# Quasicentral Approximate Units and Voiculescu's Theorem

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## 1 Quasicentral Units

The results in this section are primarily from [1]. The original proof of Voiculescu's theorem, as given in [3], doesn't require quasicentral approximate identities, but Arveson in [1] presented a simplified proof using this abstract framework. The proof we present is the one found in Davidson [2].

**Definition 1.0.1.** Let $A$ be a C*-algebra and $K \subseteq A$ a two-sided ideal (not necessarily norm-closed or self-adjoint). An **approximate unit** (or **approximate identity**) in $K$ is an increasing net $(e_\lambda)_\lambda \subseteq K$ of positive operators of norm $\leq 1$ such that $\lim_\lambda \|e_\lambda k - e_\lambda\| = 0$ for all $k \in K$ (notice that since $\|x\| = \|x^*\|$ for all $x \in A$, $\lim_\lambda \|ke_\lambda - e_\lambda\| = 0$ automatically).

**Definition 1.0.2.** Let $A$ be a C*-algebra and $K \subseteq A$ a two-sided ideal (not necessarily norm-closed or self-adjoint). A **quasicentral approximate unit** is an approximate unit $(e_\lambda)_\lambda \subseteq K$ such that for all $a \in A$, $\lim_\lambda \|ae_\lambda - e_\lambda a\| = 0$.

Of course, every approximate unit $(e_\lambda)_\lambda \subseteq K$ is quasicentral on $K$, since $\|ke_\lambda - e_\lambda k\| \leq \|ke_\lambda - k\| + \|k - e_\lambda k\|$, with both terms converging to zero. What is crucial to our definition is that the quasicentral approximate unit is quasicentral on *all of the ambient algebra $A$*.

**Lemma 1.0.3.** If $K \subseteq A$ are as in definition 1.0.2 and $(e_\lambda)_\lambda \subseteq K$ is an approximate unit, then for all $f \in A^*$ and $a \in A$, $\lim_\lambda f(ae_\lambda - e_\lambda a) = 0$ (so every approximate unit is "weakly quasicentral").

*Proof.* By the Jordan Decomposition, we can assume without loss of generality that $f \in S(A)$. Consider the GNS representation $(\pi_f, \mathcal{H}_f, \xi_f)$ corresponding to $f$. Since $e_\lambda$ is an approximate unit, $\lim_\lambda \pi_f(e_\lambda)\pi_f(k)\xi_f = \pi_f(k)\xi_f$ for all $k \in K$, so $\pi_f(e_\lambda)$ converges strongly to the identity on $\overline{\pi_f(K)\xi_f}$. Of course since $(e_\lambda)_\lambda \subseteq K$, we can be sure that $\xi_f = \lim_\lambda \pi_f(e_\lambda)\xi_f \in \overline{\pi_f(K)\xi_f}$, and moreover $\overline{\pi_f(K)\xi_f}$ is clearly $\pi_f(A)$-invariant,so $\pi_f(a)\xi_f \in \overline{\pi_f(K)\xi_f}$. Thus

$$\lim_\lambda \pi_f(e_\lambda a)\xi_f = \pi_f(a)\xi_f, \qquad \lim_\lambda \pi_f(ae_\lambda)\xi_f = \pi(a)\xi_f$$

whence $\lim_\lambda \langle \pi_f(ae_\lambda - e_\lambda a)\xi_f, \xi_f\rangle = 0$. ∎

**Proposition 1.0.4.** If $(e_\lambda)_\lambda \subseteq K$ is an approximate unit, then $\text{conv}\{e_\lambda : \lambda\}$ can be viewed as an approximate unit for $K$ as well.

*Proof.* We can view $\text{conv}\{e_\lambda : \lambda\}$ as a net, indexed "by itself", and with a preorder given by the standard order on a C*-algebra. To be explicit, we let $\mathrm{M} := \text{conv}\{e_\lambda : \lambda\}$ be our index set, and for $\mu, \nu \in \mathrm{M}$, we define $\mu \leq \nu$ if and only if $\nu - \mu$ is positive in $A$. As $0 \leq e_\lambda \leq 1$ for each $\lambda$, and since $A_+$ is a real cone, we also clearly have $0 \leq \mu \leq 1$ for all $\mu \in \mathrm{M}$. Moreover, given $\mu = \sum_i t_i e_{\lambda_i}$ and $nu = \sum_j t'_j e_{\lambda'_j}$ both in M, there exists some $\lambda$ such that $e_{\lambda_i}, e_{\lambda'_j} \leq e_\lambda$ for all $i$ and $j$, whence $\mu, \nu \leq e_\lambda \in \mathrm{M}$, which implies M is upwards directed. We then define a new net $(f_\mu)_{\mu \in \mathrm{M}}$ be letting $f_\mu = \mu$ (yes its quite redundant, but this is the pedantic formalism). What remains to be shown is that $f_\mu k \to k$ for all $k \in K$. Notice that for a fixed $\lambda$ and any $\mu \in M$, $\mu \geq e_\lambda$, and $k \in K$, that

$$k^*(1-\mu)^2 k \leq k^*(1-\mu)k \leq k^*(1-e_\lambda)k$$

whence

$$\|k - \mu k\|^2 = \|k^*(1-\mu)^2 k\| \leq \|k^*(1-e_\lambda)k\| \leq \|k\|\|k - e_\lambda k\|$$

and of course, since $\lambda$ can be chosen large enough such that the final quantity is arbitrarily small. So $(f_\mu)_{\mu \in \mathrm{M}}$ is indeed an approximate unit. □

**Lemma 1.0.5.** If $(e_\lambda)_\lambda \subseteq K$ is a convex approximate unit (which is to say that the set $\{e_\lambda : \lambda\}$ is convex), then $\inf_\lambda \|ae_\lambda - e_\lambda a\| = 0$ for all $a \in A$.

*Proof.* We argue by contradiction. The statement we'd like to prove is equivalent to showing that $0 \in \overline{\{ae_\lambda - e_\lambda a : \lambda\}} =: C$ for all $\in A$, so assume that there exists $a$ for which 0 is separated from this closed set. Since $(e_\lambda)_\lambda$ is convex, so too is this set $C$, and so the geometric Hahn-Banach theorem allows us to choose a functional $f \in A^*$ for which $|f(ae_\lambda - e_\lambda a)| > 0$ for all $\lambda$, which contradicts lemma 1.0.3. □

**Theorem 1.0.6.** Let $A$ be a C*-algebra. Then every (norm-closed, two-sided) ideal $K \trianglelefteq A$ contains a quasicentral approximate unit.

*Proof.* We'll use a familiar trick from class (which is now, by definition, a "technique": a trick used more than once). Suppose for any finite subset $F := \{a_1, ..., a_n\} \subseteq A$ and $\epsilon > 0$, we can find some $\lambda := \lambda_{(F,\epsilon)}$ for which $\|e_\lambda a - ae_\lambda\| < \epsilon$ for all $a \in F$. Then we can let $\mathrm{M} := \{(F, \epsilon) \ : \ F \subset A \text{ finite}, \epsilon > 0\}$ be a new index set, ordered by letting $(F, \epsilon) \leq (F', \epsilon')$ if and only if $F \subseteq F'$ and $\epsilon \geq \epsilon'$, and extract a subnet $(e_{\lambda_{(F,\epsilon)}})_{(F,\epsilon)\in \mathrm{M}}$ which remains an approximate unit, but is now clearly quasicentral as well.

Consider the algebra $A^n := A \oplus \cdots \oplus A$, with $n$ copies of $A$, with operations defined termwise and norm equal to the sup-norm. Then clearly $K^n := K \oplus \cdots \oplus K$ is also a norm-closed two-sided ideal in $A^n$, and $f_\lambda := (e_\lambda, ..., e_\lambda)_\lambda$ is an approximate identity in $K^n$, which is moreover still convex. Thus by lemma 1.0.5, letting $a := (a_1, ..., a_n) \in A^n$, we see that $\inf_\lambda \|f_\lambda a - af_\lambda\| = 0$, and so there exists $\lambda$ such that

$$\begin{aligned} \epsilon > \|f_\lambda a - af_\lambda\| &= \|(e_\lambda, ..., e_\lambda)(a_1, ..., a_n) - (a_1, ..., a_n)(e_\lambda, ..., e_\lambda)\| \\ &= \sup_i \|e_\lambda a_i - a_i e_\lambda\| \end{aligned}$$

and so we've achieved our task in finding $\lambda$. □

**Corollary 1.0.7.** If $A$ is a *separable* C*-algebra and $K \trianglelefteq A$ a (norm-closed, two-sided) ideal, then $K$ admits a quasicentral approximate unit $(e_n)_{n\in\mathbb{N}}$ which is also a sequence. We can do the same if $J \subseteq A$ is a self-adjoint, but not necessarily closed, ideal.

