AER210
Vector Calculus and Fluid Mechanics

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Contents

Part I

Vector Calculus

1 Review of Multivariable Calculus

1.1 Vectors and Vector-Valued Functions

- A vector is a quantity that has magnitude and direction

- Vectors having the same length and direction are equivalent

- <u>Position Vector</u> - Initial point of the vector at the origin of the coordinate system

1.2 Dot Product

- $\vec{a} \cdot \vec{b} \rightarrow$ scalar product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad (algebraic)$$
$$\vec{a} \cdot \vec{b} = |a||b|cos\theta \qquad (geometric)$$

1.3 Cross Product

- $\vec{a} \times \vec{b} \rightarrow$ another vector

- $\vec{c} = \vec{a} \times \vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \qquad (algebraic)$$

$$|\vec{c}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|sin\theta \qquad (geometric)$$

- $|\vec{c}|$ is the area of a parallelogram formed by $\vec{a}$ and $\vec{b}$

1.4 Triple Scalar Product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

- <u>Volume of Parallelpiped</u> $= |\vec{a}||\vec{b} \times \vec{c}|cos\theta = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

1.5 Vector Functions

A vector function is a function hose input is a scalar (such as time) and output is a vector.

$$r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

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1.5.1 Derivatives of Vector Functions

$$\frac{d\vec{r}(t)}{dt} = \vec{r}'(t) = \lim_{n \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

At $h = 0$, the vector $\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$ approaches a vector that is tangent to the curve at point P.

1.5.2 Unit Tangent Vector

$$\vec{T}(\theta) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

1.5.3 Arc Length

$$S = \int_a^b |\vec{r}'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

1.5.4 Partial Derivatives

For functions of multiple variables, select the variable to differentiate with respect to, and treat all others as constants.

$$f(x, b) = g(x) \quad \to \quad f_x(x, b) = g'(x)$$

example:

Set $f(x, y) = 3x^4 y^2$. Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = 12x^3 y^2 \qquad f_y(x, y) = 6x^4 y$$

1.5.5 Higher Order Partial Derivatives

$$(f_x)_x = f_{xx} \qquad (f_y)_y = f_{yy} \qquad (f_x)_y = (f_y)_x = f_{xy} = f_{xy}$$

1.5.6 Gradient Vector

The gradient vector is the rate of change of a function $f(x, y, z)$ composed of its partial derivatives.

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

1.5.7 Chain Rule

$$h = f(x, y, z) \quad \to \quad x = x(t) \quad y = y(t) \quad z = z(t)$$

$$\frac{df(\vec{r}(t))}{dt} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

$$\frac{df(\vec{r}(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

1.5.8   Tangent Planes

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

1.5.9   Linearization

If $\Delta x \to 0$, $\Delta y \to 0$, then $\Delta z \approx dz$:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

2 Multiple Integrals

2.1 Double Integrals Over Rectangles

We can define a region, R, over which the integration will take place:

$$R = [a, b] \times [c, d] = \{(x, y) \ \epsilon \ R^2 \mid a \leq x \leq b \ , \ c \leq y \leq d\}$$

The **double integral** of f over the rectangle R is thus

$$\iint_R f(x, y) dA = \lim_{\infty, \ a \to 0} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta A$$

If $f(x, y) \geq 0$, then the double integral could be interpreted as the volume in the defined region:

$$V = \iint_R f(x, y) dA$$

An iterated integral is defined as

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

To evaluate the above, solve the inner integral with respect to y treating x as a constant. Then, solve the outer integral with respect to x treating y as a constant.

Fubini's Theorem: States if f is continuous on region R, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

If $f(x, y)$ can be expressed as $g(x)h(y)$, then we can express its integration as

$$\iint_R f(x, y) dA = \int_a^b \int_c^d g(x)h(y) dy dx = \int_a^b g(x) dx \int_c^d h(y) dy$$

2.2 Double Integrals Over General Regions

Double integrals do not always cover rectangular regions. They often cover general regions bounded by several functions.

If f is integrable over R, then we define the double integral of f over D by

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA =$$

In the case that $f(x,y) \geq 0$, we can still interpret $\iint\limits_D f(x,y)dA$ as the volume of the solid that lies above D and under the surface $z = f(x,y)$.

Similarly to rectangular regions, we can evaluate iterated integrals with functions as parameters.

If f is a continuous function such that $D = \{(x,y) \mid a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x)\}$ (**Type I**), then

$$\iint\limits_D f(x,y)dA = \int_b^a \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If f is a continuous function such that $D = \{(x,y) \mid c \leq y \leq d, \; h_1(y) \leq x \leq h_2(y)\}$ (**Type II**), then

$$\iint\limits_D f(x,y)dA = \int_d^c \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

If regions do not overlap, they can also be summed:

$$\iint\limits_D f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$$

Finally, if we integrate the constant function $f(x,y) = 1$, we'll simply get the area of D.

$$\iint\limits_D 1 dA = A(D)$$

2.3 Double Integrals in Polar Coordinates

Recall that polar coordinates (r,θ) relate to rectangular coordinates (x,y) with the following equations:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

We can define the region $R = \{(r,\theta) \mid a \leq r \leq b, \; \alpha \leq \theta \leq \beta\}$. If f is continuous on a polar rectangle defined by this region R, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$$

2.4 Applications of Double Integrals

2.4.1 Density and Mass

Consider a lamina with variable density $\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$. We can obtain its total mass as the limiting value of its Riemann approximation:

$$m = \lim_{k,l \to \infty} \sum_{i=1}^l \sum_{j=1}^l \rho(x_{ij}^\star, y_{ij}^\star)\Delta A = \iint\limits_D \rho(x,y)dA$$

2.4.2 Moments and Center of Mass

Consider a lamina that occupies a region D and has density function $\rho(x, y)$. We can define the moments about the $x-axis$ and moments about the $y-axis$ as follows:

$$M_x = \iint\limits_{D} y\rho(x, y)dA \qquad M_y = \iint\limits_{D} x\rho(x, y)dA$$

We can then say the coordinates of the centre of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_{D} y\rho(x, y)dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_{D} x\rho(x, y)dA$$

where m is given by $m = \iint\limits_{D} \rho(x, y)dA$

2.4.3 Moment of Inertia

To find the moment of inertia about an area, the following integrals apply:

$$I_x = \iint\limits_{D} y^2\rho(x, y)dA \qquad I_y = \iint\limits_{D} x^2\rho(x, y)dA$$

The moment of inertia about the origin is given as

$$I_0 = \iint\limits_{D} (x^2 + y^2)\rho(x, y)dA$$

2.5 Surface Area

The area of a surface with equation $z = f(x, y)$, $(x, y) \in D$ where f_x and f_y are continuous , is

$$A(S) = \iint\limits_{D} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \; dA$$

We can also use an alternative notation to express the same surface area function:

$$A(S) = \iint\limits_{D} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dA$$

2.6 Triple Integrals

The same way that single and double integrals were defined for one variable and two variable functions, we can define triple integrals for functions of three variables.

We can consider f which is defined as a rectangular box.

$$B = \{(x, y, z) \mid a \leq \; x \leq b, \, c \leq y \leq d, \, r \leq z \leq s \}$$

The triple integral then, of f over box B is

$$\iiint\limits_{R} f(x,y,z)\ dA = \lim_{l,\ m,\ n \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^{\star}, y_{ijk}^{\star}, z_{ijk}^{\star}) \Delta V$$

Fubini's Theorem (For Triple Integrals): If f is continuous over the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint\limits_{R} f(x,y,z)\ dV = \int_{r}^{s} \int_{c}^{d} \int_{a}^{b} f(x,y,z)\ dx\ dy\ dz$$

The same rules for double integrals can also be applied to triple integrals

2.6.1   Triple Integrals in Cylindrical Coordinates

We can convert to cylindrical coordinates (r, θ, z) from rectangular coordinates (x, y, z) with the following equations:

$$r^2 = x^2 + y^2 \qquad x = rcos\theta \qquad y = rsin\theta \qquad z = z$$

The formula for triple integrals in cylindrical coordinates then is:

$$\iiint\limits_{R} f(x,y,z)\ dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) \cdot r\ dz\ dr\ d\theta$$

To solve for center of mass questions, apply the following formulas:

$$m = \iiint\limits_{E} \rho(x,y,z)\ dV \qquad (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{V}(M_{yz}, M_{xz}, M_{xy})$$

2.6.2   Triple Integrals in Spherical Coordinates

The spherical coordinates (ρ, θ, ϕ) of a point P in space is where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi$$

We can convert from rectangular to spherical coordinates with the following equations

$$\rho^2 = x^2 + y^2 + z^2 \qquad x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\theta$$

The formula for triple integrals in cylindrical coordinates then is:

$$\iiint\limits_{R} f(x,y,z)\ dV = \int_{c}^{d} \int_{\alpha}^{\beta} \int_{a}^{b} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \cdot \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

2.7 Change of Variables in Multiple Integrals

Changing integration variables in multiple integrals is a bit of different then it is in single integrals. We consider the change of variables as a transformation from the (u,v)-plane to the (x,y)-plane:

$$T(u, v) = (x, y)$$

where x and y are related by the equations

$$x = g(u, v) \qquad y = h(u, v)$$

We can define the Jacobian of the transformation given by the above equations as

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

With this definition, the formula for the change of variables in double integrals then is

$$\iint\limits_R f(x, y) \, dA = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv$$

Similarly in triple integrals, the Jacobian and change of variables are the following

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,z)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iint\limits_R f(x, y, z) \, dA = \iint\limits_S f(x(u, v, w), y(u, v, w), z(x, y, z)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \, dv \, dw$$

3 Vector Calculus

3.1 Line Integrals

Line integrals are very similar to single integrals, except instead of integrating over an interval $[a, b]$, we integrate over a curve C. To evaluate a line integral, we can use the following formula:

$$\int_C f(x, y) \ dA = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

This is called the line integral with respect to arc length. We can also evaluate a line integral with respect to x and y using the following formulas:

$$\int_C f(x, y) \ dx = \int_a^b f(x(t), y(t)) \ x'(t) \ dt \qquad \int_C f(x, y) \ dy = \int_a^b f(x(t), y(t)) \ y'(t) \ dt$$

We can also parametrize a line segment which begins at $\vec{r_0}$ and ends at $\vec{r_1}$ using a vector representation:

$$\vec{r}(t) = (1 - t)\vec{r_0} + t\vec{r_1}$$

This helps to reduce the parameters of the integral to one variable t.

Line integral apply in vector fields in important equations such as that for work. If F is a continuous vector field on a smooth curve C given by vector function $\vec{r}(t)$, $a \leq t \leq b$, then the line integral of F along C is

$$\int_C F \cdot dr = \int_a^b F(\vec{r}(t)) \cdot r'(t) \ dt = \int_C F \cdot T \ ds$$

3.2 Fundamental Theorem for Line Integrals

Theorem: Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variable whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))$$

We can say that $\int_C \nabla f \cdot dr$ is independent of path in D if and only if $\int_C \nabla f \cdot dr = 0$ for every closed path C in D. The line integrals of conservative vector fields are independent of path.

Theorem: Supposing that F is a vector field that is continuous on an open connected region D. If $\int_C \nabla f \cdot dr$ is independent of path on D, then F is a conservative vector field on D, meaning there exists a function f such that $\nabla f = F$

Theorem: If $F(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is a conservative vector field on, where P and Q have continuous first order partial derivatives on a domain D, then throughout D:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

If the above is true, then F is conservative

3.3 Green's Theorem

Theorem: Let C is a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

3.4 Curl and Divergence

3.4.1 Curl

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on R^3 and the partial derivatives of P, Q, and R all exist, then the curl of F is the vector field defined by

$$curl\ F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k} = \nabla \times F$$

Theorem: If F is a function that has continuous second-order partial derivatives, then

$$curl(\nabla F) = 0$$

This also tells us that if F is a conservative vector field, the $curl\ F = 0$

Theorem: If F is a vector field defined on all of R^3, whose component functions all have continuous partial derivatives and $curl\ F = 0$, then F is conservative.

3.4.2 Divergence

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on R^3 and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ all exist then the divergence of F is the function of three variables defined by:

$$div\ F = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial y}\vec{j} + \frac{\partial R}{\partial z}\vec{k} = \nabla \cdot F$$

If F is a function that has continuous second-order partial derivatives, then

$$div\ curl\ F = 0$$

3.4.3   Vector Form of Green's Theorem

The curl and divergence operators now allow us to write Green's theorem in vector form as follows:

$$\oint_C F \cdot dr = \iint_D (curl\ F) \cdot k\ dA \qquad \oint_C F \cdot n\ ds = \iint_D div\ F(x,y)\ dA$$

3.5   Parametric Surfaces and Their Areas

Similar to how we can define parametric functions of one variable t, we can describe a surface $r(u,v)$ of two parameters, u, and v, where

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

If a smooth parametric surface S is given by the equation $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\vec{r_u} \times \vec{r_v}|\ dA$$

where $\vec{r_u}$ and $\vec{r_v}$ are

$$\vec{r_u} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \qquad \vec{r_v} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

3.6   Surface Integrals

Using the parametrized function $\vec{r}(u,v)$ from before, if the components are all non-zero and non-parallel, then the surface integral is

$$\iint_S f(x,y,z)\ dS = \iint_D f(\vec{r}(u,v))\ |\vec{r_u} \times \vec{r_v}|\ dA$$

If F is continuous vector field defined by a unit normal vector $\vec{n}$, then the surface integral of F over S is

$$\iint_S F \cdot ds = \iint_S F \cdot \vec{n}\ ds$$

where $\vec{n}$ is

$$\frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|}$$

This is also known as the **flux** of F along S.

3.7   Stoke's Theorem

Stoke's Theorem can be considered a higher order version of Green's Theorem, where it relates a surface integral over a region S to a line integral around the boundary of S.

Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_C F \cdot dr = \iint_S curl\ F \cdot dS = \iint_S curl\ F \cdot k\ dA$$

Part II

Fluid Mechanics

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