MIE375
Financial Engineering

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Contents

1 Theory of Interest

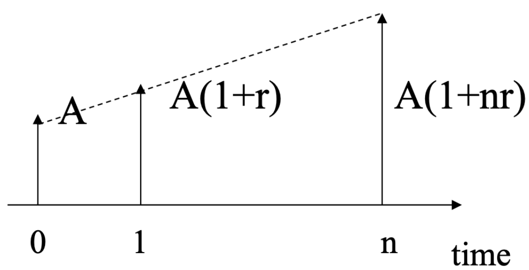
1.1 Principle and Interest

- **Principle:** amount invested
- **Interest:** "rent" paid on an investment
- Amount received after one period is $A(1+r)$
- Two compounding rules: (i) simple (ii) compound

1.1.1 Simple Interest

- Interest is proportional to the time invested
- At time n the value is $A(1+rn)$; money grows linearly
- Not often used in practice

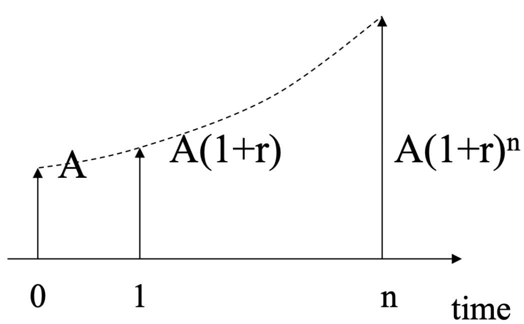
$$V = A(1+rn)$$



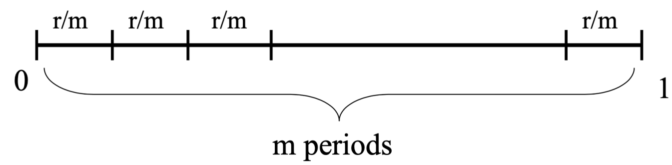
1.1.2 Compound Interest

- Interest on interest
- At time n the value is $A(1+r)^n$; money grows geometrically

$$V = A(1+r)^n$$



- **Rule of 72:** Money invested at $r\%$ a year doubles in approximately $72/r$ years
- **Nominal Interest:** At $r\%$ compounded m periods per year for k periods would cause money to grow by the factor $(1 + \frac{r}{m})^k$



- **Continuous Compounding:** Take the limit of m periods as m approached infinity:

$$V = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m = e^{rt}$$

1.1.3 Effective Interest Rate

- Interest rates usually quotes as nominal rates
- Amount paid depends on compounding periods
- When we compute equivalent rate with a single compounding period over the time interval of interest, we refer to this as the effective rate:

$$r_{eff} = \left(1 + \frac{r_{nom}}{m}\right)^m - 1$$

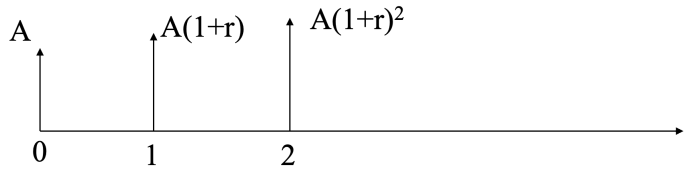
1.1.4 Commonly Referred to Rates

- **Annual Percentage Rate (APR):** Nominal annual rate of interest including all fees and charges
- **Annual Percentage Yield (APY):** Effective annual rate of interest including all fees and charges
- **Discount Rate:** Rate at which member banks may borrow short term funds directly from federal reserve bank
- **Federal Funds Rate:** Rate that banks charge each other for use of federal funds
- **Prime Rate:** Rate that commercial banks charge their most creditworthy borrowers
- **London Interbank Offer Rate (LIBOR):** The interest rate banks charge each other for loans (usually euros)

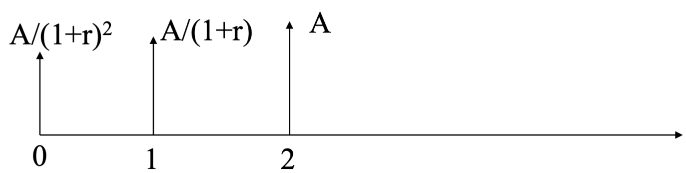
1.2 Present Value and Internal Rate of Return

1.2.1 Ideal Bank

- Applies the same rate of interest to deposits and loans and has no transaction costs
- Two main operations:
  - **Compounding -** Every time a cash flow is moved *forward* one period, it is *multiplied* by $(1+r)$



  - **Discounting -** Every time a cash flow is moved *backward* one period, it is *divided* by $(1+r)$



1.2.2 Present / Future Value

- **Present Value:** Cash flows moved to present time

$$PV = \sum_{k=0}^{n} \frac{x_k}{(1+r)^k}$$

- **Future Value:** Cash flows moved forward in time to the time of the final cash flow

$$FV = PV(1+r)^n$$

- Two cash flow streams are equivalent for an ideal bank if and only if they have the same present value

1.2.3 Internal Rate of Return

- The return on your investment which would have the present value equal to 0
- Let $x = (x_0, x_1, x_2, \cdots x_n)$ be a cash flow stream, the IRR of this stream is a number r satisfying $PV(x) = 0$

$$x_0 + \frac{x_1}{(1+r)} + \frac{x_3}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} = 0$$

2 Fixed Income Securities

2.1 Terminology

- **Financial Instrument:** A legal obligation or claim having a monetary structure *(e.g. stocks, bonds, mortgages, futures, insurance, etc.)*
- **Security:** A tradable financial instrument satisfying legal and regulatory requirements
- **Fixed Income Security:** Securities that promise a fixed income to the holder over some span of time (*e.g. bonds, mortgages, etc.)*

2.2 Annuities

A contract that pays the holder money periodically according to a fixed income

- **Perpetuity:** Pays a fixed sum periodically forever

$$PV = \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{r}$$

- **Finite Life Streams:** Pays an annuity A for n periods

$$PV = \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

  Alternatively, if solving for the value of the annuity :

$$A = \frac{rP}{\left(1 - \frac{1}{(1+r)^n}\right)}$$

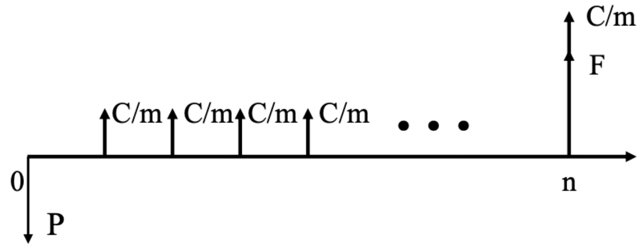
2.3 Bonds

An agreement to pay money according to the rules of the issue

- **Face Value / Par Value / Principle:** An amount to be paid at the maturity date
- **Coupon Payments:** Amount paid periodically expressed as a percentage of face value, last paid at maturity
- **Zero-Coupon Bond:** No coupons are paid out, only payoff is face value at maturity
- Bond price quotes ignore accrued interest, which must be added to the price; must pay the previous owner their portion of the next coupon payment
- Market bond prices are referred to as the *clean* price. When accrued interest is included, then it is referred to as the *dirty* or *cash* price.

2.3.1 Price and Yield

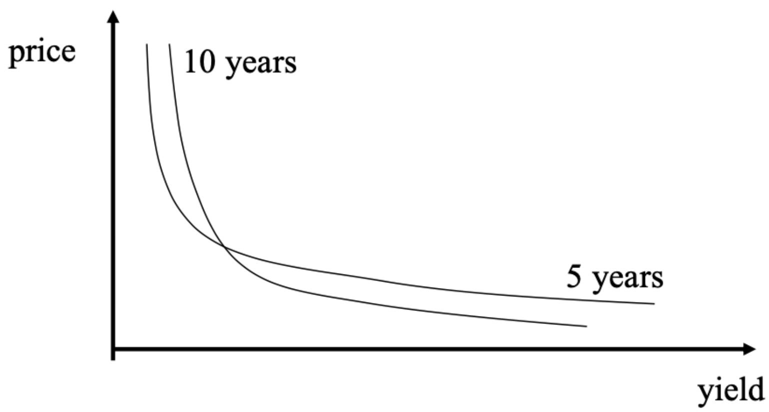
The internal rate of return of a bond cash flow stream is referred to as the yield (λ). The yield captures the "rate of return" that one would obtain by purchasing the bond and holding it to maturity.



