Module 10: Introduction to Discrete Probability

Reading from the Textbook: Chapter 6 Counting Methods

Introduction

We present a brief and informal treatment of discrete probability in this module. The counting methods of the previous module are applied for this development. We start with the basic sum rule and product rule, and then study conditional probability. Finally, we apply the counting methods and discrete probability for games such as poker and lottery.

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Basic Definitions

An **experiment** is a process that produces an **outcome**. For example, an experiment may be the roll of a six-sided die, which has 6 possible outcomes.

The set of possible outcomes is called **sample space.**

An **event** is an outcome or a set of outcomes. For example, the event that the die is ≥ 5 consists of two outcomes $\{5, 6\}$.

For a roll of a die, if we assume that all 6 possible outcomes are equally likely, then they each have a probability of 1/6, and the sum of the probabilities for all possible outcomes equals to 1.

If E is an event from a finite sample space S, and if all outcomes have equal probability, then the probability of event E is the ratio of the number of outcomes in event E over number of outcomes in sample space S.

$$\boxed{P(E) = \frac{|E|}{|S|}}$$

For example, consider the event that a roll of a die produces a face ≥ 5. Since $E = \{5,6\}$, and $S = \{1,2,3,4,5,6\}$, this event has the probability $2/6$.

Another immediate rule applies to complements:

$$\boxed{P(E) + P(\bar{E}) = 1}$$

Two events E_1 and E_2 are **disjoint**, or **mutually exclusive**, if $E_1 \cap E_2 = \emptyset$.

Two events E_1 and E_2 are **independent** if the outcome on one event does not change the probability for the other event.

Given two sets of events E_1 and E_2,

- $(E_1 \cup E_2)$ may also be viewed as the event (E_1 **or** E_2),
- $(E_1 \cap E_2)$ may also be viewed as the event (E_1 **and** E_2).

Sum Rule

Recall the addition principle for disjoint sets E_1 and E_2, which is $|E_1 \cup E_2| = |E_1| + |E_2|$. We may express a similar sum-of-probability rule for two disjoint events.

Sum Rule: If two events E_1 and E_2 are disjoint (also called mutually exclusive), then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Note: $(E_1 \cup E_2)$ may also be interpreted as $(E_1 \text{ or } E_2)$.

Example: Consider the roll of a pair of 6-sided dice. What is the probability that the sum is 6 or 7? The sample space consists of $6 \times 6 = 36$ outcomes. Let E_1 be the event that the sum equals 6, and $E_2$the event that the sum equals 7.

$$E_1 = \{(1,5), (2,4), (3,3), (4,2), (5,1)\},$$

$$E_2 = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

So, $P(E_1) = 5/36$ and $P(E_2) = 6/36$. Therefore,

$$P(E_1 \cup E_2) = \frac{5}{36} + \frac{6}{36} = \frac{11}{36}$$

■

If the two events are not disjoint, then following the addition principle, the following obvious rule applies.

$$\boxed{P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)}$$

Example: A pair of dice is rolled. What is the probability that one die is 5? (This includes the case when both are 5.)

$$P(D_1 = 5 \vee D_2 = 5) = P(D_1 = 5) + P(D_2 = 5) - P(D_1 = 5 \wedge D_2 = 5) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$$

Another way to compute this probability is as follows. First observe that the probability that neither die is 5 becomes $25/36$, because each die has 5 possible values. So,

$$P(D_1 = 5 \vee D_2 = 5) = 1 - P(D_1 \neq 5 \wedge D_2 \neq 5) = 1 - \frac{25}{36} = \frac{11}{36}$$

This probability may also be seen by enumerating the entire sample space, as below. Note that (5,5) is removed in the second set of 6, because it is repeated, thus leaving 11 outcomes in the sample space.

$D_1 D_2$
51
52
53
54
55
56
15
25
35
45
~~55~~
65

Example: A coin is flipped 4 times. What is the probability of exactly two heads (and two tails)? The sample space has $2^4 = 16$ possible outcomes. The number of outcomes with exactly two heads equals the number of subsets of size 2 out of 4, which is $C(4,2) = 6$. So the probability of exactly 2 heads is

$$\frac{C(4,2)}{16} = \frac{6}{16} = \frac{3}{8}$$

We may also list all 16 outcomes to see that 6 of them have exactly two heads.

