Percents, Percent changes, Percentage point changes, and logarithms

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Goals of this unit

- Understand the correct and distinct use of the concepts "percent," "percent change" (or "percent difference"), and "percentage-point change"
- Convert a long-term change into a long-term percent change and an average annual growth rate.
- Understand exponential growth processes (values that grow at a growth rate expressed in percent)
  - Express these processes as either recursively, $y_t = A \cdot (1+g)y_{t-1}$, or exponentially, $y_t = A \cdot e^{gt}$
- Understand the use of logarithm to convert changes (or differences) into percent changes or differences or to compute an average annual growth rate.

$$\ln(Y_1) - \ln(Y_0) \approx \frac{Y_1 - Y_0}{Y_0}$$

$$g \approx= \frac{(\ln(Y_T) - \ln(Y_0))}{T}$$

- Convert time-series values to logarithms to show exponential processes as linear percent-growth process

Percents

Percents or percentages describe a quantity in relation to a whole. For example, the Unemployment Rate expresses the number Unemployed as a percent, or share, of the Labor Force at a point in time. The Employment-Population Ratio expresses the number Employed as a percent, or share, of the Population at a point in time.

Use FRED for the following exercise. Choose a point in time. Retrieve the number of Unemployed and the number in Labor Force. Compute the Unemployment Rate. Compare your calculation to the published Unemployment Rate.

Changes

A percent change describes a change in a quantity from an initial period to a final period *in relation to* its value in the initial period.

$$\%\Delta X = \frac{X_f - X_i}{X_i}$$

Changes in GDP (economic growth), in population (population growth), or in the price level (inflation) are expressed in percent changes.

Changes

The essence of exponential growth is that the new growth (in the next one year, for example) depends on the *level* that we are starting from.

Suppose you have a job that pays \$100,000 per year.

Consider the difference between these two alternative growth rules for your career ladder:

1. Your income grows by \$1,000 per year.
2. Your income each year is 1% higher than the previous year (a 1% growth rate)

Year	Growth rule 1	Growth rule 2
0	\$100,000	\$100,000
1	\$101,000	\$101,000
2	\$102,000	\$102,010
⋮		
45	\$145,000	\$156,481

In the first year, the two rules give the same increase, a \$1,000 increase.

But in the second year, the first rule simply gives an additional \$1,000 while the second rule gives an additional \$1,010 because the growth builds on the previous salary. Over 45 years, the difference in raises (and in the end salary) is rather large.

- Why do percentage growth rates frequently appear in the world?
- Because the growth process depends on the level that is growing.
  - The size of an economy determines how much of its output can be allocated for new growth.
  - The size of a population determines how many individuals will be available to reproduce.
  - The size of an infected group determines how many entities will be spreading a disease. . .

A percent change $\%\Delta X$ that occurs over a longer period of T years is sometimes expressed as the *average annual growth rate*, g, the one-year growth that compounded for T years leads to a total growth of $\%\Delta X$.

Growing by g% for T years in a row compounds the growth T times.

Given the total percent change over T years we can find the average annnual growth rate that makes it so:

$$
\begin{aligned}
(1+g)^T &= 1 + \%\Delta X \\
g &= (1 + \%\Delta X)^{1/T} - 1
\end{aligned}
$$

Recall that exponentiating to the $1/T$ is equivalent to taking the Tth root.

Differences

The same formula can be used to describe differences (as well as changes) between a base entity and a comparison entity *in relation to* the value for the base entity.

$$\%\Delta X = \frac{X_c - X_b}{X_b}$$

Examples

- Differences in GDP per capita between countries are often expressed in percent differences.
- Consider the wage gap in average earnings between African American men ($621/week) and African-American women ($582/week)

$$\%\text{Difference} = \frac{582 - 621}{621} = -0.0628019$$

Changes

Use the percent change formula,

$$\%\Delta X = \frac{X_f - X_i}{X_i}$$

to compute changes in GDP or in population between 1990 and today.

Use the average annual growth rate formula,

$$g = (1 + \%\Delta X)^{1/T} - 1$$

to compute the average annual growth rate in GDP or in population between 1990 and today.

Changes and differences

Percentage-point change: describes a change in a quantity measured in percent from an initial period to a final period.

Percentage-point difference: describes a differene in quantities measured in percent.

Find the unemployment rate at its most recent peak in October 2009, and find the unemployment rate today. What was the percentage-point change in the unemployment rate over this period?

Find the growth rate for China and for India for some recent period of time. What was the percentage-point difference in growth rates between China and India?