We use the following formula to relate the price and yield with n remaining coupon payments:

$$P = \frac{F}{\left(1+\frac{\lambda}{m}\right)^n} + \frac{C/m}{\lambda/m}\left(1 - \frac{1}{\left(1+\frac{\lambda}{m}\right)^n}\right)$$

The longer the time to maturity, the more sensitive the price of the bond is to the yield. We can see this in the Price-Yield curve:



2.3.2 Duration

The maturity time of a bond is related to the sensitivity of the bond price to the yield. With coupons, the maturity time does not exactly correspond to the sensitivity.

We define the **Macaulay Duration**, which is a weighted average of times to cash flows:

$$D = \frac{PV(t_1)t_1 + PV(t_2)t_2 + \cdots + PV(t_n)t_n}{PV_{TOTAL}}$$

This is a weighted average of times, quoted in years. For a coupon bond, *macaulay duration* $<$

maturity date, while for a zero-coupon bond *macaulay duration* = *maturity date*.

The Macaulay Duration for a bond with coupon rate per year, C, yield, λ, periods per year, m, and periods remaining, n, is given by:

$$D = \frac{1+\frac{\lambda}{m}}{\lambda} - \frac{1+\frac{\lambda}{m}+n(\frac{C}{m}-\frac{\lambda}{m})}{C[(1+\frac{\lambda}{m})^n - 1]+\lambda}$$

We then define the **Modified Duration** to calculate the sensitivity of a bond to yield exactly.

$$D_m = -\frac{1}{P(\lambda_0}\frac{dP(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_0} \approx -\frac{1}{P}\frac{\Delta P}{\Delta \lambda}$$

The Macaulay Duration and Modified Duration are then related through the following equation:

$$D_m = \frac{D}{1+\frac{\lambda}{m}}$$

In the case of continuous compounding, $D_m = D$.

Finally, if composing a portfolio of bonds, given fixed income securities with prices P_i, durations, D_i, and weights, $w_i = P_i/P$, $i = 1, 2, \cdots m$, the aggregate portfolio of all of these has price P and duration D given by

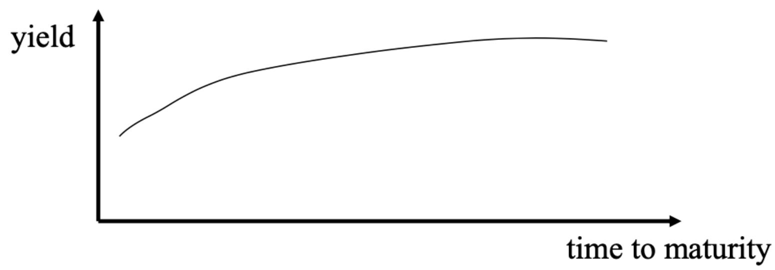
$$P = P_1 + P_2 + \cdots + P_m$$

$$D = D_1 w_1 + D_2 w_2 + \cdots + D_m w_m$$

3 Term Structure of Interest Rates

3.1 Yield Curve

In general, yields on bonds of different maturities are different. If we plot *yield* vs *time* to maturity for bonds, we call the resulting curve the yield curve. The yield curve is not exactly what we want since there may be coupons, etc. We would really like to know the interest rate on a cash flow at time t, with no intermediate cash flows.



3.2 Spot Rates

Interest rates for a specific time period are called spot rates. They can be quted as being compounded (i) yearly, (ii) m periods / year, or (iii) continuously. When computing PV the spot rate corresponding to the time of the cash flow must be used. If we know the price of a zero coupon bond, it's yield will be the spot rate:

$$P = Fe^{-st}$$

There are 3 equations we can use from a portfolio of bonds to determine the spot rate.

$$\begin{aligned} xP_1 + yP_2 &= P_0 \\ xC_1 + yC_2 &= C_0 \\ xF_1 + yF_2 &= F_0 \end{aligned}$$

We solve the system of equations to determine x and y and from there we can find the price and spot rate.

