EECS 203: Discrete Mathematics
Natural Deduction Practice Problems

Note: Each of these problems requires only the 12 basic intro/elim rules + the 4 intro/elim rules for quantifiers.

1.

Using Natural Deduction, prove

$$\frac{(p \to q) \land \neg(p \land q)}{\neg p}$$

Solution:

1	$(p \to q) \land \neg(p \land q)$	Premise
2	$p \to q$	$\land -$ elim 1
3	$\neg(p \land q)$	$\land -$ elim 1
4	p	Assumption
5	q	$\to -$elim $2, 4$
6	$p \land q$	$\land -$ intro $4, 5$
7	F	$\neg -$ elim $6, 3$
8	$\neg p$	$\neg -$ intro $4 - 7$

2.

Using Natural Deduction, prove

$$\frac{(p \leftrightarrow q),\ (q \to s), \neg s}{\neg p \wedge \neg q}$$

Solution:

1	$p \leftrightarrow q$	Premise
2	$q \to s$	Premise
3	$\neg s$	Premise
4	q	Assumption
5	s	$\to -$elim $2, 4$
6	F	$\neg -$ elim $3, 5$
7	$\neg q$	$\neg -$ intro $4 - 6$
8	p	Assumption
9	q	$\leftrightarrow -$elim $1, 8$
10	F	$\neg -$ elim $7, 9$
11	$\neg p$	$\neg -$ intro $8 - 10$
12	$\neg p \wedge \neg q$	$\wedge -$ intro $7, 11$

*Note that once we have $p \leftrightarrow q$ and $\neg q$, we cannot directly use a Natural Deduction rule to conclude $\neg p$, but can conclude this using a negation introduction (see steps 8-10).

3.

Using Natural Deduction, prove

$$\frac{\neg p \to j,\, j \to m}{\therefore \quad m \vee p}$$

Solution:

1	$\neg p \to j$	Premise
2	$j \to m$	Premise
3	$\neg(m \vee p)$	Assumption
4	p	Assumption
5	$m \vee p$	$\vee -$ intro(4)
6	F	$\neg -$ elim$(3, 5)$
7	$\neg p$	$\neg -$ intro$(4 - 6)$
8	j	$\to -$elim$(1, 7)$
9	m	$\to -$elim$(2, 8)$
10	$m \vee p$	$\vee -$ intro(9)
11	F	$\neg -$ elim$(3, 10)$
12	$\neg\neg(m \vee p)$	$\neg -$ intro$(3 - 11)$
13	$m \vee p$	$\neg -$ elim(12)

4.

Using Natural Deduction, prove

$$\frac{\forall x(P(x) \rightarrow Q(x)) \text{ and } \forall x \neg Q(x)}{\forall x \neg P(x)}$$

Solution:

1	$\forall x(P(x) \rightarrow Q(x))$	Premise
2	$\forall x \neg Q(x)$	Premise
3	c	
4	$P(c) \rightarrow Q(c)$	$\forall -$ elim 1
5	$\neg Q(c)$	$\forall -$ elim 2
6	$P(c)$	assumption
7	$Q(c)$	$\rightarrow -$elim $6, 4$
8	F	$\neg -$ elim $7, 5$
9	$\neg P(c)$	$\neg -$ intro $6 - 8$
10	$\forall x \neg P(x)$	$\forall -$ intro $3 - 9$

5.

Using Natural Deduction, prove

$$\frac{(P(x) \to Q(x)) \land (R(x) \to S(x))}{(P(x) \lor R(x)) \to (Q(x) \lor S(x))}$$

Solution:

1	$(P(x) \to Q(x)) \land (R(x) \to S(x))$	Premise
2	$P(x) \to Q(x)$	$\land -$ elim 1
3	$R(x) \to S(x)$	$\land -$ elim 1
4	$P(x) \lor R(x)$	Assumption
5	$P(x)$	Assumption
6	$Q(x)$	$\to -$elim $2, 5$
7	$Q(x) \lor S(x)$	$\lor -$ intro 6
8	$R(x)$	Assumption
9	$S(x)$	$\to -$elim $3, 8$
10	$Q(x) \lor S(x)$	$\lor -$ intro 9
11	$Q(x) \lor S(x)$	$\lor -$ elim $4, 5 - 7, 8 - 10$
12	$(P(x) \lor R(x)) \to (Q(x) \lor S(x))$	$\to -$intro $4 - 11$

6.

Using Natural Deduction, prove

$$\frac{\forall x(P(x) \to Q(x))}{\forall x[(Q(x) \to R(x)) \to (P(x) \to R(x))]}$$

Solution:

1	$\forall x(P(x) \to Q(x))$	Premise
2	x_0	
3	$P(x_0) \to Q(x_0)$	$\forall -$ elim 1
4	$Q(x_0) \to R(x_0)$	Assumption
5	$P(x_0)$	Assumption
6	$Q(x_0)$	$\to -$elim $3, 5$
7	$R(x_0)$	$\to -$elim $4, 6$
8	$P(x_0) \to R(x_0)$	$\to -$intro $5 - 7$
9	$(Q(x_0) \to R(x_0)) \to (P(x_0) \to R(x_0))$	$\to -$intro $4 - 8$
10	$\forall x[(Q(x_0) \to R(x_0)) \to (P(x_0) \to R(x_0))]$	$\forall -$ intro $2 - 9$

7.

Using Natural Deduction, prove

$$\frac{\forall x \exists y (P(x) \to Q(y)), \quad \exists x P(x)}{\therefore \quad \exists y Q(y)}$$

Solution:

1	$\forall x \exists y (P(x) \to Q(y))$	Premise 1
2	$\exists x P(x)$	Premise 2
3	$a \quad P(a)$	Assumption
4	$\exists y P(a) \to Q(y)$	$\forall -$ elim(1)
5	$b \quad P(a) \to Q(b)$	Assumption
6	$Q(b)$	$\to -$elim$(3, 5)$
7	$\exists y Q(y)$	$\exists -$ intro(6)
8	$\exists y Q(y)$	$\exists -$ elim$(4, 5-7)$
9	$\exists y Q(y)$	$\exists -$ elim$(2, 3-8)$