Module 8: Classes of Recurrence Equations

Reading from the Textbook: Chapter 7 Recurrences

Introduction

In the last module, we studied several divide-and-conquer algorithms and the recurrence equations for their analysis. In this module, we derive the solution of a more general class of recurrence equations which commonly arise from divide-and-conquer algorithms. The solution form of this class of recurrences is known as the Master-Theorem. We also study the class of linear recurrences. The well-known Fibonacci sequence is an example of a linear recurrence. We will find the exact solution of this sequence and relate the solution to the famous Golden Ratio, which is claimed to appear in many places in nature!

Contents

Divide-and-Conquer Recurrences

The following recurrence is a general form of recurrences which commonly arise in divide-and-conquer algorithms. ($a, b, c, d,$ and β are constants determined by the specific algorithm, and $n = b^k$ for some integer *k*.)

$$T(n) = \begin{cases} a \cdot T\left(n/_b\right) + c \cdot n^\beta, & n > 1 \\ d, & n = 1 \end{cases}$$

This recurrence may correspond to a divide-and-conquer algorithm which divides a problem of size n into $'a'$ subproblems, each of size $n/_b$. The term $T\left(n/_b\right)$ represents the time for solving each subproblem. And the term cn^β represents the additional work for "combining" the solutions of the subproblems to find the overall solution. For example, the recurrence for Mergesort is $T(n) = 2T\left(n/_2\right) + cn$. The term $T\left(n/_2\right)$ is the time for sorting each half of the array, and cn is the time for merging the two sorted halves.

We will use repeated-substitution to find the general form of the solution.

$$T(n) = cn^\beta + a \cdot T\left(n/_b\right)$$
$$= cn^\beta + a \cdot \left[c\left(n/_b\right)^\beta + aT\left(n/_{b^2}\right)\right]$$
$$= cn^\beta + cn^\beta\left(\frac{a}{b^\beta}\right) + a^2\, T\left(n/_{b^2}\right)$$
$$= cn^\beta + cn^\beta\left(\frac{a}{b^\beta}\right) + a^2 \cdot \left[c\left(n/_{b^2}\right)^\beta + aT\left(n/_{b^3}\right)\right]$$
$$= cn^\beta + cn^\beta\left(\frac{a}{b^\beta}\right) + cn^\beta\left(\frac{a^2}{(b^2)^\beta}\right) + a^3\, T\left(n/_{b^3}\right)$$ Use the equality $(b^2)^\beta = (b^\beta)^2$

$$= cn^\beta + cn^\beta\left(\frac{a}{b^\beta}\right) + cn^\beta\left(\frac{a^2}{(b^\beta)^2}\right) + a^3\, T\left(n/_{b^3}\right)$$ Factor out cn^β

$$= cn^\beta\left[1 + \frac{a}{b^\beta} + \left(\frac{a}{b^\beta}\right)^2\right] + a^3\, T\left(n/_{b^3}\right)$$
$\vdots$
$$= cn^\beta\left[1 + \frac{a}{b^\beta} + \left(\frac{a}{b^\beta}\right)^2 + \cdots + \left(\frac{a}{b^\beta}\right)^{k-1}\right] + a^k T\left(n/_{b^k}\right)$$ Recall $n = b^k$

$$= cn^\beta\left[1 + \frac{a}{b^\beta} + \left(\frac{a}{b^\beta}\right)^2 + \cdots + \left(\frac{a}{b^\beta}\right)^{k-1}\right] + a^k T(1)$$ Use $T(1) = d$

And use the equality

$$a^k = a^{\log_b n} = n^{\log_b a}$$

(The reason for the equality $a^{\log_b n} = n^{\log_b a}$ is seen if we take the log of both sides, which becomes $\log_b n \log_b a$.) Let $h = \log_b a$. Then,

$$T(n) = cn^\beta \left[1 + \frac{a}{b^\beta} + \left(\frac{a}{b^\beta}\right)^2 + \cdots + \left(\frac{a}{b^\beta}\right)^{k-1}\right] + dn^h$$

We now consider two cases, depending on whether the ratio $\frac{a}{b^\beta}$ equals 1 or not.

1. The ratio $\frac{a}{b^\beta} \neq 1$, which means $a \neq b^\beta$, or $\log_b a \neq \beta$. That is, $h \neq \beta$.
   In this case, the summation inside the brackets is geometric sum. So,

$$T(n) = cn^\beta \cdot \frac{\left(\frac{a}{b^\beta}\right)^k - 1}{\left(\frac{a}{b^\beta}\right) - 1} + dn^h$$
$$= \frac{c}{\left(\frac{a}{b^\beta}\right) - 1} n^\beta \left[\left(\frac{a}{b^\beta}\right)^k - 1\right] + dn^h$$

