Module 2: Propositional Logic

Introduction

A proposition is a statement that is either true or false. Propositional Logic is the rules for combining propositions into compound statements and valid arguments. In this module, we study logic and its relation to set algebra.

Contents

Basic Logic Operations

A proposition is a statement that is either true or false. For example,

- It is cold.
- 2 is greater than 3.
- A prime number is an integer that is divisible only by 1 and itself.
- 7 is a prime number.

Not all statements are propositions. For example, consider the highway sign that warns the drivers: "Don't text and drive."

We use connectives (logic operations) such as {and, or, not} to combine propositions into compound propositions. For example, "It is raining and it is not cold."

There are three basic logic operations:

Name	Modern Name	Symbol	Definition
Conjunction	AND	$A \wedge B$	True if A and B are both true. False otherwise.
Disjunction	OR	$A \vee B$	True if A or B (or both) is true. False if neither one is true.
Negation	NOT	$\neg A$	True if A is false; False if A is true.

A formal definition of these operations is by *truth table*. Below are the truth table definitions for $\wedge$ and $\vee$. (Since there are 2 variables A and B, there are 4 rows in the truth table, for every possible combination of truth values of A and B.)

A B	$A \wedge B$ AND	$A \vee B$ OR
T T	T	T
T F	F	T
F T	F	T
F F	F	F

And below is the truth table definition of negation.

A	$\neg A$ NOT A
T	F
F	T

Example (Web Search): Search engines allow the user to enter keywords to be used to match with Web pages on the internet. These keywords may be combined with operators *and, or,* and *not* in order to narrow the search. For example, in Google,

- Space is interpreted as *and* operator,
- OR is used for *or* operator,
- Minus sign - is used for *not.*
- Additionally, a phrase may be enclosed inside double quotation marks to be treated as a single keyword.

Let us enter the following query in Google:

Nassimi ("Discrete Mathematics" OR Algorithms)

It produces the following four hits on the top of the list:

Prof. D. Nassimi, CS 610
https://web.njit.edu/~**nassimi**/cs610/ ▾ New Jersey Institute of Technology ▾
Prof. David **Nassimi** CS 610, Data Structures & **Algorithms** Spring 2015. Section 104, Thurs 6-9 pm, KUPF 205. Office Hours (Mine and TA's). Syllabus and Class ...

Prof. D. Nassimi, CS 506
https://web.njit.edu/~**nassimi**/cs506/ ▾ New Jersey Institute of Technology ▾
Prof. David **Nassimi** CS 506, Foundations of Computer Science (**Discrete Mathematics**) Spring 2014. M,W 11:30-12:55. TIER 113. Office Hours (Mine and TA's).

Prof. D. Nassimi, CS 241
https://web.njit.edu/~**nassimi**/cs241/ ▾ New Jersey Institute of Technology ▾
Feb 3, 2015 - David **Nassimi** CS 241, Foundations of Computer Science (**Discrete Mathematics**) Spring 2015. Mon 6-9 pm, KUPF 209. Office Hours: My Office ...

[PDF] Calvin-Gerbessiotis-Nassimi Data Structures and Algorith...
cs.njit.edu/**nassimi**/phd/qual/prev/13/610.pdf ▾
Calvin-Gerbessiotis-**Nassimi**. Data Structures and **Algorithms**. May 2013. AY 2012-2013. Time: 2hours 30mins. CS PhD Qualifying Exam. CODE: .

Logic Rules:

All the rules of set algebra (Table 1 in Module 1) also apply to propositional logic in a similar way.

Example: De Morgan's Laws in Logic are:

$$\neg(A \wedge B) = \neg A \vee \neg B \qquad (1)$$

$$\neg(A \vee B) = \neg A \wedge \neg B \qquad (2)$$

Let us use a truth table to prove the first equality (1). The truth table is used to evaluate each side of the equality step-by-step. For the left-half of rule (1), we first evaluate the column $(A \wedge B)$ and then the negation of it, $\neg(A \wedge B)$. For the right-side, we first evaluate the column ¬*A*, then ¬*B*, and finally $(\neg A \vee \neg B)$. We observe the equality is correct because the left-side and right-side are equal in every row of the truth table.

