Sampling and Interpolation in 1D

The notebook serves to introduce some concepts which are useful with continous signals

Some Definitions

A time continous function $f(t)$ is related to its Fourier transform $F(u)$:

$$F(u) = \int_{-\infty}^{\infty} f(t) \cdot exp[-j \cdot 2\pi \cdot u \cdot t] \cdot dt$$

The inverse Fourier transform is given by:

$$f(t) = \int_{-\infty}^{\infty} F(u) \cdot exp[j \cdot 2\pi \cdot u \cdot t] \cdot du$$

Depending on the properties of function $f(t)$ its Fourier transform $F(u)$ may either extend to

1. infinite frequencies (not bandlimited)
2. infinite frequencies but most of the signal energy is located in the finite frequency band $-\frac{B}{2} \leq u \leq \frac{B}{2}$ (nearly bandlimited)
3. all signal energy confined to frequency band $-\frac{B}{2} \leq u \leq \frac{B}{2}$ (strictly bandlimited)

For numerical evaluation only the cases *nearly bandlimited* and *strictly bandlimited* considered.

Thus the inverse Fourier transform may written with finite limits for the integral:

$$f(t) \approx \int_{-\frac{B}{2}}^{\frac{B}{2}} F(u) \cdot exp[j \cdot 2\pi \cdot u \cdot t] \cdot du$$

Periodic Repetions

In the *signal domain* the function $\tilde{f}(t)$ denotes a periodic repetition of function $f(t)$ with a period T_f.

In the *transform domain* the function $\tilde{F}(u)$ denotes a periodic repetion of the Fourier transform $F(u)$ with a period B_f.

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t - n \cdot T_f)$$

$$\tilde{F}(u) = \sum_{m=-\infty}^{\infty} F(u - m \cdot B_f)$$

Since $\tilde{f}(t)$ and $\tilde{F}(u)$ are periodic they can be written as Fourier series:

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k \cdot exp\left[j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right]$$

$$\tilde{F}(u) = \sum_{l=-\infty}^{\infty} C_l \cdot exp\left[-j \cdot 2\pi \cdot l \cdot \frac{u}{B_f}\right]$$

signal domain

For the signal domain representation the Fourier coefficients c_k are determined:

$$c_k = \frac{1}{T_f}\int_0^{T_f} \tilde{f}(t) \cdot exp\left[-j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right] \cdot dt$$

$$c_k = \frac{1}{T_f}\sum_{n=-\infty}^{\infty} \int_0^{T_f} f(t - n \cdot T_f) \cdot exp\left[-j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right] \cdot dt$$

$$c_k = \frac{1}{T_f}\int_{-\infty}^{\infty} f(t) \cdot exp\left[-j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right] \cdot dt$$

$$c_k = \frac{1}{T_f} \cdot F\left(\frac{k}{T_f}\right)$$

Apart from a scaling factor $\frac{1}{T_f}$ the Fourier series coefficient c_k related to the value of the Fourier transform at discrete frequency $\frac{k}{T_f}$. The fundamental frequency is $\frac{1}{T_f}$.

Hence we have these equations for the periodic repetition $\tilde{f}(t)$ of $f(t)$:

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t - n \cdot T_f)$$

$$\tilde{f}(t) = \frac{1}{T_f}\sum_{k=-\infty}^{\infty} F\left(\frac{k}{T_f}\right) \cdot exp\left[j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right]$$

transform domain

For the transform domain the Fourier coefficients C_k are determined:

$$C_l = \frac{1}{B_f}\int_{-B_f/2}^{B_f/2}\tilde{F}(u) \cdot exp\left[j \cdot 2\pi \cdot l \cdot \frac{u}{B_f}\right] \cdot du$$

$$C_l = \frac{1}{B_f}\sum_{m=-\infty}^{\infty}\int_{-B_f/2}^{B_f/2}F(u - m \cdot B_f) \cdot exp\left[j \cdot 2\pi \cdot l \cdot \frac{u}{B_f}\right] \cdot du$$

This equation can be re-expressed as with an integral having infinite limits.

$$C_l = \frac{1}{B_f}\int_{-\infty}^{\infty}F(u) \cdot exp\left[j \cdot 2\pi \cdot l \cdot \frac{u}{B_f}\right] \cdot du$$

$$C_l = \frac{1}{B_f} \cdot f\left(\frac{l}{B_f}\right)$$

Defining the sampling interval T_s by $T_s = \frac{1}{B_f}$ the Fourier series coefficients C_l are expressed in terms $f(t)$ taken at samples spaced T_s apart.