Changes

Percent change in percentages describes a change in a quantity measured in percent from an initial period to a final period in relation to its value in the initial period. In the percent-change expression,

$$\%\Delta X = \frac{X_f - X_i}{X_i}$$

X can be a term measured in percent. Compute the percent decline in the unemployment rate from its most recent peak in October 2009 through today.

The Council of Economic Advisers Style Guide

The symbol % is ambiguous when describing changes. Suppose that the unemployment rate is 4%. Does a "5% increase in the unemployment rate" mean that the unemployment rate increases from 4% to 9% (an increase of 5 percentage points) or that the unemployment rate increases from 4% to 4.2% (an increase of 5 percent)? (The Council of Economic Advisers style guide *always* requires writing out the terms "percent" or "percentage points" to avoid ambiguity.)

The CEA style guide also calls for using the singular "percentage point" if the change or difference is less than or equal to 1 and the plural "percentage points" if the change or differences is greater than 1.

Logarithms

Logarithm is the inverse operation of an exponential function

$$\text{If } x = b^y \text{ or } x \text{ equals } b \text{ to the } y$$

$$\text{then } y = \log_b(x)\text{read: } y \text{ equals } \log \text{ base } b \text{ of } x$$

y answers the question, "b to the what equals x?"
Because $b^0 = 1$, $\log_b(1) = 0$, or $\log$ of 1 is zero.
It is only possible to take $\log$ of positive numbers.

Some common examples

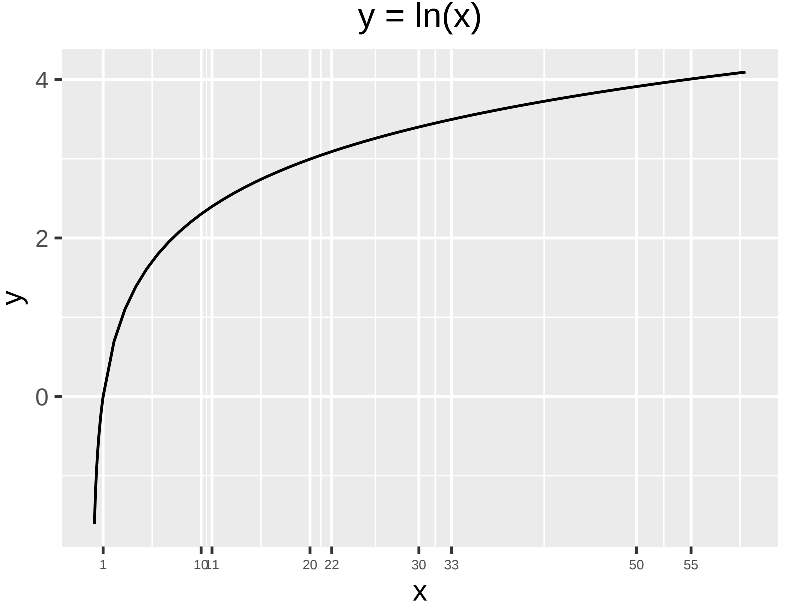
$\log_{10}$ computes orders of magnitude, e.g., $\log_{10} 1{,}000 = 3$ and $\log_{10} 1{,}000{,}000 = 6$.
$\log_e$ ("$\log$ base e" with the constant $e = 2.71828\ldots$) is called "natural log." It is so common that (1) people often mean natural log when they say "log" and (2) natural log has a special symbol $\ln$.
For example, $\ln 35 = 3.5553481$ because $e^{3.5553481} = 35$

Logarithms

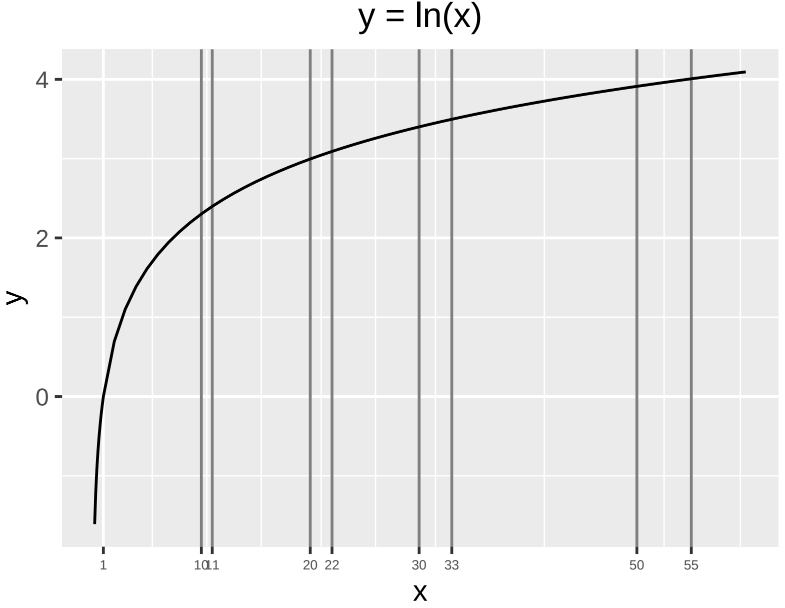
Important rules of logarithms

$$\log(m \cdot n) = \log m + \log n \qquad \text{(multiplication becomes addition)}$$
$$\log(m/n) = \log m - \log n \qquad \text{(division becomes subtraction)}$$
$$\log(m^k) = k \log m \qquad \text{(exponentiation becomes multiplication)}$$
$$\log_c b = \log_a b \cdot \log_c a \qquad \text{(change of base)}$$
$$\log_e 1000 = \log_{10} 1000 \log_e 10 \qquad \text{(change of base: example)}$$
$$\log_e 1000 = 3 \cdot 2.30 = 6.90 \qquad \text{(change of base: example)}$$

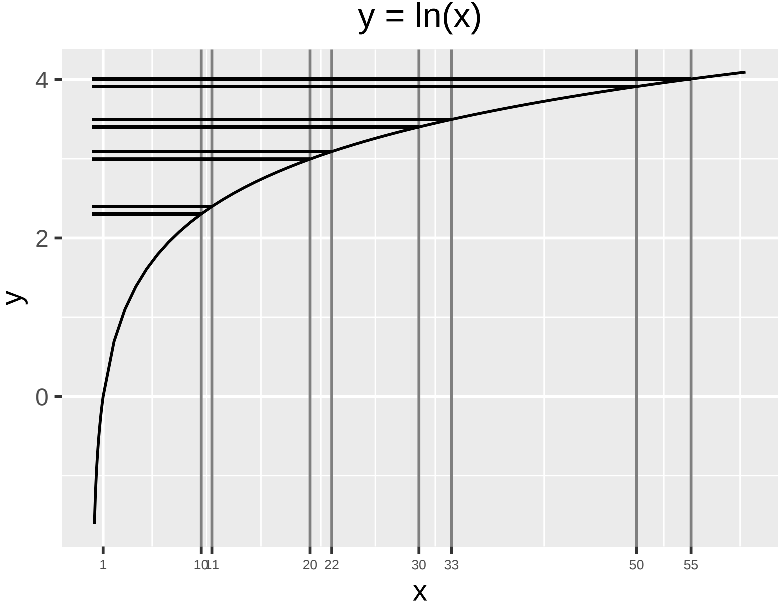
The next slide shows a plot of $y = \ln(x)$
Note that the plot is upward sloping everywhere but concave (very positive slope on the left, less positive slope on the right).
Note that $\ln(1) = 0$.
Note that $\ln(0.5) < 0$


y = ln(x)
4
2
0
y
1
10
11
20
22
30
33
50
55
x

An interesting feature of $\ln(x)$ concerns equal percent changes in x. For example, 10 and 11 (a 10% increase), 20 and 22 (a 10% increase), 30 and 33 (a 10% increase), and 50 and 55 (a 10% increase) are all marked on the following slide.


y = ln(x)
4
2
0
y
1
10
11
20
22
30
33
50
55
x

The next slide demonstrates that equal *percent* changes in x cause equal *absolute* changes in $\ln(x)$.



y = ln(x)
4
2
0
y
1
10
11
20
22
30
33
50
55
x

$$\begin{aligned}
\ln(11) - \ln(10) &= 0.0953102 \\
\ln(22) - \ln(20) &= 0.0953102 \\
\ln(33) - \ln(30) &= 0.0953102 \\
\ln(55) - \ln(50) &= 0.0953102
\end{aligned}$$

True even for big numbers that are 10% apart:

$$\ln(55000) - \ln(50000) = 0.0953102$$

More on Logarithms — fill in the missing numbers

$$\ln(1+g) \approx g \text{ when g is “small”}$$

What is “small”? $-0.15 < g < +0.15$

$$\ln(1{-}0.20) = -0.2231436 \overset{?}{\approx} -0.20$$

$$\ln(1{-}0.10) = -0.1053605 \overset{?}{\approx} -0.10$$

$$\ln(1{-}0.05) = -0.0512933 \approx -0.05$$

$$\ln(1{-}0.03) = -0.0304592 \approx -0.03$$

$$\ln(1{+}0.00) = 0$$

$$\ln(1{+}0.03) = 0.0295588 \approx +0.03$$

$$\ln(1{+}0.05) = \qquad \approx +0.05$$

$$\ln(1{+}0.10) = \qquad \overset{?}{\approx} +0.10$$

$$\ln(1{+}0.15) = \qquad \overset{?}{\approx} +0.15$$

$$\ln(1{+}0.20) = \qquad \overset{?}{\approx} +0.20$$

More on Logarithms

Why does $\ln(1+g) \approx g$ when g is "small"? The answer involves calculus. (Feel free to skip.)

We will need the derivative of $\ln(x)$ with respect to x:

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

A Taylor series expansion of $\ln(x)$ around $x_0 = 1$ is

$$\ln(x) \approx \ln(x_0) + \left.\frac{d\ln(x)}{dx}\right|_{x=x_0} \cdot (x - x_0)$$

$$\ln(x) \approx \ln(x_0) + \frac{1}{x_0} \cdot (x - x_0) \qquad \text{(Substituting derivative)}$$

$$\ln(x) \approx \ln(1) + \frac{1}{1} \cdot (x - 1) \qquad \text{(Around } x_0 = 1\text{)}$$

$$\ln(x) \approx 0 + 1 \cdot (x - 1) = x - 1 \qquad \text{(Simplify)}$$

Now consider $x = 1 + g$ with g small so that x is close to $x_0 = 1$

$$\ln(1+g) \approx (1+g) - 1 = g \qquad \text{(As claimed)}$$

Applying Logarithms

Putting together the rules for logarithms and $\ln(1+g) \approx g$, we can use logarithms to speed calculations to express percent changes. Suppose Y_0 has grown by a factor of $1+g$ into Y_1, i.e., by $g \times 100$ percent:

$$
\begin{aligned}
Y_1 &= (1+g)Y_0 \\
Y_1/Y_0 &= (1+g) && \text{(Rearrange)} \\
\ln(Y_1/Y_0) &= \ln(1+g) && \text{(Take } \ln \text{ of both sides)} \\
\ln Y_1 - \ln Y_0 &= \ln(1+g) \approx g \\
&\text{(Difference of logs is approximately the percent difference)}
\end{aligned}
$$

Keep in mind that this formula works only for small changes. Compute the percent change in real per capita GDP for 2017 and 1950 using the change in logs and using the exact percent change formula. Note the (relatively small) difference between the two methods.

Example spreadsheet on Exponential Growth

Key Lessons

- Recursive expression of exponential growth at 6%: $y_t = 1.06 y_{t-1}$
- Exponential expression of exponential growth
  - $y = 2^{0.06t}$ is too small
  - $y = 3^{0.06t}$ is too big
  - $y = e^{0.06t}$ is just right (see that it very closely matches $y_t = 1.06 y_{t-1}$
- Taking log of both sides $\ln(y) = \ln(e^{0.06t}) = 0.06t$ expresses $\ln(y)$ as a straight line in time t with slope 0.06

Applying Logarithms

Putting together the rules for logarithms and $\ln(1+g) \approx g$, we can use logarithms to speed calculations to express percent changes.

Using the example above, we can also examine average annual growth rates.

Suppose Y_0 has grown by a factor of $1+g$ per year for T years into Y_1

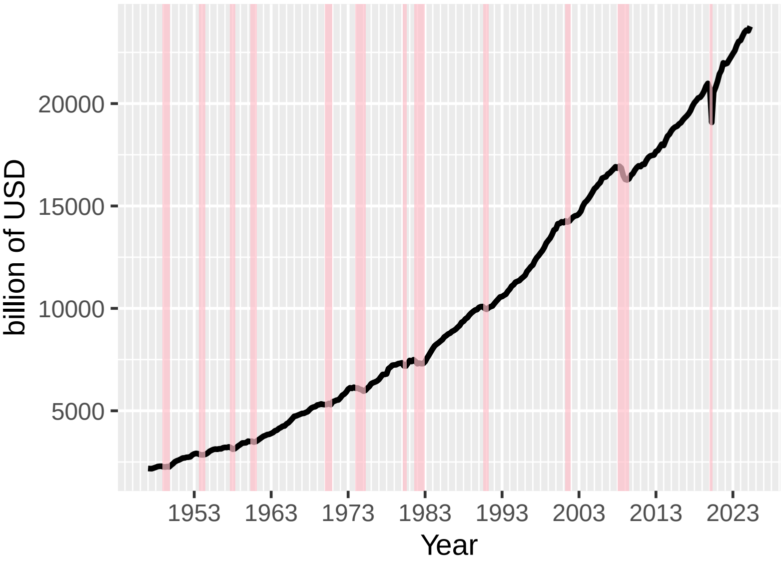
$$
\begin{aligned}
Y_1 &= (1+g)^T Y_0 \\
Y_1/Y_0 &= (1+g)^T && \text{(Rearrange)} \\
\ln(Y_1/Y_0) &= \ln(1+g)^T = T\ln(1+g) && \text{(Take } \ln \text{ of both sides)} \\
\frac{\ln Y_1 - \ln Y_0}{T} &= \ln(1+g) \approx g \\
& && \text{(Approximate average annual growth rate)}
\end{aligned}
$$

Keep in mind that this formula works only for small changes.

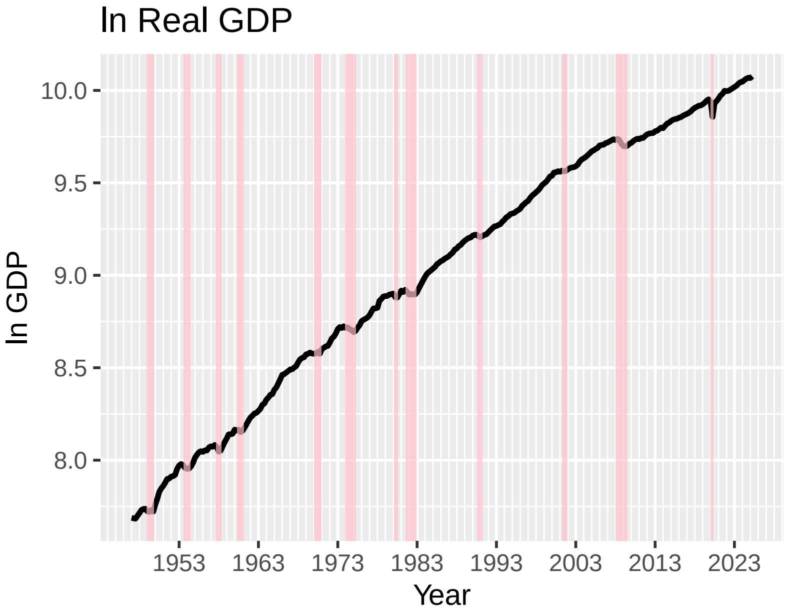
Compute the percent change in real per capita GDP for 2017 and 1950 using the change in logs and using the exact percent change formula. Note the difference between the two methods.

The next two slides show time series plots of level GDP (in billions of dollars) and $\ln$ GDP.
The level GDP series is so dominated by the enormous growth in GDP that it is impossible to identify changes in the growth rate over time, early recessions look small (because the economy was small).
Plotted in log, changes between years are interpreted as percent changes. The slope of the log GDP series is the growth rate of GDP (in percentage terms). Recessions are clearly indicated, and the growth slowdown after 1973 is evident.

Real GDP



billion of USD
20000
15000
10000
5000
1953
1963
1973
1983
1993
2003
2013
2023
Year


ln Real GDP
10.0
9.5
9.0
8.5
8.0
ln GDP
1953
1963
1973
1983
1993
2003
2013
2023
Year

Another change application

We frequently decompose one term into the product or quotient of several other terms. When the terms are products or quotients, percent changes in the outcome term are the sums and differences of percent changes of the constituent terms.
Nominal GDP $= P \cdot Y$ where P is the price level and Y is real GDP.

$$\text{nom } Y = P \cdot Y$$
$$\%\Delta \text{ nom GDP} = \%\Delta P + \%\Delta Y$$

Real productivity $= \frac{Y}{PL}$ where Y is nominal GDP (this time), L is hours worked, and P is the price level.

$$\text{Real productivity } = \frac{Y}{PL}$$
$$\%\Delta \text{ Real Producvitity} = \%\Delta Y - \%\Delta P - \%\Delta L$$

One way to derive this relationship is to use the rules for logarithms and the percent change rule.

$$
\begin{aligned}
A &= \frac{B \cdot C}{D} \\
\log(A) &= \log(B) + \log(C) - \log(D) \\
&\qquad\qquad\qquad\qquad \text{(rules for logarithms)} \\
\log(A_1) - \log(A_0) &= \log(B_1) + \log(C_1) - \log(D_1) \\
&\quad - (\log(B_0) + \log(C_0) - \log(D_0))
\end{aligned}
$$

$$
\log(A_1) - \log(A_0) = \log(B_1) - \log(B_0) + \log(C_1) - \log(C_0) - (\log(D_1) - \log(D_0))
$$

$$
\%\Delta A = \%\Delta B + \%\Delta C - \%\Delta D
$$