CS 506, Online Foundations of CS	**Homework 5** **Functions & Relations**	Dr. David Nassimi

Study: Modules 4 (Functions) and Module 5 (Relations)

1. Consider the following sets of ordered pairs.

   (a) $S_1 = \{(1,a),(2,b),(3,c),(4,d)\}$

   (b) $S_2 = \{(1,a),(2,a),(3,d),(4,d)\}$

   (c) $S_3 = \{(1,a),(1,b),(2,c),(2,d)\}$

   Determine if each set defines a *function* from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$. If the set is a function, then:

   - Determine its domain D, co-domain Y, and range R.
   - Is the function one-to-one?
   - Is the function *onto*?
   - If the function is one-to-one and onto, find its inverse.

2. The functions $f(n) = n^2$ and $g(n) = 2n$ are defined on the set of positive real numbers.

   (a) Find the composition $f \circ g$. (The composition $(f \circ g)(n)$ is defined as $f(g(n))$.)

   (b) Find the composition $g \circ f$.

   (c) Find the inverse functions f^{-1} and g^{-1}.

3. A sequence $(F_1, F_2, F_3, \cdots)$ is defined recursively as follows. (This recursive definition is called a *recurrence equation*.)

$$F_n = \begin{cases} 1, & n = 1, \\ 2F_{n-1} + 1, & n \geq 2. \end{cases}$$

   (a) Compute $F_1, F_2, \cdots, F_{10}$ and tabulate results.

   (b) Prove by induction on n that

$$F_n = 2^n - 1, \quad n \geq 1.$$

4. The Fibonacci sequence is defined recursively as follows:

$$F_n = \begin{cases} 1, & n = 1, \\ 1, & n = 2, \\ F_{n-1} + F_{n-2}, & n \geq 3. \end{cases}$$

   (a) Compute and tabulate $(F_1, F_2, \cdots, F_{12)}$.

   (b) Prove by induction that for all $n \geq 1$,

$$F_n \leq (1.62)^{n-1}.$$

   (c) Prove by induction that for all $n \geq 1$,

$$F_n \geq (1.61)^{n-2}.$$

5. Let $\{F_n\}$ be the Fibonacci sequence (defined above). Prove by induction that for all $n \geq 1$,

$$S(n) = \sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

6. Consider the following relations.

   (a) $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$

   (b) $R_2 = \{(x,y) \mid x,y \text{ are positve integers } \leq 3, \text{ and } x+y \leq 5\}$

   (c) $R_3 = \{(x,y) \mid x,y \text{ are positve integers } \leq 3, \text{ and } x \leq y \leq x+2\}$

   For each relation,

   - Show the relation as a *set of ordered pairs* (if not in that form already), in *table* form, and in the form of a *directed graph*. (Show the graph with two columns of vertices, where the domain is the left column and the range is the right column.)
   - Determine whether the relation is *reflexive, symmetric, antisymmetric, transitive.* (Justify your answers.)
   - Determine if the relation is a *partial order* or an *equivalence relation.* If the latter is true, then specify the equivalence classes.

7. For each of the following relations, show the Boolean matrix A and compute the Boolean matrix A^2. Then, by comparing A and A^2, determine whether the relation is transitive, and justify your answer.

   (a) $R_1 = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$

   (b) $R_2 = \{(1,1),(2,1),(2,3),(3,1)\}$

   (c) $R_3 = \{(1,1),(2,3),(3,1)\}$

Additional Exercises
(Not to be handed-in)

8. Let $\{F_n\}$ be the Fibonacci sequence (defined above). Prove by induction that for all $n \geq 6$,

$$\frac{1}{\sqrt{5}}1.61^n < F_n < \frac{1}{\sqrt{5}}1.62^n.$$

(Hint: You need to use $n = 6$ and $n = 7$ for base cases.)

It is interesting to see how the above lower and upper bounds were determined. In Chapter 7, we will see how to find the exact solution of the Fibonacci sequence. The exact solution is

$$F_n = \frac{1}{\sqrt{5}}(r_1)^n - \frac{1}{\sqrt{5}}(r_2)^n$$

where $r_1 = \frac{1+\sqrt{5}}{2} \approx 1.618034$ and $r_2 = \frac{1-\sqrt{5}}{2} \approx -0.618034$. The second term, $(r_2)^n$ becomes a very small fraction as n gets large,

$$|(r_2)^n| < 0.0557, \quad n \geq 6$$

and so $F_n \approx \frac{1}{\sqrt{5}}1.618^n$.

9. Let $\{F_n\}$ be the Fibonacci sequence (defined above). Prove by induction that for all $n \geq 10$,

$$\frac{1}{\sqrt{5}}1.618^n < F_n < \frac{1}{\sqrt{5}}1.6181^n.$$

(Hint: You need to use $n = 10$ and $n = 11$ for base cases.) Note that $(1.6180)^2 = 2.6179$ and $(1.6181)^2 = 2.6182$.

10. Consider the relational database defined by the tables on p. 177 in our textbook (Edition 7).

- Table 3.6.4: EMPLOYEE
- Table 3.6.5: DEPARTMENT
- Table 3.6.6: SUPPLIER
- Table 3.6.7: BUYER

(Note: In the older edition of the textbook, these are Tables 3.4.4–3.4.7, p. 141.)

Write a sequence of database operations for each of the following queries. Also, provide the answer to each query.

(a) Find the names of all managers.
(b) Find the names of all employees managed by Jones.
(c) Find all part numbers supplied by department 23.
(d) Find all buyers supplied by department 23.
(e) Find all buyers supplied by the department managed by Jones.

11. Prove each of the following simple equalities.
Hint: If n is divisible by 4 (i.e, $n = 4k$ for some integer k), these are obviously correct. When n is not divisible by 4, then $n = 4k + r$, where r is the remainder, $r \in \{1, 2, 3\}$. Prove each equality by considering each r value. (This proof method is called proof by enumeration.)

(a) $\lfloor \lfloor n/2 \rfloor /2 \rfloor = \lfloor n/4 \rfloor$

(b) $\lceil \lceil n/2 \rceil /2 \rceil = \lceil n/4 \rceil$

(c) For yourself (not to hand in this part), figure out the hybrid forms:

- $\lceil \lfloor n/2 \rfloor /2 \rceil$
- $\lfloor \lceil n/2 \rceil /2 \rfloor$

Show both of these become either $\lfloor n/4 \rfloor$ or $\lceil n/4 \rceil$.

12. Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The *composition* of R_1 and R_2, denoted as $R_2 \circ R_1$, is a relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

(a) For the following relations R_1 and R_2, find $R = R_2 \circ R_1$. Show the relation R as a set of ordered pairs, in table form, and in graph form.

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$$
$$R_2 = \{(1, a), (1, b), (2, a), (2, b), (3, a)\}$$

(b) Show the Boolean matrices A_1 and A_2 for R_1 and R_2, respectively. (The matrices are 3×3 and 3×2, respectively. For R_2, columns 1 and 2 should be labeled a and b respectively.)

Then, compute the Boolean product $A_1 \times A_2$. (The product matrix is also Boolean, with all entries 0 or 1.) Verify that the product matrix correctly represents the relation R.

13. Let R be a relation and A be its Boolean matrix. Let $A^2 = A \times A$ be the Boolean product, where all entries in the product matrix are either 0 or 1. That is,

$$A^2[i, j] = \bigvee_{k=1}^{n} (A[i, k] \wedge A[k, j]).$$

(a) Prove that R is transitive <u>**if and only if**</u>

$$\forall i, j, \quad (A^2[i, j] = 1) \rightarrow (A[i, j] = 1).$$

(This implication means that whenever $A^2[i, j]$ is nonzero, $A[i, j]$ is also nonzero.)

Hint: You may provide the proof by the following two steps:

i. Prove that if R is transitive, and $A^2[i, j] = 1$ for any i, j, then $A[i, j] = 1$.

ii. Prove that if R is not transitive, then $\exists i, j, \quad A^2[i, j] = 1$ and $A[i, j] = 0$.

(b) Prove that if R is reflexive, then

$$\forall i, j, \quad (A[i, j] = 1) \rightarrow (A^2[i, j] = 1).$$

(This implication means that whenever $A[i, j]$ is nonzero, $A^2[i, j]$ is also nonzero.)

(c) Let R be a reflexive relation. Prove that R is transitive <u>**if and only if**</u>

$$A^2 = A.$$