*Proof.* Straightforward, left as an exercise for the reader. □

# 2 Voiculescu's Theorem

Recall the traditional Weyl-von Neumann-Berg theorem.

**Theorem 2.0.1** (Weyl-von Neumann-Berg)**.** Let $\mathcal{H}$ be a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ a normal operator. Then for all $\epsilon > 0$ there exists a diagonalizable operator $D$ and a compact operator $K$ such that $N = D + K$, with $\|K\| < \epsilon$. Moreover, we can choose $D$ such that $\sigma(N) = \sigma(D)$ and $\sigma(\pi(N)) = \sigma(\pi(D))$.

Voiculescu's theorem is a "non-commutative generalization" of this theorem. In particular, we'd like to prove the following:

**Theorem 2.0.2.** Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable, unital C*-algebra on a separable Hilbert space, $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ the canonical projection into the Calkin algebra, and $\rho : \pi(A) \to \mathcal{B}(\mathcal{K})$ a representation. Then $\mathrm{id}_A$ and $\mathrm{id}_A \oplus \rho$ are *approximately unitarily equivalent modulo the compacts*, which is to say there exist unitary operators $U_k : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ such that for all $a \in A$,

$$a \oplus \rho(a) - U_k^* a U_k \text{ is compact,} \qquad \lim_k \|a \oplus \rho(a) - U_k^* a U_k\| = 0$$

It's not entirely straightforward

**Theorem 2.0.3** (Glimm's Lemma)**.** Let $A$ be a separable C*-subalgebra of $\mathcal{B}(\mathcal{H})$ on a separable C*-algebra $\mathcal{H}$, and $\varphi \in S(A)$ a state such that $\varphi(A \cap \mathcal{K}(\mathcal{H})) = 0$. Then there exists a sequence of unit vectors $\xi_n \in \mathcal{H}$, converging weakly to zero, such that the vector states $\omega_{\xi_n} \overset{w*}{\to} \varphi$.

*Remark.* Elsewhere in the literature, states $\varphi$ satisfying the property that $\varphi(A \cap \mathcal{K}(\mathcal{H})) = 0$ are called **singular** states. Singular states are those states which are *constant along cosets in the Calkin algebra $\mathcal{Q}(\mathcal{H})$ ($:= \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$*: if $a, b \in A$ differ by a compact operator then $a - b \in A \cap \mathcal{K}(\mathcal{H})$, whence $\varphi(a - b) = 0$. This tells us that there is a one-to-one correspondence between singular states on $A$ and states on $\pi(A)$ where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$.

*Proof.* Let $S_s(A)$ denote the singular states in $A$, and $\Omega \subseteq S_s(A)$ those states which are weak-* limits of $\omega_{\xi_n}$ for some sequence of unit vectors $\xi_n$ converging weakly to zero. We're going to start by showing that $\Omega$ is non-empty, convex, and weak-* closed.

**Step 1 - $\Omega$ is non-empty:** We'll use a pretty standard diagonal argument. First, let $\{a_i\}_{i \in \mathbb{N}}$ be a dense subset of $A$. Let $\xi_n$ be any sequence of unit vectors converging weakly to zero in $\mathcal{H}$. For $a_1$ we have

$$|\langle a_1 \xi_n, \xi_n \rangle| \le \|a_1\|, \qquad \forall n \in \mathbb{N}$$

and so by compactness of $\|a_1\|\mathbb{D}$ we can choose a subsequence $\xi_{n_k^{(1)}}$ such that $\lim_k \left\langle a_1 \xi_{n_k^{(1)}}, \xi_{n_k^{(1)}} \right\rangle$ exists. Now we do the same thing for $a_2$: choose a subsequence $n_k^{(2)}$ of $n_k^{(1)}$ such that $\lim_k \left\langle a_2 \xi_{n_k^{(2)}}, \xi_{n_k^{(2)}} \right\rangle$ exists (and of course, since $n_k^{(2)}$ is a subsequence of $n_k^{(1)}$, the limit $\lim_k \left\langle a_1 \xi_{n_k^{(2)}}, \xi_{n_k^{(2)}} \right\rangle$ also exists). Continue inductively, obtaining monotonically increasing sequences of integers $(n_k^{(j)})_{k \in \mathbb{N}}$ such that

$$\lim_k \left\langle a_i \xi_{n_k^{(j)}}, \xi_{n_k^{(j)}} \right\rangle \text{ exists} \quad \forall 1 \le i \le j$$

Then let $n_k := n_k^{(k)}$. By definition, $\lim_k \langle a_i \xi_{n_k}, \xi_{n_k} \rangle$ exists for all $i \in \mathbb{N}$. Thus, extending by linearity and density, we have a well-defined state $\psi := w^* - \lim_k \omega_{\xi_{n_k}} \in \Omega$.

**Step 2 - $\Omega$ is weak-* closed:** This is another diagonal argument. Normally, we would have to consider all *nets* $(\psi_\lambda)_\lambda \in \Omega$ converging to $\psi \in S(A)$, which would be problematic for our diagonal argument (not all directed sets are countably cofinal, so we can't always take a sub*sequence* of a net!). However, separability of $A$ comes in handy here: recall that if $X$ is a separable Banach space, then the unit ball of $X^*$ is *weak-* metrizable*. Thus, in being metrizable, the weak-* topology on $S(A)$ is *sequential*, meaning sequences entirely characterize the topology, and so we need only consider sequences $\psi_n \in \Omega$ converging to $\psi \in S(A)$.

Here we also need to assume without loss of generality that $\mathcal{H}$ is separable. This is not an issue: so long as $A$ is separable it can be represented faithfully on a separable hilbert space $\mathcal{H}_A$, which can always be made a direct summand of $\mathcal{H}$, and the compact operators on $\mathcal{H}$ clearly contain the compact operators on $\mathcal{H}_A$.

Follows from a tedious double diagonalization argument.

**Step 3 - $\Omega$ is convex:** First, since $\Omega$ is closed, it suffices to how that $\Omega$ is *midpoint-convex*: for all $\varphi, \psi \in \Omega$, $\frac{1}{2}(\varphi + \psi) \in \Omega$ as well. Let $\xi_n$ and $\eta_n$ be sequences of unit vectors weakly converging to zero which represent $\varphi$ and $\psi$ respectively. We're going to inductively construct new vectors $\eta_k'$, but first we have to inductively define $P_n$ to be the projection onto the space

$$\operatorname{span}\left(\{\xi_n\} \cup \{a_i\xi_k,\ a_i^*\xi_k \ : \ 1 \le i \le n\} \cup \left\{\eta_j' \ : \ 1 \le j < n\right\}\right)$$

Seeing as $P_n$ is finite rank, for any $k \in \mathbb{N}$ we can choose $n_k$ large enough that $\|P_n\eta_{n_k}\| < \frac{1}{k+1}$. Let $\eta_k' := \frac{P_n^\perp \eta_{n_k}}{\|P_n^\perp \eta_{n_k}\|}$. It's not hard to check that $\|\eta_k' - \eta_{n_k}\| < \frac{1}{k(k+1)}$, $\|\eta_k'\| = 1$, and by definition $\eta_k' \perp \operatorname{ran} P_n$.

Let $\theta_n := \frac{1}{\sqrt{2}}(\xi_n + \eta_n')$. Then $\theta_n$ converges weakly to zero, and

$$\langle a_i\theta_n, \theta_n\rangle = \frac{1}{2}\left(\langle a_i\xi_n, \xi_n\rangle + \langle a_i\xi_n, \eta_n'\rangle + \langle a_i\eta_n', \xi_n\rangle + \langle a_i\eta_n', \eta_n'\rangle\right)$$

but for fixed $i$, $\eta_n' \perp \{a_i\xi_n, a_i^*\xi_n\}$ for $n$ large enough, so eventually these terms fall off, leaving behind

$$\langle a_i\theta_n, \theta_n\rangle = \frac{1}{2}\left(\langle a_i\xi_n, \xi_n\rangle + \langle a_i\eta_n', \eta_n'\rangle\right) \qquad n > i$$

and of course, the $\eta_n'$'s are asymptotic to the original sequence $\eta_n$, so that

$$\lim_n \langle a_i\theta_n, \theta_n\rangle = \frac{1}{2}\left(\varphi(a_i) + \psi(a_i)\right)$$

from which it follows by density that $\frac{1}{2}(\varphi + \psi) \in \Omega$.

**Step 4 - $\Omega = S_s(A)$:** Now suppose we had a state $\varphi \in S_s(A)\backslash\Omega$. Since $\Omega$ is weak-* closed and convex, we can choose an element $a \in A$ such that $\varphi(a) = 1 \notin \widehat{a}(\Omega)$. We can assume $a$ is self-adjoint by replacing $a$ by $\frac{1}{2}(a + a^*)$, so that $\widehat{a}(\Omega) \subset \mathbb{R}$.

Let $\operatorname{conv}\sigma(\pi(a)) = [r_1, r_2]$ ($\pi$ the quotient map into the Calkin algebra). Consider the function $f$ given by

$$f(x) = \max(r_1, \min(r_2, x))$$

which, visually, is just the constant function $r_1$ for $x < r_1$, the constant function $r_2$ for $x > r_2$, and the function $x$ on $[r_1, r_2]$. Since $f$ is the identity when restricted to $\sigma(\pi(a))$, we have $f(\pi(a)) = \pi(a)$, but since $\pi$ is a *-homomorphism $f(\pi(a)) = \pi(f(a))$, so $\pi(f(a) - a) = 0$, or rather $f(a) - a$ is compact. Thus $\varphi(f(a)) = \varphi(a) = 1 \notin \widehat{f(a)}(\Omega) = \widehat{a}(\Omega)$. By the spectral mapping theorem $\sigma(f(a)) = f(\sigma(a)) \subseteq [r_1, r_2]$, and so $r_1 I \le f(a) \le r_2 I$, and $r_1 \le \langle f(a)\xi, \xi\rangle \le r_2$ for $\|\xi\| = 1$. Thus $\widehat{f(a)}(\Omega) \subseteq [r_1, r_2]$. Let $b = f(a)$.

Now, recall the $L^\infty$ functional calculus for normal elements of a von Neumann algebra. If $N \in \mathcal{B}(\mathcal{H})$ is normal, then there is a positive regular Borel measure supported on $\sigma(N)$ and a *-isomorphism $\Gamma_N$ between $W^*(N)$ and $L^\infty(\sigma(N), \mu)$. We can use this to define the *spectral projections* $\chi_B(N) := \Gamma_N^{-1}(\chi_B)$ for any Borel set $B \subseteq \sigma(N)$ (where $\chi_B$ is the indicator function on $B$). These are of course projections since $\Gamma_N$ is a *-isomorphism, and $\chi_B = \chi_B^* = \chi_B^2$.

For each $n \in \mathbb{N}$, the projection $\chi_{(r_2 - 1/n, r_2]}(b)$ is infinite rank. Indeed if it were finite rank, then $b = \chi_{[-\infty, r_2]}(b)b = \left(\chi_{[-\infty, r_2-1/n)}(b) + \chi_{(r_2-1/n, r_2]}(b)\right)b$, and so $\pi(b) = \pi(\chi_{[-\infty, r_2-1/n)]}(b)b)$, hence

$$\sigma(\pi(b)) = \sigma(\pi(\chi_{[-\infty, r_2-1/n)}(b)b)) \subseteq \sigma(\Gamma_b^{-1}(\chi_{[-\infty, r_2-1/n)} \cdot \mathrm{id}_{\sigma(b)})) \subseteq [-\infty, r_2 - 1/n)$$

contradicting the fact that $\operatorname{conv}\sigma(\pi(b)) = [r_1, r_2]$.

Let $\mathcal{H}_n := \chi_{[-\infty, r_2]}(b)\mathcal{H}$, and choose orthonormal sequences $\{\xi_k^{(n)}\}_{k\in\mathbb{N}} \subseteq H_n$. For fixed $n$, each $\xi_k^{(n)}$ converges weakly to zero, and for each fixed $k$,

$$r_2 - 1/n < \langle b\xi_k^{(n)}, \xi_k^{(n)}\rangle < r_2$$

so that $\lim_n \langle b\xi_k^{(n)}, \xi_k^{(n)}\rangle = r_2$ for all $k$. Let $\{h_i\}_{i\in\mathbb{N}}$ be a dense subset of $\mathcal{H}$. For all $n \in \mathbb{N}$, we can choose $k_n^{(1)}$ such that $|\langle \xi_{k_n^{(1)}}^{(n)}, h_1\rangle| < \frac{1}{n}$. Then choose a subsequence $k_n^{(2)}$ of $k_n^{(1)}$ such that $|\langle \xi_{k_n^{(2)}}^{(n)}, h_2\rangle| < \frac{1}{n}$. Continue inductively. Let $\eta_n = \xi_{k_n^{(n)}}^{(n)}$. Then

$$\lim_n \langle \eta_n, h_i\rangle = 0, \quad \forall i \in \mathbb{N} \qquad \lim_n \langle b\eta_n, \eta_n\rangle = r_2$$

Performing a second diagonalization argument on $\eta_n$ yields a sequence $\eta_n'$ which converges weakly to zero and which defines an element $\psi \in \Omega$, for which $\psi(b) = r_2$. Similarly, we can obtain a state $\psi' \in \Omega$ such that $\psi'(b) = r_1$. Thus, as $\widehat{b}(\Omega)$ is compact and convex, $[r_1, r_2] \subseteq \widehat{b}(\Omega)$. Yet $\varphi(b) \in [r_1, r_2]$, but we explicitly chose $b$ such that $\varphi(b) \notin \widehat{b}(\Omega)$. This must have induced a contradiction. In conclusion, $\Omega = S_s(A)$. $\square$

**Theorem 2.0.4.** Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable, unital C*-algebra on a separable Hilbert space $\mathcal{H}$, and $\varphi : A \to \mathbb{M}_n(\mathbb{C})$ a unital, completely positive map such that $\varphi(A \cap \mathcal{K}(\mathcal{H})) = 0$. Then there exists a sequence of isometries $V_k : \mathbb{C}^n \to \mathcal{H}$ such that $V_k \to 0$ in the weak operator topology, and $\|\varphi(a) - V_k^* a V_k\| \to 0$ for all $a \in A$.

*Remark.* Let's think about the structure of $\varphi$, which can be conceptualized as simply an $n \times n$ matrix of functionals on $A$. Since $\varphi$ is unital, positive, and vanishes on the compacts, the diagonal entries of $\varphi$ are *singular states*, and so we can extract sequences of unit vectors $(\xi_i^{(j)})_i$ for which $\omega_{\xi_i^{(j)}}$ converges weak-* to $\varphi_{jj}$. This is *almost* what we want, but we haven't thought about the off-diagonal entries yet. Fortunately we've only used *positivity* of $\varphi$ up to this point, and have yet to exploit complete positivity.

*Proof.* As $\varphi$ is completely positive, the $n^{\text{th}}$ ampliation is positive. Consider the map

$$\begin{array}{rcccc} \Phi & : & \mathbb{M}_n(A) \subseteq \mathcal{B}(\mathcal{H}^n) & \to & \mathbb{C} \\ & & [a_{ij}] & \mapsto & \frac{1}{n}\sum_{ij} \varphi_{ij}(a_{ij}) \end{array}$$

We can view $\varphi^{(n)}$ as a map from $\mathbb{M}_n(A) \to \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$, taking $[a_{ij}] \to \sum_{ij} \varphi(a_{ij}) \otimes e_{ij}$, and we can regard $\mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$ as the operators on $\mathbb{C}^n \otimes \mathbb{C}^n$. For any $\sum_k x_k \otimes y_k \in \mathbb{C}^n \otimes \mathbb{C}^n$ we have

$$\left\langle \left( \sum_{ij} \varphi(a_{ij}) \otimes e_{ij} \right) \left( \sum_k x_k \otimes y_k \right), \sum_\ell x_\ell \otimes y_\ell \right\rangle = \sum_{ijk\ell} \langle \varphi(a_{ij}) x_k, x_\ell \rangle \langle e_{ij} y_k, y_\ell \rangle$$

which, to anyone acclimated to these tensorial proofs, is strikingly similar to our functional $\Phi$. All we have to do is take $x_k = y_k = e_k$, the standard basis for $\mathbb{C}^n$, for which $\langle e_{ij} y_k, y_\ell \rangle = \delta_{i\ell}\delta_{jk}$ and $\langle \varphi(a_{ij}) x_k, x_\ell \rangle = \varphi_{ij}(a_{ij})$. Thus, denoting $e = \sum_k e_k \otimes e_k$, we have

$$\Phi([a_{ij}]) = \frac{1}{n} \langle \varphi^{(n)}([a_{ij}]) e, e \rangle$$

from which we glean that $\Phi$ is in fact a positive linear functional. The factor of $\frac{1}{n}$ is there to ensure $\Phi$ is actually a state.

We leave it as an easy exercise for the reader to check that $\mathcal{K}(\mathcal{H}^n) = \mathbb{M}_n(\mathcal{K}(\mathcal{H}))$, so that $\Phi(\mathbb{M}_n(A) \cap \mathcal{K}(\mathcal{H}^n)) = \Phi(\mathbb{M}_n(A \cap \mathcal{K}(\mathcal{H})) = 0$, so $\Phi$ is a *singular* state. Thus we can extract unit vectors $(x_k^1, ..., x_k^n) \in \mathcal{H}^n$ converging weakly to zero (equivalently each component converges weakly to zero) such that $\Phi([a_{ij}]) = \lim_k \langle [a_{ij}](x_k^1, ... x_k^n), (x_k^1, ... x_k^n) \rangle = \lim_k \sum_{ij} \langle a_{ij} x_k^j, x_k^i \rangle$, and so in particular $\varphi_{ij}(a) = \lim_k \langle a_{ij} x_k^j, x_k^i \rangle$.

Consider the operators

$$U_k := \begin{bmatrix} x_k^1 & \cdots & x_k^n \end{bmatrix} \quad : \quad \begin{array}{ccc} \mathbb{C}^n & \to & \mathcal{H} \\ (\alpha_1, ..., \alpha_n)^T & \mapsto & \sum_i \alpha_i x_k^i \end{array}$$

It's not hard to see that since the $x_k^i$'s tend weakly to zero, $U_k \overset{wot}{\to} 0$. A straightforward calculation shows that $U_k^* a U_k = [\langle a x_k^j, x_k^i \rangle]$, and so $\varphi(a) = \lim_k U_k^* a U_k$. The problem is these operators $U_k$ aren't exactly isometries, but a quick modification will do the trick. Notice that

$$\lim_k \langle x_k^i, x_k^j \rangle = n\Phi(1_A \otimes E_{ji}) = \varphi_{ij}(1_A) = \delta_{ij}$$

and so the vectors $\{x_k^1, ..., x_k^j\}$ are *approximately orthogonal*, implying $U_k^* U_k = [\langle x_k^j, x_k^i \rangle]$ converges entrywise (and hence in norm) to $I$. Let $U_k := V_k |U_k|$ be the polar decomposition of $U_k$, for some partial isometry

$V_k : \mathbb{C}^n \to \mathcal{H}$. Of course, since $U_k^* U_k \to I$, for $k$ large enough each $U_k^* U_k$ is invertible, whence $V_k^* V_k = ((U_k^* U_k)^{-1/2} U_k^*)(U_k (U_k^* U_k)^{-1/2}) = I$, so each $V_k$ is an *isometry*. Moreover,

$$\begin{aligned}\lim_k |\langle V_k x, y\rangle| &\leq \lim_k (|\langle (V_k - U_k)x, y\rangle| + \langle U_k x, y\rangle) \leq \lim_k \|U_k - V_k\| \|x\| \|y\| \\ &= \lim_k \|V_k((U_k^* U_k)^{1/2} - I)\| \leq \lim_k \|(U_k^* U_k)^{1/2} - I\| = 0\end{aligned}$$

so that $V_k \overset{wot}{\to} 0$ as well.[1] □

**Corollary 2.0.5.** Given $\varphi : A \subseteq \mathcal{B}(\mathcal{H}) \to \mathbb{M}_n(\mathbb{C})$ as above, $\mathcal{F} \subset A$ a finite subset, $\mathcal{N} \subseteq \mathcal{H}$ a finite-dimensional subspace, and $\epsilon > 0$. Then there exists an isometry $V : \mathbb{C}^n \to \mathcal{H}$ such that $\operatorname{ran} V \subseteq \mathcal{N}^\perp$, and $\|\varphi(a) - V^* a V\| < \epsilon$ for all $a \in \mathcal{F}$.

*Proof.* We first choose isometries $V_k$ as in the prior theorem, and $k_0$ large enough such that $k \geq k_0$ implies

$$\|\varphi(a) - V_k^* a V_k\| < \epsilon, \ \forall a \in \mathcal{F} \qquad \|P_{\mathcal{N}} V_k\| < \epsilon$$

To obtain this latter bound, notice that $P_{\mathcal{N}} V_k$ is a sequence of operators between finite dimensional Hilbert spaces which converges in the weak operator topology to zero, but this topology coincides with the norm topology on finite-dimensional spaces.

To obtain an isometry whose range is entirely contained in $\mathcal{N}^\perp$, consider the partial isometry $V$ obtained from the polar decomposition: $P_{\mathcal{N}^\perp} V_k = V|P_{\mathcal{N}^\perp} V_k|$. Two applications of the triangle inequality yields

$$\|\varphi(a) - V_k^* a V_k\| \leq \|\varphi(a) - V^* a V\| + 2\|a\| \|V - V_k\|$$

and

$$\begin{aligned}\|V - V_k\| &= \|V - P_{\mathcal{N}} V_k - P_{\mathcal{N}^\perp} V_k\| \leq \|V - P_{\mathcal{N}^\perp} V_k\| + \epsilon \\ &= \|V(I - |P_{\mathcal{N}^\perp} V_k|)\| + \epsilon \leq \|I - |P_{\mathcal{N}^\perp} V_k|\| + \epsilon = \||V_k| - |P_{\mathcal{N}^\perp} V_k|\| + \epsilon\end{aligned}$$

Two applications of the triangle inequality gives us

$$\||V_k|^2 - |P_{\mathcal{N}^\perp} V_k|\|^2 \leq (\|V_k\| + \|P_{\mathcal{N}^\perp} V_k\|) \|V_k - P_{\mathcal{N}^\perp} V_k\| \leq 2\|P_{\mathcal{N}} V_k\| < 2\epsilon$$

which, for “reasons beyond our comprehension” implies

$$\||V_k| - |P_{\mathcal{N}^\perp} V_k|\| \text{ is also small}$$

whence the result. □

Now let’s start thinking about the general case: $\varphi : A \to B$ a completely positive, unital map between C*-subalgebras of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively, for which $\varphi(A \cap \mathcal{K}(\mathcal{H})) = 0$. Suppose we have an increasing sequence of finite rank projections $P_n \in \mathcal{B}(\mathcal{K})$ tending strongly to the identity (i.e. an approximate unit). Each $P_n \varphi(\cdot) P_n$ is (up to *-isomorphism of the range space) a completely positive unital map into $\mathbb{M}_{d_n}(\mathbb{C})$ (where $d_n = \operatorname{rank} P_n$), which we know can be point-norm approximated by isometries. We’re heading in the right direction, but we’re missing a few pieces to the puzzle.

**Theorem 2.0.6** (Non-commutative Weyl-von Neumann-Berg Theorem)**.** Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable unital C*-algebra and $\varphi : A \to \mathcal{B}(\mathcal{K})$ a unital, completely positive map such that $\varphi(A \cap \mathcal{B}(\mathcal{K})) = 0$. Then there exists a sequence of isometries $V_k : \mathcal{K} \to \mathcal{H}$ such that $\varphi(a) - V_k^* a V_k$ is *compact* for all $a \in A, k \in \mathbb{N}$, and $\varphi(a) = \lim_k V_k^* a V_k$ for all $a \in A$.

[1] If $X_n \to I$ and $X_n$ is uniformly bounded by $M$, choose a polynomial $p$ arbitrarily close to $f(x) = \sqrt{x}$ on $[0, M]$, so that

$$\begin{aligned}\|I - X_n^{1/2}\| &\leq \|I - p(I)\| + \|p(I) - p(X_n)\| + \|p(X_n) - X_n^{1/2}\| \\ &\leq (1 + M)\|f - p\| + \|p(I) - p(X_n)\|\end{aligned}$$

which can be made arbitrarily small.

