Lecture: Almost Elementarity and $\mathcal{Z}$-Stability - Section 9

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9 $\mathcal{Z}$-Stability of $C_r^*(\mathcal{G})$

Definition 9.0.1. A castle $\mathcal{C}$ is K-extendable to $\mathcal{D}$ if

$$\mathcal{C} = \{C_{i,j}^{\ell} \ : \ i,j \in E_\ell, \ \ell \in I\}, \qquad \mathcal{D} = \{D_{i,j}^{\ell} \ : \ i,j \in F_\ell, \ \ell \in I\}$$

where $E_\ell \subseteq F_\ell$, $C_{i,j}^{\ell} = D_{i,j}^{\ell}$ if $i,j \in E_\ell$, and

$$K \cdot \bigsqcup_{i,j \in E_\ell} C_{i,j}^{\ell} \subseteq \bigsqcup_{i,j \in F_\ell} D_{i,j}^{\ell}$$

Remark. Notice that the *index sets* I are the same between $\mathcal{C}$ and $\mathcal{D}$ above, meaning $\mathcal{C}$ and $\mathcal{D}$ have the same number of multisections.

Definition 9.0.2. A multisection $\mathcal{C} = \{C_{i,j} \ : \ i,j \in E\}$ is **nested in** $\mathcal{D} = \{D_{i,j} \ : \ i,j \in F\}$ **with multiplicity** $\geq N$ if

- every $\mathcal{C}$-level is contained in a $\mathcal{D}$-level,
- every $\mathcal{D}$-level contains more than N distinct $\mathcal{C}$-levels,
- for every $m,n \in F$, the map

  $$\begin{array}{ccc} \{C \subseteq D_{n,n}\} \cap \mathcal{C}^{(0)} & \to & \{C \subseteq D_{m,m}\} \cap \mathcal{C}^{(0)} \\ C & \mapsto & r(D_{m,n}C) \end{array}$$

  is **bijective**.

Remark. There is quite a bit of hidden structure in these definitions. For instance, the maps $C \mapsto r(D_{m,n}C)$ have the following nice structure:

- $C \mapsto r(D_{n,n}C)$ is the identity
- $r(D_{n,m}r(D_{m,n}C)) = C$ (in other words, the maps $C \mapsto r(D_{m,n}C)$ and $C \mapsto r(D_{n,m}C)$ are inverses)
- $r(D_{n,k}r(D_{m,n}C)) = r(D_{m,k}C)$ (in other words, composing the maps $C \mapsto r(D_{m,n}C)$ and $C \mapsto r(D_{n,k}C)$ gives the map $C \mapsto r(D_{m,k}C)$).

Additionally, notice that the sets $\{C \subseteq D_{m,m}\} \cap \mathcal{C}^{(0)}$ all have the same cardinality, so every $\mathcal{D}$-level contains exactly the same number of $\mathcal{C}$-levels, which is $> N$.

Definition 9.0.3. Let $\mathcal{C}, \mathcal{D}$ be castles. We say $\mathcal{C}$ is **nested in $\mathcal{D}$ with multiplicity** $\geq N$ if

- every multisection in $\mathcal{C}$ nests in a *unique* multisection in $\mathcal{D}$ with multiplicity $\geq N$,
- every multisection in $\mathcal{D}$ admits at least one multisection in $\mathcal{C}$ which nests in it with multiplicity $\geq N$ (there are no "superfluous" multisections in $\mathcal{D}$)

Definition 9.0.4. Recall from Dolapo's talk, we constructed what we called a **$\mathcal{B}$-compatible collection** $\{h_B \ : \ B \in \mathcal{B}\} \subset C_c(\mathcal{G})$, and an **$\mathcal{H}^{(0)} - \mathcal{B}^{(0)}$-nesting system**.

An $\mathcal{B}$-compatible collection is a collection of functions in $C_c(\mathcal{G})$ indexed by the bisections in $\mathcal{B}$, such that

- $s(h_B) = h_{s(B)}$, and $r(h_B) = h_{r(B)}$
- $h_{r(B)} * 1_B = h_B$, and $1_B * h_{s(B)} = h_B$
- $\mathrm{supp}(h_B) \subseteq B$
- $h_B \equiv 1$ on B', where $B' \subseteq B$ is a compact bisection (the B''s arise from a compact castle $\mathcal{B}' \Subset \mathcal{B}$).