   Recall $a^k = a^{\log_b n} = n^{\log_b a} = n^h$, and $\left(b^\beta\right)^k = (b^k)^\beta = n^\beta$. So,

$$T(n) = \frac{c}{\left(\frac{a}{b^\beta}\right) - 1} n^\beta \left[\frac{n^h}{n^\beta} - 1\right] + dn^h$$
$$= \frac{c}{\left(\frac{a}{b^\beta}\right) - 1} \left(n^h - n^\beta\right) + dn^h$$
$$= \left[\frac{c}{\left(\frac{a}{b^\beta}\right) - 1} + d\right] \cdot n^h + \left[\frac{-c}{\left(\frac{a}{b^\beta}\right) - 1}\right] \cdot n^\beta$$

   Therefore,

$$\boxed{T(n) = A \cdot n^h + B \cdot n^\beta}$$

   where *A* and *B* are the constants as derived.
   The solution is a linear sum of two terms: n^h and n^β. The first term n^h comes from the term $a \cdot T\left({}^{n}/_{b}\right)$ in the recurrence. And the second term n^β comes from $c \cdot n^\beta$ in the recurrence.
2. The ratio $\frac{a}{b^\beta} = 1$, which means $a = b^\beta$, or $\log_b a = \beta$. That is, $h = \beta$.
   In this case, the summation inside the brackets is $[1 + 1 + \cdots + 1] = k = \log_b n$.

$$T(n) = cn^\beta \log_b n + dn^h$$
$$= cn^h \log_b n + dn^h \qquad (\text{since } h = \beta)$$

   Therefore,

$$\boxed{T(n) = A \cdot n^h \log_b n + B \cdot n^h}$$

   where *A* and *B* are the constants as derived.

Let us summarize the solution we derived for the above class of divide-and-conquer recurrences.

Summary (Master Theorem): Given the recurrence equation

$$T(n) = \begin{cases} a \cdot T\left({}^{n}/_{b}\right) + c \cdot n^{\beta}, & n > 1 \\ d, & n = 1 \end{cases}$$

Let $h = \log_b a$. The solution has the following general forms and bounds. (*A* and *B* are some constants for each case.)

$$T(n) = \begin{cases} An^h + Bn^\beta & = \Theta(n^h), & h > \beta \\ An^h + Bn^\beta & = \Theta\left(n^\beta\right), & h < \beta \\ An^h \log n + Bn^h & = \Theta(n^h \log n), & h = \beta \end{cases}$$

Note: This is a slightly simpler version of what is known as the ***Master Theorem*** in many textbooks on algorithms.

We derived the values of the constants *A* and *B* in terms of the given constants $a, b, c, d,$ and β. However, the values of the constants *A* and *B* need not be memorized. Normally, we may be interested only in the order of the solution. And if the exact value of the solution is desired, the constants may be easily computed as illustrated in the following examples.

Example 1: Finding Max of an array by divide-and-conquer

As discussed earlier, the recurrence for this algorithm for the case when $n = 2^k$ is:

$$f(n) = \begin{cases} 0, & n = 1 \\ 2f(n/2) + 1, & n \geq 2 \end{cases}$$

Let us apply our Master Theorem to find the solution form.

$$a = 2, b = 2, \beta = 0$$
$$h = \log_2 2 = 1$$

Since $h \neq \beta$, the solution form is

$$f(n) = An^h + Bn^\beta$$
$$= An + B$$

Finding the constants A and B

If this solution form was an initial "guess", we would apply induction to prove it correct and find the constants. But in this case, since we know the solution form is correct, we don't need to apply induction. Instead, we may use an easier method of simply plugging in two values for n.

$n = 1$:

$$f(1) = 0 \quad \text{(from the recurrence}$$
$$= A + B \quad \text{(from the solution form)}$$

$n = 2$:

$$f(2) = 2f(1) + 1 = 2 * 0 + 1 = 1$$
$$= 2A + B$$

So we have two equations:

$$\begin{cases} A + B = 0 \\ 2A + B = 1 \end{cases}$$

We find the constants: $A = 1,\ B = -1$. Therefore,

$$\boxed{f(n) = n - 1}$$

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Example 2: Binary Search Algorithm (Special case $n = 2^k$)

As we saw, the recurrence equation for the number of key-comparisons in this algorithm is

$$f(n) = \begin{cases} 1, & n = 1 \\ 1 + f\left(\frac{n}{2}\right), & n \geq 2 \end{cases}$$

By Master Theorem:

$$a = 1, b = 2, \beta = 0$$
$$h = \log_2 1 = 0$$

Since $h = \beta,$ the solution form is:

$$f(n) = An^h \log_2 n + Bn^h$$
$$= A \log n + B$$

To find the constants A and B, use $n = 1$ and $n = 2$.

$$f(1) = 1$$
$$= A \log 1 + B = B$$

$$f(2) = 1 + f(1) = 1 + 1 = 2$$
$$= A \log 2 + B = A + B$$

So we have two equations: $B = 1,\ A + B = 2,$ which give: $A = 1,\ B = 1.$ Therefore,

$$\boxed{f(n) = \log n + 1}$$

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Example 3: Mergesort (Special case $n = 2^k$)

