Module 4: Functions

Reading from the Textbook: Chapter 3 Functions, Sequences, and Relations

Introduction:

Functions are familiar to all of us. A function assigns to each element in its domain exactly one element in its range. A function in general may be characterized as a many-to-one mapping. For example, the floor function maps all real numbers x in the range $5 \leq x < 6$, to the integer $\lfloor x \rfloor = 5$. There are many useful functions, such as random number generators, hashing functions, floor, ceiling, and mod function.

Relations are a simple generalization of functions, allowing for many-to-many mapping. For example, the relation $a < b$ on the set of positive integers allows for many integers < 506 and $506 <$ many integers. The relational database model is based on the concept of relation.

We study functions in this module, and relations in the next module.

Contents

Function Definitions

A function assigns to each element in its *domain* exactly one element in its *range*. For example, the floor function $\lfloor x \rfloor$ takes a real number *x* and maps it to an integer, which is the largest integer $\leq x$. The domain of this function is the set of real numbers, and its range is the set of integers. We say the floor function is a mapping from the set of real numbers to the set of integers.

Below is a formal definition, which defines a function as a set of ordered pairs.

Definition: A function *f* from a set *X* to a set *Y* is a subset of the Cartesian product $X \times Y$ such that for every $x \in X$, there is exactly one $y \in Y$, with $(x, y) \in f$.
A function from *X* to *Y* may also be written as

$$f\colon\ X \to Y.$$

The right arrow here denotes a mapping, and is read as "goes to" or "maps to".
The set *X* is the *domain* of *f,* and the set *Y* is the *codomain* of *f.*
The range of the function *f* is the set

$$R = \{y \mid (x, y) \in f\}.$$

The range is a subset of codomain, $R \subseteq Y$.
The sets X and Y may or may not be the same. If $X = Y$, then we simply say the function is *on* the set X (rather than saying from X to X).

Examples: Suppose the domain for the following examples is $X = \{1,2,3,4\}$.

1. The following set of ordered pairs is not a function

$$f = \{(1, a), (2, b), (2, c), (3, a), (4, c)\}$$

because $f(2)$ is not unique. That is, $(2, b) \in f$ and $(2, c) \in f$.

2. The following set of ordered pairs is not a function

$$f = \{(1, a), (2,b),(3,a)\}$$

because $f(4)$ is not defined.

3. The following set of ordered pairs is a function.

$$f = \{(1, a), (2, b), (3, a), (4, b)\}$$

The range of this function is $R = \{a, b\}$.

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Next, we provide some examples to clarify the difference between range and codomain.

Example: Consider the following function, where the domain *X* is the set of integers (which includes negative integers, 0, and positive integers).

$$f(x) = x^2$$

The range *R* is the set of non-negative perfect-square integers.

$$R = \{x^2 | x \in X\} = \{0,1,4,9,16,\cdots\}$$

Depending on how we describe the function, it will determine its codomain *Y.*

1. Suppose we describe this function in a precise manner, and say the function is
   From: set of integers *X*
   To: set of non-negative perfect square integers $Y = \{0,1,4,9,\cdots\}$.
   In this case $R = Y$.
2. But suppose for convenience we say the function is
   From: set of integers *X*
   To: set of non-negative integers *Y.*
   In this case $R \subset Y$. ∎

Example: Consider the following function, where the domain *X* is the set of positive integers.

$$f(x) = 2x - 1$$

The range is the set of positive odd integers, $= \{1, 3, 5, \cdots\}$.

This function as a set of ordered pairs is:

$$f = \{(1,1), (2,3), (3,5), (4,7), \cdots\}.$$

Depending on how we describe the function, it will determine its codomain *Y.*

1. From: set of positive integers (domain *X)*
   To: set of positive odd integers (codomain Y)

   In this case, $R = Y$.

2. From: set of positive integers (domain *X*)
   To: set of positive integers (codomain *Y*)

   Since $X = Y$, we may simply say the function is on the set of positive integers.

   In this case, $R \subset Y$. ∎

Definition (Onto/Into): Consider a function *f* with domain *X*, codomain *Y*, and range *R.*
If $R = Y$, then we say the function is *onto. (*The function is from *X* onto *Y.)*
If $R \subset Y$, then we say the function is *into*. (The function is from *X* into *Y.*).

