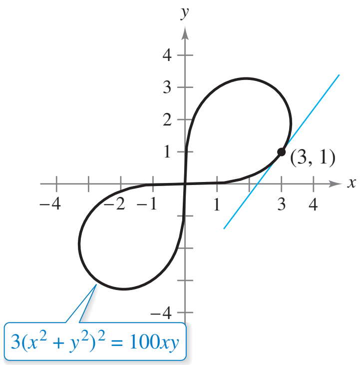
$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$



Lemniscate
Figure 2.30

At the point (3, 1), the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 2.30. This graph is called a **lemniscate.**

EXAMPLE 6 Determining a Differentiable Function

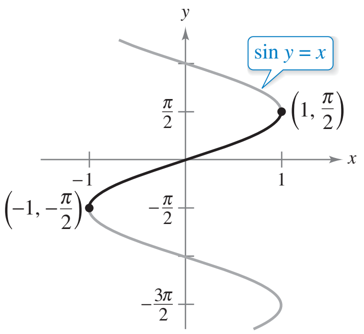
Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 2.31).

Solution

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$



The derivative is $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

Figure 2.31

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. When you restrict y to the interval $-\pi/2 < y < \pi/2$, you should be able to write dy/dx explicitly as a function of x. To do this, you can use

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

You will study this example further when inverse trigonometric functions are defined in Section 5.6.

ISAAC BARROW (1630–1677)

The graph in Figure 2.32 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.

See LarsonCalculus.com to read more of this biography.

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution Differentiating each term with respect to x produces

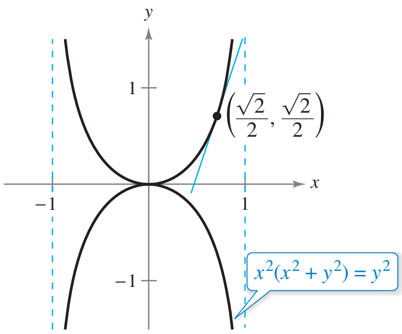
$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

Differentiating a second time with respect to x yields

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{(y)(1) - (x)(dy/dx)}{y^2} && \text{Quotient Rule} \\ &= -\frac{y - (x)(-x/y)}{y^2} && \text{Substitute } -\frac{x}{y} \text{ for } \frac{dy}{dx}. \\ &= -\frac{y^2 + x^2}{y^3} && \text{Simplify.} \\ &= -\frac{25}{y^3}. && \text{Substitute 25 for } x^2 + y^2. \end{aligned}$$

EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph of $x^2(x^2 + y^2) = y^2$ at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$, as shown in Figure 2.32.



The kappa curve
Figure 2.32

Solution By rewriting and differentiating implicitly, you obtain

$$\begin{aligned} x^4 + x^2y^2 - y^2 &= 0 \\ 4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} &= 0 \\ 2y(x^2 - 1)\frac{dy}{dx} &= -2x(2x^2 + y^2) \\ \frac{dy}{dx} &= \frac{x(2x^2 + y^2)}{y(1 - x^2)}. \end{aligned}$$

At the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$, the slope is

$$\frac{dy}{dx} = \frac{\left(\sqrt{2}/2\right)[2(1/2) + (1/2)]}{\left(\sqrt{2}/2\right)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$\begin{aligned} y - \frac{\sqrt{2}}{2} &= 3\left(x - \frac{\sqrt{2}}{2}\right) \\ y &= 3x - \sqrt{2}. \end{aligned}$$

2.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Derivative In Exercises 1–16, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 9$
2. $x^2 - y^2 = 25$
3. $x^{1/2} + y^{1/2} = 16$
4. $2x^3 + 3y^3 = 64$
5. $x^3 - xy + y^2 = 7$
6. $x^2y + y^2x = -2$
7. $x^3y^3 - y = x$
8. $\sqrt{xy} = x^2y + 1$
9. $x^3 - 3x^2y + 2xy^2 = 12$
10. $4 \cos x \sin y = 1$
11. $\sin x + 2 \cos 2y = 1$
12. $(\sin \pi x + \cos \pi y)^2 = 2$
13. $\sin x = x(1 + \tan y)$
14. $\cot y = x - y$
15. $y = \sin xy$
16. $x = \sec \dfrac{1}{y}$

Finding Derivatives Implicity and Explicitly In Exercises 17–20, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).

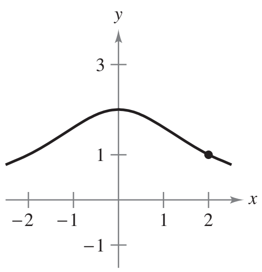
17. $x^2 + y^2 = 64$
18. $25x^2 + 36y^2 = 300$
19. $16y^2 - x^2 = 16$
20. $x^2 + y^2 - 4x + 6y + 9 = 0$

Finding and Evaluating a Derivative In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

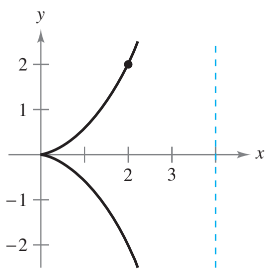
21. $xy = 6, \quad (-6, -1)$
22. $y^3 - x^2 = 4, \quad (2, 2)$
23. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}, \quad (7, 0)$
24. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$
25. $(x + y)^3 = x^3 + y^3, \quad (-1, 1)$
26. $x^3 + y^3 = 6xy - 1, \quad (2, 3)$
27. $\tan(x + y) = x, \quad (0, 0)$
28. $x \cos y = 1, \quad \left(2, \dfrac{\pi}{3}\right)$

Famous Curves In Exercises 29–32, find the slope of the tangent line to the graph at the given point.

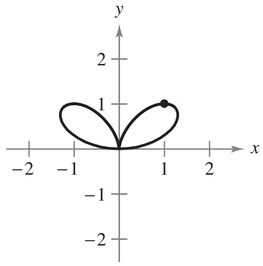
29. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



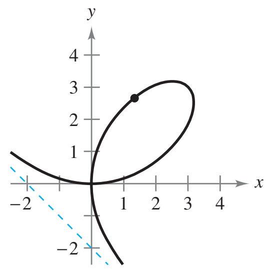
30. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



31. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$

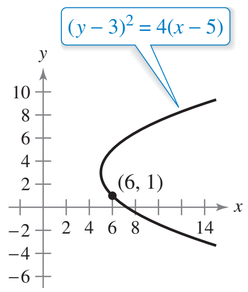


32. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$

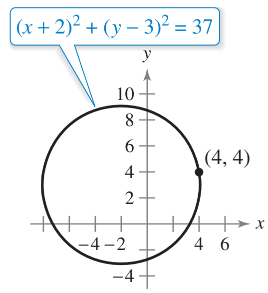


Famous Curves In Exercises 33–40, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to *MathGraphs.com*.

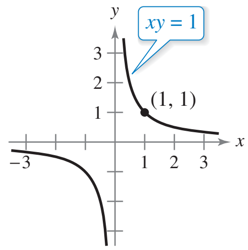
33. Parabola



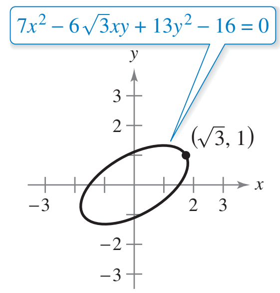
34. Circle



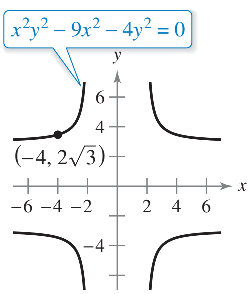
35. Rotated hyperbola



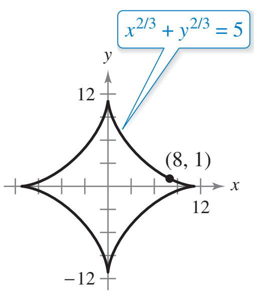
36. Rotated ellipse



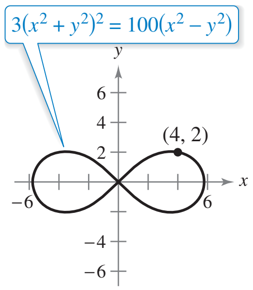
37. Cruciform



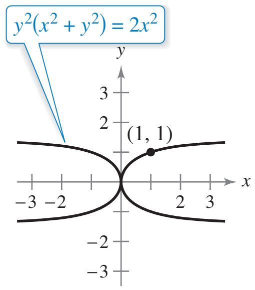
38. Astroid



39. Lemniscate



40. Kappa curve



41. Ellipse

(a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

42. Hyperbola

(a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

Determining a Differentiable Function **In Exercises 43 and 44, find dy/dx implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that y is a differentiable function of x. Write dy/dx as a function of x.**

43. $\tan y = x$ **44.** $\cos y = x$

Finding a Second Derivative **In Exercises 45–50, find d^2y/dx^2 implicitly in terms of x and y.**

45. $x^2 + y^2 = 4$ **46.** $x^2y - 4x = 5$

47. $x^2 - y^2 = 36$ **48.** $xy - 1 = 2x + y^2$

49. $y^2 = x^3$ **50.** $y^3 = 4x$

Finding an Equation of a Tangent Line **In Exercises 51 and 52, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.**

51. $\sqrt{x} + \sqrt{y} = 5$, $(9, 4)$ **52.** $y^2 = \frac{x-1}{x^2+1}$, $\left(2, \frac{\sqrt{5}}{5}\right)$

Tangent Lines and Normal Lines **In Exercises 53 and 54, find equations for the tangent line and normal line to the circle at each given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.**

53. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$

54. $x^2 + y^2 = 36$
$(6, 0), (5, \sqrt{11})$

55. Normal Lines Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

56. Circles Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

Vertical and Horizontal Tangent Lines **In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line.**

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

Orthogonal Trajectories **In Exercises 59–62, use a graphing utility to sketch the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection, their tangent lines are perpendicular to each other.]**

59. $2x^2 + y^2 = 6$
$y^2 = 4x$

60. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

61. $x + y = 0$
$x = \sin y$

62. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 63 and 64, verify that the two families of curves are orthogonal, where C and K are real numbers. Use a graphing utility to graph the two families for two values of C and two values of K.**

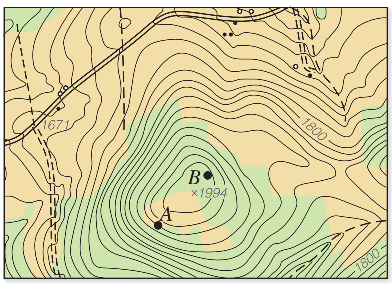
63. $xy = C$, $x^2 - y^2 = K$ **64.** $x^2 + y^2 = C^2$, $y = Kx$

WRITING ABOUT CONCEPTS

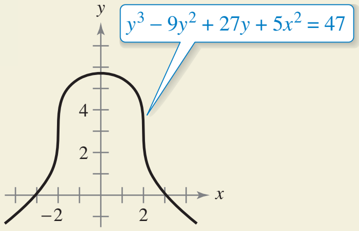
65. Explicit and Implicit Functions Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

66. Implicit Differentiation In your own words, state the guidelines for implicit differentiation.

67. Orthogonal Trajectories The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map, and they decide to follow the path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. Their goal is to reach the road along the top of the map. Which starting point should they use? To print an enlarged copy of the map, go to *MathGraphs.com*.



68. **HOW DO YOU SEE IT?** Use the graph to answer the questions.



(a) Which is greater, the slope of the tangent line at $x = -3$ or the slope of the tangent line at $x = -1$?

(b) Estimate the point(s) where the graph has a vertical tangent line.

(c) Estimate the point(s) where the graph has a horizontal tangent line.

69. Finding Equations of Tangent Lines Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

70. Tangent Lines and Intercepts Let L be any tangent line to the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{c}.$$

Show that the sum of the x- and y-intercepts of L is c.

71. Proof Prove (Theorem 2.3) that

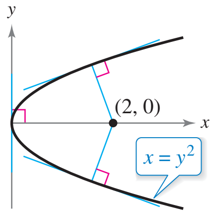
$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

72. Slope Find all points on the circle $x^2 + y^2 = 100$ where the slope is $\frac{3}{4}$.

73. Tangent Lines Find equations of both tangent lines to the graph of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that pass through the point $(4, 0)$ not on the graph.

74. Normals to a Parabola The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of x_0 are two of the normal lines perpendicular to each other?



75. Normal Lines (a) Find an equation of the normal line to the ellipse $\dfrac{x^2}{32} + \dfrac{y^2}{8} = 1$ at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

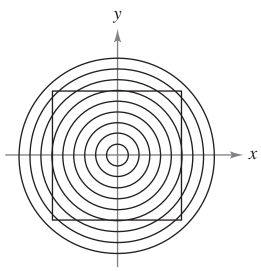
SECTION PROJECT

Optical Illusions

In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of dy/dx for the given values of x and y.

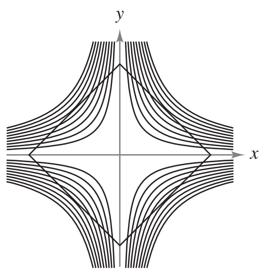
(a) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$



(b) Hyperbolas: $xy = C$

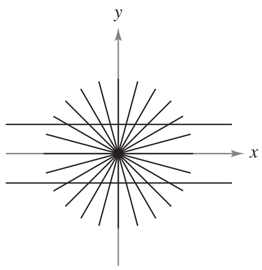
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$

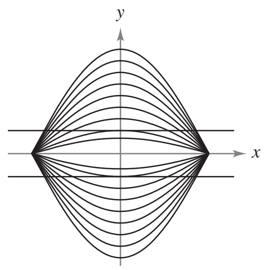
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$



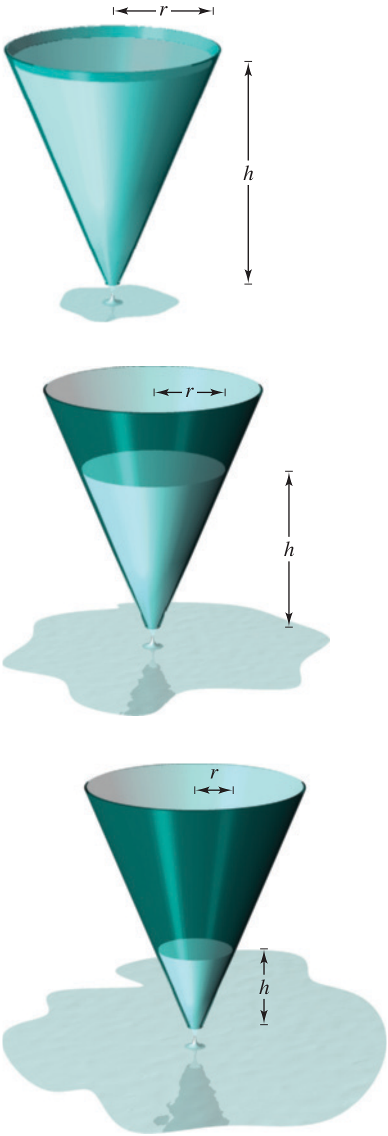
(d) Cosine curves: $y = C \cos x$

$x = \dfrac{\pi}{3}, y = \dfrac{1}{3}, C = \dfrac{2}{3}$



FOR FURTHER INFORMATION For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*.

2.6 Related Rates



Volume is related to radius and height.
Figure 2.33

- **Find a related rate.**
- **Use related rates to solve real-life problems.**

Finding Related Rates

You have seen how the Chain Rule can be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank (see Figure 2.33), the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h \qquad \text{Original equation}$$

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\begin{aligned}\frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{\pi}{3} r^2 h\right] \\ \frac{dV}{dt} &= \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t. \\ &= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right).\end{aligned}$$

From this equation, you can see that the rate of change of V is related to the rates of change of both h and r.

Exploration

Finding a Related Rate In the conical tank shown in Figure 2.33, the height of the water level is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of r and h? Explain.

EXAMPLE 1 Two Rates That Are Related

The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$\begin{aligned} y &= x^2 + 3 \qquad \text{Write original equation.} \\ \frac{d}{dt}[y] &= \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t. \\ \frac{dy}{dt} &= 2x\frac{dx}{dt} \qquad \text{Chain Rule}\end{aligned}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

FOR FURTHER INFORMATION To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

Problem Solving with Related Rates

In Example 1, you were *given* an equation that related the variables x and y and were asked to find the rate of change of y when $x = 1$.

Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2 \quad \text{when} \quad x = 1$

Find: $\frac{dy}{dt} \quad \text{when} \quad x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.

EXAMPLE 2 Ripples in a Pond

Total area increases as the outer radius increases.
Figure 2.34

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.34. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

Equation: $A = \pi r^2$

Given rate: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt} \quad \text{when} \quad r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{Chain Rule}$$

$$= 2\pi(4)(1) \qquad \text{Substitute 4 for } r \text{ and 1 for } \frac{dr}{dt}.$$

$$= 8\pi \text{ square feet per second} \qquad \text{Simplify.}$$

When the radius is 4 feet, the area is changing at a rate of 8π square feet per second.

REMARK When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\dfrac{dx}{dt} = 50$ mi/h when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\dfrac{dV}{dt} = 10\ \text{m}^3/\text{h}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\dfrac{d\theta}{dt} = 25(2\pi)$ rad/min
A population of bacteria is increasing at a rate of 2000 per hour.	$x =$ number in population $\dfrac{dx}{dt} = 2000$ bacteria per hour

EXAMPLE 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

Solution Let V be the volume of the balloon, and let r be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

Given rate: $\dfrac{dV}{dt} = \dfrac{9}{2}$ (constant rate)

Find: $\dfrac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \dfrac{4}{3}\pi r^3$ Volume of a sphere

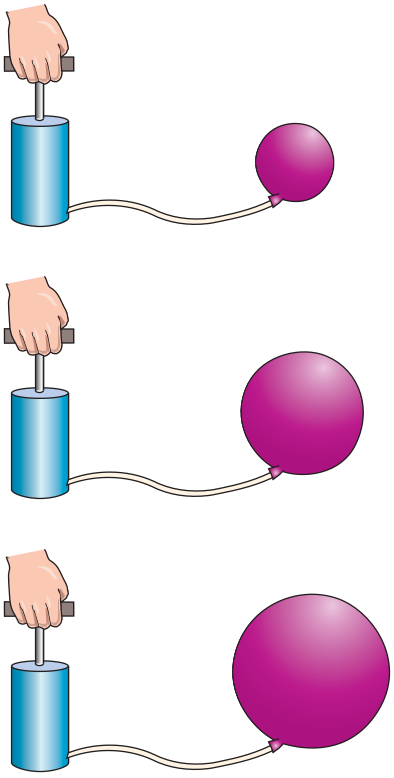
Differentiating both sides of the equation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } \frac{dr}{dt}.$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{4\pi(2)^2}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

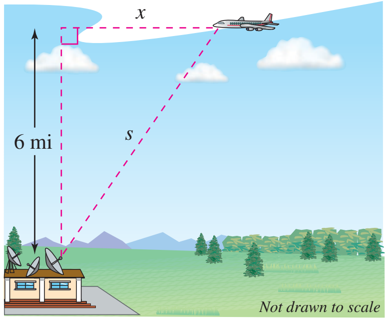


Inflating a balloon
Figure 2.35

In Example 3, note that the volume is increasing at a *constant* rate, but the radius is increasing at a *variable* rate. Just because two rates are related does not mean that they are proportional. In this particular case, the radius is growing more and more slowly as t increases. Do you see why?

EXAMPLE 4 The Speed of an Airplane Tracked by Radar

See LarsonCalculus.com for an interactive version of this type of example.



An airplane is flying at an altitude of 6 miles, s miles from the station.
Figure 2.36

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. The distance s is decreasing at a rate of 400 miles per hour when $s = 10$ miles. What is the speed of the plane?

Solution Let x be the horizontal distance from the station, as shown in Figure 2.36. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $ds/dt = -400$ when $s = 10$

Find: dx/dt when $s = 10$ and $x = 8$

You can find the velocity of the plane as shown.

Equation:

$$x^2 + 6^2 = s^2 \quad \text{Pythagorean Theorem}$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt} \quad \text{Differentiate with respect to } t.$$

$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right) \quad \text{Solve for } \frac{dx}{dt}.$$

$$= \frac{10}{8}(-400) \quad \text{Substitute for } s, x, \text{ and } \frac{ds}{dt}.$$

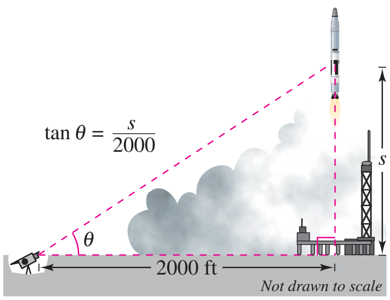
$$= -500 \text{ miles per hour} \quad \text{Simplify.}$$

Because the velocity is -500 miles per hour, the *speed* is 500 miles per hour.

REMARK The velocity in Example 4 is negative because x represents a distance that is decreasing.

EXAMPLE 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 2.37 at 10 seconds after lift-off.



A television camera at ground level is filming the lift-off of a rocket that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad.
Figure 2.37

Solution Let θ be the angle of elevation, as shown in Figure 2.37. When $t = 10$, the height s of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

Given rate: $ds/dt = 100t =$ velocity of rocket

Find: $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 2.37, you can relate s and θ by the equation $\tan\theta = s/2000$.

Equation:

$$\tan\theta = \frac{s}{2000} \quad \text{See Figure 2.37.}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right) \quad \text{Differentiate with respect to } t.$$

$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000} \quad \text{Substitute } 100t \text{ for } \frac{ds}{dt}.$$

$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2 \frac{100t}{2000} \quad \cos\theta = \frac{2000}{\sqrt{s^2 + 2000^2}}$$

When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

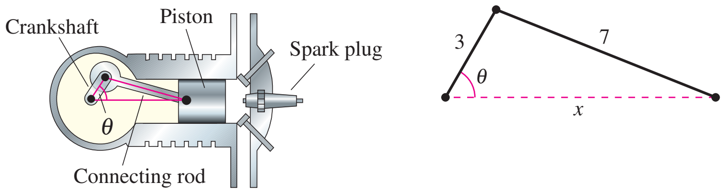
So, when $t = 10$, θ is changing at a rate of $\frac{2}{29}$ radian per second.

AP* Tips

Related rate problems, since they represent a powerful application of implicit derivatives, make frequent appearances on the free response section of the AP Exam.

EXAMPLE 6 The Velocity of a Piston

In the engine shown in Figure 2.38, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



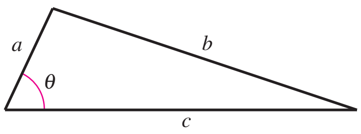
The velocity of a piston is related to the angle of the crankshaft.
Figure 2.38

Solution Label the distances as shown in Figure 2.38. Because a complete revolution corresponds to 2π radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minute.

Given rate: $\dfrac{d\theta}{dt} = 400\pi$ (constant rate)

Find: $\dfrac{dx}{dt}$ when $\theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (see Figure 2.39) to find an equation that relates x and θ.



Law of Cosines:
$b^2 = a^2 + c^2 - 2ac\cos\theta$
Figure 2.39

Equation:

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\theta$$

$$0 = 2x\frac{dx}{dt} - 6\left(-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right)$$

$$(6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)$$

When $\theta = \pi/3$, you can solve for x as shown.

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3}$$

$$49 = 9 + x^2 - 6x\left(\frac{1}{2}\right)$$

$$0 = x^2 - 3x - 40$$

$$0 = (x - 8)(x + 5)$$

$$x = 8$$ Choose positive solution.

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)\left(\sqrt{3}/2\right)}{6(1/2) - 16}(400\pi)$$

$$= \frac{9600\pi\sqrt{3}}{-13}$$

$$\approx -4018 \text{ inches per minute.}$$

REMARK The velocity in Example 6 is negative because x represents a distance that is decreasing.

2.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using Related Rates **In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.**

	Equation	Find	Given
1.	$y = \sqrt{x}$	(a) $\dfrac{dy}{dt}$ when $x = 4$	$\dfrac{dx}{dt} = 3$
		(b) $\dfrac{dx}{dt}$ when $x = 25$	$\dfrac{dy}{dt} = 2$
2.	$y = 3x^2 - 5x$	(a) $\dfrac{dy}{dt}$ when $x = 3$	$\dfrac{dx}{dt} = 2$
		(b) $\dfrac{dx}{dt}$ when $x = 2$	$\dfrac{dy}{dt} = 4$
3.	$xy = 4$	(a) $\dfrac{dy}{dt}$ when $x = 8$	$\dfrac{dx}{dt} = 10$
		(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = -6$
4.	$x^2 + y^2 = 25$	(a) $\dfrac{dy}{dt}$ when $x = 3, y = 4$	$\dfrac{dx}{dt} = 8$
		(b) $\dfrac{dx}{dt}$ when $x = 4, y = 3$	$\dfrac{dy}{dt} = -2$

Moving Point **In Exercises 5–8, a point is moving along the graph of the given function at the rate dx/dt. Find dy/dt for the given values of x.**

5. $y = 2x^2 + 1$; $\dfrac{dx}{dt} = 2$ centimeters per second

(a) $x = -1$ (b) $x = 0$ (c) $x = 1$

6. $y = \dfrac{1}{1 + x^2}$; $\dfrac{dx}{dt} = 6$ inches per second

(a) $x = -2$ (b) $x = 0$ (c) $x = 2$

7. $y = \tan x$; $\dfrac{dx}{dt} = 3$ feet per second

(a) $x = -\dfrac{\pi}{3}$ (b) $x = -\dfrac{\pi}{4}$ (c) $x = 0$

8. $y = \cos x$; $\dfrac{dx}{dt} = 4$ centimeters per second

(a) $x = \dfrac{\pi}{6}$ (b) $x = \dfrac{\pi}{4}$ (c) $x = \dfrac{\pi}{3}$

WRITING ABOUT CONCEPTS

9. Related Rates Consider the linear function

$$y = ax + b.$$

If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

10. Related Rates In your own words, state the guidelines for solving related-rate problems.

11. Area The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

12. Area The included angle of the two sides of constant equal length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin\theta$.

(b) The angle θ is increasing at the rate of $\frac{1}{2}$ radian per minute. Find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

13. Volume The radius r of a sphere is increasing at a rate of 3 inches per minute.

(a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

14. Volume A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

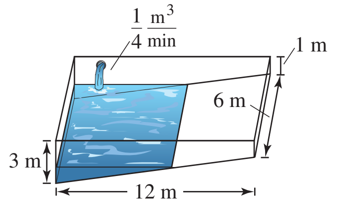
15. Volume All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

16. Surface Area All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the surface area changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

17. Volume At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high? (*Hint:* The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

18. Depth A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

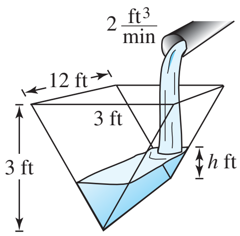
19. Depth A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.



(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

20. Depth A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.



(a) Water is being pumped into the trough at 2 cubic feet per minute. How fast is the water level rising when the depth h is 1 foot?

(b) The water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$. Determine the rate at which water is being pumped into the trough.

21. Moving Ladder A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

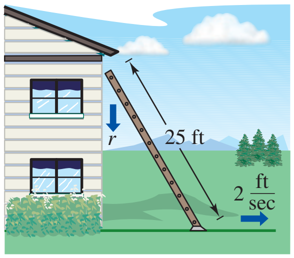


Figure for 21

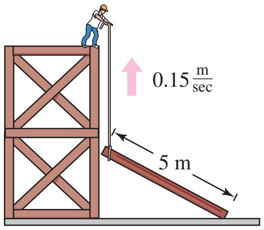


Figure for 22

FOR FURTHER INFORMATION For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

22. Construction A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meter per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

23. Construction A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.

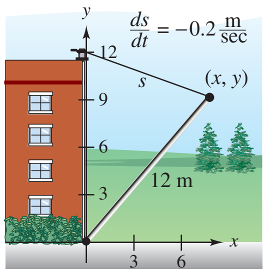


Figure for 23

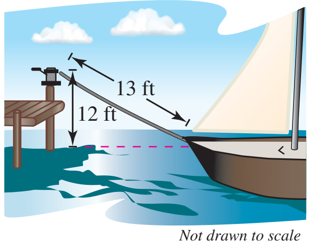


Figure for 24

24. Boating A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

25. Air Traffic Control An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 225 miles from the point moving at 450 miles per hour. The other plane is 300 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?

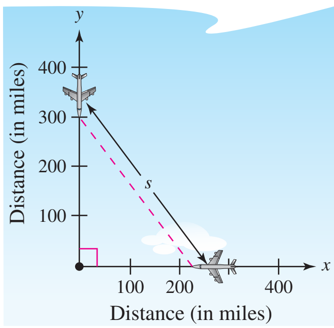


Figure for 25

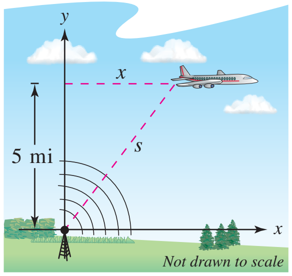


Figure for 26

26. Air Traffic Control An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

27. Sports A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 25 feet per second is 20 feet from third base. At what rate is the player's distance s from home plate changing?

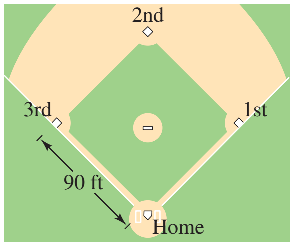


Figure for 27 and 28

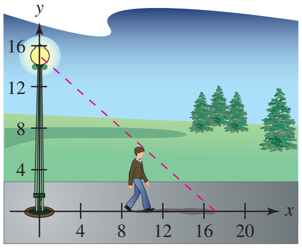


Figure for 29

28. Sports For the baseball diamond in Exercise 27, suppose the player is running from first base to second base at a speed of 25 feet per second. Find the rate at which the distance from home plate is changing when the player is 20 feet from second base.

29. Shadow Length A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure).

(a) When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?

(b) When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

30. Shadow Length Repeat Exercise 29 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

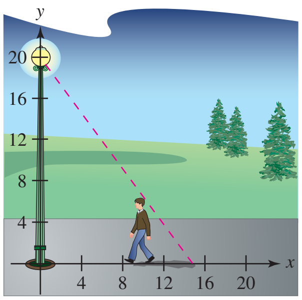


Figure for 30

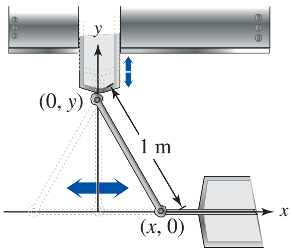


Figure for 31

31. Machine Design The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

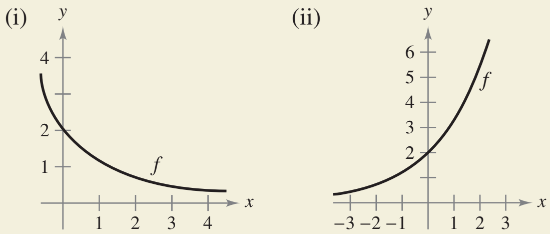
(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

32. Machine Design Repeat Exercise 31 for a position function of $x(t) = \frac{3}{5}\sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

33. Evaporation As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

34. HOW DO YOU SEE IT? Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.



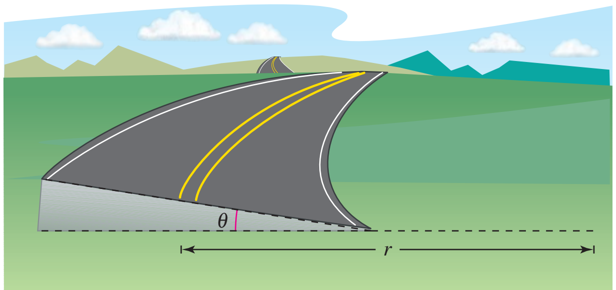
35. Electricity The combined electrical resistance R of two resistors R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

36. Adiabatic Expansion When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation $pV^{1.3} = k$, where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

37. Roadway Design Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation $rg \tan \theta = v^2$, where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.



38. Angle of Elevation A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

39. Angle of Elevation A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?

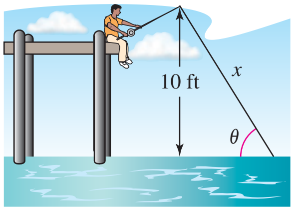


Figure for 39

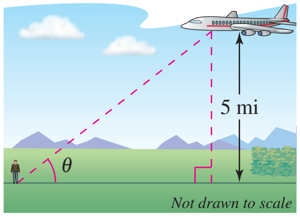


Figure for 40

40. Angle of Elevation An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.

41. Linear vs. Angular Speed A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?

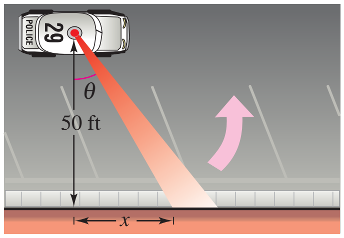


Figure for 41

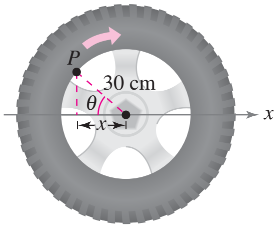


Figure for 42

42. Linear vs. Angular Speed A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

43. Flight Control An airplane is flying in still air with an airspeed of 275 miles per hour. The plane is climbing at an angle of 18°. Find the rate at which it is gaining altitude.

44. Security Camera A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $|dx/dt| = 2$ feet per second.

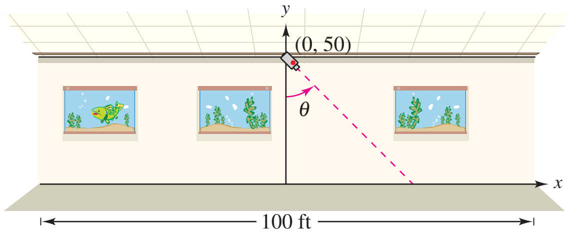


Figure for 44

45. Think About It Describe the relationship between the rate of change of y and the rate of change of x in each expression. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 46 and 47, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

46. Find the acceleration of the top of the ladder described in Exercise 21 when the base of the ladder is 7 feet from the wall.

47. Find the acceleration of the boat in Exercise 24(a) when there is a total of 13 feet of rope out.

48. Modeling Data The table shows the numbers (in millions) of single women (never married) s and married women m in the civilian work force in the United States for the years 2003 through 2010. *(Source: U.S. Bureau of Labor Statistics)*

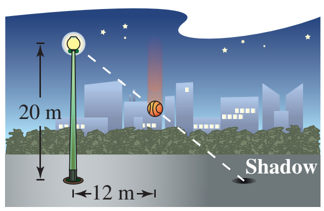
Year	2003	2004	2005	2006
s	18.4	18.6	19.2	19.5
m	36.0	35.8	35.9	36.3

Year	2007	2008	2009	2010
s	19.7	20.2	20.2	20.6
m	36.9	37.2	37.3	36.7

(a) Use the regression capabilities of a graphing utility to find a model of the form $m(s) = as^3 + bs^2 + cs + d$ for the data, where t is the time in years, with $t = 3$ corresponding to 2003.

(b) Find dm/dt. Then use the model to estimate dm/dt for $t = 7$ when it is predicted that the number of single women in the work force will increase at the rate of 0.75 million per year.

49. Moving Shadow A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*



Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Derivative by the Limit Process **In Exercises 1–4, find the derivative of the function by the limit process.**

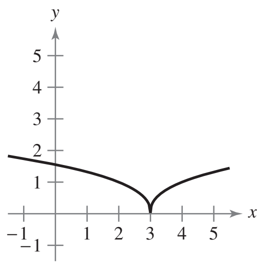
1. $f(x) = 12$
2. $f(x) = 5x - 4$
3. $f(x) = x^2 - 4x + 5$
4. $f(x) = \dfrac{6}{x}$

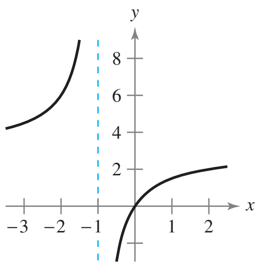
Using the Alternative Form of the Derivative **In Exercises 5 and 6, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).**

5. $g(x) = 2x^2 - 3x, \quad c = 2$
6. $f(x) = \dfrac{1}{x + 4}, \quad c = 3$

Determining Differentiability **In Exercises 7 and 8, describe the x-values at which f is differentiable.**

7. $f(x) = (x - 3)^{2/5}$
8. $f(x) = \dfrac{3x}{x + 1}$





Finding a Derivative **In Exercises 9–20, use the rules of differentiation to find the derivative of the function.**

9. $y = 25$
10. $f(t) = 4t^4$
11. $f(x) = x^3 - 11x^2$
12. $g(s) = 3s^5 - 2s^4$
13. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$
14. $f(x) = x^{1/2} - x^{-1/2}$
15. $g(t) = \dfrac{2}{3t^2}$
16. $h(x) = \dfrac{8}{5x^4}$
17. $f(\theta) = 4\theta - 5\sin\theta$
18. $g(\alpha) = 4\cos\alpha + 6$
19. $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$
20. $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

Finding the Slope of a Graph **In Exercises 21–24, find the slope of the graph of the functions at the given point.**

21. $f(x) = \dfrac{27}{x^3}, \quad (3, 1)$
22. $f(x) = 3x^2 - 4x, \quad (1, -1)$
23. $f(x) = 2x^4 - 8, \quad (0, -8)$
24. $f(\theta) = 3\cos\theta - 2\theta, \quad (0, 3)$

25. **Vibrating String** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds. Find the rates of change of F when (a) $T = 4$ and (b) $T = 9$.

26. **Volume** The surface area of a cube with sides of length ℓ is given by $S = 6\ell^2$. Find the rates of change of the surface area with respect to ℓ when (a) $\ell = 3$ inches and (b) $\ell = 5$ inches.

Vertical Motion **In Exercises 27 and 28, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

27. A ball is thrown straight down from the top of a 600-foot building with an initial velocity of -30 feet per second.
   (a) Determine the position and velocity functions for the ball.
   (b) Determine the average velocity on the interval $[1, 3]$.
   (c) Find the instantaneous velocities when $t = 1$ and $t = 3$.
   (d) Find the time required for the ball to reach ground level.
   (e) Find the velocity of the ball at impact.

28. To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. The splash is seen 9.2 seconds after the weight is dropped. What is the height (in feet) of the building?

Finding a Derivative **In Exercises 29–40, use the Product Rule or the Quotient Rule to find the derivative of the function.**

29. $f(x) = (5x^2 + 8)(x^2 - 4x - 6)$
30. $g(x) = (2x^3 + 5x)(3x - 4)$
31. $h(x) = \sqrt{x}\sin x$
32. $f(t) = 2t^5\cos t$
33. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$
34. $f(x) = \dfrac{2x + 7}{x^2 + 4}$
35. $y = \dfrac{x^4}{\cos x}$
36. $y = \dfrac{\sin x}{x^4}$
37. $y = 3x^2\sec x$
38. $y = 2x - x^2\tan x$
39. $y = x\cos x - \sin x$
40. $g(x) = 3x\sin x + x^2\cos x$

Finding an Equation of a Tangent Line **In Exercises 41–44, find an equation of the tangent line to the graph of f at the given point.**

41. $f(x) = (x + 2)(x^2 + 5), \quad (-1, 6)$
42. $f(x) = (x - 4)(x^2 + 6x - 1), \quad (0, 4)$
43. $f(x) = \dfrac{x + 1}{x - 1}, \quad \left(\dfrac{1}{2}, -3\right)$
44. $f(x) = \dfrac{1 + \cos x}{1 - \cos x}, \quad \left(\dfrac{\pi}{2}, 1\right)$

Finding a Second Derivative **In Exercises 45–50, find the second derivative of the function.**

45. $g(t) = -8t^3 - 5t + 12$
46. $h(x) = 6x^{-2} + 7x^2$
47. $f(x) = 15x^{5/2}$
48. $f(x) = 20\sqrt[5]{x}$
49. $f(\theta) = 3\tan\theta$
50. $h(t) = 10\cos t - 15\sin t$

51. Acceleration The velocity of an object in meters per second is $v(t) = 20 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$.

52. Acceleration The velocity of an automobile starting from rest is

$$v(t) = \frac{90t}{4t + 10}$$

where v is measured in feet per second. Find the acceleration at (a) 1 second, (b) 5 seconds, and (c) 10 seconds.

Finding a Derivative In Exercises 53–64, find the derivative of the function.

53. $y = (7x + 3)^4$

54. $y = (x^2 - 6)^3$

55. $y = \dfrac{1}{x^2 + 4}$

56. $f(x) = \dfrac{1}{(5x + 1)^2}$

57. $y = 5\cos(9x + 1)$

58. $y = 1 - \cos 2x + 2\cos^2 x$

59. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

60. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

61. $y = x(6x + 1)^5$

62. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

63. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

64. $h(x) = \left(\dfrac{x + 5}{x^2 + 3}\right)^2$

Evaluating a Derivative In Exercises 65–70, find and evaluate the derivative of the function at the given point.

65. $f(x) = \sqrt{1 - x^3}$, $(-2, 3)$

66. $f(x) = \sqrt[3]{x^2 - 1}$, $(3, 2)$

67. $f(x) = \dfrac{4}{x^2 + 1}$, $(-1, 2)$

68. $f(x) = \dfrac{3x + 1}{4x - 3}$, $(4, 1)$

69. $y = \dfrac{1}{2}\csc 2x$, $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$

70. $y = \csc 3x + \cot 3x$, $\left(\dfrac{\pi}{6}, 1\right)$

Finding a Second Derivative In Exercises 71–74, find the second derivative of the function.

71. $y = (8x + 5)^3$

72. $y = \dfrac{1}{5x + 1}$

73. $f(x) = \cot x$

74. $y = \sin^2 x$

75. Refrigeration The temperature T (in degrees Fahrenheit) of food in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

76. Harmonic Motion The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \frac{1}{4}\cos 8t - \frac{1}{4}\sin 8t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/4$.

Finding a Derivative In Exercises 77–82, find dy/dx by implicit differentiation.

77. $x^2 + y^2 = 64$

78. $x^2 + 4xy - y^3 = 6$

79. $x^3y - xy^3 = 4$

80. $\sqrt{xy} = x - 4y$

81. $x \sin y = y \cos x$

82. $\cos(x + y) = x$

Tangent Lines and Normal Lines In Exercises 83 and 84, find equations for the tangent line and the normal line to the graph of the equation at the given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, the tangent line, and the normal line.

83. $x^2 + y^2 = 10$, $(3, 1)$

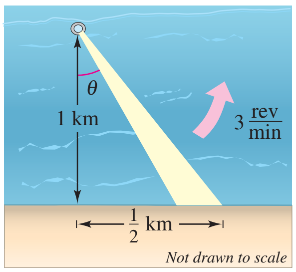
84. $x^2 - y^2 = 20$, $(6, 4)$

85. Rate of Change A point moves along the curve $y = \sqrt{x}$ in such a way that the y-value is increasing at a rate of 2 units per second. At what rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

86. Surface Area All edges of a cube are expanding at a rate of 8 centimeters per second. How fast is the surface area changing when each edge is 6.5 centimeters?

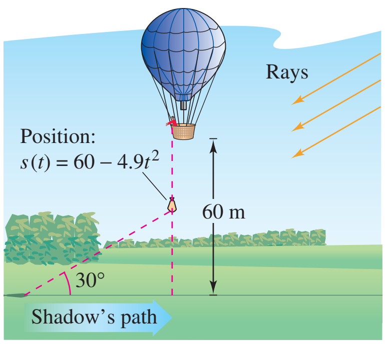
87. Linear vs. Angular Speed A rotating beacon is located 1 kilometer off a straight shoreline (see figure). The beacon rotates at a rate of 3 revolutions per minute. How fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?



88. Moving Shadow A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). The position of the sandbag is

$$s(t) = 60 - 4.9t^2.$$

Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters.



P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Finding Equations of Circles Consider the graph of the parabola $y = x^2$.

(a) Find the radius r of the largest possible circle centered on the y-axis that is tangent to the parabola at the origin, as shown in the figure. This circle is called the **circle of curvature** (see Section 12.5). Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

(b) Find the center $(0, b)$ of the circle of radius 1 centered on the y-axis that is tangent to the parabola at two points, as shown in the figure. Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

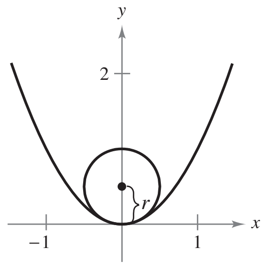


Figure for 1(a)

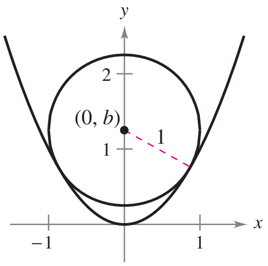


Figure for 1(b)

2. Finding Equations of Tangent Lines Graph the two parabolas

$$y = x^2 \quad \text{and} \quad y = -x^2 + 2x - 5$$

in the same coordinate plane. Find equations of the two lines that are simultaneously tangent to both parabolas.

3. Finding a Polynomial Find a third-degree polynomial $p(x)$ that is tangent to the line $y = 14x - 13$ at the point $(1, 1)$, and tangent to the line $y = -2x - 5$ at the point $(-1, -3)$.

4. Finding a Function Find a function of the form $f(x) = a + b \cos cx$ that is tangent to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

5. Tangent Lines and Normal Lines

(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

(b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The *normal line* at a point is perpendicular to the tangent line at the point.) Where does this line intersect the parabola a second time?

(c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.

(d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

6. Finding Polynomials

(a) Find the polynomial $P_1(x) = a_0 + a_1 x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.

(b) Find the polynomial $P_2(x) = a_0 + a_1 x + a_2 x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree Taylor polynomial of $f(x) = \cos x$ at $x = 0$.

(c) Complete the table comparing the values of $f(x) = \cos x$ and $P_2(x)$. What do you observe?

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$							
$P_2(x)$							

(d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

7. Famous Curve The graph of the **eight curve**

$$x^4 = a^2(x^2 - y^2), \quad a \neq 0$$

is shown below.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constant a. Describe how a affects the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

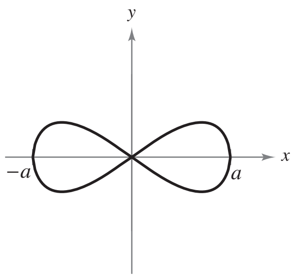


Figure for 7

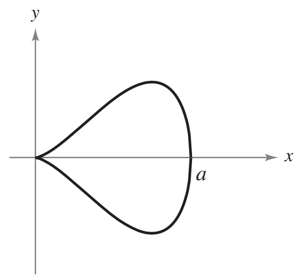


Figure for 8

8. Famous Curve The graph of the **pear-shaped quartic**

$$b^2y^2 = x^3(a - x), \quad a, b > 0$$

is shown above.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constants a and b. Describe how a and b affect the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

9. Shadow Length A man 6 feet tall walks at a rate of 5 feet per second toward a streetlight that is 30 feet high (see figure). The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

(a) Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

(d) Determine how fast the tip of the man's shadow is moving as a function of x, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

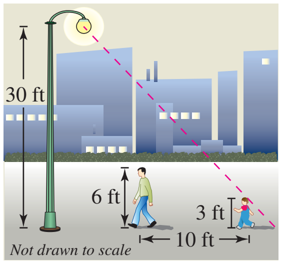


Figure for 9

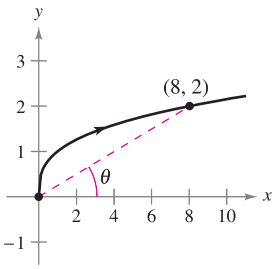


Figure for 10

10. Moving Point A particle is moving along the graph of $y = \sqrt[3]{x}$ (see figure). When $x = 8$, the y-component of the position of the particle is increasing at the rate of 1 centimeter per second.

(a) How fast is the x-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination θ changing at this moment?

11. Projectile Motion An astronaut standing on the moon throws a rock upward. The height of the rock is

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the height of the rock at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

12. Proof Let E be a function satisfying $E(0) = E'(0) = 1$. Prove that if $E(a + b) = E(a)E(b)$ for all a and b, then E is differentiable and $E'(x) = E(x)$ for all x. Find an example of a function satisfying $E(a + b) = E(a)E(b)$.

13. Proof Let L be a differentiable function for all x. Prove that if $L(a + b) = L(a) + L(b)$ for all a and b, then $L'(x) = L'(0)$ for all x. What does the graph of L look like?

14. Radians and Degrees The fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

assumes that x is measured in radians. Suppose you assume that x is measured in degrees instead of radians.

(a) Set your calculator to *degree* mode and complete the table.

z (in degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$			

(b) Use the table to estimate

$$\lim_{z \to 0} \frac{\sin z}{z}$$

for z in degrees. What is the exact value of this limit? (*Hint:* $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find

$$\frac{d}{dz} \sin z$$

for z in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate

$$\frac{d}{dz} S(z).$$

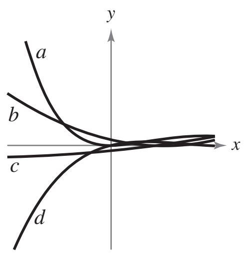
(e) Explain why calculus is made easier by using radians instead of degrees.

15. Acceleration and Jerk If a is the acceleration of an object, then the *jerk* j is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of j.

(b) Find j for the slowing vehicle in Exercise 117 in Section 2.3 and interpret the result.

(c) The figure shows the graphs of the position, velocity, acceleration, and jerk functions of a vehicle. Identify each graph and explain your reasoning.



AP* Review Questions for Chapter 2

1. (no calculator)

Let $f(x) = (x^2 - 3)^4$.

(a) Write an equation of the line tangent to the graph of f at $x = 2$.

(b) Find the values of x for which the graph of f has a horizontal tangent.

(c) Find $f''(x)$.

2. (no calculator)

Let $f(x) = \sqrt{4x - 3}$ and $g(x) = \dfrac{f(x)}{x}$.

(a) What is the slope of the graph of f at $x = 3$? Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of g at $x = 3$.

(c) What is the slope of the line normal to the graph of g at $x = 3$?

3. (no calculator)

Evaluate each limit analytically.

(*Note: Finding the answer should* <u>*not*</u> *involve a lengthy algebraic process.*)

(a) $\lim\limits_{h \to 0} \dfrac{\sin(x + h) - \sin x}{h}$

(b) $\lim\limits_{h \to 0} \dfrac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h}$

(c) $\lim\limits_{h \to 0} \dfrac{\sqrt{16 + h} - 4}{h}$

(d) $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{5 + h} - \dfrac{1}{5}}{h}$

4. Given:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	-3	1	5	-2
5	4	7	-1	2

(a) If $h(x) = \dfrac{f(x)}{g(x)}$, find $h'(2)$.

(b) If $j(x) = f(g(x))$, find $j'(2)$.

(c) If $k(x) = \sqrt{f(x)}$, find $k'(5)$.

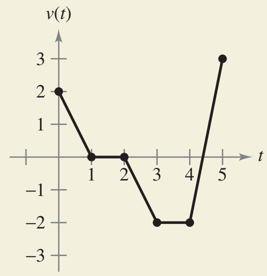
5. (no calculator)

Given: $f(x) = x^2$

(a) Find the slope of the normal line to the graph of f at $x = -3$.

(b) Two lines passing through the point $(3, 8)$ will be tangent to the graph of f. Find an equation for each of these lines.

6. The accompanying diagram shows the graph of the velocity in $\dfrac{\text{ft}}{\text{sec}}$ for a particle moving along the line $x = 4$.



(a) During which time interval is the particle:

(i) moving upward?

(ii) moving downward?

(iii) at rest?

(b) State the acceleration of the particle at the specified times. Include units.

(i) $t = 0.75$

(ii) $t = 4.2$

7. (no calculator)

Given: $g(x) = f(x) \cdot \tan x + kx$, where k is a real number.

f is differentiable for all x; $f\left(\dfrac{\pi}{4}\right) = 4$; $f'\left(\dfrac{\pi}{4}\right) = -2$.

(a) For what values of x, if any, in the interval $0 < x < 2\pi$ will the derivative of g fail to exist? Justify your answer.

(b) If $g'\left(\dfrac{\pi}{4}\right) = 6$, find the value of k.

8. The table provided below shows the position of a particle, S, at several times, t, as the particle moves along a straight line, where t is measured in seconds and S is measured in meters.

t	2.0	2.7	3.2	3.8
$S(t)$	5.2	7.8	10.6	12.2

Which of the following best estimates the velocity of the particle at $t = 3$?

(a) $9.2\,\dfrac{\text{m}}{\text{s}}$ (b) $7.8\,\dfrac{\text{m}}{\text{s}}$ (c) $5.6\,\dfrac{\text{m}}{\text{s}}$

9. (no calculator)

If $2y^3 - 3xy + x^2 = 4$, then $\dfrac{dy}{dx} =$

(A) $-\dfrac{2x}{6y^2 - 3}$ (B) $\dfrac{2x - 3y}{3x - 6y^2}$ (C) $\dfrac{2x - 3}{6y^2}$

(D) $-\dfrac{2x}{6y^2 - 3x}$ (E) $\dfrac{3y - 2x}{6y^2}$

AP2-2

10. (no calculator)

The volume of a cylinder with radius r and height h is given by $V = \pi r^2 h$. The radius and height of the cylinder are increasing at constant rates. The radius is expanding at $\frac{1}{3}\ \frac{\text{cm}}{\text{sec}}$ and the height is increasing at $\frac{1}{2}\ \frac{\text{cm}}{\text{sec}}$. At what rate, in cubic *cm* per second, is the volume of the cylinder increasing when its height is 9 cm and the radius is 4 cm?

(A) 32π

(B) 6π

(C) $\frac{8\pi}{3}$

(D) $\frac{4\pi}{3}$

(E) $\frac{\pi}{18}$

3 Applications of Differentiation

Estimation of Error *(Example 3, p. 233)*

Offshore Oil Well *(Exercise 39, p. 222)*

Engine Efficiency *(Exercise 85, p. 204)*

Path of a Projectile *(Example 5, p. 182)*

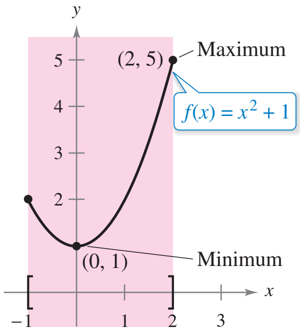
Speed *(Exercise 57, p.175)*

3.1 Extrema on an Interval

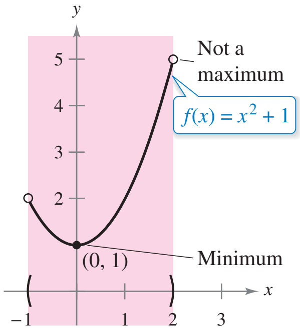
- **Understand the definition of extrema of a function on an interval.**
- **Understand the definition of relative extrema of a function on an open interval.**
- **Find extrema on a closed interval.**

Extrema of a Function

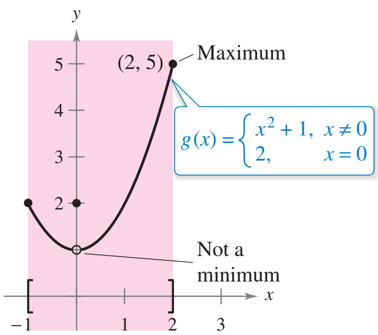
In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter, you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.



(a) f is continuous, $[-1, 2]$ is closed.



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.

Figure 3.1

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** when $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval. Extrema can occur at interior points or endpoints of an interval (see Figure 3.1). Extrema that occur at the endpoints are called **endpoint extrema.**

A function need not have a minimum or a maximum on an interval. For instance, in Figure 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 3.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval. This suggests the theorem below. (Although the Extreme Value Theorem is intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

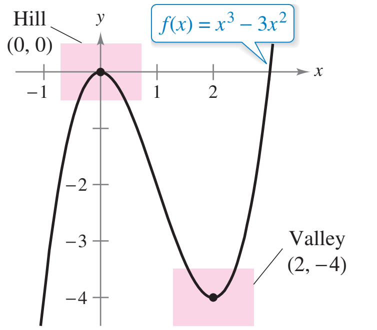
Exploration

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the *minimum* and *maximum* features of a graphing utility to find the extrema of each function. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

Relative Extrema and Critical Numbers



f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.
Figure 3.2

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point $(0, 0)$ and a **relative minimum** at the point $(2, -4)$. Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. When the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). When the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 3.3.)

EXAMPLE 1 The Value of the Derivative at Relative Extrema

Find the value of the derivative at each relative extremum shown in Figure 3.3.

Solution

a. The derivative of $f(x) = \dfrac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

$$= \frac{9(9 - x^2)}{x^4}. \qquad \text{Simplify.}$$

At the point $(3, 2)$, the value of the derivative is $f'(3) = 0$ [see Figure 3.3(a)].

b. At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ [see Figure 3.3(b)].

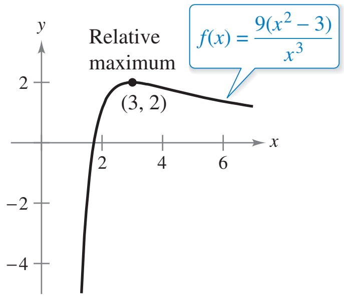
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

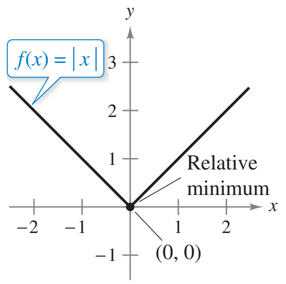
c. The derivative of $f(x) = \sin x$ is

$$f'(x) = \cos x.$$

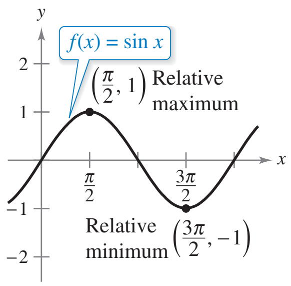
At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ [see Figure 3.3(c)].



(a) $f'(3) = 0$



(b) $f'(0)$ does not exist.



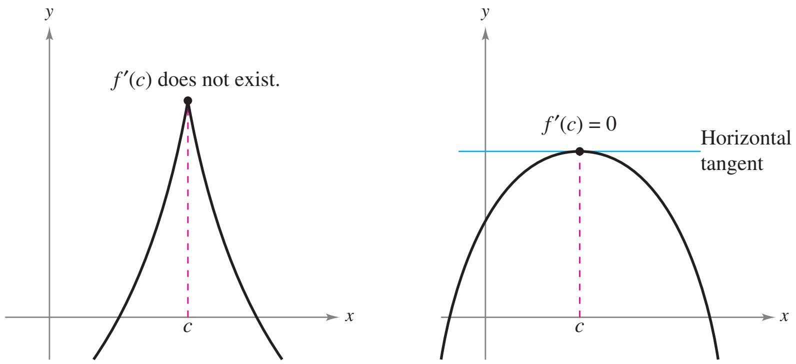
(c) $f'\left(\frac{\pi}{2}\right) = 0$; $f'\left(\frac{3\pi}{2}\right) = 0$

Figure 3.3

Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The x-values at these special points are called **critical numbers.** Figure 3.4 illustrates the two types of critical numbers. Notice in the definition that the critical number c has to be in the domain of f, but c does not have to be in the domain of f'.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.
Figure 3.4

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Proof

Case 1: If f is *not* differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0. Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b).$$

[See Exercise 78(b), Section 1.2.]

Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x-values in the interval (a, b).

Left of c: $x < c$ and $f(x) < f(c)$ ⇨ $f(c)$ is not a relative minimum.

Right of c: $x > c$ and $f(x) > f(c)$ ⇨ $f(c)$ is not a relative maximum.

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid. ■

See LarsonCalculus.com for Bruce Edwards's video of this proof.

PIERRE DE FERMAT (1601–1665)

For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, Theorem 3.2 is sometimes attributed to Fermat.
See LarsonCalculus.com to read more of this biography.

The Print Collector/Alamy

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

> **GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL**
>
> To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.
>
> **1.** Find the critical numbers of f in (a, b).
>
> **2.** Evaluate f at each critical number in (a, b).
>
> **3.** Evaluate f at each endpoint of $[a, b]$.
>
> **4.** The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

EXAMPLE 2 Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 3x^4 - 4x^3$$

on the interval $[-1, 2]$.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad \text{Write original function.}$$

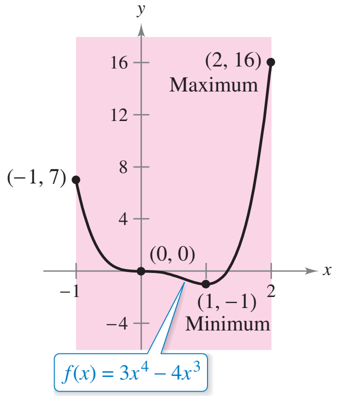
$$f'(x) = 12x^3 - 12x^2 \qquad \text{Differentiate.}$$

To find the critical numbers of f in the interval $(-1, 2)$, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

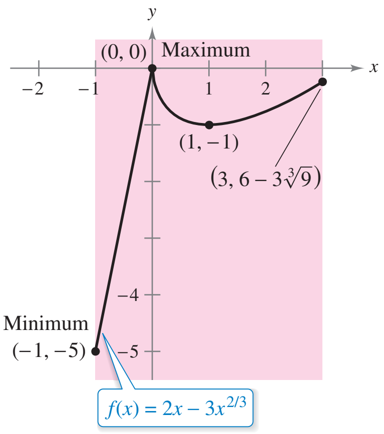


On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.
Figure 3.5

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table. The graph of f is shown in Figure 3.5.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.

Figure 3.6

EXAMPLE 3 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

Solution Begin by differentiating the function.

$$f(x) = 2x - 3x^{2/3} \qquad \text{Write original function.}$$

$$f'(x) = 2 - \frac{2}{x^{1/3}} \qquad \text{Differentiate.}$$

$$= 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) \qquad \text{Simplify.}$$

From this derivative, you can see that the function has two critical numbers in the interval $(-1, 3)$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as shown in the table. The graph of f is shown in Figure 3.6.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

AP* Tips

If an AP question asks for an absolute extremum of a function on a closed interval, be sure to check the y-coordinates at the endpoints of the interval.

EXAMPLE 4 Finding Extrema on a Closed Interval

See LarsonCalculus.com for an interactive version of this type of example.

Find the extrema of

$$f(x) = 2 \sin x - \cos 2x$$

on the interval $[0, 2\pi]$.

Solution Begin by differentiating the function.

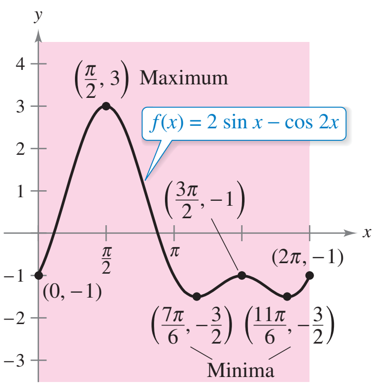
$$f(x) = 2 \sin x - \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2 \cos x + 2 \sin 2x \qquad \text{Differentiate.}$$

$$= 2 \cos x + 4 \cos x \sin x \qquad \sin 2x = 2 \cos x \sin x$$

$$= 2(\cos x)(1 + 2 \sin x) \qquad \text{Factor.}$$

Because f is differentiable for all real x, you can find all critical numbers of f by finding the zeros of its derivative. Considering $2(\cos x)(1 + 2 \sin x) = 0$ in the interval $(0, 2\pi)$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2 \sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at *two* points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as shown in the table. The graph is shown in Figure 3.7.



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.

Figure 3.7

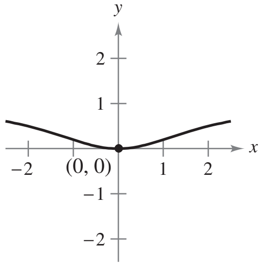
Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

3.1 Exercises

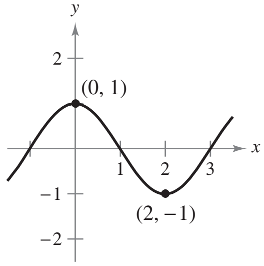
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Value of the Derivative at Relative Extrema In Exercises 1–6, find the value of the derivative (if it exists) at each indicated extremum.

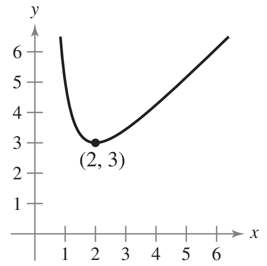
1. $f(x) = \dfrac{x^2}{x^2 + 4}$



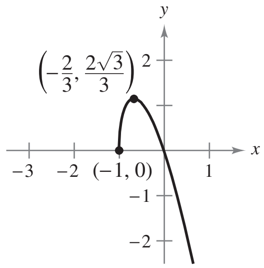
2. $f(x) = \cos \dfrac{\pi x}{2}$



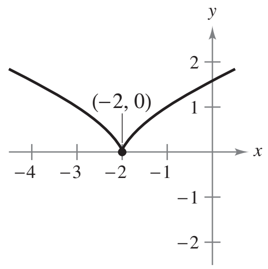
3. $g(x) = x + \dfrac{4}{x^2}$



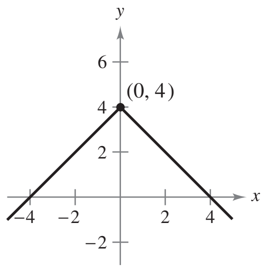
4. $f(x) = -3x\sqrt{x + 1}$



5. $f(x) = (x + 2)^{2/3}$

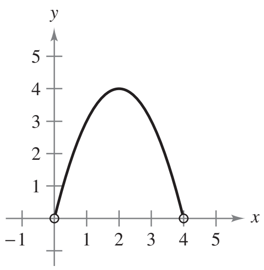


6. $f(x) = 4 - |x|$



Approximating Critical Numbers In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, a relative minimum, an absolute maximum, an absolute minimum, or none of these at each critical number on the interval shown.

7.



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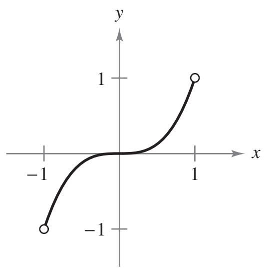

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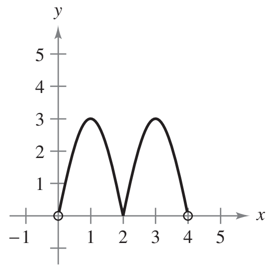

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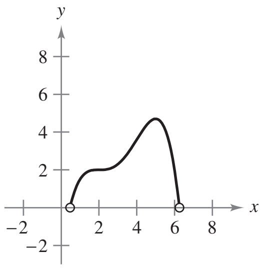


Finding Critical Numbers In Exercises 11–16, find the critical numbers of the function.

11. $f(x) = x^3 - 3x^2$

12. $g(x) = x^4 - 8x^2$

13. $g(t) = t\sqrt{4 - t},\ t < 3$

14. $f(x) = \dfrac{4x}{x^2 + 1}$

15. $h(x) = \sin^2 x + \cos x$
$0 < x < 2\pi$

16. $f(\theta) = 2 \sec \theta + \tan \theta$
$0 < \theta < 2\pi$

Finding Extrema on a Closed Interval In Exercises 17–36, find the absolute extrema of the function on the closed interval.

17. $f(x) = 3 - x,\ [-1, 2]$

18. $f(x) = \dfrac{3}{4}x + 2,\ [0, 4]$

19. $g(x) = 2x^2 - 8x,\ [0, 6]$

20. $h(x) = 5 - x^2,\ [-3, 1]$

21. $f(x) = x^3 - \dfrac{3}{2}x^2,\ [-1, 2]$

22. $f(x) = 2x^3 - 6x,\ [0, 3]$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

24. $g(x) = \sqrt[3]{x},\ [-8, 8]$

25. $g(t) = \dfrac{t^2}{t^2 + 3},\ [-1, 1]$

26. $f(x) = \dfrac{2x}{x^2 + 1},\ [-2, 2]$

27. $h(s) = \dfrac{1}{s - 2},\ [0, 1]$

28. $h(t) = \dfrac{t}{t + 3},\ [-1, 6]$

29. $y = 3 - |t - 3|, [-1, 5]$

30. $g(x) = |x + 4|,\ [-7, 1]$

31. $f(x) = [\![x]\!],\ [-2, 2]$

32. $h(x) = [\![2 - x]\!],\ [-2, 2]$

33. $f(x) = \sin x,\ \left[\dfrac{5\pi}{6}, \dfrac{11\pi}{6}\right]$

34. $g(x) = \sec x,\ \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

35. $y = 3 \cos x,\ [0, 2\pi]$

36. $y = \tan\left(\dfrac{\pi x}{8}\right),\ [0, 2]$

Finding Extrema on an Interval In Exercises 37–40, find the absolute extrema of the function (if any exist) on each interval.