A B	$A \wedge B$	Left Side $\neg(A \wedge B)$	¬*A*	¬*B*	Right Side $\neg A \vee \neg B$
T T	T	F	F	F	F
T F	F	T	F	T	T
F T	F	T	T	F	T
F F	F	T	T	T	T

$$\underbrace{\uparrow \qquad\qquad\qquad\qquad \uparrow}_{Equal}$$

Example: The statement *S* below is similar to conditions used in a program.

$$S\text{:}\ \ (x > 5) \wedge (x < 10)$$

Let us find the negation of it, ¬*S*, by using De Morgan rule.

$\neg S$: $\neg(x > 5) \vee \neg(x < 10)$
$\neg S$: $(x < 5) \vee (x = 5) \vee (x = 10) \vee (x > 10)$
$\neg S$: $(x \leq 5) \vee (x \geq 10)$

Correspondence between Set Algebra, Propositional Logic, and Boolean Algebra

These 3 algebraic systems are basically equivalent. The correspondence between them is shown below. (In this course, we do not study Boolean algebra any further.)

Set Algebra	**Propositional Logic**	**Boolean Algebra**
Intersection $A \cap B$	AND $A \wedge B$	AND $A\ B$
Union $A \cup B$	OR $A \vee B$	OR $A + B$
Complement $\bar{A}$	NOT $\neg A$	NOT $\bar{A}$
Empty set $\emptyset$	FALSE F	Zero 0
Universal set U	TRUE T	One 1
Venn diagram	Truth Table	Truth Table

Every rule that holds in set algebra, its corresponding rule holds in logic. For example:

	Set Algebra	Propositional Logic
De Morgan's Law	$\overline{A \cap B} = \bar{A} \cup \bar{B}$	$\neg(A \wedge B) = \neg A \vee \neg B$
Distributive Law	$A \cap (B \cup C) = A \cap B \ \cup A \cap C$	$A \wedge (B \vee C) = A \wedge B \vee A \wedge C$
Identity Laws	$A \cap U = A$ $A \cup \emptyset = A$	$A \wedge True = A$ $A \vee False = A$
Contraction Rule	$A \cap B \cup A \cap \bar{B} = A$	$A \wedge B \vee A \wedge \neg B = A$

If a rule was proved for set-algebra, its corresponding rule automatically holds for logic and does not need to be proved again. Nevertheless, let us prove the contraction rule for logic, which was proved earlier for set algebra. The reader will note that the proof is basically identical to the earlier one.

Example: Prove the contraction rule for logic: $A \wedge B \vee A \wedge \neg B = A$

Set Algebraic Proof (done earlier)	Logic Proof
$A \cap B \cup A \cap \bar{B}$ Distributive rule $= A \cap (B \cup \bar{B})$ $= A \cap U$ $= A$	$A \wedge B \vee A \wedge \neg B$ Distributive rule $= A \wedge (B \vee \neg B)$ $= A \wedge True$ $= A$

Next, we discuss some other logic statements.

Conditional Proposition (Implication): If-Then

Let *p* and *q* be two propositions. The following statement is a conditional proposition, where *p* is called the *hypothesis,* and *q* the *conclusion.*

If *p* then *q*.

The statement claims that supposing *p* is true, then *q* will be true. (If *p* is false, *q* may be true or false.) The statement is also called implication and is denoted as:

$$p \to q.$$

(It is read as p *implies* q.)

Before we give a formal definition of the conditional proposition, let us consider a motivating example.

Example: Let *x* and *y* be two real numbers. The following statement is a conditional proposition.

$$If \ \underbrace{(x\ > 0 \ and \ y > 0)}_{hypothesis} \ \textit{then} \ \underbrace{(xy > 0)}_{conclusion}.$$

The statement claims that whenever the *hypothesis* is true, then the *conclusion* will be true. To show that this claim is in fact valid, suppose *x* and *y* are both positive, so the hypothesis is true:

$$x\ > 0 \wedge y > 0.$$

Then, we know from our knowledge of ordinary algebra that the conclusion will be true:

$$xy\ > 0.$$

And this alone establishes that the conditional proposition is true.

But what happens if the hypothesis is false? The conclusion may be true or may be false. (The statement does not care about that!) For example, if *x* and *y* are both negative, the conclusion will be true. But if x is positive and y is negative, then the conclusion will be false. ■

We are now ready to give a formal definition of the conditional proposition.

Formal Definition of If-Then

p	*q*	If *p* then *q* $p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that in the last two rows where *p* is false, the statement is considered true, regardless of true/false value of *q*. The proposition is false only in the 2nd row, where *p* is true but *q* is false.

Example: Let *n* be a positive integer. Prove that:

If *n* is odd then n^2 is odd.

Direct Proof: In a direct proof, we start by supposing that the hypothesis is true. Then we prove that will imply the conclusion is true. So, suppose *n* is odd, which means

$n = 2k + 1$, for some integer *k*.

Then,

$$n^2 = (2k + 1)^2 = 4\,k^2 + 4\,k + 1 = 2(2\,k^2 + 2\,k\,) + 1.$$

Since $(2\,k^2 + 2\,k\,)$ is an integer, this means that n^2 is odd. ■

Examples: Consider the following conditional proposition.

"If the moon is made of blue cheese, then the earth is made of milk and honey!"

This statement is true, because the hypothesis "moon is made of blue cheese" is false. So, the statement is automatically true, regardless of the conclusion. As another example, consider the proposition

If $2 = 4$ then $100 = 1000$

This proposition is automatically true because the hypothesis "$2 = 4$" is false.

A conditional proposition that is true because its hypothesis is always false is said to be **true by default** or **vacuously true**. ■

Contrapositve Equivalent Form of Implication:

Consider the implication

$$p \to q. \qquad (1)$$

What happens if *q* is false? Then *p* must be false, because if *p* were true, then *q* would be true. Therefore, an equivalent proposition is

$$\neg q \to \neg p. \qquad (2)$$

We must emphasize that proposition (2) is *not* the negation of (1), as it may appear. It is equivalent to it. Proposition (2) is called the contrapositive equivalent form of (1). The equivalence of these two propositions may be formally proved by a truth table:

p *q*	(1) $p \to q$	$\neg q$ $\neg p$	(2) $\neg q \to \neg p$
T T	T	F F	T
T F	F	T F	F
F T	T	F T	T
F F	T	T T	T

The following example shows how contrapositive form may be very useful.

Example: Let *n* be a positive integer. Prove that

If n^2 is odd, then *n* is odd.

A direct proof of this is difficult. That is, if we start by supposing that n^2 is odd, then it is difficult to conclude that *n* must be odd. Instead, we consider the contrapositive equivalent form of this proposition:

If *n* is not odd, then n^2 is not odd.

This means that

If *n* is even, then n^2 is even.

It now becomes very easy to prove the latter proposition by direct proof. Suppose *n* is even. Then, $n = 2k$ for some integer *k*. So,

$$n^2 = (2k)^2 = 4\,k^2 = 2(2k^2).$$

Since $(2k^2)$ is an integer, n^2 is even. $\blacksquare$

Implication in terms of basic operations ∧, ∨, ¬

Let us prove that the following propositions are equivalent:

1. $p \to q$
2. $\neg(p \wedge \neg q)$
3. $\neg p \vee q$

First, let us provide an intuitive reasoning why (1) and (2) are equivalent. Remember from the truth table of implication that (1) is false only when *p* is true and *q* is false. And proposition (2) is also false only when *p* is true and *q* is false.

The following truth table formally proves that (1) and (2) are equivalent.

p q	(1) $p \to q$	$\neg q$	$(p \wedge \neg q)$	(2) $\neg(p \wedge \neg q)$
T T	T	F	F	T
T F	F	T	T	F
F T	T	F	F	T
F F	T	T	F	T

And (3) follows from (2) by De Morgan's rule. So, all three are equivalent.

Negation of Implication

Now let us figure out the negation of an implication:

$$\neg\,(p \to q).$$

At first, one may wrongly guess that an equivalent form is: $p \to \neg q$. But a truth table quickly verifies that these two are not equivalent. In fact, the negation of an implication will not be in any form of implication!

So how do we find the negation of an implication? We first express the implication by an equivalent form in terms of the basic operations (∧, ∨, ¬). Then, we negate it.

$$\begin{aligned} &\neg\,(p \to q) \\ &= \neg(\neg p \vee q) \\ &= p \wedge \neg q. \end{aligned}$$

Example: Find an equivalent form for the following proposition, in terms of ∧, ∨, ¬. Then, find the negation of it.

$$S: \ \ if \ (x > y) \ then \ (x^2 > y^2)$$

An equivalent form is

$$S: \ \neg(x > y) \vee (x^2 > y^2)$$

The negation, by De Morgan's rule, becomes:

$$\neg S: \ \ (x > y) \wedge \neg(x^2 > y^2)$$
$$\neg S: \ \ (x > y) \wedge (x^2 \leq y^2)$$

Only-If

The proposition

p only if q

is equivalent to

if $\neg q \ \ then \ \neg p$

which is equivalent to

if $p \ \ then \ q$.

Example: Taylor Swift, in her hit song *blank space*, warns the girls of the world that:

> "Boys want love only if it is torture!"

What does it mean? It means that

> "If love is not torture, then boys don't want it!"

which is the same as

> "If boys want love, then it must be torture!"

If-and-Only-If

The proposition

p if and only if *q*

Means

(1) $p \to q$, and
(2) $q \to p$.

Symbolically, if-and-only is denoted by a double-sided arrow:

$p \leftrightarrow q$.

If we replace (2) by its contrapositive form, we get an equivalent form:

- $p \to q$, and
- $\neg p \to \neg q$.

Alternatively, if we replace (1) by its contrapositive form, we get another equivalent form:

- $q \to p$, and
- $\neg q \to \neg p$.

All of the above forms are equivalent.

Let us examine the truth table for this proposition.

p q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$ $p \leftrightarrow q$
T T	T	T	T
T F	F	T	F
F T	T	F	F
F F	T	T	T

Observe that $p \leftrightarrow q$ is true if *p* and *q* are both true or both false. Otherwise, it is false.

Logical Equivalence

We say two propositions *p* and *q* are logically equivalent,

$$p \equiv q$$

if *p* and *q* are both true or both false.

Formal truth table definition:

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

Observe from this truth table that

$$p \equiv q$$

is exactly the same as

$$p \leftrightarrow q.$$

Next we give some examples of proofs involving if-and-only-if statements.

Example: Let *A* and *B* be two sets. Prove the following set equality is true if and only if *A* and *B* are disjoint. (Two sets are *disjoint* if they have no common elements.)

$$(A \cup B) - B = A\ ?$$

(The question mark is to emphasize that the equality is conditional.)

Proof: The proof is presented in two parts:

1. If the two sets are disjoint, then the set equality is true.
2. If the two sets are not disjoint, then the equality is false.

Proof of (1): Suppose sets *A* and *B* are disjoint. Then, $(A \cup B)$ adds all elements of *B* to the original elements of *A*, and the set subtraction takes away all elements of *B*, so the result is *A* again.

Proof of (2): Suppose sets *A* and *B* are not disjoint. Then, there is at least one element in their intersection. Let $x \in (A \cap B)$. The set subtraction will take away element x from $(A \cup B)$, so the resulting set will be missing element *x*. Therefore, the result will not equal set *A*.

An alternative pictorial proof is by Venn diagram, which is left to the student as an exercise. ■

Note: In spoken English, we often say "if" when "if and only if" is meant. For example, in the above example on sets, it was informally stated that

"Two sets are disjoint if they have no common elements."

What was meant is obviously if and only if. That is, if two sets have no common elements, then they are disjoint; and if they have some common elements then they are not disjoint.

Example: Let *n* be a positive integer. Prove that:

"n^2 is divisible by 3 if and only if *n* is divisible by 3."

Proof: The proof will be in two parts:

1. If *n* is divisible by 3, then n^2 is divisible by 3.
2. If *n* is not divisible by 3, then n^2 is not divisible by 3.

Proof of (1): Suppose *n* is divisible by 3. Then $n = 3k,$ for some integer *k*. So,

$$n^2 = (3k)^2 = 9\,k^2 = 3(3k^2).$$

Since $(3k^2)$ is an integer, n^2 is divisible by 3.

Proof of (2): Suppose *n* is not divisible by 3. Then

$$n = 3k + r$$

for some integer *k,* and some non-zero remainder *r.* That is, *r* is either 1 or 2, $r \in \{1,2\}$. Therefore,

$$n^2 = (3k + r)^2 = 9k^2 + 6kr + r^2 = 3(3k^2 + 2kr) + r^2.$$

This means that a division of n^2 by 3 produces a nonzero remainder,

$$r^2 mod\ 3.$$

This is indeed a nonzero remainder, since $r^2 \in \{1,4\}$, so

$$r^2 mod\ 3 = 1.$$

Therefore, n^2 is not divisible by 3. $\blacksquare$

Arguments:

An argument consists of a list of propositions $p_1, p_2, \cdots, p_n$ called the *hypotheses* (or *premises*), followed by a proposition *q*, called the *conclusion*. It is written as:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

(The symbol ∴ is read 'therefore'.)

The argument is *valid* if, supposing that the premises are all true, then the conclusion is also true. That is, the truth of all premises must imply the truth of the conclusion. (In a valid argument, we say the conclusion logically follows from the premises.)

Example: A valid argument

$$\begin{array}{l} p \\ p \to q \\ - - - \\ \therefore q \end{array}$$

Assuming that p is true, and assuming $p \to q$, then it follows that q is true. So, the argument is valid.

Example: An invalid argument

$$\begin{array}{l} q \\ p \to q \\ - - - \\ \therefore p \end{array}$$

In the implication $p \to q$, the truth of *q* does not necessarily imply the truth of *p*. That is, *p* may be false when *q* is true. Therefore, the argument is invalid.

Example: A valid argument

$$
\begin{array}{l}
2 > 3 \\
3 > 4 \\
--- \\
\therefore 2 > 4
\end{array}
$$

Supposing that it is true that 2 > 3 and 3 > 4, then it follows from transitivity that 2 > 4. So the argument is valid! (Note that the conclusion by itself is not correct. But it logically follows from the absurd premises!) ◼

Example: An invalid argument

If the class studies hard, they will learn everything.

If they will learn everything, they will do well on the exam.

The class did well on the exam.

Therefore, the class studied hard and learned everything.

Not necessarily! Maybe the exam was too simple! This is an invalid argument. ◼

Quantifiers

The following proposition, which is named *P(n),* states that *n* is an even integer.

$$P(n): \quad n \text{ is an even integer.}$$

The true/false value of this proposition depends on *n*. For example, *P(4)* is true, but *P(7)* is false. This statement is called a *propositional function*, or a *predicate.*

Let *F(x)* be a predicate. There are two types of *quantifiers* that may be used in conjunction with a predicate:

1. Universal quantifier: ∀
   "for all" or "for every"

2. Existential quantifier: ∃
   "for some" or "there exists"

Definition: The universal proposition:

$$\forall x, F(x)$$

states that for every *x* in its domain, $F(x)$ is true. If for every *x,* the proposition *F(x)* is in fact true, then the statement is true. But if there exists at least one value of *x* that makes *F(x)* false, then the statement is false.