As we saw, the recurrence equation for this algorithm is

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + c\,n, & n \ge 2 \\ d, & n = 1 \end{cases}$$

We find the solution by use of the Master Theorem:

$$a = 2, b = 2, \beta = 1$$
$$h = \log_2 2 = 1$$

Since $h = \beta$, the solution form is

$$T(n) = An^h \log_2 n + Bn^h$$
$$= An \log n + Bn$$

We find the constants $A \text{ and } B$ quickly by using two values $n = 1,\ n = 2$.

$$T(1) = d$$
$$= An \log 1 + B = B$$

$$T(2) = 2T(1) + c * 2 = 2d + 2c$$
$$= A * 2 \log 2 + B * 2 = 2A + 2B$$

So we have two equations: $B = d,\ \ A + B = c + d$.

Solving them gives: $B = d,\ A = c$. Therefore,

$$\boxed{T(n) = cn \log n + dn}$$

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Example 4: Strassen's Matrix Multiplication Algorithm

In one of the earlier modules, we discussed a straightforward way of multiplying two matrices of size $n \times n$, which takes $O(n^3)$ time. Strassen's algorithm is an asymptotically faster and more sophisticated algorithm, which is based on the divide-and-conquer strategy. We will not discuss the details of this algorithm here, but we present the recurrence equation arising from it, and analyze its time complexity.

Let $T(n)$ be the time to multiply two $n \times n$ matrices. The algorithm uses 7 multiplications of submatrices of size $\frac{n}{2} \times \frac{n}{2}$, each with time $T(n/2)$, plus some additions and subtractions of submatrices, which take $O(n^2)$ time.

$$T(n) = \begin{cases} 7\,T(n/2) + cn^2, & n > 1 \\ d, & n = 1 \end{cases}$$

We use the master theorem to find the solution form.

$$a = 7, b = 2, \beta = 2$$
$$h = \log_2 7 \cong 2.8$$

Therefore,

$$T(n) = An^h + Bn^\beta$$
$$= An^{\log_2 7} + Bn^2 = \Theta(n^{\log_2 7})$$

If we want the exact value of the constants, we plug in two values of n to get two equations for A and B.

$$T(1) = d$$
$$= A + B$$

$$T(2) = 7T(1) + 4c = 7d + 4c$$
$$= A2^{\log_2 7} + B2^2 = 7A + 4B$$

So we have:

$$A + B = d$$
$$7A + 4B = 7d + 4c$$

Therefore, $A = d + \frac{4}{3}c$, $B = -\frac{4}{3}c$. So, the exact solution is

$$T(n) = \left(d + \frac{4}{3}c\right) \cdot n^{\log_2 7} - \frac{4}{3}c \cdot n^2$$

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Linear Recurrences

Let $\{F_1, F_2, \cdots, F_n\}$ be a sequence. A linear recurrence expresses F_n as a linear function of several of its predecessors $F_{n-1}, F_{n-2}, \cdots$. Let us start with some simple examples.

Example: Compound Interest

Suppose a saving bank offers yearly interest of 5% compounded annually. Suppose the initial deposit is \$1000, and F_n is the accumulated amount at the end of n years. Then,

$$F_n = \begin{cases} 1000, & n = 0 \\ 1.05 * F_{n-1}, & n \geq 1 \end{cases}$$

It is easy to find the solution of this recurrence by repeated substitution.

$$\begin{aligned} F_n &= 1.05 * F_{n-1} \\ &= 1.05 * 1.05 * F_{n-2} \\ &= (1.05)^3 * F_{n-3} \\ &\vdots \\ &= (1.05)^n * F_0 \\ &= 1000 * (1.05)^n \end{aligned}$$

Example: Towers of Hanoi

We studied the algorithm for this problem earlier. The number of single moves to accomplish moving n disks is expressed by the recurrence equation

$$F_n = \begin{cases} 1, & n = 1 \\ 2F_{n-1} + 1, & n \geq 2 \end{cases}$$

We obtained the solution of this recurrence (by repeated substation) as

$$F_n = 2^n - 1$$

Example: Fibonacci Sequence

The recurrence equation for this well-known sequence is defined as

$$F_n = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ F_{n-1} + F_{n-2}, & n \geq 3 \end{cases}$$

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In general, a homogeneous linear recurrence of order k with constant coefficients has the following form,

$$F_n = c_1 F_{n-1} + c_2 F_{n-2} + \cdots + c_k F_{n-k}$$

together with k initial values for $F_1, F_2, \cdots, F_k$.

For first-order recurrences of the first two examples above, we saw the solution in both cases is exponential form r^n (where $r = 1.05$ and $r = 2$, respectively).

For higher-order recurrences, we start with this same form as an initial trial.

