The Connection Between the Poisson and Binomial Distributions

The Poisson distribution is actually a limiting case of a Binomial distribution when the number of trials, n, gets very large and p, the probability of success, is small. As a rule of thumb, if $n \geq 100$ and $np \leq 10$, the Poisson distribution (taking $\lambda = np$) can provide a very good approximation to the binomial distribution.

This is particularly useful as calculating the combinations inherent in the probability formula associated with the binomial distribution can become difficult when n is large.

To better see the connection between these two distributions, consider the binomial probability of seeing x successes in n trials, with the aforementioned probability of success, p, as shown below.

$$P(x) = {}_nC_x p^x q^{n-x}$$

Let us denote the expected value of the binomial distribution, np, by λ. Note, this means that

$$p = \frac{\lambda}{n}$$

and since $q = 1 - p$,

$$q = 1 - \frac{\lambda}{n}$$

Now, if we use this to rewrite $P(x)$ in terms of λ, n, and x, we obtain

$$P(x) = {}_nC_x \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Using the standard formula for the combinations of n things taken x at a time and some simple properties of exponents, we can further expand things to

$$P(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \cdot \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Notice that there are exactly x factors in the numerator of the first fraction. Let us swap denominators between the first and second fractions, splitting the n^x across all of the factors of the first fraction's numerator.

$$P(x) = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Finally, let us split the last factor into two pieces, noting (for those familiar with Calculus) that one has a limit of $e^{-\lambda}$.

$$P(x) = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

It should now be relatively easy to see that if we took the limit as n approaches infinity, keeping x and λ fixed, the first x fractions in this expression would tend towards 1, as would the last factor in the expression. The second to last factor, as was mentioned before, tends towards $e^{-\lambda}$, and the remaining factor stays unchanged as it does not depend on n. As such,

$$\lim_{n \to \infty} P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Which is what we wished to show.