Homework 5 - Functions

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CS 506 - Online Foundations of CS

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1. Consider the following sets of ordered pairs.

(a) $S_1 = \{(1, a), (2, b), (3, c), (4, d)\}$
(b) $S_2 = \{(1, a), (2, b), (3, d), (4, d)\}$
(c) $S_3 = \{(1, a), (1, b), (2, c), (2, d)\}$

Determine if each set defines a *function* from $X = 1, 2, 3, 4$ to $Y = a, b, c, d$. If the set is a function, then:

Determine its domain D, co-domain Y and range R.
Is the function one-to-one?
Is the function *onto*
If the function is one-to-one and onto, find its inverse.

1(a). Describes a function where the domain $D = \{1, 2, 3, 4\}$, the co-domain $Y = \{a, b, c, d\}$, and the range $R = \{a, b, c, d\}$. The function is in fact one-to-one, because no two values in the domain map to the same value in the range. The function can be described as being *onto*, because both the range as well as the co-domain are equivalent($R = Y$). Given that the function is both one-to-one as well as onto, we can find the inverse, expressed as follows in ordered pairs:

$$f^{-1} = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$$

1(b). Describes a function where the domain $D = \{1, 2, 3, 4\}$, the co-domain $Y = \{a, b, c, d\}$, and the range $R = \{a, b, d\}$. The function is many-to-one, given that two values from the domain map to a same value in the range (specifically $f(3)$ and $f(4)$ which both map to d). The function can be described as being into, because the co-domain is a proper sub-set of the range ($R \subset Y$), meaning that all values of the range can be found in the co-domain but not all values in the co-domain are represented in the range. Because this function is many-to-one, there is no inverse.

1(c). Does not describe a function, it does not define $f(3)$ or $f(4)$, as well as neither $f(1)$ nor $f(2)$ being unique. (That is, $f(1, a) \in f$ and $f(1, b) \in f$)

2. The functions $f(n) = n^2$ and $g(n) = 2n$ are defined on the set of positive real numbers.

(a) Find the composition of $f \circ g$. (The composition $(f \circ g)(n)$ is defined as $f(g(n))$.)
(b) Find the composition $g \circ f$
(c) Find the inverse functions f^{-1} and g^{-1}.

2(a).
$$(f \circ g)(n) = f(g(n)) = (2n)^2 = 4n^2$$

2(b).
$$(g \circ f)(n) = g(f(n)) = 2(x^2) = 2x^2$$

2(c).
$$f^{-1}(x) = \sqrt{x}$$
$$g^{-1}(x) = \frac{x}{2}$$

3. A sequence $(F1, F2, F3, ...)$ is defined recursively as follows. (this recursive definition is called a *recurrance equation.*)

$$f(x) = \begin{cases} 1, & \text{n} = 1 \\ 2F_{n-1} + 1, & \text{c} \geq 2 \end{cases} \tag{1}$$

(a) Compute $F_1, F_2, ...F_10$ and tabulate results.
(b) Prove by induction on n that

$$F_n = 2^n - 1, n \geq 1.$$

3(a).

$f(x)$	Output
$F1$	1
$F2$	2(1) + 1 = 3
$F3$	2(3) + 1 = 7
$F4$	2(7) + 1 = 15
$F5$	2(15) + 1 = 31
$F6$	2(31) + 1 = 63
$F7$	2(63) + 1 = 127
$F8$	2(127) + 1 = 255
$F9$	2(255) + 1 = 511
$F10$	2(511) + 1 = 1023

3(b). Provided the hypothesis $P(n) : F_n = 2^n - 1 \forall n \geq 1$, to use simple induction, we must begin by proving a base case. In this instance I have chosen the case of $n = 1$, which as shown here

$$F_1 = 2^1 - 1 \Rightarrow 1 \geq 1$$

proves to be true. We are now to assume that $P(1) \wedge P(2) \wedge ... \wedge P(k-1)$ for some integer k, which we will use to prove $P(k)$. We assume, given our base case, that the following is also true:

$$P(k-1) : F_{k-1} = 2^{k-1} - 1$$

$$P(k) : F_k = 2^k - 1$$

Using the following substitutions and algebra we can show through induction that:

$$F_k = 2F_{k-1} + 1$$
$$\Rightarrow 2(2^{k-1} - 1) + 1$$
$$\Rightarrow 2^k - 2 + 1$$
$$\Rightarrow 2^k - 1$$

4. the Fibonacci sequence is defined recursively as follows:

$$f(x) = \begin{cases} 1, & \text{n} = 1 \\ 2, & \text{n} = 2 \\ F_{n-1} + F_{n-2}, & \text{n} \geq 3 \end{cases} \tag{2}$$

(a) Compute and tabulate $F_1, F_2, ...F_12$.
(b) Prove by induction that for all $n \geq 1$,

$$F_n \leq (1.62)^{n-1}$$

(c) Prove by induction that for all $n \geq 1$,

$$F_n \geq (1.61)^{n-2}$$

4(a).

$f(x)$	Output
$F1$	1
$F2$	2
$F3$	2 + 1 = 3
$F4$	3 + 2 = 5
$F5$	5 + 3 = 8
$F7$	8 + 5 = 13
$F8$	13 + 8 = 21
$F9$	21 + 13 = 34
$F10$	34 + 21 = 55
$F11$	55 + 34 = 89
$F12$	89 + 55 = 144

4(b). Provided the hypothesis $P(n) : F_n \leq (1.62)^{n-1}\ \forall n \geq 1$ to use the necessary strong induction, we must begin by providing some base cases. For these instances I have chosen the values $n = 1$ as well as $n = 2$

$$P(1) : F_1 \leq (1.62)^0 \Rightarrow 1 \leq 1$$

$$P(2) : F_2 \leq (1.62)^1 \Rightarrow 1 \leq 1.62$$

which proves to be true. We are now to assume that $P(1) \wedge P(2) \wedge ... \wedge P(k-1)$ for some integer k, which we will use to prove $P(k)$. We assume, given our base cases and other information, that the following is also true:

$$P(k-2) : F_{k-2} \leq (1.62)^{k-3}$$

$$P(k-1) : F_{k-1} \leq (1.62)^{k-2}$$

$$P(k) : F_k \leq (1.62)^{k-1}$$

Now, with substitutions, information from the recurrance equation, and some inequality properties, we can manipulate the equations as follows:

$$F_k = F_{k-1} + F_{k-2} \leq F_{k-1} + (1.62)^{k-3}$$
$$F_k = F_{k-1} + F_{k-2} \leq (1.62)^{k-2} + (1.62)^{k-3}$$
$$\Rightarrow F_k \leq (1.62)^{k-2} + (1.62)^{k-3}$$
$$\Rightarrow F_k \leq (1.62)^{k-3}[1.62 + 1]$$
$$\Rightarrow F_k \leq (1.62)^{k-3}[2.62] \overset{?}{\leq} (1.62)^{k-3}(1.62)^2$$

$$(1.62)^2 = 2.6244 \geq 2.62)$$
$$\Rightarrow F_k \leq (1.62)^{k-3}(1.62)^2$$
$$\Rightarrow F_k \leq (1.62)^{k-1}$$

4(c).Provided the hypothesis $P(n) : F_n \geq (1.61)^{n-2}\ \forall n \geq 1$ to use the necessary strong induction, we must begin by providing some base cases. For these instances I have chosen the values $n = 2$ as well as $n = 3$

$$P(1) : F_1 \geq (1.61)^{1-2} \Rightarrow 1 \geq \frac{1}{1.61}$$
$$P(2) : F_2 \geq (1.61)^{2-2} \Rightarrow 1 \geq 1$$

which prove to be true. We now assume that $P(1) \wedge P(2) \wedge ... \wedge P(k-1)$ for some integer k, which we will use to prove $P(k)$. We assume given our base cases and other information that the following is true:

$$P(k-2) : F_{k-2} \geq (1.61)^{k-4}$$
$$P(k-1) : F_{k-1} \geq (1.61)^{k-3}$$

$$P(k) : F_k \geq (1.61)^{k-2}$$

Now, with substitutions, information from the recurrance equation, and some inequality properties, we can manipulate the equations as follows:

$$F_k = F_{k-1} + F_{k-2} \geq F_{k-1} + (1.61)^{k-4}$$
$$F_k = F_{k-1} + F_{k-2} \geq (1.61)^{k-3} + (1.61)^{k-4}$$
$$\Rightarrow F_k \geq (1.61)^{k-3} + (1.61)^{k-4}$$
$$\Rightarrow F_k \geq (1.61)^{k-4}[1.61 + 1]$$
$$\Rightarrow F_k \geq (1.61)^{k-4}[2.61] \stackrel{?}{\geq} (1.61)^{k-4}(1.61)^2$$

$$(1.61)^2 = 2.5921 \leq 2.61$$
$$\Rightarrow F_k \geq (1.61)^{k-4}(1.61)^2$$
$$\Rightarrow F_k \leq (1.61)^{k-2}$$

5. Let F_n be the Fibonacci sequence (defined above). Prove by induction that for all $n \geq 1$,

$$S(n) : \sum_{k=1}^{n} F_k = F_{n+2} - 1$$

Provided the hypothesis$S(n) : \sum_{k=1}^{n} F_k = F_{n+2} - 1$ to use induction, we must first prove the base case. For this instance, I have chosen the value $n = 1$

$$S(1) : \sum_{k=1}^{1} F_k = 1 = F_{n+2} - 1 = F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$$

Thus the base case holds true for the hypothesis. We now assume that $S(1) \wedge S(2) \wedge ... \wedge S(n-1)$ for some integer n, which we will use to prove $S(n)$. We assume given our base cases and other information that the following is true:

$$S(n) : \sum_{k=1}^{n} F_k = F_1 + F_2 + ... + F_n = F_{n+2} - 1$$

$$S(n+1) : \sum_{k=1}^{n+1} F_k = F_1 + F_2 + ... + F_n + F_{n+1} = F_{(n+1)+2} - 1$$

Now, with substitutions and information from the recurrance equation, we can manipulate the equations as follows:

$$S(n+1) : \sum_{k=1}^{n+1} F_k = F_1 + F_2 + ... + F_n + F_{n+1} = F_{(n+1)+2} - 1$$

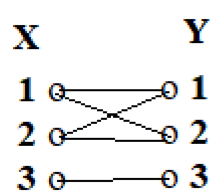
$$S(n+1): \sum_{k=1}^{n} F_k + F_{n+1} = F_{n+3} - 1$$

$$S(n+1): \sum_{k=1}^{n} F_k = F_{n+2} - 1$$

6. Consider the following relations.
(a) $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$
(b) $R_2 =$$\{(x,y)|x,y$ are positive integers≤ 3, and $x+y \leq 5\}$
(c) $R_3 =$$\{(x,y)|x,y$ are positive integers≤ 3, and $x \leq y \leq x+2\}$
For each relation,
Show the relation as a *set of ordered pairs* (if not in that form already), in *table* form, and in the form of a *directed graph.* (Show the graph with two columns of vertices, where the domain is the left column and the range is the right column.)
Determine whether the relation is *reflexive, symmetric, antisymmetric, transitive.* (Justify your answers.)
Determine if the relation is a *partial order* or an *equivalence relation.* If the latter is true, then specify the equivalence classes.
6(a). The relation comes to us in the form of ordered pairs. In the form of a table, it is as follows:

1	1
1	2
2	1
2	2
3	3

As a graph it looks as follows:



Reflexive: Given that the function is on the domain of $X = \{1,2,3\}$The above relation is reflexive, because for every X there exsits the ordered pair (x,x).
Symmetric: The above relation is in fact symmetric, because for all values of x and y there exsists (x,y) as well as a cooresponding (y,x) in the same relation.
Anti-Symmetric: As explained above, because of the symmetric properties of the relation, it is impossible to be anti-symmetric, for no relation can have both properties at the same time.

