Nuclear C*-Algebras and the Weak Expectation Property

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1 Introduction

1.1 A Brief History of Tensor Products in Operator Theory

When one considers the tensor product of two C*-algebras, a natural expectation is that the resulting object is itself a C*-algebra, as is the case for tensor products of other "discrete" entities (vector spaces, modules, algebras, involutive algebras, etc.). However, one immediately runs into a troubling issue: there seem to be a couple of different C*-norms one can define on the tensor product of C*-algebras. Do they coincide? If not, which norm is the "right" norm? Each norm results in a *different* C*-algebraic tensor product, with a radically different collection of properties. Were different tensor products more "suited" to different situations? What was an operator theorist to do?

Takasi Turumaru initiated the study of tensor products of C*-algebras in 1952 in [40], and in this paper introduced the first natural C*-tensor norm. Guichardet in [18] introduced another natural C*-tensor norm, which could easily be seen to be the *maximal* norm among all C*-tensor norms, and so induced the *maximal tensor product*: $A \otimes_{\max} B$. Over a decade after Turumaru's paper, Takesaki would prove a suprising result: Turumaru's norm is the *minimal* norm among all C*-tensor norms (see [38]), and so induced the *minimal tensor product*: $A \otimes_{\min} B$.

In this same paper, Takesaki also introduced a seemingly innocent concept: "nuclearity" (which he called "Property (T)"). A C*-algebra A is said to be *nuclear* if its tensor product with any other C*-algebra B admitted a *unique* C*-norm (or, more succinctly, $A \otimes_{\min} B = A \otimes_{\max} B$).

For two decades following Turumaru's initial investigation, little progress had been made on determining which C*-algebras possessed the elusive "property (T)". Takesaki *did* provide a healthy class of examples - type I C*-algebras and direct limits of such - but beyond this, a *non-tensorial characterization* of nuclearity remained out of reach. Identifying nuclear C*-algebras was no fruitless exercise either. It was apparent from Takesaki's paper that the different tensor products behaved wildly different from one another. When Guichardet's tensor product possessed some desirable property, the same property would often fail disastrously for Turumaru's tensor product, and vice versa. Identifying nuclearity was thus the key to resolving these various conflicts.

While direct progress may have been scant, *indirect* progress was certainly accumulating. The key figureheads here were Turumaru, Guichardet, Takesaki, Wulfsohn, Tomiyama, and their contemporaries, often in tangential areas of study. Arveson and Stinespring, for instance, studied *completely positive maps*, whose relationship to tensor products was realized by Choi and Effros in [7], as well as Lance in [26].

Through the joint work of Lance, Effros, Choi, and Kirchberg, from 1975 to 1978, these precursory tools were synthesized into the first equivalent characterization of nuclearity: the "completely positive approximation property", discovered by Choi and Effros in [10], and independently by Kirchberg in [24]. The CPAP, roughly speaking, allowed one to "approximately factor the identity map id_A through finite dimensional matrix algebras $M_n(\mathbb{C})$", which demonstrated that nuclear C*-algebras were, in a certain sense, "small", with some authors comparing nuclearity to compactness of topological spaces.

Lance and Effros also related nuclearity of a C*-algebra A to various properties of A's *enveloping von Neumann algebra* A^{**}. In particular, they were able to show that if A^{**} is a so-called "*semidiscrete*" von Neumann algebra, then A must be nuclear. With the help of Haagerup and some results from Tomita-Takesaki theory, they also showed that if A is nuclear then A^{**} is an *injective* von Neumann algebra.

The definition of “semidiscreteness” was highly reminiscent of the CPAP - you could even call it an “adaptation” of the CPAP to the category of von Neumann algebras. Having already established that nuclearity was equivalent to the CPAP, this posed a natural question: if A is nuclear when A^{**} is semidiscrete, does the converse also hold? Lance and Effros also came incredibly close to proving another conjecture in their work: was A nuclear if and *only if* A^{**} was *injective*? In summation, we have the following question:

Question 1.1.1. For a C*-algebra A with enveloping von Neumann algebra A^{**}, are the following three properties equivalent?

1. A is nuclear.
2. A^{**} is semidiscrete.
3. A^{**} is injective.

In 1976 Choi and Effros received a preprint from Alain Connes, in which he proved the following (using some fairly advanced von Neumann algebra theory):

Theorem 1.1.2 (Connes)**.** If $M \subseteq \mathcal{B}(H)$ is a *factor* (a von Neumann algebra with trivial center) on a separable Hilbert space H, then M is injective if and only if M is semidiscrete.

With this result, and a little extra work (well, a lot of extra work), Choi and Effros proved that injectivity and semidiscreteness were equivalent *in general*, thus giving an affirmative answer to the question above:

Theorem 1.1.3 (Choi-Effros)**.** If A is a C*-algebra, then A is nuclear if and only if A^{**} is injective, if and only if A^{**} is semidiscrete.

At the same time, Lance was working on a question *related* to nuclearity, but somewhat more general. Whenever you have C*-algebras $A \subseteq B, C \subseteq D$, there is a “natural inclusion” $A \otimes_{\min} C \subseteq B \otimes_{\min} D$, which fails in general: the “min” here is *crucial* to the validity of this statement. For instance, Lance observed that the same phenomenon *doesn't* always hold for the max tensor product:

$$A \otimes_{\max} C \not\subseteq B \otimes_{\max} D$$

If A were nuclear, then we would at least have $A \otimes_{\max} C = A \otimes_{\min} C \subseteq A \otimes_{\min} D = A \otimes_{\max} D$, but it's unclear how one would get the *other* inclusion $A \otimes_{\max} D \subseteq B \otimes_{\max} D$. Lance's goal was to determine *precisely* when this would occur: for which algebras A do we always have a natural inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} C$, whenever $A \subseteq B$?

His work led him to develop a new property of C*-algebras: the “weak expectation property” (or WEP), which posits the existence of a certain completely positive bimodule map (called a “weak expectation”) from $\mathcal{B}(H) \to \overline{A}^{wot}$, whenever $A \subseteq \mathcal{B}(H)$ for a Hilbert space H. In particular, Lance proved the following seminal theorem about the WEP:

Theorem 1.1.4 (Lance)**.** A C*-algebra A has the WEP if and only if $A \otimes_{\max} C \subseteq B \otimes_{\max} C$ for any C*-algebras $B \supseteq A$ and C.

One corollary of this result is that *nuclear* C*-algebras are automatically “WEP algebras” (C*-algebras possessing the WEP), a result which is actually quite quick to prove with the right preliminaries. In the von Neumann algebra case, one can even prove the following:

Proposition 1.1.5. If M is a von Neumann algebra, then M is injective if and only if M has the WEP. Thus A is nuclear if and only if A^{**} has the WEP.

The WEP and nuclearity are quite intimately related, and this was well known to Lance and Effros, who at one point even conjectured that the WEP was *equivalent* to nuclearity. The two notions, however, are decidedly *not* coincident. Still, nuclearity seemed only a stone's throw away from the WEP in many instances, tantalizingly close to being equivalent. As such, there are a few *additional* properties WEP algebras can possess which *guarantee* they're nuclear. Two such properties are “exactness” and “local reflexivity”, the latter of which we will explore later on.

Despite the enticing parallels between the WEP and nuclearity, the latter remained the zeitgeist in operator theory over the next fifty years, whereas the WEP somewhat fell into obscurity by comparison, largely ignored as a property to be studied in its own right. It certainly is harder to visualize the "merit" of the WEP compared to nuclearity: that a C*-algebra is nuclear tells us there's a unique way of constructing the C*-algebra tensor product of two C*-algebras, and moreover that this tensor product enjoys all the delightful properties of *every* C*-algebraic tensor product ($\otimes_{\min}$, $\otimes_{\max}$, etc.). Coming up with such a nice intuition behind the WEP, which *doesn't* return to Lance's seminal theorem, is comparatively difficult.

It wasn't until the early 90s when Kirchberg published [25], which established many of the results on the WEP which are used today, including the following brilliant result:

Theorem 1.1.6 (Kirchberg)**.** A C*-algebra A has the WEP if and only if $A \otimes_{\min} \mathscr{C} = A \otimes_{\max} \mathscr{C}$, where $\mathscr{C}$ is the full group C*-algebra of the free group $\mathbb{F}_\infty$ of countable rank.

Modern work on the WEP by Farah, Paulsen, Pisier, Goldbring, and their contemporaries has relied heavily on this result to prove other interesting properties of WEP algebras.

Part of the problem with the WEP is that many of its equivalent formulations require long and tortuous treks through other areas of mathematics to prove. One seemingly innocent equivalent formulation of the weak expectation property is the following unpublished result due to Haagerup, with published proof due to Pisier:

Theorem 1.1.7 (Haagerup-Pisier)**.** Let A be a C*-algebra, and $\overline{A}$ denote the *conjugate* C*-algebra (the same underlying set, but with multiplication reversed). Then A has the WEP if and only if for all $a_i \in A$,

$$\left\| \sum_i a_i \otimes \overline{a_i} \right\|_{A \otimes_{\min} \overline{A}} = \left\| \sum_i a_i \otimes \overline{a_i} \right\|_{A \otimes_{\max} \overline{A}}$$

The proof, however, requires results about interpolations of Banach spaces, and operator-valued Hardy spaces, both of which are tangents we will not have the space for (though the interested reader can find a proof in [36], section 23.3).

The WEP, though perhaps fringe by comparison to nuclearity, remains a fascinating topic of study to this day. One of the central problems in operator theory - the Connes embedding problem - is related to another conjecture called the QWEP conjecture, which asks whether or not all C*-algebras are quotients of WEP algebras (see [32] for more). The WEP is also a sort of "dual property" to another property explored by Kirchberg in [25]: the "local lifting property" (or LLP). The WEP and LLP both intimately intertwine with the theory of nuclear C*-algebras.

1.2 Thesis

The purpose of *this* paper is to whet the reader's appetite for the world of nuclear C*-algebras and the weak expectation property. To do so, we'll walk through a number of the most important proofs in the history of this subfield of operator theory, and establish some of the results mentioned above. In doing so, we also hope to provide the reader with the necessary prerequisite knowledge to understand some of the more recent developments in these areas (both nuclearity and the WEP being active areas of research today).

In section 2 we will review the basics of enveloping von Neumann algebras, and prove the celebrated Sherman-Takeda theorem. Much of what will be covered in this section can be found in different forms in texts like [5, 36, 37].

In section 3, we will begin our discussion of nuclear C*-algebras. We'll introduce the *algebraic* tensor product, followed by a collection of the most important C*-norms one can place on this tensor product, and finish up with a number of theorems relating to the behaviour of these tensor products, including some preliminary results about nuclearity.

Much of the interesting literature on nuclearity requires different formulations than the "tensor product" definition, so we'll devote section 4 to exploring these equivalent characterizations of nuclearity. In particular, we'll prove that nuclearity is equivalent to the CPAP and the "approximate factorization of the identity map". We'll also prove

$$A^{**} \text{ semidiscrete} \implies A \text{ nuclear} \implies A^{**} \text{ injective}$$

however we *won't* be proving that injectivity and semidiscreteness are equivalent, the proof of which requires deep knowledge of von Neumann algebras, which isn't particularly useful to our exploration (see [11] for the proof).

Finally in section 5 we will look at the weak expectation property. We'll establish the results of Lance and Kirchberg above, and we'll finish up by looking at "local reflexivity", a property introduced by Archbold and Batty in [2], which will provide us with a satisfying "converse" to the fact that "nuclear implies WEP".

1.3 Prerequisites and Notation

We expect the reader to be familiar with the basics of Banach algebras and C*-algebras, which itself necessitates a familiarity with Banach spaces and basic functional analysis. Since much of our work requires von Neumann algebra theory, we will omit a lot of the *very basic* theory of von Neumann algebras, but anything even slightly more complex than this will be included in the appendix. Many of the results on von Neumann algebras can be found in Dixmier's second opus, [15], though some of the results we'll require are outside the scope of even this textbook.

Throughout the text, we will let H, K, L denote Hilbert spaces, $\mathcal{B}(H)$ the space of bounded linear operators on H, and $\mathcal{K}(H) \subseteq \mathcal{B}(H)$ the ideal of *compact* operators on H. Operators in $\mathcal{B}(H)$ are typically denoted T, and operators in $\mathcal{K}(H)$ are typically denoted K, modulo subscripts, superscripts, etc. The weak, strong, ultraweak and ultrastrong operator topologies on $\mathcal{B}(H)$ will be denoted *wot*, *sot*, $\sigma - wot$ and $\sigma - sot$ respectively (e.g., $wot{-}\lim_\lambda \xi_\lambda$ is the limit of ξ_λ in the weak operator topology, and $\overline{S}^{\sigma-sot}$ is the closure of S in the ultrastrong operator topology). We will also say "σ-weak" (resp. σ-strong) to equivalently mean "ultraweak" (resp. "ultrastrong").

We typically won't have a fixed notation for general Banach spaces, other than X, Y, Z, etc. When X is a Banach space, $\mathbb{B}(X)$ will denote the *closed unit ball* of X (the set of those $x \in X$ with $\|x\| \leq 1$). The weak topology on X and the weak-* topology on X^* will typically be denoted wk and w^* respectively (e.g., $wk{-}\lim_\lambda x_\lambda$ or $\overline{C}^{w^*}$, denoting the weak limit of x_λ and the weak-* closure of C respectively). We also let X' denote the *algebraic dual* of X (the space of all linear functionals on X), and X^* the *topological dual* (the Banach space of all continuous linear functionals on X).

C*-algebras will be denoted A, B, C, D, and von Neumann algebras M, N, R, S. Operator spaces will be denoted E, F, G. In the case of von Neumann algebras, M' will refer *exclusively* to the *commutant* of M (rather than the algebraic dual), unless specified otherwise. When A (resp. E, M) is a C*-algebra (resp. von Neumann algebra, operator space), A_+ (resp. E_+, M_+) will denote the *positive elements* in the respective space. We will also be punctilious in our usage of the term "representation" to mean a linear map $\rho : A \to \mathcal{B}(H)$ such that $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$ (so merely an *algebra* homomorphism), and "*-representation" to mean a representation which also satisfies $\rho(a^*) = \rho(a)^*$.

1.4 Acknowledgements

We blame Effros, Lance, Choi, Connes, Haagerup, Kirchberg, Brown, Ozawa, and Pisier for all our pain and suffering.

In all seriousness, this paper relies heavily on the work of these mathematicians and their colleagues. Throughout the paper I will do my best to give adequate credit wherever it is due, but it is an understatement to say that I'm standing on the shoulders of giants.

I'd also like to take this time to thank Prof. Laurent Marcoux of the University of Waterloo, who oversaw my work on this project. With his help, not only have I learned a tremendous amount about this area of mathematics, but I also rediscovered a lot of my lost passion for math, learned some things about life, learned some things about myself, and, perhaps most importantly, found in Laurent a new good friend. I owe to him in equal amount what I owe the mathematicians above.

2 The Bidual and the Enveloping von Neumann Algebra

2.1 The Universal Representation and Enveloping von Neumann Algebra of a C*-Algebra

Given a C*-algebra A, it is well-known that A admits a *universal representation* $\pi_u : A \to \mathcal{B}(H_u)$ - a faithful, nondegenerate *-representation which isometrically embeds A as a C*-subalgebra of bounded operators on a Hilbert space. To construct π_u, let $S(A)$ denote the *state space* of A, the set of positive, norm 1 linear functionals on A. To each state $\rho \in S(A)$, we can assign to ρ a cyclic, non-degenerate representation $\pi_\rho : A \to \mathcal{B}(H_\rho)$ known as the *GNS representation.* The universal representation is then defined as the direct sum of all GNS representations:

$$\pi_u := \bigoplus_{\rho \in S(A)} \pi_\rho$$

The "universality" of this representation is made clearer in the context of the following result.

Theorem 2.1.1 ([12], Theorem 5.9)**.** Let A be a C*-algebra, and $\pi : A \to \mathcal{B}(H)$ a non-degenerate *-representation. Then π is the equivalent to the direct sum of cyclic, non-degenerate *-representations.

Cyclic representations of A are particularly nice to work with, in part because they correspond to positive linear functionals. Indeed, every cyclic representation $\pi : A \to \mathcal{B}(H)$ with unit cyclic vector $e \in H$ induces a state $\rho_\pi \in S(A)$ given by

$$\rho_\pi(a) := \langle \pi(a)e, e\rangle$$

This transformation is in fact the "inverse" to the GNS representation: if π_{ρ_π} denotes the GNS representation constructed from ρ_π, then π_{ρ_π} is *equivalent* to π. Similarly if π_ρ is the GNS representation corresponding to $\rho \in S(A)$, then $\rho_{\pi_\rho} = \rho$). In combination with 2.1.1 we have established the following first "universality property":

Corollary 2.1.2 (First Universality Property of π_u)**.** Every *-representation of a C*-algebra A is equivalent to a direct summand of the universal representation of A.

Additionally, now that we've embedded A as a subalgebra of $\mathcal{B}(H)$ (in a universal way), we can also speak sensibly about the "largest von Neumann algebra" generated by A.

Definition 2.1.3. Each C*-algebra A induces an *enveloping von Neumann algebra*, the von Neumann algebra $\overline{\pi_u(A)}^{wot}$ (or $\pi_u(A)''$).

For reasons which will become clear momentarily, we will denote this von Neumann algebra by A^{**} (do not worry about needing to distinguish this from A's bidual - although perhaps I've said too much!)

Theorem 2.1.4 (Second Universality Property of π_u)**.** Let A be a C*-algebra, and let π_u denote its universal representation. Given any other faithful, non-degenerate representation $\sigma : A \to \mathcal{B}(H)$, there is a normal *-homomorphism $\rho : \overline{\pi_u(A)}^{wot} \to \overline{\sigma(A)}^{wot}$ such that $\sigma = \rho \circ \pi_u$. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & \pi_u(A) \arrow[r, "\subseteq"] & \overline{\pi_u(A)}^{wot} \arrow[r, "\subseteq"] \arrow[dd, dashed, "\rho"] & \mathcal{B}(H_u) \\
A \arrow[ur, "\pi_u"] \arrow[dr, "\sigma"'] & & & \\
 & \sigma(A) \arrow[r, "\subseteq"] & \overline{\sigma(A)}^{wot} \arrow[r, "\subseteq"] & \mathcal{B}(H)
\end{tikzcd}$$

Remark. Another way of interpreting 2.1.4 is that every von Neumann algebra generated by A is in fact a "quotient" of $\overline{\pi_u(A)}^{wot}$.

Remark. We refer the reader to appendix A.3 for information regarding "normal" maps of von Neumann algebras.

Let us briefly recall why *faithful, non-degenerate* representations are of particular interest. Faithfulness is clear: an injective *-representation between C*-algebras is automatically isometric (perhaps a shocking fact to the uninitiated). The primary reason we work solely with *non-degenerate* representations σ is because non-degeneracy is what guarantees that $\overline{\sigma(A)}^{wot}$ is a *von Neumann algebra.* Indeed, one form of von Neumann's bicommutant theorem states that if $A \subseteq \mathcal{B}(H)$ is a self-adjoint subalgebra of $\mathcal{B}(H)$, which is *not necessarily unital* but has *trivial kernel* - where the kernel of A is defined as the set

$$\ker A := \{\xi \in H \mid T\xi = 0 \; \forall T \in A\}$$

- then $\overline{A}^{wot} = A''$, and thus $\overline{A}^{wot}$ is a von Neumann algebra. The triviality of the kernel is easily seen to be equivalent to non-degeneracy:

$$\begin{aligned} \ker \pi(A) = 0 \quad &\iff \quad \langle \pi(T)\xi, \eta \rangle = 0 \; \forall T \in A, \; \eta \in H \implies \xi = 0 \\ &\iff \quad \langle \xi, \pi(T)\eta \rangle = 0 \; \forall T, \eta \implies \xi = 0 \\ &\iff \quad (\pi(A)H)^{\perp} = 0 \\ &\iff \quad \overline{\pi(A)H} = H \end{aligned}$$

So it is precisely the non-degenerate representations which generate von Neumann algebras.

Alternatively, given a *degenerate* representation ς (denoted with the unprincipled "variant σ" character, representing it's inherently dissolute and contemptible nature), it is not hard to see that ς decomposes as the direct sum of a *non-degenerate* representation σ and a zero representation:

$$\varsigma = \sigma \oplus 0$$

So $\overline{\varsigma(A)}^{wot}$ is simply the direct sum of a von Neumann algebra and a trivial algebra - in essence just a von Neumann algebra with extra symbolic cruft.

Fact 2.1.5. If A is a C*-algebra and $\pi_u : A \to \mathcal{B}(H_u)$ its universal representation, then every state on A is a vector state on $\pi_u(A)$, and every normal state on $\overline{\pi_u(A)}^{wot}$ is also a vector state.

Proof. If $\rho \in S(A)$, then ρ induces a GNS representation π_ρ with unit cyclic vector ξ_ρ. Let $\widetilde{\xi_\rho}$ denote the vector in H_u equal to ξ_ρ in the ρ'th coordinate, and 0 elsewhere (recall H_u is the direct sum of H_ρ, $\rho \in S(A)$). Then

$$\rho(a) = \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle_{H_\rho} = \langle \pi_u(a)\widetilde{\xi_\rho}, \widetilde{\xi_\rho} \rangle_{H_u}$$

Now if $\omega \in S(\overline{\pi_u(A)}^{wot})$ is normal, ω is ultraweakly continuous, and so is determined by its values on $\pi_u(A)$. ω's restriction to $\pi_u(A)$ is a state, whence a vector state, say $\omega|_{\pi_u(A)}(a) = \langle a\xi, \xi \rangle$. Then $\omega(x) = \langle x\xi, \xi \rangle$ is a well-defined normal state on $\overline{\pi_u(A)}^{wot}$, and by uniqueness of ω it follows that ω is a vector state. ☐

2.2 Normalization of Maps and the Sherman-Takeda Theorem

Given a Banach space X, we let $J_X : X \to X^{**}$ (or simply J if X is implicit from context) denote the canonical evaluation map which embeds X isometrically in X^{**}, and denote by $\widehat{X}$ the range of X. Similarly, given $x \in X$ we will occasionally write $\hat{x}$ for the image of x under J_X. Sometimes this map will be referred to as merely the *inclusion* of X in X^{**}.

Definition 2.2.1. Let X and Y be Banach algebras, and let $u \in \mathcal{B}(X, Y^*)$. Then there exists a *unique* map $\ddot{u} \in \mathcal{B}(X^{**}, Y^*)$ such that $\ddot{u}|_{\widehat{X}} \circ J_X = u$, called the *normalization of* u. The map $\ddot{u}$ is given explicitly by

$$\ddot{u} = (u^*|_{\widehat{Y}} \circ J_Y)^* \tag{1}$$

Moreover, $\|\ddot{u}\| = \|u\|$ and $\operatorname{ran} \ddot{u} = \overline{\operatorname{ran} u}^{w*}$.

Walking through the expression in (1) step by step, first we take the adjoint $u^* : Y^{**} \to X^*$, which we then restrict to $\widehat{Y}$. By composing on the right with J_Y, we obtain a map $u^*|_{\widehat{Y}} \circ J_Y : Y \to X^*$. Finally, taking the adjoint yields the desired map from X^{**} into Y^*.

Let us first evaluate $\ddot{u}$ on $\widehat{X}$.

$$\ddot{u}(\widehat{x}) = \widehat{x} \circ u^*|_{\widehat{Y}} \circ J_Y$$

Now $\ddot{u}(\widehat{x})$ is in Y^*, so we can evaluate at $y \in Y$:

$$\begin{aligned} \ddot{u}(\widehat{x})(y) &= (\widehat{x} \circ u^*|_{\widehat{Y}} \circ J_Y)(y) \\ &= \widehat{x}(u^*(\widehat{y})) \\ &= u^*(\widehat{y})(x) \end{aligned}$$

By definition of the adjoint, $u^*(\widehat{y}) = \widehat{y} \circ u$, so

$$u^*(\widehat{y})(x) = \widehat{y}(u(x)) = u(x)(y)$$

In conclusion

$$\ddot{u}(\widehat{x})(y) = u(x)(y)$$

and since this holds for all $y \in Y$, we conclude that $\ddot{u}|_{\widehat{X}} \circ J_X = u$.

To see that $\ddot{u}$ is unique, note that since $\ddot{u}$ is the adjoint of a continuous linear operator, $\ddot{u}$ is weak-*-to-weak-* continuous. Then by Goldstine's theorem - which states $\overline{\mathbb{B}(\widehat{X})}^{w*} = \mathbb{B}(X^{**})$ (see [29] 2.6.26 for instance) - any $x^{**} \in \mathbb{B}(X^{**})$ is the weak-*-limit of a net $\widehat{x}_i \in \mathbb{B}(\widehat{X})$. Thus

$$\ddot{u}(x^{**}) = \ddot{u}(w^*\text{–}\lim_i \widehat{x}_i) = w^*\text{–}\lim_i \ddot{u}(\widehat{x}_i) = w^*\text{–}\lim_i u(x_i)$$

Thus $\ddot{u}$ is uniquely determined by u. This line of reasoning also tells us $\operatorname{ran} \ddot{u} = \overline{\operatorname{ran} u}^{w^*}$.

Goldstine's theorem is also the main ingredient in establishing that $\|\ddot{u}\| = \|u\|$. Let $p : X^{**} \to \mathbb{R}_{\geq 0}$ be given by $p(x) := \|\ddot{u}(x)\|$. Then p is weak-*-continuous, so

$$p(\mathbb{B}(X^{**})) = p\left(\overline{\mathbb{B}(\widehat{X})}^{w^*}\right) \subseteq \overline{p(\mathbb{B}(\widehat{X}))}$$

implying

$$\|\ddot{u}\| = \sup p(\mathbb{B}(X^{**})) \leq \sup \overline{p(\mathbb{B}(\widehat{X}))} = \|u\|$$

Since $\|u\| \leq \|\ddot{u}\|$, equality follows.

Remark. The notation $\ddot{u}$ is borrowed from Pisier ([36]). When the symbol u is an unwieldy expression, we will use $u^{\cdot\cdot}$ instead of $\ddot{u}$ to refer to the normalization of u (for instance, $(\Phi_\lambda \circ T)^{\cdot\cdot}$ would denote the normalization of $\Phi_\lambda \circ T$).

Example 2.2.2. Given a linear map $u : X \to Y$, precomposing with J_Y yields a map $v := J_Y \circ u : X \to Y^{**}$. It is not hard to check that $\ddot{v} = u^{**}$, and so normalization is a sort of generalization of the bi-adjoint.

Normalization plays an important role in the realm of C*-algebras when considering maps $u : A \to M$ from a C*-algebra A into a von Neumann algebra M. As von Neumann algebras admit preduals (see appendix A.3), $u : A \to (M_*)^*$ is precisely the kind of map which we can normalize to obtain a natural extension $\ddot{u} : A^{**} \to M$. This normalized map has many desirable properties, as we will see, but first we need to discuss why the bidual of a C*-algebra is even an object of interest.

Given a C*-algebra A, the universal representation acts as a map from A into the von Neumann algebra $M := \overline{\pi_u(A)}^{wot}$ (A's enveloping von Neumann algebra), and so $\pi_u : A \to M$ thus extends to a map $\ddot{\pi}_u : A^{**} \to M$. More surprisingly, the map $\ddot{\pi}_u$ is in fact an *isometric isomorphism* of Banach spaces.

This is a highly-celebrated result known as the *Sherman-Takeda theorem.* It has become so ubiquitous since it's announcement in 1950 and subsequent proof in 1954 that nowadays we speak unambiguously of A^{**} as the enveloping von Neumann algebra of A, without reference to this remarkable fact that it truly does coincide with the bidual of A.

Proving Sherman-Takeda relies on a few lemmata. The core ingredient of the proof will be to take advantage of the universality of π_u, along with a Kaplansky-style density argument.

Lemma 2.2.3 (Jordan Decomposition)**.** If A is a C*-algebra, and M a von Neumann algebra, then $\operatorname{span} S(A) = A^*$ and $\operatorname{span}(S(M) \cap M_*) = M_*$.

This is a well-known result in the theory of C*-algebras, which we refer the reader to theorem 3.3.10 in [31] (or the original paper, [34]) for more information.

Lemma 2.2.4. Let A be a C*-algebra, and $\pi_u : A \to \mathcal{B}(H_u)$ its universal representation. Let $M := \overline{\pi_u(A)}^{wot}$. Then every state $\rho \in S(A)$ extends to a normal state $\widetilde{\rho} \in S(M)$.

Proof. First, ρ induces a (non-degenerate) GNS representation $\pi_\rho : A \to \mathcal{B}(H_\rho)$ such that $\rho(a) = \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle$ for some unit vector $\xi_\rho \in H_\rho$. Consider the map

$$\begin{array}{cccc} \rho_1 & : & \overline{\pi_\rho(A)}^{wot} & \to & \mathbb{C} \\ & & x & \mapsto & \langle x\xi_\rho, \xi_\rho \rangle \end{array}$$

Then by definition, ρ_1 is an ultraweakly continuous state which coincides with ρ.

Next, by theorem 2.1.4, there exists a normal (hence ultraweakly continuous) *-homomorphism $\sigma : \overline{\pi_u(A)}^{wot} \to \overline{\pi_\rho(A)}^{wot}$, and we can define

$$\widetilde{\rho} := \rho_1 \circ \sigma$$

Then $\widetilde{\rho}$ is the desired extension. $\widetilde{\rho}$ is indeed normal, since normality and ultraweak continuity are equivalent for positive linear functionals. ☐

Theorem 2.2.5 (Sherman-Takeda)**.** Let A be a C*-algebra, and $\pi_u : A \to \mathcal{B}(H_u)$ its universal representation. Then $A^{**} \cong M := \overline{\pi_u(A)}^{wot}$ isometrically, via the isometric isomorphism

$$\ddot{\pi}_u := (\pi_u^*|_{M_*})^* : A^{**} \to M$$

Proof. We present an argument similar to the one that appears in [37].

A bounded linear map $T : X \to Y$ between Banach spaces is an isometric isomorphism if and only if $T^* : Y^* \to X^*$ is an isometric isomorphism. Thus, to prove our desired result, we can look instead at the map $\pi_u^*|_{M_*} : M_* \to A^*$.

First we establish that this map is isometric. Given $\omega \in M_*$, we evaluate

$$\|\pi_u^*|_{M_*}\omega\| = \sup_{a \in \mathbb{B}(A)} |\omega(\pi_u(a))| \leq \sup_{m \in \mathbb{B}(M)} |\omega(m)| = \|\omega\|$$

By Kaplansky's density theorem, given any $m \in \mathbb{B}(M)$ there exists a sequence $a_\lambda \in \mathbb{B}(A)$ such that $\pi_u(a_\lambda)$ converges strongly to m. Since the strong and ultrastrong topologies coincide on bounded sets, $\pi_u(a_\lambda) \to m$ ultrastrongly, and hence ultraweakly. By ultraweak continuity of ω, we thus have that

$$|\omega(m)| = \lim_\lambda |\omega(\pi_u(a_\lambda))|$$

and so in turn

$$\sup_{m \in \mathbb{B}(M)} |\omega(m)| = \sup_{a \in \mathbb{B}(A)} |\omega(\pi_u(a))|$$

thus giving us that $\pi_u^*|_{M_*}$ is indeed isometric.

To prove that this map is *surjective*, all we need to show is that every state on $\pi_u(A)$ extends to a *normal state* on $\overline{\pi_u(A)}^{wot}$. This will be sufficient for our proof because $A^* = \operatorname{span} S(A)$, and similarly $M_* = \operatorname{span}\{S(M) \cap M_*\}$. This is where universality becomes necessary. By lemma 2.2.4, every state on $\pi_u(A)$ extends to a normal state on M, and so $\pi_u^*|_{M_*}$ is surjective, thus completing the proof. ☐

Corollary 2.2.6. The space A^{**} admits an abstract von Neumann algebra structure, given by

$$a \cdot b := \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a)\ddot{\pi}_u(b)), \qquad a^* := \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a)^*)$$

for any $a, b \in A^{**}$.

Remark. When we say a space W admits an "abstract von Neumann algebra" structure, we primarily mean that W is isometrically isomorphic to a *concrete* von Neumann algebra (a unital, self-adjoint wot-closed subalgebra of $\mathcal{B}(H)$). Another well-known "abstract" quantifier of von Neumann algebra structures is a theorem of Sakai: a C*-algebra is an abstract von Neumann algebra if and only if it admits a (unique) predual (see [37]).

Corollary 2.2.7. The weak-* topology on A^{**} is identical to the *ultraweak* topology on $\overline{\pi_u(A)}^{wot}$.

Proof. This is more of a remark than a corollary, but an incredibly important remark nonetheless. Let us first recall that the weak-* topology on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ (induced by its predual M_*) is called the *ultraweak* topology (and in fact coincides with the restriction of the ultraweak topology on $\mathcal{B}(H)$). The map $\ddot{\pi}_u : A^{**} \to \overline{\pi_u(A)}^{wot}$ is not only an isometric isomorphism, it's also *weak-*-to-weak-** continuous (since it's the adjoint of a bounded linear map), or rather weak-*-to-ultraweak continuous. Moreover, letting $M := \overline{\pi_u(A)}^{wot}$, since $\pi_u^*|_{M_*}$ was an isomorphism, we have

$$\ddot{\pi}_u^{-1} = ((\pi_u^*|_{M_*})^*)^{-1} = (\pi_u^*|_{M_*}^{-1})^*$$

that is, if $T : X \to Y$ is an isometric isomorphism of Banach spaces then $(T^*)^{-1} = (T^{-1})^*$. So $\ddot{\pi}_u^{-1}$ is also ultraweak-to-weak* continuous (again, since it's an adjoint of a map). In conclusion, $\ddot{\pi}_u$ is a weak-*-to-ultraweak *homeomorphism*, and so the weak-* topology on A^{**} is equivalent to the ultraweak topology on $\overline{\pi_u(A)}^{wot}$. ☐

Corollary 2.2.8. The inclusion $J_A : A \to A^{**}$ is an isometric *-homomorphism, and hence is completely positive.

Proof. It is easy to see that $J_A = \ddot{\pi}_u^{-1} \circ \pi_u$, and so $J_A(ab) = \ddot{\pi}_u^{-1}(\pi_u(a)\pi_u(b)) = a \cdot b$, and similarly $J_A(a^*) = \ddot{\pi}_u^{-1}(\pi_u(a)^*) = a^*$. ☐

In order to establish some of the main results about A^{**}, we'll first need to prove the following Kaplansky-style density theorem, a useful result when dealing with positive elements in A^{**}.

Lemma 2.2.9 (Kaplansky's Density Theorem for A^{**})**.** Let A be a C*-algebra, and treat A^{**} as a von Neumann algebra. Recall $\widehat{A}$ denotes the image of A under $J_A : A \to A^{**}$. Then

$$\overline{\mathbb{B}\left(\widehat{A}_+\right)}^{w*} = \mathbb{B}\left((A^{**})_+\right)$$

where a subscript $+$ denotes the set of positive elements in the respective set.

Proof. Recall the usual Kaplansky's Density theorem, which states that if A is a C*-subalgebra of $\mathcal{B}(H)$, then $\overline{\mathbb{B}(A_+)}^{sot} = \mathbb{B}((\overline{A}^{sot})_+)$. Now, the strong operator topology is a little foreign to our present milieu of weak operator, weak-*, and ultraweak topologies. Fortunately for us, these familiar operator topologies afford some flexibility. Unfortunately for us, clearly expressing what's going on with these topologies is not exactly straightforward. This proof is more "abstruse jargon" than it is "difficult ideas", so do not be daunted by the imminent symbol-juggling.

Consider the C*-algebra $\widehat{A}$ as contained in A^{**}, treated as a subalgebra of $\mathcal{B}(H_u)$. For the sake of notational convenience, we will write A in place of $\widehat{A}$. In what follows, *sot*, *wot*, and $\sigma - wot$ refer to the operator topologies on $\mathcal{B}(H_u)$ and the closures are being taken in $\mathcal{B}(H_u)$, unless specified otherwise (in which case we make it *excruciatingly* clear what we're trying to say, often at the expense of brevity).

First, $\overline{\mathbb{B}(A_+)}^{sot} = \overline{\mathbb{B}(A_+)}^{wot}$ since $\mathbb{B}(A_+)$ is convex.[1] Next, $\overline{\mathbb{B}(A_+)}^{wot} = \overline{\mathbb{B}(A_+)}^{\sigma-wot}$ since $\mathbb{B}(A_+)$ is bounded.[2] Notice that A^{**} is $\sigma - wot$ closed in $\mathcal{B}(H_u)$, and so

$$\overline{\mathbb{B}(A_+)}^{\sigma-wot-A^{**}} = \overline{\mathbb{B}(A_+)}^{\sigma-wot-\mathcal{B}(H_u)} \cap A^{**} = \overline{\mathbb{B}(A_+)}^{\sigma-wot-\mathcal{B}(H_u)}$$

[1] The *sot* and *wot* closures of convex sets agree.
[2] The *wot* and $\sigma - wot$ closures of bounded sets agree.

where $\sigma - wot - A^{**}$ refers to the ultraweak topology on A^{**}, and $\sigma - wot - \mathcal{B}(H_u)$ the ultraweak topology on $\mathcal{B}(H_u)$. Here we have simply used the formula $\mathrm{cl}_Y(U) = \mathrm{cl}_X(U) \cap Y$, where $U, Y \subseteq X$, and cl_X, cl_Y are the closure operators in X and Y respectively. Of course, by corollary 2.2.7 we know the $\sigma - wot$ topology on A^{**} is just the weak-* topology, so really we have $\overline{\mathbb{B}(A_+)}^{\sigma - wot - A^{**}} = \overline{\mathbb{B}(A_+)}^{w*}$, the weak-* closure being taken in A^{**}. To conclude this first round of reductions:

$$\overline{\mathbb{B}(A_+)}^{sot} = \overline{\mathbb{B}(A_+)}^{w*}$$

Next, by von Neumann's density theorem the *sot* and $\sigma - wot$ closures of a C*-algebra coincide, so $\mathbb{B}((\overline{A}^{sot})_+) = \mathbb{B}((\overline{A}^{\sigma - wot})_+)$, and for the same reason as above we don't have to differentiate between the $\sigma - wot$ topology on $\mathcal{B}(H_u)$ and the $\sigma - wot$ topology on A^{**}, so we also have $\mathbb{B}((\overline{A}^{\sigma - wot})_+) = \mathbb{B}((\overline{A}^{w*})_+)$. To finish off, by Goldstine's theorem we know $\overline{A}^{w*} = A^{**}$, the closure again being taken in A^{**}. In conclusion,

$$\overline{\mathbb{B}(A_+)}^{w*} = \mathbb{B}((A^{**})_+)$$

□

The *really* important part of this lemma is that if $a^{**} \in \mathbb{B}(A^{**})$ is positive, then a^{**} is the weak-*-limit of a net $\widehat{a_\lambda} \in \mathbb{B}(A)$, for which each $\widehat{a_\lambda}$ is *also* positive. We'll see how this is useful in the following corollaries and theorems.

Another important realization this lemma inspires is that positivity in A^{**} has a concrete meaning:

Fact 2.2.10. If an element $a^{**} \in \mathbb{B}(A^{**})$ is positive then $a^{**}(f) \geq 0$ for all positive $f \in A^*$.

Proof. Assume that $a^{**} \geq 0$, and without loss of generality that $\|a^{**}\| \leq 1$. Then there exists a sequence $\widehat{a_\lambda} \in \mathbb{B}(\widehat{A}_+)$ converging weak-* to a^{**}. Then for all $f \in A^*$ we have

$$a^{**}(f) = \lim_\lambda \widehat{a_\lambda}(f) = \lim_\lambda f(a_\lambda)$$

so that if $f \in A^*$ is positive, each $f(a_\lambda) \geq 0$, so $a^{**}(f) \geq 0$. □

One might wonder whether the converse of the fact above is true, but unfortunately at this point it eludes the author.

Corollary 2.2.11. Let A be a C*-algebra, and treat A^{**} as a von Neumann algebra. Then $(A^{**})_* \cong A^*$, via the map

$$\begin{array}{cccc} \Theta & : & (A^{**})_* & \to & A^* \\ & & \omega & \mapsto & \omega \circ J_A \end{array}$$

Moreover, both Θ and Θ^{-1} are completely positive (see appendix A.2 and definition A.2.9 for clarification on "complete positivity" of maps between C*-algebra duals).

Proof. To start, notice that the map Θ is really just $\pi_u^*|_{M_*}$ in disguise, where $M := \overline{\pi_u(A)}^{wot}$ as before. Indeed, there is first and foremost an isometric isometry between $(A^{**})_*$ and M_* given by taking $\omega \mapsto \omega \circ \ddot{\pi}_u^{-1}$. Then

$$\pi_u^*|_{M_*}(\omega \circ \ddot{\pi}_u^{-1}) = \omega \circ \ddot{\pi}_u^{-1} \circ \pi_u = \omega \circ J_A$$

and so the map Θ is equal to $\pi_u^*|_{M_*} \circ (\ddot{\pi}_u^{-1})^*|_{(A^{**})_*}$. Notice that since $\ddot{\pi}_u$ is a *-isomorphism, that $\ddot{\pi}_u^{-1}$, and in turn $(\ddot{\pi}_u^{-1})^*$, are both isometric isomorphisms. Moreover, it isn't hard to see that $(\ddot{\pi}_u^{-1})^*|_{(A^{**})_*}$ has M_* as its range since $*$-isomorphisms are normal, so if $\omega \in (A^{**})^*$ is normal then $\omega \circ \ddot{\pi}_u^{-1} \in M^*$ is normal as well. I isn't hard to check then that $(\ddot{\pi}_u^{-1})^*|_{(A^{**})_*} : (A^{**})_* \to M_*$ is an isometric isomorphism. From this, we see that Θ is indeed itself an isometric isomorphism.

Seeing that Θ is completely positive is easy. Given $[\omega_{ij}] \geq 0$, then $\Theta_n([\omega_{ij}]) = [\omega_{ij} \circ J_A]$ is positive if and only if the corresponding functional in $M_n(A)^*$ is positive, that is:

$$[\omega_{ij} \circ J_A] \geq 0 \quad \iff \quad \sum_{ij} \omega_{ij} \circ J_A(a_i^* a_j) \geq 0 \quad \forall a_i \in A$$

but $\sum_{ij} \omega_{ij} \circ J_A(a_i^* a_j)$ is just equal to $\sum_{ij} \omega_{ij}(a_i^* a_j)$ (since J_A is a *-homomorphism), which is ≥ 0 by assumption. Thus Θ is completely positive.

To finish the proof, we need to show that Θ^{-1} is completely positive. Let $f \in A_+^*$, and let $\omega = \Theta^{-1}(f) \in (A^{**})_*$. Then ω is the unique weak-*-continuous linear functional on A^{**} such that $\omega(\hat{a}) = f(a)$ for all $a \in A$. To see that ω is also positive, let $a^{**} \in \mathbb{B}(A^{**})_+$. By lemma 2.2.9 there is a net $\widehat{a_\lambda} \in \mathbb{B}(\widehat{A})_+$ converging weak-* to a^{**}, and so

$$\omega(a^{**}) = \lim_\lambda \omega(\widehat{a_\lambda}) = \lim_\lambda f(a_\lambda) \geq 0$$

This tells us that Θ^{-1} is positive. Checking complete positivity is a similar routine, but with a couple more hoops to jump through.

First, we remark that if $M \subseteq \mathcal{B}(H)$ is a von Neumann algebra, then $M_n(M)$ can be realized as a *wot*-closed subalgebra of $\mathcal{B}(H^n)$, and so $M_n(M)$ is itself a von Neumann algebra. Moreover, recall that there is an isomorphism between $M_n(A^*)$ and $M_n(A)^*$, which endows $M_n(A^*)$ with an order structure (see fact A.2.8). Unfortunately we don't quite have the same nice correspondence between $M_n(M_*)$ and $M_n(M)_*$, but this won't matter.

Let $[f_{ij}] \in M_n(A^*)$, and suppose $[f_{ij}] \geq 0$. Let $[\omega_{ij}] := (\Theta^{-1})_n([f_{ij}])$. In order to show that $[\omega_{ij}] \geq 0$, we need to show that for all $[a_{ij}^{**}] \in M_n(A^{**})_+$, that $\sum_{ij} \omega_{ij}(a_{ij}^{**}) \geq 0$. Fortunately since $M_n(A^{**})$ is a von Neumann algebra with $M_n(\widehat{A})$ as a *wot*-dense subalgebra, we can obtain a Kaplansky-style density result:

$$\mathbb{B}(M_n(A^{**})_+) = \overline{\mathbb{B}\left(M_n(\widehat{A})_+\right)}^{\sigma-wot}$$

The details of how you derive this result are very, very similar to that of the proof of lemma 2.2.9, and so we leave them to the reader as an (easy) exercise.

So assuming without loss of generality that $\|[a_{ij}^{**}]\| \leq 1$, we can choose a net $[\widehat{a}_{ij}^\lambda] \in \mathbb{B}\left(M_n(\widehat{A})_+\right)$ converging ultraweakly to $[a_{ij}^{**}]$. Notice that this implies each $\widehat{a}_{ij}^\lambda$ itself converges ultraweakly to a_{ij}^{**} (for each fixed i, j). Thus, by ultraweak continuity of the ω_{ij}'s:

$$\sum_{ij} \omega_{ij}(a_{ij}^{**}) = \lim_\lambda \sum_{ij} \omega_{ij}(\widehat{a}_{ij}^\lambda) = \lim_\lambda \sum_{ij} f(a_{ij}^\lambda) \geq 0$$

thus proving that $[\omega_{ij}] \geq 0$, so Θ^{-1} is completely positive. □

Proposition 2.2.12. Let A be a C*-algebra, M a von Neumann algebra, and $u : A \to M$ a *-homomorphism. Then $\ddot{u} : A^{**} \to M$ is a normal *-homomorphism.

Proof. We will postpone the proof of normality of $\ddot{u}$ until after the proof of proposition 2.2.14. The rest follows easily from a density argument (not even a Kaplansky-style argument, just Goldstine's theorem). First take $a \in A, b \in \mathbb{B}(A^{**})$. Choose a sequence $b_\lambda \in \mathbb{B}(\widehat{A})$ converging weak-* to b. Observe that

$$\begin{aligned} ab &:= \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a)\ddot{\pi}_u(b)) \\ &= \lim_\lambda \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a)\ddot{\pi}_u(b_\lambda)) \\ &= \lim_\lambda ab_\lambda \end{aligned}$$

which follows from the facts that $\ddot{\pi}_u$ is weak-*-to-ultraweakly continuous, and that multiplication by a fixed element on the left/right is ultraweakly-continuous in the unit sphere. Then

$$\ddot{u}(ab) = \lim_\lambda \ddot{u}(ab_\lambda) = \lim_\lambda u(ab_\lambda) = \lim_\lambda u(a)u(b_\lambda) = \ddot{u}(a)\ddot{u}(b)$$

which again follows from the fact that $\ddot{u}$ is weak-*-to-ultraweakly continuous. Next, given $a \in \mathbb{B}(A^{**}), b \in A^{**}$, choose $a_\lambda \in \mathbb{B}(\widehat{A})$ converging weak-* to a, so that

$$\ddot{u}(ab) = \lim_\lambda \ddot{u}(a_\lambda b) = \lim_\lambda \ddot{u}(a_\lambda)\ddot{u}(b) = \ddot{u}(a)\ddot{u}(b)$$

Similarly, given $a \in \mathbb{B}(A^{**})$ and a sequence $a_\lambda \in \mathbb{B}(\widehat{A})$ converging weak-* to a, we have

$$a^* := \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a)^*) = \lim_\lambda \ddot{\pi}_u^{-1}(\ddot{\pi}_u(a_\lambda)^*) = \lim_\lambda a_\lambda^*$$

since $T \mapsto T^*$ is ultraweakly continuous in the unit sphere. Thus

$$\ddot{u}(a^*) = \lim_\lambda \ddot{u}(a_\lambda^*) = \lim_\lambda \ddot{u}(a_\lambda)^* = \ddot{u}(a)^*$$

□

Proposition 2.2.13. If A is a C*-algebra, then for all $n \in \mathbb{N}$, $M_n(A^{**}) \cong M_n(A)^{**}$ isometrically.

Proof. This is another proof where the complexity of the notation belies the simplicity of the idea, so we'll try to make clear what the *goal* is before we begin with the formalization. Start with the injective *-homomorphism $\iota : M_n(A) \to M_n(A^{**})$ (which is just the n-th ampliation of J_A). Since $M_n(A^{**})$ is a von Neumann algebra, we can normalize ι to obtain a *-homomorphism $\ddot{\iota} : M_n(A)^{**} \to M_n(A^{**})$.

The idea here is to exploit the *uniqueness* of the extension $\ddot{\iota}$. We'll construct a different map T between $M_n(A^{**})$ and $M_n(A)^{**}$ which we *know* is an isomorphism, is weak-*-to-weak-* continuous, and coincides with ι on when restricted to $M_n(A)$. Thus, since $\ddot{\iota}$ is the unique weak-*-to-weak-* map extending ι, we'll conclude that $T = \iota$, and so ι is in fact a *-isomorphism.

