Sampling and Interpolation in 2D

The notebook serves to introduce some concepts which are useful with the processing of 2D signals (eg. images)

Some Definitions

A continous 2D function $f(x,\ y)$ is related to its 2D-Fourier transform $F(u,\ v)$:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot exp[-j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot dx \cdot dy$$

The inverse Fourier transform is given by:

$$f(x,\ y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,\ v) \cdot exp[j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot du \cdot dv$$

Let us assume that Fourier transform $F(u,\ v)$ is bandlimited to bandwidth $B_u,\ B_v$. Then $F(u,\ v)$ is defined for a finite range

$$-\frac{B_u}{2} \leq u \leq \frac{B_u}{2}$$
$$-\frac{B_v}{2} \leq v \leq \frac{B_v}{2}$$

Thus the inverse Fourier transform may written with finite limits for the integral:

$$f(x,\ y) = \int_{-\frac{B_u}{2}}^{\frac{B_u}{2}}\int_{-\frac{B_v}{2}}^{\frac{B_v}{2}} F(u,\ v) \cdot exp[j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot du \cdot dv$$

Periodic Repetions

In the *2D-signal domain* the function $\tilde{f}(x,\ y)$ denotes a periodic repetition of function $f(y,\ y)$ with a periods $T_x,\ T_y$ for variables $x,\ y$.

In the *2D-transform domain* the function $\tilde{F}(u,\ v)$ denotes a periodic repetion of the Fourier transform $F(u)$ with a periods $B_u,\ B_v$ for the transform variables $u,\ v$.

$$\tilde{f}(x,\ y) = \sum_{n_y=-\infty}^{\infty}\sum_{n_x=-\infty}^{\infty} f(x - n_x \cdot T_x,\ y - n_y \cdot T_y)$$

$$\tilde{F}(u,\ v) = \sum_{m_v=-\infty}^{\infty}\sum_{m_u=-\infty}^{\infty} F(u - m_u \cdot B_u,\ v - m_v \cdot B_v)$$

Since $\tilde{f}(x,\ y)$ and $\tilde{F}(u,\ v)$ are periodic they can be written as Fourier series:

$$\tilde{f}(x,y) = \sum_{k_y=-\infty}^{\infty} \sum_{k_x=-\infty}^{\infty} c_{k_x,k_y} \cdot exp\left[j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{T_x} + k_y \cdot \frac{y}{T_y}\right)\right]$$

$$\tilde{F}(u,\ v) = \sum_{l_v=-\infty}^{\infty} \sum_{l_u=-\infty}^{\infty} C_{l_u,l_v} \cdot exp\left[-j \cdot 2\pi \cdot \left(l_u \cdot \frac{u}{B_u} + l_v \cdot \frac{v}{B_v}\right)\right]$$

Computing Fourier series coefficients

signal domain

For the signal domain representation the Fourier coefficients c_{k_x,k_y} from this equation:

$$\int_{y=0}^{T_y}\int_{x=0}^{T_x} \tilde{f}(x,y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x^{'} \cdot \frac{x}{T_x} + k_y^{'} \cdot \frac{y}{T_y}\right)\right] \cdot dx \cdot dy$$

Inserting the 2D-Fourier series representation of $\tilde{f}(x,y)$ yields:

$$\sum_{k_y=-\infty}^{\infty} \sum_{k_x=-\infty}^{\infty} c_{k_x,k_y} \cdot \int_{y=0}^{T_y}\int_{x=0}^{T_x} exp\left[j \cdot 2\pi \cdot \left((k_x - k_x^{'}) \cdot \frac{x}{T_x} + (k_y - k_y^{'}) \cdot \frac{y}{T_y}\right)\right] \cdot dx \cdot dy$$

The double integral has non-zero contribution only if $k_x = k_x^{'}\ and\ k_y = k_y^{'}$:

$$c_{k_x,k_y} = \frac{1}{T_x \cdot T_y} \cdot \int_{y=0}^{T_y}\int_{x=0}^{T_x} \tilde{f}(x,y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{T_x} + k_y \cdot \frac{y}{T_y}\right)\right] \cdot dx \cdot dy$$

For a special and important case the double integral can be expressed by the D-Fourier transform.

special case

Function $f(x,\ y)$ shall be defined for the finite ranges

$$0 \le x \le T_x$$
$$0 \le y \le T_y$$

Within this range the periodic repetition $\tilde{f}(x,y)$ equal $f(x,\ y)$. For the Fourier series coefficients we may write:

$$c_{k_x,k_y} = \frac{1}{T_x \cdot T_y} \cdot \underbrace{\int_{y=0}^{T_y}\int_{x=0}^{T_x} f(x,y) \cdot exp\left[-j \cdot 2\pi \cdot \left(k_x \cdot \frac{x}{T_x} + k_y \cdot \frac{y}{T_y}\right)\right] \cdot dx \cdot dy}_{F\left(u=\frac{k_x}{T_x},\ v=\frac{k_y}{T_y}\right)}$$

Thus the Fourier series coefficients c_{k_x,k_y} are just (apart from a scaling factor) directly related to samples of 2D-Fourier transform $F(u,\ v)$:

$$c_{k_x,k_y} = \frac{1}{T_x \cdot T_y} \cdot F\left(u = \frac{k_x}{T_x},\ v = \frac{k_y}{T_y}\right)$$

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transform domain

For the transform domain representation the Fourier series coefficients C_{l_u,l_v} are computed from this equation:

$$\int_{-B_v/2}^{B_v/2}\int_{-B_u/2}^{B_u/2} \tilde{F}(u,\ v) \cdot exp\left[j \cdot 2\pi \cdot \left(l_u^{'} \cdot \frac{u}{B_u} + l_v^{'} \cdot \frac{v}{B_v}\right)\right] \cdot du \cdot dv$$

Inserting the 2D-Fourier series representation of $\tilde{F}(u,\ v)$ yields:

$$\sum_{l_v=-\infty}^{\infty}\sum_{l_u=-\infty}^{\infty} C_{l_u,l_v} \cdot \int_{-B_v/2}^{B_v/2}\int_{-B_u/2}^{B_u/2} exp\left[-j \cdot 2\pi \cdot \left((l_u - l_u^{'}) \cdot \frac{u}{B_u} + (l_v - l_v^{'}) \cdot \frac{v}{B_v}\right)\right] \cdot du \cdot dv$$

The double integral has non-zero contribution only if $l_u = l_u^{'}\ and\ l_v = l_v^{'}$:

$$C_{l_u,l_v} = \frac{1}{B_u \cdot B_v} \cdot \int_{-B_v/2}^{B_v/2}\int_{-B_u/2}^{B_u/2} \tilde{F}(u,\ v) \cdot exp\left[j \cdot 2\pi \cdot \left(l_u \cdot \frac{u}{B_u} + l_v \cdot \frac{v}{B_v}\right)\right] \cdot du \cdot dv$$

For a special and important case the double integral can be expressed by the D-Fourier transform.

special case

Function $F(u,\ v)$ shall be defined for the finite ranges (bandlimited)

$$-\frac{1}{B_u} \le u \le \frac{1}{B_u}$$
$$-\frac{1}{B_v} \le v \le \frac{1}{B_v}$$

Within this range the periodic repetition $\tilde{F}(u,\ v)$ equal $F(u,\ v)$. For the Fourier series coefficients we may then write:

$$C_{l_u,l_v} = \frac{1}{B_u \cdot B_v} \cdot \int_{-B_v/2}^{B_v/2} \int_{-B_u/2}^{B_u/2} F(u,\ v) \cdot exp\left[j \cdot 2\pi \cdot \left(l_u \cdot \frac{u}{B_u} + l_v \cdot \frac{v}{B_v}\right)\right] \cdot du \cdot dv$$

$$C_{l_u,l_v} = \frac{1}{B_u \cdot B_v} \cdot \underbrace{\int_{-B_v/2}^{B_v/2} \int_{-B_u/2}^{B_u/2} F(u,\ v) \cdot exp\left[j \cdot 2\pi \cdot \left(l_u \cdot \frac{u}{B_u} + l_v \cdot \frac{v}{B_v}\right)\right] \cdot du \cdot dv}_{f(x=\frac{l_u}{B_u},\ y=\frac{l_v}{B_v})}$$

