Math Template

Michael Francesco Ala

1 Intro

Give some motivation for the talk.

2 Twists

Definition 2.0.1. A **twist** over a locally compact Hausdorff étale groupoid $\mathcal{G}$ is a groupoid $\mathcal{E}$ with unit space $\mathcal{G}^{(0)}$, along with two continuous groupoid homomorphisms $\iota : \mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E}$, and $\pi : \mathcal{E} \to \mathcal{G}$, such that

- ι and π descend to *homeomorphisms* on the unit spaces.
- ι is injective and π is surjective.
- $\iota(\mathcal{G}^{(0)} \times \mathbb{T}) = \pi^{-1}(\mathcal{G}^{(0)})$
- $\iota(\mathcal{G}^{(0)} \times \mathbb{T})$ is *central* in $\mathcal{E}$, in the sense that for all $\varepsilon \in \mathcal{E}$ and $z \in \mathbb{T}$, $\iota(r(e), z)e = e\iota(s(e), z)$.
- $\mathcal{E}$ is a $\mathbb{T}$-**bundle**: for all $\gamma \in \mathcal{G}$, there exists an open bisection $U \ni \gamma$ such that $\pi^{-1}(U) \cong U \times \mathbb{T}$.

Example. Let $\mathcal{E} = \mathcal{G} \times \mathbb{T}$, with ι and π the obvious maps.

Example. Let $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ be a **continuous, normalized** 2**-cocycle**: a continuous map satisfying $\sigma(r(\gamma), \gamma) = \sigma(\gamma, s(\gamma)) = 1$, and the following cocycle identity

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$$

Let $\mathcal{E}$ have the same underlying set as $\mathcal{G} \times \mathbb{T}$, but with the following "twisted multiplication":

$$(\alpha, w)(\beta, z) = (\alpha\beta, \sigma(\alpha, \beta)wz) \qquad \sigma(\alpha,$$

Fact 2.0.2. There is a natural $\mathbb{T}$-action on $\mathcal{E}$, given by letting $z \cdot \varepsilon := \iota(r(\varepsilon), z)\varepsilon$. Then this action preserves fibres ($\pi(z \cdot \varepsilon) = \pi(\varepsilon)$), and moreover whenever $\varepsilon, \delta \in \mathcal{E}$ occupy the same fibre ($\pi(\varepsilon) = \pi(\delta)$), then there is a *unique* $z \in \mathbb{T}$ such that $\delta = z \cdot \varepsilon$. Finally, under this $\mathbb{T}$-action, $\mathcal{E}/\mathbb{T} \cong \mathcal{G}$ (algebraically, and homeomorphically if π is an open map).

The goal of twisted groupoids is to expand the class of C*-algebras arising from groupoids.

2.1 Convolution Algebras

Definition 2.1.1. Let $\Sigma_c(\mathcal{G};\mathcal{E}) := \{f \in C_c(\mathcal{E}) \mid f(z \cdot \varepsilon) = zf(\varepsilon)\}$. We'd like to define some sort of convolution operation.

For any $\gamma \in \mathcal{G}$, $\mathbb{T} \cong \pi^{-1}(\{\gamma\})$, via the map $z \mapsto z \cdot \delta$ for any choice $\delta \in \pi^{-1}(\{\gamma\})$. Thus we can obtain a $\mathbb{T}$-invariant measure on the fibre of γ by pushing forward the Haar measure on $\mathbb{T}$ via this homeomorphism. For each $x \in \mathcal{G}^{(0)}$, define a measure λ^x supported on $\mathcal{E}^x$ via

$$\lambda^x(U) = \sum_{\gamma \in \mathcal{G}^x} \lambda^{\pi^{-1}(\{\gamma\})}(U \cap \pi^{-1}(\{\gamma\})$$

Definition 2.1.2. $\Sigma_c(\mathcal{G};\mathcal{E})$ becomes a $*$-algebra with the following operations:

$$(f * g)(\varepsilon) = \int_{\mathcal{E}^{r(\varepsilon)}} f(\delta)g(\delta^{-1}\varepsilon)\,\mathrm{d}\,\lambda^{r(\varepsilon)}(\delta)$$

and

$$f^*(\varepsilon) = \overline{f(\varepsilon^{-1})}$$

Notice that when $\pi(\delta) = \pi(\delta')$, then there exists a unique $z \in \mathbb{T}$ such that $z \cdot \delta = \delta'$, implying

$$f(\delta)g(\delta^{-1}\varepsilon) = f(\delta')g(\delta'^{-1}\varepsilon)$$

so that the integrand in the convolution formula is fibrewise constant. Thus, for any section $S : \mathcal{G}^{r(\varepsilon)} \to \mathcal{E}^{r(\varepsilon)}$ (not even necessarily continuous),

$$(f * g)(\varepsilon) = \sum_{\alpha\beta = \pi(\varepsilon)} f(S(\alpha))g(S(\beta))$$