3.3 Forward Rates

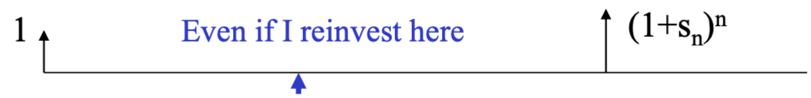
Interest rates for money to be borrowed between two dates in the future, but under terms agreed upon today. f_{t_1,t_2} represents the forward rate between t_1 and t_2

$$(1 + s_i)^i (1 + f_{i,j})^{j-i} = (1 + s_j)^j$$

When we consider rates that only span one period, we refer to those as spot rates

3.4 Expectation Dynamics

We don't know what actual future rates will be when the future time arrives. Using forward rates to predict is called expectation dynamics. According to the **Invariance Theorem**, if interest rates evolve according to expectation dynamics then money invested invested for n years will grow by $(1+s_n)^n$ regardless of strategy.



3.5 Immunization

We construct a portfolio of bonds which is protected against changes in interest rates. The portfolio is constructed with present value P and duration D equal to those of your obligation.

$$xP_1 + yP_2 = P$$
$$x\frac{P_1}{P}D_1 + y\frac{P_2}{P}D_2 = D$$

Solving for x and y we can determine our portfolio. The idea behind this immunization is Taylor expansion

$$P(\lambda) = P(\lambda_0) + P'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2}P''(\lambda_0)(\lambda - \lambda_0)^2 + \cdots$$

4 Applied Interest Rate Analysis

4.1 Dynamic Programming

Running present value is a recursive method for calculating present value. This satisfies the recursive formula:

$$PV(k) = x_k + \frac{PV(k+1)}{1 + f_{k,k+1}}$$

Dynamic decisions are problems where cash flows depend on decisions that will be made at future times. These problems are often similar to running present value calculations, except we have a decision to make at each time. See the fishing problem for an example of how this is used. Remember to "solve the problem backwards"!

4.2 Capital Budgeting

- Deciding on which projects to invest in
- Need to choose a portfolio which maximizes the net present value
- Either select entire project or not at all
- Optimal capital budgeting:
  - $\max \sum_{i=1}^{n} b_i x_i$ (present value)
  - $\max \sum_{i=1}^{n} c_i x_i < c$ (budget constraint)
  - $x_i = 0$ or $x_i = 1$ (all or nothing)

5 Mean-Variance Portfolio Theory

5.1 Probability

Review of some important probability concepts that are used in portfolio theory

- **Expectation / Mean**

$$\bar{x} = E[x] = \sum_{i=0}^{n} x_i p_i$$

- **Variance**

$$VAR[X] = E[X^2] - (E[X])^2$$

$$VAR[aX \pm bY] = a^2 VAR[X] + b^2 VAR[Y]$$

$$\sigma_x = \sqrt{VAR[X]}$$

- **Covariance**

$$COV(X_1, X_2) = E[X_1 X_2] - \bar{x}_1 \bar{x}_2 = \sigma_{12}$$

$$COV(X_1, X_2) = 0 \text{ if } x_1 \text{ and } x_2 \text{ are independent}$$

$$COV(X_1, X_1) = VAR[X_1]$$

$$COV(aX + bY + f, cX + dY + e) = acVAR[X] + (ad - bc)COV[X, Y] + bdVAR[Y]$$

- **Correlation**

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

5.2 Portfolio Returns

Assume there is an asset whose current value is X_0 and its value one year later is X_1. It's rate of return and total return are expressed as:

$$r = \frac{X_1 - X_0}{X_0} \qquad R = \frac{X_1}{X_0}$$

Now, assuming we had a portfolio of n assets, we can describe this through the following table:

Asset	**$ Invested**	**Weights**	**Return**
1	X_0^1	$w_1 = \frac{X_0^1}{X_0}$	r_1
2	X_0^2	$w_2 = \frac{X_0^2}{X_0}$	r_2
...	...	...	...
n	X_0^n	$w_n = \frac{X_0^n}{X_0}$	r_n
$TOTAL$	$X_0 = \sum_{i=1}^{n} X_0^i$	$\sum_{i=1}^{n} w_i = 1$	$r_p = \sum_{i=1}^{n} w_i r_i$

As seen in the table above, the return of the portfolio is given by:

$$r_p = \sum_{i=1}^{n} w_i r_i$$

- The returns, r_i, on the individual assets are random
- r_p is a weighted sum of the random variables r_i
- To determine the expected return of a portfolio, or the variance / standard deviation of the return, must start with this

