Continous and discrete 2D Signals and how they are related in the 2D-signal- and 2D-frequency domain

The notebook serves to introduce some concepts which are useful with the processing of 2D signals (eg. images)

Some Definitions

A continous 2D function $f(x,\ y)$ is related to its 2D-Fourier transform $F(u,\ v)$:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot exp[-j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot dx \cdot dy$$

The inverse Fourier transform is given by:

$$f(x,\ y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,\ v) \cdot exp[j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot du \cdot dv$$

For later use we look at the 2D Fourier transform of the shifted function $f(x - x_0,\ y - y_0)$:

$$F_{[x_0,\ y0]}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - x_0,\ y - y_0) \cdot exp[-j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot dx \cdot dy$$

Changing variables we get:

$$F_{[x_0,\ y0]}(u,v) = F(u,v) \cdot exp\left[j \cdot 2\pi \cdot \left(u \cdot x_0 + v \cdot y_0\right)\right]$$

For a real function $f(x,\ y)$ the 2D Fourier transform has these symmetry properties:

$$F(u,\ v) = F^*(-u,\ -v)$$

If we specifically define $f(x,y)$ as an image it shall have these properties:

1. $f(x,\ y)$ has only real values
2. $f(x,\ y)$ shall be defined for a finite range with $x_l \le x \le x_u$ and $y_l \le y \le y_u$
3. The width of these ranges shall be denoted $A_x = x_u - x_l$ and $A_y = y_u - y_l$

The 2D Fourier transform is then evaluated only with finite integration limits like this:

$$F(u,v) = \int_{y_l}^{y_u}\int_{x_l}^{x_u} f(x,y) \cdot exp[-j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot dx \cdot dy$$

Periodic Repetions

In the *2D-signal domain* the function $\tilde{f}(x,\ y)$ denotes a periodic repetition of function $f(y,\ y)$ with a periods $A_x,\ A_y$ for variables $x,\ y$.

$$\tilde{f}(x,\ y) = \sum_{n_y=-\infty}^{\infty} \sum_{n_x=-\infty}^{\infty} f(x - n_x \cdot A_x,\ y - n_y \cdot A_y)$$

Since $\tilde{f}(x,\ y)$ was defined as periodic it can be written as Fourier series:

$$\tilde{f}(x, y) = \sum_{k_y=-\infty}^{\infty} \sum_{k_x=-\infty}^{\infty} c_{k_x,k_y} \cdot exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{A_x} + k_y \cdot \frac{y}{A_y}\right)\right]$$

Since $f(x,\ y)$ has been defined with a finite range we have also $\tilde{f}(x, y) = f(x,\ y)$ within this range.

Computing Fourier series coefficients

signal domain

For the signal domain representation the Fourier coefficients c_{k_x,k_y} from this equation:

$$\int_{y_l}^{y_u}\int_{x_l}^{x_u}\tilde{f}(x, y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x^{'} \cdot \frac{x}{A_x} + k_y^{'} \cdot \frac{y}{A_y}\right)\right] \cdot dx \cdot dy = \int_{y_l}^{y_u}\int_{x_l}^{x_u} f(x, y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x^{'} \cdot \frac{x}{A_x} + k_y^{'}\right.\right.$$

Inserting the 2D-Fourier series representation of $\tilde{f}(x, y)$ yields:

$$\int_{y_l}^{y_u}\int_{x_l}^{x_u} f(x, y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x^{'} \cdot \frac{x}{A_x} + k_y^{'} \cdot \frac{y}{A_y}\right)\right] \cdot dx \cdot dy = \sum_{k_y=-\infty}^{\infty} \sum_{k_x=-\infty}^{\infty} c_{k_x,k_y} \cdot \int_{y=y_l}^{y_u}\int_{x_l}^{x_u} exp\left[j \cdot 2\pi \cdot \left(\right.\right.$$

The double integral on the right hand side of the equation has non-zero contribution only if $k_x = k_x^{'}\ and\ k_y = k_y^{'}$:

$$c_{k_x,k_y} = \frac{1}{A_x \cdot A_y} \cdot \int_{y_l}^{y_u}\int_{x_l}^{x_u}\tilde{f}(x, y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{A_x} + k_y \cdot \frac{y}{A_y}\right)\right] \cdot dx \cdot dy$$

For a special and important case the double integral can be expressed by the D-Fourier transform.