For each $(\mathcal{H}^p)^{(0)}$, let I_p denote those ℓ for which $\mathcal{B}^\ell$ nests in $\mathcal{D}^p$ with multiplicity $\geq nN$.

$$\begin{tikzcd}
\mathcal{B}^{\ell_r} \arrow[dr, bend left, "\geq nN"] & \\
\vdots \arrow[r, "\geq nN"] & \mathcal{D}^p \\
\mathcal{B}^{\ell_1} \arrow[ur, bend right, "\geq nN"'] &
\end{tikzcd} \qquad\qquad I_p = \{\ell_1, ..., \ell_r\}$$

Let $D \in (\mathcal{H}^p)^{(0)}$ and $\ell \in I_p$. We began by letting

$$P^{D,\ell} = \{B \in (\mathcal{B}^\ell)^{(0)} \ : \ B \subseteq D\}, \qquad \text{implying} \ \ |P^{D,\ell}| \geq nN$$
$$P_i^{D,\ell} \subset P^{D,\ell}, \ i = 1, ..., n \qquad |P_i^{D,\ell}| = \lfloor |P^{D,\ell}|/n \rfloor \geq N$$

We then obtained bijections $\Theta_{i,j}^{D,\ell} : P_j^{D,\ell} \to P_i^{D,\ell}$ such that

- $\Theta_{i,i}^{D,\ell} = \mathrm{id}, \qquad \Theta_{i,j}^{D,\ell \, -1} = \Theta_{j,i}^{D,\ell}, \qquad \Theta_{i,j}^{D,\ell} \circ \Theta_{j,k}^{D,\ell} = \Theta_{i,k}^{D,\ell}$
- $\forall D \in \mathcal{D}$ such that $s(D), r(D) \in (\mathcal{H}^p)^{(0)}$,
$$r(D\Theta_{i,j}^{s(D),\ell}(B)) = \Theta_{i,j}^{r(D),\ell}(r(DB)) \qquad \forall B \in P_{i,j}^{s(D),\ell}$$

Given $D \in (\mathcal{H}^p)^{(0)}$, let

$$R_{i,j}^{D,\ell} = \{B \in \mathcal{B}^\ell \ : \ s(B) \in P_j^{D,\ell}, \ \text{ and } \ r(B) = \Theta_{i,j}^{D,\ell}(s(B))\}, \qquad i, j = 1, ..., n, \ \ell \in I_p$$

and finally set

$$Q_{i,j}^{D} := \bigsqcup_{\ell \in I_p} R_{i,j}^{D,\ell}$$

With this setup, and a function $\kappa : \mathcal{H}^{(0)} \to [0,1]$, we defined $\psi : M_n(\mathbb{C}) \to C_r^*(\mathcal{G})$ by

$$\psi(e_{ij}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^{D}} \kappa(D) h_B$$

Theorem 9.0.5. The maps ψ are c.p.c. order zero maps.

Proof. Define $\varphi : M_n(\mathbb{C}) \to C_r^*(\mathcal{G})^{**}$ by

$$\varphi(e_{ij}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^{D}} 1_B$$

By the properties of the maps $\Theta_{i,j}$, one can check that φ is in fact a $*$-homomorphism.

Next let $h_0 := \psi(1_n)$. We calculate

$$h_0 * \varphi(e_{i,j}) = \sum_{k=1}^{n} \sum_{D, D' \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^{D}} \sum_{B' \in Q_{k,k}^{D'}} \kappa(D') h_{B'} * 1_B$$

but by the definition of a $\mathcal{B}$-compatible system,

$$h_{B'} * 1_B = \begin{cases} h_B & \text{if } \ B' = r(B), \ D' = D, \ i = k \\ 0 & \text{otherwise} \end{cases}$$

and so our quintuple sum above collapses to simply

$$h_0 * \varphi(e_{i,j}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^{D}} \kappa(D) h_B = \psi(e_{i,j})$$

A similar calculation yields

$$\psi(e_{i,j}) = h_0 * \varphi(e_{i,j}) = \varphi(e_{i,j}) * h_0$$

which tells us that $\varphi : M_n(\mathbb{C}) \to C_r^*(\mathcal{G})^{**} \cap \{h_0\}'$, and a well-known structure theorem for cpc order zero maps tells us that ψ is indeed order zero. $\square$

Theorem 9.0.6. Let $\mathcal{G}$ be lcHé, minimal, σ-compact, with $\mathcal{G}^{(0)}$ compact metrizable. If $\mathcal{G}$ is *almost elementary*, then $C_r^*(\mathcal{G})$ is tracially $\mathcal{Z}$-stable.

Proof. Recall from Dolapo's talk that we've reduced the problem as follows. Given

- $\epsilon > 0, \quad n \in \mathbb{N}, \quad g \in C(\mathcal{G}^{(0)})_+$

- $F \subset C_c(\mathcal{G})$ finite, such that $\forall f \in F$
  - $\operatorname{supp}(f)$ is a compact subset of an open bisection V_f
  - $\mu(\partial s(\operatorname{supp}(f))) = 0$ for all $\mu \in M(\mathcal{G})$

we need to find cpc order zero maps $\psi : M_n(\mathbb{C}) \to C_r^*(\mathcal{G})$ such that

1. $1_{C_r^*(\mathcal{G})} - \psi(1_n) \precsim g$
2. $\forall x \in M_n(\mathbb{C})$ with $\|x\| = 1$ and $f \in F$, $\|[\psi(x), f]\|_r < \epsilon$.

Part 1: Broad Setup

Let $m = |F|$. We will start by writing $\mathcal{O}_f := \operatorname{supp}^\circ f$ for each $f \in F$. We will also use the symbols U_f to denote *either* $\mathcal{O}_f$ or $\mathcal{O}_f^{-1}$: that is, any statement with an occurrence of a symbol U_f should be thought of as multiple separate statements with U_f replaced by each of $\mathcal{O}_f$ and $\mathcal{O}_f^{-1}$.

Since $\mathcal{G}$ is minimal, there is $\eta > 0$ such that $\mu(\operatorname{supp}^\circ g) > 2\eta$ for all $\mu \in M(\mathcal{G})$ (and $\eta < 1/2$ necessarily). Choose $N > \max\{2n^2/\epsilon, 1/\eta\}$.

Let
$$S := \bigcup_{f \in F} (\partial s(\mathcal{O}_f) \cup \partial r(\mathcal{O}_f))$$
Since S has measure zero, by a lemma from Paweł's talk, there is $\delta > 0$ such that
$$\mu\left(\overline{B_\delta}(S)\right) < \frac{\eta}{(2|F|)^{N+1}}$$
Define the compact set
$$K := \mathcal{G}^{(0)} \cup \bigcup_{i=1}^{N+1} \left(\bigcup_{f_1, \ldots, f_i \in F} \overline{U_{f_1}} \cdot \overline{U_{f_2}} \cdots \overline{U_{f_i}} \right)$$
Finally, choose an open cover $\mathcal{V}$ of $\mathcal{G}^{(0)}$ such that every member of $\mathcal{V}$ has diameter less than δ, and for any $V \in \mathcal{V}$, $u, v \in V$, and $f \in F$,
$$|s(f)(u) - s(f)(v)| < \varepsilon/2n^2$$

Since $\mathcal{G}$ is almost elementary, given our compact set K and open cover $\mathcal{V}$, by the theorem given in Paweł's talk, we can choose **open castles** $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (aptly named a *kingdom*) such that:

- $\overline{\mathcal{C}}$ and $\overline{\mathcal{D}}$ are **compact**,
- $\mathcal{A}$ is $\mathcal{K}$-extendable to $\mathcal{B}$, and $\mathcal{C}$ is K-extendable to $\mathcal{D}$,
- $\mathcal{A}$ is nested in $\mathcal{C}$ with multiplicity nN,
- $\mathcal{B}$ is nested in $\mathcal{D}$ with multiplicity nN,
- $\mu\left(\mathcal{G}^{(0)} \setminus \bigsqcup \mathcal{A}^{(0)}\right) < \eta$, for all $\mu \in M(\mathcal{G})$.