Row	$F_1 F_2 F_3 F_4$	Number of Heads
0	T T T T	0
1	T T T H	1
2	T T H T	1
3	T T H H	2
4	T H T T	1
5	T H T H	2
6	T H H T	2
7	T H H H	3
8	H T T T	1
9	H T T H	2
10	H T H T	2
11	H T H H	3
12	H H T T	2
13	H H T H	3
14	H H H T	3
15	H H H H	4

Let us compare the above probability with the probability of exactly k heads, for different values of k.

k	$C(4,k)$	Probability of k heads
0	1	1/16
1	4	4/16
2	6	6/16
3	4	4/16
4	1	1/16

The probability is the highest for $k = 2$, as expected. That is, if we flip a coin n times, the probability of exactly k heads is the highest for $k = n/2$. This means that it is most likely that half of the outcomes are head and half tail, which basically follows from the definition of probability. ■

Product Rule

Consider a random flip of a coin. It has two outcomes: head and tail, each with the probability 1/2. Now consider two flips of a coin. The outcome of the first flip does not change the probability for the second flip. So we say the two events are independent.

What is the probability that both flips are head? There are 4 possible outcomes in the sample space. (Each flip has 2 possible outcomes. So, by the multiplication principle, the number of outcomes for two flips is $2 \times 2 = 4.$) The 4 outcomes have equal probability 1/4. So, the probability that a head occurs on both flips is 1/4.

Flip1 Flip2
H H
H T
T H
T T

Another way is by using the product rule for independent events. Let H_1 be the event that a head occurs on the first flip, and H_2 the event that a head occurs on the second flip.

$$P(H_1 \; and \; H_2) = P(H_1) * P(H_2) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

Let us state the rule formally.

Product Rule: Given two independent events E_1 and E_2,

$$P(E_1 \cap E_2) = P(E_1) * P(E_2)$$

Note: $E_1 \cap E_2$ may also be interpreted as $E_1 and\ E_2$.

Example: Consider an experiment where we keep flipping a coin until we get a head.

(a) What is the probability that the first head occurs on the 3rd flip?
(b) What is the probability that the first head occurs on the k^{th} flip?
(c) What is the **expected** number of flips before the first head occurs?

Solution:

(a) Probability of getting the first head on the 3rd flip is found by using the product rule.

$$P(T_1 \wedge T_2 \wedge H_3) = P(T_1) * P(T_2) * P(H_3) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}.$$

(b) Probability of getting the first head on the k^{th} flip is similarly found by the product rule.

$$P(\text{first head occurs on } k^{th} \text{ flip}) = P(T_1 \wedge T_2 \wedge \cdots \wedge T_{k-1} \wedge H_k) = \left(\frac{1}{2}\right)^k = \frac{1}{2^k}$$

(c) The expected number of flips before getting the first head is

$$\sum_{k=1}^{\infty} k * P(\text{first head occurs on } k^{th} \text{ flip}) = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

The above summation has a value of 2. To prove this sum value, let us first consider the sum for the finite series. We leave it to the student to prove by induction the following formula.

$$S(n) = \sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Then, it is immediate that

$$\lim_{n \to \infty} S(n) = 2.$$

This result (expected value of 2) also follows directly from the definition of probability. Since the probability of a head on a single flip is 1/2, then it follows that on the average, it takes two flips before getting the first head.

Conditional Probability

We now consider the case when the outcome of one event effects the probability of another event. That is, when the events are not independent. As a very simple example, suppose a pair of dice is rolled. The probability that the sum is 11 is $2/36$, since there are two outcomes {56, 65} with the sum $= 11$. But suppose we already know the first die is 6. This knowledge changes the probability to $1/6$ because the second die has to be 5, which is one of six possible outcomes. The following example will further motivate the concept of conditional probability, and the formula for it.

Example: A pair of dice is rolled. Suppose we know that at least one die is 6. Compute the probability that the sum of the two is ≥ 9, **given that** one or both die is 6. This is called **conditional probability,** and it is written as:

$$P(sum \geq 9 \mid one\ die\ is\ 6).$$

(The vertical line | is read as “given that”.)

Solution: The original unconditional sample space, S, has 36 outcomes. The sum value for each outcome is shown below. Since we know that one die is 6, the sample space becomes limited to only the last row and last column, as shaded, which has 11 outcomes. Let us call this the conditional sample space S'.

	$D_2 = 1$	$D_2 = 2$	$D_2 = 3$	$D_2 = 4$	$D_2 = 5$	$D_2 = 6$
$D_1 = 1$	2	3	4	5	6	7
$D_1 = 2$	3	4	5	6	7	8
$D_1 = 3$	4	5	6	7	8	9
$D_1 = 4$	5	6	7	8	9	10
$D_1 = 5$	6	7	8	9	10	11
$D_1 = 6$	7	8	9	10	11	12

There are 7 outcomes (out of 11) in sample space S' with the sum ≥ 9, as highlighted in blue. Therefore, the conditional probability of this event is the ratio $7/11$.

$$P(sum \geq 9 \mid \text{one die is } 6) = \frac{7}{11}$$

Now, observe that the 7 highlighted outcomes in the original unconditional sample space S constitute the event

$$(sum \geq 9) \cap (\text{one die is } 6)$$

This event has the probability of 7/36. Therefore,

$$P(sum \geq 9 \mid \text{one die is } 6) = \frac{P((sum \geq 9) \cap (\text{one die is } 6))}{P(\text{one die is } 6)} = \frac{7/36}{11/36} = \frac{7}{11}$$

The above example motivates the formula for the general case.

Conditional Probability: The probability of event A, given that event B has occurred, is

$$P(A\,|B) = \frac{P(A \cap B)}{P(B)}$$

Therefore,

$$\begin{aligned} P(A \cap B) &= P(A\,|B) * P(B) \\ &= P(B \mid A) * P(A) \end{aligned}$$

If events A and B are independent, then $P(A\,|B) = P(A)$, and the above formula becomes

$$P(A \cap B) = P(A) * P(B)$$

which is the product rule stated earlier for independent events.

Example: A pair of dice is rolled. Compute the probability of each event.

(a) Sum ≥ 8.
(b) One die is 6. (This includes the case when both are 6.)
(c) Sum ≥ 8 and one die is 6.
(d) Sum ≥ 8, given that one die is 6. (Use the formula for conditional probability)
(e) One die is 6, given that Sum ≥ 8. (Use the formula for conditional probability)

Solution:

(a) Sum ≥ 8: The following table shows all 36 outcomes in the sample space.

	$D_2 = 1$	$D_2 = 2$	$D_2 = 3$	$D_2 = 4$	$D_2 = 5$	$D_2 = 6$
$D_1 = 1$	2	3	4	5	6	7
$D_1 = 2$	3	4	5	6	7	8
$D_1 = 3$	4	5	6	7	8	9
$D_1 = 4$	5	6	7	8	9	10
$D_1 = 5$	6	7	8	9	10	11
$D_1 = 6$	7	8	9	10	11	12

There are 15 outcomes with the sum ≥ 8, as shaded in pink. (There are 5 outcomes with the sum equal 8, four outcomes with the sum 9, three outcomes with sum 10, two outcomes with sum 11, and 1 with sum 12.) So,

$$P(Sum \geq 8) = \frac{5+4+3+2+1}{36} = \frac{15}{36}$$

(b) One die is 6: There are 11 outcomes in this event (the last row and last column).

$$P(\text{one die is } 6) = \frac{11}{36}$$

(c) Sum ≥ 8 and one die is 6: There are 9 outcomes in this event, as highlighted in blue.

$$P(\text{Sum } \geq 8 \text{ and one die is } 6) = \frac{9}{36}$$

(d) Sum ≥ 8, given that one die is 6.

$$P(\text{Sum } \geq 8 \mid \text{one die is } 6) = \frac{P(\text{Sum } \geq 8 \text{ and one die is } 6)}{P(\text{one die is } 6)} = \frac{9/36}{11/36} = \frac{9}{11}$$

(e) One die is 6, given that Sum ≥ 8.

$$P(\text{One die is } 6 \mid \text{Sum } \geq 8) = \frac{P(\text{Sum } \geq 8 \text{ and one die is } 6)}{P(Sum \geq 8)} = \frac{9/36}{15/36} = \frac{9}{15}$$

Example (Conditional probability of full house): A poker hand is a drawing of 5 cards out of a deck of 52, and *full-house* is a poker hand with 3 cards out of one kind, and 2 cards out of another kind. Compute the probability of each of the following.