37. $f(x) = 2x - 3$
(a) $[0, 2]$ (b) $[0, 2)$
(c) $(0, 2]$ (d) $(0, 2)$

38. $f(x) = 5 - x$
(a) $[1, 4]$ (b) $[1, 4)$
(c) $(1, 4]$ (d) $(1, 4)$

39. $f(x) = x^2 - 2x$
(a) $[-1, 2]$ (b) $(1, 3]$
(c) $(0, 2)$ (d) $[1, 4)$

40. $f(x) = \sqrt{4 - x^2}$
(a) $[-2, 2]$ (b) $[-2, 0)$
(c) $(-2, 2)$ (d) $[1, 2)$

Finding Absolute Extrema **In Exercises 41–44, use a graphing utility to graph the function and find the absolute extrema of the function on the given interval.**

41. $f(x) = \dfrac{3}{x-1}, \quad (1, 4]$

42. $f(x) = \dfrac{2}{2-x}, \quad [0, 2)$

43. $f(x) = x^4 - 2x^3 + x + 1, \quad [-1, 3]$

44. $f(x) = \sqrt{x} + \cos\dfrac{x}{2}, \quad [0, 2\pi]$

Finding Extrema Using Technology **In Exercises 45 and 46, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the given interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).**

45. $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

46. $f(x) = \dfrac{4}{3}x\sqrt{3-x}, \quad [0, 3]$

Finding Maximum Values Using Technology **In Exercises 47 and 48, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 4.6.)**

47. $f(x) = \sqrt{1 + x^3}, \quad [0, 2]$

48. $f(x) = \dfrac{1}{x^2+1}, \quad \left[\dfrac{1}{2}, 3\right]$

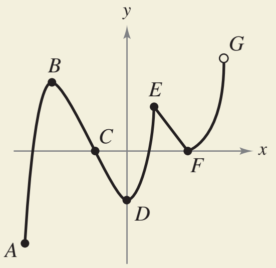
Finding Maximum Values Using Technology **In Exercises 49 and 50, use a computer algebra system to find the maximum value of $|f^{(4)}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 4.6.)**

49. $f(x) = (x+1)^{2/3}, \quad [0, 2]$

50. $f(x) = \dfrac{1}{x^2+1}, \quad [-1, 1]$

51. Writing Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of such a function.

52. HOW DO YOU SEE IT? Determine whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum, or none of these.



WRITING ABOUT CONCEPTS

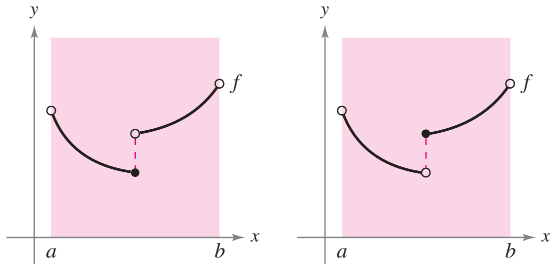
Creating the Graph of a Function **In Exercises 53 and 54, graph a function on the interval $[-2, 5]$ having the given characteristics.**

53. Absolute maximum at $x = -2$
Absolute minimum at $x = 1$
Relative maximum at $x = 3$

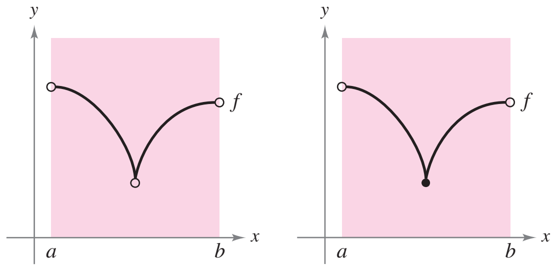
54. Relative minimum at $x = -1$
Critical number (but no extremum) at $x = 0$
Absolute maximum at $x = 2$
Absolute minimum at $x = 5$

Using Graphs **In Exercises 55–58, determine from the graph whether f has a minimum in the open interval (a, b).**

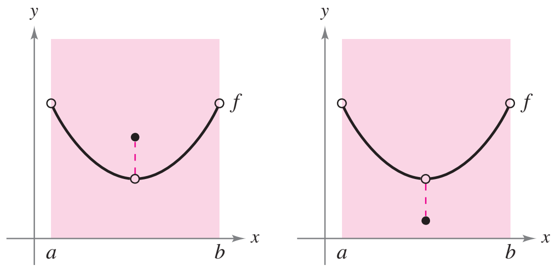
55. (a) (b)



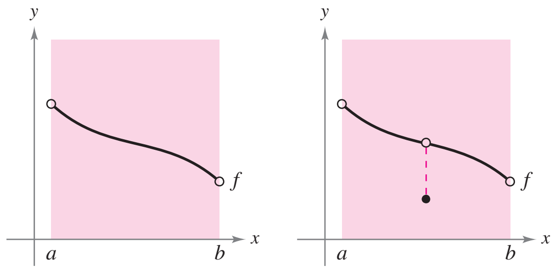
56. (a) (b)



57. (a) (b)



58. (a) (b)



59. Power The formula for the power output P of a battery is

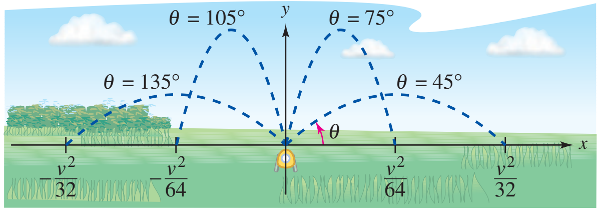
$$P = VI - RI^2$$

where V is the electromotive force in volts, R is the resistance in ohms, and I is the current in amperes. Find the current that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohm. Assume that a 15-ampere fuse bounds the output in the interval $0 \le I \le 15$. Could the power output be increased by replacing the 15-ampere fuse with a 20-ampere fuse? Explain.

60. Lawn Sprinkler A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \quad 45° \le \theta \le 135°$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?



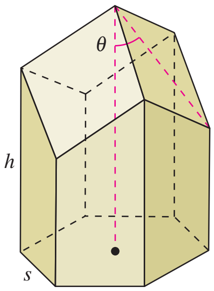
Water sprinkler: $45° \le \theta \le 135°$

FOR FURTHER INFORMATION For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

61. Honeycomb The surface area of a cell in a honeycomb is

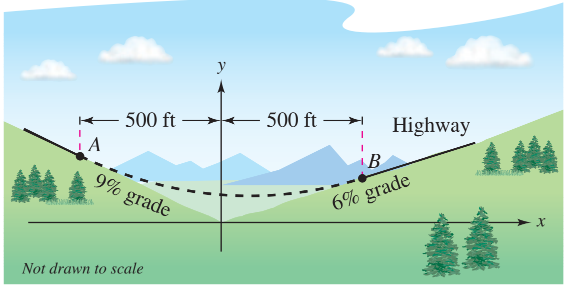
$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell (see figure). Find the angle θ $(\pi/6 \le \theta \le \pi/2)$ that minimizes the surface area S.



FOR FURTHER INFORMATION For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA.

62. Highway Design In order to build a highway, it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B. The horizontal distances from A to the y-axis and from B to the y-axis are both 500 feet.



(a) Find the coordinates of A and B.

(b) Find a quadratic function $y = ax^2 + bx + c$ for $-500 \le x \le 500$ that describes the top of the filled region.

(c) Construct a table giving the depths d of the fill for $x = -500, -400, -300, -200, -100, 0, 100, 200, 300, 400,$ and 500.

(d) What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?

True or False? **In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

64. If a function is continuous on a closed interval, then it must have a minimum on the interval.

65. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

66. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

67. Functions Let the function f be differentiable on an interval I containing c. If f has a maximum value at $x = c$, show that $-f$ has a minimum value at $x = c$.

68. Critical Numbers Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, where $a \ne 0$. Show that f can have zero, one, or two critical numbers and give an example of each case.

PUTNAM EXAM CHALLENGE

69. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region $R = \{(x, y);\ 0 \le x \le a,\ 0 \le y \le f(x)\}$ has perimeter k units and area k square units for some real number k.

This problem was composed by the Committee on the Putnam Prize Competition.

3.2 Rolle's Theorem and the Mean Value Theorem

- **Understand and use Rolle's Theorem.**
- **Understand and use the Mean Value Theorem.**

Rolle's Theorem

The Extreme Value Theorem (see Section 3.1) states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints. **Rolle's Theorem,** named after the French mathematician Michel Rolle (1652–1719), gives conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Proof Let $f(a) = d = f(b)$.

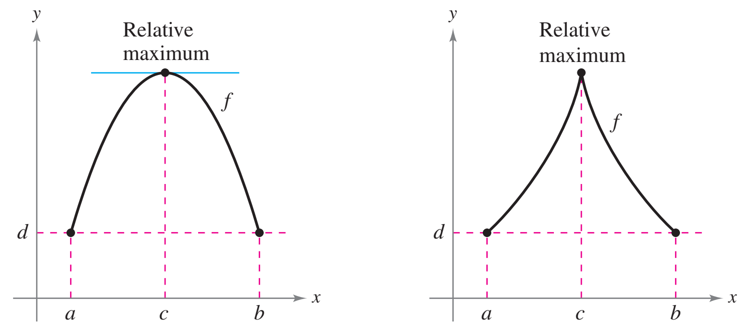
Case 1: If $f(x) = d$ for all x in $[a, b]$, then f is constant on the interval and, by Theorem 2.2, $f'(x) = 0$ for all x in (a, b).

Case 2: Consider $f(x) > d$ for some x in (a, b). By the Extreme Value Theorem, you know that f has a maximum at some c in the interval. Moreover, because $f(c) > d$, this maximum does not occur at either endpoint. So, f has a maximum in the *open* interval (a, b). This implies that $f(c)$ is a *relative* maximum and, by Theorem 3.2, c is a critical number of f. Finally, because f is differentiable at c, you can conclude that $f'(c) = 0$.

Case 3: When $f(x) < d$ for some x in (a, b), you can use an argument similar to that in Case 2, but involving the minimum instead of the maximum.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

From Rolle's Theorem, you can see that if a function f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there must be at least one x-value between a and b at which the graph of f has a horizontal tangent [see Figure 3.8(a)]. When the differentiability requirement is dropped from Rolle's Theorem, f will still have a critical number in (a, b), but it may not yield a horizontal tangent. Such a case is shown in Figure 3.8(b).



(a) f is continuous on $[a, b]$ and differentiable on (a, b).

(b) f is continuous on $[a, b]$.

Figure 3.8

Exploration

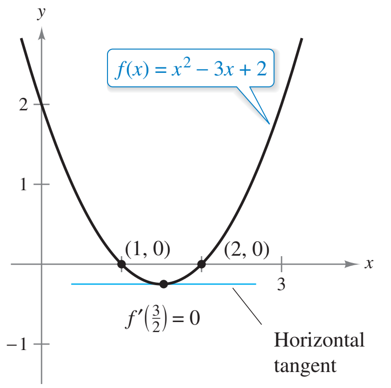
Extreme Values in a Closed Interval Sketch a rectangular coordinate plane on a piece of paper. Label the points $(1, 3)$ and $(5, 3)$. Using a pencil or pen, draw the graph of a differentiable function f that starts at $(1, 3)$ and ends at $(5, 3)$. Is there at least one point on the graph for which the derivative is zero? Would it be possible to draw the graph so that there *isn't* a point for which the derivative is zero? Explain your reasoning.

ROLLE'S THEOREM

French mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that it gave erroneous results and was based on unsound reasoning. Later in life, Rolle came to see the usefulness of calculus.

AP* Tips

On some AP free response questions, there may be more than one way of applying derivatives and the theorems of Chapters 1 and 3 to justify your answer.



The x-value for which $f'(x) = 0$ is between the two x-intercepts.
Figure 3.9

EXAMPLE 1 Illustrating Rolle's Theorem

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

Solution Note that f is differentiable on the entire real number line. Setting $f(x)$ equal to 0 produces

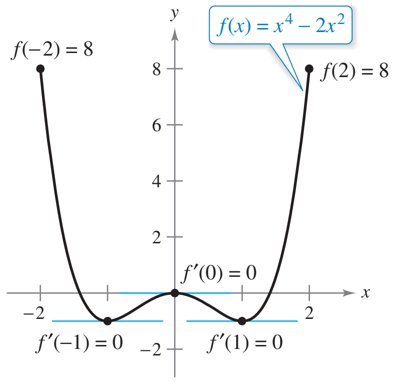
$$x^2 - 3x + 2 = 0 \qquad \text{Set } f(x) \text{ equal to 0.}$$
$$(x - 1)(x - 2) = 0 \qquad \text{Factor.}$$
$$x = 1, 2. \qquad x\text{-values for which } f'(x) = 0$$

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there *exists* at least one c in the interval $(1, 2)$ such that $f'(c) = 0$. To *find* such a c, differentiate f to obtain

$$f'(x) = 2x - 3 \qquad \text{Differentiate.}$$

and then determine that $f'(x) = 0$ when $x = \frac{3}{2}$. Note that this x-value lies in the open interval $(1, 2)$, as shown in Figure 3.9.

Rolle's Theorem states that when f satisfies the conditions of the theorem, there must be *at least* one point between a and b at which the derivative is 0. There may, of course, be more than one such point, as shown in the next example.



$f'(x) = 0$ for more than one x-value in the interval $(-2, 2)$.
Figure 3.10

EXAMPLE 2 Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

Solution To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover, because $f(-2) = f(2) = 8$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Because

$$f'(x) = 4x^3 - 4x \qquad \text{Differentiate.}$$

setting the derivative equal to 0 produces

$$4x^3 - 4x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$
$$4x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1, -1. \qquad x\text{-values for which } f'(x) = 0$$

So, in the interval $(-2, 2)$, the derivative is zero at three different values of x, as shown in Figure 3.10.

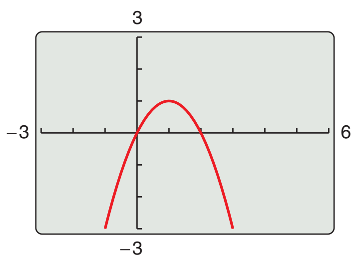


Figure 3.11

TECHNOLOGY PITFALL A graphing utility can be used to indicate whether the points on the graphs in Examples 1 and 2 are relative minima or relative maxima of the functions. When using a graphing utility, however, you should keep in mind that it can give misleading pictures of graphs. For example, use a graphing utility to graph

$$f(x) = 1 - (x - 1)^2 - \frac{1}{1000(x - 1)^{1/7} + 1}.$$

With most viewing windows, it appears that the function has a maximum of 1 when $x = 1$ (see Figure 3.11). By evaluating the function at $x = 1$, however, you can see that $f(1) = 0$. To determine the behavior of this function near $x = 1$, you need to examine the graph analytically to get the complete picture.

The Mean Value Theorem

Rolle's Theorem can be used to prove another theorem—the **Mean Value Theorem.**

REMARK The "mean" in the Mean Value Theorem refers to the mean (or average) rate of change of f on the interval $[a, b]$.

> **THEOREM 3.4 The Mean Value Theorem**
>
> If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

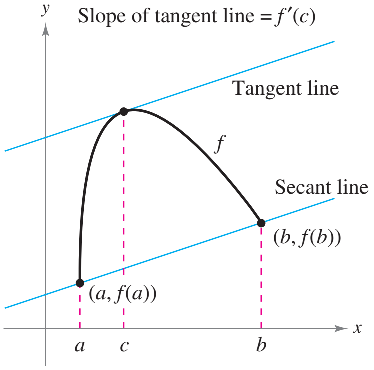


Figure 3.12

Proof Refer to Figure 3.12. The equation of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a). \end{aligned}$$

By evaluating g at a and b, you can see that

$$g(a) = 0 = g(b).$$

Because f is continuous on $[a, b]$, it follows that g is also continuous on $[a, b]$. Furthermore, because f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to the function g. So, there exists a number c in (a, b) such that $g'(c) = 0$, which implies that

$$\begin{aligned} g'(c) &= 0 \\ f'(c) - \frac{f(b) - f(a)}{b - a} &= 0. \end{aligned}$$

So, there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

AP* Tips

Be able to apply the Mean Value Theorem on the AP Exam. It may be referred to directly, or it may be necessary to use the theorem to justify your answer.

JOSEPH-LOUIS LAGRANGE (1736–1813)

The Mean Value Theorem was first proved by the famous mathematician Joseph-Louis Lagrange. Born in Italy, Lagrange held a position in the court of Frederick the Great in Berlin for 20 years.
See LarsonCalculus.com to read more of this biography.

Although the Mean Value Theorem can be used directly in problem solving, it is used more often to prove other theorems. In fact, some people consider this to be the most important theorem in calculus—it is closely related to the Fundamental Theorem of Calculus discussed in Section 4.4. For now, you can get an idea of the versatility of the Mean Value Theorem by looking at the results stated in Exercises 77–85 in this section.

The Mean Value Theorem has implications for both basic interpretations of the derivative. Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points

$$(a, f(a)) \quad \text{and} \quad (b, f(b)),$$

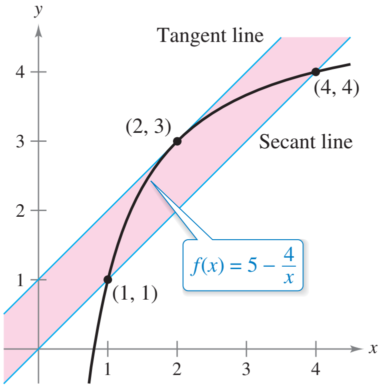
as shown in Figure 3.12. Example 3 illustrates this geometric interpretation of the Mean Value Theorem. In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in Example 4.

EXAMPLE 3 Finding a Tangent Line

See LarsonCalculus.com for an interactive version of this type of example.

For $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$



The tangent line at (2, 3) is parallel to the secant line through (1, 1) and (4, 4).
Figure 3.13

Solution The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1. \qquad \text{Slope of secant line}$$

Note that the function satisfies the conditions of the Mean Value Theorem. That is, f is continuous on the interval $[1, 4]$ and differentiable on the interval $(1, 4)$. So, there exists at least one number c in $(1, 4)$ such that $f'(c) = 1$. Solving the equation $f'(x) = 1$ yields

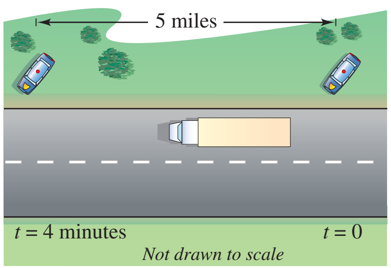
$$\frac{4}{x^2} = 1 \qquad \text{Set } f'(x) \text{ equal to 1.}$$

which implies that

$$x = \pm 2.$$

So, in the interval $(1, 4)$, you can conclude that $c = 2$, as shown in Figure 3.13.

EXAMPLE 4 Finding an Instantaneous Rate of Change



At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.
Figure 3.14

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.

Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s\left(\frac{1}{15}\right) = 5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity} = \frac{s(1/15) - s(0)}{(1/15) - 0} = \frac{5}{1/15} = 75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes.

A useful alternative form of the Mean Value Theorem is: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c). \qquad \text{Alternative form of Mean Value Theorem}$$

When doing the exercises for this section, keep in mind that polynomial functions, rational functions, and trigonometric functions are differentiable at all points in their domains.

3.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Writing **In Exercises 1–4, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.**

1. $f(x) = \left|\frac{1}{x}\right|$, $[-1, 1]$ **2.** $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$

3. $f(x) = 1 - |x - 1|$, $[0, 2]$ **4.** $f(x) = \sqrt{(2 - x^{2/3})^3}$, $[-1, 1]$

Intercepts and Derivatives **In Exercises 5–8, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two x-intercepts.**

5. $f(x) = x^2 - x - 2$ **6.** $f(x) = x^2 + 6x$

7. $f(x) = x\sqrt{x + 4}$ **8.** $f(x) = -3x\sqrt{x + 1}$

Using Rolle's Theorem **In Exercises 9–22, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.**

9. $f(x) = -x^2 + 3x$, $[0, 3]$

10. $f(x) = x^2 - 8x + 5$, $[2, 6]$

11. $f(x) = (x - 1)(x - 2)(x - 3)$, $[1, 3]$

12. $f(x) = (x - 4)(x + 2)^2$, $[-2, 4]$

13. $f(x) = x^{2/3} - 1$, $[-8, 8]$ **14.** $f(x) = 3 - |x - 3|$, $[0, 6]$

15. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$

16. $f(x) = \dfrac{x^2 - 1}{x}$, $[-1, 1]$

17. $f(x) = \sin x$, $[0, 2\pi]$ **18.** $f(x) = \cos x$, $[0, 2\pi]$

19. $f(x) = \sin 3x$, $\left[0, \frac{\pi}{3}\right]$ **20.** $f(x) = \cos 2x$, $[-\pi, \pi]$

21. $f(x) = \tan x$, $[0, \pi]$ **22.** $f(x) = \sec x$, $[\pi, 2\pi]$

Using Rolle's Theorem **In Exercises 23–26, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.**

23. $f(x) = |x| - 1$, $[-1, 1]$ **24.** $f(x) = x - x^{1/3}$, $[0, 1]$

25. $f(x) = x - \tan \pi x$, $\left[-\frac{1}{4}, \frac{1}{4}\right]$

26. $f(x) = \dfrac{x}{2} - \sin \dfrac{\pi x}{6}$, $[-1, 0]$

27. Vertical Motion The height of a ball t seconds after it is thrown upward from a height of 6 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 6$.

(a) Verify that $f(1) = f(2)$.

(b) According to Rolle's Theorem, what must the velocity be at some time in the interval $(1, 2)$? Find that time.

28. Reorder Costs The ordering and transportation cost C for components used in a manufacturing process is approximated by

$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$$

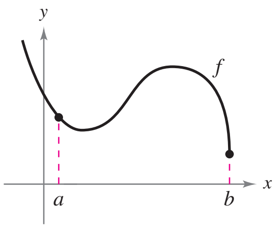
where C is measured in thousands of dollars and x is the order size in hundreds.

(a) Verify that $C(3) = C(6)$.

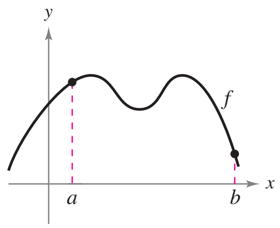
(b) According to Rolle's Theorem, the rate of change of the cost must be 0 for some order size in the interval $(3, 6)$. Find that order size.

Mean Value Theorem **In Exercises 29 and 30, copy the graph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

29.

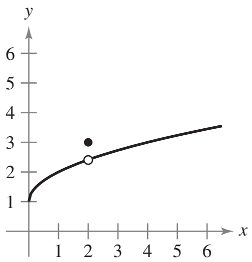


30.

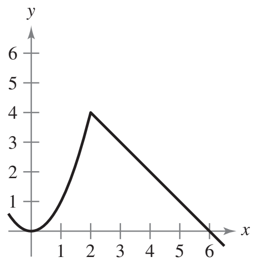


Writing **In Exercises 31–34, explain why the Mean Value Theorem does not apply to the function f on the interval $[0, 6]$.**

31.



32.



33. $f(x) = \dfrac{1}{x - 3}$ **34.** $f(x) = |x - 3|$

35. Mean Value Theorem Consider the graph of the function $f(x) = -x^2 + 5$ (see figure on next page).

(a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line.

(c) Find the equation of the tangent line through c.

(d) Then use a graphing utility to graph f, the secant line, and the tangent line.

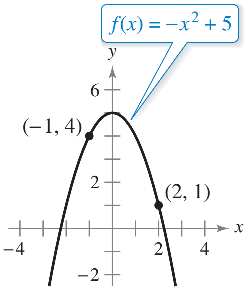


Figure for 35

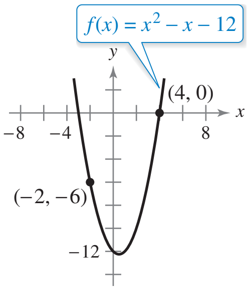


Figure for 36

36. **Mean Value Theorem** Consider the graph of the function $f(x) = x^2 - x - 12$ (see figure).

(a) Find the equation of the secant line joining the points $(-2, -6)$ and $(4, 0)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-2, 4)$ such that the tangent line at c is parallel to the secant line.

(c) Find the equation of the tangent line through c.

(d) Then use a graphing utility to graph f, the secant line, and the tangent line.

Using the Mean Value Theorem **In Exercises 37–46, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

If the Mean Value Theorem cannot be applied, explain why not.

37. $f(x) = x^2, \quad [-2, 1]$

38. $f(x) = 2x^3, \quad [0, 6]$

39. $f(x) = x^3 + 2x, \quad [-1, 1]$

40. $f(x) = x^4 - 8x, \quad [0, 2]$

41. $f(x) = x^{2/3}, \quad [0, 1]$

42. $f(x) = \dfrac{x + 1}{x}, \quad [-1, 2]$

43. $f(x) = |2x + 1|, \quad [-1, 3]$

44. $f(x) = \sqrt{2 - x}, \quad [-7, 2]$

45. $f(x) = \sin x, \quad [0, \pi]$

46. $f(x) = \cos x + \tan x, \quad [0, \pi]$

Using the Mean Value Theorem **In Exercises 47–50, use a graphing utility to (a) graph the function f on the given interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.**

47. $f(x) = \dfrac{x}{x + 1}, \quad \left[-\dfrac{1}{2}, 2\right]$

48. $f(x) = x - 2\sin x, \quad [-\pi, \pi]$

49. $f(x) = \sqrt{x}, \quad [1, 9]$

50. $f(x) = x^4 - 2x^3 + x^2, \quad [0, 6]$

51. **Vertical Motion** The height of an object t seconds after it is dropped from a height of 300 meters is

$$s(t) = -4.9t^2 + 300.$$

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall, the instantaneous velocity equals the average velocity. Find that time.

52. **Sales** A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2 + t}\right)$$

where t is the time in months.

(a) Find the average rate of change of $S(t)$ during the first year.

(b) During what month of the first year does $S'(t)$ equal the average rate of change?

WRITING ABOUT CONCEPTS

53. **Converse of Rolle's Theorem** Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

54. **Rolle's Theorem** Let f be continuous on $[a, b]$ and differentiable on (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

(a) $g(x) = f(x) + k$ (b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

55. **Rolle's Theorem** The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

56. **Mean Value Theorem** Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

57. **Speed**

A plane begins its take-off at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.

58. Temperature When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later, the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

59. Velocity Two bicyclists begin a race at 8:00 A.M. They both finish the race 2 hours and 15 minutes later. Prove that at some time during the race, the bicyclists are traveling at the same velocity.

60. Acceleration At 9:13 A.M., a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.

61. Using a Function Consider the function

$$f(x) = 3\cos^2\left(\frac{\pi x}{2}\right).$$

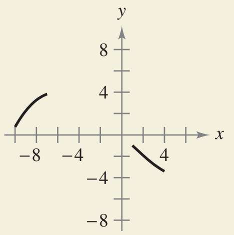
(a) Use a graphing utility to graph f and f'.

(b) Is f a continuous function? Is f' a continuous function?

(c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.

(d) Evaluate, if possible, $\lim_{x\to 3^-} f'(x)$ and $\lim_{x\to 3^+} f'(x)$.

62. HOW DO YOU SEE IT? The figure shows two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Explain why f must have at least one zero in $[-10, 4]$.

(b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?

(c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.

Think About It **In Exercises 63 and 64, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.**

63. f is continuous on $[-5, 5]$.

64. f is not continuous on $[-5, 5]$.

Finding a Solution **In Exercises 65–68, use the Intermediate Value Theorem and Rolle's Theorem to prove that the equation has exactly one real solution.**

65. $x^5 + x^3 + x + 1 = 0$ **66.** $2x^5 + 7x - 1 = 0$

67. $3x + 1 - \sin x = 0$ **68.** $2x - 2 - \cos x = 0$

Differential Equation **In Exercises 69–72, find a function f that has the derivative $f'(x)$ and whose graph passes through the given point. Explain your reasoning.**

69. $f'(x) = 0,\quad (2, 5)$ **70.** $f'(x) = 4,\quad (0, 1)$

71. $f'(x) = 2x,\quad (1, 0)$ **72.** $f'(x) = 6x - 1,\quad (2, 7)$

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. The Mean Value Theorem can be applied to

$$f(x) = \frac{1}{x}$$

on the interval $[-1, 1]$.

74. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

75. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

76. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.

77. Proof Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

78. Proof Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

79. Proof Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

80. Using Rolle's Theorem

(a) Let $f(x) = x^2$ and $g(x) = -x^3 + x^2 + 3x + 2$. Then $f(-1) = g(-1)$ and $f(2) = g(2)$. Show that there is at least one value c in the interval $(-1, 2)$ where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$. Identify c.

(b) Let f and g be differentiable functions on $[a, b]$ where $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one value c in the interval (a, b) where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$.

81. Proof Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.

82. Fixed Point Use the result of Exercise 81 to show that $f(x) = \frac{1}{2}\cos x$ has at most one fixed point.

83. Proof Prove that $|\cos a - \cos b| \le |a - b|$ for all a and b.

84. Proof Prove that $|\sin a - \sin b| \le |a - b|$ for all a and b.

85. Using the Mean Value Theorem Let $0 < a < b$. Use the Mean Value Theorem to show that

$$\sqrt{b} - \sqrt{a} < \frac{b - a}{2\sqrt{a}}.$$

3.3 Increasing and Decreasing Functions and the First Derivative Test

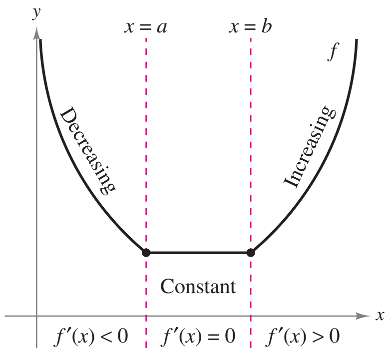
- **Determine intervals on which a function is increasing or decreasing.**
- **Apply the First Derivative Test to find relative extrema of a function.**

Increasing and Decreasing Functions

In this section, you will learn how derivatives can be used to *classify* relative extrema as either relative minima or relative maxima. First, it is important to define increasing and decreasing functions.

> **Definitions of Increasing and Decreasing Functions**
>
> A function f is **increasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
>
> A function f is **decreasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function is increasing when, *as x moves to the right*, its graph moves up, and is decreasing when its graph moves down. For example, the function in Figure 3.15 is decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b), and is increasing on the interval (b, ∞). As shown in Theorem 3.5 below, a positive derivative implies that the function is increasing, a negative derivative implies that the function is decreasing, and a zero derivative on an entire interval implies that the function is constant on that interval.



The derivative is related to the slope of a function.
Figure 3.15

> **THEOREM 3.5 Test for Increasing and Decreasing Functions**
>
> Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
>
> **1.** If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
> **2.** If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
> **3.** If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

REMARK The conclusions in the first two cases of Theorem 3.5 are valid even when $f'(x) = 0$ at a finite number of x-values in (a, b).

Proof To prove the first case, assume that $f'(x) > 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know that there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) > 0$ and $x_2 - x_1 > 0$, you know that $f(x_2) - f(x_1) > 0$, which implies that $f(x_1) < f(x_2)$. So, f is increasing on the interval. The second case has a similar proof (see Exercise 97), and the third case is a consequence of Exercise 78 in Section 3.2.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

AP* Tips

Sign charts are useful organization tools, but will not be graded on the AP Exam. To receive full credit, you must justify your responses in complete sentences.

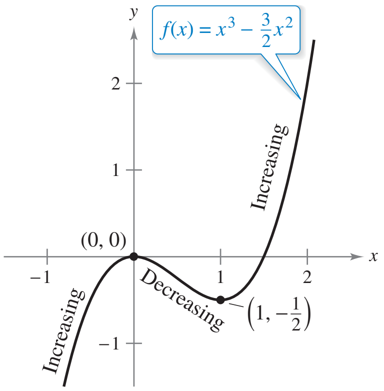


Figure 3.16

EXAMPLE 1 Intervals on Which f Is Increasing or Decreasing

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution Note that f is differentiable on the entire real number line and the derivative of f is

$$f(x) = x^3 - \tfrac{3}{2}x^2 \qquad \text{Write original function.}$$
$$f'(x) = 3x^2 - 3x. \qquad \text{Differentiate.}$$

To determine the critical numbers of f, set $f'(x)$ equal to zero.

$$3x^2 - 3x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$
$$3(x)(x-1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1 \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

By Theorem 3.5, f is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 3.16. ■

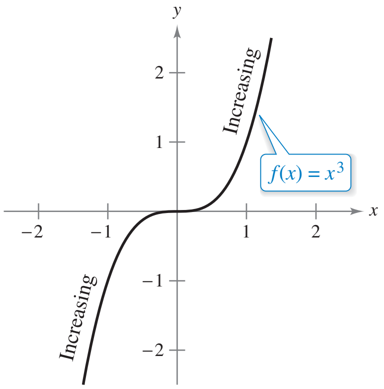
Example 1 gives you one instance of how to find intervals on which a function is increasing or decreasing. The guidelines below summarize the steps followed in that example.

GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

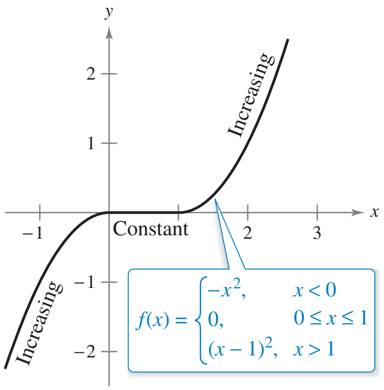
Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid when the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.



(a) Strictly monotonic function



(b) Not strictly monotonic

Figure 3.17

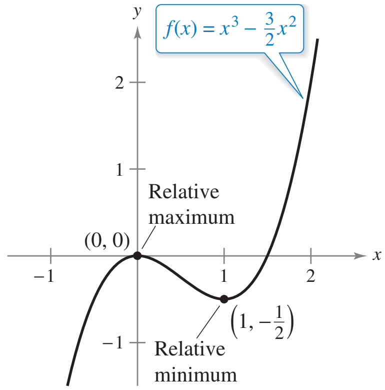
A function is **strictly monotonic** on an interval when it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 3.17(a). The function shown in Figure 3.17(b) is not strictly monotonic on the entire real number line because it is constant on the interval $[0, 1]$.

The First Derivative Test

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. For instance, in Figure 3.18 (from Example 1), the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

has a relative maximum at the point $(0, 0)$ because f is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$. Similarly, f has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because f is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$. The next theorem makes this more explicit.

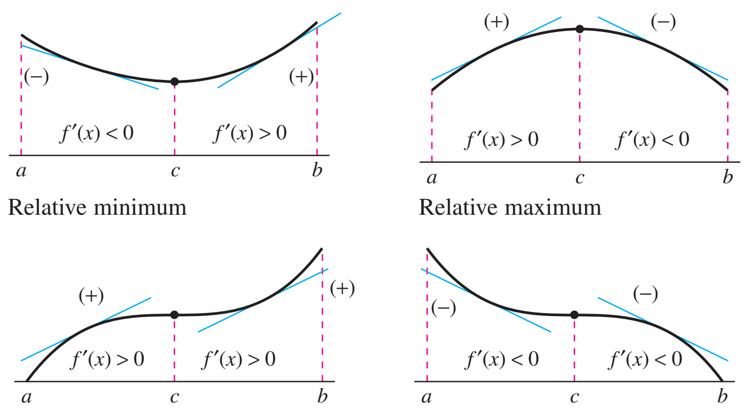


Relative extrema of f
Figure 3.18

THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Neither relative minimum nor relative maximum

Proof Assume that $f'(x)$ changes from negative to positive at c. Then there exist a and b in I such that

$$f'(x) < 0 \text{ for all } x \text{ in } (a, c) \quad \text{and} \quad f'(x) > 0 \text{ for all } x \text{ in } (c, b).$$

By Theorem 3.5, f is decreasing on $[a, c]$ and increasing on $[c, b]$. So, $f(c)$ is a minimum of f on the open interval (a, b) and, consequently, a relative minimum of f. This proves the first case of the theorem. The second case can be proved in a similar way (see Exercise 98).

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 2 Applying the First Derivative Test

Find the relative extrema of $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

Solution Note that f is continuous on the interval $(0, 2\pi)$. The derivative of f is $f'(x) = \frac{1}{2} - \cos x$. To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

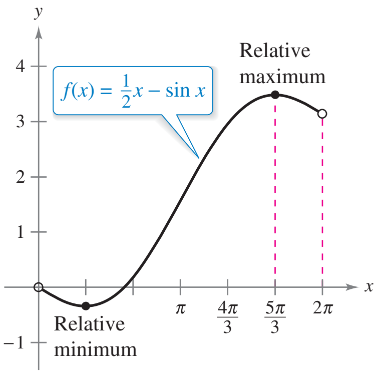
$$\frac{1}{2} - \cos x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers. By applying the First Derivative Test, you can conclude that f has a relative minimum at the point where $x = \pi/3$ and a relative maximum at the point where $x = 5\pi/3$, as shown in Figure 3.19.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'\left(\frac{\pi}{4}\right) < 0$	$f'(\pi) > 0$	$f'\left(\frac{7\pi}{4}\right) < 0$
Conclusion	Decreasing	Increasing	Decreasing



A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.
Figure 3.19

EXAMPLE 3 Applying the First Derivative Test

Find the relative extrema of $f(x) = (x^2 - 4)^{2/3}$.

Solution Begin by noting that f is continuous on the entire real number line. The derivative of f

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{General Power Rule}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2$, $x = 0$, and $x = 2$. The table summarizes the testing of the four intervals determined by these three critical numbers. By applying the First Derivative Test, you can conclude that f has a relative minimum at the point $(-2, 0)$, a relative maximum at the point $\left(0, \sqrt[3]{16}\right)$, and another relative minimum at the point $(2, 0)$, as shown in Figure 3.20.

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

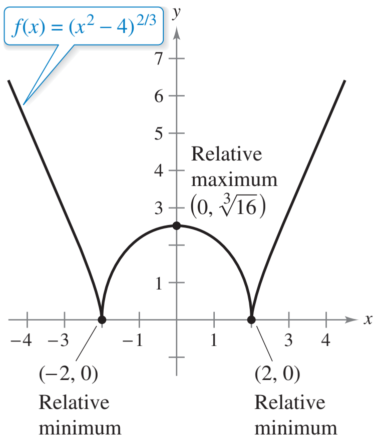


Figure 3.20

Note that in Examples 1 and 2, the given functions are differentiable on the entire real number line. For such functions, the only critical numbers are those for which $f'(x) = 0$. Example 3 concerns a function that has two types of critical numbers—those for which $f'(x) = 0$ and those for which f is not differentiable.

When using the First Derivative Test, be sure to consider the domain of the function. For instance, in the next example, the function

$$f(x) = \frac{x^4 + 1}{x^2}$$

is not defined when $x = 0$. This x-value must be used with the critical numbers to determine the test intervals.

EXAMPLE 4 Applying the First Derivative Test

See LarsonCalculus.com for an interactive version of this type of example.

Find the relative extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$.

Solution Note that f is not defined when $x = 0$.

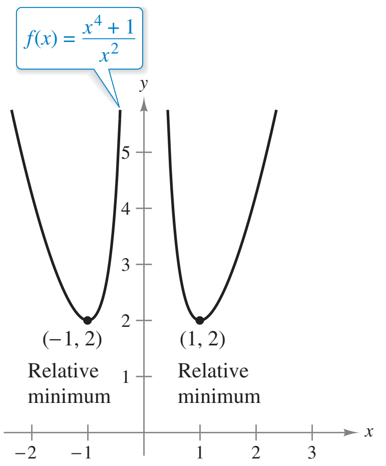
$$\begin{aligned} f(x) &= x^2 + x^{-2} && \text{Rewrite original function.} \\ f'(x) &= 2x - 2x^{-3} && \text{Differentiate.} \\ &= 2x - \frac{2}{x^3} && \text{Rewrite with positive exponent.} \\ &= \frac{2(x^4 - 1)}{x^3} && \text{Simplify.} \\ &= \frac{2(x^2 + 1)(x - 1)(x + 1)}{x^3} && \text{Factor.} \end{aligned}$$

So, $f'(x)$ is zero at $x = \pm 1$. Moreover, because $x = 0$ is not in the domain of f, you should use this x-value along with the critical numbers to determine the test intervals.

$$\begin{aligned} x &= \pm 1 && \text{Critical numbers, } f'(\pm 1) = 0 \\ x &= 0 && 0 \text{ is not in the domain of } f. \end{aligned}$$

The table summarizes the testing of the four intervals determined by these three x-values. By applying the First Derivative Test, you can conclude that f has one relative minimum at the point $(-1, 2)$ and another at the point $(1, 2)$, as shown in Figure 3.21.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'\left(-\frac{1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing



x-values that are not in the domain of f, as well as critical numbers, determine test intervals for f'.

Figure 3.21

TECHNOLOGY The most difficult step in applying the First Derivative Test is finding the values for which the derivative is equal to 0. For instance, the values of x for which the derivative of

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

is equal to zero are $x = 0$ and $x = \pm\sqrt{\sqrt{2} - 1}$. If you have access to technology that can perform symbolic differentiation and solve equations, use it to apply the First Derivative Test to this function.

When a projectile is propelled from ground level and air resistance is neglected, the object will travel farthest with an initial angle of 45°. When, however, the projectile is propelled from a point above ground level, the angle that yields a maximum horizontal distance is not 45° (see Example 5).

AP* Tips

Questions that involve velocity or position functions are common on the AP Exam.

EXAMPLE 5 The Path of a Projectile

Neglecting air resistance, the path of a projectile that is propelled at an angle θ is

$$y = \frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h, \quad 0 \le \theta \le \frac{\pi}{2}$$

where y is the height, x is the horizontal distance, g is the acceleration due to gravity, v_0 is the initial velocity, and h is the initial height. (This equation is derived in Section 12.3.) Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h = 9$ feet. What value of θ will produce a maximum horizontal distance?

Solution To find the distance the projectile travels, let $y = 0$, $g = -32$, $v_0 = 24$, and $h = 9$. Then substitute these values in the given equation as shown.

$$\frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h = y$$

$$\frac{-32 \sec^2 \theta}{2(24^2)}x^2 + (\tan \theta)x + 9 = 0$$

$$-\frac{\sec^2 \theta}{36}x^2 + (\tan \theta)x + 9 = 0$$

Next, solve for x using the Quadratic Formula with $a = -\sec^2 \theta/36$, $b = \tan \theta$, and $c = 9$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-\tan \theta \pm \sqrt{(\tan \theta)^2 - 4(-\sec^2 \theta/36)(9)}}{2(-\sec^2 \theta/36)}$$

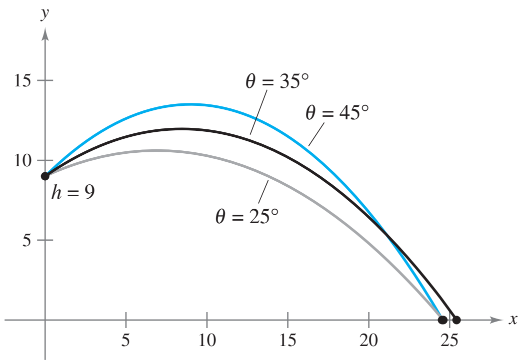
$$x = \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + \sec^2 \theta}}{-\sec^2 \theta/18}$$

$$x = 18 \cos \theta\left(\sin \theta + \sqrt{\sin^2 \theta + 1}\right), \quad x \ge 0$$

At this point, you need to find the value of θ that produces a maximum value of x. Applying the First Derivative Test by hand would be very tedious. Using technology to solve the equation $dx/d\theta = 0$, however, eliminates most of the messy computations. The result is that the maximum value of x occurs when

$$\theta \approx 0.61548 \text{ radian}, \quad \text{or} \quad 35.3°.$$

This conclusion is reinforced by sketching the path of the projectile for different values of θ, as shown in Figure 3.22. Of the three paths shown, note that the distance traveled is greatest for $\theta = 35°$.



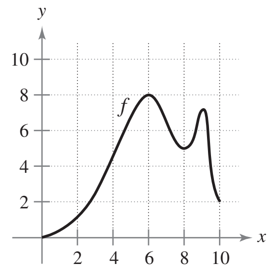
The path of a projectile with initial angle θ
Figure 3.22

3.3 Exercises

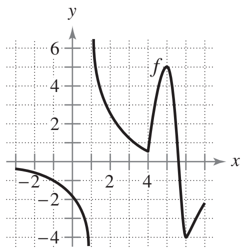
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using a Graph In Exercises 1 and 2, use the graph of f to find (a) the largest open interval on which f is increasing, and (b) the largest open interval on which f is decreasing.

1.

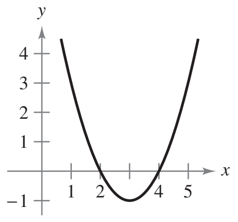


2.

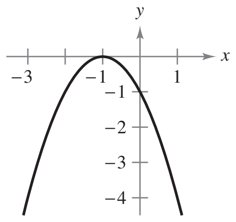


Using a Graph In Exercises 3–8, use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.

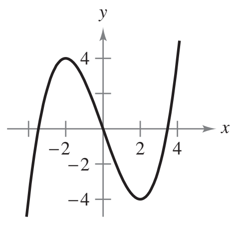
3. $f(x) = x^2 - 6x + 8$



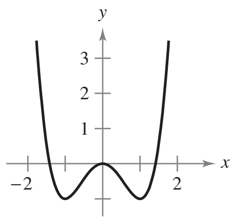
4. $y = -(x + 1)^2$



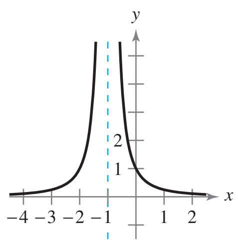
5. $y = \dfrac{x^3}{4} - 3x$



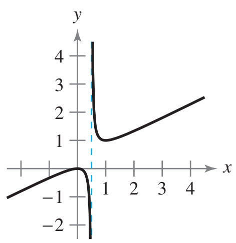
6. $f(x) = x^4 - 2x^2$



7. $f(x) = \dfrac{1}{(x + 1)^2}$



8. $y = \dfrac{x^2}{2x - 1}$



Intervals on Which f Is Increasing or Decreasing In Exercises 9–16, identify the open intervals on which the function is increasing or decreasing.

9. $g(x) = x^2 - 2x - 8$

10. $h(x) = 12x - x^3$

11. $y = x\sqrt{16 - x^2}$

12. $y = x + \dfrac{9}{x}$

13. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

14. $h(x) = \cos \dfrac{x}{2}, \quad 0 < x < 2\pi$

15. $y = x - 2\cos x, \quad 0 < x < 2\pi$

16. $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

Applying the First Derivative Test In Exercises 17–40, (a) find the critical numbers of f (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.

17. $f(x) = x^2 - 4x$

18. $f(x) = x^2 + 6x + 10$

19. $f(x) = -2x^2 + 4x + 3$

20. $f(x) = -3x^2 - 4x - 2$

21. $f(x) = 2x^3 + 3x^2 - 12x$

22. $f(x) = x^3 - 6x^2 + 15$

23. $f(x) = (x - 1)^2(x + 3)$

24. $f(x) = (x + 2)^2(x - 1)$

25. $f(x) = \dfrac{x^5 - 5x}{5}$

26. $f(x) = x^4 - 32x + 4$

27. $f(x) = x^{1/3} + 1$

28. $f(x) = x^{2/3} - 4$

29. $f(x) = (x + 2)^{2/3}$

30. $f(x) = (x - 3)^{1/3}$

31. $f(x) = 5 - |x - 5|$

32. $f(x) = |x + 3| - 1$

33. $f(x) = 2x + \dfrac{1}{x}$

34. $f(x) = \dfrac{x}{x - 5}$

35. $f(x) = \dfrac{x^2}{x^2 - 9}$

36. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

37. $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

38. $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

39. $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

40. $f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

Applying the First Derivative Test In Exercises 41–48, consider the function on the interval $(0, 2\pi)$. For each function, (a) find the open interval(s) on which the function is increasing or decreasing, (b) apply the First Derivative Test to identify all relative extrema, and (c) use a graphing utility to confirm your results.

41. $f(x) = \dfrac{x}{2} + \cos x$

42. $f(x) = \sin x \cos x + 5$

43. $f(x) = \sin x + \cos x$

44. $f(x) = x + 2\sin x$

45. $f(x) = \cos^2(2x)$

46. $f(x) = \sin x - \sqrt{3}\cos x$

47. $f(x) = \sin^2 x + \sin x$

48. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

Finding and Analyzing Derivatives Using Technology

In Exercises 49–54, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of f and f' on the same set of coordinate axes over the given interval, (c) find the critical numbers of f in the open interval, and (d) find the interval(s) on which f' is positive and the interval(s) on which it is negative. Compare the behavior of f and the sign of f'.

49. $f(x) = 2x\sqrt{9 - x^2}, \quad [-3, 3]$

50. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), \quad [0, 5]$

51. $f(t) = t^2 \sin t, \quad [0, 2\pi]$

52. $f(x) = \frac{x}{2} + \cos\frac{x}{2}, \quad [0, 4\pi]$

53. $f(x) = -3\sin\frac{x}{3}, \quad [0, 6\pi]$

54. $f(x) = 2\sin 3x + 4\cos 3x, \quad [0, \pi]$

Comparing Functions

In Exercises 55 and 56, use symmetry, extrema, and zeros to sketch the graph of f. How do the functions f and g differ?

55. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1}$

$g(x) = x(x^2 - 3)$

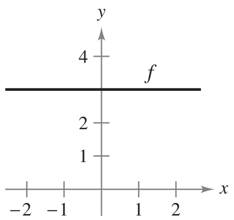
56. $f(t) = \cos^2 t - \sin^2 t$

$g(t) = 1 - 2\sin^2 t$

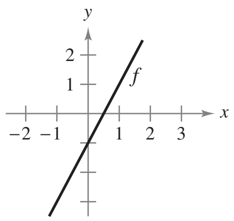
Think About It

In Exercises 57–62, the graph of f is shown in the figure. Sketch a graph of the derivative of f. To print an enlarged copy of the graph, go to *MathGraphs.com*.

57.



58.



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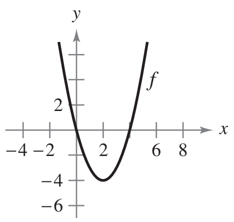

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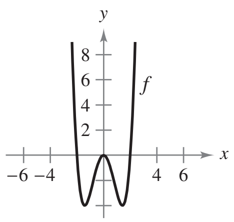

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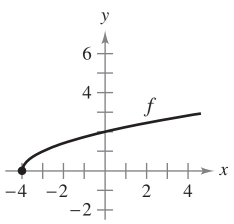
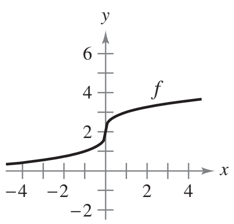


62.



WRITING ABOUT CONCEPTS

Transformations of Functions

In Exercises 63–68, assume that f is differentiable for all x. The signs of f' are as follows.

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Supply the appropriate inequality sign for the indicated value of c.

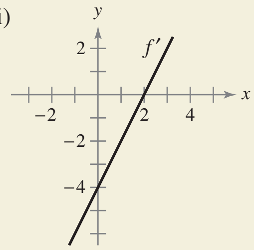
	Function	Sign of $g'(c)$
63.	$g(x) = f(x) + 5$	$g'(0)$ ▢ 0
64.	$g(x) = 3f(x) - 3$	$g'(-5)$ ▢ 0
65.	$g(x) = -f(x)$	$g'(-6)$ ▢ 0
66.	$g(x) = -f(x)$	$g'(0)$ ▢ 0
67.	$g(x) = f(x - 10)$	$g'(0)$ ▢ 0
68.	$g(x) = f(x - 10)$	$g'(8)$ ▢ 0

69. Sketching a Graph Sketch the graph of the arbitrary function f such that

$$f'(x)\begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4. \\ < 0, & x > 4 \end{cases}$$

70. HOW DO YOU SEE IT? Use the graph of f' to (a) identify the critical numbers of f, (b) identify the open interval(s) on which f is increasing or decreasing, and (c) determine whether f has a relative maximum, a relative minimum, or neither at each critical number.

(i)



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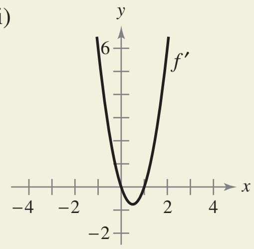

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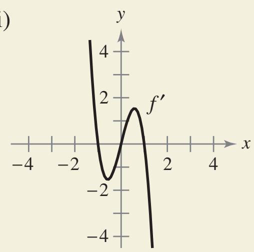
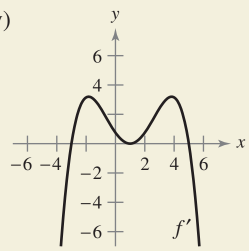


(iv)



71. Analyzing a Critical Number A differentiable function f has one critical number at $x = 5$. Identify the relative extrema of f at the critical number when $f'(4) = -2.5$ and $f'(6) = 3$.

72. Analyzing a Critical Number A differentiable function f has one critical number at $x = 2$. Identify the relative extrema of f at the critical number when $f'(1) = 2$ and $f'(3) = 6$.

Think About It **In Exercises 73 and 74, the function f is differentiable on the indicated interval. The table shows $f'(x)$ for selected values of x. (a) Sketch the graph of f, (b) approximate the critical numbers, and (c) identify the relative extrema.**

73. f is differentiable on $[-1, 1]$.

x	-1	-0.75	-0.50	-0.25	0
$f'(x)$	-10	-3.2	-0.5	0.8	5.6

x	0.25	0.50	0.75	1
$f'(x)$	3.6	-0.2	-6.7	-20.1

74. f is differentiable on $[0, \pi]$.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f'(x)$	3.14	-0.23	-2.45	-3.11	0.69

x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$f'(x)$	3.00	1.37	-1.14	-2.84

75. Rolling a Ball Bearing A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ. The distance (in meters) the ball bearing rolls in t seconds is $s(t) = 4.9(\sin\theta)t^2$.

(a) Determine the speed of the ball bearing after t seconds.

(b) Complete the table and use it to determine the value of θ that produces the maximum speed at a particular time.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$							

76. Modeling Data The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1999 through 2010 are shown.

1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;
2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4
2007: 326.0; 2008: 321.3; 2009: 304.2; 2010: 271.9

(Source: U.S. Centers for Medicare and Medicaid Services)

(a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data. (Let $t = 9$ represent 1999.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the maximum value of the model and compare the result with the actual data.

77. Numerical, Graphical, and Analytic Analysis The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

(a) Complete the table and use it to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

(c) Use calculus to determine analytically the time when the concentration is greatest.

78. Numerical, Graphical, and Analytic Analysis Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint*: Show that $h'(x) > 0$, where $h = f - g$.]

79. Trachea Contraction Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. The velocity of the air during coughing is

$$v = k(R - r)r^2, \quad 0 \leq r < R$$

where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

80. Electrical Resistance The resistance R of a certain type of resistor is

$$R = \sqrt{0.001T^4 - 4T + 100}$$

where R is measured in ohms and the temperature T is measured in degrees Celsius.

(a) Use a computer algebra system to find dR/dT and the critical number of the function. Determine the minimum resistance for this type of resistor.

(b) Use a graphing utility to graph the function R and use the graph to approximate the minimum resistance for this type of resistor.

Motion Along a Line **In Exercises 81–84, the function $s(t)$ describes the motion of a particle along a line. For each function, (a) find the velocity function of the particle at any time $t \geq 0$, (b) identify the time interval(s) in which the particle is moving in a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction, and (d) identify the time(s) at which the particle changes direction.**

81. $s(t) = 6t - t^2$

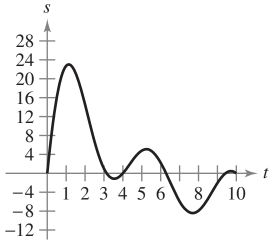
82. $s(t) = t^2 - 7t + 10$

83. $s(t) = t^3 - 5t^2 + 4t$

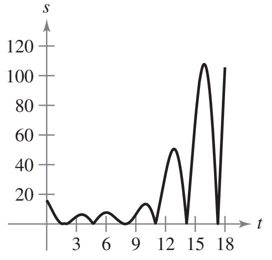
84. $s(t) = t^3 - 20t^2 + 128t - 280$

Motion Along a Line **In Exercises 85 and 86, the graph shows the position of a particle moving along a line. Describe how the particle's position changes with respect to time.**

85.



86.



Creating Polynomial Functions **In Exercises 87–90, find a polynomial function**

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the x-coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.

87. Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$

88. Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$

89. Relative minima: $(0, 0)$, $(4, 0)$; Relative maximum: $(2, 4)$

90. Relative minimum: $(1, 2)$; Relative maxima: $(-1, 4)$, $(3, 4)$

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. The sum of two increasing functions is increasing.

92. The product of two increasing functions is increasing.

93. Every nth-degree polynomial has $(n - 1)$ critical numbers.

94. An nth-degree polynomial has at most $(n - 1)$ critical numbers.

95. There is a relative maximum or minimum at each critical number.

96. The relative maxima of the function f are $f(1) = 4$ and $f(3) = 10$. Therefore, f has at least one minimum for some x in the interval $(1, 3)$.

97. Proof Prove the second case of Theorem 3.5.

98. Proof Prove the second case of Theorem 3.6.

99. Proof Use the definitions of increasing and decreasing functions to prove that $f(x) = x^3$ is increasing on $(-\infty, \infty)$.

100. Proof Use the definitions of increasing and decreasing functions to prove that

$$f(x) = \frac{1}{x}$$

is decreasing on $(0, \infty)$.

PUTNAM EXAM CHALLENGE

101. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

This problem was composed by the Committee on the Putnam Prize Competition.

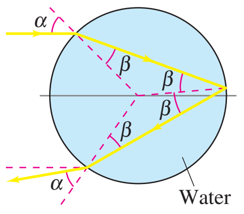
SECTION PROJECT

Rainbows

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that

$$\frac{\sin \alpha}{\sin \beta} = k$$

where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.



(a) Use a graphing utility to graph

$$D = \pi + 2\alpha - 4\sin^{-1}\left(\frac{\sin \alpha}{k}\right), \quad 0 \leq \alpha \leq \frac{\pi}{2}.$$

(b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

For water, what is the minimum angle of deflection $D_{\min}$? (The angle $\pi - D_{\min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α, is called a *rainbow ray*.)

FOR FURTHER INFORMATION For more information about the mathematics of rainbows, see the article "Somewhere Within the Rainbow" by Steven Janke in *The UMAP Journal*.

3.4 Concavity and the Second Derivative Test

- **Determine intervals on which a function is concave upward or concave downward.**
- **Find any points of inflection of the graph of a function.**
- **Apply the Second Derivative Test to find relative extrema of a function.**

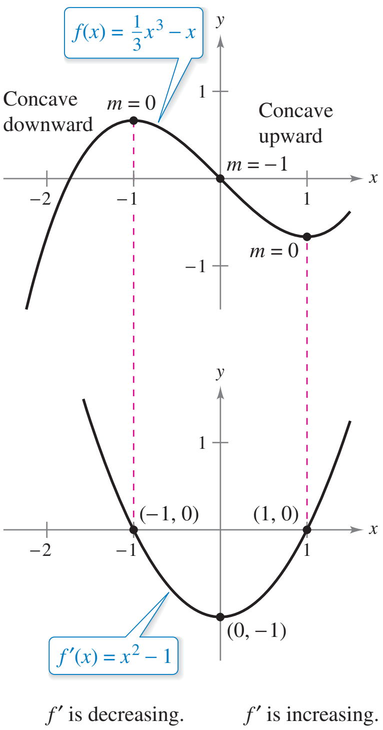
Concavity

You have already seen that locating the intervals in which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward*.

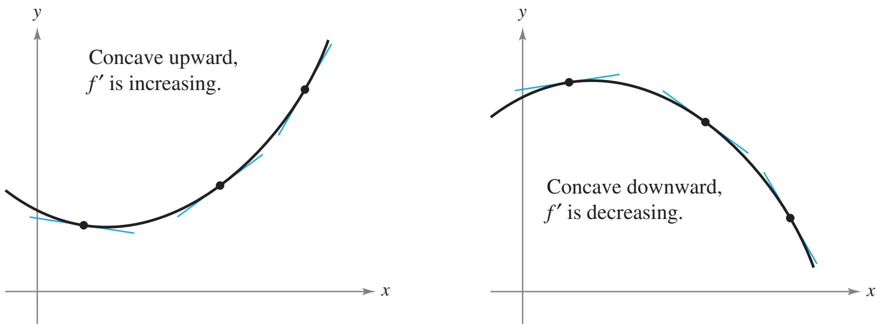
> **Definition of Concavity**
>
> Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.

The following graphical interpretation of concavity is useful. (See Appendix A for a proof of these results.) *See LarsonCalculus.com for Bruce Edwards's video of this proof.*

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I. [See Figure 3.23(a).]
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I. [See Figure 3.23(b).]



The concavity of f is related to the slope of the derivative.
Figure 3.24



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.
Figure 3.23

To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because

$$f'(x) = x^2 - 1$$

is decreasing there. (See Figure 3.24.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.

The next theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward. A proof of this theorem follows directly from Theorem 3.5 and the definition of concavity.

REMARK A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

EXAMPLE 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.

Solution Begin by observing that f is continuous on the entire real number line. Next, find the second derivative of f.

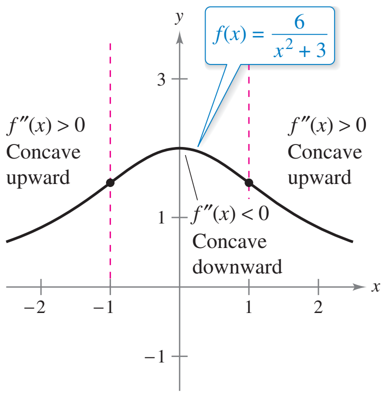
$$f(x) = 6(x^2 + 3)^{-1} \qquad \text{Rewrite original function.}$$

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x) \qquad \text{Differentiate.}$$

$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4} \qquad \text{Differentiate.}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Second derivative}$$



From the sign of f'', you can determine the concavity of the graph of f.
Figure 3.25

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real number line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.25.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

The function given in Example 1 is continuous on the entire real number line. When there are x-values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

EXAMPLE 2 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

is concave upward or concave downward.

Solution Differentiating twice produces the following.

$$f(x) = \frac{x^2 + 1}{x^2 - 4} \qquad \text{Write original function.}$$

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2} \qquad \text{Differentiate.}$$

$$= \frac{-10x}{(x^2 - 4)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \qquad \text{Differentiate.}$$

$$= \frac{10(3x^2 + 4)}{(x^2 - 4)^3} \qquad \text{Second derivative}$$

There are no points at which $f''(x) = 0$, but at $x = \pm 2$, the function f is not continuous. So, test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table. The graph of f is shown in Figure 3.26.

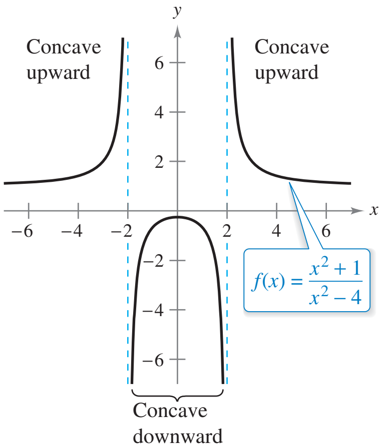
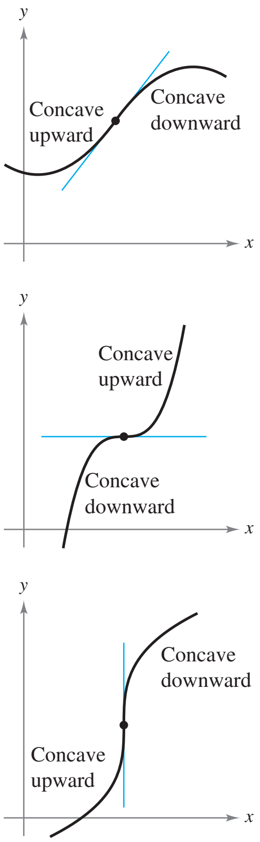


Figure 3.26

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of Inflection

The graph in Figure 3.25 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, then that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.27.



The concavity of f changes at a point of inflection. Note that the graph crosses its tangent line at a point of inflection.
Figure 3.27

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

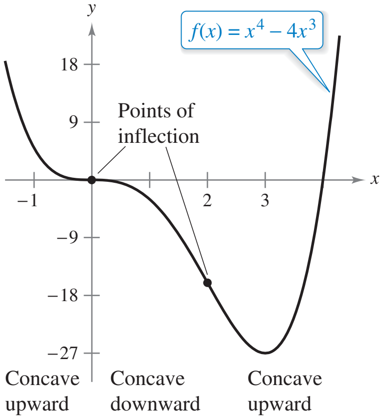
REMARK The definition of *point of inflection* requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \geq 0 \end{cases}$$

to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward.

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f.

> **THEOREM 3.8 Points of Inflection**
>
> If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.
Figure 3.28

EXAMPLE 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3.$$

Solution Differentiating twice produces the following.

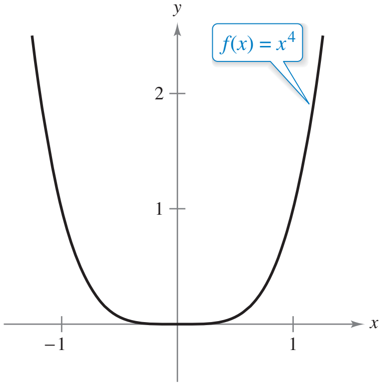
$$f(x) = x^4 - 4x^3 \qquad \text{Write original function.}$$
$$f'(x) = 4x^3 - 12x^2 \qquad \text{Find first derivative.}$$
$$f''(x) = 12x^2 - 24x = 12x(x - 2) \qquad \text{Find second derivative.}$$

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.28.

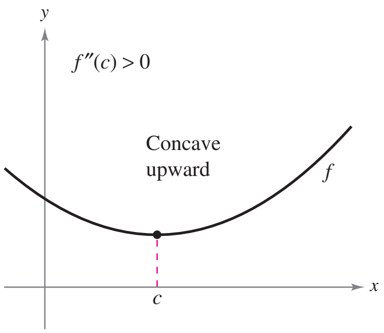
Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

> **Exploration**
>
> Consider a general cubic function of the form
>
> $$f(x) = ax^3 + bx^2 + cx + d.$$
>
> You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x. Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c. Then give a graphical explanation of why changes in c do not affect the values of the second derivative.

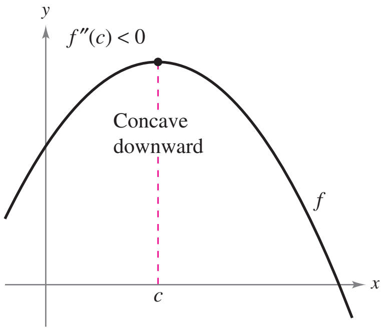
The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.29. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.
Figure 3.29



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

Figure 3.30

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative maximum of f (see Figure 3.30).

> **THEOREM 3.9 Second Derivative Test**
>
> Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.
>
> 1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
> 2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.
>
> If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Proof If $f'(c) = 0$ and $f''(c) > 0$, then there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. So, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you. *See LarsonCalculus.com for Bruce Edwards's video of this proof.*

AP* Tips

On the AP Exam, be prepared to apply the second derivative test to justify a point being a local maximum or minimum.

EXAMPLE 4 Using the Second Derivative Test

See LarsonCalculus.com for an interactive version of this type of example.

Find the relative extrema of

$$f(x) = -3x^5 + 5x^3.$$

Solution Begin by finding the first derivative of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2)$$

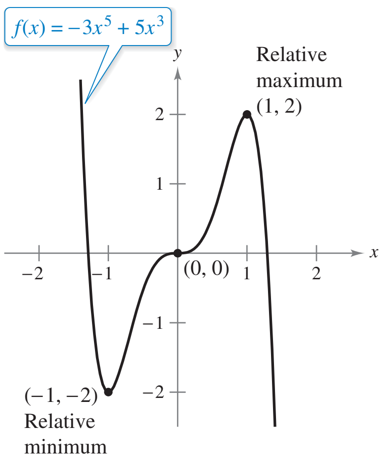
From this derivative, you can see that $x = -1, 0,$ and 1 are the only critical numbers of f. By finding the second derivative

$$f''(x) = -60x^3 + 30x = 30x(1 - 2x^2)$$

you can apply the Second Derivative Test as shown below.

Point	$(-1, -2)$	$(0, 0)$	$(1, 2)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) = 0$	$f''(1) < 0$
Conclusion	Relative minimum	Test fails	Relative maximum

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.31.



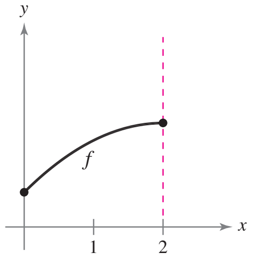
(0, 0) is neither a relative minimum nor a relative maximum.

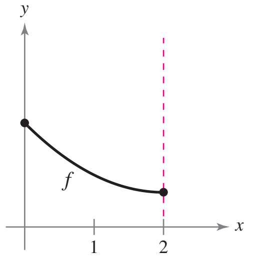
Figure 3.31

3.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using a Graph In Exercises 1 and 2, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.

1. 

2. 

Determining Concavity In Exercises 3–14, determine the open intervals on which the graph is concave upward or concave downward.

3. $y = x^2 - x - 2$
4. $g(x) = 3x^2 - x^3$
5. $f(x) = -x^3 + 6x^2 - 9x - 1$
6. $h(x) = x^5 - 5x + 2$
7. $f(x) = \dfrac{24}{x^2 + 12}$
8. $f(x) = \dfrac{2x^2}{3x^2 + 1}$
9. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
10. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$
11. $g(x) = \dfrac{x^2 + 4}{4 - x^2}$
12. $h(x) = \dfrac{x^2 - 1}{2x - 1}$
13. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
14. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

Finding Points of Inflection In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.

15. $f(x) = x^3 - 6x^2 + 12x$
16. $f(x) = -x^3 + 6x^2 - 5$
17. $f(x) = \frac{1}{2}x^4 + 2x^3$
18. $f(x) = 4 - x - 3x^4$
19. $f(x) = x(x - 4)^3$
20. $f(x) = (x - 2)^3(x - 1)$
21. $f(x) = x\sqrt{x + 3}$
22. $f(x) = x\sqrt{9 - x}$
23. $f(x) = \dfrac{4}{x^2 + 1}$
24. $f(x) = \dfrac{x + 3}{\sqrt{x}}$
25. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$
26. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$
27. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$
28. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$
29. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$
30. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

Using the Second Derivative Test In Exercises 31–42, find all relative extrema. Use the Second Derivative Test where applicable.

31. $f(x) = 6x - x^2$
32. $f(x) = x^2 + 3x - 8$
33. $f(x) = x^3 - 3x^2 + 3$
34. $f(x) = -x^3 + 7x^2 - 15x$
35. $f(x) = x^4 - 4x^3 + 2$
36. $f(x) = -x^4 + 4x^3 + 8x^2$
37. $f(x) = x^{2/3} - 3$
38. $f(x) = \sqrt{x^2 + 1}$
39. $f(x) = x + \dfrac{4}{x}$
40. $f(x) = \dfrac{x}{x - 1}$
41. $f(x) = \cos x - x, \quad [0, 4\pi]$
42. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

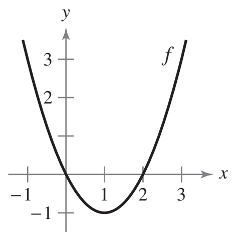
Finding Extrema and Points of Inflection Using Technology In Exercises 43–46, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

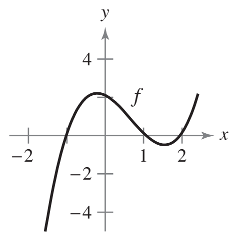
43. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$
44. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$
45. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$
46. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

47. **Sketching a Graph** Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
48. **Sketching a Graph** Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
49. **Sketching a Graph** Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.
50. **Think About It** S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?
    (a) The rate of change of sales is increasing.
    (b) Sales are increasing at a slower rate.
    (c) The rate of change of sales is constant.
    (d) Sales are steady.
    (e) Sales are declining, but at a slower rate.
    (f) Sales have bottomed out and have started to rise.

Sketching Graphs In Exercises 51 and 52, the graph of f is shown. Graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to *MathGraphs.com*.

51. 

52. 

Think About It **In Exercises 53–56, sketch the graph of a function f having the given characteristics.**

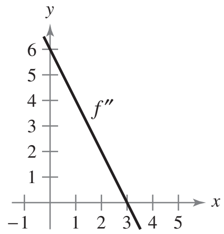
53. $f(2) = f(4) = 0$
$f'(x) < 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ for $x > 3$
$f''(x) < 0, x \neq 3$

54. $f(0) = f(2) = 0$
$f'(x) > 0$ for $x < 1$
$f'(1) = 0$
$f'(x) < 0$ for $x > 1$
$f''(x) < 0$

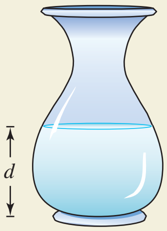
55. $f(2) = f(4) = 0$
$f'(x) > 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ for $x > 3$
$f''(x) > 0, x \neq 3$

56. $f(0) = f(2) = 0$
$f'(x) < 0$ for $x < 1$
$f'(1) = 0$
$f'(x) > 0$ for $x > 1$
$f''(x) > 0$

57. Think About It The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.) To print an enlarged copy of the graph, go to *MathGraphs.com*.



58. HOW DO YOU SEE IT? Water is running into the vase shown in the figure at a constant rate.



(a) Graph the depth d of water in the vase as a function of time.

(b) Does the function have any extrema? Explain.

(c) Interpret the inflection points of the graph of d.