The expected return of a portfolio is then given by :

$$\bar{r}_p = E[r_p] = \sum_{i=1}^{n} w_i \bar{r}_i$$

The variance of the return of a portfolio is given by:

$$\sigma_p^2 = VAR[r_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

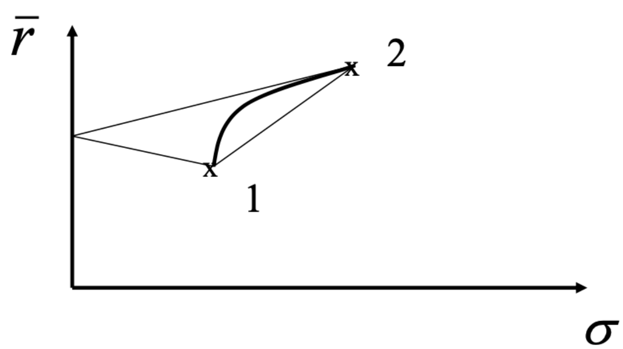
When the assets are uncorrelated, this becomes:

$$\sigma_p^2 = VAR[r_p] = \sum_{i=1}^{n} w_i^2 \sigma_i^2$$

5.3 Short Selling

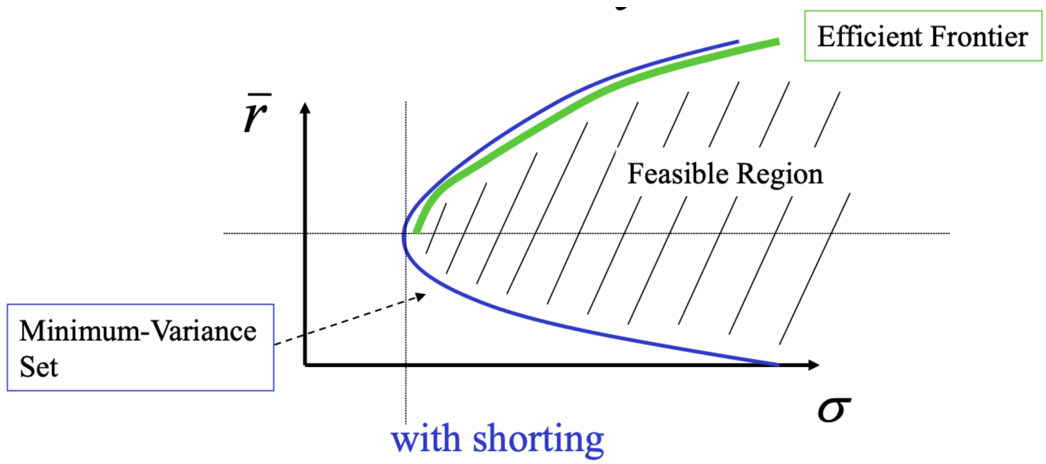
Short selling corresponds to borrowing an asset and selling it with the understanding that you'll buy it back at some point in the future. In portfolio terms, the asset is given a negative weight w_i. Money is made if the price drops.

5.4 Portfolio Diagrams



5.5 Minimum-Variance Set

For a given mean return, you would like to minimize your risk or variance



If you're risk averse. then:

- for a given mean return, you will minimize the variances or "risk"
- always want your portfolio to be on the minimum variance set
- for a given variance, you will maximize your mean return
- hence, you will want to be on the top portion of the minimum variance set, known as the efficient frontier

5.6 Markowitz Theory

Markowitz formulated the problem of being on the efficient frontier as an optimization problem. Assuming there are n risky assets with mean returns $\bar{r}_i, \bar{r}_2, \cdots, r_n$ and covariances σ_{ij} for $i, j = 1 \cdots n$, we want to minimize (1) subject to (2) and (3). Note that we allow short selling and we assume all assets are risky.