Examples: Suppose the domain of n is positive integers. The following proposition is true:

$\forall n$, *(2n)* is even

But the following proposition is false:

$$\forall n, (n^2 - 1 = 0)$$

A ***counterexample*** is $n = 5,$ because $n^2 - 1 \neq 0$. In fact, any *n* other than ± 1 makes the predicate false. So the universal statement is false. ■

Definition: The existential proposition
$$\exists x, F(x)$$
states that for some *x*, the predicate $F(x)$ is true. If there is at least one value of *x* that makes $F(x)$ true, then the statement is true. But if no such value exists, then the statement is false.

Examples: Suppose the domain of *x* is real numbers. The following statement

$$\exists x, (x^2 - 1 = 0)$$

is true because there exists some values of *x* (namely $x = \pm 1$) to make the predicate true. The following statement

$$\exists x, (x^2 < 0)$$

is false because no *real* value of *x* makes $x^2 < 0$. ■

Negation of Quantifiers

Negation of a quantified proposition follows a simple rule as shown in the table below.

(1) For a universal case, the quantifier ∀ becomes ∃, and $F(x)$ is negated.
(2) For an existential case, the quantifier ∃ becomes ∀, and $F(x)$ is negated.

S: $\forall x, F(x)$	$\neg S$: $\exists x, \neg F(x)$
T: $\exists x, F(x)$	$\neg T$: $\forall x, \neg F(x)$

The reasoning is simple:

(1) The universal statement says that for every *x*, the predicate $F(x)$ is true.
$$S: \quad \forall x, F(x)$$
The negation of it says for some *x*, $F(x)$ is false, which means $\neg F(x)$ is true.

$$\neg S: \quad \exists x, \neg F(x)$$

(2) The existential statement says that for some *x*, $F(x)$ is true.
$$T: \quad \exists x, F(x)$$
The negation of it says for every *x*, $F(x)$ is false.
$$\neg T: \quad \forall x, \neg F(x)$$

Examples:

S: $\forall n, (2n) is\ even$	$\neg S$: $\exists n, (2n) is\ odd$
T: $\exists x, (x^2 - 1 = 0)$	$\neg T$: $\forall x, (x^2 - 1 \neq 0)$

Example:

English statements sometimes have more than one possible interpretation. For example, consider the following quote from Shakespeare's "The merchant of Venice":

All that glitters is not gold.

A first possible interpretation is "Everything that glitters is not gold." But obviously this is not what was meant. The correct interpretation may be phrased in several equivalent ways:

- Not everything that glitters is necessarily gold.
- It is not true that if the object glitters, then it implies gold.
- There are objects that glitter but are not gold.

Symbolically, let us use the following predicates:

(1) $S(x)$: Object *x* shines (glitters),
(2) $G(x)$: Object *x* is gold.

Then, the quote may be stated as follows:

$$\neg\{\forall x,\ \text{if } S(x) \text{ then } G(x)\}$$

By using the negation rule, we get:

$$\exists x,\ \neg\ \{\text{if S(x) then G(x)}\}$$

$$\exists x,\ S(x) \wedge \neg G(x).$$

Nested Quantifiers

Quantifiers may be generalized to more than one level in a straightforward way. For example, if you have a predicate with two variables, *F(x, y),* then there will be two quantifiers, one for *x* and one for *y.* For example, consider the statement

The product of any two positive real numbers is positive.

Symbolically, this statement may be expressed as follows.

$$\forall x\ \forall y,\ \text{if } (x > 0 \wedge y > 0) \text{ then } (xy > 0).$$

Let *F(x, y)* be a predicate. Let us examine each possible way for the two quantifiers and how they work together. **Important Note: The quantifiers are read from left-to-right.**

1. $\forall x\ \forall y,\ F(x, y)$

   For every x and every y, the predicate $F(x, y)$ is true.

2. $\forall x\ \exists y,\ F(x, y)$

   For every *x,* there is some **corresponding** value of *y,* so that $F(x, y)$ is true. For every *x*, you may pick a *y* (which may depend on x). For example, for each x, you may have $y = x + 2$.

3. $\exists x\ \forall y,\ F(x, y)$

   For some **fixed** *x*, and every *y*, $F(x, y)$ is true. Here, once you pick some x, that fixed value has to work for all y.

4. $\exists x\ \exists y,\ F(x, y)$

   For some x and some y, $F(x, y)$ is true.

Remember, we read and execute the quantifiers from left to right. The order is very important. (Nested quantifiers are similar to nested loops in a program.)

Examples: Let the domain of x and y be positive integers (which start with 1).

1. $\forall x\ \forall y, (x \le y)$
   This is false. When $x > 1$, not all values of y will be greater than it. As a **counterexample**, pick an $x > 1$, and then $y = x - 1$.

2. $\forall x\ \exists y, (x \le y)$
   This is true. For every x, there is some $y \ge x$. For example, we may pick $y = x$.

3. $\exists x\ \forall y, (x \le y)$
   This is true. For $x = 1$, which is the smallest positive integer, it will be $\le$ every y.

4. $\exists x\ \exists y, (x \le y)$
   This is obviously true. For example, $x = 5, y = 6$. ■

Example: Let the domain of x and y be real numbers.

$$\forall x \forall y,\ \text{if}\ \ x > y\ \ \text{then}\ x^2 > y^2$$

This proposition is false. As a counterexample, suppose $x = -1,\ \ y = -5$.

Then, $x > y$, but $x^2 < y^2$. ■

Negation of Nested Quantifiers

This is straightforward and follows the same rule as was given for single quantifier.

General Rule:

- Change every ∀ to ∃,
- Change every ∃ to ∀,
- Negate the predicate.

For example, the negation of

$$S:\ \forall x\ \exists y, F(x, y)$$

is

$$\neg S:\ \exists x\ \forall y, \neg F(x, y)$$

Reasoning: The reasoning is simple. We already know how to negate a single quantifier. (The reasoning for single quantifier was given earlier.) Based on that, we can negate any number of quantifiers, one at a time, while we treat the remaining part as the predicate. For example, to find $\neg S$,

$\neg \left\{ \underbrace{\forall x}_{quantifier} \left\{ \underbrace{\exists y, F(x, y)}_{predicate} \right\} \right\}$ Treat it as if a single quantifier

$=\quad \exists x\quad \neg\ \{\exists y, F(x, y)\}$ Next, negate the second quantifier.

$=\quad \exists x\quad \forall y, \neg F(x, y).$

Examples: Negate each proposition:

(1) $S:\ \forall x\ \forall y\ \exists z, (x^2 + y^2 = z^2)$
$\neg S:\ \exists x\ \exists y\ \forall z, (x^2 + y^2 \neq z^2)$

(2) $T:\ \forall x \forall y$, if $x > y$ then $x^2 > y^2$
$\neg T:\ \exists x \exists y,\ \ x > y\ \wedge x^2 \leq y^2$.

Examples: Let the domain of *S* be NJIT students, and domain of *C* be NJIT courses. Let the predicate

$$Ace(S, C)$$

mean student *S* earns an A in course *C*.

Find the symbolic form for each of the following propositions. Then find the negation of each, both in English and in symbolic form.

(1) There are students with all A's. (There are students who get A in every course.)

$$\exists S \forall C, \quad Ace(S, C)$$

Negation:

Every student gets some grades lower-than A.

$$\forall S \exists C, \quad \neg\, Ace(S, C)$$

(2) There are students with no A's.

$$\exists S \forall C, \quad \neg\, Ace(S, C)$$

Negation:

Every student gets some A's.

$$\forall S \exists C, \quad Ace(S, C).$$

■

End of Module.