# Derivatives of 1D functions

A review of differentiation learned at school

This time with some proofs ...

# Difference Quotient

The definition of the difference quotient is fundamental to the definition of the differential of a function.

Let $f(x)$ denote some univariate function and assume that it is a real valued function.

The difference quotient is defined as the ratio:

$$D_f(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

At a point $x_0 + e$ in the interval $x_0 \leq x_0 + e \leq x$ the slope of the tangent to the function equals the difference quotient.

The slope or differential quotient of the tangent is denoted $f^{'}(x_0 + e)$ . In the limit $x \longrightarrow x_0$ the quantity $e$ approaches $0$ and the difference quotient $D_f(x)$ approaches the derivative $f^{'}(x_0)$ of function $f(x)$.

Some useful relations are provided here:

$$f(x) = f(x_0) - (x - x_0) \cdot D_f(x) = f(x_0) - (x - x_0) \cdot f^{'}(x_0 + e)$$

$$\lim_{x \longrightarrow x_0} D_f(x) = f^{'}(x_0)$$

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# Rules for Diffentiation

It is assumed that all function can be differentiated in the vicinity of $x_0$.

## Differentiation of a sum of two functions

Define function $f(x)$ as the sum of functions $f_1(x)$ and $f_2(x)$:

$$f(x) = f(x_0) + (x - x_0) \cdot f'(x_0 + e) \tag{1}$$
$$f(x) = f_1(x_0) + f_2(x_0) + (x - x_0) \cdot [f_1'(x_0 + e_1) f_2'(x_0 + e_1)] \tag{2}$$
$$f'(x_0 + e) = f_1'(x_0 + e_1) f_2'(x_0 + e_1) \tag{3}$$
$$\tag{4}$$
$$x \to x_0;\ e,\ e_1,\ e_2 \to 0 \tag{5}$$
$$f'(x_0) = f_1'(x_0) + f_2'(x_0) \tag{6}$$

# Differentiation of a product of two function

Define function $f(x)$ as the product of functions $f_1(x)$ and $f_2(x)$:

$$f(x) = f_1(x) \cdot f_2 x$$

$$f(x) = f(x_0) + (x - x_0) \cdot f'(x_0 + e)$$
$$f(x) = f_1(x) \cdot f_2(x) = (f_1(x_0) + (x - x_0) \cdot f_1'(x_0 + e_1)) \cdot (f_2(x_0) + (x - x_0) \cdot f_2'(x_0 + e_2$$
$$f(x) = f_1(x_0) \cdot f_2(x_0) + (x - x_0) \cdot (f_1'(x_0 + e_1) \cdot f_2(x_0) + f_1(x_0) \cdot f_2'(x_0 + e_2))$$
$$f'(x_0 + e) = f_1'(x_0 + e_1) \cdot f_2(x_0) + f_1(x_0) \cdot f_2'(x_0 + e_2)$$

$$x \to x_0;\ e,\ e_1,\ e_2 \to 0$$
$$f'(x_0) = f_1'(x_0) \cdot f_2(x_0) + f_1(x_0) \cdot f_2'(x_0)$$

# Differentiation of the reciprocal of a function

Define function $f(x)$ as the reciprocal of function $f_1(x)$:

$$f(x) = \frac{1}{f_1(x)}$$

$$f(x) = f(x_0) + (x - x_0) \cdot f'(x_0 + e) \tag{14}$$
$$\frac{1}{f_1(x)} - \frac{1}{f_1(x_0)} = -\frac{f_1(x) - f_1(x_0)}{f_1(x) \cdot f_1(x_0)} = (x - x_0) \cdot f'(x_0 + e) \tag{15}$$
$$-\frac{f_1(x_0) + (x - x_0) \cdot f_1'(x_0 + e_1) - f_1(x_0)}{f_1(x) \cdot f_1(x_0)} = (x - x_0) \cdot f'(x_0 + e) \tag{16}$$
$$f'(x_0 + e) = -\frac{f_1'(x_0 + e_1)}{f_1(x) \cdot f_1(x_0)} \tag{17}$$
$$\tag{18}$$
$$x \to x_0;\ e,\ e_1 \to 0 \tag{19}$$
$$f'(x_0) = -\frac{f_1'(x_0)}{f_1^2(x_0)} \tag{20}$$

# Differentiation of the ratio of two functions

Define function $f(x)$ as the ratio of functions $f_1(x)$ and $f_2(x)$:

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

By application of the *product rule* we get:

$$f^{'}(x) = f_1^{'}(x) \cdot \frac{1}{f_2(x)} - f_1(x) \cdot \frac{f_2^{'}(x)}{f_2^2(x)} \tag{21}$$

$$f^{'}(x) = \frac{f_1^{'}(x) \cdot f_2(x) - f_1(x) \cdot f_2^{'}(x)}{f_2^2(x)} \tag{22}$$

## Differentiation of *concatenated* functions / Chain Rule

We define a function $g(y)$ which where $y$ depends on another function $f(x)$. We want to differentiate function $g(f(x))$ with respect to variable $x$.

$$g(y) = g(y_0) + (y - y_0) \cdot g^{'}(y_0 + e) \tag{23}$$

$$y = f(x) \tag{24}$$

$$y_0 = f(x_0) \tag{25}$$

$$f(x) = f(x_0) + (x - x_0) \cdot f^{'}(x_0 + e_1) \tag{26}$$

$$g(y) - g(y_0) = (f(x_0) + (x - x_0) \cdot f^{'}(x_0 + e_1) - f(x_0)) \cdot g^{'}(y_0 + e) \tag{27}$$

$$g(y) - g(y_0) = (x - x_0) \cdot f^{'}(x_0 + e_1) \cdot g^{'}(y_0 + e) \tag{28}$$

$$\frac{g(y) - g(y_0)}{x - x_0} = f^{'}(x_0 + e_1) \cdot g^{'}(y_0 + e) \tag{29}$$

$$\tag{30}$$

$$y \rightarrow y_0;\ x \rightarrow x_0;\ e,\ e_1 \rightarrow 0 \tag{31}$$

$$g^{'}(x) = f^{'}(x) \cdot g^{'}(y) \tag{32}$$

# Differentiation of frequently used functions

Differentiation of $f(x) = x^n$

The difference quotient is defined by

$$D(x) = \frac{(x + h)^n - x^n}{h}$$

$(x + h)^n$ can be expressed by a binomial series

$$D(x) = \frac{\sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} \cdot h^k - x^n}{h} = \frac{x^n + n \cdot h \cdot x^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x^{(n-k)} \cdot h^k - x^n}{h} = n \cdot h \cdot x$$

In the limit $h \rightarrow 0$ we get:

$$f^{'}(x) = n \cdot h \cdot x^{n-1}$$

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