*Proof.* We're going to start by just finding *one* isometry $V$ such that $\varphi(a) - V^*aV$ is compact for all $a \in A$. Once we know we can do this, the trick is to do the same to $\varphi^{(\infty)} : A \to \mathcal{B}(\mathcal{K}^{(\infty)})$ taking $a \mapsto \oplus_{\mathbb{N}}\varphi(a)$, which remains a unital, completely positive map which vanishes on compacts. Let $V$ be the isometry in $\mathcal{B}(K^{(\infty)}, \mathcal{H})$ which accomplishes our task, and write $V(\xi_i)_i = \sum_i V_i\xi_i$ for operators $V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, which we observe are also necessarily isometries. Let $P_k : \mathcal{K}^{(\infty)} \to \mathcal{K}$ take $\oplus_i\xi_i \mapsto \xi_k$ so that $P_k^*\xi = 0 \oplus \cdots \oplus \xi \oplus \cdots$ ($\xi$ placed in the $k$th position). Then

$$\varphi(a) - V_k^*aV_k = P_k\left(\varphi^{(\infty)}(a) - V^*aV\right)P_k^*$$

whence

$$\|\varphi(a) - V_k^*aV_k\| \leq \|P_k(\varphi^{(\infty)}(a) - V^*aV)\| \underbrace{\|P_k^*\|}_{=1}$$

However, $P_k$ converges strongly to zero, and so for any compact $T$, $P_kT$ converges to zero *uniformly.* Thus since $\varphi^{(\infty)}(a) - V^*aV$ is compact

$$\lim_k \|\varphi(a) - V_k^*aV_k\| = 0$$

So let's see how to construct such a $V$.

Consider the C*-algebra $B := C^*(\varphi(A)) + \mathcal{K}(\mathcal{K})$, and the ideal of finite rank operators $\mathcal{F}(\mathcal{K}) \subseteq B$. We can extract a sequential, quasicentral approximate unit $(e_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}(\mathcal{K})$ relative to $B$. The construction of $B$ is just so we can obtain a sequence of positive, finite rank operators converging strongly to the identity, such that

$$\|e_nk - k\| \to 0, \quad \forall k \in \mathcal{K}(\mathcal{K}), \qquad \|\varphi(a)e_n - e_n\varphi(a)\| \to 0, \quad \forall a \in A$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a dense subset of $A$. Choose a decreasing sequence $\delta_n > 0$ tending to zero such that for all $e \geq 0$ and $\|a\| \leq 1$ in $A$, $\|ea - ae\| < \delta_n$ implies $\|e^{1/2}a - ae^{1/2}\| < 2^{-n}$. By dropping to a subsequence, we can assume without loss of generality that our quasicentral approximate unit $(e_n)_{n\in\mathbb{N}}$ is such that

$$\|e_n\varphi(a_i) - \varphi(a_i)e_n\| < \frac{1}{2}\delta_{n+1}, \qquad \forall i = 1, ..., n+1$$

from which $\|(e_n - e_{n-1})\varphi(a_i) - (e_n - e_{n-1})\varphi(a_i)\| < \delta_n$ for all $i = 1, ..., n$ (by the triangle inequality), whence

$$\|(e_n - e_{n-1})^{1/2}\varphi(a_i) - \varphi(a_i)(e_n - e_{n-1})^{1/2}\| < 2^{-n}, \quad \forall i = 1, ..., n$$

We'll let $f_n := (e_n - e_{n-1})^{1/2}$ (with $e_0 = 0$), which we remark are also finite rank operators.[2] The bound we've obtained on $\|f_n\varphi(a_i) - \varphi(a_i)f_n\|$ tells us that $\sum_n \|f_n\varphi(a_i) - \varphi(a_i)f_n\| < \infty$ for all $i \in \mathbb{N}$, and so the operator $\sum_n(f_n\varphi(a_i) - \varphi(a_i)f_n)$ converges in norm. Of course each $f_n\varphi(a_i) - \varphi(a_i)f_n$ is finite rank, so the sum of the series is compact. We'll need this in our construction of $V$.

Now let $P_n$ denote the (finite rank) orthogonal projection onto the range of $f_n$. Since the map $a \mapsto P_n\varphi(a)P_n$ is unital, completely positive, and maps into a finite matrix algebra, by the corollary above, we can inductively define isometries $L_n : P_n\mathcal{K} \to \mathcal{H}$ such that

$$\|P_n\varphi(a_i)P_n - L_n^*a_iL_n\| < 2^{-n}, \qquad \forall i = 1, ..., n$$

and additionally

$$\operatorname{ran} L_n \quad \perp \quad \mathcal{N}_n := \operatorname{span}\{\operatorname{ran}(L_i), a_j \operatorname{ran}(L_i) \ : \ 1 \leq i < n, \quad 1 \leq j \leq n\}$$

First off, we see that $\sum_n \|P_n\varphi(a_i)P_n - L_n^*a_iL_n\| < \infty$ for all $i$, and so $\sum_n(P_n\varphi(a_i)P_n - L_n^*a_iL_n)$ converges in norm. Again, each term is finite rank, so the resulting operator is compact. We've also constructed the finite-dimensional space $\mathcal{N}_n$ in such a way that $\operatorname{ran} L_n \perp \operatorname{ran} L_m$ (so the isometries are *mutually orthogonal*) whenever $m \neq n$, and $\operatorname{ran} L_n \perp a_i \operatorname{ran} L_m$ whenever $1 \leq i \leq \max\{m, n\}$. These conditions can also be expressed

$$L_m^*L_n = \delta_{mn}I, \qquad L_m^*a_iL_n = 0, \qquad 1 \leq i \leq \max\{m, n\}$$

[2] If $F$ is a finite rank operator, then $Fx = \sum_{i=1}^n \langle x, y_i\rangle e_i$ for some vectors $y_i$ and some linearly independent vectors $e_i$, from which we see that $\operatorname{codim}\ker F \leq n$, and so $F|_{(\ker F)^\perp}$ is a positive map between finite dimensional Hilbert spaces, i.e. a positive semidefinite matrix, which admits a unique positive semidefinite *square root* matrix $F^{1/2}$. Extending $F^{1/2}$ back to all of $\operatorname{dom} F$ gives us a finite rank square root.

Now, consider the operator $V := sot-\sum_n L_n f_n$. We must first check that this is even well-defined. Let $V^{(N)} := \sum_{n=1}^N L_n f_n$. Notice that since the $L_n$'s are mutually orthogonal isometries, we have

$$(V^{(N)})^*(V^{(N)}) = \sum_{n=1}^N f_n^2 = e_N$$

and so $(V^{(N)})^* V^{(N)} \to I$ strongly. Additionally, for $M < N$ we can calculate

$$\left(V^{(N)} - V^{(M)}\right)^* \left(V^{(N)} - V^{(M)}\right) = e_N - e_M$$

whence

$$\|(V^{(N)} - V^{(M)})x\|^2 = \langle (e_N - e_M)x, x\rangle \le \|(e_N - e_M)x\|\|x\|$$

which can be made arbitrarily small for $M, N$ large enough, so $V^{(N)}x$ is Cauchy, whence convergent, and the strong operator limit of the partial sums exists.

However, we can't quite conclude that $V^*V = sot-\lim_N (V^{(N)})^*(V^{(N)})$. Typically, multiplication of operators is not *jointly* continuous with respect to the strong operator topologies, however when restricted to the unit ball of $\mathcal{B}(\mathcal{H})$ (or any bounded set of operators) it *is* jointly continuous. Since $\|V^{(N)}\| \le 1$, we thus get $V^*V = sot-\lim_N (V^{(N)})^*(V^{(N)}) = I$, so $V$ is an isometry.

We also have $V^* a_i V = sot-\lim_N (V^{(N)})^* a_i V^{(N)}$ (again by uniform boundedness of our sequence). We simulatneously calculate

$$\begin{aligned}
\varphi(a_i) - (V^{(N)})^* a_i V^{(N)} &= \varphi(a_i) - \sum_{m,n=1}^N f_m L_m^* a_i L_n f_n \\
&= sot-\sum_{n\ge 1} \varphi(a_i) f_n^2 - \sum_{m,n=1}^N f_m L_m^* a_i L_n f_n \\
&= sot-\sum_{n>N} \varphi(a_i) f_n^2 + \sum_{n=1}^N (\varphi(a_i) f_n - f_n\varphi(a_i) + f_n\varphi(a_i)) f_n - \sum_{m,n=1}^N f_m L_m^* a_i L_n f_n \\
&= \left(I - (V^{(N)})^* V^{(N)}\right)\varphi(a_i) + \sum_{n=1}^N (\varphi(a_i) f_n - f_n \varphi(a_i)) f_n \\
&\quad + \sum_{n=1}^N f_n (P_n \varphi(a_i) P_n - L_n^* a_i L_n) f_n - \sum_{\substack{m,n=1 \\ m\ne n}}^N f_m L_m^* a_i L_n f_n
\end{aligned}$$

where we have used the fact that $f_n = P_n f_n$ (and of course both $P_n$ and $f_n$ are self-adjoint, so $f_n = f_n P_n$).

The first term above tends strongly to zero (as $N \to \infty$). The next two terms, as we saw above, are *norm-convergent* series of *finite rank operators*, and whence strongly continuous with strong limit compact. Finally, notice that for a *fixed* $i$, for $m, n$ large enough $L_m^* a_i L_n = 0$, and so the third sum is always a *finite sum* of finite rank operators, with the number of terms being *fixed* as $N \to \infty$. Thus, $\varphi(a_i) - V^* a_i V$ is compact. Of course the $a_i$'s were dense in $A$, so if $a_{n_i} \to a \in A$, then $\varphi(a) - V^* a V = \lim_i (\varphi(a_{n_i}) - V^* a_{n_i} V)$ is the norm-limit of compact operators, whence compact. □

**Corollary 2.0.7.** If $\rho$ is a (non-degenerate) *-representation of a separable unital C*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$, which vanishes on $A \cap \mathcal{K}(\mathcal{H})$, then there are a sequence of isometries $V_k : \mathcal{K} \to \mathcal{H}$ such that $V_k \rho(a) - aV_k$ is compact, and $\lim_k \|V_k \rho(a) - aV_k\| = 0$.

This is strikingly close to what we want from Voiculescu's theorem: approximate *unitary* equivalence modulo the compacts. You might think we're almost done, since we have isometries, and isometries can be dilated to unitaries. Unfortunately this line of thought turns out to be a little naive. The path forward is not nearly so simple.

**Theorem 2.0.8** (Voiculescu's Theorem)**.** Let $\rho : A \to \mathcal{B}(\mathcal{K})$ be a unital *-representation of a separable unital C*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$, which vanishes on $A \cap \mathcal{K}(\mathcal{H})$. Then there is a sequence of unitaries $U_k : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ such that $U_k(a \oplus \rho(a)) - aU_k$ is compact for all $a \in A, k \in \mathbb{N}$, and $\lim_k \|U_k(a \oplus \rho(a)) - aU_k\| = 0$. In other words, $\mathrm{id}_A \oplus \rho \sim_{\mathcal{K}(\mathcal{H})} \mathrm{id}_A$.

*Proof.* First, let $V \in \mathcal{B}(\mathcal{K}^{(\infty)}, \mathcal{H})$ be an isometry such that for all $a \in A$, $V\rho^{(\infty)}(a) - aV$ is compact. Recall we can write $V = \sum_i V_i J_i$, where $J_i : \mathcal{K} \to \mathcal{K}^{(\infty)}$ are the canonical injection maps (onto the $i$th copy of $\mathcal{K}$), and $V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ are isometries. Note that $J_i^*$ is the canonical *projection* (which extracts the $i$th coordinate). Consider the shift-operator

$$\begin{aligned} S_n &= \sum_{i=1}^{n-1} J_i J_i^* + sot\text{-}\sum_{i \geq n} J_{i+1} J_i^* \\ &: (\xi_1, \xi_2, ...) \mapsto (\xi_1, ..., \xi_{n-1}, 0, \xi_{n+1}, ...) \end{aligned}$$

Notice that $S_n$ is an isometry. An explicit calculation yields

$$S_n^*(\xi_1, \xi_2, ...) = (\xi_1, ..., \xi_{n-1}, \xi_{n+1}, \xi_{n+2}, ...)$$

so that

$$S_n^* S_n = I \text{ (as expected)}, \qquad S_n S_n^* = I_{\mathcal{K}^{(\infty)}} - J_n J_n^*$$

Now we'll begin defining our unitaries. Suppose that $U_n : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ are unitaries such that $U_n(a \oplus \rho(a)) - aU_n$ is compact and tends to zero. How might we define our $U_n$? Well, we already have a sequence of *isometries* $V_n : \mathcal{K} \to \mathcal{H}$ which approximately commute with $\rho$ relative to the compacts, so we might expect $U_n$ to look something like

$$U_n(h \oplus k) = W_n h + V_n k$$

for some aptly chosen operator $W_n \in \mathcal{B}(\mathcal{H})$. If we want $U_n$ to be unitary, then in particular we need

$$W_n^* W_n = I, \qquad W_n^* V_n = 0, \qquad W_n W_n^* + V_n V_n^* = I$$

Looking at this last equation, we might suppose

$$\begin{aligned} W_n W_n^* = I - V_n V_n^* &= I - VV^* + VV^* - V_n V_n^* \\ &= (I - VV^*)(I - VV^*)^* + VV^* - V_n V_n^* \end{aligned}$$

which looks close to

$$(I - VV^* + X)(I - VV^* + X)^*$$

for some operator $X$. If we expand this, we're left with $I - VV^* +$ a bunch of awkward terms involving $X$ and $X^*$. If we choose $X = VYV^*$ for some operator $Y$, then a lot of these terms disappear, and we have

$$(I - VV^* + X)(I - VV^* + X)^* = I - VV^* + VYY^*V^*$$

Another reason this is a smart choice is because $(I - VV^* + X)^*(I - VV^* + X) = I - VV^* + VY^*YV^*$, which is equal to $I$ if we can choose $Y$ to be an isometry. In order for $I - VV^* + VYY^*V^*$ to equal $I - V_n V_n^*$, we merely need

$$VYY^*V = VV^* - V_n V_n^* = V(I - J_n J_n^*)V^*$$

or even more simply $YY^* = I - J_n J_n^*$. Above we observed that $Y = S_n$ accomplishes this task. What remains to check is that if we plug in $Y = S_n$, we *indeed* obtain a unitary.

Define the operators $U_n : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ via

$$U_n(h \oplus k) := (I - VV^* + VS_n V^*)\, h + VJ_n k$$

Then one readily checks that each $U_n$ is in fact a *unitary*. We have

$$U_n^* U_n = \begin{bmatrix} (I - VV^* + VS_n^* V^*)(I - VV^* + VS_n V^*) & (I - VV^* + VS_n^* V^*)VJ_n \\ J_n^* V^*(I - VV^* + VS_n V^*) & J_n^* V^* V J_n \end{bmatrix}$$

Clearly $J_n^* V^* V J_n = I$, and

$$(I - VV^* + VS_n^* V^*)VJ_n = VJ_n - VJ_n + VS_n^* J_n = VS_n^* J_n$$

but $J_n k$ has zeros everywhere except the $n$th-slot, and $S_n^*(k_i)_i$ takes everything *except* the $n$th-slot, so $S_n^* J_n = 0$. Thus the off-diagonal elements are zero. Finally, expanding and simplifying the top term (a straightforward procedure with no tricks) gives that it is equal to $I$. Additionally

$$\begin{aligned} U_n U_n^* &= (I - VV^* + VS_nV^*)(I - VV^* + VS_n^*V^*) + VJ_nJ_n^*V^* \\ &= I - VV^* + VS_nS_n^*V^* + VJ_nJ_n^*V^* \\ &= I - VV^* + V(S_nS_n^* + J_nJ_n^*)V^* \\ &= I \end{aligned}$$

Now, we calculate

$$\begin{aligned} U_n(a \oplus \rho(a)) - aU_n &= \begin{bmatrix}(I - VV^* + VS_nV^*)a & VJ_n\rho(a)\end{bmatrix} - \begin{bmatrix}a(I - VV^* + VS_nV^*) & aVJ_n\end{bmatrix} \\ &= \begin{bmatrix}(aVV^* - VV^*a) - (aVS_nV^* - VS_nV^*a) & VJ_n\rho(a) - aVJ_n\end{bmatrix} \end{aligned}$$