First, we know there is a bijective correspondence between $M_n(V')$ and $M_n(V)'$ for any vector space V (where a $'$ here denotes the *algebraic* dual space, rather than the usual topological dual denoted by a *), given by the map $\Lambda_V : M_n(V') \to M_n(V)'$ defined by

$$\Lambda_V([f_{ij}])([v_{ij}]) = \sum_{ij} f_{ij}(v_{ij})$$

This means there's a bijective correspondence between $M_n(V'')$ and $M_n(V)''$, namely $\Lambda_{V'}^{-1} \circ \Lambda_V^\dagger$, where $\Lambda_V^\dagger : M_n(V)'' \to M_n(V')'$ takes $F \in M_n(V)''$ to $F \circ \Lambda_V$ (it's just the "Banach space adjoint", but without the Banach spaces). We'll denote $T_V := \Lambda_{V'}^{-1} \circ \Lambda_V^\dagger$. A routine (but annoying) calculation gives us that

$$T_V(F) = [v_{ij}^{**}], \quad \text{where} \quad v_{ij}^{**}(f) = F(f \circ e_{ij}^*) \tag{†}$$

where $e_{ij}^* : M_n(V) \to V$ takes $[v_{ij}] \mapsto v_{ij}$. Given $[v_{ij}] \in M_n(V)$, define the evaluation functionals $\widehat{v_i j}$ and $\widehat{[v_{ij}]}$ by

$$\begin{array}{ccccc} \widehat{v_{ij}} & : & V' & \to & \mathbb{C} \\ & & f & \mapsto & f(v_{ij}) \end{array} \qquad \begin{array}{ccccc} \widehat{[v_{ij}]} & : & M_n(V)' & \to & \mathbb{C} \\ & & f & \mapsto & f([v_{ij}]) \end{array}$$

Notice that

$$T_V(\widehat{[v_{ij}]}) = [v_{ij}^{**}] \qquad \text{where} \quad \begin{aligned} v_{ij}^{**}(f) &= \widehat{[v_{ij}]}(f \circ e_{ij}^*) \\ &= f(e_{ij}^*([v_{ij}])) \\ &= f(v_{ij}) = \widehat{v_{ij}}(f) \end{aligned}$$

and so

$$T_V(\widehat{[v_{ij}]}) = [\widehat{v_{ij}}] \tag{††}$$

Now, let $V = A$ be a C*-algebra. Now we have a topological structure, and ideally we'd like to translate this topological structure to our work above. Consider restricting T_A to only *continuous* functionals: $T_A|_{M_n(A)^{**}}$. If $F \in M_n(A)^{**}$, then it isn't hard to check that the functionals v_{ij}^{**} defined above in (†) are *themselves* continuous, and so $T_A(F) \in M_n(A^{**})$, which is to say that

$$T_A|_{M_n(A)^{**}} : M_n(A^{**}) \to M_n(A)^{**}$$

The next question is whether $T_A|_{M_n(A)^{**}}$ is *surjective*. Given $[a_{ij}^{**}] \in M_n(A^{**})$, there exists $F \in M_n(A)''$ such that $T_A(F) = [a_{ij}^{**}]$. Expanding out the definition of T_A, this tells us that

$$F \circ \Lambda_A([f_{ij}]) = \sum_{ij} v_{ij}^{**}(f_{ij})$$

or rather, letting $\pi^*_{ij} : M_n(A^*) \to A^*$ be the map which grabs the ij^{th} component - $\pi^*_{ij}([f_{ij}]) = f_{ij}$ - then

$$F(f) = \sum_{ij} v^{**}_{ij}(\pi^*_{ij}(\Lambda_A^{-1}(f)))$$

and so

$$F = \sum_{ij} v^{**}_{ij} \circ \left(\pi^*_{ij} \circ \Lambda_A^{-1}\right)$$

The important point here is that each $\pi^*_{ij} \circ \Lambda_A^{-1}|_{M_n(A)^*} : M_n(A)^* \to A^*$ is *continuous*. Indeed, we really just have

$$(\pi_{ij} \circ \Lambda_A^{-1}(f))(a) = f(a \otimes e_{ij})$$

where $\{e_{ij}\}$ are the matrix units for $M_n(\mathbb{C})$ (note that this doesn't require A to be unital, rather $a \otimes e_{ij}$ is essentially just notation for the matrix with a in the ij position and 0's elsewhere). Clearly this map, as a map from $M_n(A)^*$ to A^*, is continuous. Thus, under the assumption that the v^{**}_{ij}'s were continuous, we see that F is the sum of continuous functionals, and so is itself continuous. Thus, $F \in M_n(A)^{**}$, implying $T_A|_{M_n(A)^{**}}$ is indeed surjective, and hence a vector space isomorphism. The formula (††) also tells us that $T_A|_{M_n(A)^{**}} \circ J_{M_n(A)} = \iota$.

So, we now have two ways of going from $M_n(A)^{**}$ to $M_n(A^{**})$: the first is the map $\ddot{\iota}$ defined at the start of the proof, and the second is the map $T_A|_{M_n(A)^{**}}$.

Let F_λ be a net in $M_n(A)^{**}$ converging weak-* to $F \in M_n(A)^{**}$. Let $[a^{**}_{ij,\lambda}] := T_A(F_\lambda)$, and $[a^{**}_{ij}] = T_A(F)$. First off, we have that

$$a^{**}_{ij,\lambda}(f) = F_\lambda(f \circ e^*_{ij}) \to F(f \circ e^*_{ij}) = a^{**}_{ij}(f)$$

for all $f \in A^*$, so that $a^{**}_{ij,\lambda}$ converges weak-* to a^{**}_{ij}. Now suppose that $T_{ij,\lambda}$ are n^2 nets of operators in a von Neumann algebra $M \subseteq \mathcal{B}(H)$, converging ultraweakly to T_{ij} respectively. Then for sequences $\xi^k := \begin{bmatrix} \xi^k_1 \\ \vdots \\ \xi^k_n \end{bmatrix}$ and $\eta^k := \begin{bmatrix} \eta^k_1 \\ \vdots \\ \eta^k_n \end{bmatrix}$ in H^n such that $\sum_{k=1}^\infty \|\xi^k\|^2, \sum_{k=1}^\infty \|\eta^k\|^2 < \infty$, we observe that

$$\begin{aligned}
\sum_{k=1}^{\infty} \left\langle [T_{ij,\lambda}] \begin{bmatrix} \xi^k_1 \\ \vdots \\ \xi^k_n \end{bmatrix}, \begin{bmatrix} \eta^k_1 \\ \vdots \\ \eta^k_n \end{bmatrix} \right\rangle &= \sum_{k=1}^{\infty} \sum_{ij} \langle T_{ij,\lambda}\xi^k_j, \eta^k_i \rangle \\
&= \sum_{ij} \sum_{k=1}^{\infty} \langle T_{ij,\lambda}\xi^k_j, \eta^k_i \rangle \\
&\to \sum_{ij} \sum_{k=1}^{\infty} \langle T_{ij}\xi^k_j, \eta^k_i \rangle = \sum_{k=1}^{\infty} \left\langle [T_{ij}] \begin{bmatrix} \xi^k_1 \\ \vdots \\ \xi^k_n \end{bmatrix}, \begin{bmatrix} \eta^k_1 \\ \vdots \\ \eta^k_n \end{bmatrix} \right\rangle
\end{aligned}$$

where interchanging of the sums above is justified by the fact that each series converges absolutely. This is to say that $[T_{ij,\lambda}]$ converges ultraweakly (in $M_n(M)$) to $[T_{ij}]$. Thus, since we have componentwise weak-* convergence of each $a^{**}_{ij,\lambda} \to a^{**}_{ij}$, we also have weak-* convergence of $[a^{**}_{ij,\lambda}] \to [a^{**}_{ij}]$. In other words, the map $T_A|_{M_n(A)^{**}}$ is weak-*-to-weak-* continuous.

Since $T \circ J_{M_n(A)} = \iota$, and T is weak-*-to-weak-* continuous, by 2.2.1 we conclude that $T = \ddot{\iota}$, and so $\ddot{\iota}$, a map which we know is a *-homomorphism, is in fact also an *isomorphism*. □

Proposition 2.2.14. Let A be a C*-algebra, M a von Neumann algebra, and $u : A \to M$ a completely positive map. Then $\ddot{u} : A^{**} \to M$ is normal and completely positive.

Proof. Recall lemma 2.2.9, which says that $\overline{\mathbb{B}(\widehat{A}_+)}^{w^*} = \mathbb{B}((A^{**})_+)$, and so any $a^{**} \in \mathbb{B}(A^{**})_+$ is the weak* limit of a net $(\widehat{a_\lambda})_\lambda \subseteq \mathbb{B}(A)_+$, so

$$\ddot{u}(a^{**}) = \lim_\lambda \ddot{u}(\widehat{a_\lambda}) = \lim_\lambda u(a_\lambda) \geq 0$$

implying $\ddot{u}$ is positive.

Applying this logic to $(\ddot{u})_n : M_n(A^{**}) \to M_n(M)$ - utilizing the fact that $M_n(A^{**}) \cong M_n(A)^{**}$ - yields that each ampliation $(\ddot{u})_n$ is also positive, so $\ddot{u}$ is completely positive.

In being completely positive, by theorem A.3.2, $\ddot{u}$ is normal if and only if it is continuous with respect to the $\sigma - wot$ topology on its domain/codomain. This is just weak-*-to-weak-* continuity of $\ddot{u}$, which follows from its definition. □

Remark. To complete the proof of proposition 2.2.12, simply note that since u is a *-homomorphism then $\ddot{u}$ is a *-homomorphism, and *-homomorphisms are completely positive, whence $\ddot{u}$ is normal.

One of the most powerful use cases of the enveloping von Neumann algebra is as a means of extending maps between non-unital C*-algebras to maps between their unitizations. This is because A^{**}, in being a von Neumann algebra, is necessarily *unital*, allowing us to regard A's unitization $\widetilde{A}$ as being a norm-closed subalgebra of A^{**}. To make this explicit:

Fact 2.2.15. Let A be a non-unital C*-algebra, and $\widetilde{A}$ it's unitization. Let $1_{A^{**}}$ denote the identity of A^{**}. Then there is an isometric *-isomorphism between $\widetilde{A}$ and the subalgebra

$$\widehat{A}_1 := \{\widehat{a} + \alpha 1_{A^{**}} \ : \ a \in A, \ \alpha \in \mathbb{C}\} \subseteq A^{**}$$

Proof. The isomorphism in question is the unital extension of the natural embedding J_A:

$$\begin{array}{cccc} \widetilde{J}_A & : & \widetilde{A} & \to & \widehat{A}_1 \\ & & a + \alpha 1_{\widetilde{A}} & \mapsto & J_A(a) + \alpha 1_{A^{**}} \end{array}$$

This is clearly a bijection, and is easily checked to be a *-homomorphism, whence $\widetilde{J}_A$ is continuous and isometric. □

Corollary 2.2.16. Let A, B be non-unital C*-algebras with unitizations $\widetilde{A}, \widetilde{B}$ respectively, and $\varphi : A \to B$ a completely positive map. Then φ extends a completely positive map $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$.

Proof. First, we can regard φ as a map from A into B^{**} by precomposing with the evaluation map $J_B : B \to B^{**}$ (that is, $J_B \circ \varphi : A \to B^{**}$ is also completely positive). We can then normalize this map to obtain $(J_B \circ \varphi)^{\cdot\cdot} : A^{**} \to B^{**}$ (which, as you may recall from 2.2.2 coincides with the bi-adjoint map φ^{**}).

Consider the map

$$\varphi^\circ := (J_B \circ \varphi)^{\cdot\cdot}|_{\widehat{A}_1} \circ \widetilde{J}_A : \widetilde{A} \to B^{**}$$

This map is completely positive, and $\varphi^\circ|_A = \varphi$. All that remains to be proven is that the codomain of φ° is indeed $\widehat{B}_1$, in which case

$$\widetilde{\varphi} := \widetilde{J}_B^{-1} \circ \varphi^\circ : \widetilde{A} \to \widetilde{B}$$

is the desired completely positive extension of φ.

We have

$$\begin{aligned} \varphi^\circ(a + \alpha 1_{\widetilde{A}}) &= \widehat{\varphi(a)} + \alpha \varphi^\circ(1_{\widetilde{A}}) \\ &= \widehat{\varphi(a)} + \alpha (J_B \circ \varphi)^{\cdot\cdot}(1_{A^{**}}) \end{aligned}$$

Let e_λ be an approximate identity for A. Let $\ddot{\pi}_u : A^{**} \to M \subseteq \mathcal{B}(H_u)$ denote the *-isomorphism obtained by the Sherman-Takeda theorem. Then for all $a \in A$, $h \in H_u$,

$$\lim_\lambda \pi_u(e_\lambda)\pi_u(a)h = \pi_u(a)h$$

Since π_u is a non-degenerate *-representation, $\pi_u(A)H_u$ is dense in H_u, which tells us that $\pi_u(e_\lambda)h \to h$ for all $h \in H_u$. In other words, $\pi_u(e_\lambda)$ converges strongly to the identity in $\mathcal{B}(H_u)$. Of course $\pi_u(e_\lambda)$ lives inside $\mathbb{B}(M)$, which is a bounded, convex set, and so strong convergence in $\mathbb{B}(M)$ is equivalent to ultraweak convergence in $\mathbb{B}(M)$, which in turn tells us that $\widehat{e_\lambda}$ converges weak-* to $1_{A^{**}}$ in A^{**}. Thus, using weak-* continuity of the normalization $(J_B \circ \varphi)^{\cdot\cdot}$ we have

$$(J_B \circ \varphi)^{\cdot\cdot}(1_{A^{**}}) = w^* - \lim_{\lambda} \widehat{\varphi(e_\lambda)}$$

$\square$

To really see how powerful the enveloping von Neumann algebra is as a tool, we implore the reader to try to find proof of the corollary above which *doesn't* make reference to the enveloping von Neumann algebra (if you do attempt this, I wish you good luck).

Now that we've met A^{**} and begun to feel comfortable around it, and moreover that we've exhausted everything we can possibly say about A^{**} (not really) in meticulous detail, in the coming sections we will begin to relax our anal adherence to notation and treat these objects like human beings. Most of the time, instead of writing $\widehat{A}$ to refer to the image of A in A^{**}, we'll simply say $A \subseteq A^{**}$. We'll only refer back to the map J_A when it is absolutely necessary to clarify when something strange or unclear is happening. Moreover, if A is non-unital and $\widetilde{A}$ is its unitization, instead of referring to this cumbersome $\widehat{A}_1$, we'll just say $A \subseteq \widetilde{A} \subseteq A^{**}$.

Additionally, the reader may have noticed just how confusing Dunford-Schwarz notation (that is, letting $x \in X$, $x^* \in X^*$, $x^{**} \in X^{**}$, etc.) would be given that our bidual A^{**} is also a von Neumann algebra, and so possesses an involution *also* denoted with a * (is $a^{**} \in A^{**}$ just any old element, or are we applying the involution * to a twice, arriving back at a?). For this reason, we typically won't be using Dunford-Schwarz notation, instead opting for aptly chosen letters to denote elements of dual spaces.

3 Tensor Products of C*-Algebras

3.1 Algebraic Tensor Product

While it is unlikely that the reader has opened this document without first having been exposed to algebraic tensor products in one form or another, we will nonetheless go over the basics in our own words for our own sanity, and as an opportunity to refresh our memory.

Given two vector spaces V, W over $\mathbb{C}$ (this can be done over any field, but we really don't care about $\mathbb{F}_p(t)$ now do we), their *algebraic tensor product*, denoted $V \odot W$ (or in other texts $V \otimes W$, though we won't use this notation) is a new vector space consisting of "bilinear combinations" of elements from both V and W. More precisely, $V \otimes W$ is the vector space spanned by elements of the form $v \otimes w$, where $v \in V$ and $w \in W$, subject to the condition that this "thing" $\otimes$ is *bilinear*:

$$\begin{aligned}(\lambda u + \mu v) \otimes w &= \lambda(u \otimes w) + \mu(v \otimes w) \qquad \forall u, v \in V, w \in W, \quad \lambda, \mu \in \mathbb{C}\\ u \otimes (\lambda v + \mu w) &= \lambda(u \otimes v) + \mu(u \otimes w) \qquad \forall u \in V, v, w \in W, \quad \lambda, \mu \in \mathbb{C}\end{aligned}$$

Even more precisely, we can explicitly construct the vector space $V \otimes W$. Let $\mathcal{F}(V \times W)$ denote the vector space of finitely supported functions from the Cartesian product $V \times W$ into $\mathbb{C}$, and let $\delta_{(v,w)}$ denote the indicator function for the pair (v, w). Define a vector subspace $\mathcal{L}(V \times W)$ as the space spanned by the two sets

$$\begin{aligned}\{\delta_{(\lambda u+\mu v,w)} - \lambda\delta_{(u,w)} - \mu\delta_{(v,w)} &\ :\ u, v \in V, w \in W,\ \lambda, \mu \in \mathbb{C}\}\\ \{\delta_{(u,\lambda v+\mu w)} - \lambda\delta_{(u,v)} - \mu\delta_{(u,w)} &\ :\ u \in V, v, w \in W,\ \lambda, \mu \in \mathbb{C}\}\end{aligned}$$

Then we set $V \odot W := \mathcal{F}(V \times W)/\mathcal{L}(V \times W)$, and we denote the equivalence class of $\delta_{(v,w)}$ by $v \otimes w$. What we've done by quotienting by $\mathcal{L}(V \times W)$ is we've "forced" certain expressions to be zero, which ensure the bilinearity identities are satisfied by $\otimes$.

Elements of the form $v \otimes w$ in $V \otimes W$ will be called "simple tensors". Of course, the property of being a "simple tensor" is not an intrinsic property of the object in question, but rather a property of the *expression*: given a "simple tensor" $v \otimes w$, we can always write $v \otimes w = v \otimes tw + v \otimes (1-t)w$ for any $t \in \mathbb{C}$, an expression which is "not simple". Nonetheless, since $V \odot W$ is *spanned* by simple tensors, we will often define things only for simple tensors and then "extend by linearity", so the concept of a simple tensor is indeed useful.

Now why on earth would we care about such a space $V \odot W$? For one, the tensor product constructions is one of the fundamental ways of combining two vector spaces to get a new one, and mathematics is replete with examples where "making new things" is the key to a proof. Indeed, tensor products show up naturally in a number of different settings. One of the reasons for their ubiquity is their *universality property*: a *bilinear map* from $V \times W$ into some third vector space X always naturally factors through a *linear map* from $V \odot W$ into X.

Proposition 3.1.1 (Universality of Tensor Products)**.** Given vector spaces V, W, X and a bilinear map $B : V \times W \to X$, there exists a unique linear map $\widetilde{B} : V \odot W \to X$ such that

$$\widetilde{B}(v \otimes w) = B(v, w) \qquad \forall v \in V, w \in W$$

To prove this, let us first introduce a clever little tool which will be used time and time again in the coming notes.

Lemma 3.1.2. Let V, W be vector spaces, e_i be a set of linearly independent vectors in V, and $x := \sum_i e_i \otimes w_i \in V \odot W$. Then $x = 0$ if and only if $w_i = 0$ for all i.

Proof. Let $\{f_j\}$ be a Hamel basis for W. Consider the linear functionals $e_i^* \otimes f_j^* \in (V \odot W)'$ taking $v \otimes w$ to $e_i^*(v) f_j^*(w)$ (where e_i^*, f_j^* are the *dual functionals*, $e_i^*(\sum_j c_j e_j) = c_i$). If $x = 0$, then $0 = e_i^* \otimes f_j^*(x) = f_j^*(w_i)$ for all i, j, and so $w_i = 0$ for all i, as expected. $\square$

Proof of 3.1.1. The map $\widetilde{B}$ in question is first defined on simple tensors as $\widetilde{B}(v \otimes w) := B(v, w)$, and then "extended by linearity". That is, we'd like to say

$$\widetilde{B}\left(\sum_i v_i \otimes w_i\right) := \sum_i B(v_i, w_i)$$

but we have to check that this is well-defined. That is, assuming $\sum_{i=1}^m v_i \otimes w_i = \sum_{j=1}^n x_j \otimes y_j$, is it true that $\widetilde{B}\left(\sum_{i=1}^m v_i \otimes w_i\right) = \widetilde{B}\left(\sum_{j=1}^n x_j \otimes y_j\right)$? Assume without loss of generality that $\sum_i v_i \otimes w_i = 0$. Assume, again without loss of generality, that the w_i's are linearly independent (we can always rewrite $\sum_i v_i \otimes w_i$ to be so). Then by lemma 3.1.2 each of the v_i's is zero, and so $B(v_i, w_i) = 0$ for all i, whence

$$\widetilde{B}\left(\sum_i v_i \otimes w_i\right) = \sum_i B(v_i, w_i) = 0$$

as desired (this is enough to prove that $\widetilde{B}$ is well-defined). Uniqueness of $\widetilde{B}$ is obvious. $\square$

This proof exemplifies one of the problems with tensor products, which is that an arbitrary element $u \in V \odot W$ can be written in many different ways:

$$u = \sum_{i=1}^m v_i \otimes w_i = \sum_{j=1}^n x_j \otimes y_j$$

In other words, there are a lot of expressions of the form $\sum_{i=1}^m v_i \otimes w_i$ which are actually just equal to zero. Because of this, defining operations on $V \odot W$ isn't always as straightforward as it seems.

Tangent. At first, it wasn't entirely obvious to me why proposition 3.1.1 was useful at all. Sure it's conceptually kind of cool that we can replace bilinear things with linear things, with simple tensors acting as a sort of "bilinear pairing" of arguments. But do we really gain anything from this?

It is only in the context of other algebraic objects when this desire to "linearize bilinear things" becomes so intuitive. Allow me to take you on a brief tangent. If we look at the definition of an algebra A, the proposition above tells us that multiplication on A - a bilinear map from $A \times A \to A$ - can be realized as a *linear map* $\nabla : A \odot A \to A$ subject to some extra conditions (associativity, distributivity). The associativity condition is expressed as

$$\nabla \circ (\mathrm{id}_A \odot \nabla) = \nabla \circ (\nabla \odot \mathrm{id}_A)$$

or equivalently by commutativity of the following diagram

$$\begin{tikzcd}
A \odot (A \odot A) \arrow[rr, "\tau"] \arrow[d, "\mathrm{id}_A \odot \nabla"] & & (A \odot A) \odot A \arrow[d, "\nabla \odot \mathrm{id}_A"] \\
A \odot A \arrow[dr, "\nabla"] & & A \odot A \arrow[dl, "\nabla"] \\
& A &
\end{tikzcd}$$

where τ is the unique isomorphism taking $x \otimes (y \otimes z)$ to $(x \otimes y) \otimes z$ (and extended by linearity). We haven't really talked about what the "tensor product of maps" looks like yet (that is, $\mathrm{id}_A \odot \nabla$), which we'll discuss later on in section 3.3, but rest assured that a thorough understanding of them isn't terribly important for this tangent.

One may then "reverse the arrows" (as category theorists are wont to do) to obtain a mathematical structure called a *coalgebra*. A coalgebra is a vector space C over a field K equipped with a *comultiplication map* $\Delta : C \to C \odot C$, satisfying a "coassociativity" condition:

$$(\mathrm{id}_C \odot \Delta) \circ \Delta = (\Delta \odot \mathrm{id}_C) \circ \Delta$$

which is precisely what we get when we reverse the direction of the arrows in our associativity diagram

$$\begin{tikzcd}
C \odot (C \odot C) & & (C \odot C) \odot C \arrow[ll, "\tau^{-1}"'] \\
C \odot C \arrow[u, "\mathrm{id}_C \odot \Delta"] & & C \odot C \arrow[u, "\Delta \odot \mathrm{id}_C"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Coalgebras aren't the only structures built off of these "maps between tensor produts". We also have *bialgebras*, which are vector spaces with a compatible algebra and coalgebra structure (compatibility being a property expressed by a commutating diagram). There are also *Hopf algebras*, which are bialgebras with an extra "antipode" map. And don't even get me started on anyonic Lie algebras, quasitriangular Hopf algebras, Hopf algebroids, Courant algebroids, Drinfeld Doubles...

What's the point of this tangent? To provide some real world evidence as to why tensor products, and "linearizing bilinear things", can be so powerful. We could *try* to express all of the things above in terms of bilinear maps, but it would be excruciatingly painful. Indeed, thinking about multiplication as a "bilinear map" doesn't lend well to thinking about comultiplication: the map Δ is *not* just a map (Δ_1, Δ_2) from C to $C \times C$ (as one might expect the "opposite" of a bilinear map to be). We really do need tensor products to express all of this stuff.

End of Tangent

In light of proposition 3.1.1, it should come as no surprise (or perhaps only mildly surprising, as opposed to full-on consternating) that this operation of taking tensor products of vector spaces is both associative and commutative.

Proposition 3.1.3. Given any vector spaces V, W, X, there are canonical vector space isomorphisms

$$V \odot (W \odot X) \cong (V \odot W) \odot X, \qquad V \odot W \cong W \odot V$$

given by

$$v \otimes (w \otimes x) \mapsto (v \otimes w) \otimes x, \qquad v \otimes w \mapsto w \otimes v$$

Proof. Let $\Gamma : V \odot (W \odot X) \to (V \odot W) \odot X$ be the map defined above. First, note that by bilinearity of the tensor product, we have

$$\sum_i v_i \otimes \left(\sum_j w_{ij} \otimes x_{ij} \right) = \sum_{ij} v_i \otimes (w_{ij} \otimes x_{ij})$$

and so every vector in $V \odot (W \odot X)$ can be expressed as $\sum_i v_i \otimes (w_i \otimes x_i)$ (and similarly for $(V \odot W) \odot X$). With this, it's apparent that Γ is surjective, so all we have to check is that Γ is injective.

Suppose $\sum_i v_i \otimes (w_i \otimes x_i)$ is such that $\Gamma\left(\sum_i v_i \otimes (w_i \otimes x_i)\right) = \sum_i (v_i \otimes w_i) \otimes x_i = 0$. We can assume without loss of generality that the x_i's are linearly independent, in which case we conclude that $v_i \otimes w_i = 0$ for all i. By the note following lemma 3.1.2, we thus see that either $v_i = 0$ or $w_i = 0$ for each i. If $v_i = 0$, then $v_i \otimes (w_i \otimes x_i) = 0$. If $w_i = 0$, then $w_i \otimes x_i = 0$, and so again $v_i \otimes (w_i \otimes x_i) = 0$. Thus, $\sum_i v_i \otimes (w_i \otimes x_i) = 0$, implying Γ is surjective.

The proof of $V \odot W \cong W \odot V$ is similar but easier, and left as an exercise to the reader. $\square$

Why does proposition 3.1.1 in particular provide us the intuition for this proposition above? Well for one, bilinear maps $B : V \times W \to X$ are in one-to-one correspondence with the "swapped" map $B^s : W \times V \to X$ defined by $B^s(w, v) := B(v, w)$. Moreover a *trilinear map* $B : V \times W \times X \to Y$ is *bilinear* when thought of as a map from $(V \odot W) \times X \to Y$ *or* a map from $V \times (W \odot X) \to Y$, and these maps should *agree*.

So far, we've defined the tensor product of vector spaces, which itself is a vector space. Given algebras A, B over $\mathbb{C}$ (e.g. C*-algebras), their algebraic tensor product $A \odot B$ is also a vector space, but moreover can be endowed with its *own* algebra structure. Indeed, we start by defining

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (a_1 a_2) \otimes (b_1 b_2)$$

and extending by linearity (on the left and right). Again, it's not so clear that this operation is well-defined. Fortunately lemma 3.1.2 comes in handy again. Assuming $\sum_i a_i \otimes b_i = 0$, we have

$$\left(\sum_i a_i \otimes b_i \right) \left(\sum_j c_j \otimes d_j \right) = \sum_{i,j} a_i c_j \otimes b_i d_j$$

We can assume without loss of generality that the b_i's are linearly independent, in which case $\sum_i a_i \otimes b_i = 0$ implies $a_i = 0$ for all i. Thus, $a_i c_j = 0$ for all i, j, and so $\sum_{i,j} a_i c_j \otimes b_i d_j = 0$. Thus

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right) = 0$$

as desired. An identical argument proves the same result for $\sum_j c_j \otimes d_j$. As in the proof of proposition 3.1.1, this is enough to prove well-definedness of the operation.

If both A and B are involutive algebras, possessing involutions $*_A : A \to A$ and $*_B : B \to B$, we can equip $A \odot B$ with its own involution

$$(a \otimes b)^* := a^{*_A} \otimes b^{*_B}$$

which we leave as an exercise to the reader to check is well-defined. Below we list some other basic constructions, which we leave to the reader to verify are well-defined (once you've done one of them, you get the gist of them all).

Fact 3.1.4. Let A, B, C, D be (involutive) algebras, and $\varphi : A \to C$, $\psi : B \to D$ (*-)homomorphisms. Then there is a unique (*-)homomorphism defined by

$$\begin{array}{rcrcl} \varphi \odot \psi & : & A \odot B & \to & C \odot D \\ & & a \otimes b & \mapsto & \varphi(a) \otimes \psi(b) \end{array}$$

Fact 3.1.5. Let A, B, C be (involutive) algebras, and $\varphi : A \to C$, $\psi : B \to C$ (*-)homomorphisms. If φ and ψ have *commuting ranges* (that is, $\varphi(a)\psi(b) = \psi(b)\varphi(a)$ for all $a \in A, b \in B$) then there exists a unique (*-)homomorphism defined by

$$\begin{array}{rcrcl} \varphi \cdot \psi & : & A \odot B & \to & C \\ & & a \otimes b & \mapsto & \varphi(a)\psi(b) \end{array}$$

3.2 A Bestiary of Tensor Product Norms

Given C*-algebras A, B, we've seen how to form the algebraic tensor product $A \odot B$, but ultimately we'd like to work with some form of C*-algebra constructed "from" $A \odot B$ (so that we remain within the category of C*-algebras). To accomplish this task, we need to endow $A \odot B$ with a C*-algebra norm: a norm which is both an algebra norm (i.e. its submultiplicative $\|xy\| \leq \|x\|\|y\|$), and a C*-norm (it satisfies the C*-equation $\|x^*x\| = \|x\|^2$).

Here we run into a bit of a roadblock - there are, surprisingly, quite a few possible options of norms to choose from. What is the right norm with which to turn $A \odot B$ into a C*-algebra? One thing we could try is, for $x \in A \odot B$, define

$$\|x\| := \sup\{\|\pi(x)\| \ : \ \pi : A \odot B \to \mathcal{B}(H) \text{ a *-representation}\}$$

Wherever this expression is defined, it's certainly submultiplicative, and satisfies both the C*-identity and the triangle inequality. However, we're not sure yet if it's even well-defined (i.e. finite) for all $x \in A \odot B$, or if $\|x\|_1 = 0$ implies $x = 0$. Clearly there's some work left to do.

Let's try something else for now. If we embed $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$, then we can embed $A \odot B$ faithfully as a *-subalgebra of $\mathcal{B}(H \otimes K)$ (see appendix A.1), and this latter algebra comes equipped with a norm: the operator norm. Closing up $A \odot B$ with respect to this norm gives a tried-and-true C*-algebra: $\overline{A \otimes B}^{\|\cdot\|_{\mathcal{B}(H \otimes K)}}$. This is good because we now know there is *at least one* C*-norm on $A \odot B$. In fact, it looks like there are a lot, corresponding to different embeddings $A \subseteq \mathcal{B}(H)$, $B \subseteq \mathcal{B}(K)$.

Surprisingly, this way of constructing a C*-norm on the tensor product $A \odot B$ is incredibly stable: it actually doesn't depend on the choice of embeddings at all! Moreover, this unique norm is actually the *minimal* norm among all other C*-norms one can place on the algebraic tensor product. Neither of these results are all too easy to prove, and we'll refer the reader to [38] or section 3.4 of [5] for more details.

Theorem 3.2.1 (Takesaki)**.** The norm obtained by embedding $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ and considering $A \odot B \subseteq \mathcal{B}(H \otimes K)$ is *independent* of the chosen embeddings, and so is uniquely defined. Moreover, this unique norm is the *minimal* C*-norm with which one can equip $A \odot B$, and as a result is referred to as the *minimal norm* (also called the *spatial norm*), denoted $\|\cdot\|_{\min}$. The completion of $A \odot B$ with respect to $\|\cdot\|_{\min}$ is denoted $A \otimes_{\min} B$.

Definition 3.2.2. A seminorm ρ on the tensor product of normed vector spaces $V \odot W$ is called a *cross norm* if $\|v \otimes w\| = \|v\|\|w\|$ for all $v, w \in V, W$, and *subcross* if $\|v \otimes w\| \leq \|v\|\|w\|$.

Example. The min-norm is clearly a cross norm by lemma A.1.2.

Lemma 3.2.3 (Vowden; [41])**.** Let A, B be (possibly non-unital) C*-algebras, and ρ a C*-seminorm on $A \odot B$. Then ρ is *subcross.*

Proof. Let I denote the kernel of ρ, which (using the triangle inequality and C*-identity) one can prove is a two-sided *-ideal, whence we can construct the algebraic quotient $A \odot B/I$, which is itself an involutive algebra. Define $\widetilde{\rho}$ on $A \odot B/I$ by $\widetilde{\rho}(x+I) := \rho(x)$, which we see is well-defined since if $x + I = y + I$, then $\rho(x-y) = 0$, whence

$$\rho(x) \leq \rho(x-y) + \rho(y) = \rho(y), \qquad \rho(y) \leq \rho(y-x) + \rho(x) = \rho(x)$$

and so $\widetilde{\rho}(x+I) = \rho(x) = \rho(y) = \widetilde{\rho}(y+I)$. Moreover, $\widetilde{\rho}$ is a C*-norm (no longer a seminorm), and we can take the completion $C := \overline{A \odot B/I}^{\widetilde{\rho}}$.

Now recall that for a positive element x in a C*-algebra, $\|x\| \leq 1$ if and only if $x^2 \leq x$ (which follows from the continuous functional calculus). Supposing $a, b \in \mathbb{B}(A)_+, \mathbb{B}(B)_+$, then $a^2 \leq a$ and $b^2 \leq b$, so there exists $x \in A, y \in B$ such that $x^*x = a - a^2$ and $y^*y = b - b^2$, and thus

$$(a - a^2) \otimes (b - b^2) + I = (x \otimes y + I)^*(x \otimes y + I) \geq 0$$

or rather $(a \otimes b)^2 + I \leq a \otimes b + I$, so $\rho(a \otimes b) = \widetilde{\rho}(a \otimes b + I) \leq 1$ (the whole reason we needed to construct the quotient is so that we could be sure we're working in a C*-algebra: $(a \otimes b)^2 \leq a \otimes b$ doesn't mean anything in an arbitrary involutive algebra, and so we couldn't simply reason that $\rho(a \otimes b) \leq 1$). Scaling to arbitrary $a \geq 0, b \geq 0$ we have $\rho(a \otimes b) \leq \|a\|\|b\|$. For the general case, given $a \in A, b \in B$, the C*-identity finishes the job:

$$\rho(a \otimes b)^2 = \rho(a^*a \otimes b^*b) \leq \|a^*a\|\|b^*b\| = \|a\|^2\|b\|^2$$

□

Corollary 3.2.4. Given (possibly non-unital) C*-algebras A, B, and a C*-norm $\|\cdot\|_\alpha$ on $A \odot B$, the maps $a \mapsto a \otimes b_0$ and $b \mapsto a_0 \otimes b$ (for fixed $a_0 \in A, b_0 \in B$ respectively) are both continuous with respect to $\|\cdot\|_\alpha$.

Lemma 3.2.5. Let A, B be C*-algebras. Then for all $x \in A \odot B$, the quantity

$$\rho_{\max}(x) := \sup\{\|\pi(x)\| \ : \ \pi : A \odot B \to \mathcal{B}(H) \text{ a *-representation}\}$$

is finite, and hence is a C*-seminorm.

Proof. For any *-representation $\pi : A \odot B \to \mathcal{B}(H)$, the map $x \mapsto \|\pi(x)\|$ is a C*-seminorm, and so by lemma 3.2.3 is subcross, and so

$$\left\|\pi\left(\sum_i a_i \otimes b_i\right)\right\| \leq \sum_i \|\pi(a_i \otimes b_i)\| \leq \sum_i \|a_i\|\|b_i\|$$

but since this is true for arbitrary π, we conclude

$$\rho_{\max}\left(\sum_i a_i \otimes b_i\right) \leq \sum_i \|a_i\|\|b_i\| < \infty$$

So $\rho_{\max}$ is a well-defined C*-seminorm. □

Lemma 3.2.6. Let A, B be C*-algebras, and let $\|\cdot\|_\alpha$ be a C*-norm on $A \odot B$. Then $\|\cdot\|_\alpha \leq \rho_{\max}(\cdot)$.

Proof. Let $A \otimes_\alpha B$ be the closure of $A \odot B$ with respect to $\|\cdot\|_\alpha$. Since this is a C*-algebra, it admits a faithful representation $\pi : A \otimes_\alpha B \to \mathcal{B}(H)$, and $\pi|_{A \odot B}$ is a *-representation of $A \odot B$. Thus for all $x \in A \odot B$, $\|x\|_\alpha = \|\pi(x)\| \leq \rho_{\max}(x)$ (by definition). $\square$

Theorem 3.2.7. The C*-seminorm $\rho_{\max}$ is in fact a C*-norm, denoted $\|\cdot\|_{\max}$. The closure of $A \odot B$ with respect to this norm is called the *max tensor product*, and is denoted $A \otimes_{\max} B$.

Proof. Since $\rho_{\max}$ dominates any other C*-norm $\|\cdot\|_\alpha$ on $A \odot B$, we have $\rho_{\max}(x) \geq \|x\|_\alpha > 0$ for all nonzero $x \in A \odot B$. $\square$

Now that this busy-work is out of the way, and we're truly sure $\|\cdot\|_{\max}$ is a C*-norm (and the maximal one at that), we can easily verify a few corollaries.

Corollary 3.2.8. Every *-representation $\pi : A \odot B \to \mathcal{B}(H)$ extends uniquely to a *-representation $\widetilde{\pi} : A \otimes_{\max} B \to \mathcal{B}(H)$.

Corollary 3.2.9. The max norm is also given by

$$\begin{aligned}\|x\|_{\max} &= \sup\{\|x\|_\alpha \ : \ \|\cdot\|_\alpha \text{ a C*-norm on } A \odot B\} \\ &= \sup\{\|\pi(x)\| \ : \ \pi : A \odot B \to \mathcal{B}(H) \text{ a } \textit{cyclic} \text{ *-representation}\}\end{aligned}$$

Hint: For the second formula, recall theorem 2.1.1.

Theorem 3.2.10. Every C*-norm on $A \odot B$ is a cross norm.

Proof. Since the min norm is a cross norm, and every C*-norm is subcross, we have

$$\|a\|\|b\| = \|a \otimes b\|_{\min} \leq \|a \otimes b\|_{\max} \leq \|a\|\|b\|$$

from which we conclude that $\|a \otimes b\|_\alpha = \|a\|\|b\|$ for any C*-norm $\|\cdot\|_\alpha$ on $A \odot B$. $\square$

In the case where one of the tensorants is a *von Neumann algebra*, we might not be as interested in arbitrary *-representations of $A \odot B$ as we are in those *-representations which are *normal* (in the respective tensorant). That is, if A is a unital C*-algebra, M a von Neumann algebra, and $\pi : A \odot M \to \mathcal{B}(H)$ a *-representation, we would like the *-representation

$$\begin{array}{rccc}\sigma & : & M & \to & \mathcal{B}(H) \\ & & x & \mapsto & \pi(1 \otimes x)\end{array}$$

to be normal. This additional requirement of normality is a classic structural feature in the category of von Neumann algebras, which we would like our morphisms to maintain (see appendix A.3).

This leads us naturally into the definitions of the *normal* and *binormal* norms. Before we get there, we'd like to make a point about *restrictions* of *-representations π of $A \odot B$ to each tensorant. If A, B are unital, it is easy to see how we obtain restrictions: simply define $\pi|_A(a) := \pi(a \otimes 1_B)$, and $\pi|_B(b) := \pi(1_A \otimes b)$. We can obtain restrictions in the *non-unital* case too, but we define these restrictions more abstractly, in particular by *how they act on the range space* $\pi(A \odot B)$:

Theorem 3.2.11. Let A, B be (potentially non-unital) C*-algebras, and $\pi : A \odot B \to \mathcal{B}(H)$ a non-degenerate *-representation. Then there is a *-representation $\pi|_A : A \to \mathcal{B}(H)$ called the *restriction of* π *to* A, uniquely defined by the requirement that

$$\pi|_A(a)\pi\left(\sum a_i \otimes b_i\right)\xi = \pi\left(\sum aa_i \otimes b_i\right)\xi, \qquad \forall a_i \in A, b_i \in B, \text{ and } \xi \in H \tag{2}$$

The restriction $\pi|_B$ to B is defined analogously.

Proof. Since π is non-degenerate, the vectors $\{\pi(x)\xi \mid x \in A \odot B, \xi \in H\}$ are dense in H, so equation (2) certainly defines the operators $\pi|_A(a)$ on all of H, moreover if this expression is well-defined then we can be sure it defines a *-homomorphism. The hard part here is showing that this expression is well-defined. If we can show that $\pi\left(\sum_i a_i \otimes b_i\right) = 0$ implies $\pi\left(\sum aa_i \otimes b_i\right) = 0$, then we're done.

Suppose $\pi\left(\sum_i a_i \otimes b_i\right) = 0$, and let e_λ be an approximate identity for B. Then by continuity of π with respect to the max-norm, along with continuity of the map $b \mapsto a_0 \otimes b$ (for fixed $a_0 \in A$) with respect to any C*-norm on the tensor product (corollary 3.2.4), we have

$$\pi\left(\sum_i aa_i \otimes b\right) = \lim_\lambda \pi\left(\sum aa_i \otimes e_\lambda b_i\right) = \lim_\lambda \pi(a \otimes e_\lambda)\pi\left(\sum a_i \otimes b_i\right) = 0$$

□

Remark. Note that the restrictions $\pi|_A$ and $\pi|_B$ of π have commuting ranges (this is easy to see in the unital case, and the non-unital case follows from equation (2)). On the other hand, the opposite of restrictions of a *-homomorphism is *concatenation* of *-homomorphisms. Given *-representations $\rho : A \to \mathcal{B}(H)$ and $\sigma : B \to \mathcal{B}(K)$, we can consider these as commuting *-representations of A and B on $H \oplus K$, by defining

$$\widetilde{\rho}(a) = \begin{bmatrix} \rho(a) & 0 \\ 0 & 1 \end{bmatrix}, \qquad \widetilde{\sigma}(b) = \begin{bmatrix} 1 & 0 \\ 0 & \sigma(b) \end{bmatrix}$$

These commuting *-representations can be "concatenated" (as in fact 3.1.5) to obtain a *-representation

$$\begin{array}{rcccccc} \pi & := & \rho \cdot \sigma & : & A \odot B & \to & \mathcal{B}(H \oplus K) \\ & & & & a \otimes b & \to & \widetilde{\rho}(a)\widetilde{\sigma}(b) \end{array}$$

The notation $\rho \cdot \sigma$ will be used continuously throughout this text. If ρ, σ are already commuting *-representations on the same Hilbert space H (that is, we don't have to dilate them to $\widetilde{\rho}$ and $\widetilde{\sigma}$), $\rho \cdot \sigma$ will simply denote the *-representation taking $a \otimes b$ to $\rho(a)\sigma(b)$.

Definition 3.2.12. Let A be a C*-algebra, and M, N von Neumann algebras. The *normal norm* on $A \odot M$ is defined by

$$\|x\|_{\text{nor}} := \sup\left\{ \|\pi(x)\| \;:\; \pi = \rho \cdot \sigma, \text{ with } \begin{array}{l} \rho : A \to \mathcal{B}(H) \text{ a *-representation} \\ \sigma : M \to \mathcal{B}(K) \text{ a } \textit{normal} \text{ *-representation} \end{array} \right\}$$

The *binormal norm* on $M \odot N$ is defined by

$$\|x\|_{\text{bin}} := \sup\left\{ \|\pi(x)\| \;:\; \pi = \rho \cdot \sigma, \text{ with } \begin{array}{l} \rho : M \to \mathcal{B}(H) \text{ a } \textit{normal} \text{ *-representation} \\ \sigma : N \to \mathcal{B}(K) \text{ a } \textit{normal} \text{ *-representation} \end{array} \right\}$$

The completions with respect to these norms are denoted $A \otimes_{\text{nor}} M$ and $M \otimes_{\text{bin}} N$ respectively.

By theorem 3.2.11, we see that $\|\cdot\|_{\text{nor}}$ and $\|\cdot\|_{\text{bin}}$ are also given by

$$\|x\|_{\text{nor}} = \sup\left\{ \|\pi(x)\| \;:\; \begin{array}{l} \pi : A \odot M \to \mathcal{B}(H) \text{ a *-representation} \\ \text{such that } \pi|_M \text{ is } \textit{normal} \end{array} \right\}$$

$$\|x\|_{\text{bin}} = \sup\left\{ \|\pi(x)\| \;:\; \begin{array}{l} \pi : M \odot N \to \mathcal{B}(H) \text{ a *-representation} \\ \text{such that } \pi|_M \text{ and } \pi|_N \text{ are } \textit{normal} \end{array} \right\}$$

Occasionally we may wish to consider the normal norm on *one side* of the tensor product of two von Neumann algebras, and so to distinguish which side we wish to restrict to normal representations we may write $M \otimes_{\text{lnor}} N$ for the *left-normal norm* or $M \otimes_{\text{rnor}} N$ for the *right-normal norm.* In most instances, it will be discernable from context which of lnor or rnor the "nor" is referring to.

Remark. It might seem strange that we're considering tensor products of von Neumann algebras, but we haven't paid attention yet to whether the resulting tensor product is *itself* a von Neumann algebra. Much of our work is *dependent* on von Neumann algebras (particularly the enveloping von Neumann algebra), but operates within the realm of C*-algebras, and it turns out we don't really need $M \otimes N$ to be a von Neumann algebra itself.

There is a canonical way of forming the *von Neumann tensor product*: if $M \subseteq \mathcal{B}(H)$ and $N \subseteq \mathcal{B}(K)$, the von Neumann tensor product, denoted $M \overline{\otimes} N$, is simply the weak operator closure of $M \odot N \subseteq \mathcal{B}(H \otimes K)$. This type of tensor product is important in many structure theorems related to the decomposition of von Neumann algebras (see for instance [15] or [22]). The von Neumann tensor product will not appear further in this text.

It is not hard to check that the min, max, and binormal norms are all *commutative* and *associative*.

Proposition 3.2.13. For each of $\alpha \in \{\min, \max, \text{bin}\}$, and C*-algebras, A, B, we have

$$A \otimes_\alpha B = B \otimes_\alpha A$$

and for any third C*-algebra C, and $\beta \in \{\min, \max\}$ we have

$$A \otimes_\beta (B \otimes_\beta C) = (A \otimes_\beta B) \otimes_\beta C$$

(this doesn't work for the bin norm since $B \otimes_{\text{bin}} C$ isn't a von Neumann algebra).[3]

For each of $\alpha \in \{\min, \max, \text{bin}\}$, and C*-algebras A, B, C, we naturally have

$$A \otimes_\alpha B = B \otimes_\alpha A, \qquad A \otimes_\alpha (B \otimes_\alpha C) = (A \otimes_\alpha B) \otimes_\alpha C$$

Moreover, we have

$$A \otimes_{\text{rnor}} M = M \otimes_{\text{lnor}} A$$

Nuclearity

Definition 3.2.14. A C*-algebra A is said to be *nuclear* if for any other C*-algebra B, the min and max norms on $A \odot B$ coincide, and so $A \otimes_{\min} B = A \otimes_{\max} B$. In other words, A is nuclear if $A \odot B$ possesses a *unique* C*-norm.

Definition 3.2.15. More generally, a pair of C*-algebras (A, B) is called a *nuclear pair* if there is a unique norm on $A \odot B$.

Example. Finite-dimensional C*-algebras are nuclear. To see this, first note that we have a *-isomorphism $M_n \odot A \cong M_n(A)$, and since the latter is a C*-algebra we obtain a C*-algebra norm on $M_n \odot A$, with respect to which $M_n \odot A$ is already closed, so $M_n \odot A$ necessarily admits a unique norm.

As for the general case, it is well-known that a finite-dimensional C*-algebra is of the form $M_{n_1} \oplus \cdots \oplus M_{n_k}$ for some positive integers $n_1, ..., n_k$ (see [13]; theorem III.1.1). It is easy to check that

$$(A_1 \oplus \cdots \oplus A_k) \odot B \cong (A_1 \odot B) \oplus (A_2 \odot B) \oplus \cdots \oplus (A_k \odot B)$$

and so

$$A \odot B \cong (M_{n_1} \odot B) \oplus \cdots \oplus (M_{n_k} \odot B)$$

Since the RHS is already complete with respect to a unique C*-norm, so too is the LHS, and thus A is nuclear.

Example. Nuclear C*-algebras are plentiful. For instance,

- Commutative C*-algebras are nuclear.
- Direct limits of nuclear C*-algebras are nuclear, and so AF-algebras are nuclear.
- For any Hilbert space H, the algebra of compact operators $\mathcal{K}(H)$ is nuclear.
- If A is nuclear, then $M_n(A)$ is nuclear for every n. This follows by shuffling algebraic tensor products:

  $$M_n(A) \odot B = (M_n \odot A) \odot B = M_n \odot (A \odot B)$$

  Thus if $A \odot B$ comes with a unique norm, since M_n is itself nuclear, a moment of thought shows that $M_n(A) \odot B$ also comes equipped with this unique norm. Another proof of this fact will be provided later on.

[3] An interesting result of Pfitzner [35] and Kania [23], unrelated to our work in this paper, is that a von Neumann algebra can *never be written as a non-trivial C*-algebraic tensor product*, with respect to *any C*-tensor norm.* Von Neumann algebras, as Banach spaces, are *Grothendieck spaces*, which cannot be written as a tensor product of C*-algebras with each tensorant infinite-dimensional.