Linear Recurrence of Order 2

Let us find the solution for the following second-order linear recurrence.

$$F_n = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 5F_{n-1} - 6F_{n-2}, & n \geq 2 \end{cases}$$

We first concentrate on the recursive line of the recurrence. (Later, we deal with the initial values for $n = 0, n = 1$.)

$$F_n - 5F_{n-1} + 6F_{n-2} = 0$$

We start with the following initial "guess", where r is a constant to be determined.

$$F_n = r^n$$

Substitute back in the recurrence:

$$\begin{aligned} & r^n - 5r^{n-1} + 6r^{n-2} = 0 \\ & r^{n-2}(r^2 - 5r + 6) = 0 \\ & r^2 - 5r + 6 = 0 \qquad \text{(Characteristic Equation)} \end{aligned}$$

The latter quadratic equation is called the characteristic equation of the recurrence. The roots of the characteristic equation may be found by either factoring out the polynomial, or using the quadratic equation formula.

$$\begin{aligned} & (r-3)(r-2) = 0 \\ & r_1 = 3, \;\; r_2 = 2 \end{aligned}$$

This means that each of the solution forms $F_n = 3^n \text{ and } F_n = 2^n$ satisfies the recursive line of the recurrence equation (but not the initial values for $n = 0, n = 1$).

In fact, a linear sum of the two,

$$F_n = A3^n + B2^n$$

(which is called the *complete solution*) also satisfies the recurrence equation. This is easily verified by substituting the complete solution form back in the recurrence:

$$F_n - 5F_{n-1} + 6F_{n-2} = 0$$

$$(A3^n + B2^n) - 5(A3^{n-1} + B2^{n-1}) + 6(A3^{n-1} + B2^{n-2}) = 0$$

$$A(3^n - 5 * 3^{n-1} + 6 * 3^{n-2}) + B(2^n - 5 * 2^{n-1} + 6 * 2^{n-2}) = 0$$

$$A\,3^{n-2}\left(\underbrace{\underbrace{3^2 - 5 * 3 + 6}_{=0}}_{since\ 3\ is\ a\ root}\right) + B2^{n-2}\left(\underbrace{\underbrace{2^2 - 5 * 2 + 6}_{=0}}_{since\ 2\ is\ a\ root}\right) = 0$$

The quantity inside each parenthesis is 0, because 3 and 2 are the two roots of the characteristic equation ($r^2 - 5r + 6 = 0$). Therefore, the complete solution satisfies the recursive line of the recurrence, as claimed earlier.

It remains to satisfy the initial conditions of the recurrence (for $n = 0 \text{ and } n = 1$). We use these initial values to find the constants $A \text{ and } B$.

Finding the constants A and B

The complete solution form is:

$$F_n = A3^n + B2^n$$

And the two base cases of the recurrence (also called initial values) are $F_0 = 0, F_1 = 1$. So,

$$F_0 = 0 = A + B$$
$$F_1 = 1 = 3A + 2B$$

Solving these two equations for $A \text{ and } B$ gives:

$$A = 1, \qquad B = -1$$

Therefore,

$$\boxed{F_n = 3^n - 2^n}$$

Example: Find the complete solution of the following second-order recurrence, with the initial values $F_0 = 1,\ F_1 = 7$.

$$F_n = 8F_{n-1} - 15F_{n-2}, \quad n \geq 2$$

Substitute the solution $F_n = r^n$.

$$\begin{aligned}
&F_n - 8F_{n-1} + 15F_{n-2} = 0 \\
&r^n - 8r^{n-1} + 15r^{n-2} = 0 \\
&r^{n-2}(r^2 - 8r + 15) = 0 \\
&r^2 - 8r + 15 = 0 \\
&(r-5)(r-3) = 0 \\
&r_1 = 5, \qquad r_2 = 3
\end{aligned}$$

The complete solution is

$$F_n = A\, 5^n + B\, 3^n$$

Use the initial conditions to find A and B:

$$\begin{aligned}
F_0 &= 1 = A + B \\
F_1 &= 7 = 5A + 3B
\end{aligned}$$

Solve for A and B:

$$A = 2, \ \ B = -1$$

$$\boxed{F_n = 2 * 5^n - 3^n}$$

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Example: A recurrence equation of order-3 has the following initial values

$$F_0 = 0, \qquad F_1 = 7, \qquad F_2 = 25$$

and the following characteristic equation (in factored form)

$$(r-3)(r-2)(r-1) = 0$$

(a) Find the complete solution.
(b) Work backward from the characteristic equation to find the recurrence equation.