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \frac{1}{2} \tag{1}$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p \tag{2}$$

$$\sum_{i=1}^{n} w_i = 1 \tag{3}$$

5.6.1 Solving a General Optimization

Given the following optimization problem:

$$\min f(x, u)$$

$$\text{subject to } g_1(x, u) = c_1 \quad \text{and} \quad g_2(x, u) = c_2$$

We go through the following steps to solve this:

1. Convert all constraints to 0 on the RHS

$$g_1(x, u) - c_1 = 0$$

$$g_2(x, u) - c_2 = 0$$

2. Associate a Lagrange multiplier with each constraint

$$g_1(x, u) - c_1 = 0 \rightarrow \lambda_1$$

$$g_2(x, u) - c_2 = 0 \rightarrow \lambda_2$$

3. Form a Lagrangian by subtracting from the objective each constraint multiplied by its Lagrange multiplier

$$\mathcal{L}(x, u, \lambda_1, \lambda_2) = f(x, u) - \lambda_1(g_1(x, u) - c_1) - \lambda_2(g_2(x, u) - c_2)$$

4. Compute the partial derivatives and set them to 0

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda_1 \frac{\partial g_1}{\partial_x} - \lambda_2 \frac{\partial g_2}{\partial_x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{\partial f}{\partial u} - \lambda_1 \frac{\partial g_1}{\partial_u} - \lambda_2 \frac{\partial g_2}{\partial_u} = 0$$

$$-\frac{\partial \mathcal{L}}{\partial \lambda_1} = g_1(x, u) - c_1$$

$$-\frac{\partial \mathcal{L}}{\partial \lambda_2} = g_2(x, u) - c_2$$

5. Solve the equations for x, u, λ to find the optimal solution

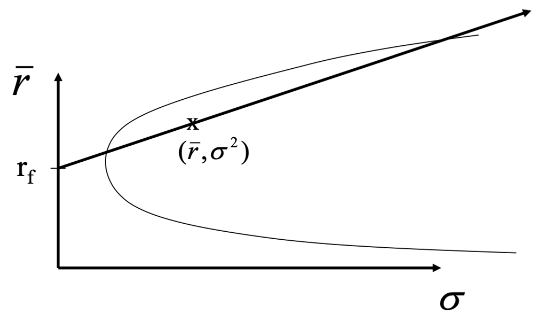
5.6.2 Two-Fund Theorem

Assuming (i) short selling is allowed, (ii) all assets are risky, and (iii) all investors have the same estimates of mean, variance, and covariance, investors seeking minimum variance portfolios need only invest in linear combinations of two minimum variance funds. Only two efficient funds need to exists and everyone else can invest in them!

5.6.3 Inclusion of a Risk Free Asset

Now assume there is a risk free asset with return r_f. We want to combine the risk free asset with the risky portfolio. This results in the following:

$$\begin{aligned} mean &= \alpha r_f + (1-\alpha)\bar{r} \\ variance &= (1-\alpha)^2\sigma^2 \\ standard\ deviation &= (1-\alpha)\sigma \end{aligned}$$



5.6.4 One-Fund Theorem

Assuming (i) short selling is allowed, (ii) all assets are risky, and (iii) all investors have the same estimates of mean, variance, and covariance, there is a single fund F of risky assets, such that any efficient portfolio can be constructed as a combination of the fund F and the risk free asset.

The one-fund is the fund of risky assets that results in the maximum slope with the risk-free rate. The slope is given by

$$slope = \frac{\bar{r}_{fund} - r_f}{\sigma_{fund}}$$

Getting the weights of the one fund theorem is just the following optimization problem:

$$\begin{aligned} (\bar{r}_k - r_f) - \lambda\left(\sum_{j=1}^{n} w_j\sigma_{kj}\right) &= 0 \\ \sum_{j=1}^{n} \lambda w_j\sigma_{kj} &= (\bar{r}_k - r_f) \\ \sum_{j=1}^{n} v_j\sigma_{kj} &= (\bar{r}_k - r_f) \\ w_j &= \frac{v_j}{\sum_{i=1}^{n} v_i} \end{aligned}$$

6 Capital Asset Pricing Model (CAPM)

6.1 Market Equilibrium

A **market portfolio** is defined to be a portfolio of every stock in the market in proportion to the that stock's representation on the entire market. Each assets weight in the market portfolio is

$$w_i = \frac{\$ \text{ value of asset } i}{\$ \text{ value of market}}$$

These weights are called **capitalization weights**.