Transitive: The above relation is in fact transitive, because for every (x, y) and (y, z) there exsists (x, z)
Given the nature of the relation (that it is **Reflexive**,**Symmetric**, and **Transitive**) this means that it is an *equivalence relation.* Its equivalence classes are as follows:

Block 1: {1,2}
Block 2: {1,2}
Block 3: {3}

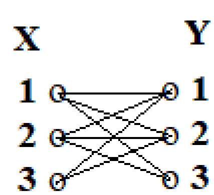
6(b). The relation in as a set of ordered pairs looks as follows:

$$R_2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}$$

As a table the relation looks as follows:

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2

Reflexive: The above relation is not reflexive, due to the fact that on the set of $\{1, 2, 3\}$ there exists no relationship $(3, 3)$
Symmetric: The above relation is in fact Symmetric because each relationship (x, y) has within the same relation a cooresponding relationship (y, x)
Anti-Symmetric: By definition, because the relation is symmetric, this relation cannot also be anti-symmetric
Transitive: This relation is not transitive, this is due to that there exists an x, y, z that does not fulfill the transitive definition, specifically in this relation $(3, 1) \in R, (1, 3) \in R, (3, 3) \notin R$
Due to the fact that the relation is not reflexive, it cannot be an equivalence relation or a partial order.
As a graph it looks as follows:



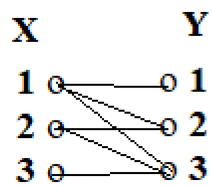
6(c).The relation in a set of ordered pairs looks as follows:

$$R_3 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

As a table the relation looks as follows:

1	1
1	2
1	3
2	2
2	3
3	3

As a graph it looks as follows:



Reflexive: This relation is in fact transitive, because on the set of X where $x \leq 3$ there exsist all instances of (x, x)
Symmetric: This relation is not symmetric, due to the fact that there exists a (x, y) for which there is no cooresponding (y, x), in this relation speficically: $(1, 2) \in R$ and $(2, 1) \notin R$
Anti-Symmetric: This relation is in fact anti-symmetric, thanks to the fact that for all (x, y) there exist no instance of (y, x)
Transitive: This relation is also Transitive, because for each and every $x, y, z \in R$ there exist $(x, y) \in R$, $(y, z) \in R$, and $(x, z) \in R$
In addition to being **Reflexive**,**Anti-Symmetric**, and **Transitive**, this list is also a partial order, because the pairs of integers are not related. An example of such an instance would be (2,1), because it does not fulfill the definition of $x \leq y$ and therefore is not a part of the relation.
7. For each of the following relations show the Boolean matrix A and compute the Boolean matrix A^2. Then, by comparing A and A^2, determine whether the relation is transitive, and justify your answer.
(a) $R_1 = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$
(b) $R_2 = \{(1,1),(2,1),(2,3),(3,1)\}$
(c) $R_3 = \{(1,1),(2,3),(3,1)\}$
To test for transitivity, we need just compare the initial matrix A with that of the A^2. Using substitution of the definition of transitivity, as well as the construction of Boolean matrices, transitivity can be defined as being having the following quality: $\forall Ai\ \forall Aj$, if $A^2[i, j]$ then $A[i, j] = 1$. we simply

administer this same test to the following matrices and can quickly find their transitivity or lack thereof.

7(a).

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \; A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Transitive (see above explanation)

7(b).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \; A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Not-transitive ($A[2,3] \neq A^2[2,3]$

7(c).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \; A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Not-transitive ($A[2,3] \neq A^2[2,3]$ AND ($A[2,1] \neq A^2[2,1]$