Eigenvalues

Sources:

1. `Linear Algebra : Theory, Intuition, Code` author: Mike X Cohen, publisher: sincXpress
2. `No bullshit guide to linear algebra` author: Ivan Savov
3. `Matrix Methods for Computational Modeling and Data Analytics` author: Mark Embree, Virginia Tech

Motivation

An understanding of this topics seems necessary to deal with methods to iteratively solve matrix equations, singular value decomposition, principal component analysis

Properties

A square matrix $\mathbf{B}$ is multiplied by some vector $\mathbf{v}$. If the following condition holds

$$\mathbf{B} \cdot v = \lambda \cdot \mathbf{v}$$

the vector $\mathbf{v}$ is an eigenvector and the associated scalar λ is the corresponding eigenvalue.

If $\mathbf{v}$ is an eigenvector than than any scaled vector $\alpha \cdot \mathbf{v}$ is also an eigenvector:

$$\mathbf{B} \cdot \alpha \cdot v = \alpha \cdot \mathbf{B} \cdot v = \alpha \cdot \lambda \cdot \mathbf{v}$$

Usually numerical software packages compute eigenvectors which have unit length ($||\mathbf{v}|| = 1$).

A $m \times m$ matrix has `m` eigenvalues and `m` eigenvectors. Eigenvalues are not required to be distinct. The can even be complex for real valued matrices.

The equation $\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}$ may be written in another form:

$$\mathbf{A} \cdot \mathbf{v} - \lambda \mathbf{v} = (\mathbf{A} - \lambda \cdot \mathbf{I}) \cdot \mathbf{v} = \mathbf{0}$$

Thus matrix $(\mathbf{A} - \lambda \cdot \mathbf{I})$ must be singular and its determinant $|\mathbf{A} - \lambda \cdot \mathbf{I}| = 0$.

numerical example

Compute eigenvalues and eigenvector of a random 3 x 3 matrix.

In general there may be complex eigenvalues

In [10]:
```
import numpy as np
import math

# a 3 x 3 matrix which is not singular (choosing an random matrix increases the lik
Amat = np.random.randn(3,3)

eigen_values, eigen_vectors_mat = np.linalg.eig(Amat)

print(f"Amat :\n{Amat}\n")
print(f"eigen_values      :\n{eigen_values}\n")
print(f"eigen_vectors_mat :\n{eigen_vectors_mat}\n")
```

```
Amat :
[[ 0.6591868   0.54580176 -0.4612541 ]
 [ 0.64048071  1.21914289 -0.63622908]
 [ 0.67373345 -0.78855268  0.37384276]]

eigen_values      :
[0.23718098+0.25641578j 0.23718098-0.25641578j 1.7778105 +0.j        ]

eigen_vectors_mat :
[[ 0.27652084+0.24836548j  0.27652084-0.24836548j -0.50132726+0.j        ]
 [ 0.38246447-0.06212397j  0.38246447+0.06212397j -0.83462347+0.j        ]
 [ 0.84363046+0.j          0.84363046-0.j          0.22819871+0.j        ]]
```

In [3]:
```
Amat_times_eigvector =  Amat @ eigen_vectors_mat[:,0]
eigvalue_times_eigvector = eigen_vectors_mat[:,0] * eigen_values[0]

# should be identical (apart from numeric artefacts)
print(f"Amat_times_eigvector    :\n{Amat_times_eigvector}\n")
print(f"eigvalue_times_eigvector:\n{eigvalue_times_eigvector}\n")
```

```
Amat_times_eigvector    :
[ 0.40441327+0.1672811j  -0.05590885+0.0782878j  -0.17806591-0.40974717j]

eigvalue_times_eigvector:
[ 0.40441327+0.1672811j  -0.05590885+0.0782878j  -0.17806591-0.40974717j]
```

Eigendecomposition & Diagonalization

The eigenvalues of a $m \times m$ matrix $\mathbf{A}$ are denoted $\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)$ and the corresponding eigenvectors $\left(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\right)$.

This results in `m` equations:

$$\begin{gathered}\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1 \\ \mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2 \\ \vdots \\ \mathbf{A} \cdot \mathbf{v}_m = \lambda_m \cdot \mathbf{v}_m\end{gathered}$$

Taking the eigenvectors as column vectors of matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

a more compact notation is obtained:

$$\mathbf{A} \cdot \mathbf{V} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \lambda_1 \cdot \mathbf{v}_1 & \lambda_2 \cdot \mathbf{v}_2 & \cdots & \lambda_m \cdot \mathbf{v}_m \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \mathbf{V} \cdot \mathbf{\Lambda}$$

The matrix $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues as its diagonal entries.

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \ddots & \cdots & \cdots & 0 \\ \cdots & \ddots & \cdots & \cdots & \lambda_m \end{bmatrix}$$

If all eigenvectors form an independent set the matrix $\mathbf{V}$ has an inverse $\mathbf{V}^{-1}$. It then follows:

$$\mathbf{V}^{-1} \cdot \mathbf{A} \cdot \mathbf{V} = \mathbf{V}^{-1} \cdot \mathbf{V} \cdot \mathbf{\Lambda} = \mathbf{\Lambda}$$

Another interesting property is obtained from right multiplication by $\mathbf{V}^{-1}$.

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{V} \cdot \mathbf{V}^{-1} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$$

A numerical example computes the eigenvalues and eigenvectors of a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The eigenvalues are returned in a list while the eigenvectors are returned as column vectors of a 3×3 matrix $\mathbf{V}$.

Then the inverse matrix $\mathbf{V}^{-1}$ is computed to demonstrate that the eigenvector decomposition

$$\mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$$

yields the original matrix $\mathbf{A}$ (well almost, apart from numerical inaccuracies).

In [7]:

```
Amat = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
lambda_list, Vmat = np.linalg.eig(Amat)
# convert list of eigenvalues into a diagonal matrix
lambda_mat = np.diag(lambda_list)

# compute the inverse of the eigen vector matrix (required to demonstrate the valid
Vmat_inv = np.linalg.inv(Vmat)
Amat_from_diagonalisation = Vmat @ lambda_mat @ Vmat_inv

print(f"Amat        :\n{Amat}\n")
print(f"lambda_mat  :\n{lambda_mat}\n")
print(f"Vmat        :\n{Vmat}\n")
print(f"Vmat_inv    :\n{Vmat_inv}\n")
print(f"Amat_from_diagonalisation :\n{Amat_from_diagonalisation}\n")
```

```
Amat        :
[[1 2 3]
 [4 5 6]
 [7 8 9]]

lambda_mat  :
[[ 1.61168440e+01  0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00 -1.11684397e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00 -1.30367773e-15]]

Vmat        :
[[-0.23197069 -0.78583024  0.40824829]
 [-0.52532209 -0.08675134 -0.81649658]
 [-0.8186735   0.61232756  0.40824829]]

Vmat_inv    :
[[-0.48295226 -0.59340999 -0.70386772]
 [-0.91788599 -0.24901003  0.41986593]
 [ 0.40824829 -0.81649658  0.40824829]]

Amat_from_diagonalisation :
[[1. 2. 3.]
 [4. 5. 6.]
 [7. 8. 9.]]
```

Special properties of symmetric matrices

1. orthogonal eigenvectors

2. real eigenvalues

proof (orthogonality of eigenvectors)

λ_1 and λ_2 denote distinct eigenvalues with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$\mathbf{A} = \mathbf{A}^T$

$$
\begin{aligned}
\lambda_1 \cdot \mathbf{v}_1^T \cdot \mathbf{v}_2 = \left(\mathbf{A} \cdot \mathbf{v}_1\right)^T \cdot \mathbf{v}_2 = \mathbf{v}^T \cdot \mathbf{A}^T \cdot \mathbf{v}_2 \\
= \mathbf{v}^T \cdot \lambda_2 \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}^T \cdot \mathbf{v}_2 \\
\rightarrow \\
\left(\underbrace{\lambda_1 - \lambda_2}_{\neq 0}\right) \cdot \underbrace{\mathbf{v}_1^T \cdot \mathbf{v}_2}_{0} = 0
\end{aligned}
$$

Using these orthogonal eigenvector (of a symmetric matrix) as column vectors of a matrix $\mathbf{V}$ the matrix product

$$\mathbf{V}^T \cdot \mathbf{V} = \mathbf{D}$$

is a diagonal matrix. Moreover if the eigenvectors are normalised the matrix product yields the identity matrix.

$$\mathbf{V}^T \cdot \mathbf{V} = \mathbf{I}$$

Obviously

$$\mathbf{V}^T = \mathbf{V}^{-1}$$

---

proof (all eigenvalues of a real symmetric matrix are real)

Due to its symmetry $\mathbf{A}^T = \mathbf{A}$. Since the matrix is real we have $\mathbf{S}^* = \mathbf{S}$. Moreover we have $\mathbf{S}^{*T} = \mathbf{S}$

Let $\lambda, \mathbf{v}$ denote an eigenvalue and its eigenvector (unit length).

$$
\begin{aligned}
\lambda = \lambda \cdot \mathbf{v}^{*T} \cdot \mathbf{v} = \mathbf{v}^{*T} \cdot \lambda \cdot \mathbf{v} = \mathbf{v}^{*T} \cdot \mathbf{A} \cdot \mathbf{v} \\
\mathbf{v}^{*T} \cdot \mathbf{A}^{*T} \cdot \mathbf{v} = (\mathbf{A} \cdot \mathbf{v})^{*T} \cdot \mathbf{v} = \lambda^* \cdot \mathbf{v}^{*T} \cdot \mathbf{v} \rightarrow \\
\lambda = \lambda^*
\end{aligned}
$$

So eigenvalue λ must be real.

Summary : Properties of symmetrix real matrix

Matrix $\mathbf{A}$ shall be real and symmetric. The matrix is of type $n \times n$. There exist `n` eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and unit length eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$

For each pair of eigenvalues / eigenvectors we have:

$$\mathbf{A} \cdot \mathbf{v}_j = \lambda_j \cdot \mathbf{v}_j$$

$\mathbf{A} \cdot \mathbf{v}_j$ are column vectors which can be interpreted as columns of a $n \times n$ matrix.

$$\begin{bmatrix} | & | & & | \\ \mathbf{A} \cdot \mathbf{v}_1 & \mathbf{A} \cdot \mathbf{v}_2 & \dots & \mathbf{A} \cdot \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 \cdot \mathbf{v}_1 & \lambda_2 \cdot \mathbf{v}_2 & \dots & \lambda_n \cdot \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

which can be expressed by a multiplication of matrices:

$$\mathbf{A} \cdot \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Defining matrices $\mathbf{V}$ and $\mathbf{\Lambda}$ as:

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \lambda_1 \cdot \mathbf{v}_1 & \lambda_2 \cdot \mathbf{v}_2 & \dots & \lambda_n \cdot \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

we get the compact form:

$$\mathbf{A} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{\Lambda}$$

All column vectors of matrix $\mathbf{V}$ are mutually orthonormal. This condition is equivalent to

$$\mathbf{V}^T \cdot \mathbf{V} = \mathbf{I}$$

and we get:

$$\mathbf{A} \cdot \mathbf{V} \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T$$
$$\rightarrow$$
$$\mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T$$

The last equation is also known as the `diagonalisation of a symmetric matrix`.

The equation can be written in different form:

$$\mathbf{A} = (\mathbf{V} \cdot \mathbf{\Lambda}) \cdot \mathbf{V}^T$$

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \lambda_1 \cdot \mathbf{v}_1 & \lambda_2 \cdot \mathbf{v}_2 & \cdots & \lambda_n \cdot \mathbf{v}_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} - & \mathbf{v}_1^T & - \\ - & \mathbf{v}_2^T & - \\ & \vdots & \\ - & \mathbf{v}_n^T & - \end{bmatrix}$$

$$\mathbf{A} = \sum_{j=1}^{n} \lambda_j \cdot \mathbf{v}_j \cdot \mathbf{v}_j^T$$

Matrix $\mathbf{A}$ can thus be expressed as sum `n` of partial matrices. Each partial matrix is the defined as outer product $\mathbf{v}_j \cdot \mathbf{v}_j^T$. Each partial matrix is of type $n \times n$.

Matrix power

Multiplying again by matrix $\mathbf{B}$ we get.

$$\mathbf{B} \cdot \mathbf{B} \cdot \mathbf{v} = \mathbf{B} \cdot \lambda \cdot \mathbf{v} = \lambda \cdot \mathbf{B} \cdot \mathbf{v} = \lambda^2 \cdot \mathbf{v}$$

Thus the scaled vector $\lambda \cdot \mathbf{v}$ is again an eigenvector.

If matrix multiplication by $\mathbf{B}$ is applied `n` times the result is:

$$\mathbf{B}^n \cdot \mathbf{v} = \lambda^n \cdot \mathbf{v}$$

If $N \times N$ matrix $\mathbf{B}$ is symmetric there are N linear independent eigenvectors. If we express a vector as a linear combination of eigenvectors the multiplication of a matrix with a vector is simply a weighted addition of the eigenvectors. Thus the matrix multiplication can be replaced by a less computational demanding addition of vectors.

Another useful properties is stated for a *positive definite* matrix $\mathbf{B}$ -> ($\mathbf{v}^T \cdot \mathbf{B} \cdot \mathbf{v} > 0$):

$$\begin{gathered} \mathbf{B} \cdot \mathbf{v} = \lambda \cdot \mathbf{v} \\ \mathbf{v}^T \cdot \underbrace{\mathbf{B}}_{> 0} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}^T \underbrace{\cdot}_{> 0} \mathbf{v} \\ \rightarrow \lambda > 0 \end{gathered}$$

All eigenvalues are strictly positive.

Special case / symmetric matrix

For a real and symmetric matrix $\mathbf{A}$ we want to get $\mathbf{A}^2$.

Since $\mathbf{A}$ can be `diagonalized`

$$\mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T$$

we can express $\mathbf{A}^2$ by the product

$$\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T \cdot \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T$$
$$\mathbf{A} \cdot \mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{\Lambda}^2 \cdot \mathbf{V}^T$$

$$\mathbf{A}^2 = \mathbf{V} \cdot \mathbf{\Lambda}^2 \cdot \mathbf{V}^T$$

and for the general case $\mathbf{A}^k$:

$$\mathbf{A}^k = \mathbf{V} \cdot \mathbf{\Lambda}^k \cdot \mathbf{V}^T$$

$$\mathbf{A}^k = \sum_{j=1}^{n} \lambda_j^k \cdot \mathbf{v}_j \cdot \mathbf{v}_j^T$$

matrix inverse

If real symmetric matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$ it can be evaluated like this:

$$\mathbf{A}^{-1} = \left(\mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T\right)$$

$$\mathbf{A}^{-1} = \left(\mathbf{V}^T\right)^{-1} \cdot \mathbf{\Lambda}^{-1} \cdot \mathbf{V}^{-1}$$

$$\mathbf{A}^{-1} = \left(\mathbf{V}^{-1}\right)^{-1} \cdot \mathbf{\Lambda}^{-1} \cdot \mathbf{V}^T$$
$$\mathbf{A}^{-1} = \mathbf{V} \cdot \mathbf{\Lambda}^{-1} \cdot \mathbf{V}^T$$

where matrix $\mathbf{\Lambda}^{-1}$ is just another diagonal matrix:

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix}$$

Symmetric positive definite matrix

Consider the expression

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = f(\mathbf{x})$$

Since matrix $\mathbf{A}$ is real symmetric the quantity $\mathbf{x}$ is real.

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = \sum_{j=1}^{n} \lambda_j \cdot \mathbf{x}^T \cdot \mathbf{v}_j \cdot \mathbf{v}_j^T \cdot \mathbf{x}$$

$$= \sum_{j=1}^{n} \lambda_j \cdot \mathbf{v}_j^T \cdot \mathbf{x} \cdot \mathbf{v}_j^T \cdot \mathbf{x} = \sum_{j=1}^{n} \lambda_j \cdot \left(\mathbf{v}_j^T \cdot \mathbf{x}\right)^2$$

We assume that eigenvalues are ordered in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$.

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = f(\mathbf{x}) \leq \lambda_1 \cdot \sum_{j=1}^{n} \left(\mathbf{v}_j^T \cdot \mathbf{x}\right)^2$$

$$\sum_{j=1}^{n} \left(\mathbf{v}_j^T \cdot \mathbf{x}\right)^2 = ||\mathbf{V} \cdot \mathbf{x}||^2 = (\mathbf{V} \cdot \mathbf{x})^T \cdot (\mathbf{V} \cdot \mathbf{x}) = \mathbf{x}^T \cdot \mathbf{V}^T \cdot \mathbf{V} \cdot \mathbf{x} = \mathbf{x}^T \cdot \mathbf{I} \cdot \mathbf{x} = \mathbf{x}^T \cdot \mathbf{x} = ||\mathbf{x}||^2$$

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} \leq \lambda_1 \cdot \sum_{j=1}^{n} \left(\mathbf{v}_j^T \cdot \mathbf{x}\right)^2 = \lambda_1 \cdot ||\mathbf{x}||^2$$

For any vector $\mathbf{x}$ with init length we have

$$\max_{||\mathbf{x}||^2=1} \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} \leq \lambda_1$$

For the specific case of $\mathbf{x}$ being an eigenvector $\mathbf{x} = \mathbf{v}_j$ we get:

$$\mathbf{v}_j^T \cdot \mathbf{A} \cdot \mathbf{v}_j = \lambda_j \cdot \mathbf{v}_j^T \cdot \mathbf{v}_j = \lambda_j \cdot ||\mathbf{v}_j||^2 = \lambda_j$$

from which it follows

$$\max_{||\mathbf{x}||^2=1} \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = \lambda_1$$

For the minimum value a similar equation can be found:

$$\min_{||\mathbf{x}||^2=1} \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = \lambda_n$$

positive definite

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} > 0$$

All eigenvalues are <u>strictly</u> positive > 0.

positive semi-definite

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} \geq 0$$

All eigenvalue are non-negative ≥ 0

Example of eigenvector decomposition

Matrix $\mathbf{A}$ is defined as a 2×3 matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

A rectangular matrix has no eigendecomposition. However for the matrix product $\mathbf{A}^T \cdot \mathbf{A}$ which is a symmetric 3×3 matrix the eigendecomposition exists.

1. its eigenvalues are real-valued
2. the eigenvector are mutually orthogonal

So lets compute (manually):

$$\mathbf{B} = \mathbf{A}^T \cdot \mathbf{A}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The eigen decomposition of $\mathbf{B}$ can be expressed like this:

$$\mathbf{B} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$$

computing the eigenvalues

The eigenvalues are computed from the roots of the characteristic polynomial $p(\lambda)$.

$$p(\lambda) = det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = \lambda \cdot (1-\lambda) \cdot (3-\lambda)$$

There are three real-valued distinct eigenvalues $\lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = 0$. For each the corresponding eigenvector is computed:

computing eigenvectors

For $\lambda_1 = 3$ the elements of eigenvector $\mathbf{v}_1$ are found solving this equation:

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$$\mathbf{v}_1 = t \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = t \cdot \sqrt{6} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

For $\lambda_2 = 1$ the elements of eigenvector $\mathbf{v}_2$ are found solving this equation:

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$$\mathbf{v}_2 = t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t \cdot \sqrt{2} \cdot \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\lambda_3 = 0$ the elements of eigenvector $\mathbf{v}_3$ are found solving this equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$$\mathbf{v}_3 = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = t \cdot \sqrt{3} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ have been computed as vectors each with length 1 and a unknown scaling factor.

Finally the orthogonal matrix $\mathbf{V}$ can be setup with the orthonormal eigenvectorrs as column vectors.

$$\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

The column vectors of $\mathbf{V}$ are mutually orthonormal. The inverse matrix $\mathbf{V}^{-1}$ is just the transpose of $\mathbf{V}$.

Then the eigenvector decomposition of matrix $\mathbf{B}$ may be written like this:

$$\mathbf{B} = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{V} \cdot \Lambda \cdot \mathbf{V}^T$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{\sqrt{6}} & \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

And here comes the numerical computation (much easier than the manual work).

In [16]:
```python
Amat = np.array([[1, 1, 0], [0, 1, 1]])
Bmat = Amat.T @ Amat
lambda_list, Vmat = np.linalg.eig(Bmat)
lambda_mat = np.diag(lambda_list)

Bmat_from_decomposition = Vmat @ lambda_mat @ Vmat.T

print(f"Amat :\n{Amat}\n")
print(f"Bmat :\n{Bmat}\n")
print(f"lambda_list :\n{lambda_list}\n")
print(f"Vmat :\n{Vmat}\n")
print(f"Bmat_from_decomposition :\n{Bmat_from_decomposition}")
```

```
Amat :
[[1 1 0]
 [0 1 1]]

Bmat :
[[1 1 0]
 [1 2 1]
 [0 1 1]]

lambda_list :
[ 3.00000000e+00  1.00000000e+00 -3.36770206e-17]

Vmat :
[[-4.08248290e-01  7.07106781e-01  5.77350269e-01]
 [-8.16496581e-01  2.61239546e-16 -5.77350269e-01]
 [-4.08248290e-01 -7.07106781e-01  5.77350269e-01]]

Bmat_from_decomposition :
[[ 1.00000000e+00  1.00000000e+00 -2.33270278e-16]
 [ 1.00000000e+00  2.00000000e+00  1.00000000e+00]
 [-1.77759127e-16  1.00000000e+00  1.00000000e+00]]
```

In [ ]: