Gauss-Jordan Elimination

In this notebook I follow the steps described in the book

`No Bullshit guide to Linear Algebra` , Ivan Savov

The three equations

$$\begin{aligned} 1x + 2y + 3z &= 14 \\ 2x + 5y + 6z &= 30 \\ -1x + 2y + 3z &= 12 \end{aligned}$$

which is cast to the augmented matrix equation:

$$\begin{bmatrix} 1 & 2 & 3 & | & 14 \\ 2 & 5 & 6 & | & 30 \\ -1 & 2 & 3 & | & 12 \end{bmatrix}$$

The next steps are to obtain the equation in `REF` (row-echelon form).

Step#1: row operation $R_2 - 2R_1 \rightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 & | & 14 \\ 0 & 1 & 0 & | & 2 \\ -1 & 2 & 3 & | & 12 \end{bmatrix}$$

Step#2: row operation $R_3 + R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & | & 14 \\ 0 & 1 & 0 & | & 2 \\ 0 & 4 & 6 & | & 26 \end{bmatrix}$$

Step#3: row operation $R_3 - 4R_2 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & | & 14 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 6 & | & 18 \end{bmatrix}$$

Step#4: $(1/6)R_3 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & | & 14 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

At this point the equation system has been transformed into `REF` form.

In the next steps the equation system is transformed into `RREF` (reduced row echelon form).

Starting with the 3'rd row we perform these steps:

Step#1: $R_1 - 3R_3 \rightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 0 & | & 5 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

Step#2: $R_1 - 2R_2 \rightarrow R_1$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

At this point the equation system has been transformed into `RREF` form from which the solution can be directly obtained:

$x_1 = 1,\ x_2 = 2,\ x_3 = 3$

Number of solutions

For a equation systems that can be represented by a 3×3 matrix there several cases to consider:

1. there is one solution
2. there infinitely many solutions in 1 dimensions
3. there infinitely many solution in 2 dimensions
4. there are no solutions

One Solution

The augmented matrix shall be `RREF` form.

$$\begin{bmatrix} 1 & 0 & 0 & | & c_1 \\ 0 & 1 & 0 & | & c_2 \\ 0 & 0 & 1 & | & c_3 \end{bmatrix}$$

with solution $x_1 = c_1,\ x_2 = \ c_2,\ x_3 = \ c_3.$

Infinitely many solutions / 1 dimensions

We assume that the transformation to `REF` yields the 3'rd row with all zeros.

$$\begin{bmatrix} 1 & 0 & a_1 & | & c_1 \\ 0 & 1 & a_2 & | & c_2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We are then free to choose $x_3 = t$ with t being an arbitrary number.

The system of equations is now expressed like this:

$$\begin{aligned} x_1 + a_1 \cdot t &= 14 \\ 2 \cdot x_1 + 5 \cdot x_2 + 6 \cdot t &= 30 \\ x_3 &= t \end{aligned}$$

or in vector notation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -a_1 \\ -a_2 \\ 1 \end{bmatrix}$$

This the vectorial representation of a line passing through point $[c_1,\ c_2,\ 0]$ in the direction of vector $[-a_1,\ -a_2,\ 1]$.

No solutions

$$\begin{bmatrix} 1 & 0 & 0 & | & c_1 \\ 0 & 1 & 0 & | & c_2 \\ 0 & 0 & 0 & | & c_3 \end{bmatrix}$$

While $x_1 = c_1,\ x_2 = c_2$ this choice of $x_1,\ x_2$ does not satisfy the equation from the 3'rd row

$$x_1 \cdot 0 + x_2 \cdot 0 + x_3 \cdot 0 = c_3$$

(assuming $c_3 \neq 0$)

Solving Excercises

E.3.1

augmented matrix:

$$\begin{bmatrix} 3 & 3 & | & 6 \\ 2 & 1.5 & | & 5 \end{bmatrix}$$

Step: $(1/3) * R_1 \rightarrow R_1$

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 2 & 1.5 & | & 5 \end{bmatrix}$$

Step: $R_2 - 2 * R_1 \rightarrow R_2$

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 0 & -0.5 & | & 1 \end{bmatrix}$$

Step: $-2 * R_2 \rightarrow R_2$

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -2 \end{bmatrix}$$

Step: $R_1 - R_2 \rightarrow R_1$

$$\begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -2 \end{bmatrix}$$

Solution: $x_1 = 4,\ x_2 = -2$

E.3.3a

augmented matrix:

$$\begin{bmatrix} 3 & 3 & | & 6 \\ 1 & 1 & | & 5 \end{bmatrix}$$

Step:

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 1 & 1 & | & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 0 & | & 3 \end{bmatrix}$$

Solution to excersises

Excersises are from the book

`No Bullshit guide to Linear Algebra` , Ivan Savov; Chapter 3.6 (Computational problems)

P.3.4 Find the solution sets for the augemented matrices.

a)

$$\begin{bmatrix} -1 & -2 & | & -2 \\ 3 & 6 & | & 6 \end{bmatrix}$$

row operations:

$$\begin{bmatrix} 1 & 2 & | & 2 \\ 1 & 2 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

which gives the solution set

$x_1 = -2 \cdot t + 2$ and $x_2 = t$

which is expressed as a vectorial description of a straight line.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

P.3.5 Find the solution sets for the augemented matrices.

a)

$$\begin{bmatrix} 1 & -1 & -2 & | & 1 \\ -2 & 2 & 4 & | & -2 \\ 3 & -3 & -6 & | & 3 \end{bmatrix}$$

row operations: R2/(-2) -> R2; R3/(3) -> R3

$$\begin{bmatrix} 1 & -1 & -2 & | & 1 \\ 1 & -1 & -2 & | & 1 \\ 1 & -1 & -2 & | & 1 \end{bmatrix}$$

row operations: R2 - R1 -> R2; R3 - R1 -> R3

$$\begin{bmatrix} 1 & -1 & -2 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_2,\ x_3$ are free variables: $x_2 = t_2,\ x_3 = t_3$

$$\begin{gathered} x_1 = 1 + t_2 + t_3 \cdot 2 \\ x_2 = t_2 \\ x_3 = t_3 \end{gathered}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_3 \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

P.3.8

Solve for $\mathbf{C}$ in the matrix equation:

$$\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D}$$

step#1

$$\begin{gathered} \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{D}^{-1} = \mathbf{A} \cdot \mathbf{D} \cdot \mathbf{D}^{-1} \\ \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \end{gathered}$$

step#2

$$\begin{gathered} \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{A}^{-1} \cdot \mathbf{A} \\ \mathbf{B} \cdot \mathbf{C} = \mathbf{1} \end{gathered}$$

step#3

$$\begin{gathered} \mathbf{B}^{-1} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{B}^{-1} \cdot \mathbf{1} \\ \mathbf{C} = \mathbf{B}^{-1} \end{gathered}$$

In [ ]: