Module 9: Counting Methods, Permutations, and Combinations

Reading from the Textbook: Chapter 6 Counting Methods

Introduction

The counting methods studied in this module are useful in many areas of computer science, including hashing techniques and probability theory. This material is commonly known as combinatorics. We start by two basic principles called multiplication principle and addition principle, and then study permutations and combinations. We use these concepts to derive binomial coefficients and Pascal's triangle. In the next module, we will see an important application of combinatorics, namely discrete probability.

Contents

Basic Principles

Multiplication Principle

Consider the example of a diner menu. Suppose the menu for “dinner special” consists of

1. A choice of 3 possible appetizers
2. A choice of 4 main entrées
3. A choice of 5 available desserts

The total number of possible orders becomes

$$3 * 4 * 5 = 60$$

This may also be expressed as the Cartesian product of sets. Let A be the set of appetizers, E the set of entrees, and D the set of desserts. Then, the number of possible orders is

$$|A \times E \times D| = |A| * |E| * |D|.$$

This is an example of the multiplication principle.

Multiplication Principle: Suppose a process consists of k independent steps:

Step 1: Choose one out of n_1 possible ways;
Step 2: Choose one out of n_2 possible ways;
⋮
Step k: Choose one out of n_k possible ways.

Then, the total number of possible choices is

$$n_1 * n_2 * \cdots * n_k$$

Note that the number of choices available in each step is independent of the choices made in previous steps.

Example: Let us compute the total number of n-bit integers. Each bit has two possible values {0, 1}. So the total number is

$$\underbrace{2 * 2 * 2 * \cdots * 2}_{n \text{ times}} = 2^n$$

(An alternative proof is by induction.)

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Example: What is the number of 8-bit integers which do not start with 11?

Since the number does not start with 11, there are 3 possible values for the leftmost two bits: 00, 01, 10. Therefore, the total number is

$$3 * 2^6 = 3 * 64 = 192$$

Example: What is the number of subsets of an n-element set $X = \{1,2,3, \cdots, n\}$?

A subset is formed by a process with n steps. In step i, $1 \le i \le n$, decide if element i is to be included in the subset or not. So, the total number of subsets is

$$\underbrace{2 * 2 * 2 * \cdots * 2}_{n \text{ times}} = 2^n$$

This problem may also be related to the number of n-bit integers. Each subset may be represented by an n-bit integer, where bit i equals 1 if and only if element i is in the subset. So the total number of subsets equals the number of n-bit integers.

Example: A college consists of the following three departments.

- Computer Science (10 faculty)
- Computer Technology (6 faculty)
- Computer Arts (4 faculty)

The college wants to select one faculty from each department to form a committee. How many ways are there to form such a committee? By the multiplication principle, the number is: $\boxed{10 * 6 * 4 = 240}$

Example: Suppose we roll a pair of 6-sided dice, one black and one white.
(a) What is the total number of outcomes?
(b) How many of the outcomes have one or both dice equal 1?

Solution:
(a) By the multiplication principle, the number of outcomes is: $\boxed{6 * 6 = 36}$
(b) One way to do this is to first find the number of outcomes with neither dice = 1. Each dice has 5 possible values. So the total is $5 * 5 = 25$. Therefore, the number of outcomes with one or both dice equal 1 is: $\boxed{36 - 25 - 11}$

Addition Principle

Consider rolling a pair of 6-sided dice. Each dice has 6 possible values, so by the multiplication principle, there are a total of 36 outcomes. Let us find the number of ways the sum of the two dice will be either 6 or 7. We consider each case separately.

- Sum $= 6$: There are 5 possible outcomes with this sum, as listed below.

$D_1 D_2$
1 5
2 4
3 3
4 2
5 1

- Sum $= 7$: There are 6 possible outcomes with this sum, as listed below.

$D_1 D_2$
1 6
2 5
3 4
4 3
5 2
6 1

Since the two sets of outcomes are disjoint, then the total number of possible outcomes is sum of the two:

$$5 + 6 = 11$$

This is an example of the addition principle.

Addition Principle: Given a number of sets $S_1,\ S_2, \cdots, S_k$ with the number of elements in them $n_1,\ n_2, \cdots, n_k$ respectively. Suppose the sets are pairwise disjoint. (That is, no pair of sets have any common elements.) The number of elements in the union of the sets is the sum of elements in each set. That is,

$$|S_1 \cup S_2 \cup \cdots \cup S_k| = |S_1| + |S_2| + \cdots + |S_k| = n_1 + n_2 + \cdots + n_k$$

The next two examples make use of both the addition principle and the multiplication principle.

Example: College of Computing has three academic departments:

- CS with 10 faculty members
- IS with 7 faculty members
- IT with 4 faculty members

(Each faculty is a member of only one department.) Each year, the college selects two faculty members to receive Teaching Awards. What is the number of ways for the college to make its selection if the college decides that one faculty must be from CS, and the other from either IS or IT?

Solution: We may view the selection as a 2-step process:

(a) Select one faculty from CS. There are 10 possible choices.
(b) Select one faculty from either IS or IT. By the addition principle, the number of choices is $7+4=11$.

So, by the multiplication principle, the total number of possible choices is $10*11=110$.

Let CS, IS, and IT denote the sets of faculty from the respective departments. Then the number of choices may also be expressed as the cardinality (size) of the Cartesian product

$$|CS \times (IS \cup IT)| = |(CS \times IS) \cup (CS \times IT)|$$

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Example: In the above example, suppose the college decides that the awards must be made to two faculty from two different departments, with no other restriction. What is the number of choices?

Solution: Now there are three possible ways to choose the two departments:

(a) CS and IS: By the multiplication principle, the number of choices is $10*7=70$.
(b) CS and IT: The number of choices is $10*4=40$.
(c) IS and IT: The number of choices is $7*4=28$.

So, by the addition principle, the total number of choices is $70+40+28=138$.

In terms of set operations (Cartesian product) this number may be expressed as

$$|(CS \times IS) \cup (CS \times IT) \cup (IS \times IT)|$$

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Inclusion-Exclusion Principle

We now consider how the addition principle is revised if the sets are not disjoint. Let us start with an example.

Example: What is the number 4-bit binary integers that either start with 0 or end with 0, or both? The count is computed as follows.

(a) Let A be the set of integers that start with 0. Then, $|A| = 2^3 = 8$
(b) Let B be the set of integers that end with 0. Then, $|B| = 2^3 = 8$.
(c) Let $C = A \cap B$ be the set of integers that start and end with 0, $|C| = 2^2 = 4$.

So the total number of integers that start with 0 or end with 0, or both, is $8 + 8 - 4 = 12$. That is,

$$|A \cup B| = |A| + |B| - |A \cap B| = 8 + 8 - 4 = 12$$

The reasoning is simple. Set A includes those integers that start and end with 0. And set B also includes those integers that start and end with 0. So $|A| + |B|$ counts those integers in the intersection twice, thus we subtract it once to get the correct count.

(Another way to find the above count is to first find the count of those integers that start and end with 1, which is $2^2 = 4$. And we know the total number of 4-bit integers is $2^4 = 16$. Therefore, the desired number is $16 - 4 = 12$.)

Inclusion-Exclusion Principle: Given two sets $A \text{ and } B$ which may not be disjoint. The number of elements in their union is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The above principle may be generalized to more than two sets in an obvious way. Below is the formula for 3 sets.

$$\begin{aligned} &|A \cup B \cup C| = |A| + |B| + |C| \\ &-|A \cap B| - |A \cap C| - |B \cap C| \\ &+|A \cap B \cap C| \end{aligned}$$

Again, the reasoning is simple. Those elements that are in the intersection of only two sets, but not in the intersection of all three, are counted twice in line 1, and subtracted once in line 2, so they are counted correctly. But those that are in the intersection of all three are counted three time in line 1, discounted (subtracted) three time in line 2, thus they still need to be counted once in line 3.

Permutations

Definition: A permutation of n distinct elements is an ordering of the n elements.

Let us find the number of permutations of n distinct elements $\{1,2,3,\cdots,n\}$. There are n positions, 1 through n.

1	Choose one of n possible elements for position 1.
2	Choose one of the remaining $n-1$ elements for position 2.
3	Choose one of the remaining $n-2$ elements for position 3.
$\vdots$	$\vdots$
$n-1$	Choose one of the remaining 2 elements for position $n-1$.
n	Place the only remaining element in position n.

By the multiplication principle, the total number of permutations is

$$\boxed{n*(n-1)*(n-2)*\cdots*2*1=n!}$$

Example: Enumerate all permutation of $\{A,B,C\}$.
There are $3!=3*2*1=6$ possible orderings.

$$ABC$$
$$ACB$$
$$BAC$$
$$BCA$$
$$CAB$$
$$CBA$$

Example: How many permutations of $\{A,B,C,D\}$ have 'AB' as a substring?
Since $'AB'$ must stay together, the substring is treated as one object. So, we need to find all permutations of 3 objects$\{AB,C,D\}$. The number of permutations is $3!=3*2=6$. Below is an enumeration of them.

$$\boldsymbol{AB}CD$$
$$\boldsymbol{AB}DC$$
$$C\boldsymbol{AB}D$$
$$D\boldsymbol{AB}C$$
$$CD\boldsymbol{AB}$$
$$DC\boldsymbol{AB}$$

Example: A girl volleyball team has four players: Genie, Ruth, Sally, and Susan. The coach wants to assign them to four distinct positions: Setter, Spiker, Server, and Bumper. How many ways are there for the assignment?

The number of possible assignments is

$$4! = 4 * 3 * 2 * 1 = 24$$

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Example: Four people (Bill, Jill, Liam, and Pat) need to be seated around a round table. How many ways are there to order them? (You may consider the ordering in clockwise direction, starting with one particular person, say Bill.)

We may seat one particular person, say Bill, and then consider all orderings of the remaining 3 (in clockwise direction). Thus the number is:

$$3! = 3 * 2 * 1 = 6$$

Suppose Bill is seated as a reference. Then, in clockwise direction, to the left of Bill, one of the remaining 3 people may be seated, and then one of the remaining 2. Finally, the last remaining person will be seated in clockwise direction (to the right of Bill).

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Permutations of Subsets

Next, we consider permutations of an r-element subset of n distinct elements, rather than permutations of all elements. That is, we pick a subset of r elements and consider all orderings of each subset.

Definition: An r-permutation of n distinct elements is an ordering of an r-element subset of n distinct elements. In other words, an r-permutation of n distinct elements is an **ordered subset** of size r.

To find all r-permutations of n distinct elements:

(a) Find all subsets of size r.
(b) For each r-element subset, find all $(r!)$ permutations of it.

The number of r-permutations of n distinct elements is denoted as $P(n, r)$.

Example: List all 2-permutations of 4 distinct elements $\{A, B, C, D\}$.

AB BA
AC CA
AD DA
BC CB
BD DB
CD DC

The total number is $4 * 3 = 12$, because any of the 4 elements may be selected for position 1, and then any of the remaining 3 elements may be selected for position 2.

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In the above listing, all orderings of a given subset are grouped together in a sub-box. For example, the subset $\{A, B\}$ generates 2 orderings: AB and BA. In general, each subset of size r generates $(r!)$ orderings.

The number of r-permutations of n distinct elements is

$$P(n,r) = n\,(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof:

- For position 1, select any of the n distinct elements.
- For position 2, select any of the remaining $n-1$ elements.

  ⋮
- For position r, select any of the remaining $n-r+1$ elements.

By the multiplication principle, the total number is

$$n\,(n-1)(n-2)\cdots(n-r+1).$$

Example: A student-club of 12 persons wants to select a president, vice-president, and a secretary. What is the number of ways to do the selection? The number is

$$P(12,3) = 12 * 11 * 10 = 1320.$$

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Example: A volleyball team has 4 players: Rebecca, Rachael, Sally, and Sue. The coach wants to assign one player for Setter and another for Spiker. What is the number of ways to make the assignment? Give the number and then list them.

The number of possible assignments is $P(4,2) = 4 * 3 = 12$. Below is the listing.

Setter	**Spiker**
Rebecca	Rachael
Rebecca	Sally
Rebecca	Sue
Rachael	Rebecca
Rachael	Sally
Rachael	Sue
Sally	Rebecca
Sally	Rachael
Sally	Sue
Sue	Rebecca
Sue	Rachael
Sue	Sally

Combinations

So far, we considered an r-permutation of n distinct elements, which is an **ordered** selection of size r. Next we consider the selection of **unordered subsets**, called c**ombination**.

Example: Enumerate all 2-combinations (i.e., all unordered subsets of size 2) out of 4 distinct elements $\{A, B, C, D\}$.

$\{A, B\}$
$\{A, C\}$
$\{A, D\}$
$\{B, C\}$
$\{B, D\}$
$\{C, D\}$

Definition: An r-combination of a set of n distinct elements is an **unordered** selection of a subset of r elements out of the set. The selection may also be called "combination of r out of n" or "choose r out of n".

The number of r-combinations of n distinct elements is denoted as $C(n, r)$ or $\binom{n}{r}$.

There is a simple relation between $P(n,r)$ and $C(n,r)$. For $P(n,r)$, each subset of size r generates $r!$ possible orderings. For $C(n,r)$, each subset of size r must be counted only once. So we get the following formula.

$$\boxed{C(n,r)=\frac{P(n,r)}{r!}=\frac{n!}{r!\,(n-r)!}}$$

Note there is symmetry in this count, so $C(n,r)=C(n,n-r)$. For example,

$$C(7,2)=C(7,5)=\frac{7!}{2!\,5!}$$

This should make an intuitive sense:

- $C(7,2)$ chooses 2 out of 7 to be included in the subset.
- $C(7,5)$ chooses 5 out of 7 to be excluded, thus leaving 2 for the subset.

Example: How many 8-bit integers have exactly four 1's and four 0's? One example is 11010001.

We need to choose a subset of 4 out of 8 for 1's. The number is

$$C(8,4)=\frac{8!}{4!\,(8-4)!}=\frac{8*7*6*5}{4*3*2*1}=\frac{1680}{24}=70$$

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Example: A committee of 7 people wants to select a chair, a vice-chair, and 2 secretaries. What is the total number of ways?

(a) Choose 1 out of 7 for chair. The number of ways is $C(7,1)=7$.
(b) Choose 1 out of the remaining 6 for vice-chair. The number of ways is $C(6,1)=6$.

Alternatively, steps (a) and (b) may be combined into one step:
Choose an ***ordered subset*** of 2 out of 7 for chair and vice-chair. The number is

$$P(7,2)=7*6=42$$

The selection of 2 must be ordered because the two positions (chair and vice-chair) are different.

(c) Choose 2 out of the remaining 5 for secretaries. This is an ***unordered subset*** of 2 out of 5, and the number is

$$C(5,2) = \frac{5!}{2!\,3!} = \frac{5*4}{2} = 10$$

By the multiplication principle, the total number of ways is

$$P(7,2) * C(5,2) = 7 * 6 * 10 = 420.$$

Alternatively, we could start by picking two secretaries out of 7 (unordered), and then pick a chair and a vice-chair out of the remaining 5 (ordered). The total number becomes the same.

$$C(7,2) * P(5,2) = \frac{7*6}{2} * (5*4) = 21 * 20 = 420$$

The important thing to remember is that the selection of chair and vice-chair is ordered, because the two positions are different, but the selection of two secretaries is unordered because the two secretaries have equal status.

Example (Poker Hands): A deck of cards consists of 52 cards, with 4 *suits* (hearts, diamonds, clubs, spades) and 13 *kinds* (ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3, 2). A poker hand is a drawing of 5 cards out of 52.

(a) What is the total number of possible poker hands?
(b) What is the number of poker hands with 3 cards out of one kind, and 2 cards out of another kind (3,2)? This is called a *full-house*. (An example is 3 aces and 2 kings.)
(c) What is the number of poker hands with 2 cards out of one kind, 2 cards out of another kind, and one card out of a third kind (2,2,1)? This is called *two-pairs.* An example is (2 aces, 2 queens, and 1 jack).

Solution:
(a) The total number of poker hands is

$$C(52,5) = \frac{52!}{5!\,(47!)} = \frac{52*51*50*49*48}{5*4*3*2} = 2{,}598{,}960$$

(b) The number of distinct full-house hands (3,2) is computed as follows.

(i) Pick one kind out of 13, and choose 3 cards out of that kind.

$$C(13,1) * C(4,3) = 13 * 4 = 52$$

(ii) Pick a second kind out of the remaining 12 kinds, and choose 2 cards from it.

$$C(12,1) * C(4,2) = 12 * \frac{4!}{2!\,(4-2)!} = 12 * \frac{4*3}{2} = 12 * 6 = 72$$

By the multiplication principle, the total number is:

$$52 * 72 \;=\; 3744$$

An alternative way of computing steps (i) and (ii) above is to pick an *ordered subset* of 2 kinds, pick 3 out of the first kind, and pick 2 out of the second kind. The total count becomes the same:

$$P(13,2) * C(4,3) * C(4,2) = 13 * 12 * 4 * 6 = 3744$$

(c) The number of distinct two-pair hands (2,2,1) is computed as follows.

(i) Choose an *unordered subset* of 2 kinds, and pick a pair out of each kind.

$$C(13,2) * C(4,2) * C(4,2) = \frac{13 * 12}{2} * \frac{4 * 3}{2} * \frac{4 * 3}{2} = 78 * 6 * 6 = 2808$$

(ii) Choose a third kind and pick one card out of that kind.

$$C(11,1) * C(4,1) = 11 * 4 = 44$$

By the multiplication principle, the total number is: $2808 * 44 = 123{,}552$.

Binomial Coefficients

An interesting application of combinations $C(n, r)$ is to derive the coefficients for the binomial expansion $(a + b)^n$. For example, consider the formula for $n = 3$.

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

There are 3 factors $(a + b)$ to be multiplied. From each factor, either a or b is selected, then multiplied together to get a product term, and the product terms are added to get the final result. Since there are 2 choices from each factor, the total number of product terms is $2 * 2 * 2 = 8$. The following table shows all product terms, one per row.

	Select from $(a+b)$	Select from $(a+b)$	Select from $(a+b)$	Product
1	a	a	a	a^3
2	a	a	b	a^2b
3	a	b	a	a^2b
4	b	a	a	a^2b
5	a	b	b	ab^2
6	b	a	b	ab^2
7	b	b	a	ab^2
8	b	b	b	b^3

To get the product term a^3, all 3 factors must select a. (This means no factor selects b.) The number of ways for this selection is $C(3,0) = C(3,3) = 1$. (Note $0! = 1$.) One row gives the product a^3, and in the final result, the coefficient for a^3 is 1.

To get the product term a^2b, we must select a from 2 of the 3 factors. (This means select b from 1 of the 3 factors.) The number of ways for this selection is $C(3,1) = 3$. Three rows in the table (rows 2, 3, 4) give the product a^2b. Therefore, after adding the product terms, we get $3a^2b$.

Similarly, for the product term ab^2, we need to select a from 1 out of the 3 factors. (This means select b from 2 out of the 3 factors.) The number of ways is $C(3,2) = 3$. Three rows in the table (rows 5, 6, 7) give the product term ab^2. So after adding the product terms, we get $3ab^2$.

Finally, to get the product term b^3, all 3 factors must select b. The number of ways for this selection is $C(3,3) = 1$. So the coefficient for b^3 is 1. In summary,

$$
\begin{aligned}
(a+b)^3 &= C(3,0)*a^3 + C(3,1)*a^2b + C(3,2)*ab^2 + C(3,3)*b^3 \\
&= 1*a^3 + 3*a^2b + 3*ab^2 + 1*b^3
\end{aligned}
$$

Note the coefficients are symmetrical. For example, the coefficients for a^2b and ab^2 are both equal to 3 because $C(3,1) = C(3,2) = 3$.

As another example, let us compute the coefficients for $(a+b)^4$.

$$
\begin{aligned}
(a+b)^4 &= C(4,0)*a^4 + C(4,1)*a^3b + C(4,2)*a^2b^2 + C(4,3)*ab^3 + C(4,4)*b^4 \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}
$$

In general,

$$
\boxed{
\begin{aligned}
(a+b)^n &= C(n,0)a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b^2 + \cdots + C(n,n)b^n \\
&= \sum_{r=0}^{n} C(n,r)a^{n-r}b^r
\end{aligned}
}
$$

Recursive Formulation of Combination

The formula for $C(n,r) = \frac{n!}{r!(n-r)!}$ may be formulated recursively in an interesting way. $C(n,r)$ is the number of ways to select an unordered subset of size r out of a set of n distinct elements $\{1,2,3,\cdots,n\}$. We consider two cases depending on whether element 1 is selected to be included in the subset or not. Let $0 < r < n$.

(a) If element 1 is in the subset, then we need to recursively select $r-1$ elements out of the remaining $n-1$ elements. The number of ways for this selection is

$$C(n-1, r-1)$$

(b) If element 1 is not in the subset, then we need to recursively select all r elements out of the remaining $n-1$ elements. The number of ways for this selection is

$$C(n-1, r)$$

These two sets of selections are disjoint. So, we apply the addition principle to get the total number for $C(n, r)$.

$$C(n,r) = \begin{cases} C(n-1, r-1) + C(n-1, r), & 0 < r < n \\ 1, & r = 0 \text{ or } r = n \end{cases}$$

The actual computation of $C(n, r)$ must not be done recursively, because recursion would introduce terrible inefficiency. (There is a great deal of overlap in recursive computation of the two terms $C(n-1, r-1)$ and $C(n-1, r)$.) The recursive formulation is used to compute $C(n, r)$ in a bottom-up approach: First compute $C(1, r)$ for all r, then compute $C(2, r)$ for all r, then $C(3, r)$, and so on. Below is a simple nested-loop to perform the bottom-up computation and store the results in an array $C[1{:}n, 0{:}n]$.

```
C[1,0] = C[1,1] = 1;
for m = 2 to n {
  C[m,0] = C[m,m] = 1;
  for r = 1 to m − 1
      C[m,r] = C[m − 1, r − 1] + C[m − 1, r];
}
```

The following table illustrates the computation for $n = 6$. First, the row $C[2, r]$ is computed, then the row $C[3, r]$, then $C[4, r]$, then $C[5, r]$, and finally the row $C[6, r]$. For example, once row 5 has been computed, row 6 is computed by using the values from row 5. One such computation is highlighted: $C[6,3] = C[5,2] + C[5,3] = 10 + 10 = 20$.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
$n = 1$	1	1					
$n = 2$	1	2	1				
$n = 3$	1	3	3	1			
$n = 4$	1	4	6	4	1		
$n = 5$	1	5	10	10	5	1	
$n = 6$	1	6	15	20	15	6	1

Note that the values in row n, which are $C[n,0], C[n,1], \cdots, C[n,n]$, are the binomial coefficients for $(a+b)^n$.

Pascal's Triangle

The above table may be drawn in the form of triangle, known as *Pascal's triangle*. The boundary values in each row of the triangle are all 1, and each interior value is computed as sum of the two values directly above it. For example, the 20 at the base is computed as 10+10, as highlighted.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Generalized Permutations and Combinations

We now discuss several generalizations of permutations/combinations, including the case when the elements are **not distinct**, and the case when we make selections with repetitions allowed.

Partitioning a set into a number of disjoint subsets

The combination $C(n,r)$ is the number of ways to choose an unordered subset of r elements out of n distinct elements. This problem may be generalized into partitioning a set of n distinct elements into a number of disjoint subsets.

Example: How many distinct strings are generated by all permutations of the following string of 8 characters, which consists of 2 A's, 3 B's, 2 C's, and 1 D?

$$AABBBCCD$$

If the 8 characters were all distinct, the number of distinct permutations would be 8!. But in this case, the number will be much smaller because there are several groups of identical letters. Given any ordering of the string, if we reorder the letters within each group, it will not produce a new string.