Solution: Since the characteristic equation is already given in factored form, we know the roots are

$$r_1 = 3, \qquad r_2 = 2, \qquad r_3 = 1$$

So the complete solution is

$$F_n = A\,3^n + B\,2^n + C\,1^n$$
$$F_n = A\,3^n + B\,2^n + C$$

Use the initial conditions to find three equations for the constants.

$$F_0 = 0 = A + B + C$$
$$F_1 = 7 = 3A + 2B + C$$
$$F_2 = 25 = 9A + 4B + C$$

Solve for the constants:

$$A = 2, \qquad B = 3, \qquad C = -5$$

The exact solution is

$$\boxed{F_n = 2 * 3^n + 3 * 2^n - 5}$$

To find the original recurrence equation, we work backward from the characteristic equation.

$$(r-3)(r-2)(r-1) = 0$$
$$(r^2 - 5r + 6)(r-1) = 0$$
$$r^3 - 6r^2 + 11r - 6 = 0$$
$$r^n - 6r^{n-1} + 11r^{n-2} - 6r^{n-3} = 0$$

Therefore,

$$\boxed{F_n = 6F_{n-1} - 11F_{n-2} + 6F_{n-3}}$$

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Repeated Roots

The linear recurrences so far had characteristic equations with distinct roots. We now consider what happens if the roots are not distinct. Consider a second-order linear recurrence, with $F_0 = 2,\ F_1 = 9,$ and

$$F_n = 6F_{n-1} - 9F_{n-2}\,, \quad n \geq 2$$
$$F_n - 6F_{n-1} + 9F_{n-2} = 0$$

Again, let us substitute the solution $F_n = r^n$. Let $P(r^n)$ denote the polynomial obtained when we substitute $F_n = r^n$ in the recurrence.

$$\begin{aligned} P(r^n) &= r^n - 6r^{n-1} + 9r^{n-2} \\ &= r^{n-2}(r^2 - 6r + 9) \\ P(r^n) &= r^{n-2}(r-3)^2 = 0 \\ (r-3)^2 &= 0 \qquad\qquad \text{Reduced Characteristic Equation} \end{aligned}$$

The characteristic equation has two roots, both $r = 3$.

(1) We already know that one solution has the form $F_n = r^n,\ r = 3$.
(2) We now claim that there is a second solution of the form:

$$F_n = n\,r^n, \;\; r = 3.$$

(The proof of this claim uses some calculus, namely the concept of derivative.)

Proof: We will provide the proof in two ways.

First Proof: One way to verify the correctness of the solution is to simply substitute it back in the recurrence equation.

$$P(nr^n) = nr^n - 6(n-1)r^{n-1} + 9(n-2)r^{n-2}$$
$$P(nr^n) = r^{n-2}[nr^2 - 6(n-1)r + 9(n-2)]$$

For $r = 3,$ it is immediately verified that $P(nr^n) = 0$.

$$P(n\,3^n) = 3^{n-2}[9n - 18(n-1) + 9(n-2)] = 0$$

Although this verifies the correctness of the solution, is still leaves us wondering as how we guessed the solution in the first place. We now provide a second proof to provide the needed insight.

Alternative Proof: Since the characteristic polynomial $P(r^n)$ has a square factor, namely $(r-3)^2$, then the derivative of the characteristic equation also has a factor $(r-3),$ thus a root $r = 3$. This is immediately seen from the reduced form.

$$\frac{d}{dr}(r-3)^2 = 2(r-3) = 0, \qquad r = 3$$

It is easy to see that the same is true about the derivative of the earlier factored form, $r^{n-2}(r-3)^2$. Thus, we know that the derivative of the original polynomial form has a root $r = 3$.

$$\frac{d}{dr}(\mathrm{P}(\mathrm{r}^{\mathrm{n}})) = \frac{d}{dr}\,(r^n - 6r^{n-1} + 9r^{n-2})$$
$$= n\,r^{n-1} - 6(n-1)\,r^{n-2} + 9(n-2)\,r^{n-3} = 0 \qquad \rightarrow \;\; r = 3$$

If we multiply both sides of the above equation by $r,$ it will still have a root $r = 3$.

$$n\,r^n - 6(n-1)\,r^{n-1} + 9(n-2)\,r^{n-2} = 0 \qquad \rightarrow \;\; r = 3$$

But this latter equation is exactly what we get by directly substituting the second solution form $F_n = n\,r^n$ into the recurrence equation. That is,

$$P(nr^n) = n\,r^n - 6(n-1)\,r^{n-1} + 9(n-2)r^{n-2} = 0$$

Since we showed that this equation has a root $r = 3,$ it means that

$$F_n = n\,r^n, \quad r = 3$$

is a solution as claimed. ■

Having proved the second solution, we obtain the complete solution as a linear sum of the two solutions, as before:

$$F_n = A\,3^n + Bn\,3^n$$

The constants are found by using the initial conditions.

$$F_0 = 2 = A$$
$$F_1 = 9 = 3A + 3B$$

So, $A = 2,\; B = 1$.

$$\boxed{F_n = 2 * \; 3^n + n\,3^n = (n+2)\,3^n}$$

We will provide some more examples of repeated roots.

Example: Find the complete solution of the following second-order linear recurrence, with the initial conditions $F_1 = 16,\ F_2 = 44$.

$$F_n = 4F_{n-1} - 4F_{n-2}$$

Solution: Substitute the solution $F_n = r^n$.

$$\begin{aligned}
&F_n - 4F_{n-1} + 4F_{n-2} = 0 \\
&r^n - 4\,r^{n-1} + 4\,r^{n-2} = 0 \\
&r^{n-2}(r^2 - 4r + 4) = 0 \\
&r^{n-2}(r-2)^2 = 0 \\
&\text{Roots: } r_1 = r_2 = 2
\end{aligned}$$

Since there are repeated roots, the complete solution has the form:

$$F_n = A\,2^n + Bn\,2^n$$

The constants are found by using the initial conditions.

$$\begin{aligned}
F_1 &= 2A + 2B = 16 \\
F_2 &= 4A + 8B = 44
\end{aligned}$$

So, $A = 5,\ B = 3$.

$$\boxed{F_n = 5\,2^n + 3n\,2^n}$$

Example: Find the general solution form of a linear recurrence of order 3, which has the following factored characteristic equation.

$$(r-5)^2(r-2) = 0$$

Solution: The roots are: $r_1 = r_2 = 5,\ \ r_3 = 2$.

The complete solution has the form:

$$F_n = A\,5^n + Bn\,5^n + C\,2^n$$

The constants (A, B, C) may be computed by using the initial conditions (three base cases), not provided here.

Fibonacci Numbers

Let us now look at the Fibonacci sequence, a second-order linear recurrence, with many applications in computer science and other fields. The sequence is defined as $F_0 = 0,\ F_1 = 1,$ and

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

Below are the tabulated values up to $n = 15$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

To find the solution, we substitute $F_n = r^n$.

$$\begin{aligned} &F_n - F_{n-1} - F_{n-2} = 0 \\ &r^n - r^{n-1} - r^{n-2} = 0 \\ &r^{n-2}(r^2 - r - 1) = 0 \\ &r^2 - r - 1 = 0 \end{aligned}$$

The quadratic equation formula is used to find the roots. [Recall the general formula: given the equation $ar^2 + br + c = 0,$ the roots are $r = \frac{-b \mp \sqrt{(b^2 - 4ac)}}{2}$.]

$$r_1 = \frac{1 + \sqrt{5}}{2} \cong 1.618033989, \qquad r_2 = \frac{1 - \sqrt{5}}{2} \cong -0.618033989$$

The complete solution has the form:

$$F_n = A\, r_1^n + B\, r_2^n$$

The Initial conditions (base cases) are used to find the constants.

$$\begin{aligned} &F_0 = 0 = A + B \\ &F_1 = 1 = A\ \frac{1 + \sqrt{5}}{2} + B\ \frac{1 - \sqrt{5}}{2} \end{aligned}$$

The constants are: $A = {}^{1}/_{\sqrt{5}}\ ,\ B = {}^{-1}/_{\sqrt{5}}$, and the exact solution is:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

It is interesting to note that although F_n is a purely integer function, the closed-form expression for it is so complex-looking!

Approximation of Fibonacci Numbers

Observe that $r_2 \cong -0.618$ is a negative fraction and for large n, the term r_2^n becomes very negligible. (For $n \geq 10$, the absolute value of $r_2^n < 0.008131.$) So, for large n,

$$F_n \cong \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}} (1.618)^n$$

The following table shows how close this approximation is to the actual value of F_n. Also shown is the ratio F_n/F_{n-1}, which gets very close to 1.618 for large n.

n	F_n	$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$	$\frac{1}{\sqrt{5}} (1.618)^n$	Ratio F_n/F_{n-1}
1	1	0.72361	0.72359	
2	1	1.17082	1.17077	1.00000
3	2	1.89443	1.89431	2.00000
4	3	3.06525	3.06499	1.50000
5	5	4.95967	4.95915	1.66667
6	8	8.02492	8.02391	1.60000
7	13	12.98460	12.98269	1.62500
8	21	21.00952	21.00599	1.61538
9	34	33.99412	33.98769	1.61905
10	55	55.00364	54.99208	1.61765
11	89	88.99775	88.97719	1.61818
12	144	144.00139	143.96509	1.61798
13	233	232.99914	232.93552	1.61806
14	377	377.00053	376.88967	1.61803
15	610	609.99967	609.80749	1.61804
16	987	987.00020	986.66852	1.61803
17	1,597	1,596.99987	1,596.42967	1.61803
18	2,584	2,584.00008	2,583.02321	1.61803
19	4,181	4,180.99995	4,179.33156	1.61803
20	6,765	6,765.00003	6,762.15846	1.61803
21	10,946	10,945.99998	10,941.17238	1.61803
22	17,711	17,711.00001	17,702.81692	1.61803
23	28,657	28,656.99999	28,643.15777	1.61803
24	46,368	46,368.00000	46,344.62928	1.61803
25	75,025	75,025.00000	74,985.61017	1.61803

Applications

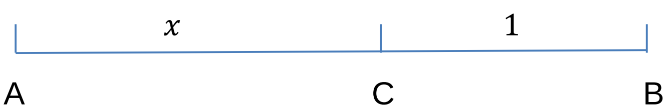
Fibonacci numbers find many applications in computer science. Below are some examples:

- Fibonacci search tree
- Analysis of AVL Tree, which is a balanced search tree. This analysis uses a sequence very close to Fibonacci numbers.
- Time complexity analysis of Euclid's Algorithm for the greatest-common-divisor.

Next, we relate Fibonacci numbers to the famous Golden Ratio, which is claimed to appear throughout nature!

The Golden Ratio

The great ancient mathematician and the founder of Euclidean Geometry, Euclid of Alexandria, apparently introduced a number around 300 B.C. which later became known as the Golden Ratio. This number was introduced by Euclid as a division of a line segment AB into two segments (AC and CB) in such a way that the ratio of the greater segment over the smaller one is the same as the ratio of the whole line segment over the greater one. (Euclid called it "extreme and mean ratio".)



Let x be the length of the larger segment, and 1 be the length of the shorter segment, thus $(x + 1)$ the total length. Then, the division must be such that

$$\frac{x}{1} = \frac{x+1}{x}$$

This produces the quadratic equation

$$x^2 - x - 1 = 0$$

This equation is exactly the same as the characteristic equation for Fibonacci sequence. The larger positive root is known as the Golden Ratio, which is the ratio of the larger segment to the smaller one.

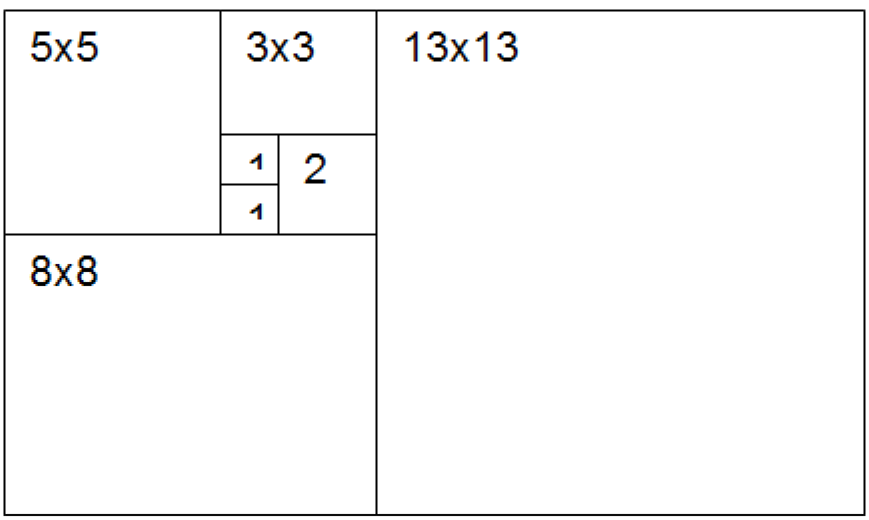
$$\boxed{Golden\ Ratio = \ x = \frac{1+\sqrt{5}}{2} \cong 1.618}$$

Examples in Nature

There is a rich body of evidence that Fibonacci numbers and the Golden Ratio show up many places in nature. Some of this material is mathematically sound and quite convincing. Some of the other examples quoted in the literature are more subjective and open to one's perception.

An interesting source is the book *The Golden Ratio*, by Mario Livio. (Broadway Books, Random House, Inc. 2002, ISBN 0-7679-0816-3) In the introductory parts, the author tries hard to excite the reader, and does not get to the meat of the matter very quickly. But, once you get passed this introductory material, you will find many intriguing examples in this book.

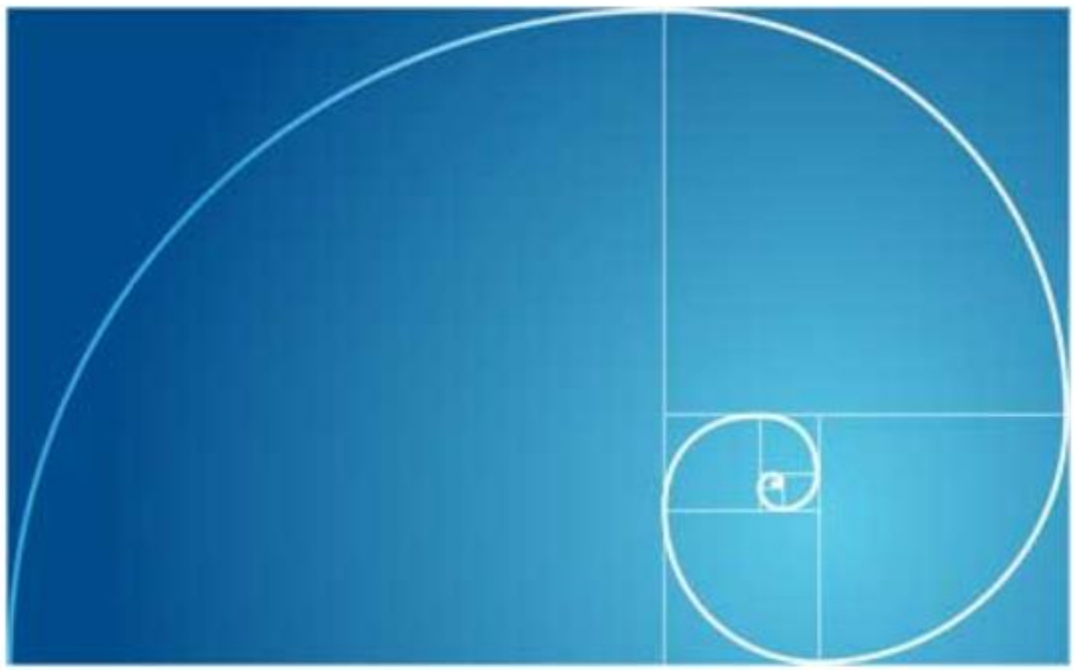
One common example is how squares with sizes that follow Fibonacci numbers $(1,1,2,3,5,8,13,\cdots)$ fit perfectly into spiraling shells. Below is a geometrical arrangement of such squares. Start at the top smallest square of size 1×1 at the center, and move in counterclockwise direction (down, right, up, left, $\cdots$) to get to the next square of size 1×1, then 2×2, then 3×3, then 5, 8, 13, and so on.



A spiraling sea shell with such squares is shown below, copied from the website:

http://science.howstuffworks.com/math-concepts/fibonacci-nature1.htm

The article by Robert Lamb on this website contains numerous examples of where Fibonacci numbers and the Golden Ratio are found in nature. The article starts with an introductory paragraph that many such observations on numbers may be plain coincidence and "numerological superstitions". But the article quickly moves on to say that Fibonacci numbers appear often enough in nature to reflect “some naturally occurring patterns.”



The golden ratio is expressed in spiraling shells. In the above illustration, areas of the shell's growth are mapped out in squares. If the two smallest squares have a width and height of 1, then the box to their left has measurements of 2. The other boxes measure 3, 5, 8 and 13.
©iStockphoto.com/Janne Ahvo

Another example is a pine cone shown below (copied from our Discrete Math textbook by Richard Johnsonbaugh). The picture claims there are 13 clockwise spirals (marked with white thread) and 8 counterclockwise spirals (marked by dark thread). I find this type of observation somewhat subjective. (If I were told that there are 14 clockwise spirals and 7 counterclockwise spirals, rather than 13 and 8, I would stare long enough until I either got dizzy or I became convinced!)

Similar observations have been made about orange, cauliflower, pineapple, and many other things in nature.

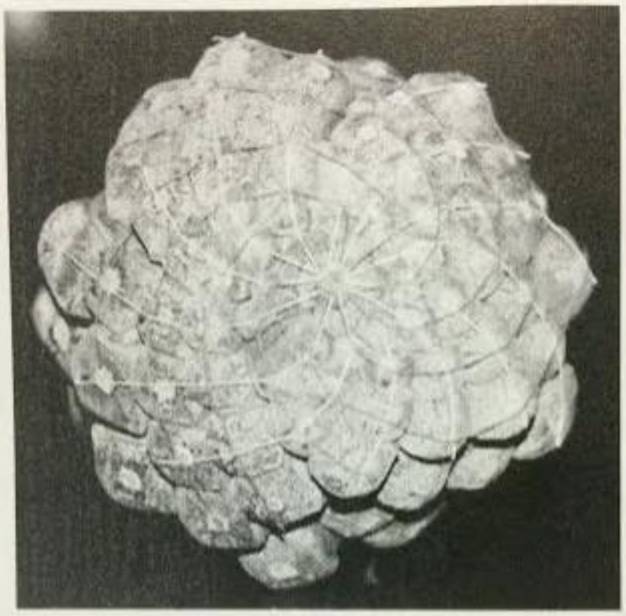


Figure 4.4.1 A pine cone. There are 13 clockwise spirals (marked with white thread) and 8 counterclockwise spirals (marked with dark thread). [*Photo by the author; pine cone courtesy of André Berthiaume and Sigrid (Anne) Settle.*]

In closing, I would like to leave you with the following intriguing thought. First observe that the specific values in Fibonacci sequence entirely depend on the two starting values (base cases), which are arbitrarily defined. Let us redefine the sequence with a slightly different base values, as shown below. (The value for $n = 2$ has been changed from 1 to 3.) The recursive definition for this new sequence, G_n, remains the same.

$$G_n = \begin{cases} 1, & n = 1 \\ 3, & n = 2 \\ G_{n-1} + G_{n-2}, & n \geq 3 \end{cases}$$

This recurrence has the same roots as before, and the Golden Ratio is still one root. But the numbers which are generated have different values. The first 10 numbers in this sequence are shown below, along with Fibonacci numbers.

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
G_n	1	3	4	7	11	18	29	47	76	123

Note that F_n and G_n are of the same order. In fact, it is easy to prove by induction that

$$F_n \leq G_n \leq 3F_n, \qquad \forall n.$$

Will nature alter its growth of sea shells, pine cones, orange, and cauliflower to match the pattern 4, 7, 11 in this sequence (rather than the current pattern of 3,5,8)? Probably not, since the base values in our sequence are not divinely inspired!