Now the periodic repetitions $\tilde{F}(u)$ can be expressed by equation:

$$\tilde{F}(u) = T_s \cdot \sum_{l=-\infty}^{\infty} f\left(\frac{l}{B_f}\right) \cdot exp\left[-j \cdot 2\pi \cdot l \cdot \frac{u}{B_f}\right]$$

$$\tilde{F}(u) = T_s \cdot \sum_{l=-\infty}^{\infty} f\left(l \cdot T_s\right) \cdot exp\left[-j \cdot 2\pi \cdot l \cdot u \cdot T_s\right]$$

If $f(t)$ is a function with bandlimited Fourier transform $F(u)$ (defined only for $[-B/2, \ldots, B/2]$) and if $B_f \leq B$ we have:

$$\tilde{F}(u) = F(u)\, for\, -B/2 \leq u \leq B/2$$

and therefore:

$$F(u) = \begin{cases} \sum_{l=-\infty}^{\infty} T_s \cdot f(l \cdot T_s) \cdot exp\left[-j \cdot 2\pi \cdot l \cdot T_s \cdot u\right] & -B/2 \leq u \leq B/2 \\ 0 & otherwise \end{cases}$$

Sampling Theorem

From the last equation some form of the `Sampling Theorem` can be derived.

We start with

$$F(u) = \sum_{l=-\infty}^{\infty} T_s \cdot f(l \cdot T_s) \cdot exp\left[-j \cdot 2\pi \cdot l \cdot T_s \cdot u\right]$$

and compute the inverse Fourier transform:

$$f(t) = \int_{-B_f/2}^{B_f/2} F(u) \cdot exp[j \cdot 2\pi \cdot t \cdot u] \cdot du$$

$$f(t) = \sum_{l=-\infty}^{\infty} T_s \cdot f(l \cdot T_s) \cdot \int_{-B_f/2}^{B_f/2} exp\left[j \cdot 2\pi \cdot \left(t - l \cdot T_s\right) \cdot u\right] \cdot du$$

$$f(t) = \sum_{l=-\infty}^{\infty} f(l \cdot T_s) \cdot sinc\left(\pi \cdot \left(\frac{t}{T_s} - l\right)\right)$$

The last equation expresses the continous function $f(t)$ as a series expansion with samples $f(l \cdot T)$ as coefficients and the $sinc()$ function as the *interpolation* function.

1D Discrete Fourier Transform

The periodic function $\tilde{f}(t)$ has been obtained from periodically repeating function $f(t)$ with a period of T_f

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t - n \cdot T_f)$$

$$\tilde{f}(t) = \frac{1}{T_f} \sum_{k=-\infty}^{\infty} F\left(\frac{k}{T_f}\right) \cdot exp\left[j \cdot 2\pi \cdot k \cdot \frac{t}{T_f}\right]$$

Now the time interval $t_0 < t < (t_0 + T_f)$ is partitioned into N subintervals of duration $T = T_f/N$. The periodic function $\tilde{f}(t)$ is evaluated/sampled for $t = t_0 + m \cdot T,\ 0 \le m \le N - 1$.

$$\tilde{f}(m \cdot T;\ t_0) = \frac{1}{T_f} \sum_{k=-\infty}^{\infty} F\left(\frac{k}{T_f}\right) \cdot exp\left[j \cdot 2\pi \cdot k \cdot \frac{t_0}{T_f}\right] \cdot exp\left[j \cdot 2\pi \cdot \frac{1}{N} \cdot k \cdot m\right]$$

Summation index k shall be expressed by:

$$k = l + r \cdot N$$
$$0 \le l \le N - 1$$
$$-\infty < r < \infty$$

Moreover let $u_0 = \frac{1}{T_f}$ be the fundamental frequency.

$$\tilde{f}(m \cdot T;\ t_0) = u_0 \cdot \sum_{l=0}^{N-1} \underbrace{\sum_{r=-\infty}^{\infty} F\left(l \cdot u_0 + r \cdot N \cdot u_0\right) \cdot exp\left[j \cdot 2\pi \cdot (l + r \cdot N) \cdot u_0 \cdot t_0\right]}_{\bar{F}\left(l \cdot u_0\right)} \cdot exp\left[j \cdot 2\pi \cdot \frac{1}{N} \cdot l \cdot \right.$$

$$\tilde{f}(m \cdot T;\ t_0) = u_0 \cdot \sum_{l=0}^{N-1} \bar{F}\left(l \cdot u_0\right) \cdot exp\left[j \cdot 2\pi \cdot \frac{1}{N} \cdot l \cdot m\right]$$

Here the coefficient $\bar{F}\left(l \cdot f_0\right)$ denotes a sample of the *aliased* value of the Fourier transform.

$$\bar{F}\left(l \cdot u_0\right) = \sum_{r=-\infty}^{\infty} F\left(l \cdot u_0 + r \cdot N \cdot u_0\right) \cdot exp\left[j \cdot 2\pi \cdot (l + r \cdot N) \cdot u_0 \cdot t_0\right]$$

It is instructing to look at the special case of $F(u)$ beeing *nearly/perfectly* bandlimited. We consider the cases

1. `N` is even
2. `N` is odd

The largest positive frequency is denoted $u_{(+)}$ and smallest negative frequency $u_{(-)}$.

N even

$$u_{(-)} = -(N/2 - 1) \cdot u_0$$
$$u_{(+)} = (N/2 - 1) \cdot u_0$$

N odd

$$u_{(-)} = -\frac{N-1}{2} \cdot u_0$$
$$u_{(+)} = \frac{N-1}{2} \cdot u_0$$

With $r = 0$ positive frequencies belong to the range of index l :

$$\bar{F}\left(l \cdot u_0\right) = F\left(l \cdot u_0\right) \cdot exp\left[j \cdot 2\pi \cdot l \cdot u_0 \cdot t_0\right]$$

$$l = \begin{cases} 0 \leq l \leq (N/2 - 1) & N\ even \\ 0 \leq l \leq \frac{N-1}{2} & N\ odd \end{cases}$$

and with $r = -1$ negative frequencies are in range of index l:

$$\overline{F}\left(l \cdot u_0\right) = F\left(-(N-l) \cdot u_0\right) \cdot exp\left[-j \cdot 2\pi \cdot (N-l) \cdot u_0 \cdot t_0\right]$$

$$l = \begin{cases} (N/2+1) \leq l \leq (N-1) & N\ even \\ -\frac{N+1}{2} \leq l \leq (N-1) & N\ odd \end{cases}$$

Application

But first a summary of what it is known already:

summary

The relationsship of samples $\tilde{f}(m \cdot T;\ t_0)$ to samples of the aliased Fourier transform $\overline{F}\left(l \cdot f_0\right)$ have been found:

$$\tilde{f}(m \cdot T;\ t_0) = u_0 \cdot \sum_{l=0}^{N-1} \overline{F}\left(l \cdot f_0\right) \cdot exp\left[j \cdot 2\pi \cdot \frac{1}{N} \cdot l \cdot m\right]$$

For the practically important case of $\tilde{f}(t)$ being bandlimited the aliased Fourier transform is related to positive frequencies $u \leq 0$ like this:

$$\overline{F}\left(l \cdot u_0\right) = F\left(l \cdot u_0\right) \cdot exp\left[j \cdot 2\pi \cdot l \cdot u_0 \cdot t_0\right]$$

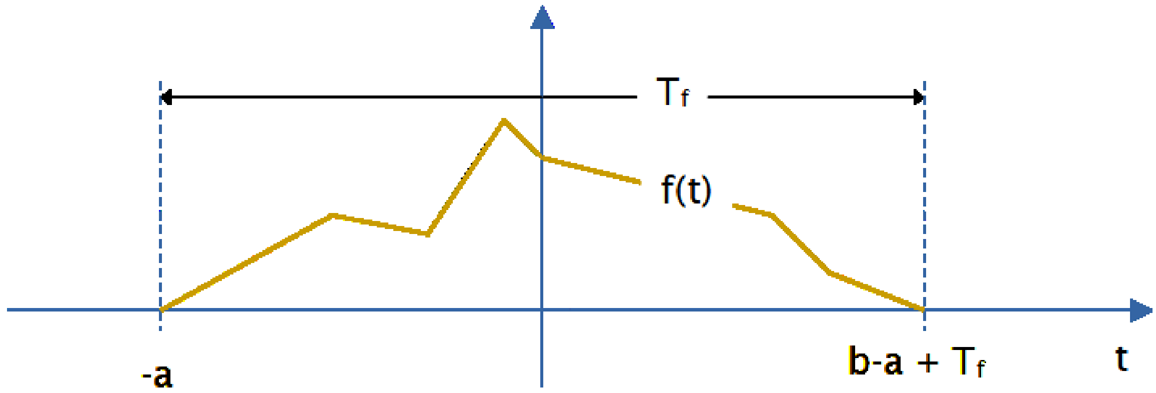
and for negative frequencies $u < 0$ like this:

