CS 506, Online Foundations of CS	**Homework 8** **Permutations/Combinations**	Dr. David Nassimi

Study Module 9 (Counting Methods)
Permutations and Combinations

1. (a) Use the multiplication principle to prove that the total number of n-bit binary integers is 2^n.
   (b) Use induction to prove the same.
2. (a) Use the addition principle (also called Incusion-Exclusion principle) to derive the number of elements in the union of three sets A, B, C, which are not disjoint.
   (b) Repeat for four sets A, B, C, D.
3. (a) Define $P(n, r)$, and give the formula for it. Compute $P(8, 4)$.
   (b) Define $C(n, r)$, and give the formula for it.
   (c) What is $C(n, r)$ in terms of $P(n, r)$? Explain.
   (d) Compute $C(8, 4)$, $C(8, 2)$, and $C(8, 6)$.
4. Consider the binomial formula:

$$(a + b)^n = c_0 a^n + c_1 a^{n-1} b + c_2 a^{n-2} b^2 + \cdots + c_n b^n.$$

   (a) Compute the coefficients for $(a + b)^4$.
   (b) Compute the coefficients for $(a + b)^6$.
5. (a) Explain the recursive formula for $C(n, r)$, which is as follows:

$$C(n, r) \quad = \quad \begin{cases} C(n-1, r) + C(n-1, r-1), & 0 < r < n, \\ 1, & r = 0, r = n. \end{cases}$$

   (b) Prove by induction that the solution of this recurrence is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

6. (a) What is the number of 6-bit binary integers with exactly 3 1's? List them in sorted order.
   (b) What is the number of n-bit binary integers with exactly r 1's?
   (c) Explain the reason for the following equality.

$$\Sigma_{r=0}^{n} C(n, r) = 2^n.$$

7. (a) A 10-member club wants to choose a president, a vice-president, and a treasurer. Derive the number of ways.
   (b) A 20-member club wants to choose a president, a vice-president, 3 secretaries, and 2 treasurers. Derive the number of ways.
   (c) A 25-member club wants to choose a 5-member committee for fund-raising. Derive the number of ways to choose the committee.
8. How many distinct strings result from all permutations of the 5 characters in string "ABBCC"? Show the combinatorial derivation. List the distinct strings in alphabetical order.

9. There are 6 identical golf balls and 3 holes A, B, and C. How many ways are there for placing the balls in the holes?

   Hint: Line up the balls in a row and place two dividers between them. For example, consider the following line up, where o is a ball and $|$ is a divider:

$$\begin{array}{rcccccccc} \text{Positions:} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{Line up:} & o & o & o & | & o & o & | & o \end{array}$$

   In this line up, the first 3 balls (to the left of the first divider) go in hole A, the next 2 balls go in hole B, and the last ball goes in hole C. (See Example 6.3.4 and the following theorem in the text.)

10. Four friends want to divide a pie of pizza that has 8 identical slices (with no leftover).

    (a) How many ways are there to divide the 8 slices between them, with no restriction?

    (b) How many ways are there if each person gets at least one slice?

Additional Exercises (Not to be handed-in)

11. Show the computations of $C(6, r)$, for $r = 0, 1, 2, 3, 4, 5, 6$. Verify that the total of these values is 64, which is 2^6. Explain briefly why.

12. Parents of three (spoiled) children bring home nine gifts to divide between the kids for Chanukah/Christmas. The gifts are:

    - A series of three different books on Hunger Games, namely: (1) The Hunger Games, (2) Catching Fire, and (3) Mockingjay.
    - Three distint boxes of chocolates.
    - Three iTunes gift cards, each with the value of $100.

    Derive the total number of distinct ways to divide the nine gifts beween the three children for each of the following cases:

    (a) Assuming an equitable division of the gifts such that each kid gets one book, one box of chocolate, and one iTunes gift card.

    (b) Assuming no restriction on how many gifts each kid receives. (A bully child may end up with all nine gifts, but that is a valuable lesson for the kids to learn.)

13. Prove by induction that an integer n is divisible by 3 if and only if sum of its decimal digits is divisible by 3. (For example, 47298 is divisible by 3 since $4 + 7 + 2 + 9 + 8 = 30$. What happens to the sum of digits when you add 3 to a number, $47298 + 3 = 47301$? How does each decimal carry effect the sum?)

14. Prove by induction that an integer n is divisible by 9 if and only if sum of its decimal digits is divisible by 9.