Module 1: Sets

Reading from Text (Johnsonbaugh):   Chapter 1 Sets and Logic

Introduction

Set terminology and concepts are widely used in computer science. For example, a graph G is defined as a set of vertices, V, and a set of edges, E.   In this module, we study sets in detail

Logic, or more specifically **Propositional Logic,** is the art of reasoning. In the next module, we study logic, and its relation to sets.

Contents:

1. Set Definitions
2. Set Operations
3. Set Algebra

1. Set Definitions

A set is an **unordered** collection of **distinct** elements.

For example, the set A

$$A = \{1, 2, 5, 7\}$$

has 4 elements in it, and is the same as the set

$$A = \{1, 5, 7, 2\}$$

(The order of elements does not matter.)

We say two sets A and B are equal, $A = B$, if they contain exactly the same elements. Two sets A and B are not equal, $A \neq B$, if there exists at least one element in A that is not in B, or at least one element in B that is not in A.

A set may not be finite. For example, the set of positive even integers is an infinite set and it may be written as:

$$B = \{2, 4, 6, \ldots\}$$

It may also be written as

$$B = \{2i \mid i \text{ is a positive integer}\}$$

The vertical line is read as 'such that', or 'where'. So B is set of integers of the form *2i*, where *i* is a positive integer.

Cardinality of a set: If a set A is finite, we use |A| to denote the number of elements in it. This number is called the *size*, or *cardinality*, of the set. For example, for the above set , $A = \{1, 2, 5, 7\}$, the size of the set is

$$|A| = 4.$$

Set **membership** $\in$:

For the above set, element 5 is an element of set A. (Element 5 is in A.)

$$5 \in A$$

Element 8 is not in A. so we write: $8 \notin A$.

Subset: A is a subset of B,

$$A \subseteq B$$

if all elements of A are also in B. (B may or may not have additional elements.)

Proper Subset:

$$A \subset C$$

All elements of A are in C, and C has some more elements in it that are not in A. That is, A ≠ C.

Conversely, B is a **superset** of A, $B \supseteq A$. And Cis a **proper superset** of A, $C \supset A$.

Universal set

$$U$$

is the set of all elements in universe. All sets are subsets of the universal set.

Empty set

$$\emptyset = \{\ \}.$$

2. Set Operations

Basic Set Operations:

1. **Union** $A \cup B$
   Set of elements that are in A **or** B, or both.

2. **Intersection** $A \cap B$
   Set of elements that are both in A **and** B. (That is, set of common elements.)

3. **Set Complement** $\bar{A}$
   Set of elements not in A. (Set of elements that are in universe and not in A.)

Example: If the universal set is U = {1,2,3,4,5,6,7,8,9}, and

$$A = \{1, 2, 5, 6, 9\}$$
$$B = \{4, 5, 6, 7\}$$

Then

$$A \cup B = \{1, 2, 4, 5, 6, 7, 9\}$$
$$A \cap B = \{5, 6\}$$
$$\bar{A} = \{3, 4, 7, 8\}$$

Other Set Operations:

Other set operations may be expressed in terms of the three basic set operations.

Set Difference:

$$A - B$$

is the set of those elements that are in A but not in B. So, we remove from A those elements of it that are also in B, and the remaining elements are *A-B*.

Example: If A and B are the above sets, then

$$A - B = \{1, 2, 9\}$$
$$B - A = \{4, 7\}$$

In terms of the basic set operations,

$$\boxed{A - B = A \cap \bar{B}}$$

Cartesian Product of two sets: $A \times B$

Set of ordered pairs (a, b) where a is in A and b is in B.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Cartesian product of several sets is similarly defined. For example,

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$$

If $|A| = n_1, |B| = n_2, |C| = n_3$, then $|A \times B \times C| = n_1 * n_2 * n_3$.

Example:

$B = \{Joe, Jack\}$
$G = \{Sue, Jill, Jen\}$
$B \times G = \{(Joe, Sue), (Joe, Jill), (Joe, Jen), (Jack, Sue), (Jack, Jill), (Jack, Jen)\}$

Example: A diner menu consists of a choice of an appetizer, a choice of an Entrée, and a choice of a Dessert.

$A = set\ of\ Appetizers\ to\ choose\ from$
$E = set\ of\ main\ Entrees\ to\ choose\ from$
$D = set\ of\ Desserts\ to\ choose\ from$

Then, the set of all possible orders is $A \times E \times D$.

Power set of a set A, $\mathcal{P}$(A), is the set of all possible subsets of A.

Example: If A = {a, b, c} then

$\mathcal{P}$(A) = {{ }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

If |A| =n, then

$$|\mathcal{P}(A)| = 2^n.$$

Notation: Let $A_1, A_2, A_3, \ldots, A_n$ be a number of sets. Then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

3. Set Algebra

Set Algebra is a set of rules for manipulating expressions on sets. In this module, we present a detailed study of set algebra. Later, we will see that the following three algebraic systems are basically equivalent:

- Set Algebra
- Propositional Logic (Propositional Algebra)
- Boolean Algebra

Thus, an in-depth study of set algebraic manipulations also helps with the latter two algebraic systems.

Precedence Rule on Set Operations:

In a set expression, the order of operations may be explicitly stated by use of parenthesis. But, in the absence of parenthesis, the following order is implied:

1. Set Complement
2. Intersection
3. Union

Example: The expression

$$A \cup B \cap \bar{C}$$

means

$$A \cup \left(B \cap (\bar{C}) \right).$$

That is, first C is complemented, then intersection is performed, and finally the union.

Note: Set difference is not part of this precedence rule. In an expression with set difference, parenthesis must be used to avoid ambiguity. For example, the order of operations in the following expression may not be clear.

$$A - B \cap \bar{C}$$

Parenthesis must be used to clarify the order, as follows:

$$A - (B \cap \bar{C})$$

The following table lists the basic rules in set algebra.

Basic Rules (Equalities) in Set Algebra	
Associative Laws:	$(X \cup Y) \cup Z = X \cup (Y \cup Z)$
	$(X \cap Y) \cap Z = X \cap (Y \cap Z)$
Commutative Laws:	$X \cup Y = Y \cup X$
	$X \cap Y = Y \cap X$
Distributive Laws:	$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
	$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
De Morgan's Laws:	$\overline{X \cap Y} = \overline{X} \cup \overline{Y}$
	$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$
Complement Laws:	$X \cup \overline{X} = U$
	$X \cap \overline{X} = \phi$
	$\overline{\overline{X}} = X$
Repetition:	$X \cup X = X$
	$X \cap X = X$
0/1 Laws:	$\overline{\phi} = U$
	$\overline{U} = \phi$
Identity:	$X \cup \phi = X$
	$X \cap U = X$
Bound Laws:	$X \cup U = U$
	$X \cap \phi = \phi$

Some of these rules are obvious and don't require a proof.   For example, the first form of the Complement Law states that

$$X \cup \overline{X} = U.$$

This should be obvious because every element in universe is either in X or not in X.

Proof Methods:

There are two methods for proving any algebraic rule (equality).

1. Venn Diagram: This is a pictorial way of proving any equality.
2. Algebraic Proof: A new rule is proved algebraically by using some of the algebra rules that have been proved already.
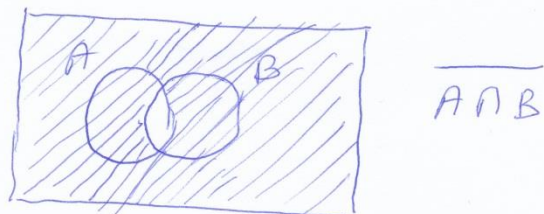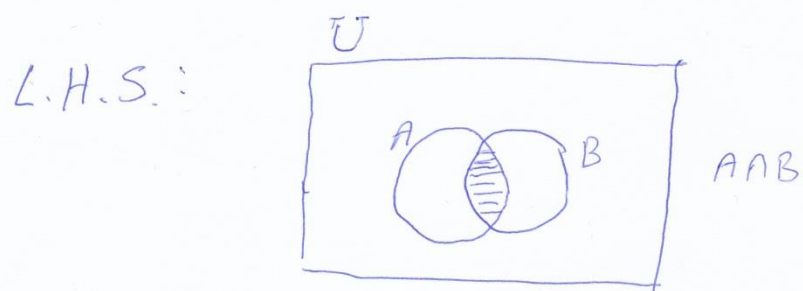
Venn Diagram:

In a Venn diagram, the universal set U is shown by a rectangle and each set is shown by a circle. The region inside a circle A represents the elements in set A, and the region outside of the circle (but inside of the rectangle) represents $\overline{A}$.

Example: Use a Venn diagram to prove the first form of De Morgan's Law
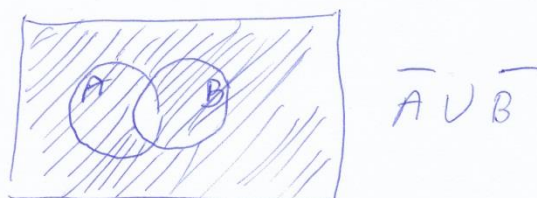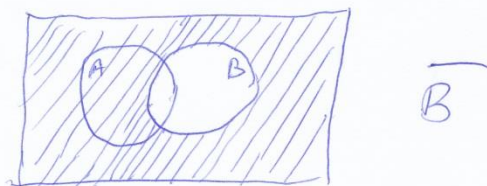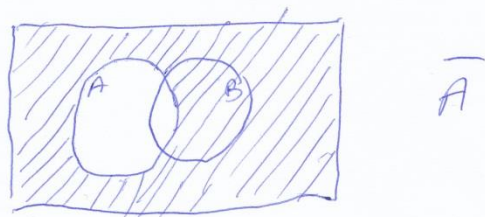
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$

L.H.S.:

$A \cap B$

$\overline{A \cap B}$ ←

R.H.S.:

$\overline{A}$

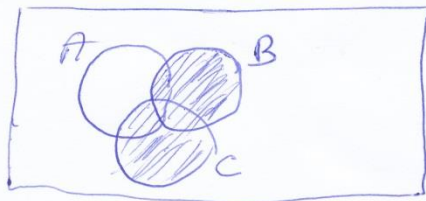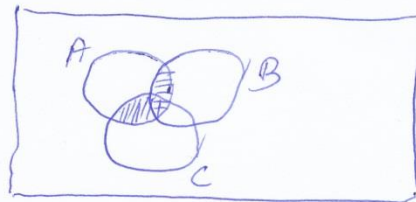$\overline{B}$

$\overline{A} \cup \overline{B}$ ←

EQUAL

Example: Use a Venn diagram to prove the first form of the Distributive Law:
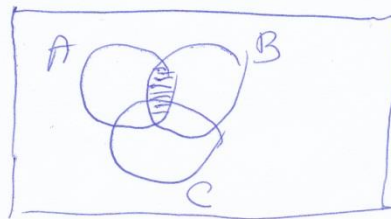
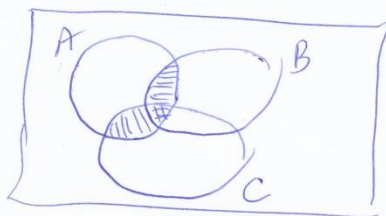$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

L.H.S.

$B \cup C$

$A \cap (B \cup C)$ ←

EQUAL

R.H.S.

$A \cap B$

$A \cap C$

$(A \cap B) \cup (A \cap C)$ ←

Algebraic Proof:

This method uses the rules that have been proven already to prove a new rule. The basic rules listed in the above table are considered proven and may be used to prove a new rule.

To prove a new rule, we start with the left-hand-side and algebraically manipulate the expression until we reach the right-hand-side. We justify each step by showing on the margin the rule that is used.

Example: Use algebraic method to prove the following rule.

Contraction Rule (also called Combination Rule):

$$(A \cap B) \cup (A \cap \bar{B}) = A$$

Proof:

$(A \cap B) \cup (A \cap \bar{B})$ Distributive in Reverse (Factor out)
$= A \cap (B \cup \bar{B})$ Complement Law
$= A \cap U$ Identity Law
$= A$

Generalization:

Any of the rules may be generalized if we replace each variable by an expression.

For example, here is a generalization of the above contraction rule:

$(A \cap B \cap C) \cup (A \cap B \cap \bar{C}) = (A \cap B)$

Duality: The rules stated in the above table (equality rules) all appear in pairs, which are dual of each other.

Definition: Given a set expression F, its dual F_d is obtained by the following changes:

- Change $\cap$ to $\cup$
- Change $\cup$ to $\cap$
- Change empty $\emptyset$ to universal U
- Change universal U to empty $\emptyset$
- Preserve the relative order of the corresponding operations (by use of parenthesis where needed).

Example: Let

$$F = A \cup B \cap C \cup \emptyset$$

The dual of the function becomes

$$F_d = A \cap (B \cup C) \cap U$$

Note that in F, the operation $(B \cap C)$ is performed first (before the other two operations).

So, in F_d, the corresponding operation $(B \cup C)$ must be done first, which requires parenthesis.

Duality Theorem: Given two set expressions (functions) F and G.

If

$$F = G$$

Then

$$F_d = G_d$$

In words, if two functions (set expressions) F and G are proved to be equal, then the theorem guarantees that the dual of the two functions will also be equal.

We will not provide a formal proof of this theorem. Rather, we will give an informal reasoning for it.

First, observe that all simple rules involving intersection and union exist in dual form. For example, in intersection,

$$A \cap \emptyset = \emptyset$$

The dual of this is true for union operation:

$$A \cup U = U$$

Suppose we use a number of the basic rules to prove an equality

$$F = G$$

Then, we can use the dual of the same basic rules to prove

$$F_d = G_d$$

We will illustrate this by the following example.

Example: Use algebraic method to prove each of the following equalities. (The second one is the dual of the first one.)

$$A \cup (\bar{A} \cap B) = A \cup B \qquad (1)$$

$$A \cap (\bar{A} \cup B) = A \cap B \qquad (2)$$

Prove (1): $A \cup (\bar{A} \cap B) = A \cup B$		Prove (2): $A \cap (\bar{A} \cup B) = A \cap B$	
$A \cup (\bar{A} \cap B)$		$A \cap (\bar{A} \cup B)$	
$= (A \cup \bar{A}) \cap (A \cup B)$	Distributive Law	$= (A \cap \bar{A}) \cup (A \cap B)$	Distributive Law
$= (U) \cap (A \cup B)$	Complement Law	$= (\emptyset) \cup (A \cap B)$	Complement Law
$= (A \cup B)$	Identity Law	$= (A \cap B)$	Identity Law

The left column proves the first rule. Starting with the left-hand-side of the rule, we apply Distributive Law, and then Complement Law ($A \cup \bar{A} = U$), and then the Identity Law ($U \cap X = X$).

The right column proves the second rule in the same way. Every step of the way, the expression on the right is dual of the expression on the left, and the basic rule applied on the right is the dual of the basic rule on the left. Therefore, the proof of (2) is redundant since it mirrors the proof of (1).

We will provide a few more examples of algebraic proof. We will prove only one form of equality and not the dual of it, since the dual of it follows from Duality Theorem.

Example: Use algebraic method to prove the Absorption Rule:

$$A \cup (A \cap B) = A$$

Failed attempt:

$A \cup (A \cap B)$	Distributive
$= (A \cup A) \cap (A \cup B)$	
$= A \cap (A \cup B)$	This is the dual of the starting expression; not helpful
$= (A \cap A) \cup (A \cap B)$	If we a apply distributive rule again
$= A \cup (A \cap B)$	We get the original expression!

Correct Proof 1:

$A \cup (A \cap B)$	Replace the first A with $A \cap U$
$= (A \cap U) \cup (A \cap B)$	Factor out A (Distributive in reverse)
$= A \cap (U \cup B)$	
$= A \cap U$	
$= A$	

An Alternative Correct Proof:

Earlier, we proved the contraction rule. The reverse of it is called expansion rule, where we start with A and expand it into $(A \cap B) \cup (A \cap \bar{B})$.

$A \cup (A \cap B)$	Expand A into $(A \cap B) \cup (A \cap \bar{B})$.
$= (A \cap B) \cup (A \cap \bar{B}) \cup (A \cap B)$	Eliminate third paren (duplicate)
$= (A \cap B) \cup (A \cap \bar{B})$	Contraction Rule
$= A$	

Example: Use algebraic method to prove the Consensus Rule:

$$(A \cap B) \cup (\bar{A} \cap C) \cup (B \cap C) = (A \cap B) \cup (\bar{A} \cap C)$$

Explanation:

In an earlier example, we proved the contraction/expansion rule. Namely, we saw the expansion $A = (A \cap B) \cup (A \cap \bar{B})$. Here we will use the expansion rule to expand the last parenthesis $(B \cap C)$ as:

$$B \cap C = (A \cap B \cap C) \cup (\bar{A} \cap B \cap C).$$

In the last step of the proof below, we use the absorption rule inside each [ ]. For example,

$$[(A \cap B) \cup (A \cap B \cap C)] = (A \cap B).$$

Proof:

$(A \cap B) \cup (\bar{A} \cap C) \cup (B \cap C)$	Start with LHS, expand $B \cap C$
$= (A \cap B) \cup (\bar{A} \cap C) \cup [(A \cap B \cap C) \cup (\bar{A} \cap B \cap C)]$	Rearrange terms
$= [(A \cap B) \cup (A \cap B \cap C)] \cup [(\bar{A} \cap C) \cup (\bar{A} \cap B \cap C)]$	Apply Absorption rule in each [ ].
$= (A \cap B) \cup (\bar{A} \cap C).$	

Next, we provide some algebraic examples for equalities involving set difference. It is important to be reminded that the basic rules, such as Distributive and Associative, do not apply to set difference. The best way to deal with set difference is to first replace it with its equivalent:

$$A - B = A \cap \bar{B}.$$

Example: Use algebraic method to prove

$$A - (B \cup C) = (A - B) \cap (A - C)$$

(This example illustrates that distributive rule does not apply to set difference.)

Proof:

$A - (B \cup C)$	Replace set difference
$= A \cap \overline{(B \cup C)}$	De Morgan
$= A \cap \bar{B} \cap \bar{C}$	Duplicate *A*. That is, $A = A \cap A$.
$= (A \cap \bar{B}) \cap (A \cap \bar{C})$	Now, replace complement with set difference.
$= (A - B) \cap (A - C)$.	

Example: Use algebraic method to prove

$$(A \cup B) - B = A - B$$

Proof:

$(A \cup B) - B$	*Replace set difference with complement*
$= (A \cup B) \cap \bar{B}$	Distributive Law
$= (A \cap \bar{B}) \cup (B \cap \bar{B})$	
$= (A \cap \bar{B}) \cup \emptyset$	
$= (A \cap \bar{B})$	
$= A - B$.	

Example: Use algebraic method to prove

$$(A \cup B) - C = (A - C) \cup (B - C)$$

Proof:

$(A \cup B) - C$	Replace set subtraction by complement
$= (A \cup B) \cap \bar{C}$	Distributive
$= (A \cap \bar{C}) \cup (B \cap \bar{C})$	Replace complement by set subtraction
$= (A - C) \cup (B - C)$.	

Generalization of Algebra Rules:

Any rule of set algebra, stated in terms of simple variables, may be generalized by replacing each variable by an arbitrary set expression.

Example: Find complement of the following function. (This is done by repeated applications of DeMorgan).

$$f = (A \cap B) \cup (\bar{A} \cap \bar{C} \cap D).$$

$\bar{f} = \overline{(A \cap B) \cup (\bar{A} \cap \bar{C} \cap D)}$
$= \overline{(A \cap B)} \cap \overline{(\bar{A} \cap \bar{C} \cap D)}$
$= (\bar{A} \cup \bar{B}) \cap (A \cup C \cup \bar{D}).$

Example: Use distributive law to prove the following equality.

$$(A \cup \bar{B}) \cap (\bar{A} \cup B) = (A \cap B) \cup (\bar{A} \cap \bar{B})$$

Proof:

$(A \cup \bar{B}) \cap (\bar{A} \cup B)$
$= (A \cap \bar{A}) \cup (A \cap B) \cup (\bar{A} \cap \bar{B}) \cup (\bar{B} \cap B)$
$= \emptyset \cup (A \cap B) \cup (\bar{A} \cap \bar{B}) \cup \emptyset$
$= (A \cap B) \cup (\bar{A} \cap \bar{B}).$

End of Module 1