- Bunce-Deddens algebras, being the direct limit of algebras of the form $M_n(C(\mathbb{T}))$ (see [13]; section V.3), are nuclear.
- If G is an amenable discrete group, then $C^*_\lambda(G)$ and $C^*(G)$ are nuclear.
- The Cuntz algebras $\mathcal{O}_n$ - the universal C*-algebra generated by n isometries $s_1, ..., s_n$ subject to the condition $\sum_i s_i s_i^* = 1$, where $n \in \{2, ..., \infty\}$ (see [13]; section V.4) - are all nuclear.

Unfortunately, we lack the tools necessary to prove these examples, and so we defer proofs to later sections. (We won't be proving that the Cuntz algebras are nuclear anywhere in this paper.)

Theorem 3.2.16. Let A be a C*-algebra. Suppose that for every finite set $F \subset A$ and $\epsilon > 0$, there exists a nuclear C*-subalgebra $B \subseteq A$ such that for all $a \in F$, $\mathrm{dist}(a, B) < \epsilon$. Then A is nuclear.

Proof. Let C be another C*-algebra, and let $x = \sum_{i=1}^n a_i \otimes c_i \in A \odot C$. Let $F = \{a_1, ..., a_n\}$, choose $\epsilon' > 0$, and let $\epsilon = \epsilon'/(2n \sup_i \|c_i\|)$. Let $B \subseteq A$ be a nuclear C*-subalgebra satisfying the property in the theorem statement for our given F and ϵ. Then

$$\begin{aligned}
\left\|\sum a_i \otimes c_i\right\|_{A \otimes_{\max} C} &\leq \epsilon'/2 + \left\|\sum b_i \otimes c_i\right\|_{A \otimes_{\max} C} \\
&\leq \epsilon'/2 + \left\|\sum b_i \otimes c_i\right\|_{B \otimes_{\max} C} \\
&= \epsilon'/2 + \left\|\sum b_i \otimes c_i\right\|_{B \otimes_{\min} C} \\
&= \epsilon'/2 + \left\|\sum b_i \otimes c_i\right\|_{A \otimes_{\min} C} \\
&\leq \epsilon' + \left\|\sum a_i \otimes c_i\right\|_{A \otimes_{\min} C}
\end{aligned}$$

where we have twice-applied the triangle inequality plus the fact that every C*-norm on the tensor product is a cross norm, once in the first line and once in the last line. What we've just shown is that $\|x\|_{A \otimes_{\max} C} \leq \epsilon' + \|x\|_{A \otimes_{\min} C}$ for every $x \in A \odot C$ and every $\epsilon' > 0$, so letting ϵ' tend to zero necessarily gives $\|x\|_{A \otimes_{\max} C} = \|x\|_{A \otimes_{\min} C}$, implying A is nuclear. $\square$

Example. For an infinite-dimensional Hilbert space H, the algebra $\mathcal{B}(H)$ is *non-nuclear.* This was demonstrated in [42], and unfortunately we won't have time to properly prove this result.

Example. It is not necessarily true that C*-subalgebras of nuclear C*-algebras are themselves nuclear - however *ideals* in nuclear C*-algebras are are nuclear, along with *quotients* of nuclear C*-algebras.

None of the claims above are particularly easy to prove, and some are downright arcane without the right formulations of nuclearity. Thus, it will be some time before we actually see any concrete examples of nuclear C*-algebras and their deeper properties.

Inheritance of Norms, and Natural Inclusions of Tensor Products

Given C*-algebras $A \subseteq B$ and $C \subseteq D$, it is natural to ask whether or not the inclusion $A \odot C \subseteq B \odot D$ is retained when completing either tensor product with respect to various norms. Before pondering what it means for there to exist a "natural inclusion", we must first check that $A \odot C \subseteq B \odot D$ makes sense. It is hypothetically possible a non-zero element $\sum_i a_i \otimes c_i$ in $A \odot C$ evaluates to zero in the larger setting of $B \odot D$.

Lemma 3.2.17. Let $\iota_1 : A \to B$, $\iota_2 : C \to D$ be injective linear maps of vector spaces. Then $\iota_1 \odot \iota_2 : A \odot C \to B \odot D$ is injective.

Proof. Since $\iota_1 \odot \iota_2 = (\iota_1 \odot \mathrm{id}_D) \circ (\mathrm{id}_A \odot \iota_2)$, it suffices to show that $\iota_1 \odot \mathrm{id}_D$ is injective. Suppose $\iota_1 \odot \mathrm{id}_D \left(\sum_i a_i \otimes d_i\right) = 0$. Write $\iota_1 \odot \mathrm{id}_D \left(\sum_i a_i \otimes d_i\right) = \sum_i \iota_1(a_i) \otimes d_i$. Assume without loss of generality that the d_i's are linearly independent. By lemma 3.1.2, $\iota_1(a_i) = 0$ for all i, and so $a_i = 0$ for all i, whence $\sum_i a_i \otimes d_i = 0$. $\square$

So we now know that $A \odot C$ is indeed a well-defined subalgebra of $B \odot D$. Now let us turn our attention to norm closures of algebraic tensor products.

Let X, Y be Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Y$ and dense subspaces E, F respectively. Suppose we have a linear map $T : E \to F$. Then T extends uniquely to a well-defined linear map $\widetilde{T} \in \mathcal{B}(X, Y)$ if $\|Tx\|_Y \leq \|x\|_X$ for all $x \in E$ - in this case, $\widetilde{T}$ is defined by density, and boundedness on E guarantees $\widetilde{T}$ is indeed *uniquely* defined. Moreover, it is not hard to show that $\|\widetilde{T}\| = \|T\|$, and if T is an *isometry* then so too is $\widetilde{T}$. In the case that X, Y are Banach (*-)algebras, E, F (*-)subalgebras, and T is a (*-)homomorphism, the extension $\widetilde{T}$ is also a (*-)homomorphism.

Definition 3.2.18. Consider the inclusion map $\iota : A \odot C \to B \odot D$. Suppose $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are norms on $A \odot C$ and $B \odot C$ respectively, such that

$$\|\iota(x)\|_\beta = \|x\|_\alpha \qquad \forall x \in A \odot C$$

Then ι extends to an isometric *-homomorphism $\widetilde{\iota} : A \otimes_\alpha C \to B \otimes_\beta D$. We can identify $A \otimes_\alpha C$ with its image under $\widetilde{\iota}$, which is a C*-subalgebra of $B \otimes_\beta D$. To denote this, we will write

$$A \otimes_\alpha C \subseteq B \otimes_\beta D$$

Whenever we place a norm on $B \odot D$, the restriction to $A \odot C$ is obviously again a norm, however we *cannot* be sure that this restricted norm inherits the same form as the parent norm.

Example. Suppose we endow $B \odot D$ with the max norm (denoted $\|\cdot\|_{B \otimes_{\max} D}$ to eliminate ambiguity), and restrict it to $A \odot C$. By maximality of the max norm on $A \odot C$, we necessarily have that

$$\|\cdot\|_{B \otimes_{\max} D} \leq \|\cdot\|_{A \otimes_{\max} C} \quad \text{restricted to } A \odot C$$

and yet it is *not always true that* these two norms are equal - in fact in many cases they aren't! To be perfectly clear,

$***$ The restriction of the max norm on $B \odot D$ to $A \odot C$ is *not* the max norm on $A \odot C$! $***$

So perhaps somewhat puzzlingly, we *do not* always have a natural inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} D$.

Example. On the other hand, the *min* norm behaves quite hospitably towards inclusions. Since the choice of embedding doesn't change the min norm (by theorem 3.2.1), we can embed $A \subseteq B \subseteq \mathcal{B}(H)$, and $C \subseteq D \subseteq \mathcal{B}(K)$, from which we clearly see that $A \odot C \subseteq B \odot D \subseteq \mathcal{B}(H \otimes K)$, and so $\|\cdot\|_{A \otimes_{\min} C} = \|\cdot\|_{B \otimes_{\min} D} =$ the operator norm on $\mathcal{B}(H \otimes K)$. Thus we have established the following fact:

Fact 3.2.19. For any C*-algebras $A \subseteq B$ and $C \subseteq D$, there is a natural inclusion

$$A \otimes_{\min} C \subseteq B \otimes_{\min} D$$

Understanding precisely when we have natural inclusions with respect to other C*-tensor products will be a centerpiece of this paper.

Example. If A is a nuclear C*-algebra, then for any other C*-algebras $C \subseteq D$, we have

$$A \otimes_{\max} C = A \otimes_{\min} C \subseteq A \otimes_{\min} D = A \otimes_{\max} D$$

If additionally D is nuclear, we can reason that whenever $A \subseteq B$, then $A \otimes_{\max} C \subseteq B \otimes_{\max} D$. However, as we will see this additional requirement on D is superfluous - we *always* have a natural inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} D$ so long as A is nuclear (this is not at all an obvious fact, and will be the subject of section 5.1).

A general theme of tensor product norms is that "what works well for the min norm fails miserably for the max norm", and vice versa. For instance, we have already seen that inclusions behave nicely with respect to the min norm, but somehow don't agree with the max norm. On the other hand, later on we will learn that $\otimes_{\max}$ is an *exact bifunctor* (it preserves exact sequences), whereas this cannot be further from the truth for $\otimes_{\min}$. All of these antinomies are reconciled by nuclearity, which is partly why it is such a coveted property.

Proposition 3.2.20. Let A, B be C*-algebras, and let α denote a tensor norm on $A \odot B$. Then there is a natural quotient map $Q_\alpha : A \otimes_{\max} B \to A \otimes_\alpha B$ (which is a surjective *-homomorphism).

Proof. This simply follows from the fact that the identity map $\mathrm{id}_{A\odot B} : A \odot B \to A \odot B$ is automatically continuous with respect to the α-norm on the codomain and the max norm on the domain (by the very definition of the max norm). Thus $\mathrm{id}_{A\odot B}$ extends by density to the map Q_α, which is also easily seen to be surjective. $\square$

This proposition perhaps makes it even stranger that something which works for $A \otimes_{\max} B$ fails for $A \otimes_{\min} B$ and vice versa, seeing as the latter is simply a quotient of the former (which, in most circumstances, we might expect to "behave nicely", even "analogously" to one another).

We also state here for good measure the following useful result, which will show up a few times in our work later on.

Lemma 3.2.21. Given any C*-algebras A, B, then

$$\|\cdot\|_{A\otimes_{\max} B} = \|\cdot\|_{A^{**}\otimes_{\max} B} \quad \text{on } A \odot B$$

Proof. By definition of the max-norm, the norm $\|\cdot\|_{A^{**}\otimes_{\max} B}$ when restricted to $A \odot B$ must be less than $\|\cdot\|_{A\otimes_{\max} B}$, so we need to show that $\|\cdot\|_{A\otimes_{\max} B} \leq \|\cdot\|_{A^{**}\otimes_{\max} B}$.

Let $\pi : A \odot B \to \mathcal{B}(H)$ be a non-degenerate *-homomorphism, and decompose $\pi = \pi|_A \cdot \pi|_B$. Note that $\pi|_A : A \to \pi|_B(B)'$, and we can extend $\pi|_A$ to a normal *-homomorphism $\sigma := \ddot{\pi}|_A : A^{**} \to \pi|_B(B)'$. Moreover, note that[4] $\sigma(A^{**}) = \pi|_A(A)''$ and so $\pi|_B(B) \subseteq \pi|_A(A)' = \pi|_A(A)''' = \sigma(A^{**})'$.

Thus, σ and $\pi|_B$ have commuting ranges, and so we can stitch them back together to obtain a *-homomorphism $\widetilde{\pi} := \sigma \cdot \pi|_B : A^{**} \odot B \to \mathcal{B}(H)$, and for which $\pi = \widetilde{\pi}|_{A\odot B}$. Thus

$$\|\pi(x)\| = \|\widetilde{\pi}(x)\| \ \forall x \in A \odot B$$

Thus, taking the supremum over all $\widetilde{\pi}$, we get $\|\pi(x)\| \leq \|x\|_{A^{**}\otimes_{\max} B}$, and since π was arbitrary, $\|x\|_{A\otimes_{\max} B} \leq \|x\|_{A^{**}\otimes_{\max} B}$ $\square$

Remark. Note that we have actually shown that

$$\|\cdot\|_{A\otimes_{\max} B} \leq \|\cdot\|_{A^{**}\otimes_{\mathrm{nor}} B} \quad \text{on } A \odot B$$

and clearly $\|\cdot\|_{A^{**}\otimes_{\mathrm{nor}} B} \leq \|\cdot\|_{A^{**}\otimes_{\max} B}$ on $A \odot B$, so in actuality all three of the norms

$$\|\cdot\|_{A\otimes_{\max} B}, \quad \|\cdot\|_{A^{**}\otimes_{\mathrm{nor}} B}, \quad \|\cdot\|_{A^{**}\otimes_{\max} B}$$

coincide on $A \odot B$.

Moreover, a simple modification of the preceding proof yields $\|\cdot\|_{A\otimes_{\max} B} = \|\cdot\|_{A^{**}\otimes_{\mathrm{bin}} B^{**}}$ when restricted to $A \odot B$, so for the sake of completeness:

Corollary 3.2.22.

$$\forall x \in A \odot B \qquad \begin{aligned} \|x\|_{A\otimes_{\max} B} &= \|x\|_{A^{**}\otimes_{\mathrm{nor}} B} &&= \|x\|_{A\otimes_{\mathrm{nor}} B^{**}} &&= \|x\|_{A^{**}\otimes_{\mathrm{bin}} B^{**}} \\ &= \|x\|_{A^{**}\otimes_{\max} B} &&= \|x\|_{A\otimes_{\max} B^{**}} &&= \|x\|_{A^{**}\otimes_{\max} B^{**}} \end{aligned}$$

3.3 Tensor Products of Maps

A running theme in mathematics is that we are not merely interested in mathematical objects alone, but also the maps *between* these objects (thus is the essence of category theory, which is arguably the essence of all mathematics). Now that we've discussed tensor products of C*-algebras, we naturally turn our attention towards tensor products of *maps* between C*-algebras.

[4]Recall that $\operatorname{ran} \ddot{u} = \overline{\operatorname{ran} u}^{w*}$. In $\mathcal{B}(H)$, the weak-* topology is just the ultra-weak topology, and since $\pi|_A(A)$ is a *-algebra, its weak operator closure coincides with its ultra-weak closure, which coincides with $\pi|_A(A)''$.

Given linear maps $\phi : A \to B$ and $\psi : C \to D$ of C*-algebras, there is a natural definition of $\phi \odot \psi : A \odot C \to B \odot D$, which we have been implicitly using in the last section without reference:

$$\phi \odot \psi \left(\sum_i a_i \otimes b_i \right) := \sum_i \phi(a_i) \otimes \psi(b_i)$$

Using lemma 3.1.2, it is not hard to check that this map is well-defined. It is also not hard to check that if both ϕ, ψ are *-homomorphisms, then so too is $\phi \odot \psi$. Another feature which will come in handy is associativity of tensor products of maps

$$\varphi \odot (\phi \odot \psi) = (\varphi \odot \phi) \odot \psi$$

under the association $A \odot (B \odot C) \cong (A \odot B) \odot C$. The algebraic tensor produt of maps also naturally gives us the *ampliation* of maps:

$$\varphi_n : M_n(A) \to M_n(B) \quad \text{is equal to} \quad \varphi \odot \mathrm{id}_{M_n} : A \odot M_n \to B \odot M_n$$

again, under the association $A \odot M_n \cong M_n(A)$.

The hard question on everyone's mind is: when does $\phi \odot \psi$ extend to a well-defined map of $A \otimes_\alpha C \to B \otimes_\beta D$?

Min-Tensor Product of Maps

The results in this section pertaining to the min-tensor product of maps are the hard work of Pisier in [36], with the addition of some explanatory details. An idiosyncrasy of Pisier's work is that it features an abundance of *operator spaces*, taking the place of C*-algebras where we would normally use them. An operator space, as we recall, is merely a norm-closed vector subspace of a C*-algebra. In many of the results that follow, we never use the fact that the objects in question are C*-algebras, only that they're operator spaces. Moreover, later on we will *need* to refer to some of these results *specifically* for operator spaces. For this reason, we state the results in this section in terms of operator spaces, but every single result will remain true if the objects of interest are restricted to C*-algebras.

First we need to clarify what the tensor product of operator spaces is. Given operator spaces $E \subseteq A$, $F \subseteq B$ in C*-algebras A, B, we have an embedding $E \odot F \subseteq A \odot B$, and so hypothetically a norm on $A \odot B$ descends to a norm on $E \odot F$. However, as we've seen, we can't always be sure that the inherited norm on $E \odot F$ has the same form as on $A \odot B$. One thing we *can* be sure of is that the *min*-norm on $A \odot B$ restricts uniquely to a norm on $E \odot F$, which we *define* to be the operator space min-norm. It is not hard to see that this norm is independent of the embedding in C*-algebras A, B, and so this min-norm is uniquely defined. In summary, we have:

Definition 3.3.1. Given operator spaces E, F, there is a unique norm on $E \odot F$ called the *min*-norm, which is just the min-norm inherited from $A \odot B$, where $A \supseteq E$ and $B \supseteq F$ are *any* C*-algebras. The completion of $E \odot F$ with respect to this norm is denoted $E \otimes_{\min} F$, and is naturally an operator space in $A \otimes_{\min} B$.

The min-norm on $E \otimes_{\min} F$, being a norm inherited from a C*-algebra, is naturally an "operator space norm", which turns $E \otimes_{\min} F$ into a *matricially-normed space* (which is to say that the norms on $M_n(E \otimes_{\min} F)$ "behave well" as matrix norms, satisfying certain desirable properties). For a more systematic exposition on the tensor products of operator spaces, which includes precise definitions of these terms, see [4].

Other norms on $E \odot F$ aren't quite as easy to come by. For instance, the max-norm on $E \odot F$ isn't exactly the restriction of the max-norm on $A \odot B$ anymore, as we can't be sure this is independent of our choice of A and B. Blecher and Paulsen in [4] also give a treatment of these other tensor products of operator spaces.

Fortunately for our purposes, our discussion of tensor products of operator spaces ends with the min-tensor product - no other tensor products are needed. If it makes the reader feel better, for the remainder of this section one can essentially replace the symbols E, F, G standing in for operator spaces with A, B, C denoting C*-algebras, and the arguments will remain unchanged.

Proposition 3.3.2. Given a completely bounded map $T : E \to F$ between operator spaces, and an operator space G, there is a unique bounded linear map

$$T \otimes_{\min} \mathrm{id}_G : E \otimes_{\min} G \to F \otimes_{\min} G$$

extending the map $T \odot \mathrm{id}_G : E \odot G \to F \odot G$, and for which both $\|T \otimes_{\min} \mathrm{id}_G\| = \|T \odot \mathrm{id}_G\|$, and $\|T \otimes_{\min} \mathrm{id}_G\| \leq \|T\|_{cb}$.

Proof. All the results in our proposition will follow once we prove that $T \odot \mathrm{id}_G$ is bounded with respect to the min norms. Proving this, however, is less obvious than one might expect - we'll need a bit of an approximation trick, a trick which may be familiar to those well-versed in approximation ideas ubiquitous in single operator theory.

For the sake of simplicity, we'll embed $F \subseteq \mathcal{B}(H)$ and $G \subseteq \mathcal{B}(K)$ for some Hilbert spaces H, K, so that $F \otimes_{\min} G \subseteq \mathcal{B}(H \otimes K)$.

Let $\mathcal{E}$ be an orthonormal basis for K, and let $\mathcal{V}(\mathcal{E}) := \{\operatorname{span} S \mid S \subseteq \mathcal{E},\ |S| < \infty\}$ denote the set of finite dimensional subspaces of H spanned by elements of $\mathcal{E}$. Given $V \in \mathcal{V}(\mathcal{E})$, let P_V denote the orthogonal projection onto V. Then for all $\xi \in H$, $\|\xi\| = \sup_{V \in \mathcal{V}(\mathcal{E})} \|P_V \xi\|$, and for any $T \in \mathcal{B}(H)$

$$\|T\| = \sup_{V \in \mathcal{V}(\mathcal{E})} \|P_V T|_V\|$$

Additionally, if we take $T \in \mathcal{B}(H \otimes K)$, then it is not hard to verify that

$$\|T\| = \sup_{V \in \mathcal{V}(\mathcal{E})} \|P_{H \otimes V} T|_{H \otimes V}\|$$

Thus we have that

$$\begin{aligned} \|T \odot \mathrm{id}_G\| &= \sup \left\{ \left\|\sum T(a_i) \otimes b_i\right\|_{F \otimes_{\min} G} \mid \left\|\sum a_i \otimes b_i\right\|_{E \otimes_{\min} G} \leq 1 \right\} \\ &= \sup \left\{ \left\|\sum T(a_i) \otimes P_V b_i|_V\right\| \mid \left\|\sum a_i \otimes b_i\right\| \leq 1,\ V \in \mathcal{V}(\mathcal{E}) \right\} \end{aligned}$$

The requirement that $\|\sum a_i \otimes b_i\| \leq 1$ automatically implies that $\|\sum a_i \otimes P_V b_i|_V\| \leq 1$ since the restriction and projection maps have norm 1, and the operators $P_V b_i|_V$ can be realized as operators in $\mathcal{B}(V)$. Thus we have

$$\begin{aligned} \|T \odot \mathrm{id}_G\| &\leq \sup \left\{ \left\|\sum T(a_i) \otimes v_i\right\| \mid \left\|\sum a_i \otimes v_i\right\| \leq 1,\ v_i \in \mathcal{B}(V),\ V \in \mathcal{V}(\mathcal{E}) \right\} \\ &= \sup_{V \in \mathcal{V}(\mathcal{E})} \|T \odot \mathrm{id}_{\mathcal{B}(V)}\| \\ &= \|T\|_{cb} \end{aligned}$$

The last equality follows from the fact that $\mathcal{B}(V) \cong M_n$, where $n = \dim V$.

Thus, $\|T \odot \mathrm{id}_G\| \leq \|T\|_{cb} < \infty$, and the result follows.[5] □

Note. If G is finite-dimensional (hence nuclear), then $E \odot G$ and $F \odot G$ are already closed with respect to their unique norms, and so $T \otimes_{\min} \mathrm{id}_G = T \odot \mathrm{id}_G$.

Corollary 3.3.3. Let $T : E \to F$ be a completely bounded map between operator spaces. Then

$$\|T\|_{cb} = \sup_G \|T \otimes_{\min} \mathrm{id}_G\| = \sup_A \|T \otimes_{\min} \mathrm{id}_A\|$$

where the supremum is taken over all operator spaces G, or all C*-algebras A.

Proof. Clearly, we have

$$\|T\|_{cb} = \sup_n \|T \odot \mathrm{id}_{M_n}\| \leq \sup_G \|T \otimes_{\min} \mathrm{id}_G\| \leq \sup_G \|T\|_{cb} = \|T\|_{cb}$$

□

[5] That $\|T \otimes_{\min} \mathrm{id}_G\| = \|T \odot \mathrm{id}_G\|$ is automatically obtained by density when we extend $T \odot \mathrm{id}_G$.

Corollary 3.3.4. Let $T : E \to F$ be a completely bounded map between operator spaces. Then $T \otimes_{\min} \mathrm{id}_G$ is completely bounded, and

$$\|T\|_{cb} = \sup_G \|T \otimes_{\min} \mathrm{id}_G\|_{cb} = \sup_A \|T \otimes_{\min} \mathrm{id}_A\|$$

where the supremum is taken over all operator spaces G, or all C*-algebras A.

Proof. We merely take advantage of associativity of tensor products, plus the fact that $\mathrm{id}_G \odot \mathrm{id}_{M_n} = \mathrm{id}_{M_n(G)}$:

$$\begin{aligned}\|T \otimes_{\min} \mathrm{id}_G\|_{cb} &= \sup_n \|(T \otimes_{\min} \mathrm{id}_G) \odot \mathrm{id}_{M_n}\| \\ &= \sup_n \|T \otimes_{\min} (\mathrm{id}_G \odot \mathrm{id}_{M_n})\| \\ &= \sup_n \|T \otimes_{\min} \mathrm{id}_{M_n(G)}\| \leq \|T\|_{cb}\end{aligned}$$

The equality $\|T\|_{cb} = \sup_G \|T \otimes_{\min} \mathrm{id}_G\|_{cb}$ then follows trivially. □

Proposition 3.3.5. Let $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ be completely bounded maps between operator spaces. Then the map

$$\begin{aligned} T_1 \odot T_2 \quad : \quad E_1 \odot E_2 &\to F_1 \odot F_2 \\ a_1 \otimes a_2 &\mapsto T_1(a_1) \otimes T_2(a_2)\end{aligned}$$

extends uniquely to a completely bounded map

$$T_1 \otimes_{\min} T_2 \quad : \quad E_1 \otimes_{\min} E_2 \to F_1 \otimes_{\min} F_2$$

for which

$$\|T_1 \otimes_{\min} T_2\|_{cb} = \|T_1\|_{cb}\|T_2\|_{cb}$$

Proof. First observe that

$$\begin{aligned}\|T_1 \otimes_{\min} T_2\| &= \|(T_1 \otimes_{\min} \mathrm{id}_{F_2}) \circ (\mathrm{id}_{E_1} \otimes_{\min} T_2)\| \\ &\leq \|T_1 \otimes_{\min} \mathrm{id}_{F_2}\| \|\mathrm{id}_{E_1} \otimes_{\min} T_2\| \\ &\leq \|T_1\|_{cb}\|T_2\|_{cb}\end{aligned}$$

where we recall by corollary 3.3.3 that $\|T_1 \otimes_{\min} \mathrm{id}_{F_2}\| \leq \|T_1\|_{cb}$, and $\|\mathrm{id}_{E_1} \otimes_{\min} T_2\| \leq \|T_2\|_{cb}$. Additionally, by commutativity and associativity of the min tensor product

$$\begin{aligned}\|T_1 \otimes_{\min} T_2\|_{cb} &= \sup_n \|(T_1 \otimes_{\min} T_2) \odot \mathrm{id}_{M_n}\| \\ &= \sup_n \|(T_1 \odot \mathrm{id}_{M_n}) \otimes_{\min} T_2\| \\ &\leq \sup_n \|T_1 \odot \mathrm{id}_{M_n}\|_{cb}\|T_2\|_{cb} \\ &= \|T_1\|_{cb}\|T_2\|_{cb}\end{aligned}$$

As for the other side of the inequality, we start by observing that

$$\begin{aligned}\|T_1 \otimes_{\min} T_2\| &= \sup \left\{ \left\| \sum T_1(a_i) \otimes T_2(b_i) \right\| \;\middle|\; \left\| \sum a_i \otimes b_i \right\| \leq 1 \right\} \\ &\geq \sup \{\|T_1(a) \otimes T_2(b)\| \mid \|a_1\| \leq 1,\ \|a_2\| \leq 1\} \\ &= \|T_1\|\|T_2\|\end{aligned}$$

which we use to obtain the following lower bound:

$$\begin{aligned}\|T_1 \otimes_{\min} T_2\|_{cb} &= \sup_n \|(T_1 \otimes_{\min} T_2) \odot \mathrm{id}_{M_n}\| \\ &\geq \sup_{m,n} \|(T_1 \otimes_{\min} T_2) \odot \mathrm{id}_{M_m} \odot \mathrm{id}_{M_n}\| \\ &= \sup_{m,n} \|(T_1 \odot \mathrm{id}_{M_m}) \otimes_{\min} (T_2 \odot \mathrm{id}_{M_n})\| \\ &\geq \sup_{m,n} \|T_1 \odot \mathrm{id}_{M_m}\|\|T_2 \odot \mathrm{id}_{M_n}\| \\ &= \|T_1\|_{cb}\|T_2\|_{cb}\end{aligned}$$

To go from the first line to the second line, we have used the fact that $M_{mn} \cong M_m \odot M_n$ (*-isomorphically), and so $\mathrm{id}_{M_{mn}} = \mathrm{id}_{M_m} \odot \mathrm{id}_{M_n}$ (see theorem A.2.15). $\square$

Note. Notice that we can't attain $\|T_1 \otimes_{\min} T_2\| = \|T_1\|\|T_2\|$, as this would imply that $\|T\|_{cb} = \sup_n \|T \otimes \mathrm{id}_{M_n}\| = \sup_n \|T\| = \|T\|$ for all cb maps T, but this is preposterous.

An important consequence of all this work we've done on extending completely bounded maps to min tensor products is the following characterization of the min tensor norm.

Proposition 3.3.6. Let E, F be operator spaces, and $t = \sum_k a_k \otimes b_k$. Then

$$\|t\|_{E\otimes_{\min}F} = \sup\left\{ \left\|\sum a_k \otimes v(b_k)\right\|_{M_n(E)} \mid v \in CB(F, M_n),\ \|v\|_{cb} \le 1,\ n \ge 1 \right\}$$

Proof. Given v as in the supremum, we have

$$\|(\mathrm{id}_E \otimes_{\min} v)(t)\| \le \|\mathrm{id}_E \otimes_{\min} v\|\|t\| \le \|\mathrm{id}_E \otimes_{\min} v\|_{cb}\|t\| = \|\mathrm{id}_E\|_{cb}\|v\|_{cb}\|t\| = \|t\|$$

so that the supremum is $\le \|t\|$. On the other hand, if we embed $E \otimes_{\min} F \subseteq \mathcal{B}(H \otimes K)$ as we did in the proof of proposition 3.3.2, then

$$\|t\| = \sup_V \left\|\sum a_k \otimes P_V b_k|_V\right\|$$

where the supremum is over all finite dimensional subspaces V spanned by elements of an orthonormal basis for K. The map $b \mapsto P_V b|_V$ is easily seen to be completely contractive, and so $\|t\|$ is $\le$ the supremum in the proposition statement. $\square$

Max-Tensor Product of Maps

The results pertaining to the max-tensor product of maps are the work of Brown and Ozawa in [5]. If it is not clear at this point, the author of this paper is standing (or perhaps piggybacking) on the shoulders of giants.

Theorem 3.3.7. Let $u_1 : A_1 \to B_1$, $u_2 : A_2 \to B_2$ be completely positive maps between C*-algebras. The map $u_1 \odot u_2 : A_1 \odot A_2 \to B_1 \odot B_2$ extends to a completely positive map $u_1 \otimes_{\max} u_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$, which moreover satisfies $\|u_1 \otimes_{\max} u_2\| \le \|u_1\|\|u_2\|$.

In order to prove this we'll first establish an auxiliary result to Stinespring's dilation theorem.

Lemma 3.3.8 ([5]; Proposition 1.5.6)**.** Let A be C*-algebra, $\varphi : A \to \mathcal{B}(H)$ a completely positive map, and let $\varphi = V^*\pi(\cdot)V$ be a Stinespring dilation, where $\pi : A \to \mathcal{B}(K)$ is a *-homomorphism, K is another Hilbert space, and $V \in \mathcal{B}(H, K)$. Then there exists a *-homomorphism $\rho : \varphi(A)' \to \pi(A)'$ such that

$$\varphi(a)x = V^*\pi(a)\rho(x)V$$

for all $a \in A$, $x \in \varphi(A)'$.

Proof. We will do this for the case of A unital. In the case that A is non-unital, we can extend φ to a completely positive map $\widetilde{\varphi} : \widetilde{A} \to \mathcal{B}(H)$ on the unitization $\widetilde{A}$ of A (see the appendix for this result, it's an incredibly difficult result to prove).

To create ρ, we have to go back and review how the Stinespring dilation is constructed in the first place. We define a sesquilinear form on the algebraic tensor product (of vector spaces) $A \odot H$ by defining

$$\left\langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \right\rangle := \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i\rangle$$

To obtain the Hilbert space K we first take the quotient of $A \odot H$ by the kernel $L := \{x \in A \odot H \ : \ \langle x, x\rangle = 0\}$, and complete $(A \odot H)/L$ with the resulting non-degenerate sesquilinear form. The image of $\sum_i a_i \otimes \xi_i$ under the quotient map will be denoted $[\sum_i a_i \otimes x_i]$. We then let

$$V\xi := [1_A \otimes \xi], \qquad \pi(a)\left(\left[\sum_i b_i \otimes \xi_i\right]\right) = \left[\sum_i ab_i \otimes \xi_i\right]$$

from which we observe that

$$V^*\left(\left[\sum_i a_i \otimes \xi_i\right]\right) = \sum_i \varphi(a_i)\xi_i$$

Checking that these maps have the desired properties is a straightforward exercise, and included in any proof of Stinespring's dilation theorem.

Next, given $x \in \varphi(A)'$ we define

$$\rho(x)\left(\left[\sum_i a_i \otimes \xi_i\right]\right) := \left[\sum_i a_i \otimes x\xi_i\right]$$

First let's check that ρ is well-defined. For any $a_i \in A, \xi_i \in H$, we calculate

$$\begin{aligned}
\left\|\rho(x)\left[\sum_i a_i \otimes \xi_i\right]\right\|_K^2 &= \left\langle \left[\sum_i a_i \otimes x\xi_i\right], \left[\sum_i a_i \otimes x\xi_i\right]\right\rangle \\
&= \sum_{i,j} \langle x^*\varphi(a_i^* a_j)x\xi_j, \xi_i\rangle \\
&= \left\langle \begin{bmatrix} x^* & & \\ & \ddots & \\ & & x^* \end{bmatrix} [\varphi(a_i^* a_j)] \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle
\end{aligned}$$

Let $\operatorname{diag}(x)$ denote the diagonal $n \times n$-matrix with x along the diagonal, and $\xi := [\xi_1, ..., \xi_n]^T \in H^n$. Note that $[\varphi(a_i^* a_j)] = \varphi_n([a_i^* a_j]) \geq 0$, and also that $\operatorname{diag}(x)$ commutes with $\varphi_n([a_i^* a_j])$. Now, if $S, T \in \mathcal{B}(K)$ are commuting operators, and T is positive, then $(\|S\|^2 - S^*S)T$ is itself a positive operator, and so $\langle(\|S\|^2 - S^*S)T\xi, \xi\rangle \geq 0$. In particular, we deduce

$$\begin{aligned}
\langle \operatorname{diag}(x)^* \operatorname{diag}(x)\varphi_n([a_i^* a_j])\xi, \xi\rangle &\leq \|x\|^2 \langle \varphi_n([a_i^* a_j])\xi, \xi\rangle \\
&= \|x\|^2 \left\|\left[\sum_i a_i \otimes \xi_i\right]\right\|_K^2
\end{aligned}$$

From this, we see that not only is $\rho(x)$ well-defined, it's also continuous. Now, by precisely the same argument that π is a *-homomorphism one may conclude that ρ is also a *-homomorphism. We can just as easily check that π and ρ commute

$$\begin{aligned}
\pi(a)\rho(x)\left[\sum_i b_i \otimes \xi_i\right] &= \pi(a)\left[\sum_i b_i \otimes x\xi_i\right] = \left[\sum_i ab_i \otimes x\xi_i\right] \\
&= \rho(x)\left[\sum_i ab_i \otimes \xi_i\right] = \rho(x)\pi(a)\left[\sum_i b_i \otimes \xi_i\right]
\end{aligned}$$

Finally, for all $\xi \in H$ we have

$$\begin{aligned}
V^*\pi(a)\rho(x)V\xi = V^*\pi(a)\rho(x)[1_A \otimes \xi] &= V^*[a \otimes x\xi] \\
&= \varphi(a)x\xi
\end{aligned}$$

so that ρ accomplishes the desired task. □

What this lemma is saying is not only can we dilate a completely positive map to a *-homomorphism, we can dilate the *commutant* of the map's range at the same time.

Proof of 3.3.7. Notice that we need only prove the result for $u_1 = u : A \to B$ and $u_2 = \mathrm{id}_C : C \to C$ for C*-algebras A, B, C, since

$$u_1 \odot u_2 = (u_1 \odot \mathrm{id}_{B_2}) \circ (\mathrm{id}_{A_1} \odot u_2)$$

If each of $u_1 \odot \mathrm{id}_{B_2}$ and $\mathrm{id}_{A_1} \odot u_2$ extends to a completely positive map on the appropriate max tensor products, then so too does $u_1 \odot u_2$.

Let $\sigma : B \otimes_{\max} C \to \mathcal{B}(H)$ be a *faithful* *-representation. Using theorem 3.2.11 we can extract restriction representations $\sigma_1 : B \to \mathcal{B}(H)$ and $\sigma_2 : C \to \mathcal{B}(H)$ such that σ_1, σ_2 have commuting ranges.

Now consider the completely positive map $\sigma_1 \circ u : A \to \mathcal{B}(H)$, and choose a Stinespring dilation for it:

$$\sigma_1 \circ u = V^* \pi(\cdot) V$$

where, again, $\pi : A \to \mathcal{B}(K)$ is a *-homomorphism, K is some Hilbert space, and $V \in \mathcal{B}(H, K)$. By lemma 3.3.8, there exists a *-homomorphism $\rho : \sigma_1(u(A))' \to \pi(A)'$ such that

$$\sigma_1(u(a))x = V^* \pi(a) \rho(x) V$$

for all $a \in A,\ x \in \sigma_1(u(A))'$, and so in particular

$$\sigma_1(u(a))\sigma_2(c) = V^* \pi(a) \rho(c) V$$

for all $c \in C$. Since π and ρ have commuting ranges, we can combine them to form a new *-homomorphism

$$\varphi(a \otimes c) := \pi(a)\rho(c) \quad \forall a \in A, c \in C$$

which we extend to $A \odot C$ by linearity. Then what we've shown is that

$$\sigma \circ (u \odot \mathrm{id}_C) = V^* \varphi(\cdot) V$$

so that $\sigma \circ (u \odot \mathrm{id}_C)$ is a dilation of a *-homomorphism. Now clearly φ, in being a *-homomorphism, extends naturally to a *-homomorphism $\widetilde{\varphi} : A \otimes_{\max} C \to \mathcal{B}(K)$. Define

$$\begin{array}{rccc} u \otimes_{\max} \mathrm{id}_C & : & A \otimes_{\max} C & \to & \mathcal{B}(H) \\ & & x & \mapsto & \sigma^{-1}\left(V^* \widetilde{\varphi}(x) V\right) \end{array}$$

(this is why we needed σ to be faithful). This map is completely positive, and moreover since $\|u\| = \|V\|^2$ we have

$$\|u \otimes_{\max} \mathrm{id}_C(x)\| = \|V^* \widetilde{\varphi}(x) V\| \le \|V\|^2 \|\widetilde{\varphi}(x)\| \le \|V\|^2 = \|u\|$$

which follows since both σ^{-1} is isometric and $\widetilde{\varphi}$ is a *-homomorphism, whence contractive. Finally we must check that the range of this map is $B \otimes_{\max} C$, but fortunately this is a simple matter. Since $u \otimes_{\max} \mathrm{id}_C$ is continuous, given an $x \in A \otimes_{\max} C$ we can approximate it by a net $(x_\lambda)_\lambda \subset A \odot C$, and so

$$u \otimes_{\max} \mathrm{id}_C(x) = \lim_\lambda \underbrace{u \otimes_{\max} \mathrm{id}_C(x_\lambda)}_{\in B \odot C} \in B \otimes_{\max} C$$

$\square$

Fact 3.3.9. Let $I \subseteq A$ be a self-adjoint *-ideal in a *-algebra. Then for any *-algebra B, the sequence

$$0 \to I \odot B \overset{\iota \odot \mathrm{id}_B}{\to} A \odot B \overset{q \odot \mathrm{id}_B}{\to} A/I \odot B \to 0$$

is exact.

Proof. This is relatively straightforward. Clearly $I \odot B \subseteq A \odot B$ is injective by lemma 3.2.17, and $q \odot \mathrm{id}_B$ is also clearly surjective. It remains to show that $\ker(q \odot \mathrm{id}_B) = J \odot B$. Certainly $J \odot B \subseteq \ker(q \odot \mathrm{id}_B)$. Suppose $\sum_i q(a_i) \otimes b_i = 0$, and assume without loss of generality that the b_i's are linearly independent. Then by 3.1.2, each $q(a_i) = 0$, so $a_i \in J$ for all i, and $\sum_i a_i \otimes b_i \in J \odot B$. $\square$

Theorem 3.3.10 ($\otimes_{\max}$ is an Exact Bifunctor)**.** Let $I \subseteq A$ be a closed ideal in a C*-algebra A, and let B be another C*-algebra. Then the sequence

$$0 \to I \otimes_{\max} B \to A \otimes_{\max} B \to (A/I) \otimes_{\max} B \to 0$$

is exact. From this, we have two immediate corollaries. First, $I \otimes_{\max} B$ is a C*-subalgebra of (and in fact an ideal in) $A \otimes_{\max} B$. Second, $(A/I) \otimes_{\max} B$ is a quotient of $A \otimes_{\max} B$, given explicitly by

$$\frac{A \otimes_{\max} B}{I \otimes_{\max} B} \cong (A/I) \otimes_{\max} B$$

Remark. Saying $\otimes_{\max}$ is an "exact bifunctor" is just a fancy way of saying the functor taking a C*-algebra A to $A \otimes_{\max} B$ (for some fixed C*-algebra B), and completely positive maps $\varphi : A \to C$ to $\varphi \otimes_{\max} \mathrm{id}_B : A \otimes_{\max} B \to C \otimes_{\max} B$ (this functor being denoted $\cdot \otimes_{\max} B$) takes exact sequences in one category to exact sequences in another, and the same is true for $A \mapsto B \otimes_{\max} A$ by symmetry.

Proof. Let $\iota : I \to A$ and $q : A \to A/I$ be the canonical inclusion and quotient maps. Seeing as these maps are *-homomorphisms, the maps $\iota \odot \mathrm{id}_B$ and $q \odot \mathrm{id}_B$ extend to *-homomorphisms

$$I \otimes_{\max} B \xrightarrow{\iota \otimes_{\max} \mathrm{id}_B} A \otimes_{\max} B \xrightarrow{q \otimes_{\max} \mathrm{id}_B} (A/I) \otimes_{\max} B$$

Let $\widetilde{\iota} := \iota \otimes_{\max} \mathrm{id}_B$ and $\widetilde{q} := q \otimes_{\max} \mathrm{id}_B$. We need to show that $\widetilde{\iota}$ is injective, $\widetilde{q}$ is surjective, and $\operatorname{ran} \widetilde{\iota} = \ker \widetilde{q}$.

Showing $\widetilde{\iota}$ is injective

Since I is an ideal, it contains an approximate identity e_λ bounded in norm by 1. Using this we can construct completely positive contractions $\phi_\lambda : A \to I$ taking $a \mapsto e_\lambda a e_\lambda$. In being completely positive, by theorem 3.3.7 we obtain completely positive contractions $\psi_\lambda := \phi_\lambda \otimes_{\max} \mathrm{id}_B : A \otimes_{\max} B \to I \otimes_{\max} B$.

Suppose now that $x_\lambda \in I \odot B$ converges to $x \in I \otimes_{\max} B$, and x is such that $\widetilde{\iota}(x) = 0$. We'd like to show that $x = 0$. By continuity, $\widetilde{\iota}(\lim_\lambda x_\lambda) = \lim_\lambda \widetilde{\iota}(x_\lambda) = \lim_\lambda x_\lambda$, so the assumption that $\widetilde{\iota}(x) = 0$ tells us that $\lim_\lambda \|x_\lambda\|_{A \otimes_{\max} B} = 0$. By several applications of the triangle inequality

$$\|x\|_{I \otimes_{\max} B} \leq \|x - x_\lambda\|_{I \otimes_{\max} B} + \|x_\lambda - \psi_\mu(x_\lambda)\|_{I \otimes_{\max} B} + \|\psi_\mu(x_\lambda)\|_{I \otimes_{\max} B}, \qquad \mu \geq \lambda$$

The first term tends to zero by definition. The last term tends to zero since $\|\psi_\mu(x_\lambda)\|_{I \otimes_{\max} B} \leq \|x_\lambda\|_{A \otimes_{\max} B} \to 0$. As for the middle term, writing $x_\lambda := \sum_i k_i^\lambda \otimes b_i^\lambda$ for some $k_i^\lambda \in I$, $b_i^\lambda \in B$, we can write

$$\begin{aligned} \|x_\lambda - \psi_\mu(x_\lambda)\|_{I \otimes_{\max} B} &:= \left\| \sum_i (k_i^\lambda - e_\mu k_i^\lambda e_\mu) \otimes b_i^\lambda \right\|_{I \otimes_{\max} B} \\ &\leq \sum_i \|k_i^\lambda - e_\mu k_i^\lambda e_\mu\| \|b_i^\lambda\| \end{aligned}$$

Then μ can be chosen large enough such that the total sum is arbitrarily small. In conclusion, $\|x\|_{I \otimes_{\max} B} = 0$, so $x = 0$, implying $\widetilde{\iota}$ is indeed injective.

Remark. Seeing as $\widetilde{\iota}$ is an injective *-homomorphism of C*-algebras, it is necessarily isometric, so we can be sure that $\|\cdot\|_{I \otimes_{\max} B} = \|\cdot\|_{A \otimes_{\max} B}$ when restricted to $I \odot B$ - another victory in the fight for equality of tensor norms.

Showing $\operatorname{ran} \widetilde{\iota} = \ker \widetilde{q}$

This half of the proof was presented to me by Laurent Marcoux, University of Waterloo professor, operator theorist extraordinaire, and strong contender for the second most interesting man in the world, and is vastly clearer than the proof in [5]. Perhaps sharper men and women than us will see that these proofs are actually the same.

First, it's obvious that $\widetilde{q}$ is surjective. This is because the range of a *-homomorphism between C*-algebras is necessarily closed. Thus, since $\operatorname{ran} \widetilde{q}$ contains $\operatorname{ran}(q \odot \mathrm{id}_B) = A/I \odot B$, a dense subset of $(A/I) \otimes_{\max} B$, we conclude that $\operatorname{ran} \widetilde{q} = (A/I) \otimes_{\max} B$.

Finally, to show that our sequence is exact, we'd like to show that $\operatorname{ran} \widetilde{\iota} = \ker \widetilde{q}$. Since $I \odot B \subseteq \ker \widetilde{q}$, we have at the very least $\operatorname{ran} \widetilde{\iota} = I \otimes_{\max} B \subseteq \ker \widetilde{q}$. Moreover, $I \otimes_{\max} B$ is an ideal in $\ker \widetilde{q}$. Thus, by the third isomorphism theorem we have

$$\begin{aligned} ((A \otimes_{\max} B)/(I \otimes_{\max} B)) \,/\, (\ker \widetilde{q}/(I \otimes_{\max} B)) &\cong (A \otimes_{\max} B)/\ker \widetilde{q} \\ &\cong A/I \otimes_{\max} B \end{aligned} \tag{$*$}$$

In other words, we have a surjective *-homomorphism

$$\sigma : (A \otimes_{\max} B)/(I \otimes_{\max} B) \to A/I \otimes_{\max} B$$

which is defined by taking

$$Q\left(\sum_i a_i \otimes b_i\right) \mapsto \sum_i q(a_i) \otimes b_i$$

where $Q : A \otimes_{\max} B \to (A \otimes_{\max} B)/(I \otimes_{\max} B)$ is the canonical quotient morphism. Since $Q(A \odot B)$ is dense in $(A \otimes_{\max} B)/(I \otimes_{\max} B)$, this definition uniquely defines σ on all of $(A \otimes_{\max} B)/(I \otimes_{\max} B)$. Moreover, this expression is well-defined *by virtue* of the isomorphisms produced in $(*)$. Just to be perfectly clear, if $Q(\sum_i a_i \otimes b_i) = 0$, then $\sum_i a_i \otimes b_i \in I \otimes_{\max} B \subseteq \ker \widetilde{q}$, so $\sum_i a_i \otimes b_i + \ker \widetilde{q} = 0$, but there's an isomorphism between $(A \otimes_{\max} B)/\ker \widetilde{q}$ and $A/I \otimes_{\max} B$ taking $\sum_i a_i \otimes b_i + \ker \widetilde{q}$ to $\sum_i q(a_i) \otimes b_i$, and so $\sum_i q(a_i) \otimes b_i = 0$, as expected.

The goal now is to show that σ is actually *injective*, in which case we would have $(A \otimes_{\max} B)/(I \otimes_{\max} B) \cong A/I \otimes_{\max} B$, and so $\ker \widetilde{q} = I \otimes_{\max} B = \operatorname{ran} \widetilde{\iota}$ as desired. Notice that

$$\widetilde{q} = \sigma \circ Q$$

which is easily verified by direct calculation. Consider the map

$$\begin{array}{cccc} \pi & : & A/I \odot B & \to & (A \otimes_{\max} B)/(I \otimes_{\max} B) \\ & & \sum_i q(a_i) \otimes b_i & \mapsto & Q(\sum_i a_i \otimes b_i) \end{array}$$

Let us check that this map is well-defined. Given $\sum_i q(a_i) \otimes b_i = 0$, we can assume without loss of generality that the b_i's are linearly independent, in which case (by lemma 3.1.2) we must have $q(a_i) = 0$ for all i, and so $a_i \in I$. Thus $\sum_i a_i \otimes b_i \in I \odot B \subset \ker Q$, implying $Q(\sum_i a_i \otimes b_i) = 0$, so that π is indeed well-defined. It is left as an easy exercise for the reader to verify that π is also a *-homomorphism.

Finally, this map π is also continuous with respect to the max-norm on its domain. To see this, note that if we embed $(A \otimes_{\max} B)/(I \otimes_{\max} B)$ in $\mathcal{B}(H)$ for some Hilbert space H, then π becomes a *-representation of $A/I \odot B$, and so by definition of the max-norm we have

$$\|\pi(x)\| \leq \|x\|_{A/I \otimes_{\max} B}$$

whence π is continuous.

Notice however that

$$\sigma \circ \pi\left(\sum_i q(a_i) \otimes b_i\right) = \sigma\left(Q\left(\sum_i a_i \otimes b_i\right)\right) = \widetilde{q}\left(\sum_i a_i \otimes b_i\right) = \sum_i q(a_i) \otimes b_i$$

and so $\sigma \circ \pi = \mathrm{id}_{A/I \odot B}$. By continuity we thus have $\sigma = \pi^{-1}$, so σ is injective, finishing up the argument. $\square$

Corollary 3.3.11. Let A be a non-unital C*-algebra, and $\widetilde{A}$ it's unitization. Then A is nuclear if there is a unique norm on $\widetilde{A} \odot \widetilde{B}$ for any *unital* C*-algebra $\widetilde{B}$.

Proof. First observe that for C*-algebras A, B, with unitizations $\widetilde{A}, \widetilde{B}$ respectively (which we let equal the original algebra if it is already unital). Then A, B sit as ideals inside $\widetilde{A}, \widetilde{B}$, and so when restricted to $A \odot B$ we have

$$\|\cdot\|_{A \otimes_{\max} B} = \|\cdot\|_{\widetilde{A} \otimes_{\max} \widetilde{B}}$$

By fact 3.2.19, we also have

$$\|\cdot\|_{A \otimes_{\min} B} = \|\cdot\|_{\widetilde{A} \otimes_{\min} \widetilde{B}}$$

So if there is a unique tensor norm on $\widetilde{A} \odot \widetilde{B}$, there is a unique norm on $A \odot B$. $\square$

It isn't terribly tractable in this form, but if A is nuclear then we can also prove $\widetilde{A}$ is nuclear (the converse to this corollary). We won't need this fact soon, however, so we'll postpone it until we have better tools. The eager reader may wish to consult [27] for another proof that $\widetilde{A}$ is nuclear.

4 Nuclearity, Biduals, Semidiscreteness and Injectivity

4.1 Algebraic States and Nuclearity

Let A, B be C*-algebras. An element of $A \odot B$ is called *algebraically positive* if it is a sum of elements of the form x^*x, for some $x \in A \odot B$. Clearly if both $a \in A$ and $b \in B$ are positive, then $a = r^*r$ and $b = s^*s$ for some $r \in A$, $s \in B$, whence

$$a \otimes b = (r \otimes s)^*(r \otimes s)$$

so that $a \otimes b$ is positive. As with C*-algebras, we can endow $A \odot B$ with an order structure by defining $x \leq y$ if and only if $y - x$ is algebraically positive. A linear map $\varphi : A \odot B \to C$, where C is a C*-algebra, is said to be *algebraically positive* if it takes algebraically positive elements in $A \odot B$ to positive elements in C. φ is *completely algebraically positive* if each ampliation $\varphi_n : M_n(A \odot B) \to M_n(C)$ takes algebraically positive elements in $M_n(A \odot B)$ to positive elements in $M_n(C)$.[6]

Fact 4.1.1. Let $\varphi : A \odot B \to C$ be an algebraically positive map, where A, B, C are C*-algebras. Let $h \in A \odot B$ be self-adjoint. Then $\varphi(h)$ is self-adjoint. Moreover, for all $x \in A \odot B$, we have $\varphi(x^*) = \varphi(x)^*$.

Proof. First off, if $a \in A$ and $b \in B$ are self-adjoint, then

$$\begin{aligned} \varphi(a \otimes b) &= \varphi((a^+ - a^-) \otimes (b^+ - b^-)) = \varphi(a^+ \otimes b^+ - a^- \otimes b^+ - a^+ \otimes b^- + a^- \otimes b^-) \\ &= \varphi(a^+ \otimes b^+) - \varphi(a^- \otimes b^+) - \varphi(a^+ \otimes b^-) + \varphi(a^- \otimes b^-) \\ &= \varphi(x_1^*x_1) - \varphi(x_2^*x_2) - \varphi(x_3^*x_3) + \varphi(x_4^*x_4) \end{aligned}$$

where

$$\begin{aligned} x_1 = (a^+)^{1/2} \otimes (b^+)^{1/2}, \quad x_2 = (a^-)^{1/2} \otimes (b^+)^{1/2} \\ x_3 = (a^+)^{1/2} \otimes (b^-)^{1/2}, \quad x_4 = (a^-)^{1/2} \otimes (b^-)^{1/2} \end{aligned}$$

The last expression above for $\varphi(a \otimes b)$ is a real linear combination of positive elements, whence it is self-adjoint, so $\varphi(a \otimes b) = \varphi(a \otimes b)^*$. Next, for arbitrary $a \in A, b \in B$, we have

$$\begin{aligned} a \otimes b &= (\Re a + i\Im a) \otimes (\Re b + i\Im b) \\ &= (\Re a \otimes \Re b - \Im a \otimes \Im b) + i\,(\Re a \otimes \Im b + \Im a \otimes \Re b) \end{aligned}$$

From the discussion above, we know that $\varphi(\Re a \otimes \Re b - \Im a \otimes \Im b)$ and $\varphi(\Re a \otimes \Im b + \Im a \otimes \Re b)$ are both self-adjoint. Thus

$$\begin{aligned} \varphi((a \otimes b)^*) &= \varphi\left((\Re a \otimes \Re b - \Im a \otimes \Im b) - i\,(\Re a \otimes \Im b + \Im a \otimes \Re b)\right) \\ &= \varphi\left(\Re a \otimes \Re b - \Im a \otimes \Im b\right) - i\varphi\left(\Re a \otimes \Im b + \Im a \otimes \Re b\right) \\ &= \left(\varphi\left(\Re a \otimes \Re b - \Im a \otimes \Im b\right) + i\varphi\left(\Re a \otimes \Im b + \Im a \otimes \Re b\right)\right)^* \\ &= \varphi(a \otimes b)^* \end{aligned}$$

By linearity, $\varphi(x^*) = \varphi(x)^*$ for all $x \in A \odot B$, and so if $h \in A \odot B$ is self-adjoint then $\varphi(h)^* = \varphi(h^*) = \varphi(h)$ is also self-adjoint. $\square$

Definition 4.1.2. Let $S(A \odot B)$ denote the set of linear functionals $f : A \odot B \to \mathbb{C}$ such that $f(x^*x) \geq 0$ for all $x \in A \odot B$ (so f is algebraically positive), and

$$\sup\{|f(a \otimes b)| \;:\; \|a\| = \|b\| = 1\} = 1$$

(so f has "norm 1"). $S(A \odot B)$ is the set of *algebraic states* on $A \odot B$.

[6] The distinction between "positive" and "algebraically positive" should not be understated! Consider the inclusion map $\iota : A \odot B \to A \otimes_{\max} B$, for unital C*-algebras A, B. This map is *algebraically positive*, to be sure, but if we consider $A \odot B$ as an operator system in $A \otimes_{\min} B$, then ι *isn't even positive* (though it's still algebraically positive). Indeed, supposing there exists an element $y \in A \odot B$ such that $\beta := \|y\|_{\min} < \|y\|_{\max}$, the element $\beta^2 - y^*y$ is *positive* in $A \otimes_{\min} B$, but *non-positive* in $A \otimes_{\max} B$. I'd like to thank University of Waterloo professor Nico Spronk (also known as "Mr. Tensor Product" himself) for pointing out this example to myself and Laurent Marcoux, and Marcoux for shedding light on the distinction between "positive" and "algebraically positive" in the first place.

A simple way of constructing algebraic states is by taking tensor products of *regular* states: given $\varphi \in S(A)$ and $\psi \in S(B)$, define

$$\varphi \otimes \psi(a \otimes b) := \varphi(a)\psi(b)$$

It's not hard to check that this map is well-defined, algebraically positive, and clearly $\sup\{|\varphi(a)\psi(b)| : \|a\| = \|b\| = 1\} = \|\varphi\|\|\psi\| = 1$, so it is indeed an algebraic state.

Algebraic states behave similar in many ways to traditional states on a C*-algebra. For instance, we have the following fact.

Fact 4.1.3. Given an algebraically positive linear functional $f : A \odot B \to \mathbb{C}$, f is automatically *completely algebraically positive.*

Proof. First, letting $f_n : M_n(A \odot B) \to M_n(\mathbb{C})$ denote the n^{th} ampliation of f, given $x = [x_{ij}] \in M_n(A \odot B)$ we calculate

$$f_n(x^*x) = f_n\left([x_{ij}]^*[x_{ij}]\right) = \left[\sum_k f(x_{ki}^* x_{kj})\right]_{i,j=1}^n$$

This matrix is positive semidefinite if and only if for all $\alpha_1, ..., \alpha_n \in \mathbb{C}$, the following inequality is satisfied:

$$\sum_{ij} \left(\sum_k f\left(x_{ki}^* x_{kj}\right)\right) \overline{\alpha_i}\alpha_j \geq 0$$

Rearranging the sums and exploiting linearity, we see this expression can be written as

$$\sum_{ij} \left(\sum_k f\left(x_{ki}^* x_{kj}\right)\right) \overline{\alpha_i}\alpha_j = \sum_k f\left(y_k^* y_k\right)$$

where $y_k := \sum_i \alpha_{ki} x_i$. Since $f(y_k^* y_k) \geq 0$ for all k, we conclude that $f_n(x^*x) \geq 0$ as well. □

Every element $f \in S(A \odot B)$ induces a positive, Hermitian form $\langle x, y\rangle := f(y^*x)$ on $A \odot B$, and so satisfies the *Cauchy-Schwarz inequality*:

Fact 4.1.4. For any $f \in S(A \odot B)$, and $x, y \in A \odot B$,

$$|f(y^*x)|^2 \leq |f(x^*x)||f(y^*y)|$$

In the case where A, B are both unital, we let 1 denote the unit $1_A \otimes 1_B$ in $A \odot B$. The following proposition demonstrates how algebraic states on unital C*-algebras are themselves unital maps - a property enjoyed by *ordinary* states on a C*-algebra.

Proposition 4.1.5. Let A, B be unital C*-algebras, and $f : A \odot B \to \mathbb{C}$ be a linear functional satisfying $f(x^*x) \geq 0$ for all $x \in A \odot B$. Then $f(1) = 1$ if and only if

$$\sup\{|f(a \otimes b)| : \|a\| = \|b\| = 1\} = 1$$

Proof. First suppose $f(1) = 1$. For any positive a, b of norm 1, $a \otimes (1_B - b)$ is positive, so $f(a \otimes 1_B) \geq f(a \otimes b)$. Similarly, $f(1_A \otimes 1_B) \geq f(a \otimes 1_B)$, so $f(a \otimes b) \leq 1$. Then, by the Cauchy-Schwarz inequality, for any a, b of norm 1,

$$|f(a \otimes b)|^2 = |f((a \otimes 1)(1 \otimes b)|^2 \leq f(a^*a \otimes 1)f(1 \otimes b^*b) \leq 1$$

Given a, b such that $\|a\| = \|b\| = 1$, by positivity there exist r, s such that $r^*r = 1_A - a^*a$ and $s^*s = 1_B - b^*b$, and a simple calculation shows that

$$(a \otimes s)^*(a \otimes s) + (x \otimes 1_B)^*(x \otimes 1_B) = 1_A \otimes 1_A - a^*a \otimes b^*b$$

so that

$$0 \leq f(1_A \otimes 1_B - a^*a \otimes b^*b)$$

Then, by the Cauchy-Schwarz inequality

$$|f(a \otimes b)|^2 = |f((a \otimes 1_B)(1_A \otimes b)|^2 \leq f(a^*a \otimes 1_B)f(1_A \otimes b^*b) \leq f(1_A \otimes 1_B)^2$$

Thus

$$\sup\{|f(a \otimes b)| \ : \ \|a\| = \|b\| = 1\} = f(1_A \otimes 1_B)$$

□

The clever little calculation we performed above to show that $1 - a^*a \otimes b^*b$ was positive is made somewhat more palpable when we generalize: supposing $0 \leq a_1 \leq a_2 \in A$ and $0 \leq b_1 \leq b_2 \in B$, we have

$$a_2 \otimes b_2 - a_1 \otimes b_1 = \underbrace{(a_2 - a_1)}_{\geq 0} \otimes \underbrace{b_2}_{\geq 0} + \underbrace{a_1}_{\geq 0} \otimes \underbrace{(b_2 - b_1)}_{\geq 0}$$

Using the Cauchy-Schwarz inequality, we can also prove a non-unital version of this proposition.

Proposition 4.1.6. Let A, B be C*-algebras, and $f : A \odot B \to \mathbb{C}$ an algebraically positive linear functional. Let $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{f_\beta\}_{\beta \in \mathcal{B}}$ be approximate identities for A and B respectively. Then

$$\lim_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} f(e_\alpha \otimes f_\beta) = 1$$

if and only if

$$\sup\{|f(a \otimes b)| \ : \ \|a\| = \|b\| = 1\} = 1$$

Remark. Our definition of "approximate identity of a C*-algebra" requires the elements of the net to be increasing, positive and have norm ≤ 1.

Proof. Assuming $\sup\{|f(a \otimes b)| \ : \ \|a\| = \|b\| = 1\} = 1$, we see that $f(e_\alpha \otimes f_\beta)$ is positive, increasing in both α and β, and bounded above by 1. For fixed α, define

$$L_\alpha := \lim_\beta f(e_\alpha \otimes f_\beta) = \sup_\beta f(e_\alpha \otimes f_\beta)$$

which exists since this net is increasing and bounded above. Suppose $\alpha \leq \alpha' \in \mathcal{A}$. Since $f(e_\alpha \otimes f_\beta) \leq f(e_{\alpha'} \otimes f_\beta)$ for all $\beta \in \mathcal{B}$, we have

$$L_\alpha = \sup_\beta f(e_\alpha \otimes f_\beta) \leq \sup_\beta f(e_{\alpha'} \otimes f_\beta) = L_{\alpha'}$$

so that L_α is itself an increasing, bounded net, whence

$$\lim_\alpha L_\alpha = \sup_\alpha L_\alpha = \sup_\alpha \left(\sup_\beta f(e_\alpha \otimes f_\beta) \right)$$

A similar, symmetric argument thus yields that $L := \lim_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} f(e_\alpha \otimes f_\beta)$ exists, regardless of the order the limits are taken in, and $L = \sup_{\alpha,\beta} f(e_\alpha \otimes f_\beta)$.

From this, we see that no matter which assumption we start with ($\sup\{|f(a \otimes b)|\} = 1$ or $\lim_{\alpha,\beta} f(e_\alpha \otimes f_\beta) = 1$), we can be sure the latter limit exists, doesn't depend on the order the limits are taken in, and converges to the supremum over α, β.

Now, supposing $a \in A, b \in B$ each have norms ≤ 1, we have

$$e_\alpha a^* a e_\alpha \otimes f_\beta b^* b_\beta \leq \|a\|^2 \|b\|^2 e_\alpha^2 \otimes f_\beta^2 \leq e_\alpha \otimes f_\beta$$

where we have used the fact that $e_\alpha^2 \leq e_\alpha$ and $f_\beta^2 \leq f_\beta$ in the last step. With this, using the Cauchy-Schwarz inequality we have

$$\begin{aligned} |f\left((e_\alpha \otimes f_\beta)^*(a e_\alpha \otimes b f_\beta)\right)|^2 &\leq |f\left((e_\alpha a^* a e_\alpha) \otimes (f_\beta b^* b f_\beta)\right)| |f(e_\alpha^2 \otimes f_\beta^2)| \\ &\leq f(e_\alpha \otimes f_\beta)^2 \end{aligned}$$

In other words

$$|f(e_\alpha a e_\alpha \otimes f_\beta b f_\beta)| \le f(e_\alpha \otimes f_\beta) \tag{*}$$

Now, fix $x \ge 0$, and consider the functional $g_x(y) := f(x \otimes y)$. Clearly g_x is itself a *positive* linear functional, whence g_x is continuous, so $\lim_\beta f(x \otimes f_\beta y f_\beta) = \lim_\beta g_x(f_\beta y f_\beta) = g_x(y) = f(x \otimes y)$. Similarly, we can fix $y \ge 0$ and define $h_y(x) := f(x \otimes y)$, which is also a positive linear functional, allowing us to do the same thing. So, given an arbitrary $a \in A, b \in B$, write $a = \sum_{j=0}^3 i^j x_j^* x_j$, $b = \sum_{k=0}^3 i^k y_k^* y_k$, and so

$$\begin{aligned}\lim_{\alpha,\beta} f(e_\alpha a e_\alpha \otimes f_\beta b f_\beta) &= \lim_{\alpha,\beta} \sum_{j,k=0}^{3} i^{j+k} f(e_\alpha x_j^* x_j e_\alpha \otimes f_\beta y_k^* y_k f_\beta) \\ &= \sum_{j,k=0}^{3} i^{j+k} f(x_j^* x_j \otimes y_k^* y_k) = f(a \otimes b)\end{aligned}$$

and moreover the limit is independent of the order of α, β. Returning now to the inequality (*), by taking the supremum over α, β, we get

$$|f(a \otimes b)| = \lim_{\alpha,\beta} |f(e_\alpha a e_\alpha \otimes f_\beta b f_\beta)| \le \sup_{\alpha,\beta} |f(e_\alpha a e_\alpha \otimes f_\beta b f_\beta)| \le \sup_{\alpha,\beta} f(e_\alpha \otimes f_\beta) = L$$

However, a, b were arbitrary norm 1 elements, so really

$$\sup\{|f(a \otimes b)| : \|a\| = \|b\| = 1\} \le L$$

Of course, L is clearly $\le$ the expression on the LHS above, so really these two quantities are equal. Thus, if either one is equal to 1, then the other exists and is also equal to 1, proving the claim. □

Proposition 4.1.7. Let A, B be non-unital C*-algebras with respective unitizations $\widetilde{A}, \widetilde{B}$, and let $f \in S(A \odot B)$. Then there exists a unique $\widetilde{f} \in S(\widetilde{A} \odot \widetilde{B})$ such that $f = \widetilde{f}|_{A \odot B}$.

Proof. Given said f, we begin by defining a new map $T_f : A \to B^*$ given by

$$T_f(a)(b) := f(a \otimes b)$$

As we will see later on, this map is called the *complete state map* corresponding to f, and is in fact *completely positive* (skip ahead to fact 4.1.17 to see a proof of this fact). Notice that since f is an algebraic state, T_f is continuous. We can then consider the *normalization* of T_f, $\ddot{T}_f : A^{**} \to B^*$, which remains completely positive. Finally using the map $\varepsilon_B : B^* \to \widetilde{B}^*$ defined in A.2.13, which is completely positive by proposition A.2.14, we obtain a map

$$\widetilde{T} := \varepsilon_B \circ \ddot{T}_f : A^{**} \to \widetilde{B}^*$$

which we can use to define our extension $\widetilde{f}$:

$$\widetilde{f}(a \otimes b) := \widetilde{T}(a)(b)$$

That $\widetilde{f}$ is algebraically positive follows from fact 4.1.19. Using proposition 4.1.5, to prove that $\widetilde{f}$ is a state we need only show that $\widetilde{f}(1_{\widetilde{A}} \otimes 1_{\widetilde{B}}) = 1$. Let e_λ and f_μ denote approximate identities for A, B respectively. We have

$$\begin{aligned}\widetilde{f}(1_{\widetilde{A}} \otimes 1_{\widetilde{B}}) &= \varepsilon_B(\ddot{T}_F(1_{\widetilde{A}}))(1_{\widetilde{B}}) = \lim_\mu \ddot{T}_f(1_A)(f_\mu) \\ &= \lim_\lambda \lim_\mu T_f(e_\lambda)(f_\mu) \\ &= \lim_{\lambda,\mu} f(e_\lambda \otimes f_\mu) = 1\end{aligned}$$

where we have used the fact that $1_{\widetilde{A}} = w^*\!-\!\lim_\lambda e_\lambda$, and $\ddot{T}_f$ is weak-* continuous, to go from the first line to the second line, and then proposition 4.1.6 to finish up the third line.

To see that $\widetilde{f}$ is unique □

GNS Construction

Given an algebraic state $f \in S(A \odot B)$, it is perhaps not surprising that we can perform a GNS construction given f, producing a cyclic *-representation of $A \odot B$. Since $A \odot B$ isn't yet a C*-algebra, we'll walk through the procedure of constructing the GNS representation, but everything is practically identical to the GNS construction on a C*-algebra.

We'll assume A and B are unital, first. Given $f \in S(A \odot B)$, we define a positive Hermitian form on $A \odot B$ via $\langle x, y\rangle := f(y^*x)$ (note that antisymmetry is guaranteed by positivity of f). This bilinear form has kernel $L := \{x \mid f(x^*x) = 0\}$. As $\langle \cdot, \cdot \rangle$ is a positive Hermitian form, it satisfies the Cauchy-Schwarz inequality, which is the key to proving that L is a left ideal in $A \odot B$.

Given $x, y \in L$, we have

$$\begin{aligned} 0 \leq f((x+y)^*(x+y)) &= f(x^*x) + f(x^*y) + f(y^*x) + f(y^*y) \\ &= f(y^*x) + f(x^*y) \\ &= 2\Re f(x^*y) \\ &\leq 2|f(x^*y)| \\ &\leq 2f(x^*x)^{1/2} f(y^*y)^{1/2} = 0 \end{aligned}$$

so $x + y \in L$. Additionally, given $x \in L$, $a \in A \odot B$,

$$0 \leq f((ax)^*(ax))^2 = f(x^*(a^*ax))^2 \leq f(x^*x) f((x^*a^*a)(a^*ax)) = 0$$

so $ax \in L$.

Ideally, we would like to define a new bilinear form on $(A \odot B)/L$ which is positive Hermitian and *non-degenerate*, thus turning $(A \odot B)/L$ into a pre-Hilbert space. Given $x + L, y + L \in (A \odot B)/L$, define

$$\langle x + L, y + L\rangle := \langle x, y\rangle$$

To check that this is well-defined, it suffices to check that $x + L = x' + L$ implies $\langle x, y\rangle = \langle x', y\rangle$. Once again, the Cauchy-Schwarz inequality comes to the rescue. Since $x - x' \in L$, letting $z = x - x'$ we have

$$|\langle z, y\rangle|^2 = |f(y^*z)|^2 \leq f(y^*y) f(z^*z) = 0$$

so our bilinear form on $(A \odot B)/L$ is well-defined. Furthermore, it is non-degenerate, since if $\langle x+L, y+L\rangle = 0$ for all y, then $f(x^*x) = 0$, whence $x + L = L$.

Let H_f denote the Hilbert space $\overline{(A \odot B)/L}$, the completion taken with respect to the norm given by our inner product. Define a *-representation of $A \odot B$ on H_f by letting

$$\pi_f(x)(y + L) := xy + L$$

One may check that $\pi_f(x)$ is indeed a bounded linear operator for all $x \in A \odot B$, π_f is a cyclic *-representation with cyclic vector $\xi_f := 1 + L$, and $f(x) = \langle \pi_f(x)\xi_f, \xi_f\rangle$.

Now lets handle the non-unital case. For A, B non-unital, we can extend $f \in S(A \odot B)$ to $\widetilde{f} \in S(\widetilde{A} \odot \widetilde{B})$ by proposition 4.1.7, perform the GNS construction to obtain $(\pi_{\widetilde{f}}, H_{\widetilde{f}}, \xi_{\widetilde{f}})$, and then define $\pi_f := \pi_{\widetilde{f}}|_{A \odot B}$. It's not hard to check that $\xi_{\widetilde{f}}$ is still a cyclic vector for π_f, and we can still recover f from π_f as $f(x) = \langle \pi_f(x)\xi_{\widetilde{f}}, \xi_{\widetilde{f}}\rangle$.

Seminorms on $A \odot B$

Each state $f \in S(A \odot B)$ induces a C*-seminorm on $A \odot B$, given by

$$\rho_f(x) := \|\pi_f(x)\|$$

(It is easily checked that ρ_f satisfies the C*-identity.) We can extend this to arbitrary subsets $\Gamma \subseteq S(A \odot B)$ by setting

$$\rho_\Gamma(x) := \sup\{\rho_f(x) \mid f \in \Gamma\}$$

Definition 4.1.8. If Γ is such that ρ_Γ is a *norm*, then Γ is said to be *separating*. If Γ is separating, the completion of $A \odot B$ with respect to ρ_Γ is denoted $A \otimes_\Gamma B$.

There is a surprising amount of power in this definition. Many of the usual C*-tensor norms can be described as ρ_Γ's for some appropriate choice of Γ. To demonstrate, let us begin with a result of Turumaru.

Lemma 4.1.9 (Turumaru)**.** Let A, B be C*-algebras, and $0 \neq x = \sum_i a_i \otimes b_i \in A \odot B$. Then there exists $\varphi \in S(A)$ and $\psi \in S(B)$ such that

$$\varphi \otimes \psi(x^*x) > 0$$

Moreover, φ and ψ can be chosen to be pure, though this is of no particular use to us.

Proof. First assume without loss of generality that the a_i's are linearly independent, and $b_1 \neq 0$. We can choose a (pure) state $\psi \in S(B)$ such that $\psi(b_1^* b_1) > 0$. Using this state, let H_ψ denote the GNS representation space corresponding to ψ, and $\Gamma_\psi : B \to H_\psi$ the map sending b to its image in H_ψ (recall that we obtain H_ψ by defining a semi-inner product $\langle b_1, b_2 \rangle_\psi := \psi(b_2^* b_1)$ on B, quotienting out the left ideal $L_\psi := \{b \in B \ : \ \psi(b^*b) = 0\}$, and letting $H_\psi := \overline{A/L_\psi}$, so Γ_ψ just sends b to its coset $b + L_\psi$).

Since $\psi(b_1^* b_1) > 0$, we know $y_1 := \Gamma_\psi(b_1) \neq 0$. Let $y_2, ..., y_k \subseteq \{\Gamma_\psi(b_2), ..., \Gamma_\psi(b_n)\}$ be a maximal linearly independent subset, so that $\operatorname{span}\{y_i\}_{i=1}^k = \operatorname{span}\{\Gamma_\psi(b_i)\}_{i=1}^n$. Write

$$\Gamma_\psi(b_i) := \sum_{j=1}^k \alpha_{ij} y_j$$

for some coefficients $\alpha_{ij} \in \mathbb{C}$.

Now, given any $\varphi \in S(A)$, we have

$$\begin{aligned}
\varphi \otimes \psi(x^*x) &= \sum_{i,j=1}^n \varphi(a_i^* a_j)\psi(b_i^* b_j) = \sum_{i,j=1}^n \langle \Gamma_\varphi(a_j), \Gamma_\varphi(a_i)\rangle_\varphi \langle \Gamma_\psi(b_j), \Gamma_\psi(b_i)\rangle_\psi \\
&= \left\langle \sum_i \Gamma_\varphi(a_i) \otimes \Gamma_\psi(b_i), \sum_j \Gamma_\varphi(a_j) \otimes \Gamma_\psi(b_j) \right\rangle_{H_\varphi \otimes H_\psi} \\
&= \left\| \sum_i \Gamma_\varphi(a_i) \otimes \Gamma_\psi(b_i) \right\|_{H_\varphi \otimes H_\psi}^2
\end{aligned}$$

where we are considering the inner product (and norm) on the *Hilbert space tensor product* of H_φ and H_ψ (as defined in appendix A.1). If we can choose $\varphi \in S(A)$ such that $\sum_i \Gamma_\varphi(a_i) \otimes \Gamma_\psi(b_i) \neq 0$, then we're done.

Write

$$\begin{aligned}
\sum_i \Gamma_\varphi(a_i) \otimes \Gamma_\psi(b_i) &= \Gamma_\varphi(a_1) \otimes y_1 + \Gamma_\varphi(a_2) \otimes \left(\sum_{j=1}^k \alpha_{2j} y_j\right) + \cdots + \Gamma_\varphi(a_n) \otimes \left(\sum_{j=1}^k \alpha_{nj} y_j\right) \\
&= \Gamma_\varphi(a_1 + \alpha_{21} a_2 + \cdots + \alpha_{n1} a_n) \otimes y_1 \\
&\quad + \Gamma_\varphi(\alpha_{22} a_2 + \cdots + \alpha_{n2} a_n) \otimes y_2 + \cdots + \Gamma_\varphi(\alpha_{nk} a_n) \otimes y_k
\end{aligned}$$

Since the y_i's are linearly independent, we need at least one of the left tensorants to be nonzero for the expression above to be nonzero. Consider $x := a_1 + \alpha_{21} a_2 + \cdots + \alpha_{n1} a_n$. Since we assumed the a_i's are linearly independent, this expression is necessarily nonzero, and so there exists a (pure) state $\varphi \in S(A)$ such that $\varphi(x^*x) > 0$, and so $\Gamma_\varphi(x) \neq 0$.

Thus, we have found states $\varphi \in S(A), \psi \in S(B)$ such that

$$\sum_i \Gamma_\varphi(a_i) \otimes \Gamma_\psi(b_i) \neq 0$$

whence

$$\varphi \otimes \psi(x^*x) > 0$$

as desired. $\square$

Theorem 4.1.10 (Turumaru)**.** Let A, B be C*-algebras, and $x \in A \odot B$. Define the quantity

$$\|x\|_{\mathrm{T}} := \sup\{\|\pi_{\varphi\otimes\psi}(x)\| \ : \ \varphi \in S(A),\ \psi \in S(B)\}$$

Then $\|\cdot\|_{\mathrm{T}}$ is a C*-norm on $A \odot B$.

Proof. It is obvious from the definition that $\|\cdot\|_{\mathrm{T}}$ is a C*-seminorm. Notice that by Cauchy-Schwarz $\|\pi_{\varphi\otimes\psi}(x)\| \geq \langle \pi_{\varphi\otimes\psi}(x^*x)\xi_{\varphi\otimes\psi}, \xi_{\varphi\otimes\psi}\rangle = \varphi\otimes\psi(x^*x)$. Thus, for $x \neq 0$, choosing φ, ψ such that $\varphi\otimes\psi(x^*x) > 0$ as in lemma 4.1.9 we have $\|\pi_{\varphi\otimes\psi(x)}\| > 0$, and so $\|x\|_{\mathrm{T}} > 0$. □

For the next few lemmas we will denote the completion of $A \odot B$ with respect to $\|\cdot\|_{\mathrm{T}}$ by $A \otimes_{\mathrm{T}} B$, as usual. Soon enough we will see that this $\|\cdot\|_{\mathrm{T}}$ notation is actually redundant, but I don't want to spoil anything.

Proposition 4.1.11. Let $(A^* \odot B^*) \cap S(A \odot B)$ denote those algebraic states on $A \odot B$ which are finite linear combinations of *simple tensors* of functionals. Then every algebraic state $f \in (A^* \odot B^*) \cap S(A \odot B)$ is continuous with respect to the norm $\|\cdot\|_{\mathrm{T}}$.

Proof. Let $f = \sum_i g_i \otimes h_i$ be an element of $(A^* \odot B^*) \cap S(A \odot B)$, where $g_i \in A^*$ and $h_i \in B^*$. Using the Jordan decomposition (2.2.3), we can assume $f = \sum_j \alpha_j \varphi_j \otimes \psi_j$, for some coefficients $\alpha_j \in \mathbb{C}$, and states $\varphi_j \in S(A)$, $\psi_j \in S(B)$ (the new index "j" here indicating that the index set will be different). Then

$$|f(x)| \leq \sum_j |\alpha_j||\varphi_j \otimes \psi_j(x)| \leq \left(\sum_j |\alpha_j|\right) \|x\|_{\mathrm{T}}$$

thus proving the claim. □

As a corollary of this, any element of $(A^* \odot B^*) \cap S(A \odot B)$ extends to a uniquely defined state on $A \otimes_{\mathrm{T}} B$. Let $\widehat{S}_{\mathrm{T}}(A \odot B)$ denote the set of algebraic states on $A \odot B$ which extend uniquely to a state on $A \otimes_{\mathrm{T}} B$. Then thusfar we have shown

$$\{\varphi \otimes \psi \ : \ \varphi \in S(A),\ \psi \in S(B)\} \subseteq (A^* \odot B^*) \cap S(A \odot B) \subseteq \widehat{S}_{\mathrm{T}}(A \odot B)$$

Theorem 4.1.12. Given any two C*-algebras A, B, the norm $\|\cdot\|_{\mathrm{T}}$ on $A \odot B$ is equal to the min-norm on $A \odot B$.

Proof. Choose faithful, non-degenerate *-representations $\pi : A \to \mathcal{B}(H)$ and $\rho : B \to \mathcal{B}(K)$ of A, B respectively. By Takesaki's theorem (3.2.1), π and ρ can be used to implement the min-norm on $A \odot B$:

$$\|x\|_{\min} = \|\pi \odot \rho(x)\|_{\mathcal{B}(H\otimes K)}, \qquad \forall x \in A \odot B$$

Now let $\omega_\xi(T) := \langle T\xi, \xi\rangle$, for $T \in \mathcal{B}(H)$ and $\xi \in H$ (or $T \in \mathcal{B}(K)$ and $\xi \in K$). For arbitrary $\xi \in \mathbb{B}(H)$, $\eta \in \mathbb{B}(K)$, and $x = \sum_i a_i \otimes b_i \in A \odot B$, we calculate

$$\begin{aligned}
((\omega_\xi \circ \pi) \otimes (\omega_\eta \circ \rho))(x) &= \sum_i \omega_\xi \circ \pi(a_i)\omega_\eta \circ \rho(b_i) = \sum_i \langle \pi(a_i)\xi, \xi\rangle\langle \rho(b_i)\eta, \eta\rangle \\
&= \sum_i \langle (\pi(a_i) \otimes \rho(b_i))(\xi \otimes \eta), \xi \otimes \eta\rangle_{H\otimes K} \\
&= \left\langle \pi \otimes \rho\left(\sum_i a_i \otimes b_i\right)(\xi \otimes \eta), \xi \otimes \eta \right\rangle_{H\otimes K} = \langle \pi \otimes \rho(x)\xi \otimes \eta, \xi \otimes \eta\rangle_{H\otimes K}
\end{aligned}$$

whence

$$|((\omega_\xi \circ \pi) \otimes (\omega_\eta \circ \rho))(x)| = |\langle \pi \otimes \rho(x)(\xi \otimes \eta), \xi \otimes \eta\rangle_{H\otimes K}| \leq \|\pi \otimes \rho(x)\|_{\mathcal{B}(H\otimes K)} = \|x\|_{\min}$$

Consider, now, the sets

$$U := \{\omega_\xi \circ \pi \ : \ \xi \in \mathbb{B}(H)\} \subseteq S(A), \qquad V := \{\omega_\eta \circ \pi \ : \ \eta \in \mathbb{B}(K)\} \subseteq S(B)$$

It's not hard to check that these sets have the following "positivity discerning property":

$$\forall x = x^* \in A, \quad f(x) \geq 0 \; \forall f \in U \implies x \geq 0$$

(stated similarly for B). If we skip ahead to lemma 4.1.21 (apologies for the poor organization, all I care about is that we don't have circular logic) we see that the convex hulls of U and V are *weak-* dense* in $S(A)$ and $S(B)$ respectively. Thus, for any $\varphi \in S(A)$ and $\psi \in S(B)$, we can choose nets $(f_\lambda)_\lambda \in \operatorname{conv} U$ and $(g_\mu)_\mu \in \operatorname{conv} V$ converging weak-* to φ and ψ respectively. Fixing λ and μ, and letting $f_\lambda = \sum_i s_i \omega_{\xi_i} \circ \pi$ and $g_\mu = \sum_j t_j \omega_{\eta_j} \circ \rho$, for some $s_i \in [0,1]$, $t_j \in [0,1]$ with $\sum_i s_i = \sum_j t_j = 1$, we have

$$|f_\lambda \otimes g_\mu(x)| \leq \sum_{i,j} s_i t_j \left| \left((\omega_{\xi_i} \circ \pi) \otimes (\omega_{\eta_j} \circ \rho) \right)(x) \right| \leq \sum_{i,j} s_i t_j \|x\|_{\min} = \|x\|_{\min}$$

and so, necessarily

$$|\varphi \otimes \psi(x)| \leq \sup_{\lambda,\mu} |f_\lambda \otimes g_\mu(x)| \leq \|x\|_{\min}$$

What we've just shown is that $\varphi \otimes \psi$ is continuous with respect to the *min* norm, and so by a similar logic to proposition 4.1.11 above, we get that everything in $(A^* \odot B^*) \cap S(A \odot B)$ is *also* continuous with respect to the min norm. Let $\widehat{S}_{\min}(A \odot B)$ denote those algebraic states on $A \odot B$ which extend uniquely to a state on $A \otimes_{\min} B$. Then $(A^* \odot B^*) \cap S(A \odot B) \subseteq \widehat{S}_{\min}(A \odot B)$, and so

$$\|x\|_{\mathrm{T}} \leq \sup\{\|\pi_f(x)\| \ : \ f \in (A^* \odot B^*) \cap S(A \odot B)\} \leq \sup\{\|\pi_f(x)\| \ : \ f \in \widehat{S}_{\min}(A \odot B)\}$$

For any $f \in \widehat{S}_{\min}(A \odot B)$, let $\widetilde{f} \in S(A \otimes_{\min} B)$ denote its unique extension. Let

$$\pi_f : A \odot B \to \mathcal{B}(H_f), \qquad \pi_{\widetilde{f}} : A \otimes_{\min} B \to \mathcal{B}(H_{\widetilde{f}})$$

be the GNS representations constructed form f and $\widetilde{f}$ respectively. It's a little tedious, but ultimately straightforward to show that H_f can be isometrically identified with a subspace of $H_{\widetilde{f}}$, and for all $x \in A \odot B$, $\pi_{\widetilde{f}}(x)|_{H_f} = \pi_f(x)$. Thus $\|\pi_f(x)\| \leq \|\pi_{\widetilde{f}}(x)\|$, and so

$$\sup\{\|\pi_f(x)\| \ : \ f \in \widehat{S}_{\min}(A \odot B)\} \leq \sup\{\|\pi_{\widetilde{f}}(x)\| \ : \ \widetilde{f} \in S(A \otimes_{\min} B)\}$$

but this latter quantity is equal to $\|\pi_u(x)\|$, where π_u is the universal representation of $A \otimes_{\min} B$, which is isometric, so this is equal to $\|x\|_{\min}$. Thus, we have shown that

$$\|x\|_{\mathrm{T}} \leq \|x\|_{\min}$$

and Takesaki's theorem tells us that these two norms are in fact equal. □

Corollary 4.1.13. The proof above demonstrates the sets

$$\Omega := \{\{\varphi \odot \psi \ : \ \varphi \in S(A), \psi \in S(B)\}, \ \operatorname{conv}\{\varphi \odot \psi \ : \ \varphi \in S(A), \psi \in S(B)\}, \ (A^* \odot B^*) \cap S(A \odot B)\}$$

are all *separating* (in the sense of definition 4.1.8), and each induce the *min*-norm:

$$\|\cdot\|_{\min} = \rho_\Gamma, \qquad \Gamma \in \Omega$$

The min norm isn't the only norm that can be expressed in terms of separating sets. The following theorem organizes all the different tensor norms we've encountered along with their representative separating sets of algebraic states.

Theorem 4.1.14. Let A, B be C*-algebras, M, N von Neumann algebras. Define

$$\begin{aligned}
\Gamma_{\max}(A \odot B) &:= S(A \odot B) \\
\Gamma_{\min}(A \odot B) &:= (A^* \odot B^*) \cap S(A \odot B) \\
\Gamma_{\mathrm{nor}}(A \odot N) &:= \{f \in S(A \odot N) \mid \forall a \in A, \ n \mapsto f(a \otimes n) \text{ is normal}\} \\
\Gamma_{\mathrm{bin}}(M \odot N) &:= \left\{ f \in S(M \odot N) \mid \begin{matrix} \forall m \in M, \ n \mapsto f(m \otimes n) \text{ is normal} \\ \forall n \in N, \ m \mapsto f(m \otimes n) \text{ is normal} \end{matrix} \right\}
\end{aligned}$$

Then

$$\|\cdot\|_{A \otimes_{\max} B} = \rho_{\Gamma_{\max}(A \odot B)}, \qquad \|\cdot\|_{A \otimes_{\min} B} = \rho_{\Gamma_{\min}(A \odot B)}$$
$$\|\cdot\|_{A \otimes_{\mathrm{nor}} N} = \rho_{\Gamma_{\mathrm{nor}}(A \odot N)}, \qquad \|\cdot\|_{M \otimes_{\mathrm{bin}} N} = \rho_{\Gamma_{\mathrm{bin}}(M \odot N)}$$

Remark. For the sake of notational simplicity, we will often write $\Gamma_{\max}$ in place of $\Gamma_{\max}(A \odot B)$ when this causes no confusion. Moreover, we will write $\rho_{\max}$ in place of $\rho_{\Gamma_{\max}}$, and similarly for min, nor and bin.

Proof. We've already proven the $\rho_{\min}$ formula in corollary 4.1.13. Let us focus first on $\rho_{\max}$. In order to prove the claim, given a *-homomorphism $\pi : A \odot B \to \mathcal{B}(H)$, we must obtain a state $\varphi \in S(A \odot B)$ for which $\|\pi_\varphi(x)\| \geq \|\pi(x)\|$. If π is cyclic, with unit cyclic vector ξ, then π induces a state φ given by

$$\varphi(x) := \langle \pi(x)\xi, \xi \rangle$$

and moreover, the GNS representation π_φ induced by this state is unitarily equivalent π, whence $\|\pi_\varphi(x)\| = \|\pi(x)\|$.

On the other hand, if π is non-cyclic, as long as π is non-degenerate, π is equivalent to the direct sum of cyclic subrepresentations $\pi \cong \oplus_e \pi_e$ (see theorem 2.1.1), in which case $\|\pi(x)\| = \sup_e \|\pi_e(x)\|$. If π is degenerate, then we can restrict π to $K := \overline{\pi(A)H}$, thus obtaining a non-degenerate representation $\pi|_K$ for which $\|\pi|_K(x)\| = \|\pi(x)\|$. Thus, we do indeed have $\|x\|_{\max} = \rho_{\max}(x)$. ☐

Now that we've established a connection between subsets of $\Gamma \subseteq S(A \odot B)$ and different tensor product norms, let us see how the states on $S(A \otimes_\Gamma B)$ (which, to be clear, are just "states" on a C*-algebra, not "algebraic states" anymore) relate back to Γ. Certainly given a state $f \in S(A \otimes_\Gamma B)$, the restriction of f to $A \odot B$ is an algebraic state, so we have a restriction map

$$\begin{array}{rcccc} \Phi_\Gamma & : & S(A \otimes_\Gamma B) & \to & S(A \odot B) \\ & & f & \mapsto & f|_{A \odot B} \end{array}$$

Moreover, it is easily seen that this map is injective - indeed if $\Phi_\Gamma(f) = \Phi_\Gamma(g)$, since $A \odot B$ is dense in $A \otimes_\Gamma B$ and each of f, g is continuous with respect to ρ_Γ, it follows that $f = g$ (since they agree on a dense subset of their domain). Since Φ_Γ is injective, it induces a bijection between $S(A \otimes_\Gamma B)$ and its image.

Definition 4.1.15. We denote by $\widehat{S}_\Gamma(A \odot B)$ the image of $S(A \otimes_\Gamma B)$ under Φ_Γ. As before, if $A \odot B$ is clear from context, we may simply write $\widehat{S}_\Gamma$. Moreover if Γ is one of $\Gamma_{\min}$, $\Gamma_{\max}$, Γ_{nor} or Γ_{bin}, we may write $\widehat{S}_{\min}$, $\widehat{S}_{\max}$, $\widehat{S}_{\text{nor}}$ or $\widehat{S}_{\text{bin}}$ respectively in place of each $\widehat{S}_\Gamma$.

Let us observe that $\widehat{S}_\Gamma$ consists precisely of those algebraic states which extend (uniquely) to states on $A \otimes_\Gamma B$ (this is easy to prove from the definitions). Moreover, every state $f \in \Gamma$ is obviously bounded above by ρ_Γ, since

$$|f(x)| = |\langle \pi_f(x)\xi_f, \xi_f \rangle| \leq \|\pi_f(x)\| \leq \sup_{f \in \Gamma} \|\pi_f(x)\| = \rho_\Gamma(x)$$

Thus, by density, each $f \in \Gamma$ extends (uniquely) to all of $A \otimes_\Gamma B$, so $\Gamma \subseteq \widehat{S}_\Gamma$. Additionally, following the same logic as in the proof of 4.1.12,

$$\rho_\Gamma(x) = \sup\{\|\pi_f(x)\| \ : \ f \in \Gamma\} \leq \sup\{\|\pi_f(x)\| \ : \ f \in \widehat{S}_\Gamma\} \leq \sup\{\|\pi_{\widetilde{f}}(x)\| \ : \ \widetilde{f} \in S(A \otimes_\Gamma B)\} = \rho_\Gamma(x)$$

and so

$$\rho_\Gamma = \rho_{\widehat{S}_\Gamma}$$

The bijection $\Phi_\Gamma : S(A \otimes_\Gamma B) \to \widehat{S}_\Gamma(A \odot B)$ also allows us to pullback a topology from $S(A \otimes_\Gamma B)$ to a topology on $\widehat{S}_\Gamma(A \odot B)$. In our investigation, we wish to endow $S(A \otimes_\Gamma B)$ with the *weak-* topology* inherited from $(A \otimes_\Gamma B)^*$. A subset $U \subseteq \widehat{S}_\Gamma(A \odot B)$ is then said to be "weak-* open" if the corresponding set $\Phi_\Gamma^{-1}(U)$ is weak-* open. It is then not hard to check that a net $\varphi_\lambda \in \widehat{S}_\Gamma$ converges to $\varphi \in \widehat{S}_\Gamma$ if and only if $\varphi_\lambda(x) \to \varphi(x)$ for all $x \in X$, mirroring weak-* convergence in $(A \otimes_\Gamma B)^*$.

States and Natural Inclusions of Tensor Products

Recall that what we mean by $A \otimes_\Gamma C \subseteq A \otimes_\Gamma D$ is that the natural inclusion $A \odot C \hookrightarrow A \odot D$ is an *isometry* with respect to the Γ-norms on $A \odot C$ and $A \odot D$, and so extends to an isometric $*$-homomorphism $A \otimes_\Gamma C \hookrightarrow A \otimes_\Gamma D$, the "inclusion" morphism, which allows us to embed $A \otimes_\Gamma C$ as a C*-subalgebra of $A \otimes_\Gamma D$.

Assume that $A \otimes_\Gamma C \subseteq A \otimes_\Gamma D$, so that we have said isometric embedding $\Psi : A \otimes_\Gamma C \subseteq A \otimes_\Gamma D$ extending the inclusion morphism. Next consider the adjoint of this map:

$$\Psi^* : (A \otimes_\Gamma D)^* \to (A \otimes_\Gamma C)^*$$

As Ψ is isometric, Ψ^* is surjective[7]. Additionally, since Ψ is positive and contractive, Ψ takes states to states, so $\Psi^*(S(A \otimes_\Gamma D)) \subseteq S(A \otimes_\Gamma C)$. Finally, given a state $f \in S(A \otimes_\Gamma C)$, embedding $A \otimes_\Gamma C$ via Ψ as a C*-subalgebra of $A \otimes_\Gamma D$, we can extend f to a state $\tilde{f} \in S(A \otimes_\Gamma D)$, thus telling us that the map

$$\Psi^*|_{S(A \otimes_\Gamma D)} : S(A \otimes_\Gamma D) \to S(A \otimes_\Gamma C)$$

is surjective.

We can further drop from the C*-algebras $A \otimes_\Gamma D$ and $A \otimes_\Gamma C$ to their underlying algebraic tensor products. Let $f \in \widehat{S}_\Gamma(A \odot C)$. Then f extends uniquely to $\tilde{f} \in S(A \otimes_\Gamma C)$, which is the restriction $\Psi^* \tilde{g}$ of some $\tilde{g} \in S(A \otimes_\Gamma D)$. Letting $g = \Psi^* \tilde{g}|_{A \odot D}$, then $g \in \widehat{S}_\Gamma(A \odot D)$ is such that $g|_{A \odot C} = f$. Thus, the restriction map

$$\widehat{S}_\Gamma(A \odot D) \to \widehat{S}_\Gamma(A \odot C)$$

is also surjective.

On the other hand, we have

$$\begin{aligned} A \otimes_\Gamma C \subseteq A \otimes_\Gamma D &\iff (A \odot C, \rho_\Gamma) \hookrightarrow (A \odot D, \rho_\Gamma) \text{ is isometric} \\ &\iff \sup\{\|\pi_f(x)\| \ : \ f \in \widehat{S}_\Gamma(A \odot C)\} \\ &\qquad = \sup\{\|\pi_f(x)\| \ : \ f \in \widehat{S}_\Gamma(A \odot D)\} \ \forall x \in A \odot C \end{aligned}$$

Clearly this equality of norms occurs if the restriction map $\widehat{S}_\Gamma(A \odot D) \to \widehat{S}_\Gamma(A \odot C)$ is surjective, and thus $A \otimes_\Gamma C \subseteq A \otimes_\Gamma D$. In conclusion, we have proven the following lemma:

Lemma 4.1.16. Given C*-algebras A, $C \subseteq D$, and a separating subset $\Gamma \subseteq S(A \odot C)$, there is a natural inclusion $A \otimes_\Gamma C \subseteq A \otimes_\Gamma D$ if and only if the restriction map $\widehat{S}_\Gamma(A \odot D) \to \widehat{S}_\Gamma(A \odot C)$ is surjective.

Complete State Maps

Each state $f \in S(A \odot B)$ gives rise to a map $T_f : B \to A^*$, given by letting

$$T_f(b)(a) := f(a \otimes b)$$

When both $a, b \geq 0$, we have that $f(a \otimes b) \geq 0$, so if $b \geq 0$, then $T_f(b)$ is a positive linear map. Moreover,

$$\|T_f\| = \sup\{\|T_f(b)\| \ : \ \|b\| = 1\} = \sup\{|T_f(b)(a)| \ : \ \|a\| = \|b\| = 1\} = 1$$

When B is unital, $T_f(1)$ is easily seen to be a state in $S(A)$. We will primarily focus on the case when A, B are unital.