(a) A full-house (unconditional case)
(b) A full-house, given that the first two face-up cards are 2 aces. (The two aces may be any two suits. Or, we may assume two specific suits such as hearts and diamonds.)
(c) A full-house, given that the first two cards are (ace of hearts & king of diamonds).

Solution:

(a) The size of sample space for all poker hands is

$$C(52,5) = \frac{52!}{5!\,(47!)} = \frac{52*51*50*49*48}{5*4*3*2*1} = 2{,}598{,}960$$

The number of possible full-house hands is

$$P(13,2)*C(4,3)*C(4,2) = 13*12*4*6 = 3744$$

Therefore,

$$P(\text{full house}) = \frac{3744}{2{,}598{,}960} \cong \frac{1}{694}$$

(It is common to express the probability as a ratio of 1 over the nearest integer.)

(b) Now let us compute the conditional probability. The size of the sample space for the conditional case is the number of ways to pick 3 more cards out of 50.

$$C(50,3) = \frac{50*49*48}{3*2*1} = 19{,}600$$

After the initial 2 aces, there are two ways to make a full-house:

- Get one more ace, and two cards out of another kind. The number of ways is
$$C(2,1)*C(12,1)*C(4,2) = 2*12*6 = 144$$
- Get 3 cards out of another kind. The number of ways is
$$C(12,1)*C(4,3) = 12*4 = 48$$

By the addition principle, the total number of ways is: $144 + 48 = 192$. So, the conditional probability becomes:

$$P(\text{full house} \mid \text{first 2 aces}) = \frac{192}{19600} \cong \frac{1}{102}$$

Using the Formula for Conditional-Probability

Now, let us find the above conditional probability by using the formula. First, we assume the first two aces may be any two suits.

$$P(\text{full house} \mid \text{first 2 aces}) = \frac{P(\text{first 2 aces} \cap \text{full house})}{P(\text{first 2 aces})}$$

$$P(\text{first 2 aces}) = \frac{C(4,2)}{C(52,2)}$$

$$P(\text{first 2 aces } \cap \text{full house}) = \frac{C(4,2) * 192}{C(52,2) * C50,3)}$$

The numerator in the latter formula is the number of ways to start with two aces of any two suits, times the number of ways to finish with a full house after the initial 2 aces. And the denominator is the size of the sample space because the initial 2 cards and the latter 3 cards are two separate groups. Therefore,

$$P(\text{full house} \mid \text{first 2 aces}) = \frac{[C(4,2) * 192]/[C(52,2) * C(50,3)]}{C(4,2)/C(52,2)} = \frac{192}{C(50,3)} \cong \frac{1}{102}$$

Now, suppose we assume the two aces are two specific suits such as hearts and diamonds. Then the term $C(4,2)$ in both places in the above computation is replaced by one, since there is one way of getting two aces of specific suits. But the final conditional probability still comes out the same. This shows that the suits of the initial two aces do not matter.

(c) A full-house, given that the first two cards are (ace of hearts, king of diamonds). There are two ways to make a full-house after these two cards:

- Get two more aces and one more king, or
- Get two more kings and one more ace.

The number of ways for each of these is: $C(3,2) * C(3,1) = 3 * 3 = 9$. By the addition principle the total number of ways is: $9 + 9 = 18$. So, the conditional probability is

$$P(\text{full house} \mid \text{first 2 are ace of hearts \& king of diamonds}) = \frac{18}{C(50,3)} \cong \frac{1}{1089}$$

It is interesting to compare these three probabilities. The unconditional probability of a full-house is $1/694$. In case (b), when we know the first two cards are two aces, it becomes much more likely to end up with a full-house. (The conditional probability becomes $1/102$.) But in the last case, when the first two cards are an ace & king, it is much less likely to get a full-house, as the probability becomes $1/1089$. ■

Example (Detecting the HIV Virus): This example is an interesting application of conditional probability formula.

Suppose 10% of patients in a particular clinic have the HIV virus. A test is used to detect the HIV virus in patients. The result of this test is *positive* for 98% of patients who have the HIV virus. Unfortunately, this test is also positive on about 1% of patients who do not have the HIV virus. What is the probability that a patient has the HIV virus if the test is positive?