Within this range of integration limits the periodic repetition $\tilde{f}(x, y)$ equals $f(x,\ y)$. For the Fourier series coefficients we may write:

$$c_{k_x,k_y} = \frac{1}{A_x \cdot A_y} \cdot \underbrace{\int_{y_l}^{y_u}\int_{x_l}^{x_u} f(x,y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{A_x} + k_y \cdot \frac{y}{A_y}\right)\right] \cdot dx \cdot dy}_{F\left(u=\frac{k_x}{A_x},\ v=\frac{k_y}{A_y}\right)}$$

Thus the Fourier series coefficients c_{k_x,k_y} are just (apart from a scaling factor) directly related to samples of 2D-Fourier transform $F(u,\ v)$:

$$c_{k_x,k_y} = \frac{1}{A_x \cdot A_y} \cdot F\left(u = \frac{k_x}{A_x},\ v = \frac{k_y}{A_y}\right)$$

The periodic repetitions of $\tilde{f}(x,\ y)$ may the be expressed by Fourier series which at this point involves infinitely many frequencies.

$$\tilde{f}(x,y) = \frac{1}{A_x \cdot A_y} \cdot \sum_{k_y=-\infty}^{\infty}\sum_{k_x=-\infty}^{\infty} F\left(u = \frac{k_x}{A_x},\ v = \frac{k_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{A_x} + k_y \cdot \frac{y}{A_y}\right)\right]$$

If we are only interested in evaluating in the ranges $x_l \leq x \leq x_u$ and $y_l \leq y \leq y_u$ we may replace $\tilde{f}(x,y)$ with $f(x,\ y)$.

$$f(x,y) = \frac{1}{A_x \cdot A_y} \cdot \sum_{k_y=-\infty}^{\infty}\sum_{k_x=-\infty}^{\infty} F\left(u = \frac{k_x}{A_x},\ v = \frac{k_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{A_x} + k_y \cdot \frac{y}{A_y}\right)\right]$$

Evaluation of $f(x,\ y)$ for discrete values

$f(x,\ y)$ shall be evaluated for discrete points $x[n_x],\ y[n_y]$ on the range $x_l \leq x \leq x_u$ and $y_l \leq y \leq y_u$. These ranges are evenly partioned with samples spaced at ΔA_x and ΔA_y.

$$\Delta A_x = (x_l - x_u)/N_x = A_x/N_x$$
$$\Delta A_y = (x_l - x_u)/N_y = A_y/N_y$$
$$0 \leq n_x \leq N_x - 1$$
$$0 \leq n_y \leq N_y - 1$$
$$x[n_x] = x_l + n_x \cdot \Delta A_x$$
$$y[n_y] = y_l + n_y \cdot \Delta A_y$$

$$f(x[n_x], y[n_x]) = \frac{1}{A_x \cdot A_y} \cdot \sum_{k_y=-\infty}^{\infty}\sum_{k_x=-\infty}^{\infty} F\left(\frac{k_x}{A_x},\ \frac{k_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{x_l}{A_x} + k_y \cdot \frac{y_l}{A_y}\right)\right] \cdot exp\left[j \cdot 2\pi \cdot \left(k\right.\right.$$

For the exponential

$$exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{n_x}{N_x} + k_y \cdot \frac{n_y}{N_y}\right)\right]$$

it is observed that it is periodic for $k_x + m_x \cdot N_x$ and $k_y + m_y \cdot N_y$. Writing indices $k_x,\ k_y$ like this:

$$\begin{gathered} k_x = l_x + r_x \cdot N_x \\ k_y = l_y + r_y \cdot N_y \\ 0 \le l_x \le N_x - 1 \\ 0 \le l_y \le N_y - 1 \end{gathered}$$

and with the definition of the *aliased* samples of the 2D Fourier transform

$$\bar{F}\left(\frac{l_x}{A_x},\ \frac{l_y}{A_y}\right) = \sum_{r_x=-\infty}^{\infty} \sum_{r_y=-\infty}^{\infty} F\left(\frac{l_x + r_x \cdot N_x}{A_x},\ \frac{l_y + r_y \cdot N_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x + r_x \cdot N_x) \cdot \frac{x_l}{A_x} + (l_y + r_y \cdot N\right.\right.$$

$$f(x[n_x], y[n_y]) = f(n_x, n_y) = \frac{1}{A_x \cdot A_y} \cdot \sum_{l_y=0}^{N_y-1} \sum_{l_x=0}^{N_x-1} \bar{F}\left(\frac{l_x}{A_x},\ \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(l_x \cdot \frac{n_x}{N_x} + l_y \cdot \frac{n_y}{N_y}\right)\right]$$

Replacing $f(x[n_x], y[n_y])$ by $f(n_x, n_y)$ is convenient if it is implicitely clear how coordinate $x,\ y$ depend of indices $n_x,\ n_y$.

Special case: $F(u,\ v)$ bandlimited

In many cases the 2D Fourier transform $F(u,\ v)$ *bandlimited* or at least *nearly bandlimited*.

We consider the special case that both $N_x,\ N_y$ are even numbers.

The discrete samples of the 2D Fourier transform $F(n \cdot \frac{1}{A_x},\ m \cdot \frac{1}{A_y})$ shall be defined for the frequency ranges:

$$\begin{gathered} -N_x/2 \le n \le (N_x/2 - 1) \\ -N_y/2 \le m \le (N_y/2 - 1) \end{gathered}$$

Looking at the definition of the *aliased* 2D Fourier transform

$$\bar{F}\left(\frac{l_x}{A_x},\ \frac{l_y}{A_y}\right) = \sum_{r_x=-\infty}^{\infty} \sum_{r_y=-\infty}^{\infty} F\left(\frac{l_x + r_x \cdot N_x}{A_x},\ \frac{l_y + r_y \cdot N_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x + r_x \cdot N_x) \cdot \frac{x_l}{A_x} + (l_y + r_y \cdot N\right.\right.$$

we observe that a few indices $r_x,\ r_y$ contribute:

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = \sum_{r_x=-1}^{0} \sum_{r_y=-1}^{0} F\left(\frac{l_x + r_x \cdot N_x}{A_x}, \frac{l_y + r_y \cdot N_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x + r_x \cdot N_x) \cdot \frac{x_l}{A_x} + (l_y + r_y \cdot N_y\right.\right.$$

For these four cases the relationship between $\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right)$ and samples of $F(u, v)$ must be found:

1. $0 \le l_x \le N_x/2 - 1$ and $0 \le l_y \le N_y/2 - 1$
2. $0 \le l_x \le N_x/2 - 1$ and $N_y/2 \le l_y \le N_y - 1$
3. $N_x/2 \le l_x \le N_x - 1$ and $0 \le l_y \le N_y/2 - 1$
4. $N_x/2 \le l_x \le N_x - 1$ and $N_y/2 \le l_y \le N_y - 1$

case#1 $0 \le l_x \le N_x/2 - 1$ and $0 \le l_y \le N_y/2 - 1$

Positive frequencies in u and positive frequencies in v.

Only summation indices $r_x = 0,\ r_y = 0$ contribute:

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = F\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(l_x \cdot \frac{x_l}{A_x} + l_y \cdot \frac{y_l}{A_y}\right)\right]$$

case#2 $0 \le l_x \le N_x/2 - 1$ and $N_y/2 \le l_y \le N_y - 1$

Positive frequencies in u and negative frequencies in v.

Only summation indices $r_x = 0,\ r_y = -1$ contribute:

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = F\left(\frac{l_x}{A_x}, \frac{l_y - N_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(l_x \cdot \frac{x_l}{A_x} + (l_y - N_y) \cdot \frac{y_l}{A_y}\right)\right]$$

case#3 $N_x/2 \le l_x \le N_x - 1$ and $0 \le l_y \le N_y/2 - 1$

Negative frequencies in u and positive frequencies in v.

Only summation indices $r_x = -1,\ r_y = 0$ contribute:

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = F\left(\frac{l_x - N_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x - N_x) \cdot \frac{x_l}{A_x} + l_y \cdot \frac{y_l}{A_y}\right)\right]$$

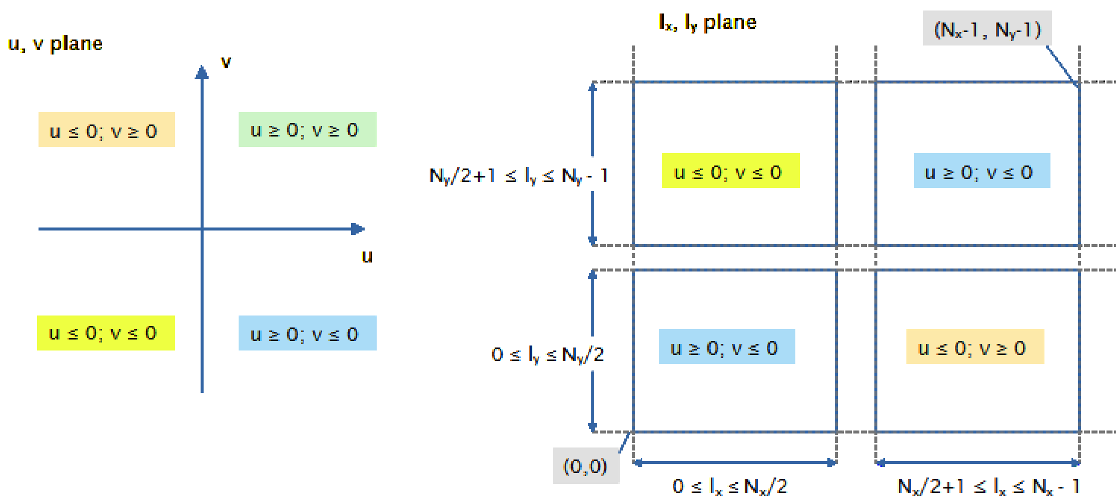
case#4 $N_x/2 \le l_x \le N_x - 1$ and $N_y/2 \le l_y \le N_y - 1$

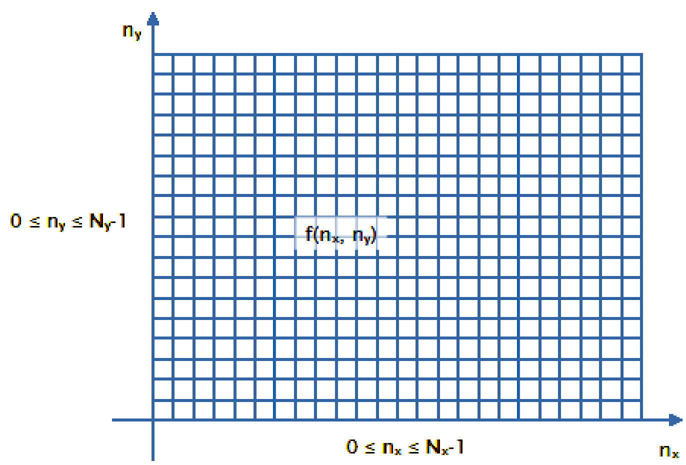
Negative frequencies in u and negative frequencies in v.

Only summation indices $r_x = -1,\ r_y = -1$ contribute:

$$\overline{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = F\left(\frac{l_x - N_x}{A_x}, \frac{l_y - N_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x - N_x) \cdot \frac{x_l}{A_x} + (l_y - N_y) \cdot \frac{y_l}{A_y}\right)\right]$$

The mapping of the $u,\ v$-plane to the $l_x,\ l_y$-plane is shown in the next figure:





Samples $f(n_x, n_y)$ can be computed from the inverse 2D Fourier transform $\overline{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right)$ using equation:

$$f(n_x, n_y) = \frac{1}{A_x \cdot A_y} \cdot \sum_{l_y=0}^{N_y-1} \sum_{l_x=0}^{N_x-1} \overline{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left(l_x \cdot \frac{n_x}{N_x} + l_y \cdot \frac{n_y}{N_y}\right)\right]$$

In the opposite direction $\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right)$ can be computed from $f(n_x, n_y)$ using the discrete 2D Fourier transform.

First calculate

$$\sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} f(n_x, n_y) \cdot exp\left[-j \cdot 2\pi \cdot \left(n_x \cdot \frac{l_x'}{N_x} + n_y \cdot \frac{l_y'}{N_y}\right)\right]$$

inserting the double sum formula for $f(n_x, n_y)$ yields:

$$\frac{1}{A_x \cdot A_y} \cdot \sum_{l_y=0}^{N_y-1}\sum_{l_x=0}^{N_x-1}\sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} \bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x - l_x') \cdot \frac{n_x}{N_x} + (l_y - l_y') \cdot \frac{n_y}{N_y}\right)\right]$$

The inner double sum

$$\sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} \bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x - l_x') \cdot \frac{n_x}{N_x} + (l_y - l_y') \cdot \frac{n_y}{N_y}\right)\right]$$

has non-zero contribution only if $l_x' = l_x, \ l_y' = l_y$:

$$\sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} \bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) \cdot exp\left[j \cdot 2\pi \cdot \left((l_x - l_x') \cdot \frac{n_x}{N_x} + (l_y - l_y') \cdot \frac{n_y}{N_y}\right)\right] = \begin{cases} N_x \cdot N_y \cdot \bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) & l_x' = l_x \\ 0 & otherv \end{cases}$$

Finally we get:

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = \frac{A_x \cdot A_y}{N_x \cdot N_y} \cdot \sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} f(n_x, n_y) \cdot exp\left[-j \cdot 2\pi \cdot \left(n_x \cdot \frac{l_x}{N_x} + n_y \cdot \frac{l_y}{N_y}\right)\right]$$

$$\bar{F}\left(\frac{l_x}{A_x}, \frac{l_y}{A_y}\right) = \Delta A_x \cdot \Delta A_y \cdot \sum_{n_y=0}^{N_y-1}\sum_{n_x=0}^{N_x-1} f(n_x, n_y) \cdot exp\left[-j \cdot 2\pi \cdot \left(n_x \cdot \frac{l_x}{N_x} + n_y \cdot \frac{l_y}{N_y}\right)\right]$$

Computation with Numpy / Scipy

Method `numpy.fft.fft2` implements the 2D discrete Fourier transform in this form:

$$A_{k,\,l} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} a_{m,\,n} \cdot exp\left[-j \cdot 2\pi \cdot \left(\frac{m \cdot k}{M} + \frac{n \cdot l}{N}\right)\right]$$

$a_{m,\,n}$ is a matrix (generally complex) with M rows and N columns. m is the row index and n is the column index. As a result matrix $A_{k,\,l}$ is return. Again this matrix has M rows and N columns.

Method `numpy.fft.ifft2` implements the 2D discrete *inverse* Fourier transform:

$$a_{m,\,n} = \frac{1}{M \cdot N} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} A_{k,\,l} \cdot exp\left[j \cdot 2\pi \cdot \left(\frac{k \cdot m}{M} + \frac{l \cdot n}{N}\right)\right]$$

In [ ]:

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