Talk about:

- $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f)\operatorname{supp}(g)$
- D_c commutative, isomorphic to $C_c(\mathcal{G}^{(0)})$.
- Even though $\mathcal{E}$ isn't étale, every function f is a linear combination of functions supported on $\pi^{-1}(U)$, $U \subseteq \mathcal{G}$ a bisection.

Definition 2.1.3. $C^*(\mathcal{G};\mathcal{E}) = \overline{\Sigma_c(\mathcal{G};\mathcal{E})}^{\|\cdot\|_{\max}}$, has a universality property. Left-regular representations

$$\begin{array}{rcccl} \pi_x & : & \Sigma_c(\mathcal{G};\mathcal{E}) & \to & \mathcal{B}(L^2(\mathcal{G}_x;\mathcal{E}_x)) \\ & & (\pi_x(f)g)(\gamma) & = & \displaystyle\int_{\mathcal{E}_{r(\gamma)}} f(\delta)g(\delta\gamma)\,\mathrm{d}\,\lambda_{r(\gamma)}(\delta) \end{array}$$

Define

$$C_r^*(\mathcal{G};\mathcal{E}) = \overline{\oplus_x \pi_x(\Sigma_c(\mathcal{G};\mathcal{E})}$$

As in the untwisted case, there is a surjective $*$-homomorphism $\pi_r : C^*(\mathcal{G};\mathcal{E}) \to C_r^*(\mathcal{G};\mathcal{E})$.

The representations $\oplus_x \pi_x$ and $\pi_{\max}$ are both injective, so both $C^*(\mathcal{G};\mathcal{E})$ and $C_r^*(\mathcal{G};\mathcal{E})$ contain $\Sigma_c(\mathcal{G};\mathcal{E})$.

We now have three norms on $\Sigma_c(\mathcal{G};\mathcal{E})$: $\|\cdot\|_{\sup} \leq \|\cdot\|_r \leq \|\cdot\|_{\max}$, and when restricted to D_c, $\|\cdot\|_{\max} \leq \|\cdot\|_{\sup}$, so all three norms coincide. Thus, both $C^*(\mathcal{G};\mathcal{E})$ and $C_r^*(\mathcal{G};\mathcal{E})$ contain isometric copies of $D_0 := \overline{D_c} \cong C_0(\mathcal{G}^{(0)})$. When $\mathcal{G}$ is amenable, $\|\cdot\|_r = \|\cdot\|_{\max}$.

3 Cartan Pairs

Definition 3.0.1. Let $B \subseteq A$. $N(B) = \{n \in A \mid nBn^*, n^*Bn \subseteq B\}$. $u \in N(B) \cap U(A)$ if and only if $uB = Bu$.

Definition 3.0.2. B is **regular** in A if $N(B)$ generates A as a C*-algebra.

Definition 3.0.3. (A, B) is a **Cartan pair** if

- B is a **masa** in A,
- B is **regular** in A,
- B contains an approximate identity in A,
- $\exists!\ \Phi : A \to B$, faithful conditional expectation.

Theorem 3.0.4. Let $\mathcal{E} \to \mathcal{G}$ be a twist over a locally compact Hausdorff, *effective* étale groupoid $\mathcal{G}$. Then $(C_r^*(\mathcal{G};\mathcal{E}), D_r)$ is a Cartan pair.

Proof. Argue exactly as in the untwisted case. Extend $\Sigma_c(\mathcal{G};\mathcal{E}) \to D_c$, taking $f \mapsto f|_{\iota(\mathcal{G}^{(0)}\times\mathbb{T})}$ to a contractive idempotent on $C_r^*(\mathcal{G},\mathcal{E}) \to D_r$, which by Tomiyama's theorem is a conditional expectation. Assuming that $\mathcal{G}$ is effective, we can show this expectation is *unique*, arguing as in $\square$

Fact 3.0.5. Let (A, B) be a Cartan pair. Then for any $n \in N(B)$, $n^*n, nn^* \in B$.

From this point forward, given a Cartan pair (A, B), since B is commutative I'd like for us to think of $B = C_0(X)$, where $X = \widehat{B}$ is the spectrum of B.