Remark. Paweł only stated that we can obtain $\mathcal{G}^{(0)} \backslash \bigsqcup \mathcal{A}^{(0)} \precsim_{\mathcal{G}} \mathcal{O}$, given an open set $\mathcal{O}$, so what gives? In the process of proving that theorem, they obtain an $\theta > 0$ such that

$$\mu(\mathcal{O}) > 2\theta \geq \mu(\mathcal{G}^{(0)} \backslash \bigsqcup \mathcal{A}^{(0)}) \qquad \forall \mu \in M(\mathcal{G})$$

from which it follows that $\mathcal{G}^{(0)} \backslash \bigsqcup \mathcal{A}^{(0)} \precsim \mathcal{O}$ by groupoid strict comparison. So here, $\operatorname{supp}^\circ g$ plays the role of $\mathcal{O}$, and η the role of θ.

Additionally, by a remark in section 8 (remark 8.10), we can also furnish

- $\forall i \leq N+1,\ f_1, ..., f_i \in F$, and $\mathcal{C}$-level $C \in \mathcal{C}^{(0)}$, *either*

$$\begin{gathered} C \subseteq s(U_{f_1} \cdot U_{f_2} \cdots U_{f_i}), \quad \text{or} \\ C \cap s(U_{f_1} \cdot U_{f_2} \cdots U_{f_i}) = \emptyset \end{gathered}$$

  and moreover, if $C \subseteq s(U_{f_1} \cdot U_{f_2} \cdots U_{f_i})$, then $s(D) = C$ and $U_{f_1} \cdot U_{f_2} \cdots U_{f_i} C = D$ for some $D \in \mathcal{D}$ (*not necessarily* a $\mathcal{D}$-level).

We remark that under these conditions, we can also obtain the following fact:

- for any $\mathcal{D}$-level $D \in \mathcal{D}^{(0)}$,

$$\begin{array}{llll} \forall f \in F & \text{either} \ \ D \subseteq s(\mathcal{O}_f) & \text{or} \ \ D \cap s(\mathcal{O}_f) = \emptyset & \text{and} \\ & \text{either} \ \ D \subseteq r(\mathcal{O}_f) & \text{or} \ \ D \cap r(\mathcal{O}_f) = \emptyset & \end{array}$$

Part 2: Obtaining a Nesting System

We're going to inductively define $\mathcal{D}$-level sets by letting $\mathcal{D}_0^{(0)} = C^{(0)}$, and for $k = 1, ..., N+1$,

$$\mathcal{D}_k^{(0)} = \left\{ D \in \mathcal{D}^{(0)} \ : \ D = r(U_{f_k} \cdots U_{f_1} C),\ f_1, ..., f_k \in F,\ C \in \mathcal{C}^{(0)} \right\} \Big\backslash \bigsqcup_{i=0}^{k-1} \mathcal{D}_i^{(0)}$$

and then define

$$\mathcal{H}^{(0)} = \bigsqcup_{k=0}^{N} \mathcal{D}_k^{(0)}$$

(notice we're using everything *except* $\mathcal{D}_{n+1}^{(0)}$). In other words, $\mathcal{H}^{(0)}$ consists of those $\mathcal{D}$-levels D which are expressible as $D = r(U_{f_i} \cdots U_{f_1} C)$, for some $i = 1, ..., N$, some $f_j \in F$, and some $C \in \mathcal{C}^{(0)}$. The $\mathcal{D}_k^{(0)}$'s are just those D-levels which are expressible this way, and for which the *minimal* such i for which such an expression exists is $i = k$.

Part 3: The Order Zero Maps

Define $\kappa : \mathcal{H}^{(0)} \to [0,1]$ by $\kappa(D) = 1 - k/N$ if $D \in \mathcal{D}_k^{(0)}$, and define $\psi : M_n(\mathbb{C}) \to C_r^*(\mathcal{G})$ by

$$\psi(e_{ij}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^D} \kappa(D) h_B$$

which we know is cpc order zero. Let $h = 1_{C_r^*(\mathcal{G})} - \psi(1_n)$. By the final remark from Dolapo's talk, we have

$$\mu(\operatorname{supp}^\circ h) < 2\eta < \mu(\operatorname{supp}^\circ g) \qquad \forall \mu \in M(\mathcal{G})$$
$$\implies \operatorname{supp}^\circ h \precsim_{\mathcal{G}} \operatorname{supp}^\circ g$$

By a result from Xin Ma's prior paper on the topic, this implies $h \precsim g$ (Cuntz subequivalence), so $1_{C_r^*(\mathcal{G})} - \psi(1_n) \precsim g$.