Fact 4.1.17. If $f \in S(A \odot B)$, then T_f is completely positive.

Proof. Let $(T_f)_n : M_n(B) \to M_n(A^*)$ be the n^{th} ampliation of T_f. Recall that we can associate $M_n(A^*)$ with $M_n(A)^*$ without fussing with notions of positivity: $[f_{ij}] \in M_n(A^*)$ is positive if and only if it's corresponding functional in $M_n(A)^*$ is positive (in fact, this is the *definition* of positivity in $M_n(A^*)$). Thus we will implicitly consider $(T_f)_n$ as a map from $M_n(B) \to M_n(A)^*$. We have

$$(T_f)_n([b_{ij}])([a_{ij}]) = [T_f(b_{ij})]([a_{ij}]) = \sum_{i,j=1}^{n} T_f(b_{ij})(a_{ij}) = f\left(\sum_{i,j=1}^{n} a_{ij} \otimes b_{ij}\right)$$

[7]It is not necessarily true that an injective linear map $T : X \to Y$ between Banach spaces has surjective adjoint. However, if T is *topologically* injective (injective with closed range), then T^* is indeed surjective. When T is an isometry, as it is in our case, T is topologically injective.

A positive element of $M_n(B)$ is a sum of elements of the form $[b_i^* b_j]$ (similarly for $M_n(A)$), so we need only check that $(T_f)_n([b_i^* b_j])([a_i^* a_j]) \geq 0$. Certainly,

$$\begin{aligned}(T_f)_n([b_i^* b_j])([a_i^* a_j]) &= f\left(\sum_{i,j=1}^{n} a_i^* a_j \otimes b_i^* b_j\right) \\ &= f\left(\left(\sum_{i=1}^{n} a_i^* \otimes b_i^*\right)\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right) \\ &= f(x^* x) \geq 0\end{aligned}$$

where $x = \sum_{i=1}^{n} a_i \otimes b_i$. Thus T_f is completely positive. □

Definition 4.1.18. The map T_f is the *complete state map* corresponding to f. The set of complete state maps from B to A^* is denoted $CS(B, A^*)$.

Of course, we could also use f to construct a complete state map $S_f : A \to B^*$ in an identical manner - the set of such complete state maps would be denoted $CS(A, B^*)$.

Fact 4.1.19. Suppose $T : B \to A^*$ is a completely positive, norm 1 map. Then T induces a state $f_T \in S(A \odot B)$.

Proof. It is not hard to guess that $f_T(a \otimes b) := T(b)(a)$. It is clear that $\sup\{|f_T(a \otimes b)| : \|a\| = \|b\| = 1\} = \|T\| = 1$. That f_T is positive follows from complete positivity of T. Take $x = \sum_{i=1}^{n} a_i \otimes b_i \in A \odot B$, and consider

$$\begin{aligned}f_T(x^* x) &= f_T\left(\sum_{i,j=1}^{n} a_i^* a_j \otimes b_i^* b_j\right) \\ &= \sum_{i,j=1}^{n} T(b_i^* b_j)(a_i^* a_j) \\ &= T_n([b_i^* b_j])([a_i^* a_j])\end{aligned}$$

where $T_n : M_n(B) \to M_n(A)^*$ denotes the n^{th} ampliation of T (as above, we've implicitly associated $M_n(A^*)$ with $M_n(A)^*$). Since T is completely positive, and each of the matrices $[b_i^* b_j]$ and $[a_i^* a_j]$ are positive, it follows that $f_T(x^* x) \geq 0$, so f_T is indeed a state. □

In other words, we have

$$CS(B, A^*) = \{T : B \to A^* \mid T \text{ c.p.}, \ \|T\| = 1\}$$

It is also easy to verify that the constructions f_T and T_f are inverses of each other: given any state $f \in S(A \odot B)$ and complete state map $T : B \to A^*$, then

$$T_{f_T} = T, \quad \text{and} \quad f_{T_f} = f$$

Thus we have a bijective correspondence between

$$S(A \odot B) \iff CS(B, A^*)$$

Recall that there is an isometric complete order isomorphism of Banach spaces $A^* \cong (A^{**})_*$ (corollary 2.2.11). Thus, given $T \in CS(B, A^*)$, we can regard T as a completely positive, norm 1 from $B \to R_*$, where $R = A^{**}$. Letting $f = f_T$ (so that $T = T_f$), what this means is that $f \in S(R \odot B)$ is such that $T_f(B) \subseteq R_*$, or equivalently

$$f \in \Gamma_{\text{nor}}(A^{**} \odot B)$$

So we actually have a bijective correspondence between the following *three* sets:

Fact 4.1.20.

$$S(A \odot B) \iff CS(B, A^*) \iff \Gamma_{\text{nor}}(A^{**} \odot B)$$

Let's try to be a bit more explicit about the correspondence between $S(A \odot B)$ and $\Gamma_{\text{nor}}(A^{**} \odot B)$. Let $\Theta : (A^{**})_* \to A^*$ denote the isometric complete order isomorphism defined in 2.2.11. Take $f \in S(A \odot B)$. Since $T_f(b) \in A^*$, we see $\Theta^{-1}(T_f(b)) \in (A^{**})_*$, and we can define a state $\omega_f \in \Gamma_{\text{nor}}(A^{**} \odot B)$ by

$$\omega_f(a^{**} \odot b) := \Theta^{-1}(T_f(b))(a^{**}), \quad a^{**} \in A^{**},\ b \in B$$

This is the (unique) state corresponding to f. It is clear from complete positivity of T_f and Θ^{-1}, and isometricity of Θ^{-1}, that ω_f is indeed a state. The inverse of this operation is merely restriction of the domain to $A \odot B$: that is,

$$f = \omega_f \circ (J_A \odot \text{id}_B)$$

At first this appears to be nothing more than a cute notational eccentricity, but with time we will see that this correspondence is actually *paramount* to establishing many of the results relating nuclearity of C*-algebras to their enveloping von Neumann algebras.

Weak-* Density of Separating Sets

Lemma 4.1.21 ([14]; Lemma 3.4.1)**.** Let A be a unital C*-algebra, and $Q \subseteq S(A)$. Suppose Q is such that for all self-adjoint $x \in A$, if $f(x) \geq 0$ for all $f \in Q$, then $x \geq 0$. Then the weak*-closure of the convex hull of Q is $S(A)$.

Remark. Sets Q with the property above are sometimes referred to as "discerning" or "discerning of positivity" throughout this document. This terminology is non-standard.

Proof. The key here is the geometric Hahn-Banach. Suppose there existed $f \in S(A)$ that wasn't in $C := \overline{\text{conv}}^{w*}Q$. Then we can exhibit an element $x \in A$ and a real number α such that $\Re f(x) \geq \alpha \geq \Re g(x)$ for all $g \in C$. By passing to $y = \Re x = (x + x^*)/2$, noting that $\Re h(x) = h(\Re x) = h(y)$ for any positive linear functional h, we thus obtain

$$f(y) > \alpha \geq g(y) \quad \forall g \in C$$

By assumption of the discerning property of Q, that $g(\alpha 1 - y) \geq 0$ for all $g \in Q$ tells us that $\alpha 1 - y \geq 0$, and so $h(\alpha 1 - y) \geq 0$ for all $h \in S(A)$, and yet clearly $f(\alpha 1 - y) < 0$, a contradiction. ☐

Fact 4.1.22. Let A be a C*-algebra, and π a representation of A. Then

$$\|\pi(x)\| = \sup\{\omega_\xi \circ \pi(x^*x)^{1/2} \ : \ \|\xi\| = 1\}$$

Proof. Straightforward. ☐

Theorem 4.1.23 ([16]; Lemma 2.4)**.** Let A, B be unital C*-algebras, and Γ a *convex, separating subset* of $S(A \odot B)$. Suppose further that for all fixed $f \in \Gamma$ and $y \in A \odot B$ such that $f(y^*y) \neq 0$, the map

$$g(x) := \frac{f(y^*xy)}{f(y^*y)}$$

is a state in Γ. Then $\widehat{S}_\Gamma(A \odot B)$ is the weak-* closure of Γ, and

$$\rho_\Gamma(x) = \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in \Gamma\}$$

Recall that the "weak-*" topology on $\widehat{S}_\Gamma$ here is the topology "inherited" from $S(A \otimes_\Gamma B)$.

Proof. First note that g, as defined above, is indeed always a state, so what is crucial to the statement of this theorem is only that g is additionally in Γ.

Let $x \in A \otimes_\Gamma B$ be self-adjoint, and assume that $f(x) \geq 0$ for all $f \in \Gamma$ (note that it makes sense to evaluate $f \in \Gamma$ at $x \in A \otimes_\Gamma B$, since $\Gamma \subseteq \widehat{S}_\Gamma(A \odot B)$, and every functional in $\widehat{S}_\Gamma(A \odot B)$ extends uniquely to a state in $S(A \otimes_\Gamma B)$). Then since g is a state for all f, y as in the lemma statement, we have that $f(y^*xy) \geq 0$, which when written in terms of f's GNS representation is

$$\langle \pi_f(y^*xy)\xi_f, \xi_f\rangle \geq 0$$

or equivalently, letting $\zeta = \pi(y)\xi_f$, we have

$$\langle \pi_f(x)\zeta, \zeta \rangle \geq 0$$

Since ξ_f is a cyclic vector for π_f, the set of all ζ is dense in the representation space of π_f, so $\langle \pi_f(x)\zeta, \zeta \rangle \geq 0$ is enough to conclude that $\pi_f(x) \geq 0$.

Letting π denote direct sum of the GNS representations π_f over all $f \in \Gamma$, π is a faithful representation of $A \otimes_\Gamma B$. As each $\pi_f(x) \geq 0$, we thus have $\pi(x) \geq 0$. Moreover, by assumption that Γ is separating, for all $x \neq 0$ there exists some f such that $\pi_f(x) \neq 0$, so π is injective. Thus we conclude that $x \geq 0$.

So, we have just shown that if $x \in A \otimes_\Gamma B$ is self-adjoint, and $f(x) \geq 0$ for all $f \in \Gamma$, then $x \geq 0$. This is the prerequisite condition from lemma 4.1.21 to conclude that the weak-* closure of Γ is $\widehat{S}_\Gamma(A \odot B)$.

The formula for ρ_Γ then follows from the fact that

$$\|\pi(x)\| = \sup\{\omega_\xi \circ \pi(x^*x)^{1/2} \ : \ \|\xi\| = 1\}$$

□

The new formulae we've obtained for the ρ_Γ norms are certainly nice, but what's *truly* important about this result is that it gives us an entirely new characterization of nuclearity.

Suppose A is nuclear, so $A \otimes_{\min} B = A \otimes_{\max} B$ for all other C*-algebras B, and thus $S(A \otimes_{\min} B) = S(A \otimes_{\max} B)$. This also means $\widehat{S}_{\min}(A \odot B) = \widehat{S}_{\max}(A \odot B)$, as these sets are simply the restrictions of $S(A \otimes B)$ to $A \odot B$. By theorem 4.1.23, $\overline{(A^* \odot B^*) \cap S(A \odot B)}^{w^*} = \widehat{S}_{\min}(A \odot B) = \widehat{S}_{\max}(A \odot B) = S(A \odot B)$. On the other hand, if we don't know A is nuclear but we *do* have $\overline{(A^* \odot B^*) \cap S(A \odot B)}^{w*} = S(A \odot B)$ for all B, then by our formulae for norms in theorem 4.1.14 we clearly have $\| \cdot \|_{\min} = \| \cdot \|_{\max}$. Thus, we've proven our first characterization of nuclearity:

Theorem 4.1.24. A unital C*-algebra A is nuclear if and only if for all unital C*-algebras B,

$$\overline{(A^* \odot B^*) \cap S(A \odot B)}^{w*} = S(A \odot B)$$

In other words, A is nuclear if and only if every state on $A \odot B$ can be pointwise approximated by "finite-rank" states. Naturally, the next question we'll be attracted to is determining precisely when we can perform such an approximation (the learned reader may already know where this is leading - the "approximate factoring of the identity" characterization of nuclearity).

4.2 Semidiscreteness and CPAP

The language of algebraic states, complete state maps, and theorem 4.1.24, leads us naturally into a new area of investigation - that of pointwise approximation of maps. In particular, we know that a unital C*-algebra A is nuclear if and only if every state can be weak-* approximated by finite-rank states. Since states correspond to complete state maps, and weak-* convergence corresponds to "pointwise-weak-*" convergence of these state maps, it is perhaps not unexpected that nuclearity should say something about approximation of these state maps. What is surprising is just *how much* magnificent structure we can gather from nuclearity alone.

Many of the results in this section, while presented solely for unital C*-algebras, have "adjustments" which make them work in the non-unital case as well. However, due to a lack of time, space, willpower, self-respect, and other things, we have chosen to omit these modifications in favour of the "big picture" results. The theorems in this section are truly deep insights by Lance, Choi, Effros, Wassermann, and others, which demonstrate the profound connection between nuclearity and the concepts in prior sections, but without getting too bogged down by minutia. Adjusting for the non-unital case - while certainly a valiant effort - doesn't shed nearly as much insight into these "deep connections" we're after.

Lemma 4.2.1. Let X, Y be Banach spaces, and $C \subseteq \mathcal{B}(X, Y)$ a convex set of operators. Then the point-norm and point-weak closures of C coincide.

Remark. The "point-τ closure" of a set $S \subseteq \mathcal{B}(X, Z)$, for a locally convex topology τ on a TVS Z, is the set of those $T \in \mathcal{B}(X, Z)$ such that there exists a net $T_\lambda \in S$ such that $T_\lambda(x) \xrightarrow{\tau} T(x)$ for all $x \in X$.

Proof. This is a standard exercise involving the Hahn-Banach theorem. Let T be in the point-weak closure of C. It suffices to show that for any finite set $F \subset X$ and $\epsilon > 0$, there is an $S \in C$ such that $\|S(x) - T(x)\| < \epsilon$ for all $x \in F$. Set $\mathcal{F} := \{(F, \epsilon) \mid F \subset X,\ |F| < \infty,\ \epsilon > 0\}$, and turn $\mathcal{F}$ into a directed set by defining

$$(F, \epsilon) \leq (F', \epsilon') \qquad \iff \qquad F \subseteq F' \text{ and } \epsilon' \leq \epsilon$$

Then for each $(F, \epsilon) \in \mathcal{F}$ there is $S_{F,\epsilon} \in C$ satisfying the condition above, and $(S_{F,\epsilon})_{(F,\epsilon)\in\mathcal{F}}$ is the required net point-norm approximating T.

Given a finite set $F \subset X$, consider the set

$$C_F := \{(S(x))_{x\in F} \mid S \in C\} \subseteq Y^n$$

where we endow Y^n with the ℓ_∞-norm: $\|(y_1, ..., y_n)\|_{Y^n} := \sup_i \|y_i\|$. The set C_F is convex, and by the geometric Hahn-Banach the weak and norm closures of C_F coincide. Since $(T(x))_{x\in F} \in \overline{C_F}^{wk}$ by assumption, it follows that for all $\epsilon > 0$ and $f \in (Y^n)^*$, there is $S \in C$ such that

$$\sup_{x\in F} \|S(x) - T(x)\| = \|(S(x))_{x\in F} - (T(x))_{x\in F}\|_{Y^n} < \epsilon$$

proving the claim. □

Theorem 4.2.2. Let A, B be unital C*-algebras, and $f \in S(A \odot B)$ an algebraic state. Then $f \in \widehat{S}_{\min}(A \odot B)$ (which is equal to $\overline{(A^* \odot B^*) \cap S(A \odot B)}^{w*}$) by theorem 4.1.23) if and only if there exist nets $\varphi_\lambda : B \to M_n$, $\psi_\lambda : M_n \to A^*$ of completely positive maps which point-weak-* approximate the complete state map $T_f : B \to A^*$, and for which $\|\varphi_\lambda \circ \psi_\lambda\| = 1$.

Proof. Let us warm up with an exercise. Suppose $A \subseteq \mathcal{B}(H)$, $B \subseteq \mathcal{B}(K)$, $h = \sum_i \xi_i \otimes \eta_i \in H \odot K$. Let ω_h denote the "algebraic vector state" $a \otimes b \mapsto \langle (a \otimes b)h, h\rangle = \sum_{i,j} \langle a\xi_i, \xi_j\rangle\langle b\eta_i, \eta_j\rangle$. In other words,

$$\omega_h(a \otimes b) = \sum_{i,j} \omega_{\xi_i,\xi_j}(a)\omega_{\eta_i,\eta_j}(b)$$

Consider the maps

$$\begin{array}{cccc} \varphi : & B & \to & M_n \\ & b & \mapsto & [\omega_{\eta_i,\eta_j}(b)] \end{array} \qquad \begin{array}{cccc} \psi : & M_n & \to & A^* \\ & [\alpha_{ij}] & \mapsto & (a \mapsto \sum_{ij} \alpha_{ij}\omega_{\xi_i,\xi_j}(a)) \end{array}$$

Then a straightforward calculation shows that

$$(\psi \circ \varphi)(a)(b) = \omega_h(a \otimes b)$$

and so $\psi \circ \varphi = T_{\omega_h}$. In other words, we have *precisely* factored the complete state map T_{ω_h} through M_n. It remains to show that φ, ψ are completely positive. Observe that, in the notation of A.2.18, we have

$$\varphi := \Psi^{-1}([\omega_{\eta_i,\eta_j}]), \qquad \psi := \Psi^{-1}([\omega_{\xi_i,\xi_j}])^*$$

To be clear, $\Psi^{-1}([\omega_{\xi_i,\xi_j}]) \in CP(A, M_n)$, so it's adjoint is (by proposition A.2.11) a map in $CP(M_n^*, A^*)$. By associating $M_n^* \cong M_n$ canonically (which is also a complete order isomorphism), we thus obtain a completely positive map from $M_n \to A^*$, as expected.

Since

$$\sum_{ij} \omega_{\eta_i,\eta_j}(x_i^* x_j) = \sum_{ij} \langle x_i^* x_j \eta_i, \eta_j\rangle = \left\langle \begin{bmatrix} x_1^*x_1 & \cdots & x_1^*x_n \\ \vdots & \ddots & \vdots \\ x_n^*x_1 & \cdots & x_n^*x_n \end{bmatrix} \begin{bmatrix} \eta_1 \\ \ddots \\ \eta_n \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \ddots \\ \eta_n \end{bmatrix} \right\rangle \geq 0$$

by A.2.9 and A.2.1, we see $[\omega_{\eta_i,\eta_j}] \geq 0$, whence both φ and ψ are completely positive.

Next, consider the set $\mathcal{C} \subseteq \mathcal{B}(B, A^*)$ consisting of maps which are compositions of completely positive maps $\varphi : B \to M_n$ and $\psi : M_n \to A^*$ for some $n \in \mathbb{N}$. It is not hard to see that $\mathcal{C}$ is a real cone, whence convex. Indeed, given

$$B \xrightarrow{\varphi_1} M_{n_1} \xrightarrow{\psi_1} A^*, \qquad B \xrightarrow{\varphi_2} M_{n_2} \xrightarrow{\psi_2} A^*$$

define

$$\begin{array}{ccccc} \varphi & : & B & \to & M_{n_1+n_2} \\ & & b & \mapsto & \begin{bmatrix} \varphi_1(b) & 0 \\ 0 & \varphi_2(b) \end{bmatrix} \end{array} \qquad \begin{array}{cccc} \psi & : & M_{n_1+n_2} & \to & A^* \\ & & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \mapsto & \psi_1(A_{11}) + \psi_2(A_{22}) \end{array}$$

ψ is easily seen to be completely positive, and the map $b \mapsto \begin{bmatrix} \varphi_1(b) & 0 \\ 0 & 0 \end{bmatrix}$ is completely positive, so certainly φ being the sum of two maps of this kind is also completely positive. Finally,

$$\psi \circ \varphi = \psi_1 \circ \varphi_1 + \psi_2 \circ \varphi_2$$

so $\mathcal{C}$ is indeed closed under addition. Positive homogeneity of $\mathcal{C}$ is obvious.

Since $\omega_h \in \mathcal{C}$, we thus see $\sum_i \omega_{h_i} \in \mathcal{C}$, so there are completely positive maps φ, ψ such that $\sum_i \omega_{h_i} = \psi \circ \varphi$. Now as we know from corollary 4.1.13, the set

$$\Gamma := \operatorname{conv}\{f \otimes g \mid f \in S(A), g \in S(B)\}$$

is a separating set which induces the min norm. Moreover, Γ is positivity discerning (by lemma 4.1.9), convex, and satisfies the final requisite condition in theorem 4.1.23. Thus, Γ is weak-* dense in $\widehat{S}_{\min}(A \odot B)$.

If we represent A and B as their images under their universal representations (that is, we choose $H = H_u$ and $K = K_u$ above), then (recalling 2.1.5) each state on A and B is a vector state. Noting that $\omega_h \otimes \omega_k = \omega_{h \otimes k}$, every element of Γ is thus of the form $\sum_i c_i \omega_{h_i \otimes k_i}$ for some $h_i \in H$, $k_i \in K$, and $c_i > 0$, $\sum_i c_i = 1$, so for every $f \in \Gamma$, T_f is of the form $\psi \circ \varphi$. When $f \in \widehat{S}_{\min}(A \odot B)$ is the weak-* limit of $f_\lambda \in \Gamma$, then the corresponding complete state map T_f is the point-weak-* limit of the complete state maps T_{f_λ}, each of which is of the form $\psi \circ \varphi$. Since $\|\psi \circ \varphi\| = \|T_{f_\lambda}\| = 1$, the result follows. $\square$

Remark. Let $\mathcal{C}_1 \subseteq \mathcal{B}(B, A^*)$ denote the set of maps which are compositions of completely positive *contractions* $\varphi : B \to M_n$ and $\psi : M_n \to A^*$ for some $n \in \mathbb{N}$. Given $t \in [0,1]$, $\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2 \in \mathcal{C}_1$, by defining φ, ψ as in the proof above, but with the modification

$$\psi\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) := t\psi_1(A_{11}) + (1-t)\psi_2(A_{22})$$

then it is clear that $\|\varphi\| = \sup_i \|\varphi_i\| \le 1$, $\|\psi\| \le t\|\psi_1\| + (1-t)\|\psi_2\| \le 1$, and so

$$t\psi_1 \circ \varphi_1 + (1-t)\psi_2 \circ \varphi_2 = \psi \circ \varphi \in \mathcal{C}_1$$

so $\mathcal{C}_1$ is also convex.

Recalling that we use the term "complete state map" to refer to a completely positive map $T : B \to A^*$ of norm 1, we can phrase the following theorem:

Theorem 4.2.3 (CPAP characterizes nuclearity)**.** Let A, B be unital C*-algebras. The following are equivalent

1. Every complete state map $T : B \to A^*$ is the point-weak-* limit of completely positive, finite rank maps of norm 1.

1'. Every complete state map $T : B \to A^*$ is the point-weak-* limit of completely positive, finite rank contractions.

2. Every complete state map $T : B \to A^*$ is the point-weak-* limit of a pair of nets of completely positive, finite rank maps $\varphi_\lambda : B \to M_n$, $\psi_\lambda : M_n \to A^*$, for which $\|\psi_\lambda \circ \varphi_\lambda\| = 1$.
3. A is nuclear.

Conditions 1 and 2 above are sometimes referred to as the "completely positive approximation property", or the acronym "CPAP". We also remark that by rescaling, the "norm 1" requirement obviously implies (and is implied by the statement) that any completely positive map $T : B \to A^*$ is the point-weak-* limit of a pair of nets $(\psi_\lambda, \phi_\lambda)$ as in 2, or a net of finite rank maps T_λ as in 1, but for which $\|\psi_\lambda \circ \varphi_\lambda\| = \|T_\lambda\| = \|T\|$.

Proof. We will show $1' \implies 1 \implies 3 \implies 2$. That $2 \implies 1$, and $1 \implies 1'$ are both obvious.

($1' \implies 1$) : Let $T : B \to A^*$ be a complete state map. By assumption, there is a net $T_\lambda : B \to A^*$ of completely positive contractions which point-weak-* approximate T_f. By a relatively simple adjustment, we can assume each T_λ is of norm 1.

Observe that

$$|T_\lambda(b)(a)/\|T_\lambda\| - T(b)(a)| \leq |T_\lambda(b)(a) - T(b)(a)|/\|T_\lambda\| + (\|T_\lambda\| - 1)|T(b)(a)|$$

and so if $\|T_\lambda\| \to 1$, then $T_\lambda/\|T_\lambda\|$ *also* point-weak-* approximates T. This, however, is not hard to show. We have

$$\begin{aligned}\|T_\lambda\| \geq |T_\lambda(b)(a)| &= |T_\lambda(b)(a) - T(b)(a) + T(b)(a)| \\ &\geq |T(b)(a)| - |T_\lambda(b)(a) - T(b)(a)| \qquad \text{(reverse triangle inequality)}\end{aligned}$$

By definition, for a given $\epsilon > 0$ there are a, b such that $|T(b)(a)| > 1 - \epsilon/2$. Moreover, we can choose λ large enough such that $|T_\lambda(b)(a) - T(b)(a)| < \epsilon/2$. Thus,

$$\|T_\lambda\| \geq 1 - \epsilon$$

Since $\|T_\lambda\| \leq 1$, we thus have $\|T_\lambda\| \to 1$. So by replacing T_λ with $T_\lambda/\|T_\lambda\|$, we can assume each T_λ is of norm 1.

($1 \implies 3$) : Let $f \in S(A \odot B)$, and $T_f : B \to A^*$ be the corresponding state map. By assumption, there is a net $T_\lambda : B \to A^*$ of complete state maps converging point-weak-* to T_f. Each T_λ induces an algebraic state $f_\lambda \in S(A \odot B)$. These states weak-* approximate f:

$$f_\lambda(a \otimes b) = T_\lambda(b)(a) \to T_f(b)(a) = f(a \otimes b)$$

Additionally, since each T_λ is finite rank, each f_λ is also "finite rank": there are functionals $f_1, ..., f_n \in A^*$, $g_1, .., g_n \in B^*$ such that

$$T_\lambda(b) = g_1(b)f_1 + \cdots + g_n(b)f_n$$

and thus

$$f_\lambda(a \otimes b) = f_1(a)g_1(b) + \cdots + f_n(a)g_n(b)$$

What we've shown is that $\overline{(A^* \odot B^*) \cap S(A \odot B)}^{w*} = S(A \odot B)$, which is equivalent to A being nuclear.

($3 \implies 2$): An algebraic state f is in $\widehat{S}_{\min}(A \odot B)$ if and only if there exist nets $\varphi_\lambda : B \to M_n, \psi_\lambda : M_n \to A^*$ which point-weak-* approximate $T_f : B \to A^*$. If A is nuclear, then $\widehat{S}_{\min}(A \odot B) = S(A \odot B)$, and so every completely positive contraction $T : B \to A^*$ is of the form T_f for some $f \in \widehat{S}_{\min}(A \odot B)$, and the result follows. □

Remark. In light of lemma 4.2.1, one might wonder why we are still referring to "point-weak-* approximating nets" rather than point-norm ones, which we clearly see both exist *and* characterize nuclearity. The only real reason we continue referring to point-weak-* convergence is because point-weak-* convegence of complete state maps corresponds to weak-* convergence of the underlying states, whereas point-norm convergence does *not* imply norm convergence of states. In other words, for complete state maps the point-weak-* is the natural topological analogue of the weak-* topology for states.

Definition 4.2.4. A von Neumann algebra M is said to be *semidiscrete* if there exists a net $\varphi_\lambda \in \mathcal{B}(M)$ of unital, normal, completely positive, finite rank maps φ_λ which "point-ultraweakly approximate the identity" - that is, for all $x \in M$, and $\omega \in M_*$,

$$\omega(\varphi_\lambda(x) - x) \to 0$$

This terminology was actually introduced in [16]. This paper was written almost contemporaneously with [10], which also studied point-weak-* approximation of the identity by finite rank maps. This simultaneous development of the two concepts by Effros is likely the best explanation for where the terminology "semidiscrete" came about, which would otherwise seem completely unprompted.

Corollary 4.2.5. If A is a unital C*-algebra such that A^{**} is semidiscrete, then A is nuclear.

Proof. Let $\varphi_\lambda \in \mathcal{B}(A^{**})$ be a net of unital, normal, completely positive, finite rank maps point-ultraweakly converging to the identity. Consider the restricted adjoints $\sigma_\lambda := \varphi_\lambda^*|_{A^*}$. Since each φ_λ is normal, if $\omega \in A^*$, then $\sigma_\lambda(\omega) = \omega \circ \sigma_\lambda$ is also normal, whence in A^* (since, seeing as $(A^{**})_* = A^*$, those functionals in A^* are precisely the normal functionals on A^{**}). Thus, $\sigma_\lambda \in \mathcal{B}(A^*)$. Moreover, each σ_λ is completely positive contractive, and since

$$\omega(\varphi_\lambda(a) - a) \to 0 \qquad \forall \omega \in A^*$$

we then have

$$\sigma_\lambda(\omega)(a) \to \omega(a)$$

so that σ_λ point-weak-* approximates the identity on A^*.

Using these maps σ_λ, given a complete state map $T : B \to A^*$, we have a net $T_\lambda := \sigma_\lambda \circ T : B \to A^*$ of completely positive contractions which point-weak-* approximate T. This is equivalent to condition 1' of theorem 4.2.3, whence A is nuclear. □

As is apparently a running theme of this text, corollary 4.2.5 augurs a more impressive result: that semidiscreteness of A^{**} *characterizes* nuclearity. Of course, we will not see this result until the next section, and we will unfortunately not present a full (or even partial) proof.[8]

Up to this point, using the language of algebraic states and complete state maps, the proofs have generally been straightforward, requiring only minimal fiddling. The next result, a fundamental result in the theory of nuclear C*-algebras, is not quite so straightforward, but is so frequently used in modern analysis that we find it germane to include the messy details of the proof (as originally conceived by Choi and Effros in [10]). It also presents us with an opportunity to introduce some concepts which will be crucial to our understanding in later sections.

Theorem 4.2.6 (Completely positive approximate factoring of the identity characterizes nuclear C*-algebras)**.** Let A be a unital C*-algebra. Then A is nuclear if and only if there exists a pair of nets $\varphi_\lambda : A \to M_n$ and $\psi_\lambda : M_n \to A$ of unital completely positive maps such that $\psi_\lambda \circ \varphi_\lambda$ point-norm approximates the identity.

The proof of the $\Leftarrow$ direction is relatively transparent - such maps will allow us to easily obtain the CPAP. The $\Rightarrow$ direction requires some preliminary lemmata.

So far, we have been working with *complete state maps*, which take C*-algebras B into *duals* of C*-algebras, A^*. In order to use these maps to construct maps between *C*-algebras*, we need a way of going *from* A^* back into a C*-algebra. Presumably such a map would need to be completely positive, so as to preserve the desirable properties of complete state maps. The following definition makes these ideas precise.

Definition 4.2.7. Let A be a C*-algebra, and $\sigma \in (A^*)^+$. We define

$$C_\sigma := \overline{\mathrm{span}}\{f \in A^* \mid 0 \le f \le \sigma\}$$

and

$$\begin{array}{cccc} \theta_\sigma & : & \pi_\sigma(A)' & \to & C_\sigma \\ & & r & \mapsto & \langle \pi_\sigma(\cdot) r \xi_\sigma, \xi_\sigma \rangle \end{array}$$

Lemma 4.2.8. Using the definitions above, the map θ_σ is a completely positive isomorphism, and θ_σ^{-1} is completely positive.

Proof. The proof that θ_σ is an isomorphism is essentially just proposition 2.5.1 in [14], which we repeat here for completeness. The remainder of the proof is from [26].

We begin by checking that the codomain of θ_σ is correct. Suppose $r \in \pi_\sigma(A)'$, and $0 \le r \le 1$. Then $r = s^*s$ for some $s \in \pi_\sigma(A)'$ also with $0 \le s \le 1$. Thus

$$\begin{aligned} \theta_\sigma(r)(a^*a) &= \langle \pi_\sigma(a^*a) s^*s \xi_\sigma, s\xi_\sigma \rangle \\ &= \|s\pi_\sigma(a)\xi_\sigma\|^2 \\ &\le \|\pi_\sigma(a)\xi_\sigma\|^2 = \sigma(a^*a) \end{aligned}$$

[8]For the learned reader who wishes to skip ahead, we will show that if A is nuclear then A^{**} is injective, and then use the (astonishing) fact that injectivity is equivalent to semidiscreteness for von Neumann algebras. It is possible to prove that if A is nuclear then A^{**} is semidiscrete directly, but we also omit this proof.

and so $\theta_\sigma(r) \leq \sigma$, so $\operatorname{ran}\theta_\sigma \subseteq C_\sigma$.

Now suppose $\theta_\sigma(r) = 0$. Then for all $a, b \in A$, $\theta_\sigma(r)(b^*a) = 0$, which if we write out explicitly yields

$$\langle r\pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)\xi_\sigma\rangle = 0$$

By cyclicity of ξ_σ, this condition necessarily forces $r = 0$, so θ_σ is injective.

Next we have to check that θ_ρ is surjective, which is not nearly so simple. To start, suppose $g \in A^*$ and $0 \leq g \leq f$. Then

$$\langle \pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)\xi_\sigma\rangle_g := g(b^*a)$$

gives us a well-defined positive, non-degenerate hermitian form on $\pi_\sigma(A)\xi_\sigma$. Well-definedness follows since

$$|g(b^*a)|^2 \leq |g(b^*b)||g(a^*a)| \leq |f(b^*b)||f(a^*a)| = \|\pi_\sigma(a)\xi_\sigma\|^2\|\pi_\sigma(b)\xi_\sigma\|^2$$

so if one of $\pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)\xi_\sigma$ is zero, then $g(b^*a) = 0$. This also tells us that $\langle\cdot,\cdot\rangle_g$ is bounded, and so for each fixed $\pi_\sigma(b)\xi_\sigma$, the map

$$\pi_\sigma(a)\xi_\sigma \mapsto \langle \pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)\xi_\sigma\rangle_g$$

is a continuous linear functional on $\overline{\pi_\sigma(A)\xi_\sigma}$, and so by the Riesz representation theorem there is $h \in \overline{\pi_\sigma(A)\xi_\sigma}$ such that

$$\langle \pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)\xi_\sigma\rangle_g = \langle \pi_\sigma(a)\xi_\sigma, h\rangle$$

(The angle brackets on the RHS represent the *standard* inner product, rather than our new sesquilinear form.) Let $T_0 \in \mathcal{B}(\overline{\pi_\sigma(A)\xi_\sigma})$ denote the linear operator taking $\pi_\sigma(b)\xi_\sigma$ to h, or in other words the (unique) linear map such that

$$\langle \pi_\sigma(a)\xi_\sigma, T_0\pi_\sigma(b)\xi_\sigma\rangle = g(b^*a)$$

Note that for all $a, b \in A$,

$$\begin{aligned}\langle \pi_\sigma(a)\xi_\sigma, T_0^*\pi_\sigma(b)\xi_\sigma\rangle &= \overline{\langle \pi_\sigma(b)\xi_\sigma, T_0\pi_\sigma(a)\xi_\sigma\rangle} \\ &= \overline{g(a^*b)} \\ &= g(b^*a)\end{aligned}$$

and so $T_0^* = T_0$. Moreover,

$$\langle T_0\pi_\sigma(a)\xi_\sigma, \pi_\sigma(a)\xi_\sigma\rangle = g(a^*a) \in [0, \|\pi(a)\xi\|^2]$$

which implies $0 \leq T_0 \leq 1$. Finally, notice that for all $a, b, c \in A$

$$\begin{aligned}\langle \pi_\sigma(a)\xi_\sigma, T_0\pi_\sigma(b)\pi_\sigma(c)\xi_\sigma\rangle &= \langle \pi_\sigma(a)\xi_\sigma, T_0\pi_\sigma(bc)\xi_\sigma\rangle \\ &= g((bc)^*a) \\ &= g(c^*(b^*a)) \\ &= \langle \pi_\sigma(b^*a)\xi_\sigma, T_0\pi_\sigma(c)\xi_\sigma\rangle \\ &= \langle \pi_\sigma(a)\xi_\sigma, \pi_\sigma(b)T_0\pi_\sigma(c)\xi_\sigma\rangle\end{aligned}$$

Since this holds for all a, c, it follows that $T_0\pi_\sigma(b) = \pi_\sigma(b)T_0$, so T_0 commutes with everything in $\pi_\sigma(A)$. We can't quite say $T_0 \in \pi_\sigma(A)'$, since T_0 is not an operator on all of H_σ. To remedy this, let P denote the projection onto $\overline{\pi_\sigma(A)\xi_\sigma}$, which by proposition A.3.5 is an operator in $\pi_\sigma(A)'$. Letting $T := T_0P$, we still have $0 \leq T \leq 1$, and $\langle \pi_\sigma(a)\xi_\sigma, T\pi_\sigma(b)\xi_\sigma\rangle$ is still equal to $g(b^*a)$, but now $T \in \pi_\sigma(A)'$. Choose an approximate identity $e_\lambda \in A$. Then

$$\begin{aligned}g(a) &= \lim_\lambda g((a^*)^*e_\lambda) \\ &= \lim_\lambda \langle \pi_\sigma(e_\lambda)\xi_\sigma, T\pi_\sigma(a^*)\xi_\sigma\rangle \\ &= \lim_\lambda \langle \pi_\sigma(ae_\lambda)T\xi_\sigma, \xi_\sigma\rangle \\ &= \theta_\sigma(T)(a)\end{aligned}$$

Thus, since $\pi_\sigma(A)'$ is spanned by its positive elements, we conclude that θ_σ is surjective.

Let's check now that θ_σ is completely positive. By proposition A.2.10, complete positivity of θ_σ is equivalent to the requirement that

$$\sum_{i,j} \theta_\sigma(r_i^* r_j)(a_i^* a_j) \geq 0$$

for all $r_i \in \pi_\sigma(A)'$, and $a_i \in A$, but this expression is easily seen to simplify as

$$\begin{aligned}\sum_{i,j} \theta_\sigma(r_i^* r_j)(a_i^* a_j) &= \sum_{i,j} \langle r_i^* \pi_\sigma(a_i^*) r_j \pi_\sigma(a_j)\xi_\sigma, \xi_\sigma\rangle \\ &= \sum_{i,j} \langle r_j \pi_\sigma(a_j)\xi_\sigma, r_i \pi_\sigma(a_i)\xi_\sigma\rangle \\ &= \left\| \sum_i r_i \pi_\sigma(a_i)\xi_\sigma \right\|^2 \geq 0\end{aligned}$$

and the result follows.

(Proof that θ_σ^{-1} is completely positive, similar in content). $\square$

We warn the reader that in general θ_σ^{-1} is *not* norm continuous (unlike our completely positive friends which map C*-algebras into C*-algebras, which are automatically continuous from birth), however it's composition with certain maps is guaranteed to be norm-continuous, as the following lemma makes clear.

Lemma 4.2.9. If A is a C*-algebra, B is a unital C*-algebra, $T : B \to A^*$ a completely positive map, and $T(1) = \sigma \in S(A)$ (that is, T is a complete state map), then $T(B) \subseteq C_\sigma$, and $\theta_\sigma^{-1} \circ \Phi : B \to \pi_\sigma(A)'$ is completely positive.

Proof. Given a positive element $b \in B^+$, since 1 is an order unit in B we have $b \leq \|b\|1$, whence $T(b) \leq \|b\|\sigma$, and thus $T(b) \in \{f \in A^* \mid 0 \leq f \leq \alpha\sigma \text{ for some } \alpha > 0\}$. Since $B = \operatorname{span} B^+$, it follows that $\operatorname{ran} T \subseteq C_\sigma$. That $\theta_\sigma^{-1} \circ T$ is completely positive is obvious, and moreover this map is unital, whence continuous. $\square$

Proof of theorem 4.2.6. ($\Leftarrow$) : If we have maps $\varphi_\lambda, \psi_\lambda$ as in the theorem statement, the composition $\tau_\lambda := \psi_\lambda \circ \varphi_\lambda$ is a net of unital, completely positive, finite rank maps point-norm approximating the identity. The adjoint maps $\sigma_\lambda := \tau_\lambda^*$ are also completely positive (see proposition A.2.11), finite rank contractions which point-weak-* approximate the identity on A^*. As in the proof of corollary 4.2.5 these maps allow us to point-weak-* approximate any completely positive contraction $T : B \to A^*$, and so A is necessarily nuclear.

($\Rightarrow$) : Assume A is nuclear. First we will show that the diagram of maps

$$\begin{tikzcd} & M_n \arrow[dr, "\psi"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "J_A"] & & A^{**} \end{tikzcd}$$

pointwise-ultra-weakly approximately commutes. To do this, we will start by identifying A with its image $\pi_u(A) \subseteq \mathcal{B}(H)$ under its universal representation, so that $A^{**} = A''$. Under this assumption, every state on A is a vector state, and every normal state on A'' is also a vector state (see fact 2.1.5).

First of all, it suffices to prove that for any $\epsilon > 0$, $a_1, ..., a_k \in A$, $\xi_1, ..., \xi_\ell \in H$, there are maps φ, ψ such that

$$|\langle(\psi(\varphi(a_i)) - a_i)\xi_j, \xi_j\rangle| < \epsilon$$

This follows from precisely the same reasoning as in the proof of lemma 4.2.1.

Consider the projections $p_i := P_{\overline{A'\xi_i}}$. By proposition A.3.5, each $p_i \in A''$, and so their least upper bound $p := \vee_{i=1}^{\ell} p_i$ is also in A''.

The projection p is countably decomposable (see A.3.6). By theorem A.3.9, that every normal state on A'' is a vector state is equivalent to the statement that every countably decomposable projection in A'' possesses a separating vector, which then implies (by A.3.4 and A.3.8) that $(pA''p)' = pA'''p = pA'p$ possesses a cyclic vector. Call this vector ξ.

Let R denote the von Neumann algebra $pA'p$. As ξ is a cyclic vector for R, we see that it now suffices to show that for all $a_1, ..., a_k \in A$, $r_1, ..., r_\ell \in R$, that there are φ, ψ

$$|\langle(\psi(\varphi(a_i)) - a_i)r_j\xi, r_j\xi\rangle| < \epsilon \tag{†}$$

Define, first, the normal state $\rho \in R_*$, $\rho(r) = \langle r\xi, \xi\rangle$. Next define $\eta : A \to R'$ by $a \mapsto pap$, and let $\theta_\rho : R' \to C_\rho \subseteq R_*$ be the completely positive isomorphism defined previously. The map $\theta_\rho \circ \eta : A \to C_\rho \subseteq R_*$ is completely positive and contractive, so under the assumption that A is nuclear, by theorem 4.2.3 there are φ, ψ such that $\|\varphi \circ \psi\| \leq 1$ and

$$\begin{tikzcd} & M_n \arrow[dr, "\psi"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\theta_\rho \circ \eta"] & & R_* \end{tikzcd}$$

pointwise weak-* approximately commutes. By lemma 4.2.1, we can obtain point-norm approximation automatically. Now it may seem that we've magically made the codomain of ψ into R_*, rather than R^* as we might expect given the statement of 4.2.3. If one goes back to the original definition of ψ, it is clear that $\psi([a_{ij}])$ is in fact the finite sum of vector states, which are automatically ultraweakly continuous. Thus, we get $\psi : M_n \to R_*$ "for free".

The next adjustment we can make is ensuring that φ is unital. By lemma A.2.24, there exists a completely positive unital map $\widetilde{\varphi} : A \to M_n$ such that $\varphi(a) = b^{1/2}\widetilde{\varphi}(a)b^{1/2}$, where $b = \varphi(1)$. Define $\psi'(a) := \psi(b^{1/2}ab^{1/2})$. Then ψ' is still completely positive, and

$$\psi'(\widetilde{\varphi}(a)) = \psi(\varphi(a))$$

so these maps still point-weak-* approximately factor $\theta_\rho \circ \eta$

Moreover, we can ensure that $\varphi(1) = 1$ (via lemma A.2.24), and $\psi(1) = \rho$, so that $\psi(M_n) \subseteq C_\rho$. Thus, the following diagram point-norm approximately commutes

$$\begin{tikzcd} & M_n \arrow[dr, "\widetilde{\psi}"] & & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\eta"] & & pA''p \arrow[r, "\subseteq"] & A'' \end{tikzcd}$$

where we have labelled $\widetilde{\psi} := \theta_\rho^{-1} \circ \psi$.

At this point it may seem as though something has gone awry - we're approximating the map η, rather than the identity. Fortunately, we're only looking for maps such that condition (†) above is satisfied. Since $1 \in A'$, we have $\xi \in \overline{A'\xi}$, so $p\xi = \xi$. Moreover, for each $r_j \in R$ we can write $r_j = pb_jp$ for some $b_j \in A'$, so $r_j = pr_j$ since $p^2 = p$. Thus,

$$\langle a_ir_j\xi, r_j\xi\rangle = \langle a_ipr_j\xi, pr_j\xi\rangle = \langle pa_ipr_j\xi, r_j\xi\rangle = \langle \eta(a)r_j\xi, r_j\xi\rangle$$

whence

$$\begin{aligned} |\langle(\widetilde{\psi}(\varphi(a_i)) - a_i)r_j\xi, r_j\xi\rangle| &= |\langle(\widetilde{\psi} \circ \varphi - \eta)(a_i)r_j\xi, r_j\xi\rangle| \\ &\leq \underbrace{\|\widetilde{\psi} \circ \varphi(a_i) - \eta(a_i)\|}_{\text{can be made arbitrarily small}} \quad \|r_j\xi\| \end{aligned}$$

So we have completed the first step of the proof. The next step is to show that a completely positive contraction $\psi : M_n \to A^{**}$ can be replaced by a net of completely positive contractions $\psi_\nu : M_n \to A$.

Let $\varphi_\lambda : A \to M_{n_\lambda}$, $\psi_\lambda : M_{n_\lambda} \to A^{**}$ be nets of unital, completely positive maps which point-ultraweakly approximate $J_A : A \to A^{**}$. Using the notation of corollary A.2.17, there is a one-to-one correspondence between $CP(M_n, A^{**})$ and $M_n(A^{**})_+$, denoted Φ. As $M_n(A)$ is weak-* dense in $M_n(A)^{**}$ (which is just $M_n(A^{**})$ by proposition 2.2.13), by Kaplansky's density theorem we can find a bounded net $\tau_\mu^\lambda \in M_{n_\lambda}(A)_+$ (indexed by μ) which converges weak-* to $\Phi(\psi_\lambda)$ (for a refresher on why this is a consequence of Kaplansky's density theorem, see the proof of proposition 2.2.14). It is not hard to see that weak-* convergence of

$\tau_\mu^\lambda \to \Phi(\psi_\lambda)$ corresponds to point-ultraweak convergence of $\Phi^{-1}(\tau_\mu^\lambda) \to \psi_\lambda$. Indeed, letting $\psi_\mu^\lambda := \Phi^{-1}(\tau_\mu^\lambda)$, we have[9]

$$\psi_\mu^\lambda([c_{ij}]) = \sum_{ij} c_{ij}(\tau_\mu^\lambda)_{ij} \xrightarrow{wk-*} \sum_{ij} c_{ij}\Phi(\psi_\lambda)_{ij} = \psi_\lambda([c_{ij}])$$

Let $\varphi_\mu^\lambda := \varphi_\lambda$ for all μ, and reindexing our nets $(\varphi_\mu^\lambda, \psi_\mu^\lambda)$ by the product of the index sets of λ and μ, we obtain new nets (φ_ν, ψ_ν) for which φ_ν is unital, and the following diagram approximately point-ultraweakly commutes.

$$\begin{tikzcd} & M_{n_\nu} \arrow[dr, "\psi_\nu"] & \\ A \arrow[ur, "\varphi_\nu"] \arrow[rr, "\mathrm{id}_A"] & & A \end{tikzcd}$$

Since the restriction of the ultraweak (equiv. weak-*) topology on A^{**} to A is simply the weak topology, we see that this diagram approximately point-weakly commutes. Of course, seeing as the set of all such compositions of completely positive maps $\varphi : A \to M_n$, $\psi : M_n \to A$ for which $\varphi(1) = 1$ is convex, by lemma 4.2.1 we can assume this diagram approximately point-norm commutes.

We make one final adjustment. Seeing as ψ_ν is completely positive, we have $\|\psi_\nu\| = \|\psi_\nu(1)\| = \|\psi_\nu \circ \varphi_\nu(1)\| \to 1$. Thus, by replacing ψ_ν with $\psi_\nu/\|\psi_\nu\|$ (noting that these maps still point-norm approximate the identity, as in the proof of $1' \implies 1$ in theorem 4.2.3), we can assume without loss of generality that the maps ψ_ν are contractive.

In conclusion, under the assumption that A is nuclear, we can exhibit nets of completely positive maps $\varphi_\lambda : A \to M_{n_\lambda}$, $\psi_\lambda : M_{n_\lambda} \to A$, such that φ_λ is unital, ψ_λ is contractive, and $\psi_\lambda \circ \varphi_\lambda$ point-norm approximates the identity on A. $\square$

Corollary 4.2.10. A unital C*-algebra A is nuclear if and only if there exists a net $\sigma_\lambda \in \mathcal{B}(A)$ of completely positive, finite rank contractions such that σ_λ point-norm approximates the identity. As prior, we can equivalently require each σ_λ have norm 1.

For the sake of completeness, let us at least state the non-unital version of this characterization of nuclearity, which as we mentioned above follows from relatively unremarkable modifications of the unital case.

Theorem 4.2.11. A C*-algebra A is nuclear if and only if there exist nets $\varphi_\lambda : A \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to A$ of completely positive contractions which point-norm approximate the identity on A.

Using this, we can now examine many of the examples of nuclear C*-algebras presented in the previous section.

Corollary 4.2.12. Commutative C*-algebras are nuclear.

