Recall Navier-Stokes

$$\frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \bar{\nabla} \bar{V} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \bar{v}$$

LHS - Acceleration terms, first term in LHS is the unsteady term and the second term is the convective acceleration, represents how velocity changes as a point of fluid moves from one place to another. First term of RHS is gradient of pressure, second term represents the viscosity of the fluid. Assume that the fluid is Newtonian, ie stress and strain are linearly related. Another assumption is that the density is constant. The $(\vec{v}\cdot\nabla)\vec{v}$ makes the pde nonlinear. If viscosity $=0$ then we have, Euler

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{\nabla p}{\rho}.$$

CFD is analysing equations numerically to get to velocity field and the associated pressure. CFD is used for vastly different reasons.

Basic ingredients of CFD

1- The mathematical model

↳ Set of partial differential equations and a set of boundary / initial conditions.

Model's target application is incompressible flow, inviscid, turbulent, 2D or 3D. The model always have assumptions

2 - Discretization methods

Method for approximating the PDEs by a system of algebraic expression

$\mathcal{L}(u(\underline{x})) = f(\underline{x})$

↳ $A\underline{x} = \underline{b}$

* finite differences
* finite volume
* finite elements

Big 3

* Spectral methods
* Boundary element method.

Two aspects of discretization

↳ Geometry → Mesh

↳ Model - All mathematical operations to be transformed into arithmetic operations on the grid.

3 <u>Analyse the numerical scheme</u>

All numerical schemes must satisfy a number of conditions to be acceptable: Consistency, stability, convergence, accuracy.

4 Solve

Obtain grid point values of all the flow values

→ Time dependent → ODE

→ Steady — Algebraic system

* Time integrators

* Linear solvers

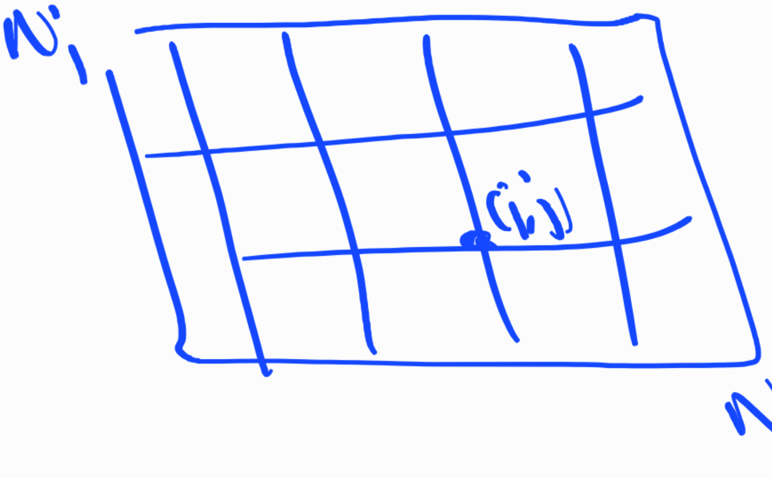
↳ Post-processing, visualization

Lecture 2

The finite difference method

↳ Approximate derivative get from Taylor expansion

↳ Fig. 1, differ grid



- Two families of lines. Grid lines of same family do not intersect. Families differ only intersect once. Standard discretisation scheme.

For each node we have an unknown value of the field variable, which depends on the neighbourhood of nodes providing an algebraic expression.

Definition of a derivative

$$\left(\frac{\partial u}{\partial x}\right)_{x_i} = \lim_{\Delta x \to 0} \frac{u(x_i+\Delta x) - u(x_i)}{\Delta x}$$

Geometrical interpretation

↳ Slope of tangent to curve $u(x)$

↳ Backward difference approximate x_i using x_{i-1}

↳ Forward difference approximate x_i using x_i, x_{i+1}

↳ Central difference, approximate derivative using, x_{i-1}, x_{i+1}.

Some approximations are better than others, quality of information improves as $\Delta x \to 0$.