In other words, a function *f* from *X* to *Y* is *onto* if and only if $\forall y \in Y$, there is at least one $x \in X$, where $(x, y) \in f$. Symbolically,

$$\forall y \in Y, \exists x \in X, \ \ (x, y) \in f.$$

Example: The following function is on the set of positive integers. (That is, from the set of positive integers to the set of positive integers.)

$$f(x) = x^2$$

The rang is $R = \{1,4,9,16, \cdots\}$ and the codomain is $Y = \{1,2,3,4, \cdots\}$, so $R \subset Y$.
Therefore, the function is *into*.

Example: The following function is from the set of positive integers to the set of positive even integers.

$$f(x) = 2x$$

This function is described in an exact manner so that the codomain is the same as the range, which is the set of positive even integers, $Y = R = \{2,4,6, \cdots\}$. So, the function is *onto*.

Definition (One-to-One): A function *f* from *X* to *Y* is *one-to-one* if no two elements in *X* map to the same element in *Y*. That is, for any pair of elements $x_1 \in X$ and $x_2 \in X$, $f(x_1) \neq f(x_2)$.

Example: The function

$$\mathrm{f(x) = x^2}$$

on the set of real numbers is not one-to-one, because $f(x) = f(-x) = x^2$. But if we define the function on the set of positive real numbers, then it becomes one-to-one.

Example: The function

$$f(x) = 2x$$

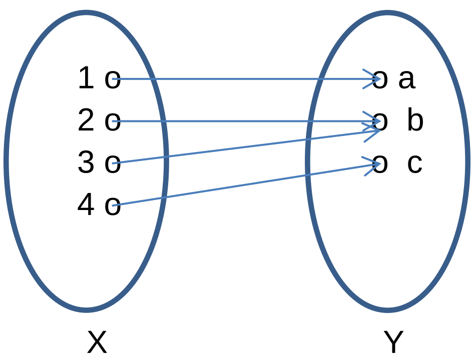
on the set of real numbers is one-to-one and onto.

Graph Representation of a Function: We may represent a function pictorially by its **graph**, where the domain *X* is shown as a set of vertices on the left, and the codomain *Y* as a set of vertices on the right. For each ordered pair $(x, y) \in f$, we draw an arrow from vertex *x* to vertex *y.*

Example: Consider a function *f* from the set $X = \{1,2,3,4\}$ to the set $Y = \{a, b, c\}$, where

$$f = \{(1, a), (2, b), (3, b), (4, c)\}.$$

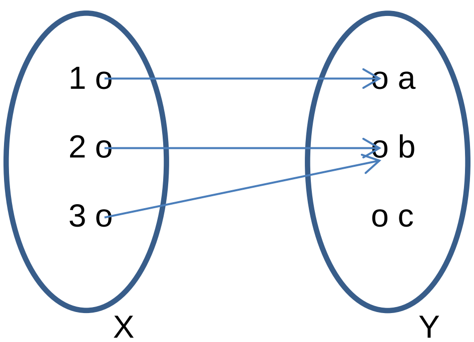
The range is $R = \{a, b, c\}$. Since $R = Y$, the function is *onto.* The function is not one-to-one because $f(2) = f(3) = b$. The graph of this function is shown below.



Example: Consider a function *f* from the set $X = \{1,2,3\}$ to the set $Y = \{a, b, c\}$, where

$$f = \{(1, a), (2, b), (3, b)\}.$$

The range is $R = \{a, b\}$ and $R \subset Y$. This function is *into*. The graph of it is shown below.



In terms of the graph of a function, it is easy to see if the function is onto, and whether it is one-to-one. Let *F* be a function from *X* to *Y*.

- F is onto if and only if every $y \in Y$ has at least one arrow coming into it.
- F is one-to-one if and only if every $y \in Y$ has at-most one arrow coming into it.
- F is one-to-one and onto if and only if $\forall y \in Y$, there is exactly one arrow coming into it.

Inverse Function

Definition: Consider a function *f* from *X* to *Y*. If the function is one-to-one and onto, then the function has an inverse, denoted as f^{-1}, defined as:

$$f^{-1} = \{(y, x) \mid (x, y) \in f\}.$$

Example: Consider the function *f* from $X = \{1,2,3\}$ to $Y = \{a, b, c, d\}$, where

$$f = \{(1, a), (2, b), (3, c)\}$$

This function is one-to-one, but not onto. It does not have an inverse.