Proof. Every commutative C*-algebra is of the form $C_0(X)$ where X is a locally compact Hausdorff space (the *spectrum* of X, or equivalently the maximal ideal space of X). This proof makes pretty heavy use of well-known topological terminology, all of which can be found in [30], in particular chapters 3 and 4.

We will construct a net of completely positive finite rank contractions $\varphi_\lambda : C_0(X) \to C_0(X)$ which point-norm approximate the identity. Seeing as these maps are finite rank, for each λ there are functionals $c_i^\lambda \in C_0(X)^*$ and functions $g_i^\lambda \in C_0(X)$ such that

$$\varphi_\lambda(f) = \sum_i c_i^\lambda(f) g_i^\lambda$$

We might take a stab in the dark that $c_i^\lambda(f) = f(x_i^\lambda)$ for some $x_i^\lambda \in X$, in which case our maps are

$$\varphi_\lambda(f) = \sum_i f(x_i^\lambda) g_i^\lambda$$

[9]We remark that if M is a von Neumann algebra, and $a_\lambda \in M_n(M)$ converges $\sigma - wot$ to $a \in M_n(M)$, then it is easy to see that each component $(a_\lambda)_{ij}$ must also converge $\sigma - wot$ in M to a_{ij}.

so the g_i^λ's sort of "interpolate" f pointwise. Perhaps, then, letting g_i^λ be an aptly chosen partition of unity would work?

Our index set will be the set of tuples

$$I := \{(F, \epsilon) \mid F \subseteq C_0(X) \text{ finite}, \ \epsilon > 0\}$$

ordered by defining

$$(F, \epsilon) \leq (F', \epsilon') \quad \iff \quad F \subseteq F', \text{ and } \epsilon \geq \epsilon'$$

Given some $F \subseteq C_0(X)$ finite and $\epsilon > 0$ there exists a compact $K \subseteq X$ such that $\|f - f|_K\| < \epsilon$ for all $f \in F$. We will restrict our attention to K for the moment.

Given a point $x \in X$, define $U_x := \{y \in X \mid |f(x) - f(y)| < \epsilon \ \forall f \in F\}$. These sets are open, and since K is compact there exist finitely many $\{x_1, ..., x_m\}$ such that $K \subseteq \bigcup_{i=1}^m U_{x_i}$. Additionally, since X is locally compact, around each point $x \in K$ there is a compact neighbourhood K_x containing an open set V_x containing x. Again, we can choose finitely many $V_x \subseteq K_x$ which cover K, relabelling them as $V_j \subseteq K_j$, $j = 1, ..., n$. Note that $K \subseteq \widetilde{K} := \bigcup_{j=1}^n K_j$, which is itself compact. Consider, finally, the open cover $\{U_{x_i} \cap V_j\}_{i=1,j=1}^{m,n}$ of K. As K is compact Hausdorff, we can construct a *partition of unity subordinate to this cover* (see [30]; Theorem 36.1): that is, a collection of functions $g_{ij} \in C(K)$ such that $\sum_{ij} g_{ij} = \mathrm{id}_K$, $0 \leq g_{ij} \leq 1$, and $\operatorname{supp} g_{ij} \subseteq U_{x_i} \cap V_j$.

The point of the extra cover V_j is so that we can assume without loss of generality that each $g_{ij} \in C_0(X)$, and $\sum_{ij} g_{ij}|_K = \mathrm{id}_K$. The following lemma makes this clear.

Lemma. Let $U, K \subseteq \widetilde{K}$, where U is open, $U \cap K \neq \emptyset$, K and $\widetilde{K}$ are compact, and $\widetilde{K}$ is Hausdorff. Given $g \in C(K)$ such that $\operatorname{supp} g \subseteq U \cap K$, and $0 \leq g \leq 1$, there exists $\widetilde{g} \in C(\widetilde{K})$ such that $\widetilde{g}|_K = g$, $0 \leq \widetilde{g} \leq 1$, and $\operatorname{supp} g \subseteq \operatorname{supp} \widetilde{g} \subseteq U$.

Proof. First remark that $\widetilde{K}$ is compact Hausdorff, whence normal, allowing us to use Tietze's extension theorem and Urysohn's lemma. By Tietze's extension theorem there exists $\widehat{g} \in C(\widetilde{K})$ such that $\widehat{g}|_K = g$, and $0 \leq \widehat{g} \leq 1$. By Urysohn's lemma, there exists $\eta \in C(\widetilde{K})$ such that $\eta|_{\operatorname{supp} g} = 1$, $\eta|_{\widetilde{K} \setminus U} = 0$ (remarking that $\operatorname{supp} g$ is closed in K which is itself compact in $\widetilde{K}$, whence $\operatorname{supp} g$ is closed in $\widetilde{K}$ as well). Then $\widetilde{g} := \widehat{g}\eta$ fits the bill. $\square$

For each g_{ij}, this lemma gives us a $\widetilde{g}_{ij} \in C(\widetilde{K})$ which is zero outside of $U_i \cap V_j$, so we might as well extend $\widetilde{g}_{ij}$ to all of X by setting it equal to zero. Thus $\widetilde{g}_{ij} \in C_0(X)$.

Given $f \in C_0(X)$, it is then easy to verify by looking at $x \in K$ and $x \notin K$ that for all $f \in F$.

$$\left\| f - \sum_{i,j} f(x_i) \widetilde{g}_{ij} \right\| < \epsilon$$

We are finally ready to construct our completely positive, finite rank contractions. For each $\lambda := (F, \epsilon) \in I$, define

$$\varphi_\lambda(f) = \sum_{i,j} f(x_i) \widetilde{g}_{ij}$$

where the x_i's and $\widetilde{g}_{ij}$'s are chosen according to our construction above. Clearly if $f \geq 0$ then $\varphi_\lambda(f) \geq 0$, so φ_λ is positive, but by theorem A.2.5 this is enough to conclude that the φ_λ's are completely positive. Finally, the inequality above tells us that for all $f \in F$,

$$\|f - \varphi_\lambda(f)\| < \epsilon$$

and so taking the limit over $\lambda \in I$ gives us the desired result: a point-norm approximation of the identity by completely positive, finite rank maps. $\square$

Remark. This example shows us that a pair of a net of completely positive finite rank contractions $\varphi_\lambda : A \to A$ point-norm approximating the identity can be thought of as a "non-commutative partition of unity". We also remark that we could have simply proven that $C(X)$ was nuclear for some compact Hausdorff space X rather than $C_0(X)$, since the unitization of $C_0(X)$ is simply $C(\alpha X)$, αX denoting the one-point compactification of X, but I prefer the proof above as I enjoy concocting bizarre and contrived topological arguments.

Lemma 4.2.13 (Local-to-Global Property of Nuclearity)**.** A C*-algebra A is nuclear if and only if for any finite set $F \subset A$ and $\epsilon > 0$ there is a nuclear C*-subalgebra $B \subseteq A$ such that for all $a \in F$, $\operatorname{dist}(a, B) < \epsilon$.

Proof. For each nuclear C*-subalgebra $B \subseteq A$, let $\varphi_\lambda^B : B \to M_{n_\lambda^B}$, $\psi_\lambda^B : M_{n_\lambda^B} \to B$ be nets of completely positive contractions point-norm approximating the identity on B. By Arveson's extension theorem we can regard φ_λ^B as a completely positive contraction on all of A. We can also regard ψ_λ^B as simply mapping into A.

Let $I := \{(F, \epsilon) \mid F \subseteq A \text{ finite}, \ \epsilon > 0\}$, ordered by $(F, \epsilon) \leq (F', \epsilon')$ if and only if $F \subseteq F'$ and $\epsilon \geq \epsilon'$. Given $\mu := (F, \epsilon) \in I$, choose $B \subseteq A$ a nuclear C*-subalgebra such that $\operatorname{dist}(a, B) < \epsilon/3$ for all $a \in F$. Next choose $G \subseteq B$ finite such that for all $a \in A$, there exists $b \in G$ such that $\|a - b\| < \epsilon/3$. Finally, set

$$\widetilde{\varphi}_\mu := \varphi_\lambda^B : A \to M_n, \qquad \widetilde{\psi}_\mu := \psi_\lambda^B : M_n \to A$$

where λ is chosen large enough such that for all $b \in G$, $\|b - \psi_\lambda^B \circ \varphi_\lambda^B(b)\| < \epsilon/3$.

Then for all $a \in F$,

$$\|a - \widetilde{\varphi}_\mu(\widetilde{\psi}_\mu(a))\| \leq \|a - b\| + \|b - \varphi_\lambda^B(\psi_\lambda^B(b))\| + \|\varphi_\lambda^B(\psi_\lambda^B(b - a))\| < \epsilon$$

so that $\widetilde{\varphi}_\mu, \widetilde{\psi}_\mu$, for $\mu \in I$, is our net approximately factoring the identity on A. □

Corollary 4.2.14. The direct limit of nuclear C*-algebras is nuclear.

Proof. Recall that a direct system consists of a sequence (or net) of objects (in our case C*-algebras) $(A_i)_{i \in I}$ and a corresponding sequence (resp. net) of *adjoining maps* $f_{ij} : A_i \to A_j$ whenever $i \leq j \in I$ (which in our case are *-homomorphisms), such that $f_{ii} = \mathrm{id}_{A_i}$ and $f_{ij} \circ f_{jk} = f_{ik}$ for all $i \leq j \leq k \in I$. Define an equivalence relation $\sim$ on $\bigsqcup_{i \in I} A_i$ by letting $a_i \in A_i$ and $a_j \in A_j$ be equivalent under $\sim$ if and only if there is some $k \in I$ with $i, j \leq k$ and $f_{ik}(a_i) = f_{jk}(a_j)$. For $a \in \bigsqcup_{i \in I} A_i$, denote $[a_i]$ it's equivalence class under $\sim$. Then $\varinjlim_{i \in I}^{\circ} A_i := \bigsqcup_{i \in I} A_i / \sim$ has a *-algebra structure. Given $[a], [b] \in \varinjlim_{i \in I}^{\circ} A_i$, there exists $i, j \in I$ such that $a \in A_i, b \in A_j$, and by upward directedness of I there is $k \in I$ with $k \geq i, j$. So, for any $\lambda \in \mathbb{C}$ we define

$$[a] + \lambda[b] := [f_{ik}(a) + \lambda f_{jk}(b)], \quad [a][b] := [f_{ik}(a) f_{jk}(b)], \quad [a]^* := [a^*]$$

We then denote by $A := \varinjlim_{i \in I} A_i := \overline{\varinjlim_{i \in I}^{\circ} A_i}^{\|\cdot\|}$, the *C*-direct limit* of this system, which is clearly a C*-algebra.

Let (A_i, f_{ij}) be the direct system for which $A = \varinjlim A_i$ is the direct limit. Given a finite set $F = \{a_1, ..., a_n\} \subseteq A$ and $\epsilon > 0$, there are elements $\{b_1, ..., b_n\} \subseteq \bigsqcup_i A_i$ such that $\|a_i - b_i\| < \epsilon$.

Essentially by definition, for every finite set $F \subseteq A$ and $\epsilon > 0$, there is an algebra A_i such that $\operatorname{dist}(a, A_i) < \epsilon$ for all $a \in F$. Since A_i is nuclear, A satisfies the conditions of lemma 4.2.13, so A is nuclear. □

Corollary 4.2.15. For any Hilbert space H, the algebra $\mathcal{K}(H)$ of compact operators is nuclear.

Proof. The case for H finite-dimensional is obvious ($\mathcal{K}(H) = M_n(\mathbb{C})$ where $n = \dim H$), so assume H is infinite-dimensional. Let $\mathcal{F} := \{P \in \mathcal{B}(H) \ : \ P \text{ a finite rank projection}\}$. For $P_1, P_2 \in \mathcal{F}$, define $P_1 \leq P_2$ iff $\operatorname{ran} P_1 \subseteq \operatorname{ran} P_2$ (for projections, this is the same as the usual order on $\mathcal{B}(H)$). This turns $\mathcal{F}$ into a directed set. Moreover this turns the family $\{P\mathcal{B}(H)P \ : \ P \in \mathcal{F}\}$ into a direct system of C*-algebras, with the adjoining maps from $P_1\mathcal{B}(H)P_1$ to $P_2\mathcal{B}(H)P_2$ taking $T \mapsto \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ (decomposing $\operatorname{ran} P_2 = \operatorname{ran} P_1 \oplus \operatorname{ran} P_1^\perp$). Then $\mathcal{K}(H)$ is equal to the direct limit of this system. Since each $P\mathcal{B}(H)P \cong M_n(\mathbb{C})$, where $n = \dim \operatorname{ran} P$, we thus see that $\mathcal{K}(H)$ is a direct limit of nuclear C*-algebras, and so is itself nuclear. □

Corollary 4.2.16. If G is an amenable group, then $C_\lambda^*(G)$ is nuclear.

Proof. The argument presented can be found in Brown & Ozawa's book [5]. Unlike Brown & Ozawa, we include lurid detail of some calculations.

We will show that $C^*_\lambda(G)$ is nuclear by exhibiting a factoring net for the identity - a pair of nets of unital, completely positive maps

$$C^*_\lambda(G) \overset{\varphi_k}{\to} M_{n_k} \overset{\psi_k}{\to} C^*_\lambda(G)$$

such that $\|x - \psi_k(\varphi_k(x))\| \to 0$ for all $x \in C^*_\lambda(G)$.

Let F_k be a Følner sequence for G, and define $P_k \in \mathcal{B}(\ell^2(G))$ to be the projection onto the space $\operatorname{span}\{\delta_g \mid g \in F_k\}$. The corner algebra $P_k\mathcal{B}(\ell^2(G))P_k$ is canonically isomorphic to $M_{|F_k|}$. We define $\varphi_k : \mathcal{B}(\ell^2(G)) \to M_{|F_k|}$ by $T \mapsto P_kTP_k$ - this completely positive contraction will be the first half of our factoring net.

Given $g, h \in F_k$, denote by $e_{g,h} \in \mathcal{B}(\ell^2(G))$ the operator

$$e_{g,h}f := \langle f, \delta_h\rangle\delta_g = f(h)\delta_g$$

The operators $e_{g,h}$ form the "canonical matrix basis" for $P_k\mathcal{B}(\ell^2(G))P_k$, since for every $T \in \mathcal{B}(\ell^2(G))$ a simple calculation reveals

$$\varphi_k(T) = P_kTP_k = \sum_{g,h\in F_k} \langle T\delta_h, \delta_g\rangle e_{g,h}$$

so that $\varphi_k(T)$ is a span of the $e_{g,h}$'s. Since $P_k = \sum_g e_{g,g}$, we also have

$$\varphi_k(T) = \sum_{g,h\in F_k} e_{g,g}Te_{h,h}$$

Which will come in handy for our calculations.

We are curious particularly about evaluating $\varphi_k(T)$ in the case $T = \lambda(s)$. Since $\operatorname{span}\{\lambda(s) \mid s \in G\}$ is dense in $C^*_\lambda(G)$, if we can check that $\lim_{k\to\infty} \|\lambda(s) - \psi_k(\varphi_k(s))\| = 0$, then by density $\psi_k \circ \varphi_k$ approximates the identity on all of $C^*_\lambda(G)$.

So let us evaluate $\varphi_k(\lambda(s))$. Given $f \in \ell^2(G)$, we evaluate for $t \in F_k$

$$\begin{aligned}
(\varphi_k(\lambda(s))f)(t) &= \sum_{g,h}(e_{g,g}\lambda(s)e_{h,h}f)(t)\\
&= \sum_{g,h}(\lambda(s)e_{h,h}f)(g)\delta_g(t)\\
&= \sum_{h}(\lambda(s)e_{h,h}f)(t)\\
&= \sum_{h}(e_{h,h}f)(s^{-1}t)\\
&= \sum_{h} f(h)\delta_h(s^{-1}t) = \begin{cases} f(s^{-1}t) & \text{if } t \in sF_k\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

By definition, $\varphi_k(\lambda(s)f \in \operatorname{span}\{\delta_g \mid g \in F_k\}$, so $(\varphi_k(\lambda(s))f)(t) = 0$ for $t \notin F_k$. Thus, we can write

$$\begin{aligned}
\varphi_k(\lambda(s))f &= \sum_{t\in F_k}(\varphi_k(\lambda(s))f)(t)\delta_t\\
&= \sum_{t\in F_k\cap sF_k} f(s^{-1}t)\delta_t\\
&= \sum_{t\in F_k\cap sF_k} e_{s^{-1}t,t}f
\end{aligned}$$

and so, as operators

$$\varphi_k(\lambda(s)) = \sum_{t\in F_k\cap sF_k} e_{s^{-1}t,t}$$

The usefulness of the Følner condition on F_k's should now start to become apparent.

On to the second half of the factoring net, which is simpler to define. Given $g, h \in F_k$, simply define $\psi_k(e_{g,h}) := \frac{1}{|F_k|}\lambda(hg^{-1})$. To see that this map is completely positive, we merely invoke Choi's theorem A.2.16. ψ_k is completely positive if and only if $[\psi_k(e_{g,h})]_{g,h\in F_k}$ is positive in $M_{n_k}(C^*_\lambda(G))$, but $[\psi_k(e_{g,h})]_{g,h\in F_k} = \frac{1}{|F_k|}[\lambda(h)\lambda(g)^*]_{g,h\in F_k}$, which by fact A.2.1 is positive.

Now we must verify that (φ_k, ψ_k) is indeed a factoring net, so that for all $x \in C^*_\lambda(G)$, $\|x - \psi_k(\varphi_k(x))\| \to 0$. Since $\{\lambda(s) \mid s \in G\}$ spans a dense subset of $C^*_\lambda(G)$, we need merely check that this is true for $x = \lambda(s)$ (the remaining x follow from density plus the triangle inequality).

Given $s \in G$, we calculate

$$\begin{aligned}\psi_k(\varphi_k(\lambda(s))) = \psi_k\left(\sum_{t\in F_k\cap sF_k} e_{s^{-1}t,t}\right) &= \frac{1}{|F_k|}\sum_{t\in F_k\cap sF_k}\lambda(t(t^1 s)) \\ &= \frac{1}{|F_k|}\sum_{t\in F_k\cap sF_k}\lambda(s)\end{aligned}$$

and so

$$\|\lambda(s) - \psi_k(\varphi_k(\lambda(s)))\| = 1 - \frac{|F_k \cap sF_k|}{|F_k|} \to 0$$

□

In time we will see that this actually characterizes amenability of G.

The final result we present in this section relating to semidiscreteness is the following "norm equivalence" characterization of semidiscreteness, which mirrors the "norm equivalence" definition of nuclearity.

Theorem 4.2.17. Suppose R is a von Neumann algebra. Then the following are equivalent:

1. R is semidiscrete.
2. $R \otimes_{\text{nor}} B = R \otimes_{\min} B$ for all C*-algebras B.
3. $R \otimes_{\text{bin}} S = R \otimes_{\min} S$ for all von Neumann algebras S.

Proof. The proof we present here is an adaptation of the proof of theorem 4.1 in [16]. We will only prove that $1 \implies 2 \implies 3$, as we will not need $3 \implies 1$ going forward.

Let us first see that if R is semidsicrete, then we have property 2. We will then use this to derive property 3.

Recall that

$$\begin{aligned}\|x\|_{\min} &= \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in (R^* \odot B^*) \cap S(R \odot B)\} \\ \|x\|_{\text{nor}} &= \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in \Gamma_{\text{nor}}(R \odot B)\}\end{aligned}$$

From this, we see that we have property 2 whenever $(R^* \odot B^*) \cap S(R \odot B)$ is weak-* dense in $\Gamma_{\text{nor}}(R \odot B)$.

If R is semidiscrete, we can exhibit a net $\varphi_\lambda \in \mathcal{B}(R)$ of normal, unital, completely positive, finite rank maps, which approximate the identity pointwise-ultraweakly. Given a state $\rho \in \Gamma_{\text{nor}}(R \odot B)$, define a net ρ_λ of positive linear functionals by setting

$$\rho_\lambda(r \otimes b) := \rho(\varphi_\lambda(r) \otimes b)$$

Then it is easily seen that each ρ_λ is a state, and since φ_λ is finite rank we also have $\rho_\lambda \in R^* \odot B^*$. Finally, since $\rho(\cdot \otimes b)$ is ultraweakly continuous for fixed b, we have

$$\lim_\lambda(\rho_\lambda(r \otimes b) - \rho(r \otimes b)) = \lim_\lambda \rho(\varphi_\lambda(r) \otimes b - r \otimes b) = 0$$

or in other words, ρ is the weak*-limit of the ρ_λ's, proving property 2.

Property 3 follows from property 2 simply by noting that the bin-norm is dominated by the nor-norm, so if R is semidiscrete then

$$\|\cdot\|_{R\otimes_{\min}S} \leq \|\cdot\|_{R\otimes_{\text{bin}}S} \leq \|\cdot\|_{R\otimes_{\text{nor}}S} = \|\cdot\|_{R\otimes_{\min}S}$$

□

4.3 Injectivity

Definition 4.3.1. A unital C*-algebra A is said to be **injective** if for any other unital C*-algebras $B \subseteq C$, and a completely positive $\varphi : B \to A$, there exists a completely positive $\widetilde{\varphi} : C \to A$ for which $\widetilde{\varphi}|_B = \varphi$. In other words, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & C \arrow[d, dashed, "\widetilde{\varphi}"] \\ & A \end{tikzcd}$$

A von Neumann algebra M is similarly said to be injective if it is injective as a C*-algebra.

Remark. Since the definition of an injective C*-algebra requires that our maps φ and $\widetilde{\varphi}$ be completely positive maps between *unital* spaces, it follows that $\|\varphi\| = \varphi(1) = \widetilde{\varphi}(1) = \|\widetilde{\varphi}\|$.

Remark. There is another definition which appears in the literature, which we might call "operator system injectivity". That is, a C*-algebra A is *operator system injective* if for any operator system E contained in a unital C*-algebra B, and a completely positive $\varphi : E \to A$, there exists a completely positive $\widetilde{\varphi} : B \to A$ which extends φ. We might call definition 4.3.1 "C*-algebra injectivity". It is clear that operator system injectivity implies C*-algebra injectivity, but the converse is not so clear, nor does it even seem apparent that it should be true. For the remainder of this text, we will be using "injective" to mean "C*-algebra injective" unless otherwise specified, which is the definition used by Choi, Lance, Effros, Connes, etc., which are the main sources for this text.

Remark. We can assume without loss of generality that the map φ is unital in definition 4.3.1. Given a completely positive $\varphi : B \to A$, let $b = \varphi(1)$. By lemma A.2.24, there exists a unital completely positive $\psi : B \to A$ such that $\varphi(a) = b^{1/2}\psi(a)b^{1/2}$. Then by assumption, ψ extends to a unital, completely positive $\widetilde{\psi} : C \to A$. Defining $\widetilde{\varphi} : C \to A$ by $\widetilde{\varphi}(a) = b^{1/2}\widetilde{\psi}(a)b^{1/2}$, we obtain a completely positive extension of φ.

Example. Arveson's extension theorem is precisely the statement that $\mathcal{B}(H)$ is injective (in fact it is "operator system injective").

Theorem 4.3.2. Let $(A_i)_{i\in I}$ be a family of unital C*-algebras. Then $A := \oplus_i A_i$ is (operator system)-injective if and only if each direct summand A_i is (operator system)-injective (respectively).

Proof. Suppose first that each A_i is injective. If $E \subseteq B$ is an operator system contained in a unital C*-algebra, and $\varphi : E \to A$ a completely positive map, then letting $\pi : A \to A_i$ be the projection maps, $\pi_i \circ \varphi : E \to A_i$ are completely positive. By injectivity of each A_i, these maps extend to completely positive maps $\widetilde{\varphi}_i : B \to A_i$. Putting them back together, we get a completely positive $\widetilde{\varphi} = \oplus_i \widetilde{\varphi}_i : B \to A$ (given by $\widetilde{\varphi}(b) = \oplus_i \widetilde{\varphi}_i(b)$), with

$$(\oplus_i \widetilde{\varphi}_i)|_E = \oplus_i (\widetilde{\varphi}_i|_E) = \oplus_i (\pi_i \circ \varphi) = \varphi$$

Now suppose A is injective. Let $\iota_i : A_i \to A$ be the inclusion map. Given $\varphi_i : E \to A_i$ a completely positive contraction, by injectivity of A, the map $\iota \circ \varphi_i : E \to A$ extends to a map $\widetilde{\varphi} : B \to A$, for which

$$(\pi_i \circ \widetilde{\varphi}_i)|_E = \pi_i \circ \iota_i \circ \varphi_i = \varphi_i$$

so that $\widetilde{\varphi}_i := \pi_i \circ \widetilde{\varphi} : B \to A_i$ is a completely positive extension of φ_i. □

Theorem 4.3.3. Let M be a von Neumann algebra. Then M is (operator system)-injective if and only if M' is (operator system)-injective (respectively).

In order to prove this, we'll borrow an extremely helpful result from an arcane school of von Neumann algebra techniques:

Theorem 4.3.4 (Tomita-Takesaki, Standard Form of a von Neumann Algebra)**.** Let R be a von Neumann algebra. Then there exists a Hilbert space H and a von Neumann algebra $M \subseteq \mathcal{B}(H)$ isometrically isomorphic to R, along with a conjugate linear isometric involution J of H, such that

1. $JMJ = M'$
2. The map $Y : M \to M'$ taking $m \mapsto JmJ$ is an isometric bijection.

Example. Consider the von Neumann algebra $\mathcal{M} = M_2 \oplus M_2 \subseteq M_4$ (that is, $\mathcal{M}$ consists of 2×2-block diagonal matrices). First, we can endow $\mathcal{M}$ with an inner product, with respect to which $\mathcal{M}$ is a Hilbert space:

$$\langle A, B\rangle = \operatorname{tr}(B^*A)$$

We can embed $\mathcal{M}$ in $\mathcal{B}(\mathcal{M})$ via the following injective *-homomorphism:

$$\begin{array}{cccc} \pi & : & \mathcal{M} & \to & \mathcal{B}(\mathcal{M}) \\ & & A & \mapsto & (X \mapsto AX) \end{array}$$

Then $\pi(A)'$ consists of those operators $L \in \mathcal{B}(\mathcal{M})$ such that $L \circ \pi(A) = \pi(A) \circ L$ for all A. In particular, $L(A) = L(\pi(A)(I)) = \pi(A)(L(I)) = AL(I)$. Thus, $\pi(A)' = \pi(A)$.

Example. In more generality, if M is a σ-finite von Neumann algebra, then M admits a faithful normal state ϕ (this is a relatively straightforward exercise following theorem A.3.7), which we can use to define an inner product on M:

$$\langle a, b\rangle := \phi(b^*a)$$

Let $H_M := \overline{M}$ denote the Hilbert space completion of M with respect to this inner product. We can then represent R as acting faithfully on H_M via the injective *-homomorphism

$$\begin{array}{cccc} \pi & : & M & \to & \mathcal{B}(H_M) \\ & & a & \mapsto & (x \mapsto ax) \end{array}$$

Note that the map $x \mapsto ax$ is defined for $x \in M$, and only extends to H_M assuming it is continuous with respect to the inner product induced norm. This, however, is not hard to check. If $\|a\| \leq 1$, then $a^*a \leq 1$, and so $x^*(1 - a^*a)x \geq 0$ for all $x \in M$. Thus

$$\phi(x^*(1 - a^*a)x) \geq 0$$

or equivalently

$$\langle x, x\rangle^{1/2} \geq \langle ax, ax\rangle^{1/2}$$

so $x \mapsto ax$ does indeed extend to a well-defined operator in $\mathcal{B}(H_M)$.

If $H_M = M$ (for instance if M is finite dimensional), by precisely the same logic as in the prior example, $\pi(M)' = \pi(M)$. In general, however, $M \subsetneq H_M$, yet one may always exhibit a conjugate linear isometric involution J on H_M for which $J\pi(M)J = \pi(M)'$. This result, the seminal work of Tomita-Takesaki, is significantly harder to prove.

For a more complete exposition on this incredibly deep area of study, see [22] section 4, or [19].

A straightforward calculation shows a *conjugate* linear map T admits a unique *adjoint* conjugate linear map T^* such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, and furthermore if T is an isometric involution then $T = T^*$. Using this it is easy to see that the maps Y and Y^{-1} are in fact "completely positive preserving" (it *cannot* be "traditionally" completely positive since it isn't even linear). If we define $Z(m) := Y(m)^*$, then Z is a "traditional" completely positive map.

Proof of Theorem 4.3.3. It suffices to show that if M is injective, then M' is injective. We can also assume without loss of generality that M is in "standard form" (that is M is as in theorem 4.3.4), and so we have a conjugate linear, completely positive preserving, isometric bijection $Y : M \to M'$.

Let $E \subseteq A$ be an operator system in a unital C*-algebra, and let $\varphi : E \to M'$ be a completely positive map. Then $Y^{-1} \circ \varphi : E \to M$ is conjugate linear, completely positive preserving. Define a completely positive map $Z : E \to M$ by $Z(e) := Y^{-1}(\varphi(e))^*$. Then Z extends to a completely positive $\widetilde{Z} : A \to M$.

Define $W : A \to M$ by $W(a) := \widetilde{Z}(a)^*$. W is conjugate linear, completely positive preserving, and for all $e \in E$,

$$W(e) = \widetilde{Z}(e)^* = Z(e)^* = (Y^{-1}(\varphi(e))^*)^* = Y^{-1}(\varphi(e))$$

Finally, define $\widetilde{\varphi} : E \to M'$ by $\widetilde{\varphi} := Y \circ W$. This is a *linear* completely positive map extending φ. □

Theorem 4.3.5. Let M be a von Neumann algebra. Then M is injective if and only if for any unital C*-algebras $C \subseteq D$, there is a natural inclusion $M \otimes_{\text{nor}} C \subseteq M \otimes_{\text{nor}} D$.

Proof. This proof, which we adapt from [16], makes heavy use of the language of algebraic states, complete state maps, and the map θ_σ as defined in 4.2.7.

First of all, by lemma 4.1.16 we have

$$M \otimes_{\text{nor}} C \subseteq M \otimes_{\text{nor}} D \quad \iff \quad \Gamma_{\text{nor}}(M \odot D) \to \Gamma_{\text{nor}}(M \odot C) \text{ is surjective}$$

It is also not hard to check that the proposition on the RHS is true if and only if every complete state map $T : C \to M_*$ extends to a complete state map $\widetilde{T} : D \to M_*$. Indeed, given such a map T, T induces a state $\rho_T \in \Gamma_{\text{nor}}(M \odot C)$ given by $\rho_T(m \otimes c) := T(c)(m)$. By assumption of surjectivity, there is a state $\widetilde{\rho} \in \Gamma_{\text{nor}}(M \odot D)$ such that $\widetilde{\rho}|_{M \odot C} = \rho_T$, and this new state defines a complete state map $T_{\widetilde{\rho}} : D \to M_*$. Then it is easily seen that $T_{\widetilde{\rho}}$ is the desired extension of T. The converse is proven identically.

Note that $M \cong \oplus_\rho \pi_\rho(M)$, where ρ ranges over all normal states on M. By 4.3.2, M is injective if and only if each $\pi_\rho(M)$ is injective, which by 4.3.3 is true if and only if each $\pi_\rho(M)'$ is injective.

$(\Rightarrow)$: First we will show that if M is injective, then every complete state map $T : C \to M_*$ extends to a complete state map $\widetilde{T} : D \to M_*$. Let $\sigma := T(1) \in M_*$. By lemma 4.2.9, $T(C) \subseteq C_\sigma$. Letting $\theta_\sigma : \pi_\sigma(M)' \to C_\sigma$ denote the map defined in 4.2.7, $\theta_\sigma^{-1} \circ T : C \to \pi_\sigma(M)'$ is completely positive. By assumption that M is injective, each $\pi_\sigma(M)'$ is injective, so $\theta_\sigma^{-1} \circ T$ extends to a completely positive map $\varphi : D \to \pi_\sigma(M)'$. Then $\widetilde{T} := \theta_\sigma \circ \varphi : D \to M_*$ is a complete state map which extends T. By the logic above, this implies $M \otimes_{\text{nor}} C \subseteq M \otimes_{\text{nor}} D$.

$(\Leftarrow)$: Now assume for any unital C*-algebras $C \subseteq D$, each complete state map $T : C \to M_*$ extends to a complete state map $\widetilde{T} : D \to M_*$. Let σ be a normal state on M, and consider a unital, completely positive map $\varphi : C \to \pi_\sigma(M)'$. Then $T := \theta_\sigma \circ \varphi : C \to C_\sigma \subseteq M_*$ is a compete state map, and by assumption extends to a complete state map $\widetilde{T} : D \to M_*$. Since $\widetilde{T}(1) = T(1) = \sigma$, we thus also have $\widetilde{T}(D) \subseteq C_\sigma$, so defining $\widetilde{\varphi} := \theta_\sigma^{-1} \circ \widetilde{T} : D \to \pi_\sigma(M)'$ we obtain a completely positive extension of φ, thus proving that $\pi_\sigma(M)'$ is injective. □

Theorem 4.3.6. Let A be a C*-algebra. Then A^{**} is injective if and only if whenever $C \subseteq D$ are C*-algebras, $A \otimes_{\max} C \subseteq A \otimes_{\max} D$.

Proof. First notice that the case for C, D non-unital can be deduced from the unital case, which follows from theorem 3.3.10. Thus we can assume without loss of generality that $C \subseteq D$ are unital.

We have

$$A \otimes_{\max} C \subseteq A \otimes_{\max} D \quad \iff \quad S(A \odot D) \to S(A \odot C) \text{ is surjective}$$

Recall that $S(A \odot C)$ (resp. $S(A \odot D)$) is in bijective correspondence with $\Gamma_{\text{nor}}(A^{**} \odot C)$ (resp. $\Gamma_{\text{nor}}(A^{**} \odot D)$). Thus, we equivalently have that the restriction map

$$\Gamma_{\text{nor}}(A^{**} \odot D) \to \Gamma_{\text{nor}}(A^{**} \odot C)$$

is also surjective, and so

$$A \otimes_{\max} C \subseteq A \otimes_{\max} D \quad \iff \quad A^{**} \otimes_{\text{nor}} C \subseteq A^{**} \otimes_{\text{nor}} D$$

Finally, by theorem 4.3.5, this is equivalent to A^{**} being injective. □

Theorem 4.3.7. Let A be a non-unital C*-algebra with unitization $\widetilde{A}$. Then $(\widetilde{A})^{**} = A^{**} \oplus \mathbb{C}$, and so A^{**} is (operator system)-injective if and only if $(\widetilde{A})^{**}$ is (operator system)-injective respectively.

In order to prove this result, we will need a preliminary lemma.

Lemma 4.3.8. Let A be a non-unital C*-algebra with unitization $\widetilde{A}$. Then $S(\widetilde{A}) = S(A) \cup \{\widehat{1}\}$, where $\widehat{1}(a + \lambda 1) := \lambda$. In other words, with the exception of $\widehat{1}$, every state on $\widetilde{A}$ is the unique extension of a state on A.

Proof. Given a state $\widetilde{f} \in S(\widetilde{A})$, if $\widetilde{f}|_A \equiv 0$, then $\widetilde{f} = \widehat{1}$. What we'd like to show is that if $\widetilde{f}|_A \not\equiv 0$, then $\widetilde{f}|_A$ is a state on A, whence every state on $S(\widetilde{A})$ other than $\widehat{1}$ is the unique extension of a state in $S(A)$ (with the extension given by proposition A.2.12). Since $\widetilde{f}|_A$ is a positive linear functional, and $\|\widetilde{f}|_A\| \leq \|\widetilde{f}\| = 1$, all we need to do is show $\|\widetilde{f}|_A\| \geq 1$. □

Proof of theorem 4.3.7. As in the previous lemma, if $\rho \in S(A)$, we will let $\widetilde{\rho}$ denote its unique extension to $\widetilde{A}$. Recall that in the case of a non-unital C*-algebra A, the GNS representation corresponding to ρ is *defined* to be the restriction of $\pi_{\widetilde{\rho}}$ to A. Moreover, letting $\widehat{1}$ be as in 4.3.8, it is not hard to check that $\pi_{\widehat{1}}(a+\lambda 1) = \lambda$ is a one-dimensional representation. So, the universal representations of A and $\widetilde{A}$ are, respectively,

$$\pi_u := \oplus_\rho \pi_{\widetilde{\rho}}|_A : A \to \mathcal{B}(H_u), \qquad \pi_{\widetilde{u}} := (\oplus_\rho \pi_{\widetilde{\rho}}) \oplus \pi_{\widehat{1}} : \widetilde{A} \to \mathcal{B}(H_u \oplus \mathbb{C})$$

and so, by 2.2.5,

$$A^{**} \cong \pi_u(A)'' = (\oplus_\rho \pi_{\widetilde{\rho}}(A))''$$
$$(\widetilde{A})^{**} \cong \pi_{\widetilde{u}}(A)'' = \left(\oplus_\rho \pi_{\widetilde{\rho}}(\widetilde{A})\right)'' \oplus \mathbb{C}$$

Suppose $a \in \mathcal{B}(H_u \oplus \mathbb{C})$ is such that $ab = ba$ for all $b \in \oplus_\rho \pi_{\widetilde{\rho}}(A)$. Then $a(b + \lambda 1) = (b + \lambda 1)a$ as well, for all $\lambda \in \mathbb{C}$. Thus, $a\widetilde{b} = \widetilde{b}a$ for all $\widetilde{b} \in \oplus_\rho \pi_{\widetilde{\rho}}(\widetilde{A})$. In other words,

$$(\oplus_\rho \pi_{\widetilde{\rho}}(A))' \subseteq \left(\oplus_\rho \pi_{\widetilde{\rho}}(\widetilde{A})\right)'$$

but we clearly also have $\supseteq$, so these two sets are equal. We thus conclude

$$(\widetilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$$

as desired. □

Corollary 4.3.9. If A is nuclear, then A^{**} is injective.

Proof. If A is nuclear, then for any C*-algebras $C \subseteq D$, we have

$$A \otimes_{\max} C = A \otimes_{\min} C \subseteq A \otimes_{\min} D = A \otimes_{\max} D$$

and by theorem 4.3.6 the result follows. □

To summarize, we have proven the following chain of implications:

$$A^{**} \text{ semidiscrete} \implies A \text{ nuclear} \implies A^{**} \text{ injective}$$

Theorem 4.3.10. A von Neumann algebra is injective if and only if it is semidiscrete.

Unfortunately, we must omit the proof of this remarkable result, which is primarily the brilliant work of Alain Connes, Man-Duen Choi, and Edward Effros. The proof relies heavily on messy von Neumann algebra structure, which we don't have time or space to cover in full. Moreover, the proof would provide little to aid our understanding of nuclearity, seeing as it almost completely eschews all mention of algebraic states, complete state maps, natural inclusions, etc. We would have to develop an entirely new tool set to properly appreciate the proof.

For a contemporaneous exposition on the program of Connes, Choi and Effros, see the introduction to [20]. In short, in [11], Alain Connes showed that for factors on a separable Hilbert space, injectivitiy implies semidiscreteness. A simplified proof of the same result was later discovered by Wassermann in [43]. This was then used by Choi and Effros in [9] to show that if A is a separable C*-algebra such that A^{**} is injective then A is nuclear. Finally in [8], Choi and Effros consider the case of a non-separable C*-algebra A. In essence, they show that every separable C*-subalgebra B of a non-separable C*-algebra A for which A^{**} is injective is contained in a separable C*-subalgebra C of A such that C^{**} is injective. Thus A, being the direct limit of its separable C*-subalgebras, is necessarily nuclear.

From hereon in, we will use theorem 4.3.10 as a black box. We immediately obtain the following:

Theorem 4.3.11. If A is a C*-algebra, then the following three statements are equivalent:

1. A^{**} is semidiscrete.
2. A is nuclear.
3. A^{**} is injective.

Since A^{**} is injective if and only if $(\widetilde{A})^{**}$ is injective, we have thus derived the following corollary:

Corollary 4.3.12. Let A be a non-unital C*-algebra with unitization $\widetilde{A}$. Then A is nuclear if and only if $\widetilde{A}$ is nuclear.

This shows the converse to corollary 3.3.11 also holds. We can also now generalize corollary 4.2.15.

Corollary 4.3.13. For any Hilbert space H, $\mathcal{K}(H)$ is nuclear.

Proof. It is well known that $\mathcal{K}(H)^* = \mathcal{B}_1(H)$ are the *trace-class operators*, and $\mathcal{B}_1(H)^* = \mathcal{B}(H)$. Since $\mathcal{K}(H)^{**} = \mathcal{B}(H)$ is injective, it follows that $\mathcal{K}(H)$ is nuclear. □

Remark. We gave $\mathcal{B}(H)$, for H a separable, infinite dimensional Hilbert space, as an example of a non-nuclear C*-algebra. The proof is attributed to Simon Wassermann in [42], where he demonstrates that $\mathcal{B}(H)^{**}$ is not injective.

By applying theorem 4.2.17, we also obtain the following corollary.

Corollary 4.3.14. If A is a C*-algebra, then A is nuclear if and only if $A^{**} \otimes_{\text{bin}} R = A^{**} \otimes_{\min} R$ for all von Neumann algebras R.

Theorem 4.3.15 (Nuclearity is Maintained Under Extensions)**.** Let A be a C*-algebra, and $I \subseteq A$ an ideal. Then A is nuclear if and only if both I and A/I are nuclear.

Proof. For the sake of brevity once more, we invoke an unproven result (proposition 3.14 in [6]): that $A^{**} \cong I^{**} \oplus (A/I)^{**}$. Then A is nuclear iff A^{**} is injective, iff both I^{**} and $(A/I)^{**}$ are injective, iff both I and A/I are nuclear. □

5 The Weak Expectation Property

5.1 Basic Properties and the Relationship to Nuclearity

Definition 5.1.1. Let A be a C*-algebra and $\pi : A \to \mathcal{B}(H)$ a faithful, non-degenerate representation. A *weak expectation for* π is a unital completely positive map $\varphi : \mathcal{B}(H) \to \overline{\pi(A)}^{wot}$ such that $\varphi(\pi(a)) = \pi(a)$ for all $a \in A$. Occasionally we might say "π *admits* a weak expectation".

Definition 5.1.2. A C*-algebra A has the *weak expectation property* (or the "WEP") if for any faithful, non-degenerate representation $\pi : A \to \mathcal{B}(H)$, π admits a weak expectation.

Ideally we'd prefer not to have to check that every faithful, non-degenerate representation π admits a weak expectation, just to check that A has the WEP. We can leverage the universality properties 2.1.2 and 2.1.4 of the universal representation to obtain a shortcut.

Fact 5.1.3. A C*-algebra A has the WEP if and only if the universal representation $\pi_u : A \to \mathcal{B}(H_u)$ admits a weak expectation.

Proof. Suppose $\pi_u : A \to \mathcal{B}(H_u)$ admits a weak expectation $\varphi : \mathcal{B}(H_u) \to \overline{\pi_u(A)}^{wot}$, and $\sigma : A \to \mathcal{B}(H)$ is another faithful, non-degenerate representation. By Arveson's extension theorem, the (completely positive, contractive) map $\pi_u \circ \sigma^{-1} : \sigma(A) \to \mathcal{B}(H_u)$ extends to a (completely positive, contractive) map $\widetilde{\sigma} : \mathcal{B}(H) \to \mathcal{B}(H_u)$ extending $\pi_u \circ \sigma^{-1}$. By universality of π_u, there exists a normal *-homomorphism $\rho : \overline{\pi_u(A)}^{wot} \to \overline{\sigma(A)}^{wot}$ such that $\sigma = \rho \circ \pi_u$.

Then

$$\rho \circ \varphi \circ \widetilde{\sigma} : \mathcal{B}(K) \to \overline{\sigma(A)}^{wot}$$

is a weak expectation for σ. □

Fact 5.1.4. A C*-algebra A has the WEP if and only if any faithful, non-degenerate representation $\pi : A \to \mathcal{B}(H)$ admits a weak expectation $\varphi : \mathcal{B}(H) \to A^{**}$ (or equivalently, the universal representation admits a weak expectation $\varphi : \mathcal{B}(H_u) \to A^{**}$).

Proof. The proof is practically identical to fact 5.1.3. We're well aware now that $\overline{\pi_u(A)}^{wot} \cong A^{**}$, so if A has the WEP then there is a weak expectation $\varphi : \mathcal{B}(H_u) \to A^{**}$. The map $\pi_u \circ \sigma^{-1} : \sigma(A) \to \mathcal{B}(H)$ extends to a completely positive contraction $\widetilde{\sigma} : \mathcal{B}(H) \to \mathcal{B}(H_u)$, and the resulting composition $\varphi \circ \widetilde{\sigma}$ is the desired weak expectation. □

Alternatively, we see that fact 5.1.4 follows simply by including $\mathcal{B}(H) \subseteq \mathcal{B}(H_u)$[10], taking a weak expectation $\varphi : \mathcal{B}(H_u) \to A^{**}$, and restricting φ's image to $\mathcal{B}(H)$.

What is primarily important about this fact above is that if $A \subseteq \mathcal{B}(H)$ is a C*-algebra with the WEP, then we can furnish a weak expectation $\varphi : \mathcal{B}(H) \to A^{**}$. This is true *even when the inclusion* $A \subseteq \mathcal{B}(H)$ *is not the universal representation* of A, even though $A^{**} \cong \overline{\pi(A)}^{wot}$ is only true when π is universal. So we can completely eschew any discussion of weak operator closures, and speak solely of C*-algebras possessing a weak expectation $\varphi : \mathcal{B}(H) \to A^{**}$, irrespective of the representation $\pi : A \to \mathcal{B}(H)$.

So when we think of the codomain of a weak expectation φ on $\mathcal{B}(H) \supseteq A$, we can work with either A^{**}, $\overline{\pi(A)}^{wot}$, or A''. Often it is this last set which is easiest to work with - in order to prove a completely positive map φ is a weak expectation, we need merely show that $\varphi|_A = \mathrm{id}_A$, and $\varphi(a)$ commutes with b whenever $b \in A'$.

A simple example of this principle is exhibited by the proof of the following fact, a useful simplification which tells us we need merely consider *unital* C*-algebras with the WEP.

Proposition 5.1.5. Let A be a non-unital C*-algebra, and $\widetilde{A}$ be its unitization. Then A has the WEP if and only if $\widetilde{A}$ has the WEP.

[10] To be perfectly clear, given $\pi : A \to \mathcal{B}(H)$ a non-degenerate representation, π decomposes as a direct sum of cyclic representations, each of which is a direct summand in the universal representation, so H is always a direct summand of H_u.

Proof. Recall that $(\widetilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$ by theorem 4.3.7. Suppose A has the WEP, and let $\varphi : \mathcal{B}(H) \to A^{**}$ be a weak expectation. By definition, φ fixes A and is unital on $\mathcal{B}(H)$, so $\varphi(a + \lambda 1) = a + \lambda 1$, hence $\varphi : \mathcal{B}(H) \to (\widetilde{A})^{**}$ is a weak expectation, whence $\widetilde{A}$ has the WEP. If, on the other hand, $\widetilde{A}$ has the WEP, and $\varphi : \mathcal{B}(H) \to (\widetilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$ is a weak expectation, then letting P denote the projection onto the A^{**} summand, $P \circ \varphi : \mathcal{B}(H) \to A^{**}$ is a weak expectation for A, so A has the WEP. $\square$

Proposition 5.1.6. Let A be a C*-algebra such that A^{**} is injective. Then A has the WEP.

Proof. Let $\pi : A \to \mathcal{B}(H)$ be a faithful non-degenerate representation, and $J : A \to A^{**}$ the canonical inclusion. By injectivity, the map $J \circ \pi^{-1} : \pi(A) \to \mathcal{A}^{**}$ extends to a completely positive, contractive $\varphi : \mathcal{B}(H) \to A^{**}$, which is a weak expectation. $\square$

Example. Since $\mathcal{K}(H)^{**} = \mathcal{B}(H)$ is injective (by Arveson's extension theorem), $\mathcal{K}(H)$ automatically has the WEP.

Proposition 5.1.6 automatically gives us a broad class of WEP algebras:

Corollary 5.1.7. Any nuclear C*-algebra A possesses the WEP.

Proof. Simply observe that if A is nuclear then A^{**} is injective. $\square$

Although technically the following construction is redundant, we find it nonetheless instructive to include for developing an intuition for weak expectations. Suppose $A \subseteq \mathcal{B}(H)$ is a nuclear C*-algebra. Then there exist nets $\varphi_\lambda, \psi_\lambda$ such that the following diagram point-norm asymptotically commutes:

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi_\lambda"'] & & A \\ & M_n \arrow[ur, "\varphi_\lambda"'] & \end{tikzcd}$$

That is, for all $a \in A$, $\lim_\lambda \|a - \varphi_\lambda(\psi_\lambda(a))\| = 0$. The net $\varphi_\lambda \circ \psi_\lambda$ lives in the set $\mathbb{B}(CP(A, A^{**}))$. Since A^{**} is injective (since A is nuclear), these maps extend to $\widetilde{\varphi}_\lambda \in \mathbb{B}(CP(\mathcal{B}(H), A^{**}))$. Equipped with the bounded-weak topology, the set $\mathbb{B}(CP(\mathcal{B}(H), A^{**}))$ is compact, and so we can extract a convergent subnet $\widetilde{\varphi}_{\lambda(\mu)} \to \varphi$. The limit point φ is in fact a weak expectation.

So the weak expectation on a nuclear C*-algebra A can in some sense be thought of as the limit of a factoring net for the identity: by approximately factoring the identity through M_n and then taking the *limit* of these approximations, what we get back is not merely the identity on A, but in fact a weak expectation.

Weak Expectations as Bimodule Maps

A somewhat different definition of a "weak expectation" appears in Lance's original paper [26]: given a C*-algebra $A \subseteq \mathcal{B}(H)$, he defines a map $\varphi : \mathcal{B}(H) \to \overline{A}^{wot}$ as a *weak expectation* if

(i) φ is unital, completely positive, and

(ii) $\varphi(atb) = a\varphi(t)b$ for all $a, b \in \mathcal{A}$, $t \in \mathcal{B}(H)$.

Property (ii) can equivalently be stated as "φ is an A-bimodule homomorphism".

Certainly Lance's definition implies our definition. To see this, note that any C*-algebra A admits a bounded approximate identity, and so meets the prerequisite criteria to apply the Cohen-Hewitt factorization theorem, which tells us that $A = A^2 := \{ab \mid a, b \in A\}$. Thus for any $c \in A$, $c = ab$ for some a, b, so if φ satisfies (ii) above, then $\varphi(c) = \varphi(ab) = \varphi(a1b) = a\varphi(1)b$, and by property (i) $a\varphi(1)b = ab = c$, so φ indeed fixes A.

On the other hand, why should we expect our definition of a weak expectation to also be an A-bimodule map? The key lies in a result known as Tomiyama's theorem.

Theorem 5.1.8 (Tomiyama; [39])**.** Let $A \subseteq B$ be C*-algebras, and $P : B \to A$ a linear map such that $P|_A = A$, and $\|P\| = 1$. Then P is a completely positive, completely bounded conditional expectation and $\|P\|_{cb} = 1$.

Unfortunately we won't have the time or space to cover Tomiyama's delicate proof of this result, so we refer the reader to either [39] or theorem 4.5 in [6] for more details.

We can use this theorem to prove the following:

Theorem 5.1.9. A unital, completely positive $\varphi : \mathcal{B}(H) \to A^{**}$ which fixes A is automatically an A-bimodule map.

Proof. Given φ as in the statement, by normalizing we obtain a map $\ddot{\varphi} : \mathcal{B}(H)^{**} \to A^{**}$ extending φ, which by proposition 2.2.14 is itself completely positive. Moreover, we claim that $\ddot{\varphi}|_{A^{**}} = \mathrm{id}_{A^{**}}$. This follows from uniqueness of normalizations: since $\varphi|_A = J_A$, and J_A uniquely normalizes to $\mathrm{id}_{A^{**}}$, it follows that

$$\ddot{\varphi}|_{A^{**}} = \ddot{J_A} = \mathrm{id}_{A^{**}}$$

By Tomiyama's theorem, $\ddot{\varphi}$ is an A^{**}-bimodule map. It is easy then to check that $\varphi = \ddot{\varphi}|_A$ is itself an A-bimodule map. ☐

Group Algebras with the WEP

Another important class of WEP algebras comes from the group algebra construction.

Theorem 5.1.10 (Brown & Ozawa)**.** For a discrete group G, the following are equivalent

1. G is amenable,
2. $C^*_\lambda(G)$ is nuclear,
3. $C^*_\lambda(G)$ has the WEP.

Proof. We have already seen ($1 \implies 2$) in corollary 4.2.16. That ($2 \implies 3$) follows from corollary 5.1.7. As in 4.2.16, the argument is due to [5].

($3 \implies 1$) : By assumption that $C^*_\lambda(G)$ has the WEP, there exists a weak expectation $\Phi : \mathcal{B}(\ell^2(G)) \to L(G)$ (since a weak expectation must exist for *every* representation of $C^*_\lambda(G)$). As we know, $L(G)$ also admits a canonical tracial vector state τ (see theorem A.4.5). Finally, note that we can represent $\ell^\infty(G)$ on $\mathcal{B}(\ell^2(G))$ via the "diagonal embedding", which takes $f \in \ell^\infty(G)$ to the operator M_f defined by

$$(M_f g)(t) := f(t)g(t)$$

Denote this faithful *-representation of $\ell^\infty(G)$ as M.

Now consider the state

$$\phi := \tau \circ \Phi \circ M \in S(\ell^\infty(G))$$

It is not hard to show that ϕ is an invariant mean. Indeed, first notice that under the representation M, the action of G on $\ell^\infty(G)$ can be represented by

$$M(s \cdot f) = \lambda(s)M(f)\lambda(s^{-1})$$

Thus

$$\begin{aligned}\phi(s \cdot f) &= \tau(\Phi(\lambda(s)M_f\lambda(s^{-1}))) \\ &= \tau(\lambda(s)\Phi(M_f)\lambda(s^{-1})) \quad \text{since } \Phi \text{ is a } C^*_\lambda(G)\text{-bimodule map} \\ &= \tau(\Phi(M_f)) = \phi(f)\end{aligned}$$

thus G is amenable. ☐

Example. So far we have seen plenty of examples of WEP algebras, but each one has also been nuclear. It is natural then to wonder if these concepts are equivalent. While the similarities between nuclearity and the WEP will become more and more uncanny as we continue our investigation, we want to make clear that nuclearity is *not* equivalent to the WEP, but examples aren't all that easy to come by.

For one, $\mathcal{B}(H)$ has the weak expectation property, but $\mathcal{B}(H)^{**}$ is not injective ([42]), and so $\mathcal{B}(H)$ cannot be nuclear. One particularly important *conjectured* example is the C*-algebra $C^*(\mathbb{F}_2)$ corresponding to the free group on two generators. Since $\mathbb{F}_2$ isn't amenable, this C*-algebra can't be nuclear. However, it is presently unknown whether or not it possesses the WEP.

While nuclearity and the WEP are decidedly *not* equivalent in general, when working in an "ambient algebra" which is nuclear, they are in fact equivalent.

Theorem 5.1.11 ([10]; Corollary 3.3)**.** Let A be a nuclear C*-algebra and $B \subseteq A$ a C*-subalgebra. Then B is nuclear if and only if B has the WEP.

Proof. Suppose we have an embedding $B \subseteq A \subseteq \mathcal{B}(H)$. Let $\Phi : \mathcal{B}(H) \to B^{**}$ be a weak expectation, and consider it's restriction to A (also a completely positive contraction). As A is nuclear, there are nets of completely positive contractions $\varphi_\lambda : A \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to A$ which point-norm approximately factor the identity. Then letting $\widetilde{\varphi}_\lambda := \varphi_\lambda|_B$, and $\widetilde{\psi}_\lambda := \Phi \circ \psi_\lambda : M_{n_\lambda} \to B^{**}$, these maps point-norm approximately factor the map $J_B : B \to B^{**}$. As we saw in the proof of theorem 4.2.6, from this starting point we can in fact arrive at new nets of completely positive contractions $\widetilde{\varphi}_\mu : B \to M_n$ and $\widetilde{\psi}_\mu : M_n \to B$ which point-norm approximately factor id_B, and so B is in fact nuclear. □

Generalized Weak Expectations

Definition 5.1.12. Suppose $A \subseteq B$ are C*-algebras. A *generalized weak expectation* is a contractive linear map $\varphi : B \to A^{**}$ such that $\varphi|_A = J_A$. If $\operatorname{ran} \varphi \subseteq \operatorname{ran} J_A$ (i.e. if we consider A as being a subset of A^{**}, then $\varphi(B) \subseteq A$), then φ is just a traditional "conditional expectation".

Remark. It is easy to see that a generalized weak expectation automatically has norm 1. Moreover by Tomiyama's theorem, generalized weak expectations are also A-bimodule maps.

Lemma 5.1.13. Suppose $A \subseteq B$ are C*-algebras. Then the following properties are equivalent:

1. There exists a generalized weak expectation $T : B \to A^{**}$.
2. There exists a normal, completely positive (completely) contractive $P : B^{**} \to A^{**}$ coinciding with the identity on A^{**}.[11]
3. For any von Neumann algebra R, and a completely positive contraction $u : A \to R$, u extends to a completely positive contraction $\widetilde{u} : B \to R$.

Remark. A few different terms exist in the literature for an inclusion of C*-algebras $A \subseteq B$ satsifying any of the three properties of lemma 5.1.13. Kirchberg, in his original paper [25] says that "B is *weakly injective relative to* A". G. Pisier in [36] refers to this property either by simply saying "$A \subseteq B$ admits a generalized weak expectation", or that "$A \subseteq B$ is *max-injective*", a term which will make more sense in due time.

Proof. ($1 \implies 2$): Given the map T, we can uniquely extend T as $P := \ddot{T} : B^{**} \to A^{**}$, which is normal and contractive. Since $T|_A = J_A$, by uniqueness of the extension to A^{**}, we have

$$P|_{A^{**}} = (T|_A)^{\cdot\cdot} = (J_A)^{\cdot\cdot} = \mathrm{id}_{A^{**}}$$

That P is also completely positive follows from theorem 5.1.8 (Tomiyama's theorem).

($2 \implies 3$): Let $u : A \to R$ be a completely positive contraction. Then u extends uniquely to $\ddot{u} : A^{**} \to R$ (also a completely positive contraction). By composing with our projection P and the canonical inclusion $J_B : B \to B^{**}$, we obtain the map

$$\widetilde{u} := \ddot{u} \circ P \circ J_B : B \to R$$

which is clearly a completely positive contraction, and extends u:

$$\widetilde{u}(a) = \ddot{u}(P(\widehat{a})) = \ddot{u}(\widehat{a}) = u(a)$$

($3 \implies 1$): The canonical inclusion $J_A : A \to A^{**}$ extends to a completely positive contraction $T : B \to A^{**}$, which is a generalized weak expectation. □

Corollary 5.1.14. If A is a C*-algebra, then A has the WEP if and only if for any inclusion $A \subseteq B$ of C*-algebras, B is weakly injective relative to A. More specifically, A has the WEP if and only if $\mathcal{B}(H)$ is weakly injective relative to A whenever $A \subseteq \mathcal{B}(H)$.

[11] Recall that for $A \subseteq B$, there is a natural inclusion of biduals $A^{**} \subseteq B^{**}$. Letting $\iota : A \subseteq B$ denote the inclusion map, $\iota^{**} : A^{**} \to B^{**}$ is injective since ι^* is surjective (a simple application of Hahn-Banach).

Proof. The only non-trivial part is showing that if A has the WEP and $A \subseteq B$, then B is weakly injective relative to A. We can assume without loss of generality that $A \subseteq B \subseteq B^{**} \subseteq \mathcal{B}(H)$ for some Hilbert space H (simply choose $\mathcal{B}(H)$ to be the universal representation of B). As A has the WEP, there is a weak expectation $\varphi : \mathcal{B}(H) \to A^{**}$. Then $\varphi|_B : B \to A^{**}$ is a generalized weak expectation. $\square$

Curiously, these results tell us that a von Neumann algebra with the WEP actually admits a *conditional* expectation, not merely a *weak* expectation.

Corollary 5.1.15. Let $R \subseteq \mathcal{B}(H)$ be a von Neumann algebra with the WEP. Then there exists a completely positive contraction $\psi : \mathcal{B}(H) \to R$.

Proof. By property 3 of lemma 5.1.13, the identity map $\mathrm{id}_R : R \to R$ extends to a completely positive contraction $\psi : \mathcal{B}(H) \to R$. $\square$

We remark in passing that this is thus an equivalent formulation of the WEP for von Neumann algebras: any conditional expectation is clearly also a weak expectation.

This then gives us the following important result.

Theorem 5.1.16. If R is a von Neumann algebra, then R is injective if and only if R has the WEP.

Proof. Suppose $R \subseteq \mathcal{B}(H)$ has the WEP. Let E be an operator space in a von Neumann algebra R, and let $u : E \to R$ a completely positive map. Consider the following diagram:

$$\begin{tikzcd}
E \arrow[r, "\subseteq"] \arrow[dr, "u"'] & A \arrow[d, dashed, "\widetilde{u}"] & & \\
& R \arrow[r, "\subseteq"] \arrow[rr, bend right, "\mathrm{id}_R"'] & \mathcal{B}(H) \arrow[r, "\psi"] & R
\end{tikzcd}$$

where ψ is a conditional expectation. We are looking to construct a map $\widetilde{u} : A \to R$ extending u.

Consider the map $u : E \to \mathcal{B}(H)$. This extends to a completely positive contraction $v : A \to \mathcal{B}(H)$, which we can then compose with our conditional expectation ψ to obtain a completely positive contraction $\widetilde{u} := \psi \circ v : A \to R$, which extends u by commutativity of the preceding diagram. Thus, R is injective.

On the other hand, if R is injective, then the identity map on R extends to a completely positive contraction $\psi : \mathcal{B}(H) \to R$, a conditional expectation. $\square$

Corollary 5.1.17. A C*-algebra A is nuclear if and only if A^{**} has the WEP.

Another simple application of the WEP emerges by comparison with injectivity. By observing the two forms of Arveson's extension theorem, it is not hard to deduce the following.

Theorem 5.1.18. Let A be a C*-algebra with the WEP.

1. Let $E \subseteq B \subseteq \mathcal{B}(H)$ be an operator system in a unital C*-algebra, and $u : E \to A$ a completely positive map. Then there exists a completely positive extension $\widetilde{u} : B \to A^{**}$ such that $\widetilde{u}|_E = u$ and $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.
2. Let $E \subseteq F \subseteq \mathcal{B}(H)$ be operator spaces, and $u : E \to A$ a completely bounded map. Then there exists a completely bounded extension $\widetilde{u} : F \to A^{**}$ such that $\widetilde{u}|_E = u$ and $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.

Proof. We can assume without loss of generality that $A \subseteq \mathcal{B}(H)$, the same $\mathcal{B}(H)$ containing our operator spaces/systems/algebras. Let $\varphi : \mathcal{B}(H) \to A^{**}$ be a weak expectation. By extending the codomain of our map u to $\mathcal{B}(H)$ (in both instances), using A.2.20 and A.2.21 respectively to obtain an extension $\widetilde{u}$ with codomain $\mathcal{B}(H)$, and precomposing with φ, we thus obtain the extensions described by the theorem statement. $\square$

Lance's Theorem

Lance introduced the weak expectation property in [26] with one purpose in mind: to characterize max-inclusions of tensor products, a long awaited result which we teased in the introduction to nuclearity.

Theorem 5.1.19 (Lance, Theorem 3.3 in [26])**.** Let $A \subseteq \mathcal{B}(H)$ be a C*-algebra. Then A has the WEP if and only if for any other C*-algebras $B \supseteq A$ and C, $A \otimes_{\max} C \subseteq B \otimes_{\max} C$ (that is, A is "max-injective").

Remark. Both of these conditions are clearly satisfied in the non-unital case if and only if they are satisfied in the unital case, so we will restrict our attention to A unital.

The proof we present is identical in content to that given by Lance, but with somewhat more modernized language.

Proof. ($\Leftarrow$) : Suppose A is max-injective. Assume $A \subseteq \mathcal{B}(H)$ is embedded in its universal representation. Consider the representation $\chi : A \odot A' \to \mathcal{B}(H)$ (where A' is the commutant of A) given by taking $\sum_i a_i \otimes b_i$ to $\sum_i a_i b_i$. By definition, χ is continuous with respect to the maximal norm on $A \odot A'$, so χ extends to a representation $\pi : A \otimes_{\max} A' \to \mathcal{B}(H)$.

Now, by max-injectivity of A, there is a natural inclusion $A \otimes_{\max} A' \subseteq \mathcal{B}(H) \otimes_{\max} A'$, and by Arveson's extension theorem we can extend π to a completely positive $\widetilde{\pi} : \mathcal{B}(H) \otimes_{\max} A' \to \mathcal{B}(H)$. Originally π was a *-homomorphism, but $\widetilde{\pi}$ is merely completely positive. To get back a *-homomorphism, we need to extend oncemore using Stinespring's dilation theorem. Namely, there exists a Hilbert space K, a bounded linear operator $V \in \mathcal{B}(H, K)$, and a *-homomorphism $\sigma : A \to \mathcal{B}(K)$ such that

$$\widetilde{\pi}(x) = V^* \sigma(x) V, \qquad \text{and} \quad \|\widetilde{\pi}(1)\| = \|V\|^2$$

Moreover, since $\widetilde{\pi}$ is unital, we can construct the operator V in such a way as to be an isometry (see Paulsen). By identifying H with $VH \subseteq K$, we thus have that

$$\widetilde{\pi}(x) = P_H \sigma(x)|_H$$

To obtain a weak expectation, consider the map $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by

$$\varphi(t) = \widetilde{\pi}(t \otimes 1)$$

Then φ is a unital, completely positive map which fixes A. To conclude that φ is a weak expectation, we must check that $\operatorname{ran} \varphi = A^{**}$. Since $A \subseteq \mathcal{B}(H)$ is assumed to be the universal representation of A, we have $A^{**} \cong \overline{A}^{wot} = A''$. Let $b \in A'$. Then

$$\begin{aligned}
\varphi(t)b &= \widetilde{\pi}(t \otimes 1)\widetilde{\pi}(1 \otimes b) \\
&= P_H \sigma(t \otimes 1)|_H P_H \sigma(1 \otimes b)|_H \\
&= P_H \sigma(t \otimes b)|_H \\
&= P_H \sigma(1 \otimes b)|_H P_H \sigma(t \otimes 1)|_H \\
&= \widetilde{\pi}(1 \otimes b)\widetilde{\pi}(t \otimes 1) = b\varphi(t)
\end{aligned}$$

so $\operatorname{ran} \varphi \subseteq A''$, and is thus a weak expectation.

Remark. There is no particular reason why we chose to work with $A \odot A'$, rather than $A \odot C$ for any other C*-algebra C, other than to show that φ maps into A''. In general, φ maps into C', a particularly useful result which Brown and Ozawa call "The Trick".

($\Rightarrow$) : Suppose A has the WEP. To prove that A is max-injective, we assume without loss of generality that $A \subseteq B \subseteq \mathcal{B}(H)$. Observe that it is enough to prove that $A \otimes_{\max} C \subseteq \mathcal{B}(H) \otimes_{\max} C$. This follows from the fact that

$$\| \cdot \|_{\mathcal{B}(H) \otimes_{\max} C} \leq \| \cdot \|_{B \otimes_{\max} C} \leq \| \cdot \|_{A \otimes_{\max} C}$$

when restricted to $A \odot C$. If $\| \cdot \|_{\mathcal{B}(H) \otimes_{\max} C} = \| \cdot \|_{A \otimes_{\max} C}$, then all three norms coincide, and $A \otimes_{\max} C \subseteq B \otimes_{\max} C$.

Recall lemma 4.1.16: to prove $A \otimes_{\max} C \subseteq \mathcal{B}(H) \otimes_{\max} C$, all we need to show is that the restriction map $S(\mathcal{B}(H) \odot C) \to S(A \odot C)$ is surjective.

Given a state $\rho \in S(A \odot C)(= \widehat{S}_{\max}(A \odot C))$, ρ induces a completely positive contraction $T_\rho : A \to C^*$. We can extend T_ρ to a completely positive contraction $\ddot{T}_\rho : A^{**} \to C^*$. Since A^{**} has the WEP, there exists a weak expectation $\varphi : \mathcal{B}(H) \to A^{**}$. Composing this with $\ddot{T}_\rho$, we arrive at yet *another* completely positive contraction $\widetilde{T} : \mathcal{B}(H) \to C^*$ given by $\widetilde{T} = \ddot{T}_\rho \circ \varphi$.

Seeing as $\widetilde{T}$ is a completely positive contraction, it too induces a state $\widetilde{\rho} \in S(\mathcal{B}(H) \odot C)$ (so that the complete state map corresponding to $\widetilde{\rho}$, $T_{\widetilde{\rho}}$, is equal to $\widetilde{T}$), and it is easily checked that $\widetilde{\rho}|_{A \odot C} = \rho$. Thus, the restriction map $S(\mathcal{B}(H) \odot C) \to S(A \odot C)$ is surjective, so there is indeed a natural inclusion $A \otimes_{\max} C \subseteq \mathcal{B}(H) \otimes_{\max} C$. $\square$

We can obtain a slight refinement of this result where instead of checking against all inclusion $A \subseteq B$, we merely have to check against a single well-chosen inclusion $A \subseteq K$.

Theorem 5.1.20. Suppose $A \subseteq K$ are C*-algebras, and K is injective. Then A has the WEP if and only if $A \otimes_{\max} C \subseteq K \otimes_{\max} C$ for any C*-algebra C.

Proof. If A has the WEP, then $A \otimes_{\max} C \subseteq K \otimes_{\max} C$ by Lance's theorem. So assume only the conclusion of this implication is true. We'd like to prove then that $A \otimes_{\max} C \subseteq B \otimes_{\max} C$ whenever $A \subseteq B$, which is equivalent to the antecedent statement.

First, we remind the reader that we can reduce this problem to the case where all algebras are assumed unital, and whenever $A \subseteq B$, all three of A, B, K can be assumed to share a unit. Recall theorem 3.3.10: whenever $I \subseteq A$ is an ideal, for any C*-algebra C we have $I \otimes_{\max} C \subseteq A \otimes_{\max} C$. Let $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{K}$ denote the respective unitizations of A, B, C, K (equal to the original algebra if it is already unital), so that each algebra sits inside its unitization as an ideal. Then if $A \otimes_{\max} C \subseteq K \otimes_{\max} C$, then

$$\widetilde{A} \otimes_{\max} \widetilde{C} \subseteq \widetilde{K} \otimes_{\max} \widetilde{C}$$

which implies $\widetilde{A} \otimes_{\max} \widetilde{C} \subseteq \widetilde{B} \otimes_{\max} \widetilde{C}$ (assuming we've proven the unital case). Restricting our attention back down to $A \odot C$, we thus arrive at

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C$$

and so the unital case implies the non-unital case.

Suppose $A \subseteq B$ are unital C*-algebras. The inclusion map $\iota : A \to K$ extends to a completely positive map $\varphi : B \to K$, which is contractive since it is unital. Being completely positive, by theorem 3.3.7 the map $\varphi \odot \mathrm{id}_C : B \odot C \to K \odot C$ extends to a completely positive contraction $\varphi \otimes_{\max} \mathrm{id}_C : B \otimes_{\max} C \to K \otimes_{\max} C$.

Since $\varphi \otimes_{\max} \mathrm{id}_C$ is contractive, for all $x \in A \odot C$ we have

$$\|x\|_{K \otimes_{\max} C} = \|\varphi \otimes_{\max} \mathrm{id}_C(x)\|_{K \otimes_{\max} C} \leq \|x\|_{B \otimes_{\max} C}$$

By assumption, however, $\|x\|_{K \otimes_{\max} C} = \|x\|_{A \otimes_{\max} C}$. Furthermore, by maximality $\|x\|_{B \otimes_{\max} C} \leq \|x\|_{A \otimes_{\max} C}$ ($\| \cdot \|_{B \otimes_{\max} C}$ is a norm on $A \odot C$, and thus automatically $\leq \| \cdot \|_{A \otimes_{\max} C}$). In conclusion, $\|x\|_{A \otimes_{\max} C} = \|x\|_{B \otimes_{\max} C}$, and so A has the WEP. $\square$

Aside from the main result, this proof exposes is that among all inclusions $A \subseteq B$ of C*-algebras, then restriction of the $B \otimes_{\max} C$ norm to $A \odot C$ is *minimal* when B is injective.

Example. If A is an injective C*-algebra, then we can take $K = A$ in theorem 5.1.20 and deduce that A has the WEP. This is not a new discovery, rather a demonstration that our new result plays nicely with our prior understanding. In fact it isn't hard to see that a C*-algebra $A \subseteq \mathcal{B}(H)$ is injective if and only if there is a *conditional* expectation $\varphi : \mathcal{B}(H) \to A$ (see exercise 7.5 in [33]).

Example. Since every C*-algebra can be considered as a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H, and $\mathcal{B}(H)$ is injective, we see that theorem 5.1.20 holds for $K = \mathcal{B}(H)$.

Example. Every C*-algebra A is contained in a certain "special" injective C*-algebra called the "injective envelope of A", denoted $I(A)$. In a sense, $I(A)$ can be thought of as the "smallest injective C*-algebra containing A", and is *uniquely* characterized by an expected universal property. If we take $K = I(A)$, we see that A has the WEP if and only if $A \otimes_{\max} B \subseteq I(A) \otimes_{\max} B$.

The restriction of the max-norm on $I(A)\otimes_{\max} B$ onto $A\odot B$ actually has a special name, the "enveloping-left" norm, denoted el. With this notation, we now have a sort of "norm-equivalence" characterization of the WEP, akin to the characterization of nuclearity so dear to our hearts:

$$A \text{ has the WEP} \quad \iff \quad A\otimes_{\mathrm{el}} B = A\otimes_{\max} B \quad \forall B$$

Generally speaking, the injective envelope can be quite difficult to calculate, and when its description is known it provides us little new information on the WEP. For instance, if H is a Hilbert space and $A\subseteq \mathcal{B}(H)$ a C*-algebra containing $\mathcal{K}(H)$, then $I(A)=\mathcal{B}(H)$ (see [21]).

If A is a commutative unital C*-algebra, then $I(A)$ is *-isomorphic to $M_{\mathrm{loc}}(A)$, the *local multiplier algebra* of A, defined as the direct limit of the multiplier algebras $M(I)$, where I ranges over all essential ideals in A. This result is proven in [17]. This commutative C*-algebra can further be described as $C(\varprojlim \beta J)$, the continuous functions on the inverse limit of the Stone-Cech compactifications of J, as J ranges over all dense open subsets of $\hat{A}$, the primitive spectrum of A (this result is known as the "local Dauns-Hoffman" theorem, and was proven by P. Ara & M. Mathieu in [1]). None of this turns out to be particularly useful in identifying the WEP - if A is commutative then it's already nuclear.

The WEP, and Injectivity of the Bidual

The similarities between the WEP and nuclearity are rather uncanny. Let us compare Lance's characterization of the WEP with Lance and Effros' characterization of C*-algebras A for which A^{**} is injective, theorem 4.3.6, (which, as we know, are simply the *nuclear* C*-algebras), which bears a striking resemblance:

$$\begin{array}{ccll} A \text{ has the WEP} & \iff & A\otimes_{\max} C\subseteq B\otimes_{\max} C & \forall B\supseteq A,\ C \\ A^{**} \text{ is injective} & \iff & A\otimes_{\max} C\subseteq A\otimes_{\max} D & \forall C\subseteq D \end{array}$$

Until shown side-by-side, these two results can easily be confused, leading one to believe (incorrectly) that A has the WEP if and only if A is nuclear. To be clear, if A has the WEP if and only if we can consider inclusions of maximal tensor products on the *same side of the* $\otimes$ *as* A. A^{**} is injective if and only if we can consider inclusions on *the opposite side of the* $\otimes$ *as* A. This was a source of considerable confusion for the author.

Also, paired with the fact that A^{**} injective implies A has the WEP, we thus see that if A^{**} is injective, and $A\subseteq B$, $C\subseteq D$ are additional C*-algebras, then

$$A\otimes_{\max} C\subseteq B\otimes_{\max} D$$

which is *also* an equivalent characterization of injectivity of A^{**}, and thus of nuclearity of A.

5.2 The Fundamental Nuclear Pair, and Kirchberg's Characterization of the WEP

We now shift gears a little bit as we head into our first non-trivial characterization of the weak expectation property, a theorem of Kirchberg from [25]. We compile the essential details of the proof as it appears in Pisier in [36], with additional expository detail as usual. Kirchberg's characterization of the WEP is *paramount* for many of the modern applications of the WEP, and any reader who wishes to experiment with WEP algebras should be familiar with this result.

Let $\mathscr{B}:=\mathcal{B}(\ell_2)$ (where $\ell_2=\ell_2(\mathbb{N})$), and $\mathscr{C}:=C^*(\mathbb{F}_\infty)$ (the free group of countably infinite rank). These two C*-algebras possess certain dual "universality" properties among C*-algebras, namely:

Proposition 5.2.1. Every separable C*-algebra embeds in $\mathscr{B}$, and every separable unital C*-algebra is a quotient of $\mathscr{C}$.

Proof. The first part is straightforward. If A is a separable C*-algebra, then the universal representation of A embeds A in $\mathcal{B}(H)$, where $\dim H$ is *at most* countably infinite. In the case where $\dim H=\infty$, $H\cong\ell_2$ (isometrically). If $\dim H<\infty$, then $H\cong\ell_2^n$ with $n=\dim H$, and $\mathcal{B}(\ell_2^n)$ embeds isometrically in $\mathcal{B}(\ell_2)$ via the identification

$$T\in\mathcal{B}(\ell_2^n) \quad\mapsto\quad \widetilde{T}:=TP_{\ell_2^n}\in\mathcal{B}(\ell_2)$$

where $P_{\ell_2^n}$ is the orthogonal projection onto ℓ_2^n.

Now suppose $A \subset \mathcal{B}(H)$ is a separable unital C*-algebra. Let $\mathbb{U}(A)$ denote the group of unitaries of A, and let $\delta : \mathbb{F} \to \mathbb{U}(A)$ be a group homomorphism of some free group $\mathbb{F}$ of at most countably infinite rank onto $\mathbb{U}(A)$, such that $\operatorname{im}\delta$ is dense in $\mathbb{U}(A)$. To convince oneself that such a map exists, note that since A is separable as a C*-algebra (i.e., a *metric space*), A is in fact *hereditarily separable*[12], so $\mathbb{U}(A)$ is separable. If $\mathcal{D}$ is a dense subset of $\mathbb{U}(A)$, we can simply map each generator of $\mathbb{F}$ one-to-one with the elements of $\mathcal{D}$, and extend to the rest of $\mathbb{F}$.

As per theorem A.4.3, δ extends uniquely to a *-representation $\pi_\delta : C^*(\mathbb{F}) \to A$. By the Russo-Dye theorem, $\overline{\operatorname{conv}}(\mathbb{U}(A)) = \mathbb{B}(A)$, so $\operatorname{ran}\pi_\delta$ is dense in A. However, the range of *-homomorphisms of C*-algebras are automatically closed, so $\operatorname{ran}\pi_\delta = A$, implying π_δ is a surjective *-homomorphism, thus establishing that A is a quotient of $C^*(\mathbb{F})$ ($A \cong C^*(\mathbb{F})/\ker\pi_\delta$). ☐

The C*-algebra $\mathscr{C}$ is thus best conceptualized as the "largest" C*-algebra generated by countably infinitely many distinct unitary elements, which correspond to the generators of $\mathbb{F}_\infty$.

The only place we used separability of A in the proof above was to guarantee that $\mathbb{F}$ had countably infinite rank. An identical argument thus yields the following corollary:

Corollary 5.2.2. Let A be a unital C*-algebra. Then there is a free group $\mathbb{F}$ such that A is isomorphic to a quotient of $C^*(\mathbb{F})$.

Indeed, if A has *density* κ (that is, the minimal cardinality of any dense subset of A is κ), then A is a quotient of $C^*(\mathbb{F}_\kappa)$. This, however, is rarely optimal.

While $\mathscr{C}$ admits these nice universality properties among unital separable C*-algebras, this is not the primary reason why we are interested in $\mathscr{C}$. Rather, the full C*-algebras of free groups admit a variety of different formulas for their min-tensor norms, making them somewhat more pleasant to tensor with than your average C*-algebra. For instance:

Theorem 5.2.3. Let $A \subseteq \mathcal{B}(H)$ be a C*-algebra and $\mathbb{F}$ a free group generated by $\{u_i\}_{i\in I}$. Treat the u_i's as elements of $C^*(\mathbb{F})$. Let $x = \sum_i x_i \otimes u_i \in A \odot C^*(\mathbb{F})$. Let $I_0 := \{i \in I \mid x_i \neq 0\}$, which is a finite set since x is an algebraic tensor. Then

$$\left\|\sum x_i \otimes u_i\right\|_{A\otimes_{\min}C^*(\mathbb{F})} = \|T\|_{cb}$$

where $T : \ell_\infty(I_0) \to A$ takes $(\alpha_i)_{i\in I_0}$ to $\sum_i \alpha_i x_i$. Moreover, $\|T\|_{cb}$ evaluates to

$$\|T\|_{cb} = \sup\left\{\left\|\sum x_i \otimes t_i\right\|_{A\otimes_{\min}G} \mid t_i \in G,\ \|t_i\| \le 1, G \text{ an operator space}\right\}$$

If $A = \mathcal{B}(H)$ where $\dim H = \infty$, then there exist $y_i, z_i \in A$ such that $x_i = y_i z_i$, and for which

$$\left\|\sum x_i \otimes u_i\right\|_{A\otimes_{\min}C^*(\mathbb{F})} = \left\|\sum y_i y_i^*\right\|^{1/2} \left\|\sum z_i^* z_i\right\|^{1/2}$$

Remark. The final expression in the theorem above is in fact true if we take the *infimum* over all such factorizations $x_i = y_i z_i$, but we will not need this going forward.

Proof. The fact that $\|x\|_{A\otimes_{\min}C^*(\mathbb{F})}$ and $\|T\|_{cb}$ coincide is moreso a matter of adventitious coincidence of equations, rather than an intuitive insight. On one hand, we see that

$$\|x\|_{A\otimes_{\min}C^*(\mathbb{F})} = \sup\left\{\left\|\sum x_i \otimes v(u_i)\right\|_{M_n(A)} \mid v \in CB(E, M_n),\ \|v\| \le 1,\ n \in \mathbb{N}\right\}$$

On the other hand, we have

$$\begin{aligned}\|T\|_{cb} &= \sup_n \|T_n\| \\ &= \sup\{\|T_n(t)\| \mid t \in \ell_\infty(I_0) \otimes M_n,\ \|t\| \le 1, n \in \mathbb{N}\}\end{aligned}$$

[12] A metric space is second countable if and only if separable, and since second countability is hereditary, so too is separability (*only* on metric spaces!).

Take some $t \in \ell_\infty(I_0) \otimes M_n$ (which, we recall, is just $\ell_\infty(I_0) \odot M_n$, which possesses a unique C*-norm, hence the usage of the bare $\otimes$ symbol). Let $(e_i)_{i \in I_0}$ be the standard basis for $\ell_\infty(I_0)$. If $t = \sum_k (\alpha_i^k)_{i \in I_0} \otimes a_k$, then we can rewrite t as

$$\begin{aligned} t &= \sum_i e_i \otimes \left(\sum_k \alpha_i^k a_k \right) \\ &= \sum_i e_i \otimes t_i \end{aligned}$$

where $t_i := \sum_k \alpha_i^k a_k$. Recall that the norm on $\ell_\infty(I_0) \otimes M_n$ has a simple form:

$$\|t\|_{\ell_\infty(I_0) \otimes M_n} = \sup_i \|t_i\|$$

and so the condition $\|t\| \leq 1$ is equivalent to $\|t_i\| \leq 1$ for all i.

Having written t in this form, we can easily calculate

$$T_n(t) = \sum_i x_i \otimes t_i$$

thus giving us the formula

$$\|T\|_{cb} = \sup \left\{ \left\| \sum x_i \otimes t_i \right\|_{M_n(A)} \mid t_i \in M_n,\ \|t_i\| \leq 1,\ n \in \mathbb{N} \right\}$$

which agrees with the formula above for $\|x\|_{A \otimes_{\min} C^*(\mathbb{F})}$ (well, only after a slight manipulation using the Russo-Dye theorem, which tells us that the convex hull of unitaries is norm-dense in the unit ball).

Now suppose that $A = \mathcal{B}(H)$. Given that T is completely bounded, it admits a Wittstock dilation, so there exists a Hilbert space $\widehat{H}$, a $*$-homomorphism $\pi : \ell_\infty(I_0) \to \mathcal{B}(\widehat{H})$, and operators $V, W \in \mathcal{B}(H, \widehat{H})$ such that $T(\alpha) = V^* \pi(\alpha) W$ for all $\alpha \in \ell_\infty(I_0)$, and for which $\|T\|_{cb} = \|V\| \|W\|$.

Since I_0 is finite, the C*-algebra $\ell_\infty(I_0)$ is separable, which allows us to choose $\widehat{H}$ to be separable too. Since $\dim H = \infty$ by assumption, we can simply set $\widehat{H} = H$.

Let e_i be the standard basis for $\ell_\infty(I_0)$. Then $1 = \sum_i e_i e_i^*$, and so $V^*V = \sum_i V^* \pi(e_i) \pi(e_i)^* V$, so that

$$\|V\| = \|V^*V\|^{1/2} = \left\| \sum_i y_i y_i^* \right\|^{1/2}$$

where $y_i = V^* \pi(e_i)$. A similar calculation yields $\|W\| = \|\sum_i z_i^* z_i\|^{1/2}$, where $z_i := \pi(e_i) W$. Finally, observe that

$$x_i = T(e_i) = T(e_i^2) = V^* \pi(e_i) \pi(e_i) W = y_i z_i$$

and the result follows. □

Theorem 5.2.4 (Kirchberg [25])**.** For any Hilbert space H, $\mathcal{B}(H) \otimes_{\min} \mathscr{C} = \mathcal{B}(H) \otimes_{\max} \mathscr{C}$, or in other words $(\mathcal{B}(H), \mathscr{C})$ is a *nuclear pair*. In particular, $(\mathscr{B}, \mathscr{C})$ is a nuclear pair, often called the *fundamental pair*.

Before getting to the pièce de résistance, we'll need the following lemmas, two of which will significantly simplify what we have to prove for theorem 5.2.4.

Lemma 5.2.5. Let $(a_i)_{i \in I}, (b_i)_{i \in I}$ be finitely supported families of operators in $\mathcal{B}(H)$. Then for any bounded family $(x_i)_{i \in I} \in \mathcal{B}(H)$

$$\left\| \sum a_i x_i b_i \right\| \leq \sup_i \|x_i\| \left\| \sum a_i a_i^* \right\|^{1/2} \left\| \sum b_i^* b_i \right\|^{1/2}$$

Proof. Observe that

$$\underbrace{\begin{bmatrix} a_1 & \cdots & a_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{:=A} \underbrace{\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}}_{:=X} \underbrace{\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \cdots & 0 \end{bmatrix}}_{:=B} = \left(\sum a_i x_i b_i \right) e_{11}$$

where e_{11} is the matrix with a 1 in the $(1,1)$-position and zero elsewhere. Thus,

$$\|\sum a_i x_i b_i\| \leq \|A\|\|X\|\|B\|$$

and it is easily checked that

$$\begin{aligned}
\|A\|^2 &= \sup\left\{\sum_i \|a_i^*\xi\|^2 \mid \|\xi\| = 1\right\} \\
&= \sup\left\{\left\langle \sum_i a_i a_i^* \xi, \xi \right\rangle \mid \|\xi\| = 1\right\} \\
&= \|\sum_i a_i a_i^*\|
\end{aligned}$$

and similarly for $\|B\|$. □

Remark. Lemma 5.2.5 is essentially a "weighted Cauchy-Schwarz inequality for Hilbert C*-modules", a concept outside the scope of this paper, which we refer the reader to [28] for more info.

Lemma 5.2.6 ([36]; Proposition 9.7)**.** Let A be a unital C*-algebra, $\{u_i\}_{i\in I} \subset A$ a set of unitaries which generate A as a C*-algebra, and $E := \overline{\operatorname{span}\{u_i\}}$ (an operator system). Given any other unital C*-algebra B, and a unital, completely contractive linear map $T : E \to B$ taking unitaries to unitaries, T extends to *-homomorphism $\widetilde{T} : A \to B$. Moreover, if T is a completely isometric, and $T(E)$ generates B, then $\widetilde{T}$ is a *-isomorphism.

Proof. Embed B in $\mathcal{B}(H)$, and consider T as a map from E to $\mathcal{B}(H)$. By Arveson's extension theorem, T extends to a map $\widetilde{T} : A \to \mathcal{B}(H)$. Since T is unital and takes unitaries to unitaries, then

$$T(u_i^* u_i) = T(1) = 1 = T(u_i)^* T(u_i)$$

and so

$$u_i \in D := \{a \in A \mid \widetilde{T}(a^*a) = \widetilde{T}(a)^*\widetilde{T}(a),\ \widetilde{T}(aa^*) = \widetilde{T}(a)\widetilde{T}(a)^*\}$$

but we know (from theorem A.2.19) that this set D is equal to

$$D = \{a \in A \mid \widetilde{T}(ab) = \widetilde{T}(a)\widetilde{T}(b),\ \widetilde{T}(ba) = \widetilde{T}(b)\widetilde{T}(a)\ \forall b \in A\}$$

which is better known as the *multiplicative domain of* $\widetilde{T}$. It's clear that D is a C*-subalgebra of A containing each u_i. Since the u_i's generate A, we thus conclude that $D = A$, and $\widetilde{T}$ is a *-homomorphism on A. Moreover, since $\widetilde{T}(E) \subseteq B$, we have that $\widetilde{T}(A) = \widetilde{T}(C^*(E)) \subseteq B$, so $\widetilde{T}$ has codomain B.

To obtain the last part, simply note that if T is completely isometric, then we can perform the same argument above to the map $T^{-1} : T(E) \to A$. We'll obtain a *-homomorphism $\widetilde{T^{-1}} : B \to A$ extending T^{-1}, and so $\widetilde{T} \circ \widetilde{T^{-1}}|_{T(E)} = \mathrm{id}_{T(E)}$. Since $B = C^*(T(E))$, and $\widetilde{T} \circ \widetilde{T^{-1}}$ is a *-homomorphism coinciding with the identity on $T(E)$, we thus conclude that $\widetilde{T} \circ \widetilde{T^{-1}} = \mathrm{id}_B$. Mirroring this argument also gives $\widetilde{T^{-1}} \circ \widetilde{T} = \mathrm{id}_A$, so $\widetilde{T^{-1}} = \widetilde{T}^{-1}$, so $\widetilde{T}$ is an isomorphism. □

Lemma 5.2.7 ([36]; Theorem 9.8)**.** Let A_1, A_2 be unital C*-algebras, $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ families of unitaries generating A_1 and A_2 as C*-algebras respectively, and let

$$E_1 := \overline{\operatorname{span}\{1_{A_1}\} \cup \{u_i\}_{i\in I}}, \qquad E_2 := \overline{\operatorname{span}\{1_{A_2}\} \cup \{v_j\}_{j\in J}}$$

Then $A_1 \otimes_{\min} A_2 = A_1 \otimes_{\max} A_2$ if and only if there is a completely isometric canonical inclusion

$$E_1 \otimes_{\min} E_2 \subseteq A_1 \otimes_{\max} A_2$$

which, as usual, is to say that the map $\iota : E_1 \odot E_2 \to A_1 \odot A_2$ is isometric with respect to the min-norm on the domain, and the max-norm on the codomain.

Proof. The only-if direction is clear. Suppose we have the aforementioned completely isometric inclusion $E_1 \otimes_{\min} E_2 \subseteq A_1 \otimes_{\max} A_2$. Note that we can view $E_1 \otimes_{\min} E_2$ as a subspace of $A_1 \otimes_{\min} A_2$ (the min-norm poses no issue for inclusions of tensor products). Then by lemma 5.2.6 the inclusion map extends to an isometric *-homomorphism T from $A_1 \otimes_{\min} A_2$ to $A_1 \otimes_{\max} A_2$.

As T is a *-homomorphism, T fixes the *-algebra generated by $E_1 \otimes 1_{A_2}$, which is dense in $A_1 \otimes 1_{A_2}$, and so T fixes $A_1 \otimes 1_{A_2}$. Similarly, T fixes $1_{A_1} \otimes A_2$. Putting this together, we clearly have that T fixes $A_1 \odot A_2$, and by density we thus get $A_1 \otimes_{\min} A_2 = A_1 \otimes_{\max} A_2$. □

Proof of Theorem 5.2.4. Let $\{u_i\}_{i\in\mathbb{N}}$ denote the generators of $\mathbb{F}_\infty$. Realized as elements of $\mathscr{C}$, the family $\{u_i\}_{i\in\mathbb{N}}$ is a family of unitary elements which generate $\mathscr{C}$ as a C*-algebra. We also let

$$E := \overline{\operatorname{span}\{1\} \cup \{u_i\}_{i\in\mathbb{N}}}$$

be an operator system in $\mathscr{C}$. The first observation we would like to prove is that the min and max norms coincide on $\mathcal{B}(H) \odot E$.

Any element of $\mathcal{B}(H) \odot E$ can be written as $x = \sum_i x_i \otimes u_i$, where x_i is a finitely supported sequence in $\mathcal{B}(H)$. Moreover, by theorem 5.2.3 we know that there are elements $y_i, z_i \in \mathcal{B}(H)$ such that $x_i = y_i^* z_i$, and for which

$$\left\|\sum x_i \otimes u_i\right\|_{\mathcal{B}(H)\otimes_{\min}\mathscr{C}} = \left\|\sum y_i^* y_i\right\|^{1/2} \left\|\sum z_i^* z_i\right\|^{1/2}$$

Now, let $\pi : \mathcal{B}(H) \odot \mathscr{C} \to \mathcal{B}(K)$ be any faithful *-homomorphism, and let $\pi_1 := \pi|_{\mathcal{B}(H)}$ and $\pi_2 := \pi|_{\mathscr{C}}$ denote the restrictions (which are *-homomorphisms with commuting ranges). Then

$$\begin{aligned}\pi(x) &= \pi\left(\sum (y_i^* z_i \otimes 1)(1 \otimes u_i)\right)\\ &= \sum \pi_1(y_i^* z_i)\pi_2(u_i)\\ &= \sum \pi_1(y_i)^*\pi_2(u_i)\pi_1(z_i)\end{aligned}$$

Using the bound from lemma 5.2.5, we obtain

$$\begin{aligned}\|\pi(x)\| &\le \left\|\sum \pi_1(y_i^* y_i)\right\|^{1/2} \sup\|\pi_2(u_i)\| \left\|\sum \pi_1(z_i^* z_i)\right\|^{1/2}\\ &= \left\|\sum y_i^* y_i\right\|^{1/2} \left\|\sum z_i^* z_i\right\|^{1/2}\\ &= \|x\|_{\mathcal{B}(H)\otimes_{\min}\mathscr{C}}\end{aligned}$$

Since π was arbitrary, this gives us

$$\|x\|_{\mathcal{B}(H)\otimes_{\min}\mathscr{C}} = \|x\|_{\mathcal{B}(H)\otimes_{\max}\mathscr{C}} \qquad \forall x \in \mathcal{B}(H) \odot E$$

By lemma 5.2.6, this is, in fact, enough to complete the argument, since the operator space E generates $\mathscr{C}$ as a C*-algebra. □

Theorem 5.2.8. Let A be a C*-algebra. The following are equivalent:

1. For any other C*-algebra $B \supseteq A$, there is a natural inclusion $A \otimes_{\max} \mathscr{C} \subseteq B \otimes_{\max} \mathscr{C}$.
2. For any other C*-algebra $B \supseteq A$, there is a natural inclusion $A \otimes_{\max} C^*(\mathbb{F}) \subseteq B \otimes_{\max} C^*(\mathbb{F})$ for any infinite rank free group $\mathbb{F}$.
3. For any other C*-algebras $B \supseteq A$ and C, there is a natural inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} C$ (i.e. A has the WEP).

Proof. We will show that 1 implies 2, and then that 2 implies 3. That 3 implies 1 is obvious.

$(1 \implies 2)$: First of all, it is not hard to see that for $t \in A \odot C^*(\mathbb{F})$, there is a copy of $\mathbb{F}_\infty$ in $\mathbb{F}$ such that $t \in A \odot C^*(\mathbb{F}_\infty)$ (this copy will remain fixed for the remainder of the proof that 1 implies 2). Then

$$\|t\|_{B\otimes_{\max}C^*(\mathbb{F})} \le \|t\|_{A\otimes_{\max}C^*(\mathbb{F})} \le \|t\|_{A\otimes_{\max}C^*(\mathbb{F}_\infty)} = \|t\|_{B\otimes_{\max}C^*(\mathbb{F}_\infty)}$$

which follows from the obvious fact that if $A \subseteq B, C$ are C*-algebras, then $\|\cdot\|_{B\otimes_{\max} C} \leq \|\cdot\|_{A\otimes_{\max} C}$ when restricted to $A \odot C$. If we can prove that $\|t\|_{B\otimes_{\max} C^*(\mathbb{F}_\infty)} \leq \|t\|_{B\otimes_{\max} C^*(\mathbb{F})}$, then we're done.

Suppose we had a *-homomorphism $\omega : C^*(\mathbb{F}) \to C^*(\mathbb{F}_\infty)$ such that $\omega|_{C^*(\mathbb{F}_\infty)} = \mathrm{id}$. By theorem 3.3.7 this map extends to a *-homomorphism on the max-tensor products:

$$\mathrm{id}_A \otimes \omega : A \otimes_{\max} C^*(\mathbb{F}) \to A \otimes_{\max} C^*(\mathbb{F}_\infty)$$

Then $\|t\|_{A\otimes_{max} C^*(\mathbb{F}_\infty)} = \|\,\mathrm{id}_A \otimes \omega(t)\|_{A\otimes_{\max} C^*(\mathbb{F}_\infty)} \leq \|t\|_{A\otimes_{\max} C^*(\mathbb{F})}$, and we're done.

Constructing ω is fortunately quite easy. Suppose $\{u_i\}_{i\in\mathbb{N}}$ are the generators of our copy of $\mathbb{F}_\infty$ inside $\mathbb{F}$. Letting u be a generator of $\mathbb{F}$, define

$$\omega(u) = \begin{cases} u, & \text{if } u = u_i \text{ for some } i \in \mathbb{N} \\ 1, & \text{otherwise} \end{cases}$$

then extend ω first to $\mathbb{C}[\mathbb{F}]$ by requiring ω be an involutive algebra homomorphism, then to $C^*(\mathbb{F})$ by density. By definition, this map is a *-homomorphism, whence completely positive, and $\omega|_{C^*(\mathbb{F}_\infty)} = \mathrm{id}$.

($2 \implies 3$): First assume we have proven that 2 implies 3 for C unital. If C is non-unital, then we can embed C as an ideal in its unitization $\widetilde{C}$, and by the first corollary of theorem 3.3.10 we thus have

$$A \otimes_{\max} C \subseteq A \otimes_{\max} \widetilde{C} \subseteq B \otimes_{\max} \widetilde{C}$$

and a second application of theorem 3.3.10 yields that for any $x \in B \odot C$, $\|x\|_{B\otimes_{\max} C} = \|x\|_{B\otimes_{\max}\widetilde{C}}$, and so really

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C$$

So we need only prove 2 implies 3 for C unital.

In the case of C unital, $C \cong C^*(\mathbb{F})/I$ for some free group $\mathbb{F}$ ideal I, by corollary 5.2.2. So by theorem 3.3.10

$$A \otimes_{\max} C \cong \frac{A \otimes_{\max} \mathscr{C}}{A \otimes_{\max} I}$$

Consider the inclusion map $\iota : A \otimes_{\max} \mathscr{C} \to B \otimes_{\max} \mathscr{C}$. Since $\iota(A \otimes_{\max} I) \subseteq B \otimes_{\max} I$ (again by theorem 3.3.10), the map ι descends to an injective *-homomorphism

$$\widetilde{\iota} : \frac{A \otimes_{\max} \mathscr{C}}{A \otimes_{\max} I} \to \frac{B \otimes_{\max} \mathscr{C}}{B \otimes_{\max} I} \cong B \otimes_{\max} C$$

thus giving us an isometric inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} C$. ☐

We now possess all the prerequisite tools to prove the following result.

Theorem 5.2.9 (Kirchberg)**.** Let $A \subseteq \mathcal{B}(H)$ be a C*-algebra. Then A has the WEP if and only if $(A, \mathscr{C})$ is a nuclear pair.

Proof. Suppose A has the WEP. Embed A^{**} into $\mathcal{B}(H)$, so that $A \subset A^{**} \subset \mathcal{B}(H)$, and let $\varphi : \mathcal{B}(H) \to A^{**}$ be a weak expectation. Since φ is a completely positive contraction, $\varphi \otimes \mathrm{id}_{\mathscr{C}}$ is a contractive map between the maximal tensor products:

$$\mathcal{B}(H) \otimes_{\max} \mathscr{C} \xrightarrow{\varphi\otimes\mathrm{id}_{\mathscr{C}}} A^{**} \otimes_{\max} \mathscr{C}$$

On the other hand, the min tensor product enjoys the property of having automatic isometric inclusions:

$$A \otimes_{\min} \mathscr{C} \hookrightarrow \mathcal{B}(H) \otimes_{\min} \mathscr{C}$$

But we know that $(\mathcal{B}(H), \mathscr{C})$ is a nuclear pair, so

$$\mathcal{B}(H) \otimes_{\min} \mathscr{C} = \mathcal{B}(H) \otimes_{\max} \mathscr{C}$$

Putting this all together, we have a map

$$A \otimes_{\min} \mathscr{C} \quad \longrightarrow \quad A^{**} \otimes_{\max} \mathscr{C}$$

which coincides with $\mathrm{id}_A \otimes \mathrm{id}_{\mathscr{C}}$ on $A \otimes_{\min} \mathscr{C}$. Since $\|\cdot\|_{A \otimes_{\max} B} = \|\cdot\|_{A^{**} \otimes_{\max} B}$ for any C*-algebras A, B, it follows that $A \otimes_{\min} \mathscr{C} = A \otimes_{\max} \mathscr{C}$.

Now suppose $(A, \mathscr{C})$ is nuclear. Then the inclusion $A \subseteq \mathcal{B}(H)$ is then seen to be max-injective:

$$A \otimes_{\max} \mathscr{C} = A \otimes_{\min} \mathscr{C} \subseteq \mathcal{B}(H) \otimes_{\min} \mathscr{C} = \mathcal{B}(H) \otimes_{\max} \mathscr{C}$$

But we know thanks to theorem 5.2.8 that this is in fact an equivalent characterization of A possessing the WEP. $\square$

Recalling that a C*-algebra A is nuclear if and only if A^{**} has the WEP, we thus have the following corollary.

Corollary 5.2.10. A C*-algebra A is nuclear if and only if $(A^{**}, \mathscr{C})$ is a nuclear pair.

In other words, nuclearity on A can be determined by tensoring A^{**} with a single other algebra - $\mathscr{C}$ - instead of having to tensor A with every other algebra B.

5.3 Local Reflexivity and a Partial Converse to Nuclear $\implies$ WEP

The material in this section is adopted from [36], primarily chapters 8 and 9, as well as [2], where many of the concepts below first emerged.

Our goal for this section is to obtain some sort of relationship between the spaces $(A \otimes_\alpha B)^* *$ and $A^* * \otimes_\alpha B^* *$, for arbitrary C*-algebras A, B and C*-tensor norm α. Starting with the inclusion morphism $\iota_\alpha : A \odot B(A \otimes_\alpha B)^{**}$, which we know (from theorem 3.2.11) can be decomposed as the tensor product of restriction maps:

$$\iota_\alpha = \iota_\alpha|_A \odot \iota_\alpha|_B, \qquad \begin{array}{ccccc} \iota_\alpha|_A & : & A & \to & (A \otimes_\alpha B)^{**} \\ \iota_\alpha|_B & : & B & \to & (A \otimes_\alpha B)^{**} \end{array}$$

which we recall have commuting ranges. We can now normalize these restrictions to obtain maps

$$\begin{array}{ccccc} (\iota_\alpha|_A)^{\cdot\cdot} & : & A^{**} & \to & (A \otimes_\alpha B)^{**} \\ (\iota_\alpha|_B)^{\cdot\cdot} & : & B^{**} & \to & (A \otimes_\alpha B)^{**} \end{array}$$

We'd like to stitch these maps back together into a map from $A^{**} \odot B^{**}$ into $(A \otimes_\alpha B)^{**}$, but in order to do this (that is, in order for the resulting "stitching" to be a homomorphism) we need to ensure $(\iota_\alpha|_A)^{\cdot\cdot}$ and $(\iota_\alpha|_B)^{\cdot\cdot}$ have commuting ranges. Fortunately this is true. To see this, recall that for Banach spaces X, Y and $u \in \mathcal{B}(X, Y^*)$, $\operatorname{ran} \ddot{u} = \overline{\operatorname{ran} u}^{w*}$. Thus

$$\operatorname{ran}(\iota_\alpha|_A)^{\cdot\cdot} = \overline{\operatorname{ran} \iota_\alpha|_A}^{\sigma-wot} \subseteq \overline{(\operatorname{ran} \iota_\alpha|_B)'}^{\sigma-wot} = (\operatorname{ran} \iota_\alpha|_B)'$$

and in turn

$$(\operatorname{ran}(\iota_\alpha|_A)^{\cdot\cdot})' \supseteq (\operatorname{ran} \iota_\alpha|_B)'' = \overline{\operatorname{ran} \iota_\alpha|_B}^{\sigma-wot} = \operatorname{ran}(\iota_\alpha|_B)^{\cdot\cdot}$$

and also $\operatorname{ran}(\iota_\alpha|_A)^{\cdot\cdot} \subseteq (\operatorname{ran}(\iota_\alpha|_B)^{\cdot\cdot})'$.

In turn, we obtain

$$\widetilde{\iota}_\alpha := (\iota_\alpha|_A)^{\cdot\cdot} \odot (\iota_\alpha|_B)^{\cdot\cdot} : A^{**} \odot B^{**} \to (A \otimes_\alpha B)^{**}$$

Proposition 5.3.1. Let $\widetilde{\iota}_{\min}$ and $\widetilde{\iota}_{\max}$ be the maps defined above, and $J : A \otimes_{\max} B \to A \otimes_{\min} B$ the canonical quotient map. Then $\widetilde{\iota}_{\min} = J^{**} \circ \widetilde{\iota}_{\max}$, and both $\widetilde{\iota}_{\min}$ and $\widetilde{\iota}_{\max}$ are injective.

Proof. That $\widetilde{\iota}_{\min} = J^{**} \circ \widetilde{\iota}_{\max}$ is relatively straightforward. First, it is routine to check that $\widetilde{\iota}_{\min}|_{A \odot B} = J^{**} \circ \widetilde{\iota}_{\max}|_{A \odot B}$. Fix $b \in B$, $a^{**} \in A^{**}$, and let $a^{**} = w^* - \lim_\lambda a_\lambda$, with $a_\lambda \in A$ and $\|a_\lambda\| \leq \|a^{**}\|$. We don't have a topology on $A^{**} \odot B^{**}$, so we don't merely have $a^{**} \otimes b = \lim_\lambda a_\lambda \otimes b$. However, following through the definitions we see

$$\widetilde{\iota}_{\min}(a^{**} \otimes b) = (\iota_{\min}|_A)^{\cdot\cdot}(a^{**}) \iota_{\min}|_B(b) = w^* - \lim_\lambda \iota_{\min}|_A(a) \iota_{\min}|_B(b) = w^* - \lim_\lambda \iota_{\min}(a_\lambda \otimes b)$$

and similarly $\widetilde{\iota}_{\max}(a^{**} \otimes b) = w^*\!-\!\lim_\lambda \iota_{\max}(a_\lambda \otimes b)$. Using weak-*-to-weak-* continuity of J^{**}, we thus have

$$\widetilde{\iota}_{\min}(a^{**} \otimes b) = w^*\!-\!\lim_\lambda \iota_{\min}(a_\lambda \otimes b) = w^*\!-\!\lim_\lambda J^{**} \circ \widetilde{\iota}_{\max}(a_\lambda \otimes b) = J^{**} \circ \widetilde{\iota}_{\max}(a^{**} \otimes b)$$

Repeating this for B yields $\widetilde{\iota}_{\min} = J^{**} \circ \widetilde{\iota}_{\max}$.

Equipped with this formula, one notices that if $\widetilde{\iota}_{\min}$ is injective then $\widetilde{\iota}_{\max}$ is also automatically injective.

Begin by embedding $A \subseteq A^{**} \subseteq \mathcal{B}(H)$ and $B \subseteq B^{**} \subseteq \mathcal{B}(K)$, so that all of $A \odot B$, $A \otimes_{\min} B$, $A^{**} \odot B^{**}$ and $A^{**} \otimes_{\min} B^{**}$ are contained isometrically in $\mathcal{B}(H \otimes K)$. Consider the identity map $\mathrm{id} : A \otimes_{\min} B \to \mathcal{B}(H \otimes K)$, and extend this by normalizing to a map $\varphi := \ddot{\mathrm{id}} : (A \otimes_{\min} B)^{**} \to \mathcal{B}(H \otimes K)$. Walking through the definitions, one can check that $\varphi(\widetilde{\iota}_{\min}(a \otimes b)) = a \otimes b$, for $a \in A, b \in B$. Moreover,

$$\begin{aligned} \varphi(\widetilde{\iota}_{\min}(a^{**} \otimes b)) = \varphi(w^*\!-\!\lim_\lambda \widetilde{\iota}_{\min}(a_\lambda \otimes b)) &= \sigma\!-\!wot\!-\!\lim_\lambda \varphi(\iota_{\min}(a_\lambda \otimes b)) \\ &= \sigma\!-\!wot\!-\!\lim_\lambda a_\lambda \otimes b = a^{**} \otimes b \end{aligned}$$

Alas, we have yet another topological conundrum here: how do we justify the last equality? For one, note that the quantities involved - $a_\lambda \otimes b$ and $a^{**} \otimes b$ - are all bounded, and so $\sigma-wot$ convergence is really equivalent to plain old *wot*-convergence. We can also write

$$\sigma\!-\!wot\!-\!\lim_\lambda a_\lambda \otimes b = wot\!-\!\lim_\lambda (a_\lambda \otimes I_K)(I_H \otimes b) = \left(wot\!-\!\lim_\lambda a_\lambda \otimes I_K\right)(I_H \otimes b)$$

and so we really only need $wot\!-\!\lim_\lambda a_\lambda \otimes I_K = a^{**} \otimes I_K$ to be sure. One should think of tensoring a^{**} with I_K as concatenating $\dim K$ copies of a^{**} along the diagonal (e.g., $T \otimes I_{\ell^2(\mathbb{N})} = T^{(\infty)}$). It is well known that the map $T \mapsto T^{(\kappa)}$), where κ is a cardinal number and $T^{(\kappa)} := \bigoplus_{i \in \kappa} T$, takes $\sigma-wot$-convergent nets to *wot*-convergent nets[13], and $\sigma-wot$ convergence of a_λ is synonymous with weak-* convergence (by virtue of our embedding $A^{**} \subseteq \mathcal{B}(\mathcal{H})$), whence the result holds.

Repeating again for B yields $\varphi(\widetilde{\iota}_{\min}(x)) = x$ for all $x \in A^{**} \odot B^{**}$, whence $\widetilde{\iota}_{\min}$ is injective. □

Remark. An unrelated, but nonetheless fun corollary of this result is the following "distributive property" of centers of tensor products of C*-algebras.

Theorem 5.3.2 (Archbold - [**archbold-center**]; Theorem 3)**.** Let A, B be C*-algebras, and let $Z(\cdot)$ denote the center of a C*-algebra. Then for any C*-tensor norm β on $A \odot B$, $Z(A \otimes_\beta B) = Z(A) \otimes_\beta Z(B)$.

Now we know there is a natural inclusion of $A^{**} \odot B^{**}$ in both $(A \otimes_{\min} B)^{**}$ and $(A \otimes_{\max} B)^{**}$. These spaces induce norms on $A^{**} \odot B^{**}$, given by letting

$$\|t\|_{(A \otimes_\alpha B)^{**}} := \|\widetilde{\iota}_\alpha(t)\| \qquad \forall t \in A^{**} \odot B^{**},\ \alpha \in \{\min, \max\}$$

In the max case, we actually already know what norm this gives.

Theorem 5.3.3. Given $t \in A^{**} \odot B^{**}$, we have $\|t\|_{(A \otimes_{\max} B)^{**}} = \|t\|_{A^{**} \otimes_{\mathrm{bin}} B^{**}}$.

Proof. It's not too hard to see that $\|t\|_{(A \otimes_{\max} B)^{**}} \le \|t\|_{A^{**} \otimes_{\mathrm{bin}} B^{**}}$: if we embed $(A \otimes_{\max} B)^{**}$ isometrically in $\mathcal{B}(\mathcal{H})$ (via some isometric *-representation π), then $\pi \circ \widetilde{\iota}_{\max}$ is a *-representation of $A^{**} \odot B^{**}$ which is normal when restricted to each tensorant, and so by definition $\|t\|_{A^{**} \otimes_{\mathrm{bin}} B^{**}} \ge \|\pi(\widetilde{\iota}_{\max}(t))\| = \|\widetilde{\iota}_{\max}(t)\| = \|t\|_{(A \otimes_{\max} B)^{**}}$.

Consider an *isometric* *-representation $\rho : A^{**} \otimes_{\mathrm{bin}} B^{**} \to \mathcal{B}(H)$ which is normal when restricted to each tensorant. To see that such a representation exists, consider the direct sum over all representations of $A^{**} \odot B^{**}$ which are normal when restricted to each tensorant. By definition of the binormal norm, this *-representation is isometric.

Suppose now that we could construct a map $\psi : (A \otimes_{\max} B)^{**} \to \mathcal{B}(H)$ such that $\psi \circ \widetilde{\iota}_{\max} = \rho|_{A^{**} \odot B^{**}}$. This would then tell us that

$$\|t\|_{A^{**} \otimes_{\mathrm{bin}} B^{**}} = \|\rho(t)\| = \|\psi(\widetilde{\iota}_{\max}(t))\| \le \|\widetilde{\iota}_{\max}(t)\| = \|t\|_{(A \otimes_{\max} B)^{**}}$$

[13] This is true even if κ is uncountable. To check this, one simply has to note that uncountable sums are convergent if and only if all but countably many terms are zero, and so we can restrict our attention to countable things.

thus establishing the other inequality. So, how can we construct ψ?

Fortunately this matter is straightforward too. Define $\varphi : A \otimes_{\max} B \to \mathcal{B}(H)$ to be the extension of $\rho|_{A\odot B}$ to $A \otimes_{\max} B$, and define $\psi := \ddot{\varphi}$. Let $b \in B$ be fixed, and suppose $a^{**} = \lim_\lambda a_\lambda$, as usual. Then

$$\begin{aligned}\psi(\widetilde{\iota}_{\max}(a^{**} \otimes b)) = \psi\left(\lim_\lambda \widetilde{\iota}_{\max}(a_\lambda \otimes b)\right) &= \lim_\lambda \psi\left(\iota_{\max}(a_\lambda \otimes b)\right) \\ &= \lim_\lambda \psi(a_\lambda \otimes b) = \lim_\lambda \pi(a_\lambda \otimes b) = \pi(a^{**} \otimes b)\end{aligned}$$

where the last equality holds since π is normal when restricted to A^{**}. Repeating for B^{**} yields $\psi \circ \widetilde{\iota}_{\max} = \pi$, completing the proof. □

Generally speaking, the min-norm admits no such simple characterization, which prompts us to introduce a definition.

Definition 5.3.4. A C*-algebra A is called *locally reflexive* if for any other C*-algebra B, we have equality of norms

$$\|\cdot\|_{A^{**}\otimes_{\min} B^{**}} = \|\cdot\|_{(A\otimes_{\min} B)^{**}}$$

Definition 5.3.5. A C*-algebra A is called *locally reflexive* if for any other C*-algebra B, we have an equality of norms

$$\|\cdot\|_{(A\otimes_{\min} B)^{**}} = \|\cdot\|_{A^{**}\otimes_{\min} B}$$

when restricted to $A^{**} \odot B$.

Theorem 5.3.6. Nuclear C*-algebras are locally reflexive.

Proof. If A is nuclear, then $A \otimes_{\min} B = A \otimes_{\max} B$. Moreover, if A is nuclear then A^{**} is semidiscrete, so $A^{**} \otimes_{\text{bin}} B^{**} = A^{**} \otimes_{\min} B^{**}$. Thus

$$\begin{aligned}\|t\|_{A^{**}\otimes_{\min} B^{**}} &= \|t\|_{A^{**}\otimes_{\text{bin}} B^{**}} \\ &= \|t\|_{(A\otimes_{\max} B)^{**}} \\ &= \|t\|_{(A\otimes_{\min} B)^{**}}\end{aligned}$$

whence A is locally reflexive. □

Before getting to the main result of this section, we need one more black box. Recall that $\mathcal{B}(H)$ and $\mathscr{C}$ is always a nuclear pair for any Hilbert space H. Kirchberg observed that this is actually a special case of a more general phenomenon, the proof of which mirrors our proof of theorem 5.2.4 (unfortunately, however, requiring modifications which we won't have time to cover).

Theorem 5.3.7 (Kirchberg)**.** Let F be a free group, and M a von Neumann algebra. Then

$$C^*(F) \otimes_{\text{nor}} M = C^*(F) \otimes_{\max} M$$

To see that this is in fact a generalization of theorem 5.2.4, note that $M = \mathcal{B}(H)$ is semidiscrete, and so the normal norm and the min norm on $C^*(F) \odot \mathcal{B}(H)$ coincide.

With this, we arrive at the following.

Theorem 5.3.8. A C*-algebra A is nuclear if and only if it is locally reflexive and has the WEP.

Proof. First, if A has the WEP, then $A \otimes_{\min} \mathscr{C} = A \otimes_{\max} \mathscr{C}$. If A is additionally locally reflexive, then $\|\cdot\|_{A^{**}\otimes_{\min}\mathscr{C}} = \|\cdot\|_{(A\otimes_{\min}\mathscr{C})^{**}}$, restricted to $A^{**} \odot \mathscr{C}$. Then

$$\|\cdot\|_{A^{**}\otimes_{\min}\mathscr{C}} = \|\cdot\|_{A^{**}\otimes_{\text{bin}}\mathscr{C}^{**}} \quad \text{on } A^{**} \odot \mathscr{C}$$

On the other hand, we know that $\|\cdot\|_{A^{**}\otimes_{\text{bin}}\mathscr{C}^{**}} = \|\cdot\|_{A^{**}\otimes_{\text{nor}}\mathscr{C}}$, but by Kirchberg's observation this is equal to $\|\cdot\|_{A^{**}\otimes_{\max}\mathscr{C}}$, thus proving that

$$\|\cdot\|_{A^{**}\otimes_{\min}\mathscr{C}} = \|\cdot\|_{A^{**}\otimes_{\max}\mathscr{C}}$$

which, as we know now, is an equivalent characterization of nuclearity of A. □

A Appendices

A.1 Tensor Products of Hilbert Spaces

Theorem A.1.1. Given Hilbert spaces H, K with inner products $\langle \cdot \rangle_H$ and $\langle \cdot \rangle_K$ respectively, the algebraic tensor product $H \odot K$ can be endowed with a *unique* inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle_H \langle k_1, k_2 \rangle_K$$

The completion of $H \odot K$ with respect to the norm induced by this inner product is denoted $H \otimes K$, and called the *Hilbert space tensor product of H and K* (or simply the tensor product).

Lemma A.1.2. Given Hilbert spaces H, K and operators $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, the operator

$$\begin{array}{rcrcl} A \odot B & : & H \odot K & \to & H \otimes K \\ & & h \otimes k & \mapsto & Ah \otimes Bk \end{array}$$

is a well-defined linear operator which is continuous with respect to the inner product norm on $H \odot K$, and so extends to a unique operator

$$A \otimes B \in \mathcal{B}(H \otimes K) \qquad A \otimes B|_{H \odot K} = A \odot B$$

Moreover, the norm of $A \otimes B$ is equal to $\|A\|\|B\|$.

Theorem A.1.3. Given Hilbert spaces H, K, there exists a canonical, faithful embedding of $\mathcal{B}(H) \odot \mathcal{B}(K) \subseteq \mathcal{B}(H \otimes K)$.

A.2 Completely Positive and Completely Bounded Maps

It is assumed that the reader is familiar with how to to endow $M_n(A)$ with a C*-algebra structure. For an introductory exposition on these ideas, see chapter 1 in [33]. We will list some relevant results here for reference in the body of the text.

Fact A.2.1. Let A be a C*-algebra, and let $[a_{ij}] \in M_n(A)$. Then the following are equivalent:

1. $[a_{ij}]$ is positive.
2. $[a_{ij}]$ is a sum of matrices of the form $[a_i^* a_j]$, $a_i \in A$.
3. For all $x_i \in A$, $\sum_{ij} x_i^* a_{ij} x_j \geq 0$.

Definition A.2.2. An *operator space* is a *norm-closed subspace* of a C*-algebra. An *operator system* is a *unital, self-adjoint* operator space.

Operator spaces, while lacking any algebra structure, inherit a notion of "positivity" from the C*-algebra they live inside: if $E \subseteq A$ is an operator space in a C*-algebra A, an element $x \in E$ is *positive* iff x is positive in A. Note that this is independent of the chosen embedding $E \subseteq A$, since for C*-subalgebras $A \subseteq B$, the spectrum of an element $x \in A$ is identical to the spectrum of x in B. Thus, if $E \subseteq A, B \subseteq A \oplus B$, then $x \in E$ is positive iff it is positive in $A \oplus B$, iff it is positive in A, iff it is positive in B.

Definition A.2.3. Given operator spaces E, F and a linear map $u : E \to F$, the n^{th}*-ampliation* of u is the map $u_n : M_n(E) \to M_n(F)$ given by $u_n([a_{ij}]) := [u(a_{ij})]$. For a given property of u, say property "X", u is said to be *"n-X"* (resp. *"completely X"*) if the "n^{th} ampliation" (resp. *every* ampliation) of u also satisfies property "X". For instance, u is *completely positive* if u_n is positive for every positive integer n, *completely contractive* if each u_n is contractive, and *completely isometric* if each u_n is an isometry.

Definition A.2.4. A map $u : E \to F$ is *completely bounded* if each ampliation u_n is bounded, and *additionally* $\sup_n \|u_n\| < \infty$. The *cb-norm* of a completely bounded map is given by $\|u\|_{cb} = \sup_n \|u_n\|$.

We denote by $CP(E, F)$ and $CB(E, F)$ the set of completely positive and completely bounded maps respectively from $E \to F$. It is easy to see that $CP(E, F) \subseteq CB(E, F)$.

Example. If A is a C*-algebra and $f \in S(A)$, then f is completely positive.

Theorem A.2.5 (Stinespring - [33]; Theorem 3.10)**.** Let A be a C*-algebra and $\varphi : C(X) \to A$ a positive map. Then φ is automatically completely positive.

Theorem A.2.6 ([28]; Lemma 5.3)**.** Let $\varphi : A \to B$ be a completely positive map of C*-algebras, and let e_λ be an approximate identity for A. Then $\|\varphi\| = \lim_\lambda \|\varphi(e_\lambda)\|$.

Proposition A.2.7 (Cauchy-Schwarz for 2-positive maps - [33]; Proposition 3.3)**.** Let A, B be unital C*-algebras, and $\phi : A \to B$ a unital 2-positive map. Then $\phi(a)^*\phi(a) \leq \phi(a^*a)$.

In the body of the text we will see that for each state $f \in S(A)$ where A is a non-unital C*-algebra, f extends to some $\widetilde{f} \in S(\widetilde{A})$, where $\widetilde{A}$ denotes the unitization of A. Thus $\widetilde{f}$ satisfies the Cauchy-Schwarz inequality above, and in turn so too does f.

In this paper we are wont to work with maps $T : B \to A^*$ between a C*-algebra and the *dual* of a C*-algebra. The space A^* comes with a natural "matricial order structure", which allows us to speak sensibly about whether or not T is "completely positive".

Fact A.2.8. There is a one-to-one correspondence Λ between $M_n(A^*)$ and $M_n(A)^*$ given by

$$M_n(A^*) \ni [f_{ij}] \mapsto \Lambda([f_{ij}]) := \left([a_{ij}] \mapsto \sum_{ij} f_{ij}(a_{ij}) \right) \in M_n(A)^*$$

Using Λ, we can now define "positivity" of elements in $M_n(A^*)$.

Definition A.2.9. An element $[f_{ij}] \in M_n(A^*)$ is *positive*, written $[f_{ij}] \geq 0$, if the corresponding functional $\Lambda([f_{ij}])$ is a positive linear functional on $M_n(A)$. A map $T : B \to A^*$ is *n-positive* if $T_n : M_n(B) \to M_n(A^*) \cong M_n(A)^*$ takes positive elements of $M_n(B)$ to positive functionals in $M_n(A)^*$. T is *completely positive* if it is n-positive for all n.

Proposition A.2.10. A map $T : B \to A^*$ is completely positive if and only if for all $b_i \in B$, $a_i \in A$, $i = 1, ..., n$, $n \in \mathbb{N}$,

$$\sum_{ij} T(b_i^* b_j)(a_i^* a_j) \geq 0$$

The proof is a simple double application of fact A.2.1.

Proposition A.2.11. Suppose $T : A \to B$ is a completely positive map between C*-algebras. Then $T^* : B^* \to A^*$ is also completely positive.

Proof. First observe that if $T : A \to B$ is positive, then T^* is positive, since if $f \in B^*$ is positive, then $T^*(f)(a^*a) = f(T(a^*a)) \geq 0$ for all $a \in A$, so $T^*(f)$ is positive.

Now, a natural question to ask is whether ampliation commutes with the Banach space adjoint - in other words, do we have $(T_n)^* = (T^*)_n$? A simple calculation shows that the claim is indeed true, under the identification $M_n(A^*) \cong M_n(A)^*$:

$$(T_n)^*(f)([a_{ij}]) = (f \circ T_n)([a_{ij}]) = f([T(a_{ij})]) = \sum_{ij} f_{ij}(T(a_{ij})) = (T^*)_n([f_{ij}])([a_{ij}])$$

This completes the argument, since as T_n is positive, then $(T_n)^*$ is positive, and so $(T^*)_n$ is positive. □

Proposition A.2.12 ([27]; Lemma 2.1)**.** Let A be a non-unital C*-algebra, $\widetilde{A}$ its unitization, e_λ an approximate identity for A, and $f \in A_+^*$ a positive linear functional. Then there exists a unique positive linear functional $\widetilde{f} \in \widetilde{A}_+^*$ such that

$$\widetilde{f}(1_{\widetilde{A}}) = \|f\| = \lim_\lambda f(e_\lambda) = \lim_\lambda f(e_\lambda^2)$$

Proof. Given $f \in A_+^*$, all we have to do is normalize f to obtain a linear functional $\ddot{f} : A^{**} \to \mathbb{C}$, which is also (completely) positive (see proposition 2.2.14). Since we can embed $A \subseteq \widetilde{A} \subseteq A^{**}$, we can define $\widetilde{f} := \ddot{f}|_{\widetilde{A}}$. Then $\widetilde{f}(1_{\widetilde{A}}) = \ddot{f}(\widetilde{1}_{\widetilde{A}}) = \lim_\lambda \ddot{f}(e_\lambda) = \lim_\lambda f(e_\lambda)$, since $\widetilde{1}_{\widetilde{A}} = w^*\!-\!\lim_\lambda e_\lambda$ and $\ddot{f}$ is weak-* continuous. □

Definition A.2.13. Let A be a non-unital C*-algebra with unitization $\widetilde{A}$, and let e_λ be an approximate identity for A. We define $\varepsilon_A : A^* \to \widetilde{A}^*$ by letting $\varepsilon_A(f)(a + \lambda 1_{\widetilde{A}}) := f(a) + \lambda\varepsilon_A(f)(1_{\widetilde{A}})$, with $\varepsilon_A(f)(1_{\widetilde{A}}) := \lim_\lambda f(e_\lambda)$.

We must of course check that this map is well-defined first. Given any $f \in A^*$, by the Jordan Decomposition ([34], Proposition 2) f can be *uniquely* decomposed as the linear combination of 4 positive linear functionals:

$$f = (f_1 - f_2) + i\,(f_3 - f_4)\,, \qquad f_j \in A^*_+$$

Since $\lim_\lambda f_j(e_\lambda)$ exists for each j, and this decomposition is unique, it follows that ε_A is well-defined. Linearity of ε_A then follows trivially from its formula.

Proposition A.2.14 ([27]; Lemma 2.2)**.** The map $\varepsilon_A : A^* \to \widetilde{A}^*$ defined in definition A.2.13 is completely positive and has norm 1.

Proof. Let $0 \le [f_{ij}] \in M_n(A^*)$, and let $[\widetilde{f}_{ij}] := [\varepsilon_A(f_{ij})] \in M_n(\widetilde{A}^*)$. Then $[\widetilde{f}_{ij}]$ is positive if and only if $\sum_{ij} \widetilde{f}_{ij}(\widetilde{a}_i^*\widetilde{a}_j) \ge 0$ for all $\widetilde{a}_i \in \widetilde{A}$. Writing $\widetilde{a}_i := a_i + \gamma_i 1_{\widetilde{A}}$ for some $\gamma_i \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{ij} \widetilde{f}_{ij}(\widetilde{a}_i^*\widetilde{a}_j) &= \sum_{ij} \widetilde{f}_{ij}\left(\left(a_i + \gamma_i 1_{\widetilde{A}}\right)^* \left(a_j + \gamma_j 1_{\widetilde{J}}\right)\right) \\
&= \sum_{ij} \widetilde{f}_{ij}\left(a_i^* a_j + \overline{\gamma_i} a_j + \gamma_j a_i^* + \overline{\gamma_i}\gamma_j 1_{\widetilde{A}}\right) \\
&= \lim_\lambda \sum_{ij} \widetilde{f}_{ij}\left(a_i^* a_j + \overline{\gamma_i} e_\lambda a_j + \gamma_j e_\lambda a_i^* + \overline{\gamma_i}\gamma_j e_\lambda^2\right) \\
&= \lim_\lambda \sum_{ij} \widetilde{f}_{ij}\left((a_i + \gamma_i e_\lambda)^* (a_j + \gamma_j e_\lambda)\right) \\
&= \lim_\lambda \sum_{ij} f_{ij}\left((a_i + \gamma_i e_\lambda)^* (a_j + \gamma_j e_\lambda)\right) \ge 0
\end{aligned}$$

whence ε_A is completely positive. That ε_A takes states to states gives us that ε_A has norm 1. $\square$

Theorem A.2.15 (Canonical Shuffle)**.** The spaces

$$M_m \odot M_n, \quad M_m(M_n), \quad M_n(M_m), \quad M_{mn}$$

are all canonically isometrically completely order isomorphic to one another. That is, between any two, there is a canonical isometric *-isomorphism (which is completely positive, and has completely positive inverse). The maps in question are simply "deleting brackets" (as in between $M_m(M_n)$ and M_{mn}) or "shuffling brackets" (as in $M_m(M_n)$ and $M_n(M_m)$).

The proof of this result is a straightforward calculation which we leave to the reader.

Theorem A.2.16 (Choi - [33]; 3.14)**.** Let $\phi : M_n \to B$ be a linear map into a C*-algebra B. Then ϕ is completely positive if and only if $[\phi(e_{ij})] \in M_n(B)$ is completely positive.

Corollary A.2.17. Let A be a C*-algebra. There is a one-to-one correspondence between completely positive maps $\psi : M_n \to A$ and $M_n(A)_+$, given by

$$\begin{array}{rccc}
\Phi & : & CP(M_n, A) & \to & M_n(A)_+ \\
 & & \psi & \mapsto & [\psi(e_{ij})]
\end{array}$$

where e_{ij} are the matrix units for M_n. We also calculate

$$\Phi^{-1}([a_{ij}])([c_{ij}]) = \sum_{ij} c_{ij} a_{ij}, \quad [a_{ij}] \in M_n(A)_+,\ [c_{ij}] \in M_n$$

Proof. A simple calculation shows that $M_n(M_n) \ni (\frac{1}{n}[e_{ij}])^2 = \frac{1}{n}[e_{ij}] = \frac{1}{n}[e_{ij}]^*$, implying the matrix $\frac{1}{n}[e_{ij}]$ is a projection, and so $[e_{ij}]$ is positive. Thus if $\psi \in CP(M_n, A)$, then $\psi_n([e_{ij}]) = [\psi(e_{ij})]$ is positive. On the other hand if $[a_{ij}] \geq 0$, then by A.2.16 the map $\Phi^{-1}([a_{ij}]$ is completely positive iff $(\Phi^{-1})_n([a_{ij}])([e_{ij}]) = [\Phi^{-1}([a_{ij}])(e_{ij})] \geq 0$, but this is clearly equal to $[a_{ij}]$, which is positive by assumption. Showing that Φ is an isomorphism is a simple matter. $\square$

Proposition A.2.18 ([5]; Proposition 1.5.14)**.** Let A be a unital C*-algebra. There is a one-to-one correspondence Ψ between $CP(A, M_n)$ and $M_n(A)^*_+$, given by

$$\Psi(\varphi) := \left([a_{ij}] \mapsto \sum_{ij} \varphi(a_{ij})_{ij} \right) \in M_n(A)^*_+$$

Proof. Suppose $\varphi \in CP(A, M_n)$. Then $\varphi_n([a_i^* a_j]) = [\varphi(a_i^* a_j)] \in (M_{n^2})_+$. Let e_i denote the standard basis vectors on $\mathbb{C}^n$, and let ξ denote the concatenation of $e_1, e_2, ..., e_n$ as a vector in $\mathbb{C}^{n^2}$. Then

$$\sum_{ij} \varphi(a_i^* a_j)_{ij} = \langle [\varphi(a_i^* a_j)]\xi, \xi \rangle \geq 0$$

Thus, by fact A.2.1, $\Psi(\varphi) \geq 0$.

The converse can be proven by a clever argument due to [5]. Given $f \in M_n(A)^*_+$, denoting $[f_{ij}] := \Lambda^{-1}(f) \in M_n(A^*)$, then $\Psi^{-1}(f)$ is the map taking $a \mapsto [f_{ij}(a)]$ (alternatively, $f_{ij}(a) := f(ae_{ij})$, where e_{ij} denote the matrix units). Consider the GNS representation $\pi_f : M_n(A) \to \mathcal{B}(H)$ (with cyclic vector ξ_f) corresponding to f. Denote by $V : \mathbb{C}^n \to H$ the linear operator taking $e_i \mapsto \pi_f(e_{1,i})\xi_f$. Then

$$\begin{aligned}
\langle V^* \pi_f(\mathrm{diag}(a, ..., a)) V e_i, e_j \rangle &= \langle \pi_f(\mathrm{diag}(a, ..., a) e_{1,i})\xi_f, \pi_f(e_{1,j})\xi_f \rangle \\
&= \langle \pi_f(e_{j,1}\,\mathrm{diag}(a, ..., a) e_{1,i})\xi_f, \xi_f \rangle \\
&= \langle \pi_f(ae_{ji})\xi_f, \xi_f \rangle \\
&= f(ae_{j,i}) = f_{ji}(a) = \langle \Psi^{-1}(f)(a)e_i, e_j \rangle
\end{aligned}$$

Thus

$$\Psi^{-1}(f)(a) = V^* \pi_f(\mathrm{diag}(a, ..., a))V$$

Since $\pi_f(\mathrm{diag}(a, ..., a))$ is a *-homomorphism, it follows that $\Psi^{-1}(f)$ is completely positive. $\square$

Theorem A.2.19 (Multiplicative Domains - [33]; Theorem 3.18)**.** Let A, B be unital C*-algebras, and $\varphi : A \to B$ a unital completely positive map. Then the sets

$$\{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a), \quad \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$
$$\{a \in A \mid \varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(ba) = \varphi(b)\varphi(a), \quad \forall b \in A\}$$

are *equal*, and are commonly referred to as the "multiplicative domain" of φ. When restricted to this set, φ becomes a *-homomorphism.

The next four results are classics in the realm of completely positive maps, and as such we refer the reader to other sources for their proofs.

Theorem A.2.20 (Arveson's Extension Theorem (CP Version) - [36]; Theorem 1.39)**.** Let $E \subseteq A \subseteq \mathcal{B}(K)$ be an operator system in a unital C*-algebra, and $u \in CP(E, \mathcal{B}(H))$. Then there exists an extension $\widetilde{u} \in CP(A, \mathcal{B}(H))$ such that $\widetilde{u}|_E = u$ and $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.

Theorem A.2.21 (Arveson's Extension Theorem (CB Version) - [36]; Theorem 1.18)**.** Let $E \subseteq F \subseteq \mathcal{B}(K)$ be operator spaces, and $u \in CP(E, \mathcal{B}(H))$. Then there exists an extension $\widetilde{u} \in CB(F, \mathcal{B}(H))$ such that $\widetilde{u}|_E = u$ and $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.

Theorem A.2.22 (Stinespring's Dilation Theorem - [33]; Theorem 4.1)**.** Let A be a unital C*-algebra, and $\phi \in CP(A \to \mathcal{B}(H))$. Then there exists a Hilbert space K, a unital *-homomorphism $\pi : A \to \mathcal{B}(K)$, and a bounded operator $V : H \to K$ such that $\phi(a) = V^*\pi(a)V$, and so in turn $\|\phi\| = \|\phi(1)\| = \|V\|^2$. Moreover, if ϕ is unital then V is an isometry, and if both H and A are separable then K can be chosen to be separable as well. The tuple (π, K, V) is referred to as a "Stinespring dilation" of ϕ.

Theorem A.2.23 (Wittstock's Dilation Theorem - [36]; Theorem 1.50)**.** Let H, K be Hilbert spaces, E an operator system, and B a unital C*-algebra such that $E \subseteq B \subseteq \mathcal{B}(K)$. Consider a completely bounded map $u : E \to \mathcal{B}(H)$. Then there is a Hilbert space $\widehat{H}$, a unital *-homomorphism $\pi : B \to \mathcal{B}(\widehat{H})$, and operators $V_1, V_2 \in \mathcal{B}(H, \widehat{H})$ such that

$$\forall x \in E \qquad u(x) = V_2^* \pi(x) V_1$$

and $\|u\|_{cb} = \|V_1\|\|V_2\|$. Moreover, if both B and K are separable, then $\widehat{H}$ can be chosen to be separable as well. The tuple $(\pi, \widehat{H}, V_1, V_2)$ is referred to as a "Wittstock dilation" of u.

Lemma A.2.24 ([7]; Lemma 2.2)**.** Let A be a unital C*-algebra, M a von Neumann algebra, and $\varphi : A \to M$ a completely positive map. Denote $b := \varphi(1)$. Then there exists a unital, completely positive map $\psi : A \to M$ such that $\varphi(a) = b^{1/2}\psi(a)b^{1/2}$.

A.3 von Neumann Algebras

Definition A.3.1. Given von Neumann algebras M, N, an (anti-)linear map $\phi : M \to N$ is *normal* if whenever $x_\lambda \in M$ is an increasing net converging strongly to $x \in M$, then $\phi(x_\lambda)$ is increasing and converges strongly to $\phi(x)$.

Theorem A.3.2 ([3]; Proposition III.2.2.2)**.** Let M, N be von Neumann algebras, and $\phi : M \to N$ a completely positive map. Then ϕ is normal if and only if ϕ is continuous with respect to the $\sigma - wot$ topologies on M and N. Thus, the normal states on M are precisely the ultraweakly-continuous states.

Definition A.3.3. Let $A \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and $X \subseteq H$ (any set). We say X is *cyclic for A* if the closure of $AX := \{ax \;:\; a \in A,\; x \in X\}$ is all of H. We say X is *separating for A* if whenever $a \in A$ is such that $ax = 0$ for all $x \in X$, then $a = 0$. We say $\xi \in H$ is a *cyclic (resp. separating) vector for A* if $\{\xi\}$ is a cyclic (resp. separating) set for A.

Theorem A.3.4 ([15]; Part I, Chapter 1, Proposition 5)**.** Let $A \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and $X \subseteq H$. Then X is cyclic for A if and only if X is separating for A'. From this we also have that X is separating for A if and only if X is cyclic for A'.

Proposition A.3.5. Let A be a *-subalgebra of $\mathcal{B}(H)$, and $\xi \in H$. Let $p \in \mathcal{B}(H)$ denote the projection onto $\overline{A\xi}$. Then $p \in A'$.

Proof. Given $h \in H$, write $h = h_1 + h_2$, where $h_1 \in \overline{A\xi}$, and $h_2 \perp \overline{A\xi}$. Then clearly for all $a \in A$, $ah_1 \in \overline{A\xi}$, and

$$\langle ah_2, \overline{A\xi}\rangle = \langle h_2, a^*\overline{A\xi}\rangle = 0$$

so $ah_2 \perp \overline{A\xi}$. Thus

$$pah = p(ah_1 + ah_2) = ah_1 = ap(h_1 + h_2) = aph$$

whence $pa = ap$ for all $a \in A$. ☐

Definition A.3.6. A von Neumann algebra A is *σ-finite* or *countably decomposable* if every family in A of non-zero mutually orthogonal projections is countable. A projection $p \in A$ is σ-finite if pAp is σ-finite.

Theorem A.3.7 ([15]; Part I, Chapter 1, Proposition 6)**.** Let A be a von Neumann algebra. Then A is σ-finite if and only if A possesses a countable separating set.

Fact A.3.8 ([15]; Part I, Chapter 2, Proposition 1 (i))**.** Let A be a von Neumann algebra and $p \in A$ a projection. Then $(pAp)' = pA'p$.

Theorem A.3.9 ([15]; Part III, Chapter 1, Corollary to Theorem 4)**.** Let A be a von Neumann algebra. Then every normal state on A is a vector state if and only if for every countably decomposable projection $p \in A$, pAp possesses a separating element.

A.4 Group C*-Algebras and Amenability

First recall the definition of the *group algebra* $k[G]$ corresponding to a group G and a field k: $k[G]$ is the algebra whose elements consist of formal sums

$$\sum_{g\in G} \alpha_g g, \qquad \alpha_g \in k,\ \alpha_g = 0 \text{ for all but finitely many } g$$

Addition and multiplication on $k[G]$ are defined as expected:

$$\begin{aligned}\left(\sum_g \alpha_g g\right)\left(\sum_h \beta_h h\right) &:= \sum_{g,h} \alpha_g \beta_h gh \\ &= \sum_g \left(\sum_k \alpha_{gk^{-1}}\beta_k\right) g\end{aligned}$$

We're primarily interested in $k = \mathbb{C}$ .The group algebra $\mathbb{C}[G]$ in particular can also be endowed with an involution:

$$\begin{aligned}\left(\sum_g \alpha_g g\right)^* &:= \sum_g \overline{\alpha_g} g^{-1} \\ &= \sum_g \overline{\alpha_{g^{-1}}} g\end{aligned}$$

If G is a discrete group, we let $\ell^2(G)$ denote the Hilbert space of square-summable functions $f : G \to \mathbb{C}$, equipped with its usual inner product $\langle f, g\rangle := \sum_{t\in G} f(t)\overline{g(t)}$. We will use δ_g to denote the standard orthonormal basis for $\ell^2(G)$ (δ_g is a point-mass of weight 1 at g). Define the *left regular representation* $\lambda : G \to \mathcal{B}(\ell^2(G))$ by

$$(\lambda(s)(f))(t) := f(s^{-1}t)$$

or in other words, $\lambda(s)\delta_t = \delta_{st}$. Note that λ is easily checked to be a unitary representation. By letting $\lambda\left(\sum_g \alpha_g g\right) := \sum_g \alpha_g \lambda(g)$, we can naturally extend λ to $\mathbb{C}[G]$ to obtain a *-representation (that λ preserves the involution on $\mathbb{C}[G]$ follows from the fact that it is a *unitary* representation on G).

Definition A.4.1. The C*-algebra

$$C^*_\lambda(G) := \overline{\lambda(\mathbb{C}[G])} \subseteq \mathcal{B}(\ell^2(G))$$

is called the *reduced group C*-algebra of G*.

This is not the only way we can turn $\mathbb{C}[G]$ into a C*-algebra.

Definition A.4.2. The *full group C*-algebra of G*, denoted $C^*(G)$, is defined as the completion of $\mathbb{C}[G]$ with respect to the "universal norm":

$$\|x\|_u := \sup\{\|\pi(x)\| \mid \pi : \mathbb{C}[G] \to \mathcal{B}(H)\}$$

where the supremum is taken over all *-representations.

Since λ is a *-representation of $\mathbb{C}[G]$, we clearly have $\|x\|_{C^*_\lambda(G)} \le \|x\|_u$, and so the map $q : \mathbb{C}[G] \to \lambda(\mathbb{C}[G])$ is contractive with respect to the universal norm on the domain and the operator norm on the codomain. Thus, q extends to a surjection $\widetilde{q} : C^*(G) \to C^*_\lambda(G)$ (that $\widetilde{q}$ is surjective follows from the fact that *-homomorphisms of C*-algebras automatically have closed range, and q's range is dense in $C^*_\lambda(G)$). Thus $C^*_\lambda(G)$ is in fact a quotient of $C^*(G)$.

Theorem A.4.3 (Universality of $C^*(G)$)**.** Suppose $\pi : G \to \mathcal{B}(H)$ is a unitary representation of a discrete group G. Then π extends uniquely to a *-representation $\widetilde{\pi} : C^*(G) \to \mathcal{B}(H)$.

Proof. Defining $\widetilde{\pi}\left(\sum_g \alpha_g g\right) = \sum_g \alpha_g \pi(g)$, we obtain a *-representation of $\mathbb{C}[G]$ onto $\mathcal{B}(H)$, and so by definition $\|\widetilde{\pi}(x)\| \le \|x\|_u$. ☐

The relationship between $C^*_\lambda(G)$ and $C^*(G)$ is in many ways analogous to the relationship between the min and max tensor norms, as already witnessed by the fact that $C^*_\lambda(G)$ is a quotient of $C^*(G)$ (mirroring the fact that $A \otimes_{\min} B$ is a quotient of $A \otimes_{\max} B$).

Finally, we also have a canonical von Neumann algebra associated to G.

Definition A.4.4. The *group von Neumann algebra of* G is defined as

$$L(G) := C^*_\lambda(G)'' = \overline{C^*_\lambda(G)}^{wot}$$

Notice that for all $s, t \in G$, we have

$$\langle \lambda(st)\delta_e, \delta_e \rangle = \begin{cases} 1 & \text{if } st = e \\ 0 & \text{otherwise} \end{cases}$$

but $st = e$ if and only if $ts = e$, so in fact

$$\langle \lambda(st)\delta_e, \delta_e \rangle = \langle \lambda(ts)\delta_e, \delta_e \rangle$$

This shows us that the vector state $\langle \cdot \delta_e, \delta_e \rangle$ is *tracial* on $\lambda(\mathbb{C}[G])$. Since it is weakly continuous, and $\lambda(\mathbb{C}[G])$ is weakly dense in $L(G)$, we thus have the following fact:

Theorem A.4.5. For any discrete group G, $L(G)$ has a *canonical tracial state* τ given by

$$\tau : T \mapsto \langle T\delta_e, \delta_e \rangle$$

A discrete group G is said to be *amenable* if it satisfies any one of a number of different equivalent conditions related to the existence of a "mean" on G - a way in which we can unparadoxically measure the sizes of sets in G. For our purposes, we only need two equivalent characterizations of amenability:

Definition A.4.6. A discrete group G is *amenable* if it satisfies one of the two following equivalent conditions:

1. There exists a norm 1 positive linear functional $\phi \in \ell^\infty(G)$ such that $\phi(s \cdot f) = \phi(f)$ for all $s \in G$, where $(s \cdot f)(t) := f(s^{-1}t)$ (ϕ is then called an *invariant mean*).
2. There exists a sequence of finite subsets of G, $\{F_k\}_{k\in\mathbb{N}}$, such that for all $s \in G$,

$$\lim_{k\to\infty} \frac{|sF_k \triangle F_k|}{|F_k|} = 0$$

where $\triangle$ denotes the symmetric difference of sets. Equivalently, via the inclusion-exclusion principle we also have

$$\lim_{k\to\infty} \left(1 - \frac{|F_k \cap sF_k|}{|F_k|}\right) = 0$$

Such a sequence $\{F_k\}_{k\in\mathbb{N}}$ is called a *Følner sequence*.

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