Here is where we can use the fact that $V\rho^{(\infty)}(a) - aV$ is compact. Observe that

$$\begin{aligned} aVV^* - VV^*a &= aVV^* - V\rho^{(\infty)}(a)V^* + V\rho^{(\infty)}(a)V^* - VV^*a \\ &= (aV - V\rho^{(\infty)}(a))V^* + V(a^*V - V\rho^{(\infty)}(a^*))^* \end{aligned}$$

which is the sum of compact operators, whence compact. The other commutator in the (1,1)-entry above can be handled similarly, but we need to remark that $S_n\rho(a)^{(\infty)} = \rho(a)^{(\infty)}S_n$ (easy to check), from which we calculate

$$\begin{aligned} aVS_nV^* - VS_nV^*a &= aVS_nV^* - V\rho(a)^{(\infty)}S_nV^* + VS_n\rho(a)S_nV^* - VS_nV^*a \\ &= (aV - V\rho^{(\infty)}(a))S_nV^* + VS_n(a^*V - V\rho^{(\infty)}(a^*))^* \end{aligned}$$

which, again, is compact. Finally, we also remark $J_n\rho(a) = \rho(a)^{(\infty)}J_n$ (another easy calculation), so that $VJ_n\rho(a) - aVJ_n = V\rho^{(\infty)}(a)J_n - aVJ_n = (V\rho^{(\infty)}(a) - aV)J_n$ is also compact.

To finish up, simply notice that $S_n$ converges strongly to the identity, and so for any compact operator $K$, both $KS_n$ and $S_nK$ converge to $K$ in *norm*. Thus,

$$\begin{aligned} aVS_nV^* - VS_nV^*a &= (aV - V\rho^{(\infty)}(a))S_nV^* + VS_n(a^*V - V\rho^{(\infty)}(a^*))^* \\ &\to (aV - V\rho^{(\infty)}(a))V^* + V(a^*V - V\rho^{(\infty)}(a^*))^* \quad \text{in norm} \\ &= aVV^* - VV^*a \end{aligned}$$

As for the (1,2)-entry, observe that $(V\rho^{(\infty)}(a) - aV)J_n = V_n\rho(a) - aV_n$, and as we know compactness of $V\rho^{(\infty)}(a) - aV$ guarantees that $\|V_n\rho(a) - aV_n\| \to 0$, concluding the argument that

$$\|U_n(a \oplus \rho(a)) - aU_n\| \to 0$$

□

We can easily rephrase this in the language of Voiculescu's original paper:

**Theorem 2.0.9.** Let $A$ be a separable unital C*-algebra, $\rho : A \to \mathcal{B}(\mathcal{H})$ a unital *-representation of $A$, and $\sigma : \pi(\rho(A)) \to \mathcal{B}(\mathcal{K})$ a unital *-representation (where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ is the canonical quotient map). Then

$$\rho \sim_{\mathcal{K}} \rho \oplus \sigma \circ \pi \circ \rho$$

*Proof.* Consider the C*-algebra $\rho(A)$ (recalling the range of *-representations is automatically closed). The representation $\sigma$ lifts to a representation $\widetilde{\sigma} = \sigma \circ \pi$, which vanishes on the compacts in $\rho(A)$. Thus

$$\rho = \mathrm{id}_{\rho(A)} \circ \rho \sim_{\mathcal{K}} (\mathrm{id}_{\rho(A)} \oplus \widetilde{\sigma}) \circ \rho = \rho \oplus \sigma \circ \pi \circ \rho$$

□

**Definition 2.0.10.** A *-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $A$ is said to be *essential* if $\pi(A)$ contains no compact operators.

*Remark.* Essential representations are surprisingly plentiful: given any *-representation $\pi$, the infinite ampliation $\pi^{(\infty)} : A \to \mathcal{B}(\mathcal{H}^{(\infty)})$ taking $a \mapsto \oplus_{\mathbb{N}}\pi(a)$ is essential. Indeed if $\oplus_{\mathbb{N}} a_i \in \mathcal{B}(\mathcal{H}^{(\infty)})$ were compact, then necessarily $\|a_i\| \to 0$ (via the same argument used in the proof of theorem 2.0.6).

**Corollary 2.0.11.** Let $A$ be a separable unital C*-algebra, $\rho : A \to \mathcal{B}(\mathcal{H})$ and $\sigma : A \to \mathcal{B}(\mathcal{K})$ unital, essential *-representations, such that $\ker\rho = \ker\sigma$. Then $\rho \sim_{\mathcal{K}} \sigma$.

*Remark.* This is sometimes phrased as $\ker\rho = \ker\pi\rho = \ker\sigma = \ker\pi\sigma$ with $\pi$ the canonical projection onto the Calkin algebra (the $\ker\pi\rho = \ker\rho$ and $\ker\pi\sigma = \ker\sigma$ conditions being the definition of "essential").

*Remark.* Since $\ker\rho = \ker\sigma$, we have $\rho(A) \cong A/\ker\rho = A/\ker\sigma \cong \sigma(A)$, so there exists a *-isomorphism $\psi : \rho(A) \to \sigma(A)$, with $\psi \circ \rho = \sigma$. Then

$$\rho = \mathrm{id}_{\rho(A)} \circ\rho \ \sim_{\mathcal{K}} \ (\mathrm{id}_{\rho(A)} \oplus\psi) \circ \rho = \rho \oplus \sigma = (\psi^{-1} \oplus \mathrm{id}_{\sigma(A)}) \circ \sigma \ \sim_{\mathcal{K}} \ \mathrm{id}_{\sigma(A)} \circ\sigma = \sigma$$

# 3 Applications

**Theorem 3.0.1.**

**Definition 3.0.2.** Let $\mathcal{Q}(\mathcal{H})$ denote the Calkin algebra, and suppose $A \subseteq \mathcal{Q}(\mathcal{H})$ is a norm-closed, unital algebra (not necessarily self-adjoint). We let

$$\mathrm{Lat}(A) := \{p \in \mathcal{Q}(\mathcal{H}) \mid p \text{ is a projection}, \ (1-p)Ap = 0\}$$

and for a subset $P \subseteq \mathcal{Q}(\mathcal{H})$ of projections, we let

$$\mathrm{Alg}(P) := \{b \in \mathcal{Q}(\mathcal{H}) \mid (1-p)bp = 0 \quad \forall p \in P\}$$

*Remark.* These notions are obviously borrowed from their non-Calkin counterparts in the theory of nest algebras, so we might expect them to behave analogously. While it *is true* that every projection $e \in \mathcal{Q}(\mathcal{H})$ lifts to a projection in $\mathcal{B}(\mathcal{H})$ (see [**lifting-projections-choi**] and the cited paper [**cited-paper**]), in general the set of projections in $\mathcal{Q}(\mathcal{H})$ is *not* a lattice (in particular you can furnish two projections on a separable $\mathcal{H}$ which lack a meet in the Calkin algebra).

**Theorem 3.0.3.** Let $A \subseteq \mathcal{Q}(\mathcal{H})$ be a separable, norm-closed, unital algebra. Then $\mathrm{Alg}(\mathrm{Lat}(A)) = A$.

First we'll need a quick lemma.

**Lemma 3.0.4.** Let $A$ be a norm-closed (not necessarily self-adjoint) proper subalgebra of a C*-algebra $B$, and $x \in B/A$. Then there exists a state $\varphi \in S(B)$ with corresponding GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ and $r > 0$ such that $\|\pi_\varphi(x-a)\| \geq r$ for all $a \in A$, and so as a consequence $\pi_\varphi(x)\xi_\varphi \notin \overline{\pi_\varphi(A)\xi_\varphi}$.