Definition 3.0.6. For $n \in N(B)$, let

$$\operatorname{dom}^\circ n = \{x \in \widehat{B} \mid n^*n(x) > 0\}, \qquad \operatorname{ran}^\circ n = \{x \in \widehat{B} \mid nn^*(x) > 0\}$$

and $\operatorname{dom} n = \overline{\operatorname{dom}^\circ n}$, $\operatorname{ran} n = \overline{\operatorname{ran}^\circ n}$. Notice that for $b \in B$, $\operatorname{dom} n = \operatorname{supp} b$, $\operatorname{dom}^\circ n^* = \operatorname{ran}^\circ n$, and $\operatorname{dom}^\circ n^*n = \operatorname{ran}^\circ n^*n = \operatorname{dom}^\circ n$.

Theorem 3.0.7. For any $n \in N(B)$, there is a homeomorphism $\alpha_n : \operatorname{dom}^\circ n \to \operatorname{ran}^\circ n$, defined by the requirement that for all $b \in C_0(\operatorname{dom}^\circ n) \subseteq B$ and $x \in \operatorname{dom}^\circ n$,

$$n^*bn(x) = b(\alpha_n(x))n^*n(x)$$

Proof. Let us work in $B \subseteq A \subseteq A^{**}$, the enveloping von Neumann algebra of A. Given any $n \in N(B)$, we can write $n = v|n|$, where $v \in A^{**}$ is a partial isometry and $|n| = (n^*n)^{1/2} \in B$. We also require v to be the *unique* isometry such that $\ker v = \ker n$ (which, if you don't like reference to the implicit Hilbert spaces, is also just the unique v for which whenever $p \in A^{**}$ is a projection for which $np = 0$, then $vp = 0$).

Consider the ideal $I_n := C_0(\mathrm{dom}^\circ n) \subseteq B$, and take f in the dense subalgebra $C_c(\mathrm{dom}^\circ n)$. Then $\mathrm{supp}(f) \subseteq \mathrm{dom}^\circ n$, which is to say that whenever $f(x)$ is nonzero, $n^*n(x)$ is nonzero (for all $x \in \widehat{B}$), so n^*n is invertible on $\mathrm{supp}(f)$. Thus, there exists $g \in I_n$ such that $|n|g|n| = f$ (just take $g = \frac{f}{n^*n}$ defined on $\mathrm{supp}(f)$, and set to zero elsewhere). Then $v|n|g|n|v^* = ngn^* = vfv^*$, but $ngn^* \in B$, so vfv^* is also in B.

You can check using the properties of the partial isometry v that v^*v is the projection onto $\mathrm{dom}\, n$, and commutes with B (B is no longer maximal abelian in A^{**}), and along with this the map $\beta_n : I_n \to B$ taking $f \mapsto vfv^*$ is a $*$-homomorphism, with range I_{n^*} and inverse β_{n^*}.

Let $\alpha_n : \mathrm{dom}^\circ n \to \mathrm{ran}^\circ n$ be the induced homeomorphism between the spectra $\widehat{I_n}$ and $\widehat{I}_{n^*}$ given by β_{n^*}. In other words, α_n is the unique continuous map such that

$$f(\alpha_n(x)) = \beta_{n^*}(f)(x) = v^*fv(x)$$

which implies

$$f(\alpha_n(x))n^*n(x) = |n|(x)f(\alpha_n(x))|n|(x) = |n|v^*fv|n|(x) = n^*fn(x)$$

as desired. $\square$

These maps α_n enjoy a sort of uniqueness guarantee as well, but it's difficult to phrase and we run short on time.

Fact 3.0.8. Some observations can be made:

- For any $b \in B$, α_b is merely the identity on $\mathrm{supp}\, b$.
- Let $m, n \in N(B)$, and note that $mn \in N(B)$ as well. We also observe

$$\begin{aligned}(mn)^*b(mn)(x) &= b(\alpha_{mn}(x))(mn)^*(mn)(x) = b(\alpha_{mn}(x))n^*(m^*m)n(x)\\ &= b(\alpha_{mn}(x))m^*m(\alpha_n(x))n^*n(x)\end{aligned}$$

$$\begin{aligned}(mn)^*b(mn)(x) &= n^*(m^*bm)n(x) = m^*bm(\alpha_n(x))n^*n(x)\\ &= b(\alpha_m \circ \alpha_n(x))m^*m(\alpha_n(x))n^*n(x)\end{aligned}$$

  So we might expect that $\alpha_m \circ \alpha_n = \alpha_{mn}$ wherever the two are defined, and indeed this is the case (though the proof is quite finnicky).
- A similar calculation yields that $\alpha_n^{-1} = \alpha_{n^*}$ (this is far easier to prove).

Definition 3.0.9 (Weyl Groupoid)**.** Let (A, B) be a Cartan pair. The collection of maps $\{\alpha_n\}_{n \in N(B)}$ forms a **pseudogroup**, and so we can look at the **groupoid of germs** $\mathcal{G}_{(A,B)}$.

More explicitly, set

$$\mathcal{N} := \{(n,x) \mid n \in N(B),\ x \in \operatorname{dom}^\circ(n)\}$$

and define an equivalence relation as follows:

$$(m,x) \sim (n,y) \quad \iff \quad x = y, \ \text{ and } \ \exists U \ni x \ \text{ s.t. } \ \alpha_m|_U = \alpha_n|_U$$

(so $[m,x]$ is the germ of α_m at x). Set

$$\mathcal{G}^{(0)}_{(A,B)} = \{[b,x] \mid b \in B,\ x \in \operatorname{supp}^\circ(b)\}$$
$$s([m,x]) = [mm^*, x], \qquad r([m,x]) = [m^*m, \alpha_m(x)]$$
$$[m, \alpha_n(x)][n,x] = [mn, x], \qquad [n,x]^{-1} = [n^*, \alpha_n(x)]$$

Equip $\mathcal{G}_{(A,B)}$ with the topology generated by the basic open sets

$$Z(n,U) = \{[n,x] \mid x \in U\}$$

where U is an open subset of $\operatorname{dom}^\circ n$.

With this topology, $\mathcal{G}_{(A,B)}$ becomes a **locally compact, Hausdorff, effective étale groupoid**. The sets $Z(n,U)$ are open bisections.

Remark. Notice that there is a natural identification of $\mathcal{G}^{(0)}_{(A,B)}$ with $\widehat{B}$, given by taking $[b,x] \mapsto x$: indeed, given any two $b, b' \in B$ with $x \in \operatorname{supp}^\circ(b) \cap \operatorname{supp}^\circ(b')$, clearly α_b and $\alpha_{b'}$ on an open neighbourhood of x, since they're both the identity map.

Proof that $\mathcal{G}_{(A,B)}$ is effective.

$$\operatorname{Iso}(\mathcal{G}_{(A,B)}) = \{[n,x] \mid x \in \operatorname{dom}^\circ(n) \cap \operatorname{ran}^\circ(n),\ \alpha_n(x) = x\}$$

In order to show that $\mathcal{G}$ is effective, given $x \in \operatorname{Iso}(\mathcal{G}) \backslash \mathcal{G}^{(0)}$ we need to show that any open neighbourhood U of x contains some other y which is not in $\operatorname{Iso}(\mathcal{G})$.

Take $[n,x] \notin \mathcal{G}^{(0)}_{(A,B)}$, so for all $b \in B$, $[b,x] \neq [n,x]$, so for every open U containing b, there is a point $y \in U$ at which $\alpha_n(y) \neq \alpha_b(y) = y$ (recall α_b is the identity). This means that for any basic open neighbourhood $Z(n,U) \ni [n,x]$, there is some $[n,y] \in Z(n,U)$ for which $\alpha_n(y) \neq y$ so $[n,y] \notin \operatorname{Iso}(\mathcal{G}_{(A,B)})$. □

Theorem 3.0.10. Let $\mathcal{E} \to \mathcal{G}$ be a twisted groupoid, and consider the Cartan pair $(A,B) = (C^*_r(\mathcal{G};\mathcal{E}), D_r)$. Then there is an isomorphism $\theta : \mathcal{G} \to \mathcal{G}_{(A,B)}$.

Proof. Let $n \in \Sigma_c(\mathcal{G};\mathcal{E})$ and $h \in D_r$. Suppose n is supported on $\pi^{-1}(U)$, for some open bisection U in $\mathcal{G}$. Then one can calculate that for any $\gamma \in \pi^{-1}(U)$:

$$(n^* * h * n)(s(\gamma)) = h(r(\gamma))|n(\gamma)|^2 = h(r(\gamma))n^*n(\gamma)$$

