

MAT292  
Calculus III

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# 1 Introduction

## 1.1 Phase Lines and Direction Fields

A first order **autonomous** differential equation is in the form

$$\frac{dy}{dt} = f(y)$$

This means that the independent variable,  $t$ , does not appear on the right side of the equation. Other examples include:

$$x' = \sin x \qquad p' = rp \left(1 - \frac{p}{k}\right)$$

The first step in qualitative analysis is to find the constant solutions, where  $f(y) = 0$ . In these cases,  $\frac{dy}{dt} = 0$  and is considered an **equilibrium** solution. These are also referred to as critical points, stationary points, and fixed points.

We can graph the  $f(y)$  vs  $y$  in what is known as the **phase line** or **one dimensional phase portrait** to see the behaviour of  $y'$ . Seeing when  $y'$  is increasing or decreasing, we can use the information to then sketch an **integral curve** where we can sample plots of  $y$  vs  $t$

## 1.2 Definitions, Classification, and Terminology

- **Differential Equation:** An equation that contains the derivatives of one or more unknown functions with respect to one or more independent variables is said to be a differential equation.
- **Ordinary Differential Equation (ODE):** The unknown function (or functions) depends on a single variable and its derivatives are ordinary derivatives.
- **Partial Differential Equation (PDE):** The unknown function (or functions) depends on more than one independent variable and partial derivatives appear in the equation
- **Order:** The order of the highest derivative, ordinary or partial, that appears in the equation
- **Linear:** An ODE is linear if it can be written in the form

$$a_0(t)y^n + a_1(t)y^{n-1} + \cdots + a_n(t)y = g(t)$$

where the functions  $a_0, a_1, \dots, a_n$  depend at most on the independent variable  $t$ .

– **Homogeneous:**  $g(t) = 0$  for all  $t$

– **Non-Homogeneous:**  $g(t) = f$

- **Non-Linear:** An ODE is non-linear if it cannot take the form of a linear equation

## 2 1st Order Differential Equations

### 2.1 Separable Equations

Consider a first order linear differential equation of form  $\frac{dy}{dx} = f(x, y)$ . If the right side of this equation,  $f(x, y)$  can be broken down to  $p(x)q(y)$ , it is classified as a separable equation.

$$\frac{dy}{dx} = f(x, y) = p(x)q(y)$$

This is easily solved by rearranging the equation to isolate the  $x$  and  $y$  variables and then integrating to find a general solution.

### 2.2 Method of Integrating Factors

A differential equation is said to be a **first order linear equation** if it can be written in the form

$$\frac{dy}{dx} + p(t)y = g(t) \quad \rightarrow \quad y' + py = g$$

The solution can be found by going through the following steps:

1. Put the equation in standard form  $y' + py = g$
2. Calculate the integrating factor  $\mu(t) = e^{\int p(t)dt}$
3. Multiply the equation by  $\mu(t)$  and write in the form  $[\mu(t)y]' = \mu(t)g(t)$
4. Integrate this equation to obtain  $\mu(t)y = \int \mu(t)g(t)dt + C$
5. Solve for  $y$

### 2.3 Differences Between Linear and Non-Linear Equations

Before attempting a problem it is important to know if there is even a solution to it. This is where we can discuss the **existence and uniqueness of solutions**. There are theorems which cover this for both linear and non-linear equations.

**For Linear Equations (Theorem 2.4.1):** If the functions  $p$  and  $g$  are continuous on an open interval  $I = (\alpha, \beta)$  containing the point  $t = t_0$ , there exists a unique function  $y = \Phi(t)$  that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each  $t$  in  $I$ , and that also satisfies the initial condition,

$$y(t_0) = y_0$$

where  $y_0$  is an arbitrary prescribed value.

A quick summary of the properties of linear differential equations:

1. Assuming the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions
2. There is an expression for the solution
3. The possible points of discontinuity or singularities can be identified by finding the points of discontinuity of the coefficients

**For Non-Linear Equations (Theorem 2.4.2):** Let the functions  $f$  and  $\frac{\partial f}{\partial y}$  be continuous in some rectangle  $\alpha < t < \beta, \gamma < y < \delta$  containing the point  $(t_0, y_0)$ . Then in some interval  $t_0 - h < t < t_0 + h$  contained in  $\alpha < t < \beta$ , there is a unique solution  $y = \Phi(t)$  of the initial value problem,

$$y'(t) = f(t, y) \quad y(t_0) = y_0$$

Looking at the properties of linear differential equations, none of them are true in general, for non-linear equations. It comes down to a case-by-case basis

## 2.4 Autonomous Equations and Population Dynamics

As previously mentioned, an autonomous equation takes the form

$$\frac{dy}{dt} = f(y)$$

### 2.4.1 Exponential Growth

The simplest analysis of population occurs with the assumption that the rate of change of the population is proportional to the current value of the population. This is modelled as

$$\frac{dP}{dt} = rP$$

where  $r$  is the rate of growth or decline depending on its sign. Given the initial condition  $P(0) = P_0$ , the solution is given as

$$P(t) = P_0 e^{rt}$$

### 2.4.2 Logistic Growth

We replace the constant  $r$  with a function  $h(P)$  from the previous equation to obtain the modified one

$$P'(t) = h(P) \cdot P$$

We choose  $h(P) = r - aP$  where  $a$  is also a positive constant. The equation now becomes

$$P'(t) = (r - aP)P = r \left(1 - \frac{P}{K}\right) P$$

where  $K = \frac{r}{a}$  and is known as the intrinsic growth rate

## 2.5 Exact Equations

Let the functions  $M$ ,  $N$ ,  $M_y$ , and  $N_x$ , where subscripts denote partial derivatives, be continuous in the rectangular region  $R$ :  $\alpha < x < \beta$ ,  $\gamma < y < \delta$ . Then,

$$M(x, y) + N(x, y)y' = 0$$

, is an exact differential equation in  $R$  if and only if

$$M_y(x, y) = N_x(x, y)$$

at each point of  $R$ . That is, there exists a function  $\psi$  satisfying

$$\psi_x(x, y) = M(x, y) \quad \psi_y(x, y) = N(x, y)$$

## 3 Systems of 2 1st Order Equations

### 3.1 Systems of Two First Order Linear Differential Equations

The general system of two first order linear differential equations is

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} P_{11}(t)x + P_{12}(t)y + g_1(t) \\ P_{21}(t)x + P_{22}(t)y + g_2(t) \end{bmatrix}$$

We can also rewrite this in vector form  $\frac{d\vec{X}}{dt} = \vec{P}(t)\vec{X} + \vec{g}(t)$ :

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} P_{11}(t) + P_{12}(t) \\ P_{21}(t) + P_{22}(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

Since we are often looking for a specific solution to a problem, we need to specify the initial conditions.

$$x(t_0) = x_0 \quad \text{and} \quad y(t_0) = y_0$$

### 3.2 Homogeneous Linear Systems with Constant Coefficients

#### 3.2.1 Reducing $\vec{x}' = A\vec{x} + \vec{b}$ to $\vec{x}' = A\vec{x}$

If the coefficient matrix  $A$  is invertible, we know that the only critical or equilibrium point of  $\vec{x}' = A\vec{x} + \vec{b}$  is  $\vec{x}_{eq} = -A^{-1}\vec{b}$ . In these cases, it is convenient to shift the origin of the phase plane to the equilibrium solution using

$$\vec{x} = \vec{x}_{eq} + \vec{\tilde{x}}$$

We can see that  $\vec{\tilde{x}} = \vec{x} - \vec{x}_{eq}$  represents the difference between  $\vec{x}$  and the equilibrium state,  $\vec{x}_{eq}$ . Thus, we get that

$$\frac{d\vec{\tilde{x}}}{dt} = \vec{\tilde{x}}' = A\vec{\tilde{x}}$$

#### 3.2.2 Principle of Superposition

If  $x_1(t)$  and  $x_2(t)$  are solutions to

$$\vec{x}'(t) = A\vec{x}(t)$$

then the expression

$$\vec{x}(t) = C\vec{x}_1(t) + D\vec{x}_2(t)$$

where  $C$  and  $D$  are arbitrary constants, is also a solution

#### 3.2.3 Wronskian and Linear Independence

Suppose that

$$x_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \quad \text{and} \quad x_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

are solutions on some interval,  $I$ . Then  $\vec{x}_1$  and  $\vec{x}_2$  are linearly independent if and only if

$$W[x_1, x_2](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} \neq 0 \quad \text{for all } t \text{ in } I$$

### 3.2.4 The Eigenvalue Method for Solving $\vec{x}' = A\vec{x}$

We will consider a system that will usually be written as

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where the elements of matrix  $A$  are real and the elements of vector  $\vec{x}$  are to be determined.

We know that

$$A\vec{v} = \lambda\vec{v} \quad \text{and} \quad (A - \lambda I)\vec{v} = \vec{0}$$

and that finding the eigenvalues and eigenvectors of this equation will yield our solution

$$\vec{x} = e^{\lambda t}\vec{v}$$

The eigenvalues  $\lambda_1$  and  $\lambda_2$  are found by solving the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - (a_{12})(a_{21})$$

After finding each eigenvalue, they can be used to find the corresponding eigenvectors  $\vec{v}_1$  and  $\vec{v}_2$ . Assuming the eigenvalues,  $\lambda_1$  and  $\lambda_2$  are real and different, the two solutions can be written as

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 \quad \text{and} \quad \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$$

Since the Wronskian of vectors  $\vec{v}_1$  and  $\vec{v}_2$  is non-zero, the general solution is

$$\vec{x}(t) = Ce^{\lambda_1 t}\vec{v}_1 + De^{\lambda_2 t}\vec{v}_2$$

## 3.3 Complex Eigenvalues

Let's consider again the system  $\vec{x}' = A\vec{x}$ . The eigenvalues of this system are the solution to  $\det(A - \lambda I) = 0$  which is the solution to  $(A - \lambda I)\vec{v} = \vec{0}$ . Since  $A$  is real valued, any complex eigenvalues must occur in conjugate pairs (*i.e.* if  $\lambda_1 = \mu + iv$  is an eigenvalue, then so is  $\lambda_2 = \mu - iv$ ).

Considering this case, if  $v_1$  is the corresponding eigenvector to  $\lambda_1$ , then it will satisfy

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

Similarly, by taking the conjugate,

$$(A - \bar{\lambda}_1 I)\vec{v}_1 = \vec{0}$$



Using these two eigenvectors, we determine the solution to the system as

$$x_1(t) = e^{(\mu+iv)t} \vec{v}_1 \quad \text{and} \quad x_2(t) = e^{(\mu-iv)t} \vec{v}_1$$

Using Euler's identity, we can then separate these equations into their real and imaginary parts

$$e^{ivt} = \cos vt + i \sin vt$$

Let's set  $\vec{v}_1 = a + ib$ . We now have

$$\vec{x}_1(t) = (a + ib)e^{\mu t}(\cos vt + i \sin vt)$$

Now by separating the parts, we obtain

$$u(t) = e^{\mu t}(a \cos vt - b \sin vt) \quad \text{and} \quad w(t) = e^{\mu t}(a \cos vt + b \sin vt)$$

We see that

$$\vec{x}_2(t) = u(t) - iw(t)$$

meaning that  $\vec{x}_1(t)$  and  $\vec{x}_2(t)$  are complex conjugates of each other.

The following is the procedure for finding a solution when  $A$  has complex eigenvalues:

1. Identify the complex conjugate eigenvalues  $\lambda = \mu \pm iv$
2. Determine the eigenvector corresponding to  $\lambda_1 = \mu + iv$
3. Express the eigenvector  $\vec{v}$  in the form  $\vec{v} = a + ib$
4. Write the general solution by separating into real and imaginary parts

$$\vec{x}_1(t) = u(t) + w(t) = e^{\mu t}(a \cos vt - b \sin vt) + ie^{\mu t}(a \cos vt + b \sin vt)$$

5. The general solution is then

$$\vec{x}_1(t) = Cu(t) + Dw(t)$$

### 3.4 Repeated Eigenvalues

We continue to look at the system  $\vec{x}' = A\vec{x}$