59. Conjecture Consider the function

$$f(x) = (x - 2)^n.$$

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.

(b) Verify your conjecture in part (a).

60. Inflection Point Consider the function $f(x) = \sqrt[3]{x}$.

(a) Graph the function and identify the inflection point.

(b) Does $f''(x)$ exist at the inflection point? Explain.

Finding a Cubic Function **In Exercises 61 and 62, find a, b, c, and d such that the cubic**

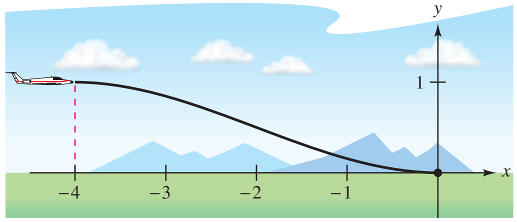
$$f(x) = ax^3 + bx^2 + cx + d$$

satisfies the given conditions.

61. Relative maximum: $(3, 3)$
Relative minimum: $(5, 1)$
Inflection point: $(4, 2)$

62. Relative maximum: $(2, 4)$
Relative minimum: $(4, 2)$
Inflection point: $(3, 3)$

63. Aircraft Glide Path A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).

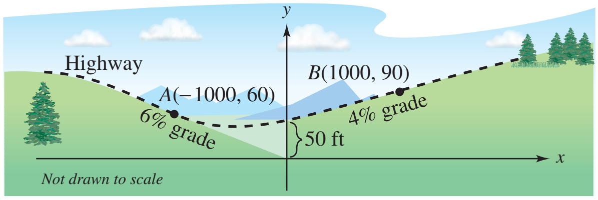


(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.

(b) The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?

FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

64. Highway Design A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.



(a) Design a section of highway connecting the hillsides modeled by the function

$$f(x) = ax^3 + bx^2 + cx + d, \quad -1000 \leq x \leq 1000.$$

At points A and B, the slope of the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.

(c) Use a graphing utility to graph the derivative of the model.

(d) Determine the grade at the steepest part of the transitional section of the highway.

65. Average Cost A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

66. Specific Gravity A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987,\ 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use the second derivative to determine the concavity of S.

(b) Use a computer algebra system to find the coordinates of the maximum value of the function.

(c) Use a graphing utility to graph the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(d) Estimate the specific gravity of water when $T = 20°$.

67. Sales Growth The annual sales S of a new product are given by

$$S = \frac{5000t^2}{8 + t^2},\quad 0 \le t \le 3$$

where t is time in years.

(a) Complete the table. Then use it to estimate when the annual sales are increasing at the greatest rate.

t	0.5	1	1.5	2	2.5	3
S						

(b) Use a graphing utility to graph the function S. Then use the graph to estimate when the annual sales are increasing at the greatest rate.

(c) Find the exact time when the annual sales are increasing at the greatest rate.

68. Modeling Data The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is

$$S = \frac{100t^2}{65 + t^2},\quad t > 0.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 69–72, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	Value of a
69. $f(x) = 2(\sin x + \cos x)$	$a = \dfrac{\pi}{4}$
70. $f(x) = 2(\sin x + \cos x)$	$a = 0$
71. $f(x) = \sqrt{1 - x}$	$a = 0$
72. $f(x) = \dfrac{\sqrt{x}}{x - 1}$	$a = 2$

73. Determining Concavity Use a graphing utility to graph

$$y = x \sin \frac{1}{x}.$$

Show that the graph is concave downward to the right of

$$x = \frac{1}{\pi}.$$

74. Point of Inflection and Extrema Show that the point of inflection of

$$f(x) = x(x - 6)^2$$

lies midway between the relative extrema of f.

True or False? **In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. The graph of every cubic polynomial has precisely one point of inflection.

76. The graph of

$$f(x) = \frac{1}{x}$$

is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

77. If $f'(c) > 0$, then f is concave upward at $x = c$.

78. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

Proof **In Exercises 79 and 80, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.**

79. Show that if f and g are concave upward on the interval (a, b), then $f + g$ is also concave upward on (a, b).

80. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

3.5 Limits at Infinity

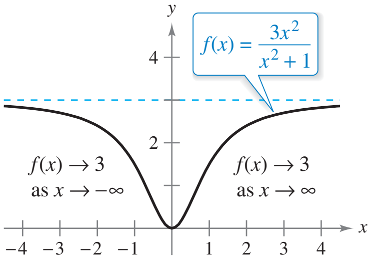
- **Determine (finite) limits at infinity.**
- **Determine the horizontal asymptotes, if any, of the graph of a function.**
- **Determine infinite limits at infinity.**

Limits at Infinity

This section discusses the "end behavior" of a function on an *infinite* interval. Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$

as shown in Figure 3.32. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.
Figure 3.32

x decreases without bound. | x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.9703	1.5	0	1.5	2.9703	2.9997	$\to 3$

$f(x)$ approaches 3. | $f(x)$ approaches 3.

The table suggests that the value of $f(x)$ approaches 3 as x increases without bound $(x \to \infty)$. Similarly, $f(x)$ approaches 3 as x decreases without bound $(x \to -\infty)$. These **limits at infinity** are denoted by

$$\lim_{x \to -\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

$$\lim_{x \to \infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

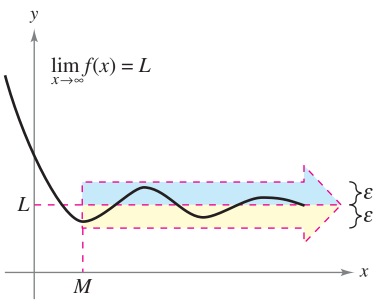
REMARK The statement $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$ means that the limit exists *and* the limit is equal to L.

To say that a statement is true as x increases *without bound* means that for some (large) real number M, the statement is true for *all* x in the interval $\{x: x > M\}$. The next definition uses this concept.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x \to -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.
Figure 3.33

The definition of a limit at infinity is shown in Figure 3.33. In this figure, note that for a given positive number ε, there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines

$$y = L + \varepsilon \quad \text{and} \quad y = L - \varepsilon.$$

Exploration

Use a graphing utility to graph

$$f(x) = \frac{2x^2 + 4x - 6}{3x^2 + 2x - 16}.$$

Describe all the important features of the graph. Can you find a single viewing window that shows all of these features clearly? Explain your reasoning.

What are the horizontal asymptotes of the graph? How far to the right do you have to move on the graph so that the graph is within 0.001 unit of its horizontal asymptote? Explain your reasoning.

Horizontal Asymptotes

In Figure 3.33, the graph of f approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of f.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ both exist, then

$$\lim_{x \to \infty} [f(x) + g(x)] = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x)$$

and

$$\lim_{x \to \infty} [f(x)g(x)] = \left[\lim_{x \to \infty} f(x)\right]\left[\lim_{x \to \infty} g(x)\right].$$

Similar properties hold for limits at $-\infty$.

When evaluating limits at infinity, the next theorem is helpful.

AP* Tips

The AP Exam frequently uses limits at infinity as a way of describing horizontal asymptotes.

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Finding a Limit at Infinity

Find the limit: $\displaystyle\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$.

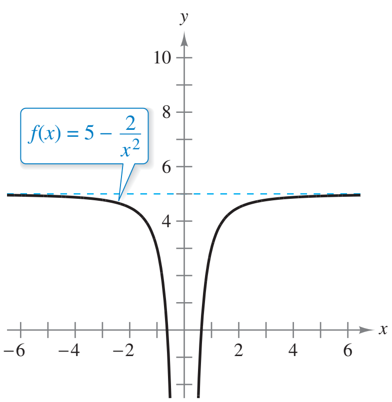
Solution Using Theorem 3.10, you can write

$$\begin{aligned}\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right) &= \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{2}{x^2} && \text{Property of limits}\\ &= 5 - 0\\ &= 5.\end{aligned}$$

So, the line $y = 5$ is a horizontal asymptote to the right. By finding the limit

$$\lim_{x \to -\infty} \left(5 - \frac{2}{x^2}\right) \qquad \text{Limit as } x \to -\infty.$$

you can see that $y = 5$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.34.



$y = 5$ is a horizontal asymptote.
Figure 3.34

EXAMPLE 2 Finding a Limit at Infinity

Find the limit: $\displaystyle\lim_{x\to\infty} \frac{2x - 1}{x + 1}$.

Solution Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} \quad \begin{array}{l} \lim_{x\to\infty} (2x - 1) \to \infty \\ \lim_{x\to\infty} (x + 1) \to \infty \end{array}$$

This results in $\frac{\infty}{\infty}$, an **indeterminate form.** To resolve this problem, you can divide both the numerator and the denominator by x. After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty} \frac{2x - 1}{x + 1} = \lim_{x\to\infty} \frac{\frac{2x-1}{x}}{\frac{x+1}{x}} \qquad \text{Divide numerator and denominator by } x.$$

$$= \lim_{x\to\infty} \frac{2 - \frac{1}{x}}{1 + \frac{1}{x}} \qquad \text{Simplify.}$$

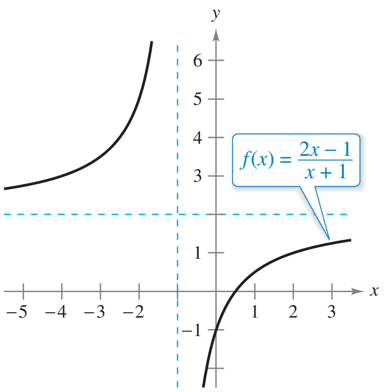
$$= \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}} \qquad \text{Take limits of numerator and denominator.}$$

$$= \frac{2 - 0}{1 + 0} \qquad \text{Apply Theorem 3.10.}$$

$$= 2$$

So, the line $y = 2$ is a horizontal asymptote to the right. By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.35.

REMARK When you encounter an indeterminate form such as the one in Example 2, you should divide the numerator and denominator by the highest power of x in the *denominator*.



$y = 2$ is a horizontal asymptote.
Figure 3.35

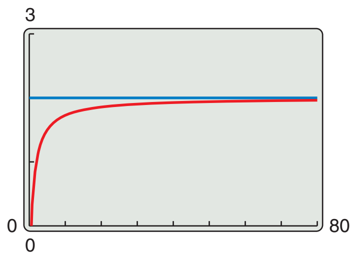
TECHNOLOGY You can test the reasonableness of the limit found in Example 2 by evaluating $f(x)$ for a few large positive values of x. For instance,

$$f(100) \approx 1.9703, \quad f(1000) \approx 1.9970,$$
$$\text{and} \quad f(10{,}000) \approx 1.9997.$$

Another way to test the reasonableness of the limit is to use a graphing utility. For instance, in Figure 3.36, the graph of

$$f(x) = \frac{2x - 1}{x + 1}$$

is shown with the horizontal line $y = 2$. Note that as x increases, the graph of f moves closer and closer to its horizontal asymptote.



As x increases, the graph of f moves closer and closer to the line $y = 2$.
Figure 3.36

EXAMPLE 3 A Comparison of Three Rational Functions

See LarsonCalculus.com for an interactive version of this type of example.

Find each limit.

a. $\lim_{x\to\infty} \dfrac{2x+5}{3x^2+1}$ **b.** $\lim_{x\to\infty} \dfrac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x\to\infty} \dfrac{2x^3+5}{3x^2+1}$

Solution In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞.

a. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1} = \lim_{x\to\infty} \frac{(2/x)+(5/x^2)}{3+(1/x^2)} = \frac{0+0}{3+0} = \frac{0}{3} = 0$$

b. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} = \lim_{x\to\infty} \frac{2+(5/x^2)}{3+(1/x^2)} = \frac{2+0}{3+0} = \frac{2}{3}$$

c. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \lim_{x\to\infty} \frac{2x+(5/x^2)}{3+(1/x^2)} = \frac{\infty}{3}$$

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3.

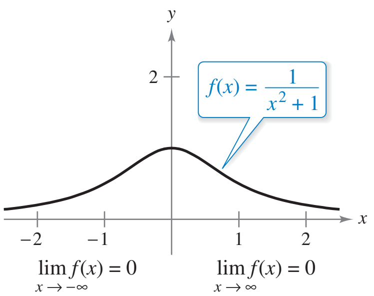
MARIA GAETANA AGNESI (1718–1799)

Agnesi was one of a handful of women to receive credit for significant contributions to mathematics before the twentieth century. In her early twenties, she wrote the first text that included both differential and integral calculus. By age 30, she was an honorary member of the faculty at the University of Bologna.
See LarsonCalculus.com to read more of this biography.
For more information on the contributions of women to mathematics, see the article "Why Women Succeed in Mathematics" by Mona Fabricant, Sylvia Svitak, and Patricia Clark Kenschaft in *Mathematics Teacher*. To view this article, go to *MathArticles.com*.

Example 3 suggests the guidelines below for finding limits at infinity of rational functions. Use these guidelines to check the results in Example 3.

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.



f has a horizontal asymptote at $y = 0$.
Figure 3.37

The guidelines for finding limits at infinity of rational functions seem reasonable when you consider that for large values of x, the highest-power term of the rational function is the most "influential" in determining the limit. For instance,

$$\lim_{x\to\infty} \frac{1}{x^2+1}$$

is 0 because the denominator overpowers the numerator as x increases or decreases without bound, as shown in Figure 3.37.

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

$$f(x) = \frac{8a^3}{x^2+4a^2} \qquad \text{Witch of Agnesi}$$

and, through a mistranslation of the Italian word *vertéré*, the curve has come to be known as the Witch of Agnesi. Agnesi's work with this curve first appeared in a comprehensive text on calculus that was published in 1748.

In Figure 3.37, you can see that the function

$$f(x) = \frac{1}{x^2 + 1}$$

approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left. This is demonstrated in Example 4.

EXAMPLE 4 A Function with Two Horizontal Asymptotes

Find each limit.

a. $\displaystyle\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$ **b.** $\displaystyle\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

Solution

a. For $x > 0$, you can write $x = \sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to\infty} \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{\sqrt{2 + 0}} = \frac{3}{\sqrt{2}}$$

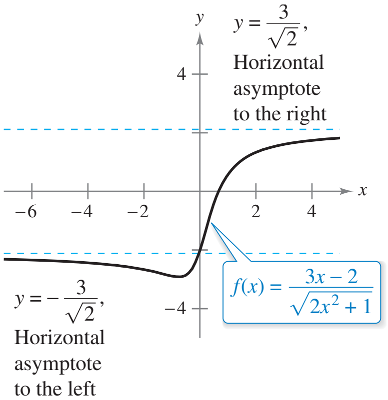
b. For $x < 0$, you can write $x = -\sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{-\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to-\infty} \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{-\sqrt{2 + 0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x - 2)/\sqrt{2x^2 + 1}$ is shown in Figure 3.38.



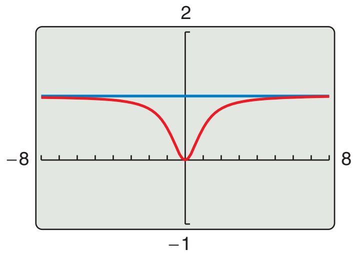
Functions that are not rational may have different right and left horizontal asymptotes.

Figure 3.38

TECHNOLOGY PITFALL If you use a graphing utility to estimate a limit, be sure that you also confirm the estimate analytically—the pictures shown by a graphing utility can be misleading. For instance, Figure 3.39 shows one view of the graph of

$$y = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}.$$

From this view, one could be convinced that the graph has $y = 1$ as a horizontal asymptote. An analytical approach shows that the horizontal asymptote is actually $y = 2$. Confirm this by enlarging the viewing window on the graphing utility.



The horizontal asymptote appears to be the line $y = 1$, but it is actually the line $y = 2$.

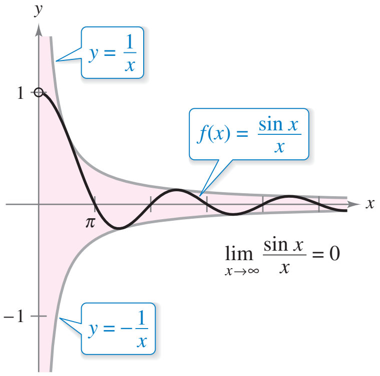
Figure 3.39

In Section 1.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

EXAMPLE 5 Limits Involving Trigonometric Functions

Find each limit.

a. $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \dfrac{\sin x}{x}$



As x increases without bound, $f(x)$ approaches 0.
Figure 3.40

Solution

a. As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

b. Because $-1 \le \sin x \le 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

where

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 \quad \text{and} \quad \lim_{x\to\infty}\frac{1}{x} = 0.$$

So, by the Squeeze Theorem, you can obtain

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0$$

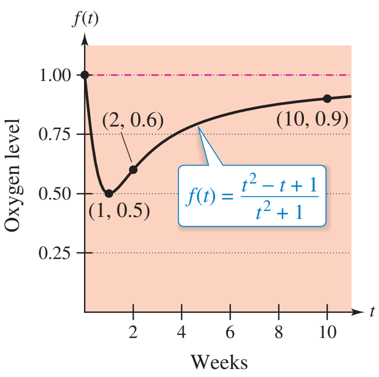
as shown in Figure 3.40.

EXAMPLE 6 Oxygen Level in a Pond

Let $f(t)$ measure the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?



The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞.
Figure 3.41

Solution When $t = 1$, 2, and 10, the levels of oxygen are as shown.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \qquad \text{1 week}$$

$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \qquad \text{2 weeks}$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \qquad \text{10 weeks}$$

To find the limit as t approaches infinity, you can use the guidelines on page 198, or you can divide the numerator and the denominator by t^2 to obtain

$$\lim_{t\to\infty}\frac{t^2 - t + 1}{t^2 + 1} = \lim_{t\to\infty}\frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

See Figure 3.41.

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The next definition is used to describe the behavior of polynomial and other functions at infinity.

REMARK Determining whether a function has an infinite limit at infinity is useful in analyzing the "end behavior" of its graph. You will see examples of this in Section 3.6 on curve sketching.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

EXAMPLE 7 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

Solution

a. As x increases without bound, x^3 also increases without bound. So, you can write

$$\lim_{x\to\infty} x^3 = \infty.$$

b. As x decreases without bound, x^3 also decreases without bound. So, you can write

$$\lim_{x\to-\infty} x^3 = -\infty.$$

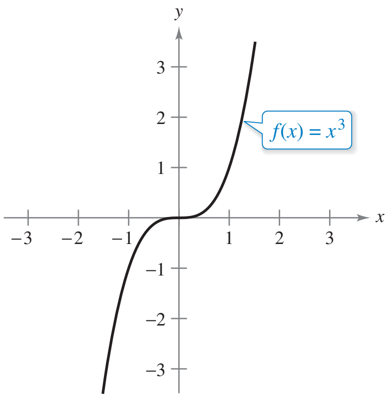


Figure 3.42

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section P.3.

EXAMPLE 8 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} \dfrac{2x^2 - 4x}{x + 1}$ **b.** $\lim_{x\to-\infty} \dfrac{2x^2 - 4x}{x + 1}$

Solution One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

a. $$\lim_{x\to\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = \infty$$

b. $$\lim_{x\to-\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to-\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = -\infty$$

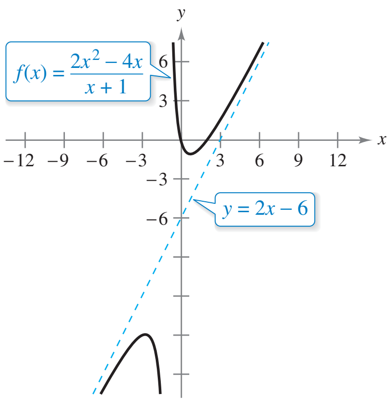


Figure 3.43

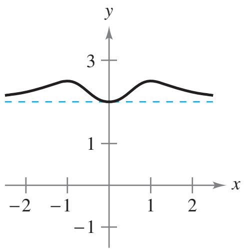
The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x) = (2x^2 - 4x)/(x + 1)$ behaves like the function $g(x) = 2x - 6$. In Section 3.6, you will see that this is graphically described by saying that the line $y = 2x - 6$ is a slant asymptote of the graph of f, as shown in Figure 3.43.

3.5 Exercises

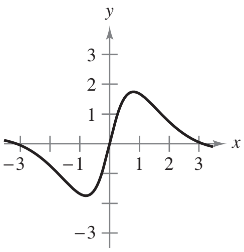
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.**

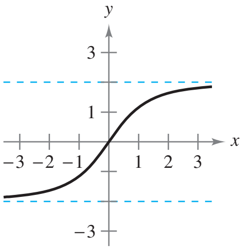
(a)



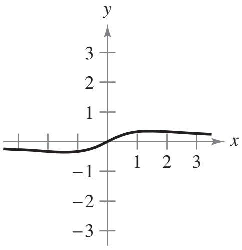
(b)



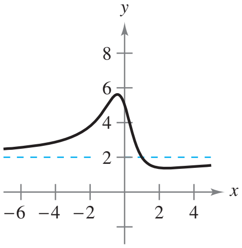
(c)



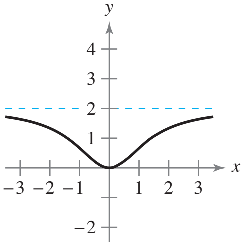
(d)



(e)



(f)



1. $f(x) = \dfrac{2x^2}{x^2+2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

3. $f(x) = \dfrac{x}{x^2+2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4+1}$

5. $f(x) = \dfrac{4\sin x}{x^2+1}$

6. $f(x) = \dfrac{2x^2-3x+5}{x^2+1}$

Numerical and Graphical Analysis **In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x+3}{2x-1}$

8. $f(x) = \dfrac{2x^2}{x+1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$

10. $f(x) = \dfrac{10}{\sqrt{2x^2-1}}$

11. $f(x) = 5 - \dfrac{1}{x^2+1}$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

Finding Limits at Infinity **In Exercises 13 and 14, find $\lim_{x\to\infty} h(x)$, if possible.**

13. $f(x) = 5x^3 - 3x^2 + 10x$
 (a) $h(x) = \dfrac{f(x)}{x^2}$
 (b) $h(x) = \dfrac{f(x)}{x^3}$
 (c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = -4x^2 + 2x - 5$
 (a) $h(x) = \dfrac{f(x)}{x}$
 (b) $h(x) = \dfrac{f(x)}{x^2}$
 (c) $h(x) = \dfrac{f(x)}{x^3}$

Finding Limits at Infinity **In Exercises 15–18, find each limit, if possible.**

15. (a) $\displaystyle\lim_{x\to\infty} \frac{x^2+2}{x^3-1}$
 (b) $\displaystyle\lim_{x\to\infty} \frac{x^2+2}{x^2-1}$
 (c) $\displaystyle\lim_{x\to\infty} \frac{x^2+2}{x-1}$

16. (a) $\displaystyle\lim_{x\to\infty} \frac{3-2x}{3x^3-1}$
 (b) $\displaystyle\lim_{x\to\infty} \frac{3-2x}{3x-1}$
 (c) $\displaystyle\lim_{x\to\infty} \frac{3-2x^2}{3x-1}$

17. (a) $\displaystyle\lim_{x\to\infty} \frac{5-2x^{3/2}}{3x^2-4}$
 (b) $\displaystyle\lim_{x\to\infty} \frac{5-2x^{3/2}}{3x^{3/2}-4}$
 (c) $\displaystyle\lim_{x\to\infty} \frac{5-2x^{3/2}}{3x-4}$

18. (a) $\displaystyle\lim_{x\to\infty} \frac{5x^{3/2}}{4x^2+1}$
 (b) $\displaystyle\lim_{x\to\infty} \frac{5x^{3/2}}{4x^{3/2}+1}$
 (c) $\displaystyle\lim_{x\to\infty} \frac{5x^{3/2}}{4\sqrt{x}+1}$

Finding a Limit **In Exercises 19–38, find the limit.**

19. $\displaystyle\lim_{x\to\infty} \left(4 + \frac{3}{x}\right)$

20. $\displaystyle\lim_{x\to-\infty} \left(\frac{5}{x} - \frac{x}{3}\right)$

21. $\displaystyle\lim_{x\to\infty} \frac{2x-1}{3x+2}$

22. $\displaystyle\lim_{x\to-\infty} \frac{4x^2+5}{x^2+3}$

23. $\displaystyle\lim_{x\to\infty} \frac{x}{x^2-1}$

24. $\displaystyle\lim_{x\to\infty} \frac{5x^3+1}{10x^3-3x^2+7}$

25. $\displaystyle\lim_{x\to-\infty} \frac{5x^2}{x+3}$

26. $\displaystyle\lim_{x\to-\infty} \frac{x^3-4}{x^2+1}$

27. $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-x}}$

28. $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$

29. $\displaystyle\lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2-x}}$

30. $\displaystyle\lim_{x\to\infty} \frac{5x^2+2}{\sqrt{x^2+3}}$

31. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{2x-1}$

32. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1}$

33. $\displaystyle\lim_{x\to\infty} \frac{x+1}{(x^2+1)^{1/3}}$

34. $\displaystyle\lim_{x\to-\infty} \frac{2x}{(x^6-1)^{1/3}}$

35. $\displaystyle\lim_{x\to\infty} \frac{1}{2x+\sin x}$

36. $\displaystyle\lim_{x\to\infty} \cos\frac{1}{x}$

37. $\displaystyle\lim_{x\to\infty} \frac{\sin 2x}{x}$

38. $\displaystyle\lim_{x\to\infty} \frac{x-\cos x}{x}$

Horizontal Asymptotes **In Exercises 39–42, use a graphing utility to graph the function and identify any horizontal asymptotes.**

39. $f(x) = \dfrac{|x|}{x+1}$

40. $f(x) = \dfrac{|3x+2|}{x-2}$

41. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

42. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

Finding a Limit **In Exercises 43 and 44, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)**

43. $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x}$

44. $\lim\limits_{x\to\infty} x \tan \dfrac{1}{x}$

Finding a Limit **In Exercises 45–48, find the limit. (*Hint*: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.**

45. $\lim\limits_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

46. $\lim\limits_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

47. $\lim\limits_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

48. $\lim\limits_{x\to\infty} \left(4x - \sqrt{16x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 49–52, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = x - \sqrt{x(x-1)}$

50. $f(x) = x^2 - x\sqrt{x(x-1)}$

51. $f(x) = x \sin \dfrac{1}{2x}$

52. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

Writing **In Exercises 53 and 54, describe in your own words what the statement means.**

53. $\lim\limits_{x\to\infty} f(x) = 4$

54. $\lim\limits_{x\to-\infty} f(x) = 2$

55. **Sketching a Graph** Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$

$f'(x) > 0$ for $x > 2$

$\lim\limits_{x\to-\infty} f(x) = 6$

$\lim\limits_{x\to\infty} f(x) = 6$

56. **Points of Inflection** Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 55 and has *no* points of inflection? Explain.

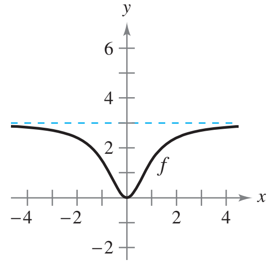
WRITING ABOUT CONCEPTS (continued)

57. **Using Symmetry to Find Limits** If f is a continuous function such that $\lim\limits_{x\to\infty} f(x) = 5$, find, if possible, $\lim\limits_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric with respect to the y-axis.

(b) The graph of f is symmetric with respect to the origin.

58. **A Function and Its Derivative** The graph of a function f is shown below. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Sketch f'.

(b) Use the graphs to estimate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

Sketching a Graph **In Exercises 59–74, sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.**

59. $y = \dfrac{x}{1-x}$

60. $y = \dfrac{x-4}{x-3}$

61. $y = \dfrac{x+1}{x^2-4}$

62. $y = \dfrac{2x}{9-x^2}$

63. $y = \dfrac{x^2}{x^2+16}$

64. $y = \dfrac{2x^2}{x^2-4}$

65. $xy^2 = 9$

66. $x^2y = 9$

67. $y = \dfrac{3x}{x-1}$

68. $y = \dfrac{3x}{1-x^2}$

69. $y = 2 - \dfrac{3}{x^2}$

70. $y = 1 - \dfrac{1}{x}$

71. $y = 3 + \dfrac{2}{x}$

72. $y = \dfrac{4}{x^2} + 1$

73. $y = \dfrac{x^3}{\sqrt{x^2-4}}$

74. $y = \dfrac{x}{\sqrt{x^2-4}}$

Analyzing a Graph Using Technology **In Exercises 75–82, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.**

75. $f(x) = 9 - \dfrac{5}{x^2}$

76. $f(x) = \dfrac{1}{x^2-x-2}$

77. $f(x) = \dfrac{x-2}{x^2-4x+3}$

78. $f(x) = \dfrac{x+1}{x^2+x+1}$

79. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$

80. $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

81. $g(x) = \sin\left(\dfrac{x}{x - 2}\right), \quad x > 3$

82. $f(x) = \dfrac{2 \sin 2x}{x}$

Comparing Functions **In Exercises 83 and 84, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)**

83. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$g(x) = x + \dfrac{2}{x(x - 3)}$

84. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

85. Engine Efficiency

The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

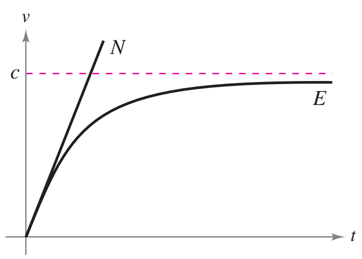
where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

86. Average Cost A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

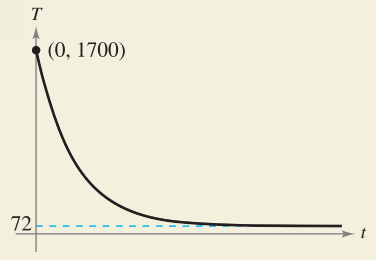
$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

87. Physics Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein, respectively. Write limit statements that describe these two theories.



88. HOW DO YOU SEE IT? The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t \to 0^+} T$. What does this limit represent?

(b) Find $\lim_{t \to \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

89. Modeling Data The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}, \; t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

90. Modeling Data A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is

$$T_2 = \frac{1451 + 86t}{58 + t}.$$

Use a graphing utility to graph T_2.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim_{t \to \infty} T_2$.

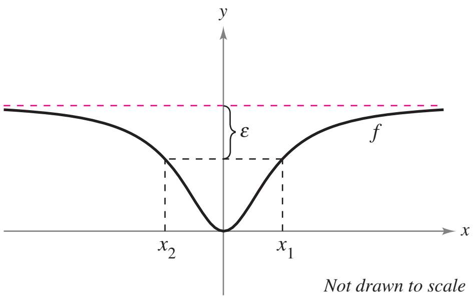
(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

Straight 8 Photography/Shutterstock.com

91. Using the Definition of Limits at Infinity The graph of

$$f(x) = \frac{2x^2}{x^2 + 2}$$

is shown.



(a) Find $L = \lim_{x\to\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

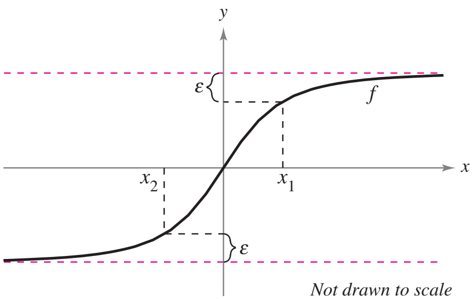
(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - L| < \varepsilon$ for $x < N$.

92. Using the Definition of Limits at Infinity The graph of

$$f(x) = \frac{6x}{\sqrt{x^2 + 2}}$$

is shown.



(a) Find $L = \lim_{x\to\infty} f(x)$ and $K = \lim_{x\to-\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - K| < \varepsilon$ for $x < N$.

93. Using the Definition of Limits at Infinity Consider

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.1$.

94. Using the Definition of Limits at Infinity Consider

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.1$.

Proof **In Exercises 95–98, use the definition of limits at infinity to prove the limit.**

95. $\lim_{x\to\infty} \frac{1}{x^2} = 0$

96. $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

97. $\lim_{x\to-\infty} \frac{1}{x^3} = 0$

98. $\lim_{x\to-\infty} \frac{1}{x - 2} = 0$

99. Distance A line with slope m passes through the point $(0, 4)$.

(a) Write the shortest distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

100. Distance A line with slope m passes through the point $(0, -2)$.

(a) Write the shortest distance d between the line and the point $(4, 2)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

101. Proof Prove that if

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$q(x) = b_m x^m + \cdots + b_1 x + b_0$$

where $a_n \neq 0$ and $b_m \neq 0$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m. \\ \pm\infty, & n > m \end{cases}$$

102. Proof Use the definition of infinite limits at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

True or False? **In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

104. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.

3.6 A Summary of Curve Sketching

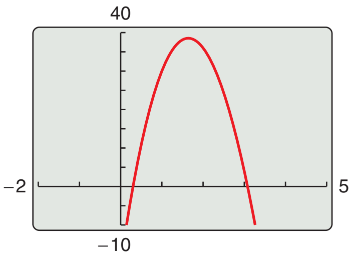
- **Analyze and sketch the graph of a function.**

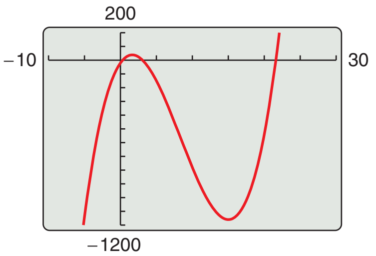
Analyzing the Graph of a Function

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- x-intercepts and y-intercepts (Section P.1)
- Symmetry (Section P.1)
- Domain and range (Section P.3)
- Continuity (Section 1.4)
- Vertical asymptotes (Section 1.5)
- Differentiability (Section 2.1)
- Relative extrema (Section 3.1)
- Concavity (Section 3.4)
- Points of inflection (Section 3.4)
- Horizontal asymptotes (Section 3.5)
- Infinite limits at infinity (Section 3.5)





Different viewing windows for the graph of $f(x) = x^3 - 25x^2 + 74x - 20$
Figure 3.44

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing windows in Figure 3.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function.

GUIDELINES FOR ANALYZING THE GRAPH OF A FUNCTION

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

REMARK In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations

$$f(x) = 0, \quad f'(x) = 0, \quad \text{and} \quad f''(x) = 0.$$

EXAMPLE 1 Sketching the Graph of a Rational Function

Analyze and sketch the graph of

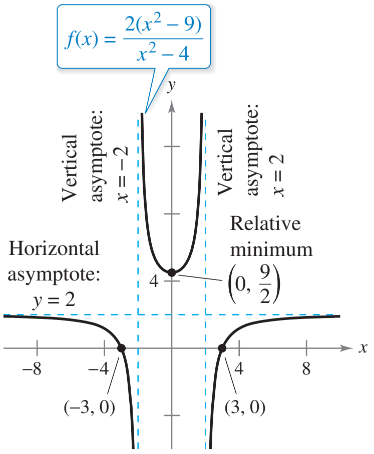
$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4}.$$

Solution

First derivative:	$f'(x) = \dfrac{20x}{(x^2 - 4)^2}$
Second derivative:	$f''(x) = \dfrac{-20(3x^2 + 4)}{(x^2 - 4)^3}$
x-intercepts:	$(-3, 0), (3, 0)$
y-intercept:	$\left(0, \frac{9}{2}\right)$
Vertical asymptotes:	$x = -2, x = 2$
Horizontal asymptote:	$y = 2$
Critical number:	$x = 0$
Possible points of inflection:	None
Domain:	All real numbers except $x = \pm 2$
Symmetry:	With respect to y-axis
Test intervals:	$(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < -2$		$-$	$-$	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		$-$	$+$	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	$+$	Relative minimum
$0 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		$+$	$-$	Increasing, concave downward



Using calculus, you can be certain that you have determined all characteristics of the graph of f.
Figure 3.45

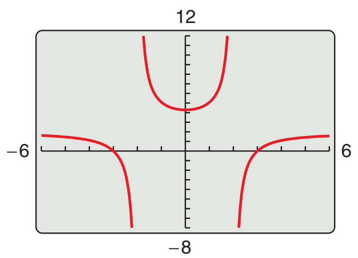
■ **FOR FURTHER INFORMATION** For more information on the use of technology to graph rational functions, see the article "Graphs of Rational Functions for Computer Assisted Calculus" by Stan Byrd and Terry Walters in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

Be sure you understand all of the implications of creating a table such as that shown in Example 1. By using calculus, you can be *sure* that the graph has no relative extrema or points of inflection other than those shown in Figure 3.45.

TECHNOLOGY PITFALL Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 3.46 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}.$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 3.45. The graphs of these two functions, however, differ significantly. Try enlarging the viewing window to see the differences.



By not using calculus, you may overlook important characteristics of the graph of g.
Figure 3.46

EXAMPLE 2 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution

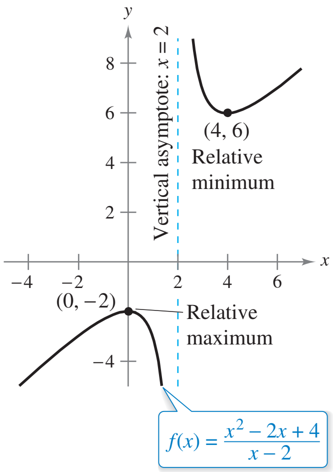


Figure 3.47

First derivative: $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

Second derivative: $f''(x) = \dfrac{8}{(x-2)^3}$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = \infty$

Critical numbers: $x = 0,\ x = 4$

Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

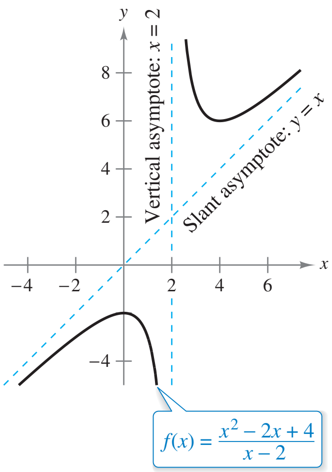
The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	-2	0	$-$	Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		$-$	$+$	Decreasing, concave upward
$x = 4$	6	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave upward

Although the graph of the function in Example 2 has no horizontal asymptote, it does have a slant asymptote. The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a **slant asymptote** when the degree of the numerator exceeds the degree of the denominator by exactly 1. To find the slant asymptote, use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} \qquad \text{Write original equation.}$$

$$= x + \frac{4}{x - 2} \qquad \text{Rewrite using long division.}$$



A slant asymptote
Figure 3.48

In Figure 3.48, note that the graph of f approaches the slant asymptote $y = x$ as x approaches $-\infty$ or ∞.

EXAMPLE 3 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$.

Solution

$$f'(x) = \frac{2}{(x^2 + 2)^{3/2}} \qquad \text{Find first derivative.}$$

$$f''(x) = -\frac{6x}{(x^2 + 2)^{5/2}} \qquad \text{Find second derivative.}$$

The graph has only one intercept, $(0, 0)$. It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49.

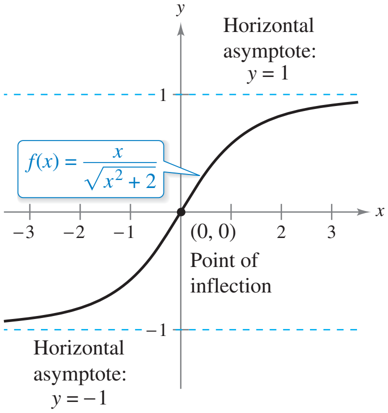


Figure 3.49

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$+$	Increasing, concave upward
$x = 0$	0	$\dfrac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		$+$	$-$	Increasing, concave downward

EXAMPLE 4 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad \text{Find first derivative.}$$

$$f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}} \qquad \text{Find second derivative.}$$

The function has two intercepts: $(0, 0)$ and $\left(\frac{125}{8}, 0\right)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.50.

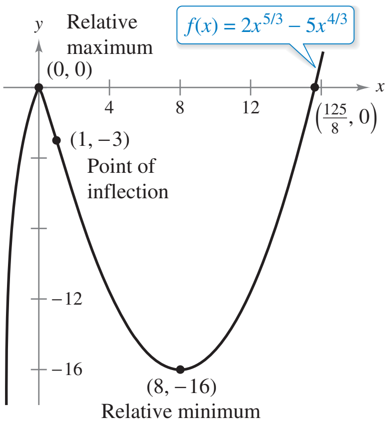


Figure 3.50

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave downward
$x = 1$	-3	$-$	0	Point of inflection
$1 < x < 8$		$-$	$+$	Decreasing, concave upward
$x = 8$	-16	0	$+$	Relative minimum
$8 < x < \infty$		$+$	$+$	Increasing, concave upward

EXAMPLE 5 Sketching the Graph of a Polynomial Function

See LarsonCalculus.com for an interactive version of this type of example.

Analyze and sketch the graph of

$$f(x) = x^4 - 12x^3 + 48x^2 - 64x.$$

Solution Begin by factoring to obtain

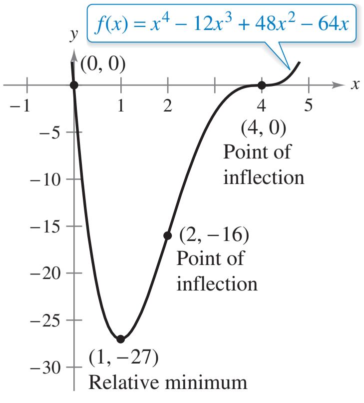
$$\begin{aligned} f(x) &= x^4 - 12x^3 + 48x^2 - 64x \\ &= x(x-4)^3. \end{aligned}$$

Then, using the factored form of $f(x)$, you can perform the following analysis.

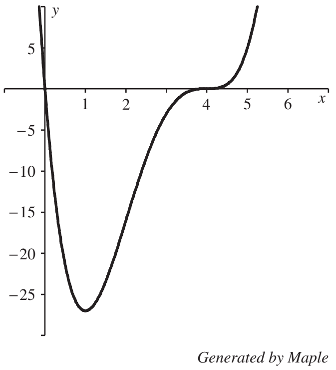
First derivative: $f'(x) = 4(x-1)(x-4)^2$
Second derivative: $f''(x) = 12(x-4)(x-2)$
x-intercepts: $(0, 0)$, $(4, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: None
Horizontal asymptotes: None
End behavior: $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$
Critical numbers: $x = 1$, $x = 4$
Possible points of inflection: $x = 2$, $x = 4$
Domain: All real numbers
Test intervals: $(-\infty, 1)$, $(1, 2)$, $(2, 4)$, $(4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.51(a). Using a computer algebra system such as *Maple* [see Figure 3.51(b)] can help you verify your analysis.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 1$		$-$	$+$	Decreasing, concave upward
$x = 1$	-27	0	$+$	Relative minimum
$1 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	-16	$+$	0	Point of inflection
$2 < x < 4$		$+$	$-$	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		$+$	$+$	Increasing, concave upward



(a)



(b)

A polynomial function of even degree must have at least one relative extremum.

Figure 3.51

The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have *at most* $n - 1$ relative extrema, and *at most* $n - 2$ points of inflection. Moreover, polynomial functions of even degree must have *at least* one relative extremum.

Remember from the Leading Coefficient Test described in Section P.3 that the “end behavior” of the graph of a polynomial function is determined by its leading coefficient and its degree. For instance, because the polynomial in Example 5 has a positive leading coefficient, the graph rises to the right. Moreover, because the degree is even, the graph also rises to the left.

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = (\cos x)/(1 + \sin x)$.

Solution Because the function has a period of 2π, you can restrict the analysis of the graph to any interval of length 2π. For convenience, choose $(-\pi/2, 3\pi/2)$.

First derivative: $f'(x) = -\dfrac{1}{1 + \sin x}$

Second derivative: $f''(x) = \dfrac{\cos x}{(1 + \sin x)^2}$

Period: 2π

x-intercept: $\left(\dfrac{\pi}{2}, 0\right)$

y-intercept: $(0, 1)$

Vertical asymptotes: $x = -\dfrac{\pi}{2},\ x = \dfrac{3\pi}{2}$ See Remark below.

Horizontal asymptotes: None

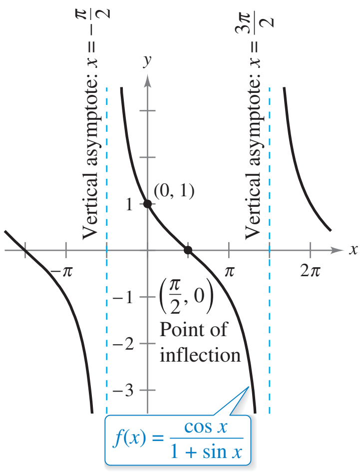
Critical numbers: None

Possible points of inflection: $x = \dfrac{\pi}{2}$

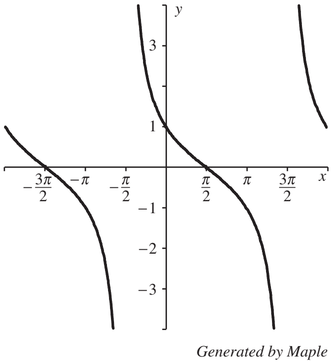
Domain: All real numbers except $x = \dfrac{3 + 4n}{2}\pi$

Test intervals: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.52(a). Compare this with the graph generated by the computer algebra system *Maple* in Figure 3.52(b).



(a)



(b)

Figure 3.52

AP* Tips

Curve sketching in and of itself is not the aim of AP free response questions. However, the exercises here will help reinforce your ability to harvest information from derivatives.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\dfrac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$		$-$	$+$	Decreasing, concave upward
$x = \dfrac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$		$-$	$-$	Decreasing, concave downward
$x = \dfrac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

REMARK By substituting $-\pi/2$ or $3\pi/2$ into the function, you obtain the form $0/0$. This is called an indeterminate form, which you will study in Section 8.7. To determine that the function has vertical asymptotes at these two values, rewrite f as

$$f(x) = \frac{\cos x}{1 + \sin x} = \frac{(\cos x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{(\cos x)(1 - \sin x)}{\cos^2 x} = \frac{1 - \sin x}{\cos x}.$$

In this form, it is clear that the graph of f has vertical asymptotes at $x = -\pi/2$ and $3\pi/2$.

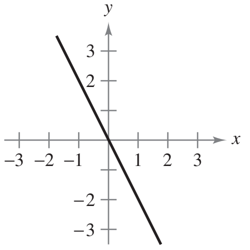
3.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

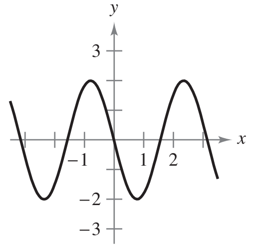
Matching **In Exercises 1–4, match the graph of f in the left column with that of its derivative in the right column.**

Graph of f **Graph of f'**

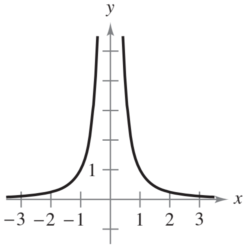
1.



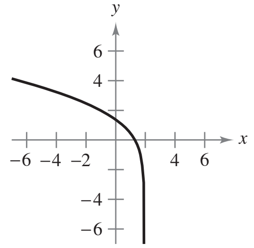
(a)



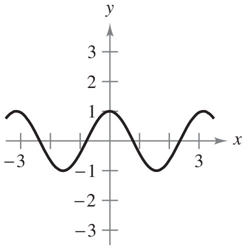
2.



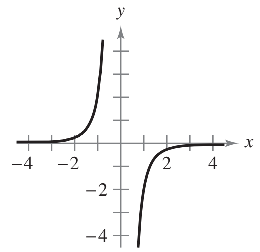
(b)



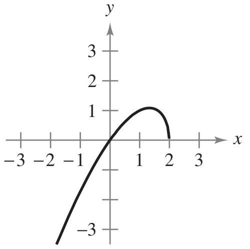
3.



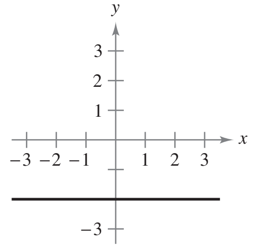
(c)



4.



(d)



Analyzing the Graph of a Function **In Exercises 5–24, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.**

5. $y = \dfrac{1}{x-2} - 3$

6. $y = \dfrac{x}{x^2+1}$

7. $y = \dfrac{x^2}{x^2+3}$

8. $y = \dfrac{x^2+1}{x^2-4}$

9. $y = \dfrac{3x}{x^2-1}$

10. $f(x) = \dfrac{x-3}{x}$

11. $f(x) = x + \dfrac{32}{x^2}$

12. $f(x) = \dfrac{x^3}{x^2-9}$

13. $y = \dfrac{x^2 - 6x + 12}{x-4}$

14. $y = \dfrac{-x^2 - 4x - 7}{x+3}$

15. $y = x\sqrt{4-x}$

16. $g(x) = x\sqrt{9-x^2}$

17. $y = 3x^{2/3} - 2x$

18. $y = (x+1)^2 - 3(x+1)^{2/3}$

19. $y = 2 - x - x^3$

20. $y = -\frac{1}{3}(x^3 - 3x + 2)$

21. $y = 3x^4 + 4x^3$

22. $y = -2x^4 + 3x^2$

23. $y = x^5 - 5x$

24. $y = (x-1)^5$

Analyzing the Graph of a Function Using Technology **In Exercises 25–34, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.**

25. $f(x) = \dfrac{20x}{x^2+1} - \dfrac{1}{x}$

26. $f(x) = x + \dfrac{4}{x^2+1}$

27. $f(x) = \dfrac{-2x}{\sqrt{x^2+7}}$

28. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

29. $f(x) = 2x - 4\sin x, \quad 0 \le x \le 2\pi$

30. $f(x) = -x + 2\cos x, \quad 0 \le x \le 2\pi$

31. $y = \cos x - \frac{1}{4}\cos 2x, \quad 0 \le x \le 2\pi$

32. $y = 2x - \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

33. $y = 2(\csc x + \sec x), \quad 0 < x < \dfrac{\pi}{2}$

34. $g(x) = x\cot x, \quad -2\pi < x < 2\pi$

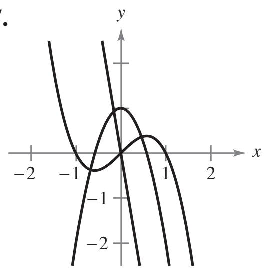
WRITING ABOUT CONCEPTS

35. **Using a Derivative** Let $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

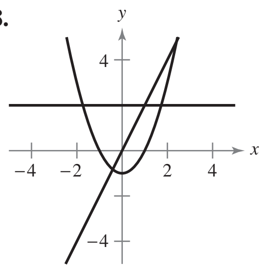
36. **Using a Derivative** Let $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

Identifying Graphs **In Exercises 37 and 38, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

37.



38.



WRITING ABOUT CONCEPTS (continued)

Horizontal and Vertical Asymptotes **In Exercises 39–42, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?**

39. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

40. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

41. $h(x) = \dfrac{\sin 2x}{x}$

42. $f(x) = \dfrac{\cos 3x}{4x}$

Examining a Function **In Exercises 43 and 44, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

43. $h(x) = \dfrac{6 - 2x}{3 - x}$

44. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

Slant Asymptote **In Exercises 45–48, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?**

45. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2}$

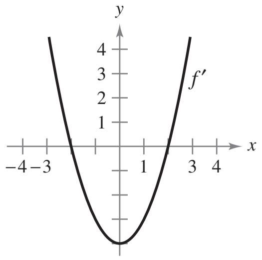
46. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5}$

47. $f(x) = \dfrac{2x^3}{x^2 + 1}$

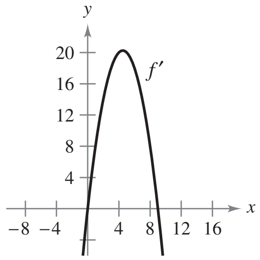
48. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2}$

Graphical Reasoning **In Exercises 49–52, use the graph of f' to sketch a graph of f and the graph of f''. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

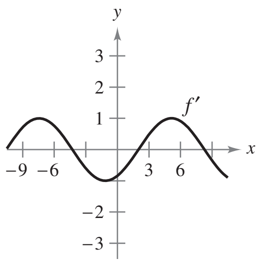
49.



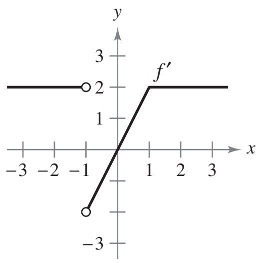
50.



51.



52.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

53. Graphical Reasoning Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad 0 < x < 4.$$

(a) Use a computer algebra system to graph the function and use the graph to approximate the critical numbers visually.

(b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

54. Graphical Reasoning Consider the function

$$f(x) = \tan(\sin \pi x).$$

(a) Use a graphing utility to graph the function.

(b) Identify any symmetry of the graph.

(c) Is the function periodic? If so, what is the period?

(d) Identify any extrema on $(-1, 1)$.

(e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It **In Exercises 55–58, create a function whose graph has the given characteristics. (There is more than one correct answer.)**

55. Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

56. Vertical asymptote: $x = -5$

Horizontal asymptote: None

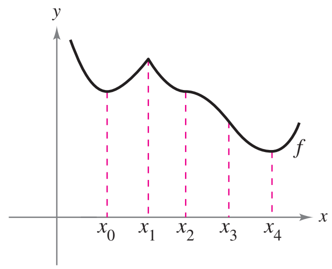
57. Vertical asymptote: $x = 3$

Slant asymptote: $y = 3x + 2$

58. Vertical asymptote: $x = 2$

Slant asymptote: $y = -x$

59. Graphical Reasoning Identify the real numbers x_0, x_1, x_2, x_3, and x_4 in the figure such that each of the following is true.



(a) $f'(x) = 0$

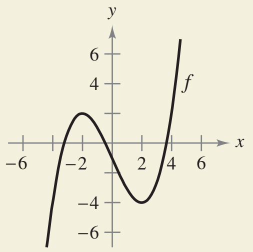
(b) $f''(x) = 0$

(c) $f'(x)$ does not exist.

(d) f has a relative maximum.

(e) f has a point of inflection.

60. **HOW DO YOU SEE IT?** The graph of f is shown in the figure.



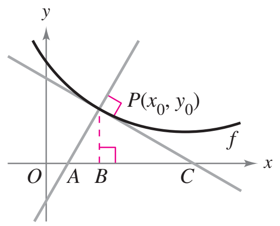
(a) For which values of x is $f'(x)$ zero? Positive? Negative? What do these values mean?

(b) For which values of x is $f''(x)$ zero? Positive? Negative? What do these values mean?

(c) On what open interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x, how does the rate of change of f compare with the rates of change of f for other values of x? Explain.

61. Investigation Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as shown in the figure. Verify each statement.



(a) The x-intercept of the tangent line is

$$\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right).$$

(b) The y-intercept of the tangent line is

$$(0, f(x_0) - x_0 f'(x_0)).$$

(c) The x-intercept of the normal line is

$$(x_0 + f(x_0) f'(x_0), 0).$$

(d) The y-intercept of the normal line is

$$\left(0, y_0 + \frac{x_0}{f'(x_0)}\right).$$

(e) $|BC| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC| = \left|\dfrac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$

(g) $|AB| = |f(x_0) f'(x_0)|$

(h) $|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$

62. Investigation Consider the function

$$f(x) = \frac{2x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

(c) For which value of n will $y = 2$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

63. Graphical Reasoning Consider the function

$$f(x) = \frac{ax}{(x - b)^2}.$$

Determine the effect on the graph of f as a and b are changed. Consider cases where a and b are both positive or both negative, and cases where a and b have opposite signs.

64. Graphical Reasoning Consider the function

$$f(x) = \frac{1}{2}(ax)^2 - ax, \quad a \neq 0.$$

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

(b) In the same viewing window, use a graphing utility to graph the function for four different values of a.

Slant Asymptotes **In Exercises 65 and 66, the graph of the function has two slant asymptotes. Identify each slant asymptote. Then graph the function and its asymptotes.**

65. $y = \sqrt{4 + 16x^2}$ **66.** $y = \sqrt{x^2 + 6x}$

PUTNAM EXAM CHALLENGE

67. Let $f(x)$ be defined for $a \le x \le b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b} = \frac{1}{2} f''(\varepsilon),$$

where ε is some number between a and b.

3.7 Optimization Problems

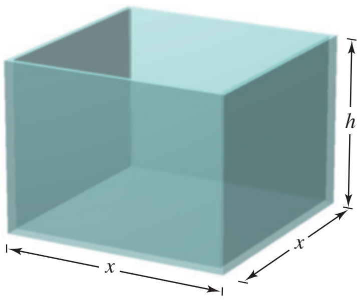
- Solve applied minimum and maximum problems.

Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Consider how frequently you hear or read terms such as greatest profit, least cost, least time, greatest voltage, optimum size, least size, greatest strength, and greatest distance. Before outlining a general problem-solving strategy for such problems, consider the next example.

EXAMPLE 1 Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume?



Open box with square base: $S = x^2 + 4xh = 108$

Figure 3.53

Solution Because the box has a square base, its volume is

$$V = x^2h. \qquad \text{Primary equation}$$

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized. The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$
$$108 = x^2 + 4xh. \qquad \text{Secondary equation}$$

Because V is to be maximized, you want to write V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$. Substituting into the primary equation produces

$$\begin{aligned} V &= x^2h && \text{Function of two variables} \\ &= x^2\left(\frac{108 - x^2}{4x}\right) && \text{Substitute for } h. \\ &= 27x - \frac{x^3}{4}. && \text{Function of one variable} \end{aligned}$$

Before finding which x-value will yield a maximum value of V, you should determine the *feasible domain*. That is, what values of x make sense in this problem? You know that $V \geq 0$. You also know that x must be nonnegative and that the area of the base $(A = x^2)$ is at most 108. So, the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

To maximize V, find the critical numbers of the volume function on the interval $(0, \sqrt{108})$.

$$\begin{aligned} \frac{dV}{dx} &= 27 - \frac{3x^2}{4} && \text{Differentiate with respect to } x. \\ 27 - \frac{3x^2}{4} &= 0 && \text{Set derivative equal to 0.} \\ 3x^2 &= 108 && \text{Simplify.} \\ x &= \pm 6 && \text{Critical numbers} \end{aligned}$$

So, the critical numbers are $x = \pm 6$. You do not need to consider $x = -6$ because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0$. So, V is maximum when $x = 6$, and the dimensions of the box are 6 inches by 6 inches by 3 inches.

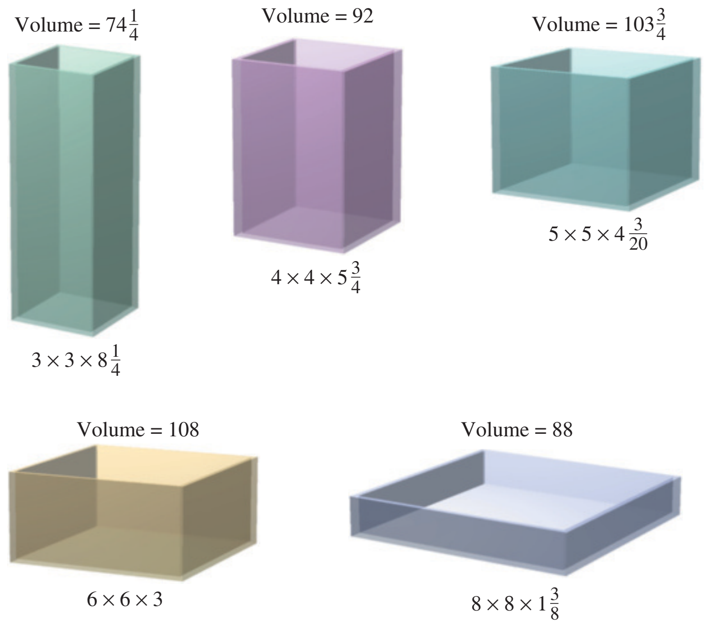
TECHNOLOGY You can verify your answer in Example 1 by using a graphing utility to graph the volume function

$$V = 27x - \frac{x^3}{4}.$$

Use a viewing window in which $0 \leq x \leq \sqrt{108} \approx 10.4$ and $0 \leq y \leq 120$, and use the *maximum* or *trace* feature to determine the maximum value of V.

In Example 1, you should realize that there are infinitely many open boxes having 108 square inches of surface area. To begin solving the problem, you might ask yourself which basic shape would seem to yield a maximum volume. Should the box be tall, squat, or nearly cubical?

You might even try calculating a few volumes, as shown in Figure 3.54, to see if you can get a better feeling for what the optimum dimensions should be. Remember that you are not ready to begin solving a problem until you have clearly identified what the problem is.



Which box has the greatest volume?
Figure 3.54

Example 1 illustrates the following guidelines for solving applied minimum and maximum problems.

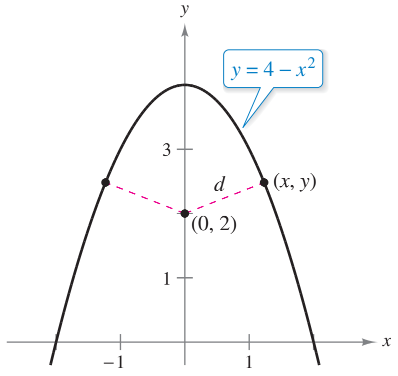
GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

REMARK For Step 5, recall that to determine the maximum or minimum value of a continuous function f on a closed interval, you should compare the values of f at its critical numbers with the values of f at the endpoints of the interval.

EXAMPLE 2 Finding Minimum Distance

See LarsonCalculus.com for an interactive version of this type of example.



The quantity to be minimized is distance: $d = \sqrt{(x-0)^2 + (y-2)^2}$.
Figure 3.55

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution Figure 3.55 shows that there are two points at a minimum distance from the point $(0, 2)$. The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is

$$d = \sqrt{(x-0)^2 + (y-2)^2}. \qquad \text{Primary equation}$$

Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$\begin{aligned} d &= \sqrt{x^2 + (4 - x^2 - 2)^2} \\ &= \sqrt{x^4 - 3x^2 + 4}. \end{aligned}$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$. Note that the domain of f is the entire real number line. So, there are no endpoints of the domain to consider. Moreover, the derivative of f

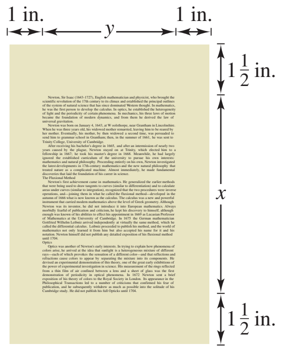
$$\begin{aligned} f'(x) &= 4x^3 - 6x \\ &= 2x(2x^2 - 3) \end{aligned}$$

is zero when

$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}.$$

Testing these critical numbers using the First Derivative Test verifies that $x = 0$ yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance. So, the closest points are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.

EXAMPLE 3 Finding Minimum Area



The quantity to be minimized is area: $A = (x + 3)(y + 2)$.
Figure 3.56

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 3.56). What should the dimensions of the page be so that the least amount of paper is used?

Solution Let A be the area to be minimized.

$$A = (x + 3)(y + 2) \qquad \text{Primary equation}$$

The printed area inside the margins is

$$24 = xy. \qquad \text{Secondary equation}$$

Solving this equation for y produces $y = 24/x$. Substitution into the primary equation produces

$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}. \qquad \text{Function of one variable}$$

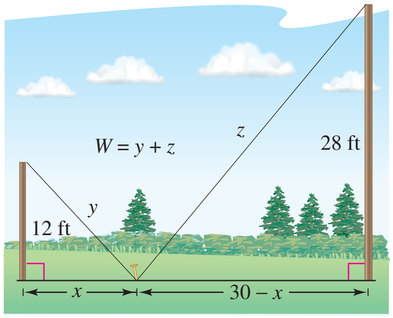
Because x must be positive, you are interested only in values of A for $x > 0$. To find the critical numbers, differentiate with respect to x

$$\frac{dA}{dx} = 2 - \frac{72}{x^2}$$

and note that the derivative is zero when $x^2 = 36$, or $x = \pm 6$. So, the critical numbers are $x = \pm 6$. You do not have to consider $x = -6$ because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x = 6$. So, $y = \frac{24}{6} = 4$ and the dimensions of the page should be $x + 3 = 9$ inches by $y + 2 = 6$ inches.

EXAMPLE 4 Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.

Figure 3.57

Solution Let W be the wire length to be minimized. Using Figure 3.57, you can write

$$W = y + z. \qquad \text{Primary equation}$$

In this problem, rather than solving for y in terms of z (or vice versa), you can solve for both y and z in terms of a third variable x, as shown in Figure 3.57. From the Pythagorean Theorem, you obtain

$$x^2 + 12^2 = y^2$$
$$(30 - x)^2 + 28^2 = z^2$$

which implies that

$$y = \sqrt{x^2 + 144}$$
$$z = \sqrt{x^2 - 60x + 1684}.$$

So, you can rewrite the primary equation as

$$\begin{aligned} W &= y + z \\ &= \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}, \quad 0 \le x \le 30. \end{aligned}$$

Differentiating W with respect to x yields

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

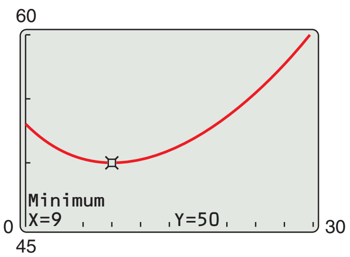
By letting $dW/dx = 0$, you obtain

$$\begin{aligned} \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} &= 0 \\ x\sqrt{x^2 - 60x + 1684} &= (30 - x)\sqrt{x^2 + 144} \\ x^2(x^2 - 60x + 1684) &= (30 - x)^2(x^2 + 144) \\ x^4 - 60x^3 + 1684x^2 &= x^4 - 60x^3 + 1044x^2 - 8640x + 129{,}600 \\ 640x^2 + 8640x - 129{,}600 &= 0 \\ 320(x - 9)(2x + 45) &= 0 \\ x &= 9, -22.5. \end{aligned}$$

Because $x = -22.5$ is not in the domain and

$$W(0) \approx 53.04, \quad W(9) = 50, \quad \text{and} \quad W(30) \approx 60.31$$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole.



You can confirm the minimum value of W with a graphing utility.

Figure 3.58

TECHNOLOGY From Example 4, you can see that applied optimization problems can involve a lot of algebra. If you have access to a graphing utility, you can confirm that $x = 9$ yields a minimum value of W by graphing

$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

as shown in Figure 3.58.

In each of the first four examples, the extreme value occurred at a critical number. Although this happens often, remember that an extreme value can also occur at an endpoint of an interval, as shown in Example 5.

EXAMPLE 5 An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution The total area (see Figure 3.59) is

$$A = (\text{area of square}) + (\text{area of circle})$$

$$A = x^2 + \pi r^2. \qquad \text{Primary equation}$$

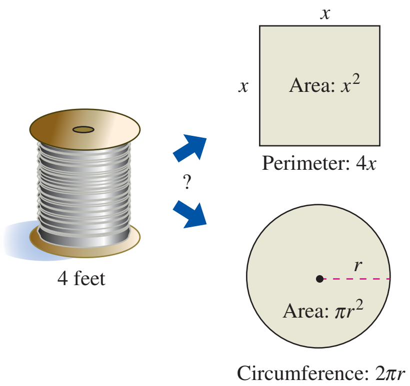
Because the total length of wire is 4 feet, you obtain

$$4 = (\text{perimeter of square}) + (\text{circumference of circle})$$

$$4 = 4x + 2\pi r.$$

So, $r = 2(1 - x)/\pi$, and by substituting into the primary equation you have

$$\begin{aligned} A &= x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2 \\ &= x^2 + \frac{4(1-x)^2}{\pi} \\ &= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]. \end{aligned}$$



The quantity to be maximized is area: $A = x^2 + \pi r^2$.
Figure 3.59

> **Exploration**
> What would the answer be if Example 5 asked for the dimensions needed to enclose the *minimum* total area?

The feasible domain is $0 \le x \le 1$, restricted by the square's perimeter. Because

$$\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi}$$

the only critical number in $(0, 1)$ is $x = 4/(\pi + 4) \approx 0.56$. So, using

$$A(0) \approx 1.273, \quad A(0.56) \approx 0.56, \quad \text{and} \quad A(1) = 1$$

you can conclude that the maximum area occurs when $x = 0$. That is, *all* the wire is used for the circle.

Before doing the section exercises, review the primary equations developed in the first five examples. As applications go, these five examples are fairly simple, and yet the resulting primary equations are quite complicated.

$$V = 27x - \frac{x^3}{4} \qquad \text{Example 1}$$

$$d = \sqrt{x^4 - 3x^2 + 4} \qquad \text{Example 2}$$

$$A = 30 + 2x + \frac{72}{x} \qquad \text{Example 3}$$

$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684} \qquad \text{Example 4}$$

$$A = \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4] \qquad \text{Example 5}$$

You must expect that real-life applications often involve equations that are *at least as complicated* as these five. Remember that one of the main goals of this course is to learn to use calculus to analyze equations that initially seem formidable.

3.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Numerical, Graphical, and Analytic Analysis Find two positive numbers whose sum is 110 and whose product is a maximum.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

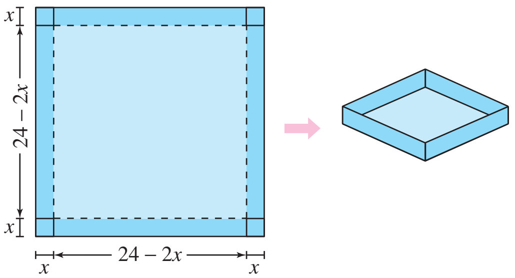
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the table feature of the graphing utility.)

(c) Write the product P as a function of x.

(d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.

(e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

2. Numerical, Graphical, and Analytic Analysis An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).



(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height, x	Length and Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

(b) Write the volume V as a function of x.

(c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

(d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.

Finding Numbers **In Exercises 3–8, find two positive numbers that satisfy the given requirements.**

3. The sum is S and the product is a maximum.

4. The product is 185 and the sum is a minimum.

5. The product is 147 and the sum of the first number plus three times the second number is a minimum.

6. The second number is the reciprocal of the first number and the sum is a minimum.

7. The sum of the first number and twice the second number is 108 and the product is a maximum.

8. The sum of the first number squared and the second number is 54 and the product is a maximum.

Maximum Area **In Exercises 9 and 10, find the length and width of a rectangle that has the given perimeter and a maximum area.**

9. Perimeter: 80 meters

10. Perimeter: P units

Minimum Perimeter **In Exercises 11 and 12, find the length and width of a rectangle that has the given area and a minimum perimeter.**

11. Area: 32 square feet

12. Area: A square centimeters

Minimum Distance **In Exercises 13–16, find the point on the graph of the function that is closest to the given point.**

13. $f(x) = x^2, \ \left(2, \frac{1}{2}\right)$

14. $f(x) = (x - 1)^2, \ (-5, 3)$

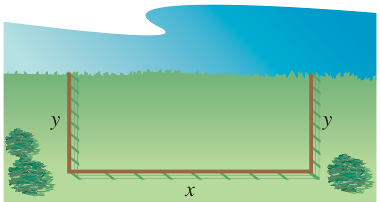
15. $f(x) = \sqrt{x}, \ (4, 0)$

16. $f(x) = \sqrt{x - 8}, \ (12, 0)$

17. Minimum Area A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

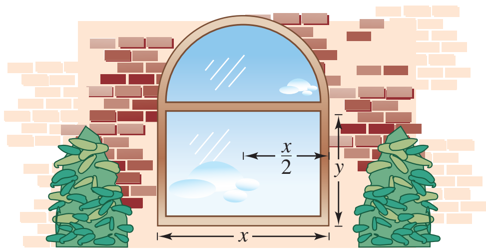
18. Minimum Area A rectangular page is to contain 36 square inches of print. The margins on each side are $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

19. Minimum Length A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?



20. Maximum Volume A rectangular solid (with a square base) has a surface area of 337.5 square centimeters. Find the dimensions that will result in a solid with maximum volume.

21. Maximum Area A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area when the total perimeter is 16 feet.



22. Maximum Area A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?

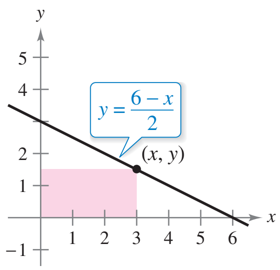


Figure for 22

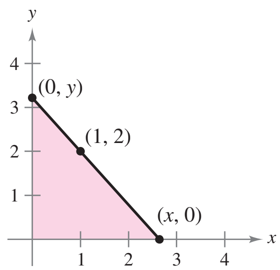


Figure for 23

23. Minimum Length and Minimum Area A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1, 2)$ (see figure).

(a) Write the length L of the hypotenuse as a function of x.

(b) Use a graphing utility to approximate x graphically such that the length of the hypotenuse is a minimum.

(c) Find the vertices of the triangle such that its area is a minimum.

24. Maximum Area Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.

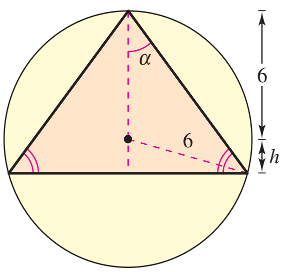


Figure for 24

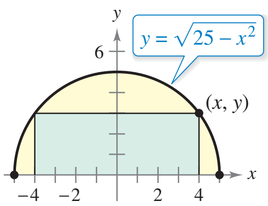


Figure for 25

25. Maximum Area A rectangle is bounded by the x-axis and the semicircle

$$y = \sqrt{25 - x^2}$$

(see figure). What length and width should the rectangle have so that its area is a maximum?

26. Maximum Area Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 25).

27. Numerical, Graphical, and Analytic Analysis An exercise room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

(a) Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.

(b) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

(c) Write the area A as a function of x.

(d) Use calculus to find the critical number of the function in part (c) and find the maximum value.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

28. Numerical, Graphical, and Analytic Analysis A right circular cylinder is designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius, r	Height	Surface Area, S
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$

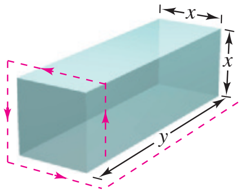
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the surface area S as a function of r.

(d) Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.

(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

29. Maximum Volume A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)



30. Maximum Volume Rework Exercise 29 for a cylindrical package. (The cross section is circular.)

WRITING ABOUT CONCEPTS

31. Surface Area and Volume A shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

32. Area and Perimeter The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

33. Minimum Surface Area A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

34. Minimum Cost An industrial tank of the shape described in Exercise 33 must have a volume of 4000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

35. Minimum Area The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

36. Maximum Area Twenty feet of wire is to be used to form two figures. In each of the following cases, how much wire should be used for each figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.}

37. Beam Strength A wooden beam has a rectangular cross section of height h and width w (see figure). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches? (*Hint:* $S = kh^2w$, where k is the proportionality constant.)

Andriy Markov/Shutterstock.com

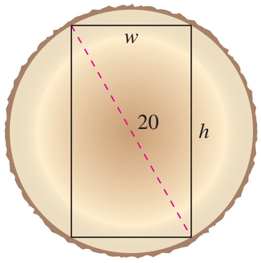


Figure for 37

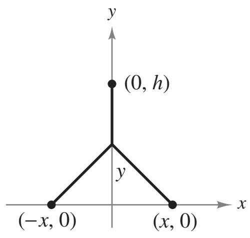


Figure for 38

38. Minimum Length Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$, and their power supply is at $(0, h)$ (see figure). Find y such that the total length of power line from the power supply to the factories is a minimum.

39. Minimum Cost An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as laying it on land. What path should the pipe follow in order to minimize the cost?

40. Illumination A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum when

$$I = \frac{k \sin \alpha}{s^2}$$

where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.

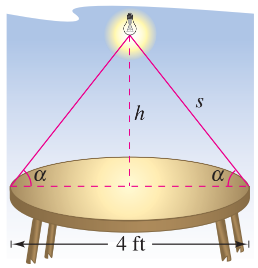


Figure for 40

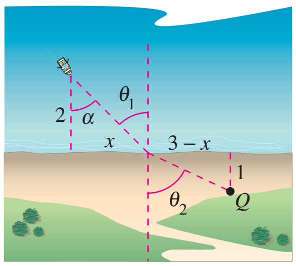


Figure for 41

41. Minimum Time A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?

42. Minimum Time The conditions are the same as in Exercise 41 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

43. Minimum Distance Sketch the graph of $f(x) = 2 - 2 \sin x$ on the interval $[0, \pi/2]$.

(a) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

(b) Write the distance d from the origin to a point on the graph of f as a function of x. Use your graphing utility to graph d and find the minimum distance.

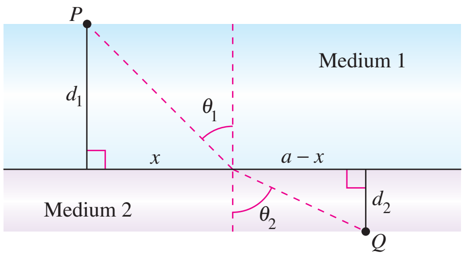
(c) Use calculus and the *zero* or *root* feature of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?

(Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO)

44. Minimum Time When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that in Exercise 42, and that light waves traveling from P to Q follow the path of minimum time.



45. Maximum Volume A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

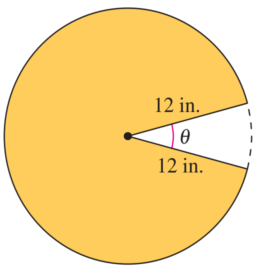


Figure for 45

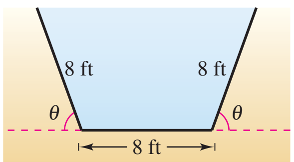


Figure for 46

46. Numerical, Graphical, and Analytic Analysis The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10°$	$8 \sin 10°$	≈ 22.1
8	$8 + 16 \cos 20°$	$8 \sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

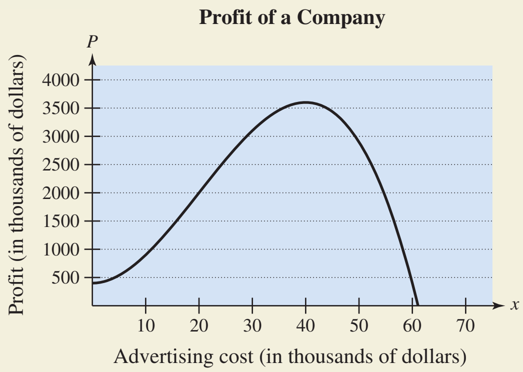
(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

47. Maximum Profit Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Furthermore, the bank can reinvest this money at 12%. Find the interest rate the bank should pay to maximize profit. (Use the simple interest formula.)

48. HOW DO YOU SEE IT? The graph shows the profit P (in thousands of dollars) of a company in terms of its advertising cost x (in thousands of dollars).



(a) Estimate the interval on which the profit is increasing.

(b) Estimate the interval on which the profit is decreasing.

(c) Estimate the amount of money the company should spend on advertising in order to yield a maximum profit.

(d) The *point of diminishing returns* is the point at which the rate of growth of the profit function begins to decline. Estimate the point of diminishing returns.

Minimum Distance **In Exercises 49–51, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures). The center supplies three factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.**

49. Minimize the sum of the squares of the lengths of the vertical feeder lines (see figure) given by

$$S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines.

50. Minimize the sum of the absolute values of the lengths of the vertical feeder lines (see figure) given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_2 and approximate the required critical number.)

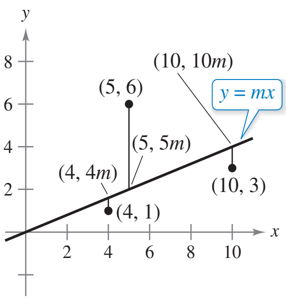


Figure for 49 and 50

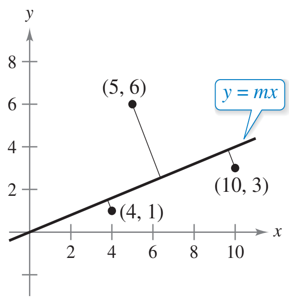


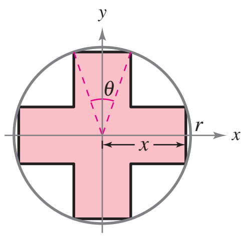
Figure for 51

51. Minimize the sum of the perpendicular distances (see figure and Exercises 83–86 in Section P.2) from the trunk line to the factories given by

$$S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_3 and approximate the required critical number.)

52. Maximum Area Consider a symmetric cross inscribed in a circle of radius r (see figure).



(a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.

(b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.

(c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?

PUTNAM EXAM CHALLENGE

53. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$.

54. Find the minimum value of

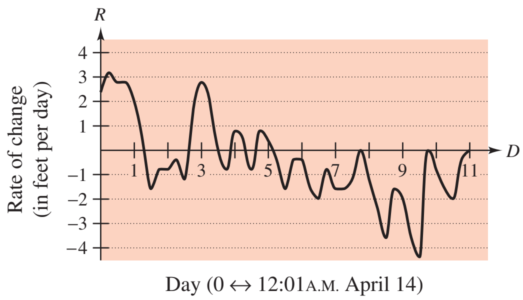
$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} \quad \text{for} \quad x > 0.$$

SECTION PROJECT

Connecticut River

Whenever the Connecticut River reaches a level of 105 feet above sea level, two Northampton, Massachusetts, flood control station operators begin a round-the-clock river watch. Every 2 hours, they check the height of the river, using a scale marked off in tenths of a foot, and record the data in a log book. In the spring of 1996, the flood watch lasted from April 4, when the river reached 105 feet and was rising at 0.2 foot per hour, until April 25, when the level subsided again to 105 feet. Between those dates, their log shows that the river rose and fell several times, at one point coming close to the 115-foot mark. If the river had reached 115 feet, the city would have closed down Mount Tom Road (Route 5, south of Northampton).

The graph below shows the rate of change of the level of the river during one portion of the flood watch. Use the graph to answer each question.



(a) On what date was the river rising most rapidly? How do you know?

(b) On what date was the river falling most rapidly? How do you know?

(c) There were two dates in a row on which the river rose, then fell, then rose again during the course of the day. On which days did this occur, and how do you know?

(d) At 1 minute past midnight, April 14, the river level was 111.0 feet. Estimate its height 24 hours later and 48 hours later. Explain how you made your estimates.

(e) The river crested at 114.4 feet. On what date do you think this occurred?

(Submitted by Mary Murphy, Smith College, Northampton, MA)

3.8 Newton's Method

■ **Approximate a zero of a function using Newton's Method.**

Newton's Method

In this section, you will study a technique for approximating the real zeros of a function. The technique is called **Newton's Method,** and it uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, f must have at least one zero in the interval (a, b). To estimate this zero, you choose

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 3.60(a). Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate of the zero of f. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

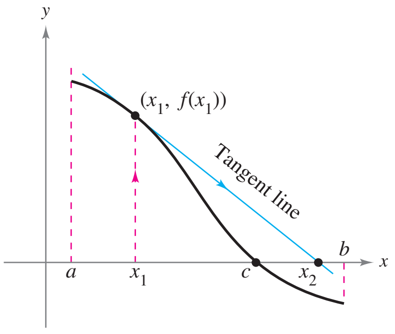
So, from the initial estimate x_1, you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate [See Figure 3.60(b).]}$$

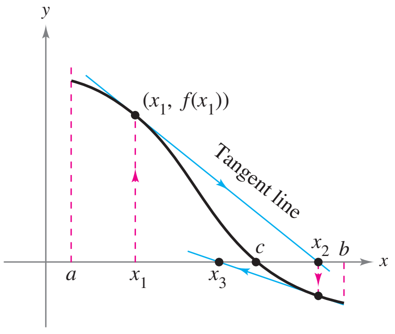
You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

Repeated application of this process is called Newton's Method.



(a)



(b)
The x-intercept of the tangent line approximates the zero of f.
Figure 3.60

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use these steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

NEWTON'S METHOD

Isaac Newton first described the method for approximating the real zeros of a function in his text *Method of Fluxions*. Although the book was written in 1671, it was not published until 1736. Meanwhile, in 1690, Joseph Raphson (1648–1715) published a paper describing a method for approximating the real zeros of a function that was very similar to Newton's. For this reason, the method is often referred to as the Newton-Raphson method.

REMARK For many functions, just a few iterations of Newton's Method will produce approximations having very small errors, as shown in Example 1.

EXAMPLE 1 Using Newton's Method

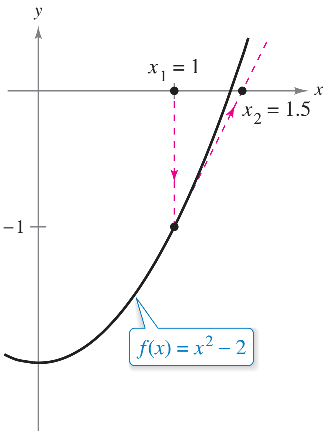
Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.000000	-1.000000	2.000000	-0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				



The first iteration of Newton's Method
Figure 3.61

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.61.

EXAMPLE 2 Using Newton's Method

See LarsonCalculus.com for an interactive version of this type of example.

Use Newton's Method to approximate the zeros of

$$f(x) = 2x^3 + x^2 - x + 1.$$

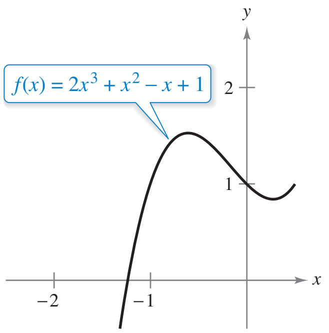
Continue the iterations until two successive approximations differ by less than 0.0001.

Solution Begin by sketching a graph of f, as shown in Figure 3.62. From the graph, you can observe that the function has only one zero, which occurs near $x = -1.2$. Next, differentiate f and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^3 + x_n^2 - x_n + 1}{6x_n^2 + 2x_n - 1}.$$

The calculations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	-1.20000	0.18400	5.24000	0.03511	-1.23511
2	-1.23511	-0.00771	5.68276	-0.00136	-1.23375
3	-1.23375	0.00001	5.66533	0.00000	-1.23375
4	-1.23375				

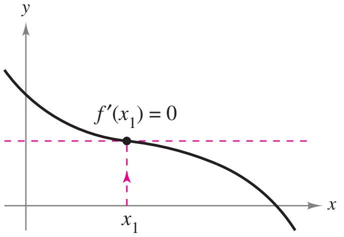


After three iterations of Newton's Method, the zero of f is approximated to the desired accuracy.
Figure 3.62

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of f to be -1.23375.

When, as in Examples 1 and 2, the approximations approach a limit, the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is said to **converge.** Moreover, when the limit is c, it can be shown that c must be a zero of f.

Newton's Method does not always yield a convergent sequence. One way it can fail to do so is shown in Figure 3.63. Because Newton's Method involves division by $f'(x_n)$, it is clear that the method will fail when the derivative is zero for any x_n in the sequence. When you encounter this problem, you can usually overcome it by choosing a different value for x_1. Another way Newton's Method can fail is shown in the next example.



Newton's Method fails to converge when $f'(x_n) = 0$.
Figure 3.63

FOR FURTHER INFORMATION For more on when Newton's Method fails, see the article "No Fooling! Newton's Method Can Be Fooled" by Peter Horton in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

EXAMPLE 3 **An Example in Which Newton's Method Fails**

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

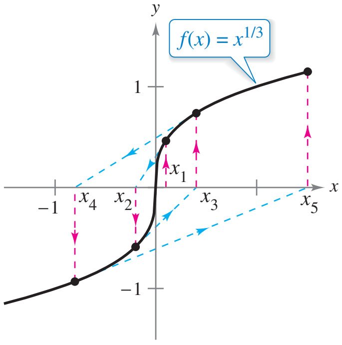
Solution Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

The calculations are shown in the table. This table and Figure 3.64 indicate that x_n continues to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	-0.20000
2	-0.20000	-0.58480	0.97467	-0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	-0.80000
4	-0.80000	-0.92832	0.3680	-2.40000	1.60000

REMARK In Example 3, the initial estimate $x_1 = 0.1$ fails to produce a convergent sequence. Try showing that Newton's Method also fails for every other choice of x_1 (other than the actual zero).



Newton's Method fails to converge for every x-value other than the actual zero of f.
Figure 3.64

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of f is that

$$\boxed{\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| < 1} \qquad \text{Condition for convergence}$$

on an open interval containing the zero. For instance, in Example 1, this test would yield

$$f(x) = x^2 - 2, \quad f'(x) = 2x, \quad f''(x) = 2,$$

and

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| = \left|\frac{(x^2 - 2)(2)}{4x^2}\right| = \left|\frac{1}{2} - \frac{1}{x^2}\right|. \qquad \text{Example 1}$$

On the interval $(1, 3)$, this quantity is less than 1 and therefore the convergence of Newton's Method is guaranteed. On the other hand, in Example 3, you have

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3}x^{-2/3}, \quad f''(x) = -\frac{2}{9}x^{-5/3}$$

and

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| = \left|\frac{x^{1/3}(-2/9)(x^{-5/3})}{(1/9)(x^{-4/3})}\right| = 2 \qquad \text{Example 3}$$

which is not less than 1 for any value of x, so you cannot conclude that Newton's Method will converge.

You have learned several techniques for finding the zeros of functions. The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring. The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by *elementary* algebraic methods. This particular function has only one real zero, and by using more advanced algebraic techniques, you can determine the zero to be

$$x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.$$

Because the *exact* solution is written in terms of square roots and cube roots, it is called a **solution by radicals.**

The determination of radical solutions of a polynomial equation is one of the fundamental problems of algebra. The earliest such result is the Quadratic Formula, which dates back at least to Babylonian times. The general formula for the zeros of a cubic function was developed much later. In the sixteenth century, an Italian mathematician, Jerome Cardan, published a method for finding radical solutions to cubic and quartic equations. Then, for 300 years, the problem of finding a general quintic formula remained open. Finally, in the nineteenth century, the problem was answered independently by two young mathematicians. Niels Henrik Abel, a Norwegian mathematician, and Evariste Galois, a French mathematician, proved that it is not possible to solve a *general* fifth- (or higher-) degree polynomial equation by radicals. Of course, you can solve particular fifth-degree equations, such as

$$x^5 - 1 = 0$$

but Abel and Galois were able to show that no general *radical* solution exists.

NIELS HENRIK ABEL (1802–1829)

EVARISTE GALOIS (1811–1832)

Although the lives of both Abel and Galois were brief, their work in the fields of analysis and abstract algebra was far-reaching.

See LarsonCalculus.com to read a biography about each of these mathematicians.

The Granger Collection, New York

3.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using Newton's Method **In Exercises 1–4, complete two iterations of Newton's Method to approximate a zero of the function using the given initial guess.**

1. $f(x) = x^2 - 5, \quad x_1 = 2.2$
2. $f(x) = x^3 - 3, \quad x_1 = 1.4$
3. $f(x) = \cos x, \quad x_1 = 1.6$
4. $f(x) = \tan x, \quad x_1 = 0.1$

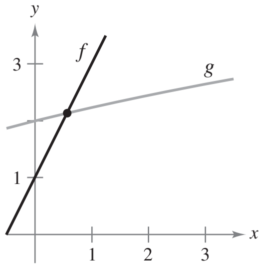
Using Newton's Method **In Exercises 5–14, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.**

5. $f(x) = x^3 + 4$
6. $f(x) = 2 - x^3$
7. $f(x) = x^3 + x - 1$
8. $f(x) = x^5 + x - 1$
9. $f(x) = 5\sqrt{x-1} - 2x$
10. $f(x) = x - 2\sqrt{x+1}$
11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$
12. $f(x) = x^4 + x^3 - 1$
13. $f(x) = 1 - x + \sin x$
14. $f(x) = x^3 - \cos x$

Finding Point(s) of Intersection **In Exercises 15–18, apply Newton's Method to approximate the x-value(s) of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]**

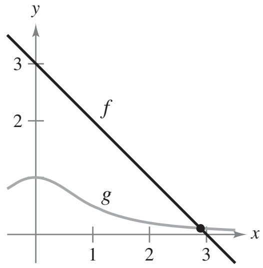
15. $f(x) = 2x + 1$

    $g(x) = \sqrt{x+4}$



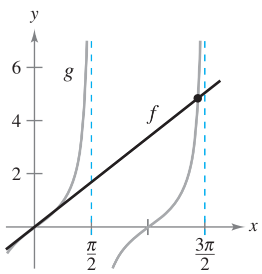
16. $f(x) = 3 - x$

    $g(x) = \dfrac{1}{x^2+1}$



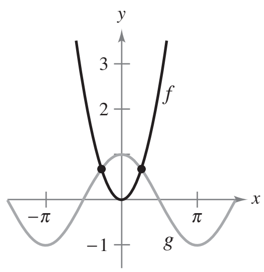
17. $f(x) = x$

    $g(x) = \tan x$



18. $f(x) = x^2$

    $g(x) = \cos x$



19. **Mechanic's Rule** The Mechanic's Rule for approximating $\sqrt{a}$, $a > 0$, is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3 \ldots$$

where x_1 is an approximation of $\sqrt{a}$.

(a) Use Newton's Method and the function $f(x) = x^2 - a$ to derive the Mechanic's Rule.

(b) Use the Mechanic's Rule to approximate $\sqrt{5}$ and $\sqrt{7}$ to three decimal places.

20. **Approximating Radicals**

(a) Use Newton's Method and the function $f(x) = x^n - a$ to obtain a general rule for approximating $x = \sqrt[n]{a}$.

(b) Use the general rule found in part (a) to approximate $\sqrt[4]{6}$ and $\sqrt[3]{15}$ to three decimal places.

Failure of Newton's Method **In Exercises 21 and 22, apply Newton's Method using the given initial guess, and explain why the method fails.**

21. $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$

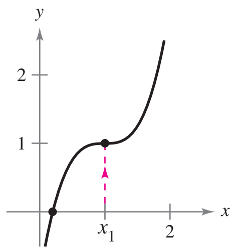


Figure for 21

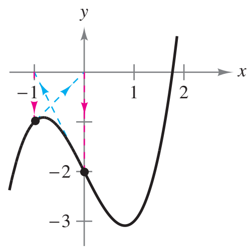


Figure for 22

22. $y = x^3 - 2x - 2, \quad x_1 = 0$

Fixed Point **In Exercises 23 and 24, approximate the fixed point of the function to two decimal places. [A *fixed point* x_0 of a function f is a value of x such that $f(x_0) = x_0$.]**

23. $f(x) = \cos x$
24. $f(x) = \cot x, \quad 0 < x < \pi$

25. **Approximating Reciprocals** Use Newton's Method to show that the equation

$$x_{n+1} = x_n(2 - ax_n)$$

can be used to approximate $1/a$ when x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. (*Hint:* Consider

$$f(x) = \frac{1}{x} - a.)$$

26. **Approximating Reciprocals** Use the result of Exercise 25 to approximate (a) $\frac{1}{3}$ and (b) $\frac{1}{11}$ to three decimal places.

WRITING ABOUT CONCEPTS

27. Using Newton's Method Consider the function $f(x) = x^3 - 3x^2 + 3$.

(a) Use a graphing utility to graph f.

(b) Use Newton's Method to approximate a zero with $x_1 = 1$ as an initial guess.

(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.

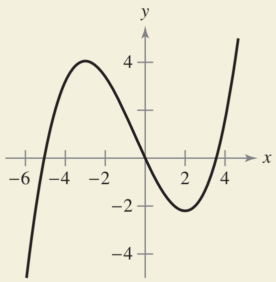
(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

28. Using Newton's Method Repeat the steps in Exercise 27 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

29. Newton's Method In your own words and using a sketch, describe Newton's Method for approximating the zeros of a function.

30. HOW DO YOU SEE IT? For what value(s) will Newton's Method fail to converge for the function shown in the graph? Explain your reasoning.



Using Newton's Method Exercises 31–33 present problems similar to exercises from the previous sections of this chapter. In each case, use Newton's Method to approximate the solution.

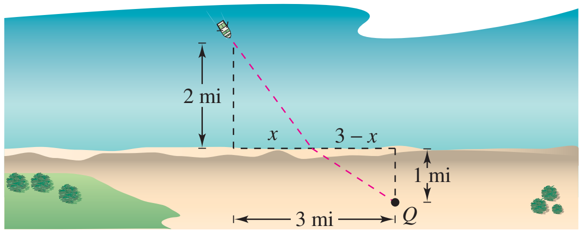
31. Minimum Distance Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

32. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{50 + t^3}.$$

When is the concentration the greatest?

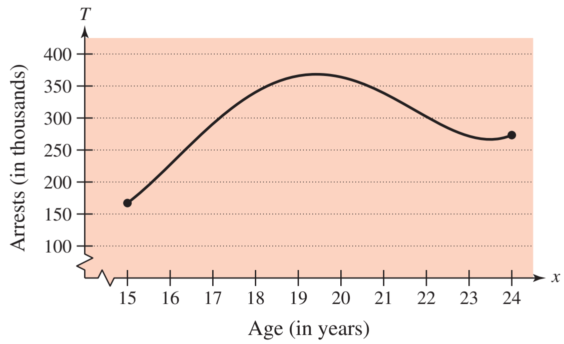
33. Minimum Time You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point Q that is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach Q in the least time?



34. Crime The total number of arrests T (in thousands) for all males ages 15 to 24 in 2010 is approximated by the model

$$T = 0.2988x^4 - 22.625x^3 + 628.49x^2 - 7565.9x + 33{,}478$$

for $15 \le x \le 24$, where x is the age in years (see figure). Approximate the two ages that had total arrests of 300 thousand. *(Source: U.S. Department of Justice)*



True or False? In Exercises 35–38, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

35. The zeros of $f(x) = \dfrac{p(x)}{q(x)}$ coincide with the zeros of $p(x)$.

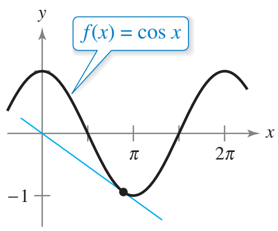
36. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

37. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

38. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

39. Tangent Lines The graph of $f(x) = -\sin x$ has infinitely many tangent lines that pass through the origin. Use Newton's Method to approximate to three decimal places the slope of the tangent line having the greatest slope.

40. Point of Tangency The graph of $f(x) = \cos x$ and a tangent line to f through the origin are shown. Find the coordinates of the point of tangency to three decimal places.



3.9 Differentials

- Understand the concept of a tangent line approximation.
- Compare the value of the differential, dy, with the actual change in y, Δy.
- Estimate a propagated error using a differential.
- Find the differential of a function using differentiation formulas.

Exploration

Tangent Line Approximation Use a graphing utility to graph $f(x) = x^2$. In the same viewing window, graph the tangent line to the graph of f at the point $(1, 1)$. Zoom in twice on the point of tangency. Does your graphing utility distinguish between the two graphs? Use the *trace* feature to compare the two graphs. As the x-values get closer to 1, what can you say about the y-values?

Tangent Line Approximations

Newton's Method (Section 3.8) is an example of the use of a tangent line to approximate the graph of a function. In this section, you will study other situations in which the graph of a function can be approximated by a straight line.

To begin, consider a function f that is differentiable at c. The equation for the tangent line at the point $(c, f(c))$ is

$$y - f(c) = f'(c)(x - c)$$

$$\boxed{y = f(c) + f'(c)(x - c)}$$

and is called the **tangent line approximation** (or **linear approximation**) **of f at c.** Because c is a constant, y is a linear function of x. Moreover, by restricting the values of x to those sufficiently close to c, the values of y can be used as approximations (to any desired degree of accuracy) of the values of the function f. In other words, as x approaches c, the limit of y is $f(c)$.

EXAMPLE 1 **Using a Tangent Line Approximation**

See LarsonCalculus.com for an interactive version of this type of example.

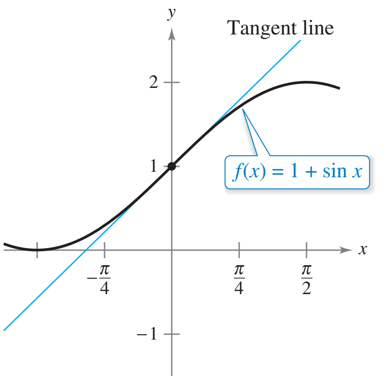
Find the tangent line approximation of $f(x) = 1 + \sin x$ at the point $(0, 1)$. Then use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution The derivative of f is

$f'(x) = \cos x.$ First derivative

So, the equation of the tangent line to the graph of f at the point $(0, 1)$ is

$$y = f(0) + f'(0)(x - 0)$$
$$y = 1 + (1)(x - 0)$$
$$y = 1 + x. \quad \text{Tangent line approximation}$$



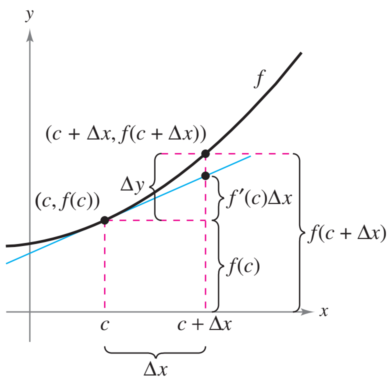
The tangent line approximation of f at the point $(0, 1)$
Figure 3.65

The table compares the values of y given by this linear approximation with the values of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation. This conclusion is reinforced by the graph shown in Figure 3.65.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

REMARK Be sure you see that this linear approximation of $f(x) = 1 + \sin x$ depends on the point of tangency. At a different point on the graph of f, you would obtain a different tangent line approximation.

Differentials



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.
Figure 3.66

When the tangent line to the graph of f at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c) \qquad \text{Tangent line at } (c, f(c))$$

is used as an approximation of the graph of f, the quantity $x - c$ is called the change in x, and is denoted by Δx, as shown in Figure 3.66. When Δx is small, the change in y (denoted by Δy) can be approximated as shown.

$$\Delta y = f(c + \Delta x) - f(c) \qquad \text{Actual change in } y$$
$$\approx f'(c)\Delta x \qquad \text{Approximate change in } y$$

For such an approximation, the quantity Δx is traditionally denoted by dx, and is called the **differential of x.** The expression $f'(x)\,dx$ is denoted by dy, and is called the **differential of y.**

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x)\,dx.$$

AP* Tips

Local linear approximation, and whether such an approximation over- or under-estimates a function value, is commonly tested on the AP free response section.

In many types of applications, the differential of y can be used as an approximation of the change in y. That is,

$$\Delta y \approx dy \quad \text{or} \quad \Delta y \approx f'(x)\,dx.$$

EXAMPLE 2 Comparing Δy and dy

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

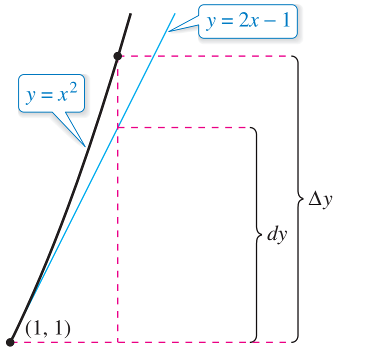
Solution Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is

$$dy = f'(x)\,dx = f'(1)(0.01) = 2(0.01) = 0.02. \qquad \text{Differential of } y$$

Now, using $\Delta x = 0.01$, the change in y is

$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201.$$

Figure 3.67 shows the geometric comparison of dy and Δy. Try comparing other values of dy and Δy. You will see that the values become closer to each other as dx (or Δx) approaches 0.



The change in y, Δy, is approximated by the differential of y, dy.
Figure 3.67

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1. \qquad \text{Tangent line to the graph of } f \text{ at } x = 1.$$

For x-values near 1, this line is close to the graph of f, as shown in Figure 3.67 and in the table.

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x) = x^2$	0.25	0.81	0.9801	1	1.0201	1.21	2.25
$y = 2x - 1$	0	0.8	0.98	1	1.02	1.2	2

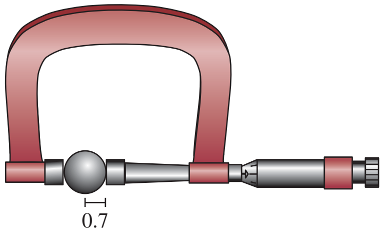
Error Propagation

Physicists and engineers tend to make liberal use of the approximation of Δy by dy. One way this occurs in practice is in the estimation of errors propagated by physical measuring devices. For example, if you let x represent the measured value of a variable and let $x + \Delta x$ represent the exact value, then Δx is the *error in measurement*. Finally, if the measured value x is used to compute another value $f(x)$, then the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error.**

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

EXAMPLE 3 **Estimation of Error**

The measured radius of a ball bearing is 0.7 inch, as shown in the figure. The measurement is correct to within 0.01 inch. Estimate the propagated error in the volume V of the ball bearing.



Ball bearing with measured radius that is correct to within 0.01 inch.

Solution The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere. So, you can write

$$r = 0.7 \qquad \text{Measured radius}$$

and

$$-0.01 \le \Delta r \le 0.01. \qquad \text{Possible error}$$

To approximate the propagated error in the volume, differentiate V to obtain $dV/dr = 4\pi r^2$ and write

$$\begin{aligned} \Delta V &\approx dV && \text{Approximate } \Delta V \text{ by } dV. \\ &= 4\pi r^2\,dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr. \\ &\approx \pm 0.06158 \text{ cubic inch.} \end{aligned}$$

So, the volume has a propagated error of about 0.06 cubic inch.

Would you say that the propagated error in Example 3 is large or small? The answer is best given in *relative* terms by comparing dV with V. The ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} && \text{Ratio of } dV \text{ to } V \\ &= \frac{3\,dr}{r} && \text{Simplify.} \\ &\approx \frac{3}{0.7}(\pm 0.01) && \text{Substitute for } dr \text{ and } r. \\ &\approx \pm 0.0429 \end{aligned}$$

is called the **relative error.** The corresponding **percent error** is approximately 4.29%.

Calculating Differentials

Each of the differentiation rules that you studied in Chapter 2 can be written in **differential form.** For example, let u and v be differentiable functions of x. By the definition of differentials, you have

$$du = u'\,dx$$

and

$$dv = v'\,dx.$$

So, you can write the differential form of the Product Rule as shown below.

$$\begin{aligned} d[uv] &= \frac{d}{dx}[uv]\,dx && \text{Differential of } uv. \\ &= [uv' + vu']\,dx && \text{Product Rule} \\ &= uv'\,dx + vu'\,dx \\ &= u\,dv + v\,du \end{aligned}$$

Differential Formulas

Let u and v be differentiable functions of x.

Constant multiple: $d[cu] = c\,du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u\,dv + v\,du$

Quotient: $d\left[\dfrac{u}{v}\right] = \dfrac{v\,du - u\,dv}{v^2}$

GOTTFRIED WILHELM LEIBNIZ (1646–1716)

Both Leibniz and Newton are credited with creating calculus. It was Leibniz, however, who tried to broaden calculus by developing rules and formal notation. He often spent days choosing an appropriate notation for a new concept.

See LarsonCalculus.com to read more of this biography.