This should look awfully familiar, compare with:

$$n^* * h * n(s(\gamma)) = h(\alpha_n(r(\gamma)))n^*n(\gamma)$$

In other words, we've just shown that on the open set $s(\pi^{-1}(U))$:

$$\alpha_n(s(\gamma)) = r(\gamma)$$

We define θ as follows: given $\gamma \in \mathcal{G}$, choose an open bisection $U \ni \pi(\gamma)$, and an $n \in \Sigma_c(\mathcal{G};\mathcal{E})$ supported on $\pi^{-1}(U)$. Then we set

$$\theta(\gamma) := [n, s(\gamma)]$$

Is θ well-defined? Well let's say we choose open bisections U, V in $\mathcal{G}$ containing $\pi(\gamma)$, and $m, n \in \Sigma_c(\mathcal{G};\mathcal{E})$ supported on $\pi^{-1}(U)$, $\pi^{-1}(V)$ respectively. Then for all $h \in D_r$ and $\delta \in \pi^{-1}(U \cap V)$,

$$m^*hm(s(\delta)) \quad = \quad h(r(\delta))m^*m(\delta)n^*hn(s(\delta)) \quad = \quad h(r(\delta))n^*n(\delta)$$

and so $\alpha_m|_{s(\pi^{-1}(U\cap V))} = \alpha_n|_{s(\pi^{-1}(U\cap V))}$. □

Definition 3.0.11 (Weyl Twist)**.** We can impose a different relation on the set $\mathcal{N}$ above as follows.

$$(m,x) \approx (n,y) \quad \iff \quad x = y, \ \text{ and } \ \exists b, b' \in B \ \text{ s.t. } \ b(x), b'(x) > 0, \ mb = nb'$$

Then $\mathcal{E}_{(A,B)} := \mathcal{N}/\approx$ is a groupoid, with precisely the same units, source, range, multiplication and inversion as defined for $\mathcal{N}/\sim\, = \mathcal{G}_{(A,B)}$. We let $[\![m,x]\!]$ denote the equivalence classes under this new relation. We equip $\mathcal{E}_{(A,B)}$ with the topology generated by the sets $\widetilde{Z}(n,U) := \{[\![n,x]\!] \mid x \in U\}$. Then $\mathcal{E}_{(A,B)}$ becomes a locally compact Hausdorff groupoid, and the sets $\widetilde{Z}(n,U)$ are open bisections.

Proof that $\mathcal{E}_{(A,B)}$ is a groupoid. First, an observation: if $[\![m,x]\!] = [\![n,x]\!]$, then $[m,x] = [n,x]$:

$$mb = nb' \quad \implies \alpha_m = \alpha_m \circ \alpha_b = \alpha_{mb} = \alpha_{m'b'} = \alpha_{m'} \circ \alpha_{b'} = \alpha_{m'} \quad \text{where defined}$$

Thus we have a well-defined surjection $\pi_{(A,B)} : \mathcal{E}_{(A,B)} \to \mathcal{G}_{(A,B)}$. When we show that $\mathcal{E}_{(A,B)}$ is a groupoid, then by definition of π it'll be easy to see that it is a continuous groupoid homomorphism and an open map.

Showing that the source and range are well-defined is straightforward. To check that multiplication is well-defined, assume $[\![n,x]\!] = [\![n',x]\!]$ and $[\![m,\alpha_n(x)]\!] = [\![m',\alpha_{n'}(x)]\!]$ (which is possible since α_n and $\alpha_{n'}$ agree on an open neighbourhood of x). Choose b, b' such that $b(\alpha_n(x)), b'(\alpha_{n'}(x)) > 0$ and $mb = m'b'$, and similarly c, c' such that $c(x), c'(x) > 0$ and $nc = n'c'$. Then

$$\begin{aligned} mn(cn^*bn) &= m(nn^*)bnc = mb(nn^*)nc = m'b'(nn^*)nc \\ &= m'(nn^*)b'nc = m'n(n^*b'n)c = m'(nc)(n^*b'n) \\ &= m'(n'c')(n^*b'n) = m'n'(c'n^*b'n) \end{aligned}$$

and so $[\![mn,x]\!] = [\![m'n',x]\!]$. □

Theorem 3.0.12. Let (A,B) be a Cartan pair. The following sequence is a twist.

$$\widehat{B} \times \mathbb{T} \xrightarrow{\iota_{(A,B)}} \mathcal{E}_{(A,B)} \xrightarrow{\pi_{(A,B)}} \mathcal{G}_{(A,B)}$$

Theorem 3.0.13. Let $\mathcal{E} \to \mathcal{G}$ be a twist, and

There is a topological groupoid isomorphism $\zeta : \mathcal{E} \to \mathcal{E}_{(A,B)}$