*Proof.* Choose a linear functional $f \in B^*$ such that $f(x) = 1$ and $f(A) = 0$ (geometric Hahn-Banach). By the Jordan decomposition, we can write $f = f_1 - f_2 + i(f_3 - f_4)$ for some positive functionals $f_i \in B^*$. Let $\psi := f_1 + f_2 + f_3 + f_4$. Recall that for a unital, 2-positive ma $\phi : A \to B$ between unital C*-algebras, $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all $a \in A$, and so

$$f_i((x-a)^*(x-a)) \geq \frac{1}{\|f_i\|}|f_i(x-a)|^2, \quad \forall a \in A$$

So

$$\psi\left((x-a)^*(x-a)\right) \geq \frac{1}{\max_i \|f_i\|}\sum_{i=1}^{4}|f_i(x-a)|^2$$

Let $\varphi = \psi/\|\psi\| \in S(B)$, with GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$. Then the above inequality simply reads $\|\pi_\varphi(x-a)\| \geq r$ with $r = \frac{1}{4\|\psi\| \max_j \|f_j\|}$, for all $a \in A$. $\square$

*Proof of 3.0.3.* We can phrase the problem as follows: for any operator $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(T) \notin A$, we have to find a projection $P \in \mathcal{B}(\mathcal{H})$ such that $(I-P)SP$ is compact whenever $\pi(S) \in A$ (so that $\pi(P) \in \mathrm{Lat}(A)$), but $(1-P)TP$ is *not* compact (so $(1-\pi(P))T\pi(P) \neq 0$, implying $T \notin \mathrm{Alg}(\mathrm{Lat}(A))$).

Consider the C*-algebra $B := C^*(\pi^{-1}(A), T)$. Since $\pi(T) \notin A \subsetneq \pi(B)$, by the lemma above there is a state $\varphi \in S(\pi(B))$ such that $\pi_\varphi(\pi(T))\xi_\varphi \notin \overline{\pi_\varphi(A)\xi_\varphi}$.

Let $E \in \mathcal{B}(\mathcal{H}_\varphi)$ denote the projection onto $\overline{\pi_\varphi(A)\xi_\varphi}$, and $Q := 0 \oplus E \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}_\varphi)$ the projection onto $0 \oplus \overline{\pi_\varphi(A)\xi_\varphi}$. For $S \in \pi^{-1}(A)$ we calculate

$$(I-Q)(S \oplus \pi_\varphi(\pi(S)))Q(h_1 \oplus h_2) = (I-Q)(0 \oplus \underbrace{\pi_\varphi(\pi(S))Eh_2}_{\in \overline{\pi_\varphi(A)\xi_\varphi}}) = 0 \quad \forall h_1 \oplus h_2 \in \mathcal{H} \oplus \mathcal{H}_\varphi$$

and yet

$$(I-Q)(T \oplus \pi_\varphi(\pi(T)))Q(0 \oplus \xi_\varphi) = (I-E)\pi_\varphi(\pi(T))\xi_\varphi \neq 0$$

which is nonzero since $\pi_\varphi(\pi(T))\xi_\varphi \notin \operatorname{ran} E$. Now let $\sigma = \pi_\varphi^{(\infty)}$, and $\widetilde{Q} := 0 \oplus E^{(\infty)} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}_\varphi^{(\infty)})$. Then

$$(I-\widetilde{Q})(S \oplus \sigma(\pi(S)))\widetilde{Q} = 0 \quad \forall S \in \pi^{-1}(A)$$

whereas

$$(I-\widetilde{Q})(T \oplus \sigma(\pi(T)))\widetilde{Q} \text{ is not just non-zero, it's non-compact}$$

Of course, Voiculescu's theorem tells us that $\mathrm{id}_B$ and $\mathrm{id}_B \oplus \sigma$ are approximately unitarily equivalent modulo the compacts, so there is a unitary operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{H} \oplus \mathcal{H}_\varphi^{(\infty)})$ such that

$$b - U^*(b \oplus \sigma(\pi(b)))U \text{ is compact } \forall b \in B$$

We check that $P := U^*\widetilde{Q}U$ is the desired projection. For any $b \in B$ we calculate

$$\begin{aligned}(I-P)bP &= (I - U^*\widetilde{Q}U)bU^*\widetilde{Q}U = U^*(I-\widetilde{Q})UbU^*\widetilde{Q}U \\ &= U^*(1-\widetilde{Q})(b \oplus \sigma(\pi(b)) + \text{compact})\widetilde{Q}U \\ &= U^*(I-\widetilde{Q})(b \oplus \sigma(\pi(b)))\widetilde{Q}U + \text{compact}\end{aligned}$$

Thus $(I-P)SP$ is compact for any $S \in \pi^{-1}(A)$, and $(1-P)TP$ is non-compact. $\square$

**Corollary 3.0.5** (Voiculescu's Double Commutant Theorem)**.** If $A \subseteq \mathcal{Q}(\mathcal{H})$ is a separable, unital C*-algebra, then $A'' = A$, where $A' := \{b \in \mathcal{Q}(\mathcal{H}) \mid ab = ba \quad \forall a \in \mathcal{A}\}$ is the *relative commutant.*

*Proof.* First, notice that if $p \in \mathrm{Lat}(A)$, then $(1-p)ap = 0$ for all $a \in A$, and so $(1-p)a^*p = 0$, whence $pa(1-p) = 0$, which altogether gives $pa = ap = pap$ for all $a \in A$. In other words, $\mathrm{Lat}(A) \subseteq A'$, whence $A'' \subseteq \mathrm{Lat}(A)' \subseteq \mathrm{Alg}(\mathrm{Lat}(A)) = A$.[3] $\square$

# 4 Further Applications

**Theorem 4.0.1.** Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable infinite dimensional C*-algebra in $\mathcal{B}(\mathcal{H})$. Then $H^1(A, \mathcal{K}(\mathcal{H})) \neq 0$.

**Theorem 4.0.2.** Let $A$ be a separable C*-algebra. Then $H^1(A, B) = 0$ for all C*-algebras $B$ containing $A$ if and only if $A$ is finite-dimensional.

[3] If $b \in (\mathrm{Lat}\, A)'$, then $bp = (bp)p = pbp$ for all $p \in \mathrm{Lat}\, A$, so $b \in \mathrm{Alg}(\mathrm{Lat}(A))$.

**Definition 4.0.3.** A set $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if for all $\epsilon > 0$, finite $\mathcal{F} \subseteq \mathcal{S}$ and finite $\Xi \subseteq \mathcal{H}$, there is a finite rank projection $P \in \mathcal{B}(\mathcal{H})$ such that $\|PT - TP\| < \epsilon$ for all $T \in \mathcal{F}$, and $\|P^{\perp}\xi\| < \epsilon$ for all $\xi \in \Xi$.

A C*-algebra $A$ is **quasidiagonal** if there is a faithful *-representation $\rho : A \to \mathcal{B}(\mathcal{H})$ such that $\rho(A)$ is a quasidiagonal set.

**Theorem 4.0.4.** A unital C*-algebra $A$ is quasidiagonal if and only if $\forall \epsilon > 0$ $\mathcal{F} \subseteq A$ finite, there exists a *-representation $\rho : A \to \mathcal{B}(\mathcal{H})$ and a finite rank projection $P \in \mathcal{B}(\mathcal{H})$ such that $\|P\rho(a)P\| \geq \|a\| - \epsilon$ and $\|P\rho(a) - \rho(a)P\| < \epsilon$ for all $a \in \mathcal{F}$.

**Theorem 4.0.5.** Let $A, B$ be C*-algebras, $\varphi_0, \varphi_1 : A \to B$ homotopic *-homomorphisms such that $\varphi_0$ is injective and $\varphi_1(A)$ is quasidiagonal. Then $A$ is quasidiagonal.

**Definition 4.0.6.** Let $A, B$ be C*-algebras. Then $A$ is said to *homotopically dominate* $B$ if there are *-homomorphisms $\pi : B \to A$ and $\sigma : A \to B$ such that $\sigma \circ \pi$ is homotopic to $\mathrm{id}_B$. $A$ and $B$ are homotopically equivalent if there are *-homomorphisms $\pi : B \to A$ and $\sigma : A \to B$ such that $\sigma \circ \pi$ is homotopic to $\mathrm{id}_B$ and $\pi \circ \sigma$ is homotopic to $\mathrm{id}_A$ (this is a stronger notion than saying $A$ homotopically dominates $B$ homotopically dominates $A$).

**Theorem 4.0.7.** Suppose $A$ homotopically dominates $B$, and $A$ is quasidiagonal. Then $B$ is quasidiagonal. Thus, quasidiagonality is invariant under homotopy-equivalence.

## References


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[2] K. R. Davidson. *C*-Algebras by Example*. American Mathematical Society, 1996.

[3] D. Voiculescu. "A Non-Commutative Weyl-Von Neumann Theorem". In: *Romanian Journal of Pure and Applied Mathematics* 21.1 (1976), pp. 97–113.