EXAMPLE 4 Finding Differentials

Function	Derivative	Differential
a. $y = x^2$	$\dfrac{dy}{dx} = 2x$	$dy = 2x\,dx$
b. $y = \sqrt{x}$	$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$	$dy = \dfrac{dx}{2\sqrt{x}}$
c. $y = 2\sin x$	$\dfrac{dy}{dx} = 2\cos x$	$dy = 2\cos x\,dx$
d. $y = x\cos x$	$\dfrac{dy}{dx} = -x\sin x + \cos x$	$dy = (-x\sin x + \cos x)\,dx$
e. $y = \dfrac{1}{x}$	$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$	$dy = -\dfrac{dx}{x^2}$

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz. The beauty of this notation is that it provides an easy way to remember several important calculus formulas by making it seem as though the formulas were derived from algebraic manipulations of differentials. For instance, in Leibniz notation, the *Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

would appear to be true because the du's divide out. Even though this reasoning is *incorrect*, the notation does help one remember the Chain Rule.

EXAMPLE 5 **Finding the Differential of a Composite Function**

$$y = f(x) = \sin 3x \qquad \text{Original function}$$

$$f'(x) = 3\cos 3x \qquad \text{Apply Chain Rule.}$$

$$dy = f'(x)\,dx = 3\cos 3x\,dx \qquad \text{Differential form}$$

EXAMPLE 6 **Finding the Differential of a Composite Function**

$$y = f(x) = (x^2 + 1)^{1/2} \qquad \text{Original function}$$

$$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}} \qquad \text{Apply Chain Rule.}$$

$$dy = f'(x)\,dx = \frac{x}{\sqrt{x^2 + 1}}\,dx \qquad \text{Differential form}$$

Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, use the formula

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

REMARK This formula is equivalent to the tangent line approximation given earlier in this section.

which is derived from the approximation

$$\Delta y = f(x + \Delta x) - f(x) \approx dy.$$

The key to using this formula is to choose a value for x that makes the calculations easier, as shown in Example 7.

EXAMPLE 7 **Approximating Function Values**

Use differentials to approximate $\sqrt{16.5}$.

Solution Using $f(x) = \sqrt{x}$, you can write

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx.$$

Now, choosing $x = 16$ and $dx = 0.5$, you obtain the following approximation.

$$f(x + \Delta x) = \sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625$$

The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 16$ is the line $g(x) = \frac{1}{8}x + 2$. For x-values near 16, the graphs of f and g are close together, as shown in Figure 3.68. For instance,

$$f(16.5) = \sqrt{16.5} \approx 4.0620$$

and

$$g(16.5) = \frac{1}{8}(16.5) + 2 = 4.0625.$$

In fact, if you use a graphing utility to zoom in near the point of tangency (16, 4), you will see that the two graphs appear to coincide. Notice also that as you move farther away from the point of tangency, the linear approximation becomes less accurate.

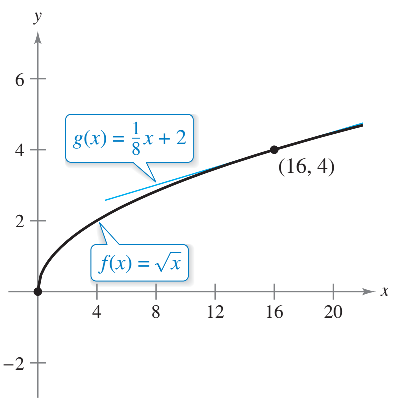


Figure 3.68

3.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using a Tangent Line Approximation **In Exercises 1–6, find the tangent line approximation T to the graph of f at the given point. Use this linear approximation to complete the table.**

x	1.9	1.99	2	2.01	2.1
$f(x)$					
$T(x)$					

1. $f(x) = x^2, \quad (2, 4)$

2. $f(x) = \dfrac{6}{x^2}, \quad \left(2, \dfrac{3}{2}\right)$

3. $f(x) = x^5, \quad (2, 32)$

4. $f(x) = \sqrt{x}, \quad (2, \sqrt{2})$

5. $f(x) = \sin x, \quad (2, \sin 2)$

6. $f(x) = \csc x, \quad (2, \csc 2)$

Comparing Δy and dy **In Exercises 7–10, use the information to evaluate and compare Δy and dy.**

	Function	x-Value	Differential of x
7.	$y = x^3$	$x = 1$	$\Delta x = dx = 0.1$
8.	$y = 6 - 2x^2$	$x = -2$	$\Delta x = dx = 0.1$
9.	$y = x^4 + 1$	$x = -1$	$\Delta x = dx = 0.01$
10.	$y = 2 - x^4$	$x = 2$	$\Delta x = dx = 0.01$

Finding a Differential **In Exercises 11–20, find the differential dy of the given function.**

11. $y = 3x^2 - 4$

12. $y = 3x^{2/3}$

13. $y = x \tan x$

14. $y = \csc 2x$

15. $y = \dfrac{x + 1}{2x - 1}$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

17. $y = \sqrt{9 - x^2}$

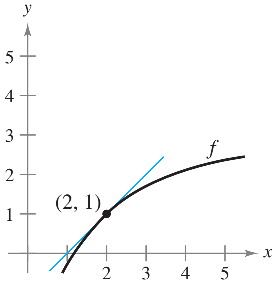
18. $y = x\sqrt{1 - x^2}$

19. $y = 3x - \sin^2 x$

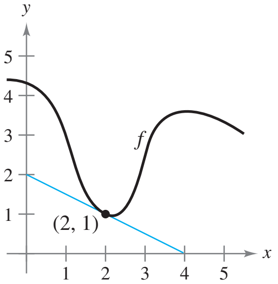
20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

Using Differentials **In Exercises 21 and 22, use differentials and the graph of f to approximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

21.

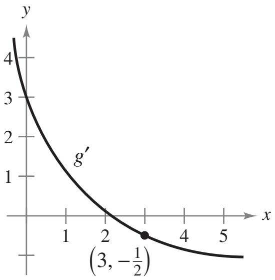


22.

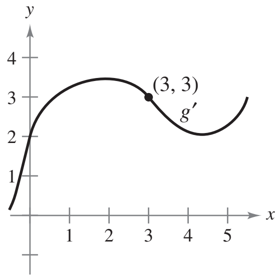


Using Differentials **In Exercises 23 and 24, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.**

23.



24.



25. Area The measurement of the side of a square floor tile is 10 inches, with a possible error of $\frac{1}{32}$ inch.

(a) Use differentials to approximate the possible propagated error in computing the area of the square.

(b) Approximate the percent error in computing the area of the square.

26. Area The measurement of the radius of a circle is 16 inches, with a possible error of $\frac{1}{4}$ inch.

(a) Use differentials to approximate the possible propagated error in computing the area of the circle.

(b) Approximate the percent error in computing the area of the circle.

27. Area The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters, respectively. The possible error in each measurement is 0.25 centimeter.

(a) Use differentials to approximate the possible propagated error in computing the area of the triangle.

(b) Approximate the percent error in computing the area of the triangle.

28. Circumference The measurement of the circumference of a circle is found to be 64 centimeters, with a possible error of 0.9 centimeter.

(a) Approximate the percent error in computing the area of the circle.

(b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.

29. Volume and Surface Area The measurement of the edge of a cube is found to be 15 inches, with a possible error of 0.03 inch.

(a) Use differentials to approximate the possible propagated error in computing the volume of the cube.

(b) Use differentials to approximate the possible propagated error in computing the surface area of the cube.

(c) Approximate the percent errors in parts (a) and (b).

30. Volume and Surface Area The radius of a spherical balloon is measured as 8 inches, with a possible error of 0.02 inch.

(a) Use differentials to approximate the possible propagated error in computing the volume of the sphere.

(b) Use differentials to approximate the possible propagated error in computing the surface area of the sphere.

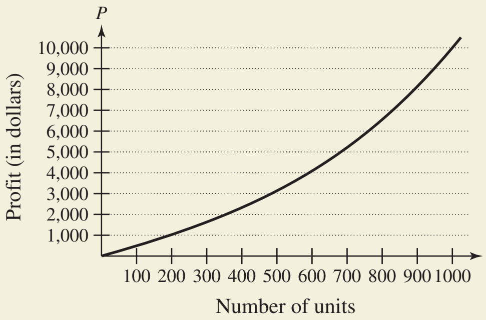
(c) Approximate the percent errors in parts (a) and (b).

31. Stopping Distance The total stopping distance T of a vehicle is

$$T = 2.5x + 0.5x^2$$

where T is in feet and x is the speed in miles per hour. Approximate the change and percent change in total stopping distance as speed changes from $x = 25$ to $x = 26$ miles per hour.

32. HOW DO YOU SEE IT? The graph shows the profit P (in dollars) from selling x units of an item. Use the graph to determine which is greater, the change in profit when the production level changes from 400 to 401 units or the change in profit when the production level changes from 900 to 901 units. Explain your reasoning



33. Pendulum The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. The pendulum has been subjected to an increase in temperature such that the length has increased by $\frac{1}{2}\%$.

(a) Find the approximate percent change in the period.

(b) Using the result in part (a), find the approximate error in this pendulum clock in 1 day.

34. Ohm's Law A current of I amperes passes through a resistor of R ohms. **Ohm's Law** states that the voltage E applied to the resistor is

$$E = IR.$$

The voltage is constant. Show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I.

35. Projectile Motion The range R of a projectile is

$$R = \frac{v_0^2}{32}(\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is the angle of elevation. Use differentials to approximate the change in the range when $v_0 = 2500$ feet per second and θ is changed from 10° to 11°.

36. Surveying A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

Approximating Function Values In Exercises 37–40, use differentials to approximate the value of the expression. Compare your answer with that of a calculator.

37. $\sqrt{99.4}$ **38.** $\sqrt[3]{26}$

39. $\sqrt[4]{624}$ **40.** $(2.99)^3$

Verifying a Tangent Line Approximation In Exercises 41 and 42, verify the tangent line approximation of the function at the given point. Then use a graphing utility to graph the function and its approximation in the same viewing window.

	Function	Approximation	Point
41.	$f(x) = \sqrt{x+4}$	$y = 2 + \frac{x}{4}$	$(0, 2)$
42.	$f(x) = \tan x$	$y = x$	$(0, 0)$

WRITING ABOUT CONCEPTS

43. Comparing Δy and dy Describe the change in accuracy of dy as an approximation for Δy when Δx is decreased.

44. Describing Terms When using differentials, what is meant by the terms *propagated error*, *relative error*, and *percent error*?

Using Differentials In Exercises 45 and 46, give a short explanation of why the approximation is valid.

45. $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$ **46.** $\tan 0.05 \approx 0 + 1(0.05)$

True or False? In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

47. If $y = x + c$, then $dy = dx$.

48. If $y = ax + b$, then $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$.

49. If y is differentiable, then $\lim\limits_{\Delta x \to 0} (\Delta y - dy) = 0$.

50. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Extrema on a Closed Interval **In Exercises 1–8, find the absolute extrema of the function on the closed interval.**

1. $f(x) = x^2 + 5x, \quad [-4, 0]$
2. $f(x) = x^3 + 6x^2, \quad [-6, 1]$
3. $f(x) = \sqrt{x} - 2, \quad [0, 4]$
4. $h(x) = 3\sqrt{x} - x, \quad [0, 9]$
5. $f(x) = \dfrac{4x}{x^2 + 9}, \quad [-4, 4]$
6. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, \quad [0, 2]$
7. $g(x) = 2x + 5\cos x, \quad [0, 2\pi]$
8. $f(x) = \sin 2x, \quad [0, 2\pi]$

Using Rolle's Theorem **In Exercises 9–12, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.**

9. $f(x) = 2x^2 - 7, \quad [0, 4]$
10. $f(x) = (x - 2)(x + 3)^2, \quad [-3, 2]$
11. $f(x) = \dfrac{x^2}{1 - x^2}, \quad [-2, 2]$
12. $f(x) = \sin 2x, \quad [-\pi, \pi]$

Using the Mean Value Theorem **In Exercises 13–18, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

If the Mean Value Theorem cannot be applied, explain why not.

13. $f(x) = x^{2/3}, \quad [1, 8]$
14. $f(x) = \dfrac{1}{x}, \quad [1, 4]$
15. $f(x) = |5 - x|, \quad [2, 6]$
16. $f(x) = 2x - 3\sqrt{x}, \quad [-1, 1]$
17. $f(x) = x - \cos x, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
18. $f(x) = \sqrt{x} - 2x, \quad [0, 4]$

19. **Mean Value Theorem** Can the Mean Value Theorem be applied to the function

$$f(x) = \frac{1}{x^2}$$

on the interval $[-2, 1]$? Explain.

20. **Using the Mean Value Theorem**
   (a) For the function $f(x) = Ax^2 + Bx + C$, determine the value of c guaranteed by the Mean Value Theorem on the interval $[x_1, x_2]$.
   (b) Demonstrate the result of part (a) for $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 4]$.

Intervals on Which f Is Increasing or Decreasing **In Exercises 21–26, identify the open intervals on which the function is increasing or decreasing.**

21. $f(x) = x^2 + 3x - 12$
22. $h(x) = (x + 2)^{1/3} + 8$
23. $f(x) = (x - 1)^2(x - 3)$
24. $g(x) = (x + 1)^3$
25. $h(x) = \sqrt{x}(x - 3), \quad x > 0$
26. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

Applying the First Derivative Test **In Exercises 27–34, (a) find the critical numbers of f (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.**

27. $f(x) = x^2 - 6x + 5$
28. $f(x) = 4x^3 - 5x$
29. $h(t) = \dfrac{1}{4}t^4 - 8t$
30. $g(x) = \dfrac{x^3 - 8x}{4}$
31. $f(x) = \dfrac{x + 4}{x^2}$
32. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$
33. $f(x) = \cos x - \sin x, \quad (0, 2\pi)$
34. $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \quad [0, 4]$

Finding Points of Inflection **In Exercises 35–40, find the points of inflection and discuss the concavity of the graph of the function.**

35. $f(x) = x^3 - 9x^2$
36. $f(x) = 6x^4 - x^2$
37. $g(x) = x\sqrt{x + 5}$
38. $f(x) = 3x - 5x^3$
39. $f(x) = x + \cos x, \quad [0, 2\pi]$
40. $f(x) = \tan\dfrac{x}{4}, \quad (0, 2\pi)$

Using the Second Derivative Test **In Exercises 41–46, find all relative extrema. Use the Second Derivative Test where applicable.**

41. $f(x) = (x + 9)^2$
42. $f(x) = 2x^3 + 11x^2 - 8x - 12$
43. $g(x) = 2x^2(1 - x^2)$
44. $h(t) = t - 4\sqrt{t + 1}$

45. $f(x) = 2x + \dfrac{18}{x}$

46. $h(x) = x - 2\cos x, \quad [0, 4\pi]$

Think About It **In Exercises 47 and 48, sketch the graph of a function f having the given characteristics.**

47. $f(0) = f(6) = 0$
$f'(3) = f'(5) = 0$
$f'(x) > 0$ for $x < 3$
$f'(x) > 0$ for $3 < x < 5$
$f'(x) < 0$ for $x > 5$
$f''(x) < 0$ for $x < 3$ or $x > 4$
$f''(x) > 0$ for $3 < x < 4$

48. $f(0) = 4, \; f(6) = 0$
$f'(x) < 0$ for $x < 2$ or $x > 4$
$f'(2)$ does not exist.
$f'(4) = 0$
$f'(x) > 0$ for $2 < x < 4$
$f''(x) < 0$ for $x \neq 2$

49. Writing A newspaper headline states that "The rate of growth of the national deficit is decreasing." What does this mean? What does it imply about the graph of the deficit as a function of time?

50. Inventory Cost The cost of inventory C depends on the ordering and storage costs according to the inventory model

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r.$$

Determine the order size that will minimize the cost, assuming that sales occur at a constant rate, Q is the number of units sold per year, r is the cost of storing one unit for one year, s is the cost of placing an order, and x is the number of units per order.

51. Modeling Data Outlays for national defense D (in billions of dollars) for selected years from 1970 through 2010 are shown in the table, where t is time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. Office of Management and Budget)*

t	0	5	10	15	20
D	81.7	86.5	134.0	252.7	299.3

t	25	30	35	40
D	272.1	294.4	495.3	693.6

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$D = at^4 + bt^3 + ct^2 + dt + e$$

for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) For the years shown in the table, when does the model indicate that the outlay for national defense was at a maximum? When was it at a minimum?

(d) For the years shown in the table, when does the model indicate that the outlay for national defense was increasing at the greatest rate?

52. Modeling Data The manager of a store recorded the annual sales S (in thousands of dollars) of a product over a period of 7 years, as shown in the table, where t is the time in years, with $t = 6$ corresponding to 2006.

t	6	7	8	9	10	11	12
S	5.4	6.9	11.5	15.5	19.0	22.0	23.6

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$S = at^3 + bt^2 + ct + d$$

for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use calculus and the model to find the time t when sales were increasing at the greatest rate.

(d) Do you think the model would be accurate for predicting future sales? Explain.

Finding a Limit **In Exercises 53–62, find the limit.**

53. $\displaystyle\lim_{x\to\infty}\left(8 + \frac{1}{x}\right)$

54. $\displaystyle\lim_{x\to-\infty}\frac{1 - 4x}{x + 1}$

55. $\displaystyle\lim_{x\to\infty}\frac{2x^2}{3x^2 + 5}$

56. $\displaystyle\lim_{x\to\infty}\frac{4x^3}{x^4 + 3}$

57. $\displaystyle\lim_{x\to-\infty}\frac{3x^2}{x + 5}$

58. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2 + x}}{-2x}$

59. $\displaystyle\lim_{x\to\infty}\frac{5\cos x}{x}$

60. $\displaystyle\lim_{x\to\infty}\frac{x^3}{\sqrt{x^2 + 2}}$

61. $\displaystyle\lim_{x\to-\infty}\frac{6x}{x + \cos x}$

62. $\displaystyle\lim_{x\to-\infty}\frac{x}{2\sin x}$

Horizontal Asymptotes **In Exercises 63–66, use a graphing utility to graph the function and identify any horizontal asymptotes.**

63. $f(x) = \dfrac{3}{x} - 2$

64. $g(x) = \dfrac{5x^2}{x^2 + 2}$

65. $h(x) = \dfrac{2x + 3}{x - 4}$

66. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

Analyzing the Graph of a Function **In Exercises 67–76, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.**

67. $f(x) = 4x - x^2$

68. $f(x) = 4x^3 - x^4$

69. $f(x) = x\sqrt{16 - x^2}$

70. $f(x) = (x^2 - 4)^2$

71. $f(x) = x^{1/3}(x + 3)^{2/3}$

72. $f(x) = (x - 3)(x + 2)^3$

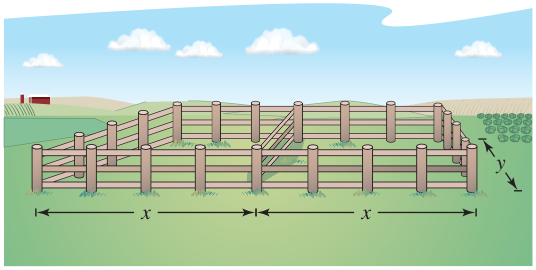
73. $f(x) = \dfrac{5 - 3x}{x - 2}$

74. $f(x) = \dfrac{2x}{1 + x^2}$

75. $f(x) = x^3 + x + \dfrac{4}{x}$

76. $f(x) = x^2 + \dfrac{1}{x}$

77. Maximum Area A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



78. Maximum Area Find the dimensions of the rectangle of maximum area, with sides parallel to the coordinate axes, that can be inscribed in the ellipse given by

$$\frac{x^2}{144} + \frac{y^2}{16} = 1.$$

79. Minimum Length A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimum.

80. Minimum Length The wall of a building is to be braced by a beam that must pass over a parallel fence 5 feet high and 4 feet from the building. Find the length of the shortest beam that can be used.

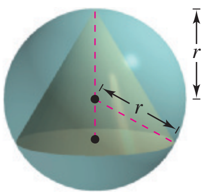
81. Maximum Length Find the length of the longest pipe that can be carried level around a right-angle corner at the intersection of two corridors of widths 4 feet and 6 feet.

82. Maximum Length A hallway of width 6 feet meets a hallway of width 9 feet at right angles. Find the length of the longest pipe that can be carried level around this corner. [*Hint:* If L is the length of the pipe, show that

$$L = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

where θ is the angle between the pipe and the wall of the narrower hallway.]

83. Maximum Volume Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.



84. Maximum Volume Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

Using Newton's Method **In Exercises 85–88, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.**

85. $f(x) = x^3 - 3x - 1$

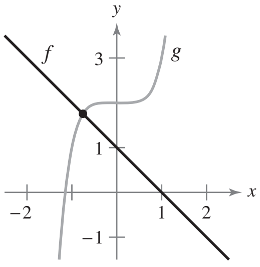
86. $f(x) = x^3 + 2x + 1$

87. $f(x) = x^4 + x^3 - 3x^2 + 2$

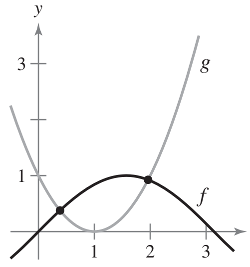
88. $f(x) = 3\sqrt{x - 1} - x$

Finding Point(s) of Intersection **In Exercises 89 and 90, apply Newton's Method to approximate the x-value(s) of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]**

89. $f(x) = 1 - x$
$g(x) = x^5 + 2$



90. $f(x) = \sin x$
$g(x) = x^2 - 2x + 1$



Comparing Δy and dy **In Exercises 91 and 92, use the information to evaluate and compare Δy and dy.**

	Function	x-Value	Differential of x
91.	$y = 0.5x^2$	$x = 3$	$\Delta x = dx = 0.01$
92.	$y = x^3 - 6x$	$x = 2$	$\Delta x = dx = 0.1$

Finding a Differential **In Exercises 93 and 94, find the differential dy of the given function.**

93. $y = x(1 - \cos x)$

94. $y = \sqrt{36 - x^2}$

95. Volume and Surface Area The radius of a sphere is measured as 9 centimeters, with a possible error of 0.025 centimeter.

(a) Use differentials to approximate the possible propagated error in computing the volume of the sphere.

(b) Use differentials to approximate the possible propagated error in computing the surface area of the sphere.

(c) Approximate the percent errors in parts (a) and (b).

96. Demand Function A company finds that the demand for its commodity is

$$p = 75 - \frac{1}{4}x$$

where p is the price in dollars and x is the number of units. Find and compare the values of Δp and dp as x changes from 7 to 8.

P.S. Problem Solving

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

1. Relative Extrema Graph the fourth-degree polynomial

$$p(x) = x^4 + ax^2 + 1$$

for various values of the constant a.

(a) Determine the values of a for which p has exactly one relative minimum.

(b) Determine the values of a for which p has exactly one relative maximum.

(c) Determine the values of a for which p has exactly two relative minima.

(d) Show that the graph of p cannot have exactly two relative extrema.

2. Relative Extrema

(a) Graph the fourth-degree polynomial $p(x) = ax^4 - 6x^2$ for $a = -3, -2, -1, 0, 1, 2,$ and 3. For what values of the constant a does p have a relative minimum or relative maximum?

(b) Show that p has a relative maximum for all values of the constant a.

(c) Determine analytically the values of a for which p has a relative minimum.

(d) Let $(x, y) = (x, p(x))$ be a relative extremum of p. Show that (x, y) lies on the graph of $y = -3x^2$. Verify this result graphically by graphing $y = -3x^2$ together with the seven curves from part (a).

3. Relative Minimum Let

$$f(x) = \frac{c}{x} + x^2.$$

Determine all values of the constant c such that f has a relative minimum, but no relative maximum.

4. Points of Inflection

(a) Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic polynomial. How many points of inflection does the graph of f have?

(b) Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, be a cubic polynomial. How many points of inflection does the graph of f have?

(c) Suppose the function $y = f(x)$ satisfies the equation

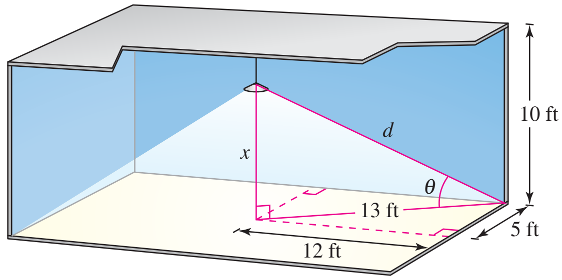
$$\frac{dy}{dx} = ky\left(1 - \frac{y}{L}\right)$$

where k and L are positive constants. Show that the graph of f has a point of inflection at the point where $y = L/2$. (This equation is called the **logistic differential equation.**)

5. Extended Mean Value Theorem Prove the following **Extended Mean Value Theorem.** If f and f' are continuous on the closed interval $[a, b]$, and if f'' exists in the open interval (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2.$$

6. Illumination The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin\theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 feet by 24 feet, with a 10-foot ceiling (see figure). Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.



7. Minimum Distance Consider a room in the shape of a cube, 4 meters on each side. A bug at point P wants to walk to point Q at the opposite corner, as shown in the figure. Use calculus to determine the shortest path. Explain how you can solve this problem without calculus. (*Hint:* Consider the two walls as one wall.)

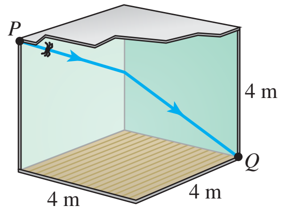


Figure for 7

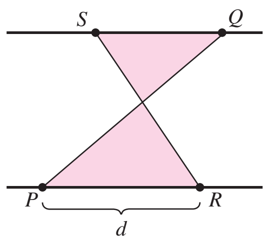


Figure for 8

8. Areas of Triangles The line joining P and Q crosses the two parallel lines, as shown in the figure. The point R is d units from P. How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?

9. Mean Value Theorem Determine the values a, b, and c such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 3]$.

$$f(x) = \begin{cases} 1, & x = 0 \\ ax + b, & 0 < x \leq 1 \\ x^2 + 4x + c, & 1 < x \leq 3 \end{cases}$$

10. Mean Value Theorem Determine the values a, b, c, and d such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 2]$.

$$f(x) = \begin{cases} a, & x = -1 \\ 2, & -1 < x \leq 0 \\ bx^2 + c, & 0 < x \leq 1 \\ dx + 4, & 1 < x \leq 2 \end{cases}$$

11. Proof Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). Prove that if $f(a) = g(a)$ and $g'(x) > f'(x)$ for all x in (a, b), then $g(b) > f(b)$.

12. Proof

(a) Prove that $\lim_{x\to\infty} x^2 = \infty$.

(b) Prove that $\lim_{x\to\infty}\left(\frac{1}{x^2}\right) = 0$.

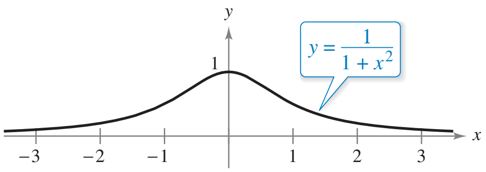
(c) Let L be a real number. Prove that if $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$

13. Tangent Lines Find the point on the graph of

$$y = \frac{1}{1 + x^2}$$

(see figure) where the tangent line has the greatest slope, and the point where the tangent line has the least slope.



14. Stopping Distance The police department must determine the speed limit on a bridge such that the flow rate of cars is maximum per unit time. The greater the speed limit, the farther apart the cars must be in order to keep a safe stopping distance. Experimental data on the stopping distances d (in meters) for various speeds v (in kilometers per hour) are shown in the table.

v	20	40	60	80	100
d	5.1	13.7	27.2	44.2	66.4

(a) Convert the speeds v in the table to speeds s in meters per second. Use the regression capabilities of a graphing utility to find a model of the form $d(s) = as^2 + bs + c$ for the data.

(b) Consider two consecutive vehicles of average length 5.5 meters, traveling at a safe speed on the bridge. Let T be the difference between the times (in seconds) when the front bumpers of the vehicles pass a given point on the bridge. Verify that this difference in times is given by

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

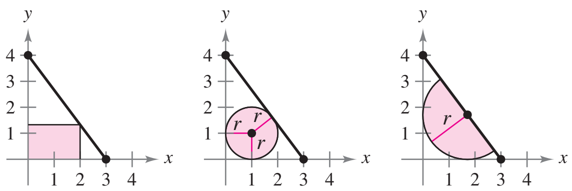
(c) Use a graphing utility to graph the function T and estimate the speed s that minimizes the time between vehicles.

(d) Use calculus to determine the speed that minimizes T. What is the minimum value of T? Convert the required speed to kilometers per hour.

(e) Find the optimal distance between vehicles for the posted speed limit determined in part (d).

15. Darboux's Theorem Prove Darboux's Theorem: Let f be differentiable on the closed interval $[a, b]$ such that $f'(a) = y_1$ and $f'(b) = y_2$. If d lies between y_1 and y_2, then there exists c in (a, b) such that $f'(c) = d$.

16. Maximum Area The figures show a rectangle, a circle, and a semicircle inscribed in a triangle bounded by the coordinate axes and the first-quadrant portion of the line with intercepts $(3, 0)$ and $(0, 4)$. Find the dimensions of each inscribed figure such that its area is maximum. State whether calculus was helpful in finding the required dimensions. Explain your reasoning.

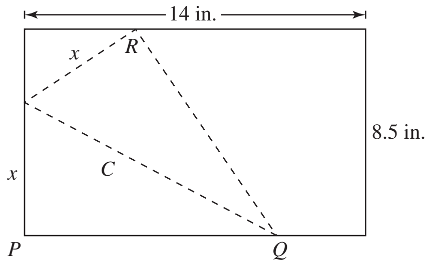


17. Point of Inflection Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

18. Minimum Length A legal-sized sheet of paper (8.5 inches by 14 inches) is folded so that corner P touches the opposite 14-inch edge at R (see figure). $\left(\textit{Note: } PQ = \sqrt{C^2 - x^2}.\right)$



(a) Show that $C^2 = \dfrac{2x^3}{2x - 8.5}$.

(b) What is the domain of C?

(c) Determine the x-value that minimizes C.

(d) Determine the minimum length C.

19. Quadratic Approximation The polynomial

$$P(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

is the quadratic approximation of the function f at $(a, f(a))$ when $P(a) = f(a)$, $P'(a) = f'(a)$, and $P''(a) = f''(a)$.

(a) Find the quadratic approximation of

$$f(x) = \frac{x}{x + 1}$$

at $(0, 0)$.

(b) Use a graphing utility to graph $P(x)$ and $f(x)$ in the same viewing window.

AP* Review Questions for Chapter 3

1. (no calculator)

Given: $g(x) = (2x + 4)^3(x - 6)$

(a) Find the critical numbers of g.

(b) For what values of x is g increasing? Justify your answer.

(c) Identify the x-coordinate of the critical points at which g has a relative minimum. Justify your answer.

2. (no calculator)

Let $f(x) = 2x + \cos(2x)$.

(a) Find the maximum value of f for $0 \le x \le \pi$. Justify your answer.

(b) Explain how the conditions of the Mean Value Theorem are satisfied by f for $0 \le x \le \pi$. Find the value of x, $0 \le x \le \pi$, whose existence is guaranteed by the Mean Value Theorem.

3. (no calculator)

Let $f(x) = \dfrac{1 - 4x^2}{x}$.

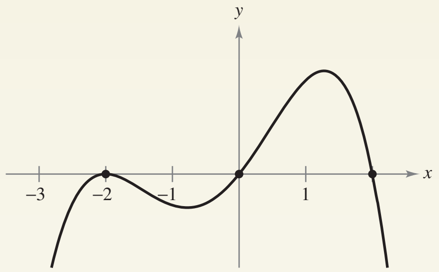
(a) State $f'(x)$ and identify the value(s) of x for which f' does not exist.

(b) For what values of x is f decreasing? Justify your answer.

(c) For what values of x is the graph of f concave downward? Show the work that leads to your answer.

(d) Does the graph of f contain an inflection point? Justify your answer.

4.



In the figure above, f', the *derivative* of function f, is shown. f is a twice differentiable function on $x \in (-\infty, \infty)$. $f''(-0.8) = 0$ and $f''(1.3) = 0$.

(a) Name the value(s) of x for which f has a relative minimum. Justify your answer.

(b) For what values of x is f increasing? Justify your answer.

(c) For what values of x is the graph of f concave downward? Justify your answer.

(d) Is $\dfrac{f(-0.5) - f(0)}{-0.5 - 0}$ positive or negative? Justify your answer.

5. The depth of the water at the end of a pier is shown in the table below and is modeled by differentiable function D for $t \ge 0$. Selected values of D are shown in the table below. D is expressed in meters, and t is the number of hours since midnight ($t = 0$).

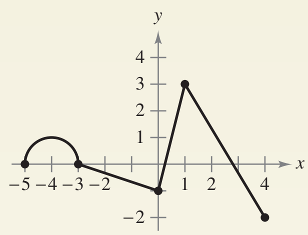
t (hours)	0	2	5	7	8	9	12
$D(t)$ (meters)	3.0	6.7	4.9	2.3	3.1	4.9	6.7

(a) Use the data in the table to estimate the rate at which the depth of the water is changing at 3:30 A.M. and 7:40 A.M. Include units.

(b) What is the least number of times in the interval $0 < t < 12$ for which $D'(t) = 0$? Justify your answer.

(c) Use the method of linear approximation to estimate the depth of the water at 2:30 A.M. ($t = 2.5$). Show the work that leads to your answer

6. (no calculator)



The graph of f', the derivative of f, is shown above. The function f is differentiable on the interval $-5 \le x \le 4$. $f''(-4) = 0$.

(a) Find $f'(-1)$.

(b) Find $f''(-1)$.

(c) Find the x-coordinate of each inflection point for the graph of f on the interval $-5 < x < 4$.

(d) If $g(x) = f(x) + \sin^2 x$, is g increasing or decreasing at $x = -\dfrac{\pi}{4}$? Justify your answer.

7. Given: f is continuous for $x \in (-\infty, \infty)$;

$f(2) = 4$; $\lim_{x \to \infty} f(x) = 0$

	$x < 4$	$x = 4$	$x > 4$
$f'(x)$	positive	does not exist	negative
$f''(x)$	negative	does not exist	positive

(A) For what values of x is f increasing?

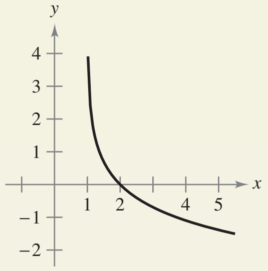
(B) Does f have a relative maximum at $x = 4$? Explain.

(C) If possible, name the x-coordinate of an inflection point on the graph of f. Justify your answer.

(D) Does the Mean Value Theorem apply over the interval $[3, 5]$? Justify your answer.

(E) Sketch a possible graph of f using the information from the table.

8.



Consider the graph of $y = f(x)$ shown above. If f is a function such that f' and f'' are defined in a region around $x = 2$, then which of the following must be true?

(A) $f''(2) < f(2)$

(B) $f''(2) < f'(2)$

(C) $f(2) = f'(2)$

(D) $f''(2) > f(2)$

(E) $f(2) = f''(2)$

9. (no calculator)

The position of an object along a vertical line is given by $s(t) = -t^3 + 3t^2 + 9t + 5$, where s is measured in feet and t in seconds. The maximum *velocity* of the object in the time interval $0 \leq t \leq 4$ is

(A) $32\dfrac{\text{ft}}{\text{sec}}$

(B) $16\dfrac{\text{ft}}{\text{sec}}$

(C) $12\dfrac{\text{ft}}{\text{sec}}$

(D) $9\dfrac{\text{ft}}{\text{sec}}$

(E) $-15\dfrac{\text{ft}}{\text{sec}}$

10. (no calculator)

Which of the following is true for the graph of $f(x) = \dfrac{4 - x}{x - 2}$?

I. $x = 2$ is a vertical asymptote of the graph of f.

II. f is decreasing for $x \in (-\infty, \infty)$.

III. f is concave down for $x \in (-\infty, 2)$.

(A) None

(B) I and II only

(C) I and III only

(D) III only

(E) I, II and III