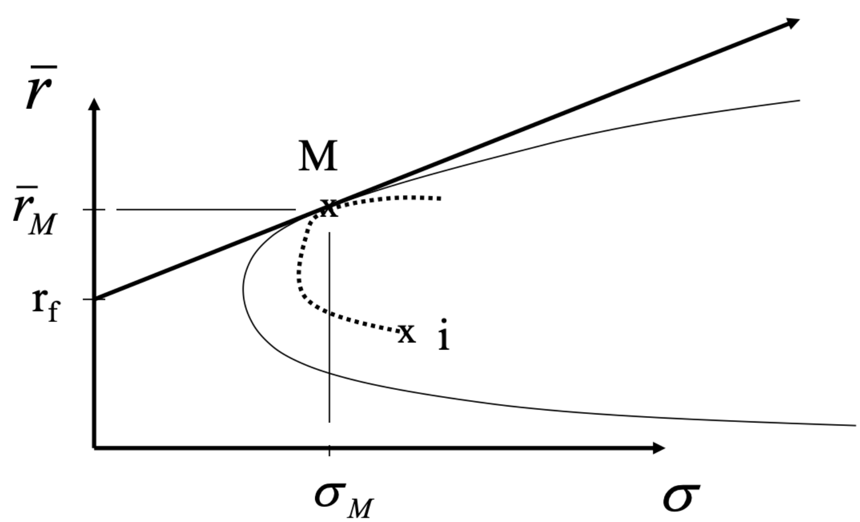
6.2 Pricing Model

The Capital Asset Pricing Model makes the following assumptions:

1. All investors are Markowitz mean-variance investors
2. Short selling is allowed
3. There exists a risk-free asset
4. Investors share the same predictions for means, variances, covariances, etc.

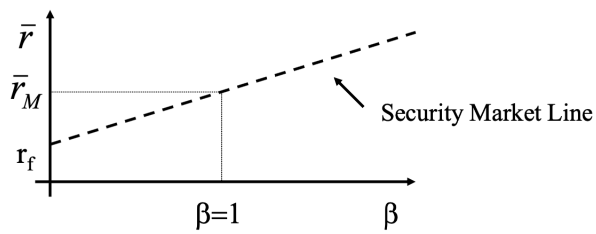
The theorem states that if the market portfolio M is efficient, the expected return r_i of any asset i satisfies

$$r_i - r_f = \beta_i(\bar{r}_m - r_f) \qquad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$



6.3 Security Market Line

- All CAPM securities lie along this line



- The correlation with market β determines the expected return
  - $\beta_i = 0 \rightarrow r_i = r_f + 0 \cdot (\bar{r}_m - r_f) = r_f$
  - $\beta_i = 1 \rightarrow r_i = r_f + 1 \cdot (\bar{r}_m - r_f) = \bar{r}_m$
  - $\beta_i = 2 \rightarrow r_i = r_f + 2 \cdot (\bar{r}_m - r_f) = 2\bar{r}_m - r_f$
- β also determines the movement with the market

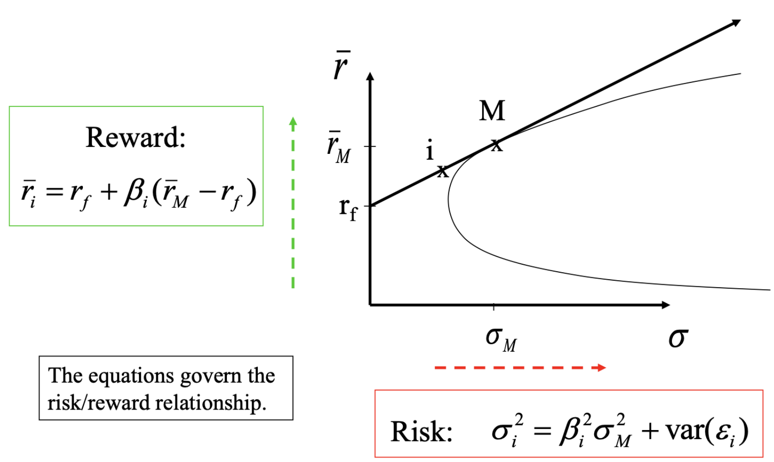
6.4 Risk

The covariance of an asset, r_i can be given as

$$\sigma_i^2 = COV(r_i, r_i) = \beta_i^2 \sigma_M^2 + var(\epsilon_i)$$

This risk is broken up into two components:

