Learning about Kalman filter

Resources:

1. `Kalman Filter from Ground Up`; author Alex Becker; https://www.kalmanfilter.net
2. A very readable account of what problems can be solved with a `Kalman Filter` is chapter 1 of `Stochastic Models: Estimation and Control V.1` by Peter S. Maybeck; publisher Academic Press. Apart from the introductory chapter the book is not easy to read without a decent background. So I might come back reading more chapters of the book once I have a better grasp of the subject.

Practically everything in this notebook is based chapter 4 of `Kalman Filter from Ground Up`

One Dimensional Kalman Filter / no process noise

Example#1

we measure the weight of a gold bar. Measurements are denoted z_n . The true (but unknown) weight is x. We can model the measurement by a sum of the true weight and some measurement error:

$$z_n = x + e_n$$

We do not know e_n but if we are lucky we know its distribution function. To simplify things furter we assume that e_n follows a gaussian distribution with known mean μ_x and standard deviation σ_x. With a gaussian distribution these to parameters/moments fully describe the distribution function.

$$\begin{aligned} E(e_n) &= \mu_e \\ E((e_n - \mu_e)^2) &= \sigma_e^2 \end{aligned}$$

We take N measurements to compute an estimatef weight w_N of the gold bar.

$$\begin{aligned} w_N &= \frac{1}{N} \cdot \sum_{n=1}^{N} z_n \\ &= x + \frac{1}{N} \cdot \sum_{n=1}^{N} e_n \end{aligned}$$

$$E(w_N) = x + \frac{1}{N} \cdot \sum_{n=1}^{N} E(e_n) = x + \mu_e$$

Assuming $\mu_e = 0$ (measurement error has zero mean) and therefore $E(e_n^2) = \sigma_e^2$ we get:

$$E(w_N) = x$$

Now the variance $Var(w_N)$:

$$Var(w_N) = E((w_N - x)^2) = E(w_N^2) - x^2$$

$$= E\left(\left(x + \frac{1}{N} \cdot \sum_{n=1}^{N} e_n\right)^2\right) - x^2$$

$$= 2 \cdot x \cdot \frac{1}{N} \cdot \sum_{n=1}^{N} E(e_n) + \frac{1}{N^2} \cdot E\left(\left(\sum_{n=1}^{N} e_n\right)^2\right)$$

$$= \frac{1}{N^2} \cdot E\left(\left(\sum_{n=1}^{N} e_n\right)^2\right)$$

$$= \frac{1}{N^2} \cdot \sum_{n=1}^{N} \sum_{m=1}^{N} E\left(e_n \cdot e_m\right)$$

$$= \frac{1}{N^2} \cdot \sum_{n=1}^{N} E\left(e_n^2\right) + \underbrace{\frac{1}{N^2} \cdot \sum_{\substack{n=1 \\ n \neq m}}^{N} \sum_{m=1}^{N} E\left(e_n \cdot e_m\right)}_{=0}$$

$$= \frac{1}{N^2} \cdot \sum_{n=1}^{N} E\left(e_n^2\right) = \frac{1}{N} \cdot \sigma_e^2$$

$$Var(w_N) = \left(\frac{\sigma_e}{\sqrt{N}}\right)^2$$

Thus averaging N measurements to obtain an estimated weight w_N has reduced the standard deviation σ_e of the measurement error e_n by a factor of $1/\sqrt{N}$.

Visualisation

The further illustrate the *weighing gold* example we shall provide a numerical example (closely related to what is presented in https://www.kalmanfilter.net).

We take $N = 10$ measurement of a gold bar with nominal weight $x = 1000g$. Since the measurement is imprecise we assume a measurement error which follows a Gaussian distribution with a standard deviation of $\sigma = 10g$.

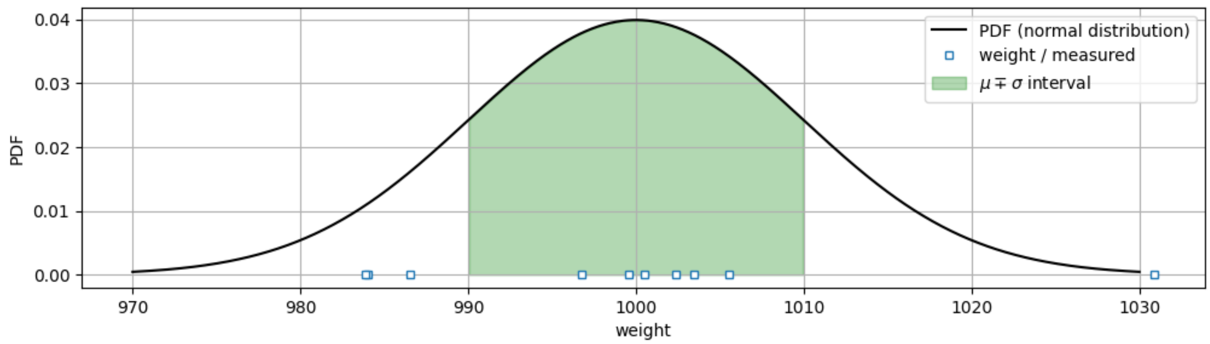
Each measurement w_n can be interpreted as a random variable drawn from normal distribution with mean $\mu = 1000g$ and standard deviation $\sigma = 10g$.

The probability density function of this distribution is then:

$$p(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot exp\left(-\frac{(w-\mu)^2}{2 \cdot \sigma^2}\right)$$

In [1]:
```
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt
```

In [2]:
```
N = 10
mean_w = 1000
sigma = 10

# get the measurements by drawing samples from the distribution
w_measurements = stats.norm.rvs(loc=mean_w, scale=sigma, size=N)
w_limit_low = mean_w - 3 * sigma
w_limit_high = mean_w + 3 * sigma

# evaluate the probability density in the interval [w_limit_low, w_limit_high]
M = 1000
wp = np.linspace(w_limit_low, w_limit_high, M)
pdf = stats.norm.pdf(wp , loc=mean_w, scale=sigma)

range = (wp > (mean_w - sigma)) & (wp < (mean_w + sigma))

# display
fig1, ax1 = plt.subplots(nrows=1, ncols=1, figsize=(12,3))

ax1.plot(wp, pdf, color='k', label='PDF (normal distribution)')
ax1.plot(w_measurements, np.zeros_like(w_measurements), linestyle='', marker='s', m
ax1.fill_between(wp[range], pdf[range], 0, color='g', alpha=0.3, label='$\mu \mp \s
ax1.set_xlabel('weight')
ax1.set_ylabel('PDF')
ax1.set_title('')
ax1.legend()
ax1.grid(True)
```



Review / Tracking constant velocity aircraft

The prediction equations were:

$$\hat{x}_{n+1,n} = \hat{x}_{n,n} + \Delta t \cdot \dot{x}_{n,n}$$
$$\dot{x}_{n+1,n} = \dot{x}_{n,n}$$

Position $\hat{x}_{n,n}$ and velocity $\dot{x}_{n,n}$ are now consider to be random variables. Moreover it shall be assumed that they are normally distributed.

Following the notation used in chapter 4 of https://www.kalmanfilter.net we will use:

$$p^x_{n,n} = Var\left(\hat{x}_{n,n}\right)$$
$$p^v_{n,n} = Var\left(\dot{x}_{n,n}\right)$$

The superscripts ^{x} and ^{v} shall denote that these variances refer to either position or velocity.

The variances for the predicted position $\hat{x}_{n+1,n}$ and velocity $\dot{x}_{n+1,n}$ follow directly from the prediction update equations:

$$p^x_{n+1,n} = p^x_{n,n} + \Delta t^2 \cdot p^v_{n,n}$$
$$p^v_{n+1,n} = p^v_{n,n}$$

Of course (just by time shifting) the equation can also be expressed as:

$$p^x_{n,n-1} = p^x_{n-1,n-1} + \Delta t^2 \cdot p^v_{n-1,n-1}$$
$$p^v_{n,n-1} = p^v_{n-1,n-1}$$

These equation are termed the `covariance extrapolation equation` .

measurement

For measurement z_n it shall be assumed that it is subject to measurement noise too. The noise variance is denoted r_n.

State update

The position prediction $p^x_{n,n-1}$ and the position measurement z_n are combined into a new position estimate.

$$x_{n,n} = w_1 \cdot z_n + w_2 \cdot x_{n,n-1}$$

with the constraint

$$w_1 + w_2 = 1$$

The weighting factors are adjusted to minimise the variance $p^x_{n,n}$.

$$\begin{aligned} p^x_{n,n} &= w_1^2 \cdot r_n + w_2^2 \cdot p^x_{n,n-1} \\ &= w_1^2 \cdot r_n + \left(1 - w_1\right)^2 \cdot p^x_{n,n-1} \end{aligned}$$

Differentiation by w_1 is used to determine w_1 such as to minimise the variance $p^x_{n,n}$:

$$\frac{dp^x_{n,n}}{w_1} = 2 \cdot w_1 \cdot r_n - 2 \cdot \left(1 - w_1\right) \cdot p^x_{n,n-1} = 2 \cdot w_1 \cdot \left(r_n + p^x_{n,n-1}\right) - 2 \cdot p^x_{n,n-1}$$

$$w_1 = \frac{p^x_{n,n-1}}{r_n + p^x_{n,n-1}}$$

$$w_2 = 1 - w_1 = \frac{r_n}{r_n + p^x_{n,n-1}}$$

Inserting the w_1 and w_2 into the state update equation yields:

$$\begin{aligned} x_{n,n} &= w_1 \cdot z_n + (1 - w_1) \cdot x_{n,n-1} \\ &= x_{n,n-1} + w_1 \cdot \left(z_n - x_{n,n-1}\right) \\ &= x_{n,n-1} + \underbrace{\frac{p^x_{n,n-1}}{r_n + p^x_{n,n-1}}}_{K_n} \cdot \left(z_n - x_{n,n-1}\right) \end{aligned}$$

With

$$K_n = \frac{p^x_{n,n-1}}{r_n + p^x_{n,n-1}}$$

the update equation becomes:

$$x_{n,n} = x_{n,n-1} + K_n \cdot \left(z_n - x_{n,n-1}\right)$$

Factor K_n is called the `Kalman-Gain`.

The result looks similar to the $\alpha - \beta$-filter. However this time the `Kalman-Gain` K_n will not be a constant but it adapts to any changes in the values of the variances.

variance $p^x_{n,n}$ of $x_{n,n}$

$$
\begin{aligned}
p^x_{n,n} &= w_1^2 \cdot r_n + w_2^2 \cdot p^x_{n,n-1} \\
&= \left(\frac{p^x_{n,n-1}}{r_n + p^x_{n,n-1}}\right)^2 \cdot r_n + \left(\frac{r_n}{r_n + p^x_{n,n-1}}\right)^2 \cdot p^x_{n,n-1} \\
&= \frac{p^x_{n,n-1}}{\left(r_n + p^x_{n,n-1}\right)^2} \cdot p^x_{n,n-1} \cdot r_n + \frac{r_n}{\left(r_n + p^x_{n,n-1}\right)^2} \cdot p^x_{n,n-1} \cdot r_n \\
&= \underbrace{\frac{r_n}{r_n + p^x_{n,n-1}}}_{1-K_n} \cdot p^x_{n,n-1} \\
p^x_{n,n} &= \left(1 - K_n\right) \cdot p^x_{n,n-1}
\end{aligned}
$$

The equation is called the `covariance update equation`.

The next section of this notebook summarizes everything we know thus far about the `Kalman-Filter` in a single dimension.

Summary / Kalman Filter in One Direction

The equation are valid for constant velocity dynamics (it won't be tracking postiton of motion with time varying accelaration).

State Update / Filtering Equation

c	Equation	Equation Name
State Update	$x_{n,n} = x_{n,n-1} + K_n \cdot \left(z_n - x_{n,n-1}\right)$	State Update /Filtering Equation
	$p^x_{n,n} = \left(1 - K_n\right) \cdot p^x_{n,n-1}$	Covariance update
	$K_n = \frac{p^x_{n,n-1}}{r_n + p^x_{n,n-1}}$	Kalman Gain
State Prediction	$\hat{x}_{n+1,n} = \hat{x}_{n,n} + \Delta t \cdot \dot{x}_{n,n}$	Position Prediction
	$\dot{x}_{n+1,n} = \dot{x}_{n,n}$	constant velocity prediction

c	Equation	Equation Name
	$p^x_{n,n-1} = p^x_{n-1,n-1} + \Delta t^2 \cdot p^v_{n-1,n-1}$	covariance extrapolation
	$p^v_{n,n-1} = p^v_{n-1,n-1}$	

Estimating the height of a building

example taken from chapter 4.2 of the book `Kalman Filter from Ground`; author Alex Becker.

The true height of a building is 50 m. A set of 10 successive measurements is available. The measurements are:

$49.03, 48.44, 55.21, 49.98, 50.6, 52.61, 45.87, 42.64, 48.26, 55.84$

But these measurement are subject to some measurement error. The standard deviation of the measurement error shall be $5m$. Its distribution shall be gaussion and its mean value 0.

initialisation

The initial assumption about the heigth $\hat{x}_{0,0}$ has been chosen to

$$\hat{x}_{0,0} = 60m$$

(My personal choice would have been to take the first measurement as an initial value)

Since the initial estimate $\hat{x}_{0,0}$ is not very accurate we assume its variance $p_{0,0}$ to be:

$$p_{0,0} = 225\ m^2$$

(corresponding to a standard deviation of 15 m).

Since the height of the bulding does not change throughout all 10 measurements out first prediction will be:

$$\hat{x}_{1,0} = \hat{x}_{0,0} = 60m$$

Accordingly the variance of the prediction does not change:

$$p_{1,0} = p_{0,0} = 225\ m^2$$

iteration#1

With the first measurement $z_1 = 49.03m$ and the measurement variance $r_1 = 25m^2$ we compute the `Kalman Gain` K_1:

$$K_1 = \frac{p_{1,0}}{p_{1,0} + r_1} = \frac{225}{225 + 25} = \frac{225}{250} = 0.9$$

The estimate $\hat{x}_{1,1}$ of the current height becomes:

$$\begin{aligned} \hat{x}_{1,1} &= \hat{x}_{1,0} + K_1 \cdot \left(z_1 - \hat{x}_{1,0}\right) \\ &= 60m - 0.9 \cdot 10.97 \\ &= 50.13m \end{aligned}$$

$$p_{1,1} = \left(1 - K_1\right) \cdot p_{1,0} = 0.1 \cdot 225\ m^2 = 22.5\ m^2$$

iteration#2

$$p_{2,1} = p_{1,1} = 22.5\ m^2$$

With the 2'nd measurement $z_2 = 48.44m$ and the measurement variance $r_2 = 25m^2$ we compute the `Kalman Gain` K_2:

$$K_1 = \frac{p_{2,1}}{p_{2,1} + r_2} = \frac{22.5}{22.5 + 25} = \frac{22.5}{47.5} = 0.47$$

The estimate $\hat{x}_{2,2}$ of the current height becomes:

$$\begin{aligned} \hat{x}_{2,2} &= \hat{x}_{2,1} + K_2 \cdot \left(z_2 - \hat{x}_{2,1}\right) \\ &= 50.13m - 0.47 \cdot 1.69m \\ &= 49.33m \end{aligned}$$

$$p_{2,2} = \left(1 - K_1\right) \cdot p_{2,1} = 0.53 \cdot 22.5\ m^2 = 11.925\ m^2$$

(in the book this value is incorrectly computed; but the basic procedure is still valid)

In [ ]:

```

```