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Assignment 2 (Exercise 3.3, 3.5)

3.3

a) Show H is symmetric.

$$H = X(X^TX)^{-1}X^T$$

$$\begin{aligned} H^T &= (X(X^TX)^{-1}X^T)^T \\ &= X^T(X^TX)^{-T}X \\ &= X^T(XX^T)^{-1}X \\ &= X(X^TX)^{-1}X^T = H \end{aligned}$$

b) Show $H^k = H$ for any positive integer k.

$$H^3 = H$$

$$H = X(X^TX)^{-1}X^T$$

$$\begin{aligned} H^3 &= [X(X^TX)^{-1}X^T \cdot X(X^TX)^{-1}X^T \cdot X(X^TX)^{-1}X^T] \\ &= X\cancel{(X^TX)^{-1}}\cancel{(X^TX)}\cancel{(X^TX)^{-1}}\cancel{(X^TX)}(X^TX)^{-1}X^T \\ &= X(X^TX)^{-1}X^T = H \end{aligned}$$

therefore if $H^3 = H$

then $H^k = H$

c) If I is the identity matrix of size N, show that $(I-H)^k = I-H$ for any positive integer k.

$$\begin{aligned} (I-H)^2 = I-H(I-H) &= II - IH - IH + HH \\ &= II - II(HH) + HH \\ &= I(I-I) - H(H-H) \\ &= I-H \end{aligned}$$

d) Show trace(H) = d+1. $X = n \times d+1$

$$\begin{aligned}\text{trace}(H) &= \text{trace}(X(X^TX)^{-1}X^T) \\ &= \text{trace}(A\,B) \quad A = X(X^TX)^{-1} \\ &= \text{trace}(BA) \quad B = X^T \\ &= \text{trace}(X^TX(X^TX)^{-1}) \\ &= \text{trace}(I) \quad I = d+1 \\ &= d+1\end{aligned}$$

3.5

a) How is tanh related to θ

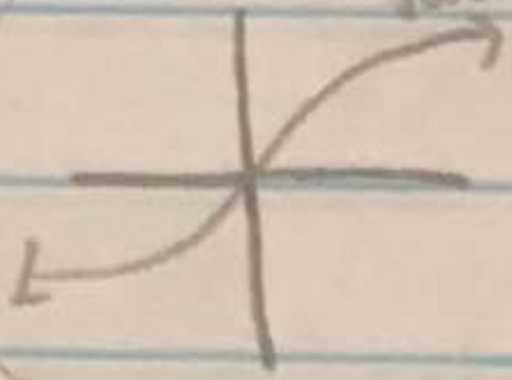
$$\tanh(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}} = \frac{e^{s+s}+1}{1+e^{s+s}} = \frac{e^{2s}-1}{1+e^{2s}} = \frac{e^{2s}}{1+e^{2s}} - \frac{1}{1+e^{2s}}$$

$$\begin{aligned} &= \theta(2s) - \frac{1}{1+e^{2s}} \\ &= \theta(2s) - (1-\theta(2s)) \\ &= \theta(2s) - 1 + \theta(2s) \\ &= 2\theta(2s) - 1\end{aligned}$$

$\theta(s) = \frac{e^s}{1+e^s}$

b) Show tanh(s) converges to a hard threshold for large |s| & converges to no threshold for small |s|.

tanh(s)



as $s \to \infty$, $\tanh(s) \to 1$

as $s \to -\infty$, $\tanh(s) \to -1$

so when $|s| \to \infty$ the hard threshold is 1

& when $|s| \to -\infty$ the hard threshold is -1

when |s| is small, tanh(s) is slowly increasing & never reaches a hard threshold so if it never reaches a hard threshold then there is no threshold when |s| is small