Now we'd like to take a look at the quantities $[\psi(e_{ij}), f]$. We're going to rearrange the expression a bit to get it into a manageable form (which, ironically, will appear at a glance less manageable).

Given $f \in F$, define

$$S_f = \left\{ D \in \mathcal{H}^{(0)} \ : \ D \subseteq s(\mathcal{O}_f),\ r(\mathcal{O}_f D) \in \mathcal{H}^{(0)} \right\}$$
$$R_f = \left\{ D \in \mathcal{H}^{(0)} \ : \ D \subseteq r(\mathcal{O}_f),\ r(\mathcal{O}_f^{-1} D) \in \mathcal{H}^{(0)} \right\}$$

and observe that the map $\sigma_f : S_f \to R_f$ defined by $\sigma_f(D) = r(\mathcal{O}_f D)$ is *bijective*.

We *also* need to know that $\sigma_f(D)$ occupies the *same multisection* of $\mathcal{D}$ as D.

Define $\pi_f : S_f \to \mathcal{D}$ by requiring $\pi_f(D)$ be the $\mathcal{D}$-ladder taking D to $\sigma_f(D)$.

Now define $\theta_{i,j}^{f,D} : Q_{i,j}^D \to Q_{i,j}^{\sigma_f(D)}$ as follows. Given $B \in Q_{i,j}^D$, choose $\theta_{i,j}^{f,D}(B)$ such that

$$s(\theta_{i,j}^{f,D}(B)) = r(\pi_f(D)s(B)), \qquad r(\theta_{i,j}^{f,D}(B)) = r(\pi_f(D)r(B))$$

from which it follows that

$$\pi_f(D)B = \theta_{i,j}^{f,D}(B)\pi_f(D)s(B)$$

With these maps in place, we can write

$$\begin{aligned}
[\psi(e_{i,j}), f] &= \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j}^D} \kappa(D)(f * h_B - h_B * f) \\
&= \sum_{D \in S_f} \sum_{B \in Q_{i,j}^D} \kappa(D) f * h_B - \sum_{D \in R_f} \sum_{B \in Q_{i,j}^D} \kappa(D) h_B * f \\
&= \sum_{D \in S_f} \sum_{B \in Q_{i,j}^D} \underbrace{\left(\kappa(D) f * h_B - \kappa(\sigma_f(D)) h_{\theta_{i,j}^{f,D}(B)} * f \right)}_{:= \alpha_{D,B}}
\end{aligned}$$

To justify the jump from the first to the second line, first recall that for all $\mathcal{D} \in \mathcal{D}^{(0)}$, either $s(\mathcal{O}_f)$ of $D \cap s(\mathcal{O}_f) = \emptyset$, but in the latter case, for any $B \in Q_{i,j}^D$, by definition we have $r(B) \subseteq D$, and so $s(\mathcal{O}_f) \cap r(B) = \emptyset$, which implies $f * h_B = 0$. So we can ignore those D for which $D \not\subset s(\mathcal{O}_f)$, and similarly those $D \not\subset r(\mathcal{O}_f)$.

Additionally, suppose $\emptyset \neq D \in \mathcal{H}^{(0)}$ is such that $D \subseteq s(\mathcal{O}_f)$, *but* $r(\mathcal{O}_f D) \notin \mathcal{H}^{(0)}$ (so $D \notin S_f$). Then **necessarily** $D \in \mathcal{D}_N^{(0)}$, in which case $\kappa(D) = 0$, so these terms can safely be ignored as well.

Finally, jumping from the second to the third line is just what the bijections σ_f and $\theta_{i,j}^{f,D}$ were designed to accomplish.