- $\beta_i^2 \sigma_M^2 \rightarrow$ systematic risk, associated with the market as a whole
- $var(\epsilon_i) \rightarrow$ non-systematic, idiosyncratic, specific risk uncorrelated with market reduced by diversification



6.5 Implications

- According to our theory, the only portfolio of risky assets you should be holding is the market
- Hence you are only rewarded the (expected return) for risk related to the market
- Risk is measured by β, not the variance of your asset
- Return on an asset is determined by how it fits into the market portfolio, not by its characteristic alone.

6.6 β of a Portfolio

Given a portfolio, $r_p = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n$, the β of the portfolio can be given as

$$\begin{aligned}
\beta_p &= \frac{COV(r_p, r_M}{VAR(r_M)} \\
&= \frac{COV(w_1 r_1 + w_2 r_2 + \cdots + w_n r_n, r_M)}{VAR(r_M)} \\
&= \frac{w_1 COV(r_1, r_M) + w_2 COV(r_2, r_M) + \cdots + w_n COV(r_n, r_M)}{VAR(r_M)} \\
&= w_1 \beta_1 + w_2 \beta_2 + \cdots + w_n \beta_n
\end{aligned}$$

7 Options

7.1 Financial Derivatives

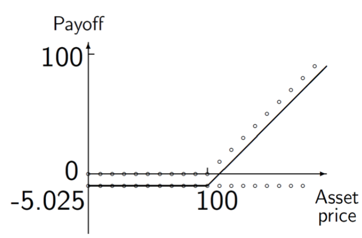
A financial derivative is a financial instrument whose value depends on the values of other more basic underlying financial instruments or variables *(i.e. options)*

7.2 Call Options

A call option is a derivative that gives the owner the right but not obligation to buy a certain asset by a certain date (maturity) for a certain price (strike price). The seller of the call option must deliver the underlying asset at the strike price if owner decides to buy. There are two main types of options:

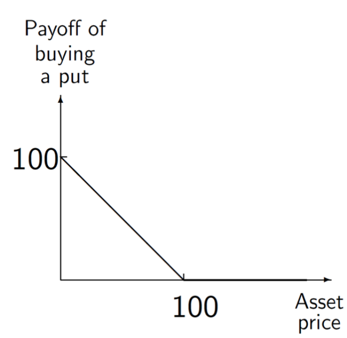
- **American:** Can be exercised at any time in its life
- **European:** Can only be exercised at maturity (we only deal with this in MIE375)

In general, option payoffs are determined by the price of the underlying asset at maturity



7.3 Put Options

A put option gives the owner the right but not the obligation to sell a certain date (maturity) for a certain price (strike price). The seller of the put option must buy the underlying asset at the strike price if the owner decides to sell the asset.



7.4 Exchange Traded vs Over-the-Counter

Options are contracts that are financial instruments. They can be traded just like stocks on an exchange. Custom options can also be constructed over the counter.

- **Exchange Traded:** Mostly electronic trading; contracts are standardized in terms of strike prices and there is virtually no credit risk.
- **Over-the-Counter:** Contracts can be non-standard and there is a small amount of credit risk still

7.5 Types of Option Traders

- **Hedgers:** Protect against adverse movements
- **Speculators:** Take position on the direction of asset

7.6 Option Portfolios

Consider a portfolio of one call option and one put option on the same underlying asset with some maturity and strike price of $100. This strategy is called a **straddle** and has a positive value as long as the underlying asset price at maturity is not a 100.

Another important application is construction portfolio overlays *(e.g. covered call portfolios)*

- **Benefit:** Receive money from selling options (will improve portfolio returns)
- **Risk:** Price of stocks may go up a lot in which case you have to deliver the stocks to the buyers of the call options at lower strike prices.

7.7 Put-Call Parity

$$C + \frac{K}{(1+r)^T} = P + S$$

- $C \to$ price of call option with strike price k
- $P \to$ price of put option with strike price k
- $S \to$ current price of one share
- $T \to$ maturity of call and put (same)
- $r \to$ interest rate