$$\overline{F}\left(l \cdot u_0\right) = F\left(-(N-l) \cdot u_0\right) \cdot exp\left[-j \cdot 2\pi \cdot (N-l) \cdot u_0 \cdot t_0\right]$$

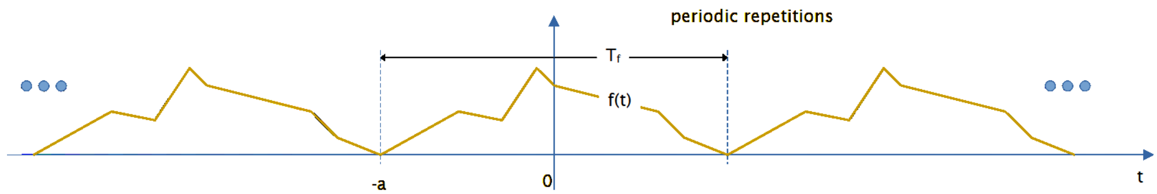
If taking samples of $\tilde{f}(t)$ starts at $t_0 = 0$ the samples of the aliased Fourier transform equal exactly the samples of Fourier transform $F(u)$. For any other starting point t_0 an additional phase component occurs.

An example shall serve as a illustration:

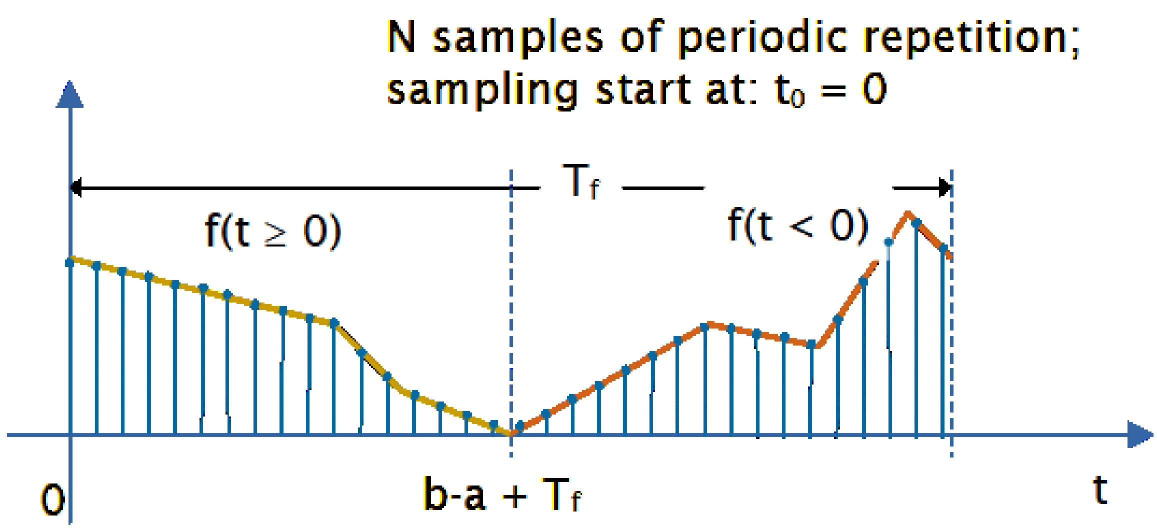
The function / signal $f(t)$ is defined on the range $-a \leq t \leq\ -a + T_f$:



Based on this the periodic repetition $\tilde{f}(t)$ is constructed.



A single period of $\tilde{f}(t)$ is sampled. N samples are taken starting at $t_0 = 0$. The figure shows that the sequence starts with samples of $f(t)$ for $t \geq 0$ and then samples of $f(t)$ for $t < 0$.



In [ ]:

In [ ]: