Matrix inverse

Mainly two resources have been used to setup this notebook:

Sources:

`Linear Algebra : Theory, Intuition, Code` author: Mike X Cohen, publisher: sincXpress

`No bullshit guide to linear algebra` author: Ivan Savov

Definitions

If it exists the inverse Matrix of a square matrix $\mathbf{A}$ is denoted by $\mathbf{A^{-1}}$. Left multiplication of $\mathbf{A}$ by $\mathbf{A^{-1}}$ yields the identity matrix $\mathbf{I}$.

$$\mathbf{A^{-1}} \cdot \mathbf{A} = \mathbf{I}$$

Solving the matrix equation

$$\mathbf{A} \cdot \mathbf{y} = \mathbf{b}$$

could be (at least in theory) done by left multiplication of both sides of the equation by the inverse matrix:

$$\begin{align} \mathbf{A^{-1}} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A^{-1}} \cdot \mathbf{b} \\ \mathbf{I} \cdot \mathbf{x} = \mathbf{A^{-1}} \cdot \mathbf{b} \\ \mathbf{x} = \mathbf{A^{-1}} \cdot \mathbf{b} \end{align}$$

If the inverse matrix exists it is **unique**.

Proof

Assume that matrices $\mathbf{B}$ **and** $\mathbf{C}$ are both inverse matrices of $\mathbf{A}$.

$$\begin{align} \mathbf{B} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{B} = \mathbf{I} \\ \mathbf{C} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{C} = \mathbf{I} \\ \\ \mathbf{B} = \mathbf{B} \cdot \mathbf{I} = \mathbf{B} \cdot (\mathbf{A} \cdot \mathbf{C}) = (\mathbf{B} \cdot \mathbf{A}) \cdot \mathbf{C} = \mathbf{I} \cdot \mathbf{C} = \mathbf{C} \\ \\ \mathbf{B} = \mathbf{C} \end{align}$$

The inverse of the matrix product $(\mathbf{A} \cdot \mathbf{B})$ is computed from:

$(\mathbf{A}\cdot\mathbf{B})^{-1}=\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$

Proof

$$(\mathbf{A}\cdot\mathbf{B})^{-1}\cdot\mathbf{A}\cdot\mathbf{B}=\mathbf{I}$$

Expressing $(\mathbf{A}\cdot\mathbf{B})^{-1}$ by the product of two matrices

$$(\mathbf{A}\cdot\mathbf{B})^{-1}=\mathbf{D}\cdot\mathbf{C}$$

yields:

$$\mathbf{D}\cdot\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{B}=\mathbf{I}$$

Choose $\mathbf{C}=\mathbf{A}^{-1}$

$$\mathbf{D}\cdot\mathbf{A}^{-1}\cdot\mathbf{A}\cdot\mathbf{B}=\mathbf{D}\cdot\mathbf{I}\cdot\mathbf{B}=\mathbf{D}\cdot\mathbf{B}=\mathbf{I}$$

Choose $\mathbf{D}=\mathbf{B}^{-1}$

$$\mathbf{D}\cdot\mathbf{B}=\mathbf{B}^{-1}\cdot\mathbf{B}=\mathbf{I}$$

Thus we have

$$\begin{align}
\mathbf{D}\cdot\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{B}&=\mathbf{I} \tag{10}\\
\underbrace{\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}}_{(\mathbf{A}\cdot\mathbf{B})^{-1}}\cdot\mathbf{A}\cdot\mathbf{B}&=\mathbf{I} \tag{11}\\
&\rightarrow \tag{12}\\
(\mathbf{A}\cdot\mathbf{B})^{-1}&=\mathbf{B}^{-1}\cdot\mathbf{A}^{-1} \tag{13}
\end{align}$$

The result can easily be generalized like this:

$$(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C})^{-1}=\mathbf{C}^{-1}\cdot\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$

The inverse matrix of the inverse matrix is the original matrix:

$$\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A}$$

Proof

To prove this equation the property $(\mathbf{A}\cdot\mathbf{B})^{-1}=\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$ is used.

$$\left(\mathbf{A}\cdot\mathbf{A}^{-1}\right)^{-1} = \mathbf{I}^{-1} = \mathbf{I} \tag{14}$$

$$\left(\mathbf{A}^{-1}\right)^{-1}\cdot\mathbf{A}^{-1} = \mathbf{I} \tag{15}$$

$$\left(\mathbf{A}^{-1}\right)^{-1}\cdot\mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{I}\cdot\mathbf{A} \tag{16}$$

$$\left(\mathbf{A}^{-1}\right)^{-1}\cdot\mathbf{I} = \mathbf{A} \tag{17}$$

$$\tag{18}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} \tag{19}$$

The inverse of a symmetric matrix is also symmetric.

If $\mathbf{A} = \mathbf{A}^T$ then $\mathbf{A}^{-1} = \left(\mathbf{A}^{-1}\right)^T = \mathbf{A}^{-T}$

Inverse Matrix of rectangular / non-square matrix

A $m \times n$ matrix $\mathbf{A}$ with $m > n$ (more rows than columns) is named a `tall` matrix.

Similarly a $m \times n$ matrix $\mathbf{A}$ with $m < n$ (more columns than rows) is named a `wide` matrix.

inverse matrix for `tall` matrix : (left inverse)

Let $\mathbf{T}$ denote the tall matrix.

The rectangular matrix has no inverse however we contruct a rectangular and symmetrix $n \times n$ matrix $\mathbf{T}^T \cdot \mathbf{T}$. The matrix has an inverse if $rank(\mathbf{T}) = n$ (full column rank).

$$\underbrace{\left(\mathbf{T}^T\cdot\mathbf{T}\right)^{-1}\cdot\mathbf{T}^T}_{left\,inverse\quad n\times m}\cdot\underbrace{\mathbf{T}}_{m\times n} = \underbrace{\mathbf{I}}_{n\times n} \tag{20}$$

inverse matrix for `wide` matrix : (right inverse)

Let $\mathbf{W}$ denote a wide $m \times n$ matrix which has no inverse. Again a square $m \times m$ matrix is obtained from $\mathbf{W}\cdot\mathbf{W}^T$. This matrix has an inverse if $rank(\mathbf{W}) = m$ (full row rank).

$$\underbrace{\mathbf{W}}_{m\times n}\cdot\underbrace{\mathbf{W}^T\cdot\left(\mathbf{W}\cdot\mathbf{W}^T\right)^{-1}}_{right\,inverse\quad n\times m} = \underbrace{\mathbf{I}}_{m\times m}$$

Examples

In [1]:
```python
import numpy as np
```

In [5]:
```python
# random square matrix
# randomness ensures in most cases that the matrix has an inverse !

Amat = np.random.randn(4,4)
Amat_inv = np.linalg.inv(Amat)

# the product (right and left)
IRight = np.matmul(Amat, Amat_inv)
ILeft = np.matmul(Amat_inv, Amat)

print(f"Amat     : {Amat}\n")
print(f"Amat_inv : {Amat_inv}\n")
print(f"IRight   : {IRight}\n")
print(f"ILeft    : {ILeft}\n")
```

```
Amat     : [[-0.58520463 -0.73070759 -0.58630732  0.04156307]
 [-0.52711811 -0.36131584 -0.1286433   0.55818633]
 [ 1.89419274 -0.93403963 -1.0470101  -0.79117395]
 [-0.89697529  0.89979277  0.76328476  0.71338095]]

Amat_inv : [[-0.97674362  0.74131682  0.38906335 -0.09164763]
 [ 2.32135608 -2.53307203  1.86761321  3.91803348]
 [-3.64213181  2.51855238 -2.64742301 -4.69456814]
 [-0.25914711  1.43234849  0.96617799  1.36766822]]

IRight   : [[ 1.00000000e+00 -2.47956856e-17  5.89000329e-18  1.17644131e-16]
 [-1.51415268e-16  1.00000000e+00 -2.23589322e-16 -5.14408172e-16]
 [-4.88741731e-16  1.13507435e-16  1.00000000e+00 -3.32550649e-17]
 [ 1.00574182e-16 -1.42015649e-16 -3.68586569e-16  1.00000000e+00]]

ILeft    : [[ 1.00000000e+00 -3.78694203e-17 -2.65358246e-17 -2.50736875e-16]
 [-8.46429403e-16  1.00000000e+00  4.84062363e-16  5.62813837e-16]
 [ 8.35466620e-16 -3.76789756e-16  1.00000000e+00 -6.38019255e-16]
 [-8.69596457e-18 -1.38890147e-16  5.94955142e-17  1.00000000e+00]]
```

In [11]:
```python
# demo of left inverse

Atall = np.random.randn(5,3)
Aleft = np.linalg.inv(np.matmul(Atall.T, Atall)) @ Atall.T
Ileft = Aleft @ Atall

print(f"Ileft : {Ileft}")
```

```
Ileft : [[ 1.00000000e+00  1.60683903e-17 -1.39496299e-17]
 [-5.44369123e-17  1.00000000e+00  2.73216249e-17]
 [ 1.17566215e-17  3.42756730e-17  1.00000000e+00]]
```

In [13]:
```python
# demo of right inverse

Awide = np.random.randn(2,6)
Aright = Awide.T @ np.linalg.inv(Awide @ Awide.T)
Iright = Awide @ Aright
```

```python
print(f"Iright: {Iright}")
```

```
Iright: [[ 1.00000000e+00 -6.35505235e-17]
 [-1.75274312e-16  1.00000000e+00]]
```

In [ ]: