CS 506, Online Foundations of CS	**Homework 9** **Discrete Probability**	Dr. David Nassimi

Study Module 10: Discrete Probability, Lottery and Games

1. A random flip of a coin gives either a "head" or "tail", with equal probabilities. Consider an experiment where we repeatedly flip a coin until we get a head. What is the probability that the first head occurs on kth flip of the coin? Explain.

2. A pair of 6-sided dice is rolled. Derive the probability that at least one of them is 5. Find the probability in three ways:

   (a) List all possible cases, where at least one dice is 5.

   (b) Use the formula,
   $$P(A \vee B) = P(A) + P(B) - P(A \wedge B).$$

   (c) Use the formula,
   $$P(A \vee B) = 1 - P(\neg A \wedge \neg B).$$

3. A pair of dice is thrown. Compute the probability that their sum equals to t for each t from 2 to 12, and tabulate them.

t	2	3	4	5	6	7	8	9	10	11	12
$P(sum = t)$											

4. A pair of dice is rolled. Find the probability for each of the following events.

   (a) The first dice is ≥ 4.

   (b) At least one of them is ≥ 4.

   (c) The sum is ≥ 8.

   (d) The sum is ≥ 8, given that the first dice is ≥ 4.

   (e) The sum is ≥ 8, given that at least one of them is ≥ 4.

   (f) At least one of them is ≥ 4, given that the sum is ≥ 8.

5. Jersey-Cash-5 is a New Jersey Lottery that "benefits education and institutions". In this lottery, a game is played by picking 5 distinct numbers out of numbers 1 to 40.

   Compute the number of ways and the probability of winning each of the following prizes. Show the combinatorial derivation. Use a calculator to perform the final step in the computation. Express the final result as a ratio $1/x$ where the denomintor x is rounded to the nearest integer.

   (a) Jackpot: Match 5 out of 5.

   (b) Second Prize: Match 4 out of 5. (You win \$500.)

   (c) Third Prize: Match 3 out of 5. (You win \$11.)

6. A poker hand is a random drawing of 5 cards out of a deck of 52 cards (with no joker). The cards have 4 **suits** (Diamond, Hearts, Spade, Club) and 13 **kinds**: (ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3, 2), where an ace counts as either the highest or lowest. (For the list of poker hands and illustrations, please see wikipedia.org.)

For the purpose of this problem, the poker hands are listed below in ranked order, along with the definition of each hand. Each hand is defined so as not to include any higher hand(s). For example, a *straight* hand means five cards in sequence, but not all in the same suite (otherwise, it would become straight flush).

Derive the number of ways $W()$, and the probability $P()$, for each poker hand. Show the combinatorial derivation. Find each probabilty in the form of a ratio $1/x$, where x is the nearest integer.

(a) Straight Flush: Five cards of the same suit and in consecutive order.

(b) Four-of-a-Kind (4,1): Four of one kind and one of another kind.

(c) Full House (3,2): Three of one kind and two of another kind.

(d) Flush: Five cards of the same suite (but not in sequence).

(e) Straight: Five cards in sequence (but not all the same suite).

(f) Three-of-a-Kind (3,1,1): Three of one kind, one of another kind, and one of a third kind.

(g) Two Pairs (2,2,1): Two of one kind, two of another kind, and one of a third kind.

(h) One Pair (2,1,1,1): Two of one kind, and three other kinds, with one card from each kind.

(i) High Card: This occurs when none of the above hands is made. (That is, the 5 cards are 5 different kinds but not all in sequence, and not all the same suit.)

Check your work by adding the number of ways for all 9 cases. The total must be $C(52, 5)$. For this check, you need to compute the number of ways to get the last hand (high card) dirctly, which is:

$$[C(13, 5) - 10] * [(C(4, 1))^5 - 4] = 1277 * 1020 = 1,302,540.$$

7. Consider a poker hand dealt with the first two cards facing up, and the remaining 3 cards face down. Compute the conditional probability of three-of-a-kind (3,1,1), given that the first two face-up cards are (Ace of hearts, King of Diamond).

   Note that the toal number of possible poker hands, given that the first two cards are (ace,king) is

   $$U = C(50, 3) = (50 * 49 * 48)/(3 * 2 * 1) = 19600.$$

   Hint: There are three ways of ending up with a 3-of-a-kind hand, given that the first two cards are (Ace,King):

   - Get two more aces, and one card of a third kind, OR
   - Get two more kings, and one card of a third kind, OR
   - Stay with one ace and one king, and get 3 cards of a third kind.

Additional Exercises (Not to be handed-in)

8. Let us look at the last problem again. We want to compute the conditional probability of 3-of-a-kind (3, 1, 1) given that the first two cards are an ace and a king.

A wrong counting method (Warning to avoid a common mistake):
Suppose a student **erronously** performs this computation as follows: Get one card from a third kind, so you have (Ace, king, and one of another kind). Then get two more cards from one of the three kinds! So, in this case,

$$W = C(11,1) * C(4,1) * C(3,1) * C(3,2) = 11 * 4 * 3 * 3 = 396$$

$$P = W/U = 396/19600 \approx 1/49$$

Why is this wrong?!

9. Consider a poker hand dealt with the **first two cards facing up**, and the remaining 3 cards face down. Compute the number of ways and the probability for each of the poker hands, given that the first two face-up cards are two aces.

   Check your work by adding the number of ways for all cases. The total must be $C(50, 3)$.

10. Consider a poker hand dealt with the **first two cards facing up**, and the remaining 3 cards face down. Compute the number of ways and the probability for each of the poker hands, given that the first two face-up cards are (Ace, King).

    Check your work by adding the number of ways for all cases. The total must be $C(50, 3)$.

11. Consider a deck of 53 cards, which includes a joker (wild card). The possible poker hands are:

    (a) Five-of-a-kind (Four of one kind, plus a joker)
    (b) Straight Flush (five cards same suit and in consecutive order).
    (c) Four-of-a-Kind (4,1)
    (d) Full House (3,2)
    (e) Flush
    (f) Straight
    (g) Three-of-a-Kind (3,1,1)
    (h) Two Pairs (2,2,1)
    (i) One Pair (2,1,1,1)
    (j) High Card: This occurs when none of the above hands is made.

    For example, a Four-of-a-Kind hand can be made in two ways:

    - Without joker: four of one kind plus one of another kind.
    - With a joker: three of one kind, plus a joder, plus one of another kind.

    Compute the number of ways to get each hand, and the probability.

    Check your work by adding the number of ways for all cases. The total must be $C(53, 5)$.