Thus the Fourier series coefficients C_{l_u,l_v} are just (apart from a scaling factor) directly related to samples of 2D function $f(x,\ y)$:

$$C_{l_u,l_v} = \frac{1}{B_u \cdot B_v} \cdot f\left(x = \frac{l_u}{B_u},\ y = \frac{l_v}{B_v}\right)$$

Introducing the following definitions

$$T_x = \frac{1}{B_u}$$

$$T_y = \frac{1}{B_v}$$

$$C_{l_u,l_v} = T_x \cdot T_y \cdot f\left(x = l_u \cdot T_x,\ y = l_v \cdot T_y\right)$$

$$F(u,\ v) = T_x \cdot T_y \cdot \sum_{l_v=-\infty}^{\infty} \sum_{l_u=-\infty}^{\infty} f\left(x = l_u \cdot T_x,\ y = l_v \cdot T_y\right) \cdot exp\left[-j \cdot 2\pi \cdot \left(l_u \cdot T_x \cdot u + l_v \cdot T_y \cdot v\right)\right]$$

2D Sampling Theorem

From the last equation some form of the `Sampling Theorem` for `2D` can be derived.

We start with

$$F(u,\ v) = T_x \cdot T_y \cdot \sum_{l_v=-\infty}^{\infty} \sum_{l_u=-\infty}^{\infty} f\left(x = l_u \cdot T_x,\ y = l_v \cdot T_y\right) \cdot exp\left[-j \cdot 2\pi \cdot \left(l_u \cdot T_x \cdot u + l_v \cdot T_y \cdot v\right)\right]$$

and compute the inverse Fourier transform:

$$f(x,\ y) = \int_{-\frac{B_u}{2}}^{\frac{B_u}{2}} \int_{-\frac{B_v}{2}}^{\frac{B_v}{2}} F(u,\ v) \cdot exp[j \cdot 2\pi \cdot (u \cdot x + v \cdot y)] \cdot du \cdot dv$$

$$f(x,\ y) = T_x \cdot T_y \cdot \sum_{l_v=-\infty}^{\infty} \sum_{l_u=-\infty}^{\infty} \int_{-\frac{B_u}{2}}^{\frac{B_u}{2}} \int_{-\frac{B_v}{2}}^{\frac{B_v}{2}} f\left(x = l_u \cdot T_x,\ y = l_v \cdot T_y\right) \cdot exp\left[j \cdot 2\pi \cdot \left(u \cdot (x - l_u \cdot T_x) + v \cdot (y\right.\right.$$

Renaming the indices $l_u,\ l_v$ to $l_x,\ l_y$:

$$f(x,\ y) = T_x \cdot T_y \cdot \sum_{l_y=-\infty}^{\infty} \sum_{l_x=-\infty}^{\infty} f\left(x = l_x \cdot T_x,\ y = l_y \cdot T_y\right) \cdot \int_{-\frac{B_u}{2}}^{\frac{B_u}{2}} \int_{-\frac{B_v}{2}}^{\frac{B_v}{2}} exp\left[j \cdot 2\pi \cdot \left(u \cdot (x - l_x \cdot T_x) + v \cdot (y\right.\right.$$

The double integral can be split like this:

$$\int_{-\frac{B_u}{2}}^{\frac{B_u}{2}} exp\left[j \cdot 2\pi \cdot u \cdot (x - l_x \cdot T_x)\right] \left(\int_{-\frac{B_v}{2}}^{\frac{B_v}{2}} exp\left[j \cdot 2\pi \cdot v \cdot (y - l_y \cdot T_y)\right] \cdot dv \right) \cdot du$$

Solving each integral separately yields an expression which relates the continous 2D function $f(x,\ y)$ to its samples:

$$f(x,\ y) = \sum_{l_y=-\infty}^{\infty} \sum_{l_x=-\infty}^{\infty} f\left(x = l_x \cdot T_x,\ y = l_y \cdot T_y\right) \cdot sinc\left(\pi \cdot (\frac{x}{T_x} - l_x)\right) \cdot sinc\left(\pi \cdot (\frac{y}{T_y} - l_y)\right)$$

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