Example: The function *f* from $X = \{1,2,3\}$ to $Y = \{a, b\}$, where

$$f = \{(1, a), (2, b), (3, b)\}$$

is onto, but not one-to-one. It does not have an inverse.

Example: The following function *f* from $X = \{1,2,3\}$ to $Y = \{a, b, c\}$, is both one-to-one and onto.

$$f = \{(1, a), (2, b), (3, c)\}$$

The inverse of it is:

$$f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$$

Example: The following function on the set of positive real numbers is both one-to-one and onto.

$$f(x) = x^2$$

The inverse function is

$$f^{-1}(x) = \sqrt{x}$$

Example: The following function on the set of positive integers

$$f(x) = x^2$$

is not onto. It does not have an inverse.

Composition of Functions

Definition: Let *f* be a function from *X* to *Y*, and let *g* be a function from *Y* to *Z.*
Then the operation

$$g\big(f(x)\big) = (g \text{ o } f)(x)$$

is function *composition* from *X* to *Z*. For every $x \in X$, first the function *f* is applied to find $f(x) = y \in Y$, and then function *g* is applied to find $g(y) = g\big(f(x)\big) = z \in Z$.
Note in the second notation, the function *f* which is on the right closer to the parameter *(x)* is the one that is applied first.

Example: Let *f* and *g* be functions on the set of real numbers, defined as:

$$f(x) = 5x + 10; \quad g(x) = x^2 - 1$$

Then,

$$(g \text{ o } f)(x) = g\big(f(x)\big) = (5x + 10)^2 - 1 = 25x^2 + 100x + 99$$
$$(f \text{ o } g)(x) = f\big(g(x)\big) = 5(x^2 - 1) + 10 = 5x^2 + 5.$$

Some Useful Functions

Hashing Functions and Random Number Generators

The theories behind these functions are rather extensive and will not be covered in this course. A brief discussion may be found in our textbook.

Floor $\lfloor x \rfloor$, Ceiling $\lceil x \rceil$, and Mod Functions

The floor and ceiling functions are both many-to-one mappings from the set of real numbers to the set of integers, as defined below.

Definitions: For any real number *x*:

- Floor of x, denoted $\lfloor x \rfloor$, is the largest integer $\leq x$.
- Ceiling of x, denoted $\lceil x \rceil$, is the smallest integer $\geq x$.

Examples:

$\lfloor 5.69 \rfloor = 5$
$\lceil 5.69 \rceil = 6$
$\lfloor 5 \rfloor = \lceil 5 \rceil = 5$
$\lfloor -5.69 \rfloor = -6$
$\lceil -5.69 \rceil = -5$

My local Shoprite this week lists the price of their Naval Orange as

3 / \$2.99

Why, I ask myself with irritation! Do they think I am that stupid to forget the pennies, floor it, and just remember it as \$2?! How insulting?! I apply a "high ceiling" and walk away, saying no thanks. (Or, as Andy Grammer would say, "Nah nah honey, I'm good!" I've got apples at home.)

Definition: Given an integer *x* and a positive integer *d,* the operation

$$x \bmod d$$

produces a **non-negative** remainder *r* when *x* is divided by *d*,

$$0 \leq r < d$$

If *q* is the integer quotient, then

$$x = q * d + r.$$

Integer *x* is called the *dividend* and *d* the *divisor*.

It is important to note that even when x is negative, the remainder is still non-negative. For example,

$17 \bmod 5 = 17 - 3 * 5 = 2$ (The quotient is +3.)

$-17 \bmod 5 = -17 + 4 * 5 = +3$ (The quotient is -4.)

The mod operation is particularly interesting, and useful, when the divisor is a power of 2. For example,

$$29 \bmod 2^3 = 29 \bmod 8 = 5$$

Let us look at the binary representation of 29, which is 11101.

$$(1\,1\ \ \underbrace{1\,0\,1}_{remainder}) \ \bmod\ 2^3 = 101$$

That is, to perform *x* mod 2^3, we retain the rightmost 3 bits of *x* in binary, while discarding the rest.

The mod operation has an important application in how negative integers are represented inside computers. The representation is called *2's complement* number system, which we briefly review next.

2's Complement Number System

Integers are normally stored inside computers using fixed-sized chunks, called one *word*. For example, if the word-size of a computer is 64-bits (which is a common size), then one-word representation of unsigned integers allows for the range

$$[0 \text{ to } 2^{64} - 1]$$

$(2^{64} \cong 1.8 * 10^{19})$. More generally, for *n*-bit word-size, the range is

$$[0 \text{ to } 2^{n} - 1].$$

To allow for signed integers, while using one-word representation, this range is cut approximately in half, with half of the range reserved for positive integers and the other half reserved for negative integers.

One possible representation for signed integers is *sign-and-magnitude*, with the leftmost bit used for the sign and the remaining $n-1$ bits used for the magnitude. Although this representation seems natural, it is not normally used because it complicates basic arithmetic. For example to add two numbers, you need to first determine if the signs are alike or opposite. If the signs are alike, then the magnitudes are added. For opposite signs, first you have to determine the smaller magnitude, and then subtract the smaller magnitude from the larger one. To avoid these complications, another representation is commonly used, known as 2's complement.

In *2's complement number system,* with *n-bit* word-size, signed integers are represented mod 2^n, and all arithmetic is also done mod 2^n. This greatly simplifies arithmetic operations, as we will see shortly.

Consider a hypothetical word-size of $n = 4$. The range of numbers that are represented is -8 to $+7$. The following table shows the 2's complement representation.

Decimal Value	Binary Representation (2's complement system)	How Binary Representation is Computed (for negative numbers)
$+7$	0111	
$+6$	0110	
$+5$	0101	
$+4$	0100	
$+3$	0011	
$+2$	0010	
$+1$	0001	
0	0000	
-1	1111	$-1\ mod\ 2^4 = 16 - 1 = 15 = 1111\ (binary)$
-2	1110	$-2\ mod\ 2^4 = 16 - 2 = 14 = 1110\ (binary)$
-3	1101	$-3\ mod\ 2^4 = 16 - 3 = 13 = 1101\ (binary)$
-4	1100	$-4\ mod\ 2^4 = 16 - 4 = 12 = 1100\ (binary)$
-5	1011	$-5\ mod\ 2^4 = 16 - 5 = 11 = 1011\ (binary)$
-6	1010	$-6\ mod\ 2^4 = 16 - 6 = 10 = 1010\ (binary)$
-7	1001	$-7\ mod\ 2^4 = 16 - 7 = 9 = 1001\ (binary)$
-8	1000	$-8\ mod\ 2^4 = 16 - 8 = 8 = 1000\ (binary)$

For example, to find the representation for -3:

$$-3 \text{ mod } 2^4 = -3 \text{ mod } 16 = -3 + 16 = +13\ (decimal) = 1101 \text{ (binary)}$$

There is also a convenient way of computing the mod.

$$-3 \text{ mod } 16 = -3 + 16 = (-3 + 15) + 1$$

Now, to compute $(-3 + 15)$, let us first look at what happens when we do the subtraction in binary:

$$\begin{array}{r} 1111 \\ -\,0011 \\ =1100 \end{array}$$

In other words, to find $(-3+15)$, we take the binary representation for +3, which is 0011, and flip all bits (complement all bits) to get 1100.

In summary, to find the 2's complement representation for -3:		
1	Find the binary representation for $+3$:	0011
2	Flip all the bits	1100
3	Add one	1101

Arithmetic in 2's Complement Number System

A nice feature of the 2's complement number system is that it greatly simplifies arithmetic operations. To perform an addition, it is simply done in mod 2^n (which means discarding a carry out of the most-significant-bit position.) The addition is done uniformly, regardless of whether the two numbers are both positive, both negative, or one positive and the other negative.

Examples:

Addition to perform (Decimal)	Binary Representation
(-5) $+(+3)$	1011 $+\,0011$
$=(-2)$	$=1110$

Addition to perform (Decimal)	Binary Representation
(-5) $+(+7)$	1011 $+\,0111$
$=(+2)$	$=0010$

Addition to perform (Decimal)	Binary Representation
(-5) $+(-2)$	1011 $+\,1110$
$=(-7)$	$=1001$