This may at first seem uglier, but a nicety we're afforded is that the summands, which we've denote $\alpha_{D,B}$, are supported on a bisection, namely

$$\mathrm{supp}(\alpha_{D,B}) \subseteq \pi_f(D)B = \theta_{i,j}^{f,D}(B)\pi_f(D)s(B)$$

Let's denote the common bisection of these two formulas as B'. Take $\gamma \in B'$. Then we can write $\gamma = \alpha_1\beta_1 = \beta_2\alpha_2$ in two different ways choosing from these two different formulas for B', but in each case the pairs (α_1, β_1) and (β_2, α_2) are unique. We choose

$$\begin{array}{cc} \alpha_1 \in \pi_f(D)r(B) & \beta_1 \in B \\ \beta_2 \in \theta_{i,j}^{f,D}(B) & \alpha_2 \in \pi_f(D)s(B) \end{array}$$

Then notice that

$$\begin{aligned} h_B(\beta_1) &= h_{s(B)}(s(\beta_1)) = h_{s(B')}(s(\gamma)) = h_{B'}(\gamma) \\ &= h_{r(B')}(r(\gamma)) = h_{r(\theta_{i,j}^{f,D}(B))}(r(\beta_2)) = h_{\theta_{i,j}^{f,D}}(\beta_2) \end{aligned}$$

and additionally, by virtue of our choice of open cover $\mathcal{V}$,

$$|f(\alpha_1) - f(\alpha_2)| = |s(f)(s(\alpha_1)) - s(f)s(\alpha_2))| < \epsilon/2n^2$$

and so

$$\begin{aligned} |f * h_B(\gamma) - h_{\theta_{i,j}^{f,D}(B)} * f(\gamma)| &= |f(\alpha_1)h_B(\beta_1) - h_{\theta_{i,j}^{f,D}(B)})(\beta_2)f(\alpha_2)| \\ &= |h(\beta_1)||f(\alpha_1) - f(\alpha_2)| < \epsilon/2n^2 \end{aligned}$$

whence $\|f * h_B - h_{\theta_{i,j}^{f,D}(B)} * f\|_\infty \leq \varepsilon/2n^2$.

We can also obtain a bound on $|\kappa(D) - \kappa(\sigma_f(D))|$. Suppose $D \in \mathcal{H}^{(0)}$, and suppose we can write $D = r(U_{f_1} \cdots U_{f_k}C)$ for some $k \leq N$. Recalling our handy fact that $r(Ar(BC)) = r(ABC)$, we see that

$$\sigma_f(D) = r(\mathcal{O}_f D) = r(\mathcal{O}_f U_{f_1} \cdots U_{f_k}C) \in \mathcal{H}^{(0)} \sqcup \mathcal{D}_{N+1}^{(0)}$$

but this formula allows us to analyze "where in $\mathcal{H}^{(0)}$" σ_f takes D.

- If $D \in \mathcal{D}_0^{(0)}$, then $\sigma_f(D) \in \mathcal{D}_0^{(0)} \sqcup \mathcal{D}_1^{(0)}$
- If $D \in \mathcal{D}_k^{(0)}$, $k = 1, ..., N-1$, then $\sigma_f(D) \in \mathcal{D}_{k-1}^{(0)} \sqcup \mathcal{D}_k^{(0)} \sqcup \mathcal{D}_{k+1}^{(0)}$.
- If $D \in \mathcal{D}_N^{(0)}$, then $\sigma_f(D) \in \mathcal{D}_{N-1}^{(0)} \sqcup \mathcal{D}_N^{(0)}$ (we can't get $\mathcal{D}_{N+1}^{(0)}$ since it's disjoint from $\mathcal{H}^{(0)}$, and we've assumed σ_f maps back into $\mathcal{H}^{(0)}$).

In any of these cases, $|\kappa(D) - \kappa(\sigma_f(D))| < 1/N < \epsilon/2n^2$.

Putting these bounds together through an application of the triangle inequality, we get

$$\begin{aligned} \|\alpha_{D,B}\|_r &= \|\alpha_{D,B}\|_\infty \\ &\leq \underbrace{|\kappa(D) - \kappa(\sigma_f(D))|}_{\leq \epsilon/2n^2} \|f * h_B\|_\infty + \kappa(\sigma_f(D)) \underbrace{\|f * h_B - h_{\theta_{i,j}^{f,D}(B)} * f\|_\infty}_{\leq \epsilon/2n^2} \\ &\leq \epsilon/n^2 \end{aligned}$$

Next, we observe that when $(D_1, B_1) \neq (D_2, B_2)$, then $\alpha^*_{D_1,B_1}\alpha_{D_2,B_2} = \alpha_{D_1,B_1}\alpha^*_{D_2,B_2} = 0$, and so $\|\alpha_{D_1,B_1} + \alpha_{D_2,B_2}\|_r = \max(\|\alpha_{D_1,B_1}\|_r, \|\alpha_{D_2,B_2}\|_r)$, and more generally

$$\|[\psi(e_{i,j}), f]\|_r = \max_{D,B} \|\alpha_{D,B}\|_r \leq \epsilon/n^2$$

Then for any $x = \sum_{i,j} x_{i,j}e_{i,j} \in M_n(\mathbb{C})$ with $\|x\| = 1$, one has

$$\|[\psi(x), f]\|_r \leq \sum_{i,j} \|[\psi(e_{ij}), f]\|_r \leq \epsilon$$

as desired. □

10 Appendix A

Fact 10.0.1. Let A, B be bisections, and $C \subseteq \mathcal{G}^{(0)}$. Then $r(Ar(BC)) = r(ABC)$.

Proof. This really just follows from writing out the explicit expressions, but we'll do one direction to be sure. Suppose $x \in r(Ar(BC))$, so there exist $a \in A$, $b \in B$ such that $r(a) = x$, $s(a) = r(b)$, and $r(b) \in C$. Then clearly $y = abr(b) \in ABC$ and $r(y) = x$, so $x \in r(ABC)$. □

Corollary 10.0.2. Recall the definition of a **nesting** of a multisection $\mathcal{C}$ into a multisection $\mathcal{D}$ with multiplicity $\geq N$. Then $r(D_{n,m}r(D_{m,n}C)) = C$, and $r(D_{n,k}r(D_{m,n}C)) = r(D_{m,k}C)$.

Fact 10.0.3. Let $S, T \in \mathcal{B}(\mathcal{H})$, and suppose $S^*T = ST^* = 0$. Then $\|S + T\| = \max(\|S\|, \|T\|)$.

Proof. First, notice that $\|S\| = \left\|(S+T)\frac{S^*}{\|S\|}\right\| \leq \sup_{\|U\|=1} \|(S + T)U\| = \|S + T\|$, and similarly $\|T\| \leq \|S + T\|$.

Notice that the conditions $S^*T = ST^* = 0$ imply $\operatorname{ran} S \perp \operatorname{ran} T$ and $\ker S^\perp \perp \ker T^\perp$. Suppose $\|x\| = 1$. Write $y = P_{\ker S^\perp}x$ and $z = P_{\ker T^\perp}x$. Then

$$\begin{aligned}\|(S + T)x\|^2 &= \|Sy + Tz\|^2 = \|Sy\|^2 + \|Tz\|^2 \qquad \text{since } \langle Sy, Tz\rangle = 0\\ &= \|(S \oplus T)(y \oplus z)\|^2\end{aligned}$$

and

$$\begin{aligned}\|y \oplus z\|^2 &= \|y\|^2 + \|z\|^2 = \|y + z\|^2 \qquad \text{since } \langle y, z\rangle = 0\\ &= \|(P_{\ker S^\perp} + P_{\ker T^\perp})x\|^2\\ &\leq \|x\|^2 = 1 \qquad \text{since } P_{\ker S^\perp} + P_{\ker T^\perp} \text{ is a projection, because } \ker S^\perp \perp \ker T^\perp\end{aligned}$$

whence $\|S + T\| \leq \|S \oplus T\| = \max(\|S\|, \|T\|)$. Combined with the lower bounds above, this proves the result. □

11 Appendix B

Old Nesting System

Define $\mathcal{C}_0^{(0)} := \{C \in \mathcal{C}^{(0)} \ : \ C \cap S = \emptyset\}$. Observe that $\forall \mu \in M(\mathcal{G})$

$$\mu\left(\bigcup\{A \in \mathcal{A}^{(0)} \ : \ A \subseteq C, \ C \in \mathcal{C}_0^{(0)}\}\right) \geq 1 - \eta - \eta/(2|F|)^{N+1}$$