Solution: Let H denote the set of patients who have the HIV virus (and $\overline{H}$ those who do not have the virus). And let T denote the set of patients who test positive. Then

$$P(H) = 0.10, \qquad P(\overline{H}) = 0.90, \qquad P(T|H) = 0.98, \qquad P(T|\overline{H}) = 0.01$$

And the probability that a patient has the HIV virus if he/she tests positive is $P(H|T)$. First, use the conditional probability formula to compute

$$P(T \cap H) = P(T|H) * P(H) = 0.98 * 0.10 = 0.098$$
$$P(T \cap \overline{H}) = P(T|\overline{H}) * P(\overline{H}) = 0.01 * 0.90 = 0.009$$

Now, $T = (T \cap H) \cup (T \cap \overline{H})$. Since $(T \cap H)$ and $(T \cap \overline{H})$ are mutually exclusive (disjoint),

$$\begin{aligned} P(T) &= P\big((T \cap H) \cup (T \cap \overline{H})\big) \\ &= P(T \cap H) + P(T \cap \overline{H}) \\ &= 0.098 + 0.009 = 0.107 \end{aligned}$$

Now compute $P(H|T)$.

$$P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{0.098}{0.107} = 0.9159$$

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The above example is a special case of Bayes' Theorem, as stated below.

Bayes' Theorem: Let: $C_1, C_2, \cdots, C_n$ be n mutually exclusive classes, and let F be a "feature set" used to classify each object into one of the classes. (Each object is classified into the class which has the highest conditional probability.)

$$P\left(C_j \middle| F\right) = \frac{P(F \cap C_j)}{\sum_{i=1}^{n} P(F \cap C_i)} = \frac{P\left(F \middle| C_j\right) P(C_j)}{\sum_{i=1}^{n} P(F|C_i)\, P(C_i)}$$

Proof: Since every object is classified into one class, $U = (C_1 \cup C_2 \cup \cdots \cup C_n)$. Thus, $F = F \cap U = F \cap (C_1 \cup C_2 \cup \cdots \cup C_n) = (F \cap C_1) \cup (F \cap C_2) \cup \cdots \cup (F \cap C_n)$. So, the above denominator becomes $P(F)$, and the equation becomes the correct formula for conditional probability.

Example: For the above example of HIV virus detection, find the probability that a patient has the HIV virus if he/she does not test positive, $P(H|\bar{T})$.

Solution:

$$P(H|\bar{T}) = \frac{P(\bar{T} \cap H)}{P(\bar{T})} = \frac{P(\bar{T} \cap H)}{P(\bar{T} \cap H) + P(\bar{T} \cap \bar{H})} = \frac{P(\bar{T}|H)P(H)}{P(\bar{T}|H)P(H) + P(\bar{T}|\bar{H})P(\bar{H})}$$

$$= \frac{(1-0.98)(0.1)}{(1-0.98)(0.1) + (1-0.01)(0.90)} = \frac{0.002}{0.002 + 0.891} = \frac{0.002}{0.893} = 0.00224$$

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Example (Black Friday Sale): The day after Thanksgiving, Best Buy has a TV sale, and has two salespersons working, Mary and Larry. Mary makes 75% of total sale, and Larry 25%. On the following Monday, 30% of all sale made by Mary are returned, and 10% of sale made by Larry are returned. (This is summarized in the following table.)

	Mary	Larry
Percent Sale	75	25
Percent Returns	30	10

Let M denote the event that Mary made the sale, and L the event that Larry made the sale. And let R denote the event that a sale item is returned. Find the probability that Mary made the sale on any returned item, $P(M|R)$.

Solution:

$$P(M) = 0.75, \qquad P(L) = 0.25, \qquad P(R|M) = 0.30, \qquad P(R|L) = 0.10$$

Recall $R = R \cap U = R \cap (M \cup L) = (R \cap M) \cup (R \cap L)$. So,

$$P(R) = P(R \cap M) + P(R \cap L)$$

Now,

$$P(M|R) = \frac{P(R \cap M)}{P(R)} = \frac{P(R|M)P(M)}{P(R|M)P(M) + P(R|L)P(L)}$$

$$= \frac{0.30 * 0.75}{0.30 * 0.75 + 0.10 * 0.25} = \frac{0.225}{0.225 + 0.025} = \frac{0.225}{0.250} = 0.90$$

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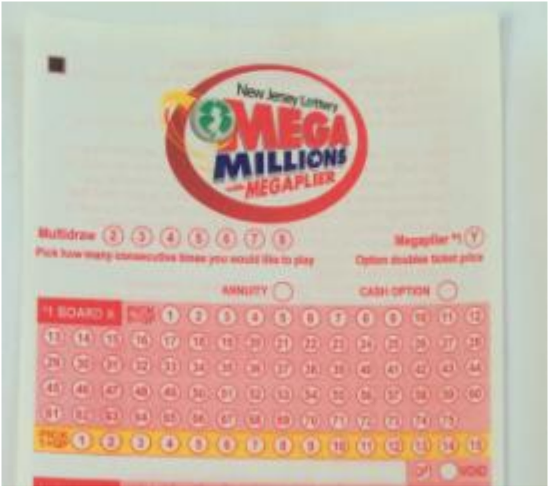
Lottery

New Jersey Mega Millions Lottery is played as follows. Each ticket consists of two sets of numbers: (A picture is attached.)

- Upper Set: Numbers 1 through 75.
  You pick 5 distinct numbers in this group. These numbers are called White Balls.
- Lower Set: Numbers 1 through 15.
  You pick 1 number (called Mega Ball) in this group.



On the day of the drawing, the winning numbers are announced (5 White-Ball numbers, and one Mega Ball number). The prizes are listed below, together with the odds of winning each prize.

Match	**Prize**	**Odds 1:x**
Match 5 white-balls, and Mega Ball	Jackpot	258,890,850
Match 5 white balls	$1,000,000	18,492,204
Match 4 White Balls, and Mega Ball	$5000	739,688
Match 4 White Balls	$500	52,835
Match 3 White Balls, and Mega Ball	$50	10,720
Match 3 White Balls	$5	766
Match 2 White Balls, and Mega Ball	$5	473
Match 1 White Balls, and Mega Ball	$2	56
Match 0 White Balls, and Mega Ball	$1	21

We will show the computation for a few of these and leave the rest for the student as exercise.

Sample Space: The size of the sample space is the total number of distinct tickets.

$$U = C(75{,}5) * C(15{,}1) = \frac{75 * 74 * 73 * 72 * 71}{5 * 4 * 3 * 2 * 1} * 15 = 258{,}890{,}850$$

Match 5 white-balls and Mega Ball (Jackpot):

To win the jackpot, your white-ball numbers must match all 5 of the winning white-balls, and your mega-ball number must match the winning mega-ball. There is only one possible way for this to happen.

$$W = C(5{,}5) * C(1{,}1) = 1$$

So the probability of winning the jackpot is:

$$P = \frac{W}{U} = \frac{1}{258{,}890{,}850}$$

Match 3 White Balls, and Mega Ball:

The drawing announces 5 winning white balls, and the remaining $(75 - 5)$ are non-winning white-ball numbers. Similarly, there is one winning mega ball, and the remaining $(15 - 1)$ non-winning mega ball numbers. To win this prize category, your ticket must have 3 out of 5 winning white balls, and 2 out of 70 non-winning white balls. And your mega-ball number must match the winning mega-ball. The number of ways is

$$W = C(5, 3) * C(70, 2) * C(1{,}1) = \frac{5 * 4}{2 * 1} * \frac{70 * 69}{2 * 1} * 1 = 10 * 2415 = 24{,}150$$

The probability is

$$P = \frac{W}{U} = \frac{24{,}150}{258{,}890{,}850} \cong \frac{1}{10{,}720}$$

Match 3 White Balls:

To win this prize, your white balls must match 3 out of 5 winning white-balls, and 2 out of 70 non-winning white-balls. And your mega-ball must be 1 of the 14 non-winning mega-balls. Therefore,

$$W = C(5{,}3) * C(70{,}2) * C(14{,}1) = 10 * 2415 * 14 = 338{,}100$$

$$P = \frac{W}{U} = \frac{338{,}100}{258{,}890{,}850} \cong \frac{1}{766}$$

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Poker Hands

We have looked at some of poker hands in our earlier examples. Here, we list all poker hands in ranked order. We show the computation of the probabilities for a select few, and leave the rest for the student as exercise.

A poker hand is a random drawing of 5 cards out of a deck of 52 cards (with no jokers). There are 4 *suits,* and 13 *ordered kinds* (ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3, 2). An ace may be counted as either the highest (above king) or lowest (as 1).

Below is the list of all poker hands in ranked order. Each hand is defined so that it does not include any of the higher hands. For example, a *straight* poker hand is five cards in sequence, but not all in the same suite. (Otherwise, it would become straight flush.) For the list of poker hands with illustrations, please see:

https://en.wikipedia.org/wiki/List_of_poker_hands

	Poker Hand	**# Ways (W)**	**Prob. (P)**
1	Straight Flush Five cards of same suit and in consecutive order	40	$1/64974$
2	Four-of-a-Kind (4,1) Four of one kind, and 1 of another kind	624	$1/4165$
3	Full House (3,2) 3 of one kind, and 2 of another kind	3,744	$1/694$
4	Flush Five cards in the same suit, but not all in sequence	5,108	$1/509$
5	Straight Five cards all in sequence, but not all in same suit	10,200	$1/255$
6	Three-of-a-Kind (3,1,1) 3 of one kind, 1 of another kind, and 1 of a third kind	54,912	$1/47$
7	Two-Pairs (2,2,1) 2 of one kind, 2 of another kind, and 1 of a third kind	123,552	$1/21$
8	One Pair (2,1,1,1) Two of one kind, and one from each of 3 other kinds	1,098,240	$1/2.36$
9	High Card (None of the above hands) Five different kinds, not all same suit, and not all in order	1,302,540	$1/1.99$
	Total	2,598,960	1.00

The number of ways for each poker hand is listed in the table, and the sum of these values equals the total sample size, $C(52,5)$. This provides a good check that the computations are done correctly. But the probabilities have been mostly truncated to 1/x, where x is the nearest integer. For this reason, sum of the probabilities cannot be used as a check. (They add up to $1.00 \cdots$.)

The sample space size is

$$U = C(52,5) = \frac{52 * 51 * 50 * 49 * 48}{5 * 4 * 3 * 2 * 1} = 2{,}598{,}960$$

Below is the computation for a few of the hands. (Earlier we showed the computation for full-house and two-pairs.)

Straight Flush (SF):

Since five cards must be in sequence, there is 10 possible values (kinds) for the highest card in the hand {ace, king, queen, jack, 10, 9, 8, 7, 6, 5}.   For example, if the high card is jack, the hand is (jack, 10, 9, 8, 7). If the high card is 5, the hand is (5, 4, 3, 2, ace). But the high card cannot be any of {4, 3 , 2}. And the five cards must be all the same suit, so there are 4 possible suits. Thus, the total number of ways W and the probability P for a straight flush is

$$W(SF) = C(10,1) * C(4,1) = 10 * 4 = 40$$

$$P(SF) = \frac{W}{U} = \frac{40}{2{,}598{,}960} \cong \frac{1}{64{,}974}$$

Flush:

Five cards must be the same suit, and five different kinds but not in sequence. This means the five cards can be any 5 kinds, except for the 10 that makes straight. So,

$$W(flush) = C(4,1) * [C(13,5) - 10] = 4 * \left[\frac{13 * 12 * 11 * 10 * 9}{5 * 4 * 3 * 2 * 1} - 10\right] = 5108$$

$$P(flush) = \frac{5108}{2{,}598{,}960} \cong \frac{1}{509}$$

One Pair:

Pick 1 out of 13 kinds, and choose a pair out of it. Then make an *unordered* selection of 3 other kinds, and pick 1 card out of each kind.

$$W(1\, pair) = C(13,1) * C(4,2) * C(12,3) * C(4,1) * C(4,1) * C(4,1)$$
$$= 13 * 6 * \frac{12 * 11 * 10}{3 * 2 * 1} * 4 * 4 * 4 = 1{,}098{,}240$$
$$P(1\, pair) = \frac{1{,}098{,}240}{2{,}598{,}960} \cong \frac{1}{2.36}$$

We leave the computation of the remaining hands to the student as an exercise.

■