CS 506, Online Foundations of CS	**Homework 3** **Proofs**	Dr. David Nassimi

Study Module 3: Proofs
Proof by Contrapositive, Contradiction, and Induction

1. Use **contrapositive proof** method for each of the following.
   (a) There are 10 boxes. Prove that if 40 balls are placed in the boxes, then at least one box has four or more balls.
   (b) Let x be a real number. Prove that if x^2 is irrational, then x must be irrational.

2. Use **contrapositive proof** for each of the following, where the domain of n is positive integers.
   (a) Prove that if n^2 is not divisible by 3, then n is not divisible by 3.
   (b) Prove that if n^2 is divisible by 3, then n is divisible by 3. (Hint: If n is not divisible by 3, then $n = 3k + r$, where k is an integer quotient and r is a non-zero remainder, $r \in \{1, 2\}$.)

3. Let x and y be two real numbers and let $A = (x + y)/2$. We want to formally prove that if $(x < y)$ then
$$x < A < y.$$
   You are not allowed to state it as a known fact that the average of two values fall between those two values! Rather, you must provide a formal proof in two ways:
   (a) **Direct Method**; and
   (b) **Contrapositive method**.

   Hints: For direct proof, assume $x < y$, and show that $2x < x + y < 2y$.
   For contrapositive proof, assume $\neg(x < A < y)$, which means $\neg[(x < A) \wedge (A < y)]$, which is
$$(A \leq x) \vee (A \geq y).$$
   Then provide the proof for each of the two cases in the OR statement.

4. Use **proof by contradiction** for each of the following, where x and y are positive real numbers.
   (a) Suppose $xy \geq 400$. Prove that at least one of the two numbers must be ≥ 20.
   (b) Suppose x is rational and y is irrational. Prove that $x * y$ is irrational.
   (c) Suppose x is rational and y is irrational. Prove that $x + y$ is irrational.

5. Use proof by **contradiction** to show that $\sqrt{3}$ is irrational.
   Hint: The proof is similar to the proof we did in class for $\sqrt{2}$. Here, use the fact that if n^2 is divisible by 3, then n is divisible by 3. (This was proved in one of the problems above.)

6. Prove by induction that all integers of the following form are divisible by 4, for all integers $n \geq 1$.
$$f(n) = 5^n - 1$$

7. Use **induction** to prove each of the following formulas.

(a) Arithmetic series sum:

$$S(n) = \sum_{i=1}^{n} (i) = \frac{n(n+1)}{2}$$

(b)

$$S(n) = \sum_{i=1}^{n} (i^2) = \frac{n(n+1)(2n+1)}{6}$$

(c) Geometric series sum, $a \neq 1$:

$$S(n) = \sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}.$$

8. A saving bank pays interest rate of 5%, compounded annually. Consider an initial deposit of \$1000. Let F_n be the total amount at the end of year n. This function may be expressed recursively as follows:

$$F_0 = 1000,$$

$$F_n = 1.05 * F_{n-1}, \quad n \geq 1.$$

(This recursive definition is called a *recurrence equation*.)

(a) Compute $F_1, F_2, \cdots, F_{10}$ and tabulate results (to get a feel for how the amount compounds).

(b) Prove by induction on n that

$$F_n = 1000 * (1.05)^n, \quad n \geq 0.$$

Additional Exercises (Not to be handed-in)

9. Let P denote the set of positive integers ≥ 2. For $i \geq 2$, define X_i as the set of integers that are greater than i and divisible by i.

$$X_i = \{ik \mid k \text{ is an integer } \geq 2\}.$$

Describe in plain words what the following set is and justify your answer.

$$P - \cup_{i=2}^{\infty} X_i.$$

10. Let the average of n real numbers $(x_1, x_2, \cdots, x_n)$ be

$$A = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Prove by **contradiction** that

$$S: \quad \exists i \ (x_i \leq A) \wedge \exists j \ (x_j \geq A).$$

Hint: In order to prove S is true, start by supposing that S is false, and show that will lead to a contradiction (which cannot be true), thus concluding that S must be true.