Next, let

$$\begin{aligned}
\mathcal{C}'^{(0)}_0 &= \{C \in \mathcal{C}_0^{(0)} \ : \ \exists f \in F, \ D \in \mathcal{D}^{(0)} \text{ s.t. } D = r(U_f C) \text{ and } D \cap S \neq \emptyset\} \\
\mathcal{C}_1^{(0)} &= \mathcal{C}_0^{(0)} \backslash \mathcal{C}'^{(0)}_0
\end{aligned}$$

A calculation reveals that for all $\mu \in M(\mathcal{G})$

$$\begin{aligned}
\mu\left(\bigcup \mathcal{C}_1^{(0)}\right) &\geq \mu\left(\bigcup\left\{A \in \mathcal{A}^{(0)} \ : \ A \subseteq C, \ C \in \mathcal{C}_1^{(0)}\right\}\right) \\
&\geq 1 - \eta - \eta/(2|F|)^{N+1} - \eta/(2|F|)^N
\end{aligned}$$

We can continue this process inductively, obtaining collections $\mathcal{C}_k^{(0)}$, $k < N$, such that

1. $\forall C \in \mathcal{C}_k^{(0)}$, if $U_{f_i} \cdots U_{f_1} C$ is equal to some $D \in \mathcal{D}$ for some $f_1, ..., f_i \in F$, $i \leq k$, then $r(D) \cap S = \emptyset$,
2. for all $\mu \in M(\mathcal{G})$,

$$\mu\left(\bigcup \mathcal{C}_k^{(0)}\right) \geq 1 - \eta\left(1 + \sum_{i=0}^{k} (2|F|)^{-(N+1-i)}\right)$$

In general, we have

$$\mathcal{C}_k^{(0)} = \left\{C \in \mathcal{C}^{(0)} \ : \ \text{if } U_{f_1} \cdots U_{f_i} C = D \ \text{ for some } i \leq k \text{ and } f_j \in F, \text{ then } \ r(D) \cap S = \emptyset\right\}$$

In particular, we obtain $\mathcal{C}_N^{(0)}$, and the second condition above gives

$$\mu\left(\bigcup \mathcal{C}_N^{(0)}\right) > 1 - \eta \sum_{k=0}^{N+1} (2|F|)^{-k}$$

$$\sum_{k=0}^{N+1} (2|F|)^{-k} \leq \sum_{k=0}^{\infty} (2|F|)^{-k} = \frac{1}{1 - 1/(2|F|)} \leq 2$$

$$\mu\left(\bigcup \mathcal{C}_N^{(0)}\right) > 1 - 2\eta > 0$$

This is all to say that $\mathcal{C}_N^{(0)}$ is non-empty.

Next define $\mathcal{D}_0^{(0)} = \mathcal{C}_N^{(0)}$, and inductively define or $k = 1, ..., N+1$

$$\mathcal{D}_k^{(0)} = \left\{D \in \mathcal{D}^{(0)} \ : \ D = r(U_{f_k} \cdots U_{f_1} C), \ f_1, ..., f_k \in F, \ C \in \mathcal{C}_N^{(0)}\right\} \Big\backslash \bigsqcup_{i=0}^{k-1} \mathcal{D}_i^{(0)}$$

Note that some $\mathcal{D}_k^{(0)}$ may be empty, but this is not a problem. Define

$$\mathcal{H}^{(0)} = \bigsqcup_{k=0}^{N} \mathcal{D}_k^{(0)}$$

(notice we're using everything *except* $\mathcal{D}_{n+1}^{(0)}$). In other words, $\mathcal{H}^{(0)}$ consists of those $\mathcal{D}$-levels D which are expressible as $D = r(U_{f_i} \cdots U_{f_1} C)$, for some $i = 1, ..., N$, some $f_j \in F$, and some $C \in \mathcal{C}_N^{(0)}$. The $\mathcal{D}_k^{(0)}$'s are just those D-levels which are expressible this way, and for which the *minimal* such i for which such an expression exists is $i = k$.

Additionally, for every $D \in \mathcal{H}^{(0)}$, $D \cap S = \emptyset$.

Then we have

$$\mathcal{C}_N^{(0)} \subseteq \mathcal{H}^{(0)} \subseteq \mathcal{D}^{(0)}$$