One way to find the number of distinct permutation is in terms of $C(n, r)$, as follows. Let the positions in the string be numbered 1 to 8.

(a) Pick 2 out of 8 positions for letter A. The number of ways is: $C(8,2)$.
(b) Pick 3 out of the remaining 6 positions for letter B. The number of ways is $C(6,3)$.
(c) Pick 2 out of the remaining 3 positions for letter C. The number of ways is $C(3,2)$.
(d) Use the remaining single position for letter D. The number of ways is 1.

By the multiplication principle, the total number is

$$C(8,2) * C(6,3) * C(3,2) = \frac{8!}{2!\,6!} * \frac{6!}{3!\,3!} * \frac{3!}{2!\,1!} = \frac{8!}{2!\,3!\,2!\,1!}$$

After cancelling the highlighted terms, the reduced result has 8! in the numerator, corresponding to the length of the string, and a factorial in the denominator for each group size (2,3,2,1).

There is also a way to derive the above formula directly. The number of permutations for a string of n *distinct* characters is $n!$. But if the characters are not distinct, then for each group of r identical characters, there are $r!$ permutations of the letters within that group which must be counted as only one. Thus, $n!$ must be divided by $r!$ for each group of r identical characters.

General Formula: Suppose we want to partition a set of n distinct elements into a number of disjoint subsets of sizes $n_1, n_2, \cdots, n_k$, where $n = \sum_{i=1}^{k} n_i$. The number of ways to do the partitioning is

$$\boxed{\frac{n!}{n_1!\,n_2!\cdots n_k!}}$$

Example: How many distinct strings are produced by all permutations of the following string?

$$MISSISSIPPI$$

There are 4 groups of identical letters: $1\ M, 4\ I, 4\ S,\ 2\ P$. The number of distinct strings is

$$\frac{11!}{1!\,4!\,4!\,2!} = 34{,}650$$

Selection with Repetitions (Introducing Walls/Dividers)

Given a set of n distinct objects $\{1,2,\cdots,n\}$, let us find the number of ways to pick an unordered selection of r elements out of the n objects, when **repetitions are allowed**. (That is, each object may be selected several times.) There is no restriction on the value of r compared to n.

Theorem: The number of ways to make an r-selection of n distinct objects (with repetitions allowed) is

$$\boxed{C(r+n-1,r) = C(r+n-1,n-1)}$$

Basically, **we need to assign a number to each object** to specify how many times that object is selected. Before explaining how the counts are incorporated, let us consider a specific small example with

$$n = 3, r = 4.$$

This means we have three distinct objects $\{1,2,3\}$ and we want to make a selection 4 times. According to the above formula, the number of ways to make the selection is

$$C(n+r-1,n-1) = C(6,2) = \frac{6*5}{2} = 15.$$

And in this case, since the problem is small enough, we can enumerate all cases as follows. (This, of course, is not proof of the above formula in general.)

Object	1	2	3
Number of times selected	0	0	4
	0	1	3
	0	2	2
	0	3	1
	0	4	0
	1	0	3
	1	1	2
	1	2	1
	1	3	0
	2	0	2
	2	1	1
	2	2	0
	3	0	1
	3	1	0
	4	0	0

This method of enumeration is not practical in general. So we need a way to prove the above formula with an eloquent combinatorial argument.

Proof of the Formula (n distinct objects, and r selection)

$$\boxed{C(r+n-1,r)=C(r+n-1,n-1)}$$

We will use a line-up of

$r+n-1$ Positions

This line-up will consist of an arrangement of

r Tokens

$n-1$ Objects (Each object is also called a **divider**)

(They may appear in any order.)

Interpretation of the line-up: The number of tokens lined-up before each object will denote the number of times that object is selected.

Let us consider the above example again ($n=3, r=4$). The particular case highlighted in yellow in the table selects object 1 twice. object 2 once, and object 3 once. The line-up of tokens (shown as 0) and objects (shown as 1) for this case is shown below.

Position	1	2	3	4	5	6
	Token	Token	Divider 1 (Object 1)	Token	Divider 2 (Object 2)	Token
	0	0	1	0	1	0

In this line up,

- There are two tokens before the first divider, so object 1 is selected twice.
- There is one token before the second divider, so object 2 is selected once.
- There is one token **after** the second (and last) divider, so object 3 is selected once.

The proof of the general formula is now apparent. The number of positions is $r+n-1$. We need to choose $n-1$ positions to place the dividers. Therefore, the number of ways is as claimed before,

$$C(r+n-1,n-1)$$

Example: What is the number of ways to place 8 identical golf balls into three distinct holes (A, B, C)?

In terms of the above theorem, this is an 8-selection of 3 distinct objects ($n = 3, r = 8.$) Use a line-up of 8 balls and 2 dividers, thus a total of 10 slots. Find the number of ways to choose 2 slots out of 10 for the dividers. The number of ways is:

$$C(n + r - 1, n - 1) = C(10,2) = \frac{10 * 9}{2} = 45$$

Example: How many ways are there to divide 8 identical slices of pizza among 3 friends, with no leftover, in each of the following cases?

(a) No restriction on how many slices each person gets
(b) Each person gets at least two slices
(c) Each person gets at least two slices and at most 3 slices

Solution:

(a) One way to compute the number of ways is by enumerating all possible number of slices for each person. This may be done by a nested summation, where the first person has a slices, the second person has b slices, and the third person has the leftover number $(8 - a - b)$.

$$\sum_{a=0}^{8} \sum_{b=0}^{8-a} (1) = \sum_{a=0}^{8} (8 - a + 1) = 9 + 8 + \cdots 1 = 9 * \frac{10}{2} = 45$$

This method is not easy to generalize to a larger number of friends.
A more general and eloquent approach is to use "dividers" as explained in the above theorem. In terms of the theorem, this is an 8-selection of 3 distinct elements. Since there are $n = 3$ friends, we use $n - 1 = 2$ dividers. We use a line-up of $8 + 2 = 10$ positions, where each position can hold either a slice of pizza (shown as O) or a divider (shown as |). All slices of pizza positioned before the first divider goes to the first person (A), all slices between the two dividers goes to person B, and all slices after the second divider goes to the third person (C).
In the following illustration, the two dividers are in positions 3 and 7. So 2 slices goes to person A, 3 slices goes to person B, and 3 slices to person C.

Position	1	2	3	4	5	6	7	8	9	10
Pitzza/Divider	O	O	|	O	O	O	|	O	O	O

So we need the number of ways to pick 2 positions out of 10 for dividers, which is

$$C(10,2) = \frac{10 * 9}{2} = 45$$

(b) After giving each person 2 slices, there are 2 slices remaining to be divided among three friends with no restriction. Again use 2 dividers and 2 slices. So the problem is the number of ways to pick 2 out of 4 positions for the dividers, which is

$$\boxed{C(4,2) = 6}$$

Below is an explicit listing of all 6 possibilities for dividing the remaining 2 slices between 3 friends, with no restriction on how many slices each person gets.

A	B	C
0	0	2
0	1	1
0	2	0
1	0	1
1	1	0
2	0	0

(c) Give each person a minimum of 2 slices. Then, there are 2 slices remaining to be divided among 3 friends, with each person getting at most one. So pick 2 out of 3 friends to get one slice each. The number of ways is: $C(3,2) = 3$.

Example: What is the number of 3-digit decimal integers with the sum of the 3 digits equal 8? One example is 215.

Solution: We use the idea of dividers used in the previous problem, in conjunction with the "unary representation" of a decimal number. For example, 215 is represented as:

$$11\_1\_11111$$

A decimal digit of 2, for example, is shown as 11 (which is two 1's), and dividers are used in between digits.

Position	1	2	3	4	5	6	7	8	9	10
1 or divider	1	1	_	1	_	1	1	1	1	1

So the problem becomes a line-up of 10 positions (8 1's plus 2 dividers). And we need the number of ways to pick 2 out of 10 positions for the two dividers.

$$C(10,2) = \frac{10 * 9}{2 * 1} = 45.$$