Finite difference approximation:

Assume Δx uniform

Backward difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x_i) - u(x_{i-1})}{\Delta x}$$

Forward difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x_{i+1}) - u(x_i)}{\Delta x}$$

Central difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x}$$

Taylor expand $u(x)$ near x_i

$$u(x) = u(x_i) + (x - x_i)\frac{\partial u}{\partial x}\Big|_i$$

$$+ \ldots \frac{(x-x_i)^n}{n!}\frac{\partial^n u}{\partial x^n}\Big|_i + \ldots$$

$x \to x_{i+1}$

$\;\;\to x_{i-1}$. get $u(x_{i\pm 1})$ in

term of $u(x_i)$ and it's

derivatives.

$$\frac{\partial u}{\partial x}\Big|_i = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}$$

$$- \frac{(x_{i+1} - x_i)}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i - \frac{(x_{i+1} - x_i)}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i$$

$$- \ldots$$

If $x_{i+1} - x_i$ is small, higher

order terms can be neglected. Then we get the forward difference approximation of order $O(\Delta x)$ truncation errors. As $\Delta x \to 0$, error $\to 0$ and the approximation error converges. Similar results for forward and central, although the error for central difference is Δx^2.

Lecture 3

Definition

The power of Δx with which the truncation error $\to 0$ is called the order of accuracy of the finite-difference approx

FD, BD Both are first order accurate i.e. $O(\Delta x)$

CD accurate to $O(\Delta x^2)$.

FD, BD are one-sided.

Consider 1D domain (0,1)

11 mesh points

$\Delta x = 0.1$

1st order approximation $O(\Delta x)$
$\sim O(10\%)$

2nd order approximation $O(\Delta x^2) \sim O(1\%)$

you need more mesh points if you want to use one-sided difference methods.

Note

first order finite difference can be considered a central difference wrt a midpoint

i.e.

$$\left.\frac{\partial u}{\partial x}\right|_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x^2)$$

gained an order of accuracy using the same formula but considering mid points.

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec{V}$$

* Diffusion term appear as a second order derivative
* Convective fluxes appear as first order derivatives

Various modelling assumption
↳ Model equation yields a system of PDEs where highest spatial derivative is second order and term is 1st order.

Numerical discretisation must be adequate to the physical process.

Lecture 4 (Should set

[illegible]

... for lessons 1–4 in the code).

① 1d, linear convection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad u(x,0) = u_0(x)$$

$$u(x,t) = u_0(x-ct)$$

Behaves similarly to a wave propagation. Relationship between convection and wave propagation. Space-time discretization i - index in x

n - " " t

1- Numerical scheme FD in T

BD in x

Discrete equation

u^{n+1} u^{n} ...

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(c\left(u_i^n - u_{i-1}^n\right)\right)$$

This function gives the function forward in time. So you need an I.C.

I.C, $u = 2$ at $0.5 \leq x \leq 1$

$u = 1$ at $x = 0$, $x = 2$

$u = 1$ everywhere else.

Code this. Pseudocode is available in the lecture series.

② 1D convection

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad \text{Inviscid Burgers}$$

$\frac{\partial x}{} + u \frac{}{\partial x} = 0$, equation

can generate discontinuous behavior for smooth initial conditions

Discretization: FD in x
BD in t

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

I.C. $u = 2$ @ $0.5 \leq x \leq 1$
$u = 1$ everywhere else
$u = 1$ at $(0, 2)$

Code this

Second Order derivatives

use approximation to $\frac{\partial u}{\partial x}$ at 2 locations

Eg Central derivative of

2nd order: Combine FD & BD for 1st derivative. Taylor

FD
$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x}\Big|_i + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i + \ldots$$

$$u_{i-1} = u_i - \Delta x \frac{\partial u}{\partial x}\Big|_i + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i + \ldots$$

Add

Isolate second deriv

$$\frac{\partial^2 u}{\partial x^2}\Big|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \ldots \quad O(\Delta x^2)$$

3: 1D Diffusion (Heat equation)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2},$$ solution known if ν constant

typically $u = \hat{u} e^{i(kx - \omega t)}$, $i\omega = \nu k^2$

$\Rightarrow u = \hat{u} e^{ikx} e^{-\nu k^2 t}$ ← exponential damping

Discretisation, physics of diffusion is isotropic the FD thus is approximate in CD

FD → t

CD → x

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2}$$

$$u_i^{n+1} = u_i^n + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Same IC, BC as prev

$u(x) = 2 \quad 0.5 \leq x \leq 1$

$\quad\quad\; 1$ everywhere else

4 Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n + 2u_i - u_{i-1}^n}{\delta x^2}$$

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) + \nu \frac{\Delta t}{\Delta x^2}\left(u\right.$$

$$\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)$$

IC

$$u = -2\nu \frac{\frac{\partial \phi}{\partial x}}{\phi} + 4 \qquad \boxed{t=0}$$

$$\phi = \exp\left(\frac{-x^2}{4\nu}\right) + \exp\left(\frac{-(x-2\pi)^2}{4\nu}\right)$$

BC, $u(0,t) = u(2\pi,t)$

Analytical solution

$$u = -2\nu \frac{\frac{\partial \phi}{\partial x}}{\phi} + 4$$

$$\phi = \exp\left(\frac{-(x-4t)^2}{4\nu(t+1)}\right) + \exp\left(\frac{-(x-4t-2\pi)^2}{4\nu(t+1)}\right)$$

Big pseudo-code available in video

Lecture 5

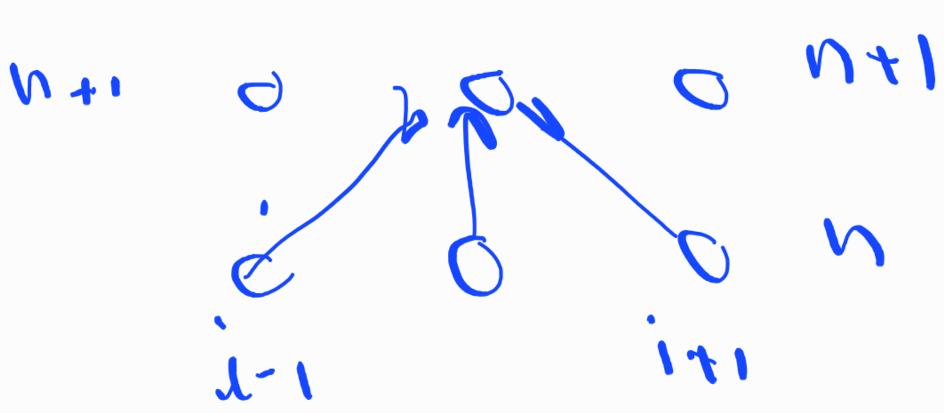
Moving to 2D

Recalling 1D diffusion, FD in time and col in space, which gives only one unknown

↳ Computed values depend only on the past history

Start a solution, IC and 2 BC

A formulation of a continuum equation into a FD eqn. that expresses one unknown is called an <u>explicit method</u>



← Stencil

Suppose 8, n+4. The informa

at the boundaries do not feed into the unknown ever, contrary to the physics.

In an explicit formula the BCs lag behind the computation at one step.

How can we have a scheme that includes the BCs at every time level?

→ BD approximation for the derivative i.e

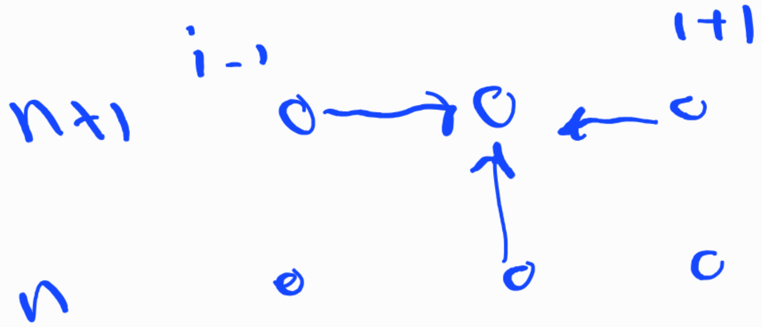
$$\frac{u_i^{n+1} - u_i^{n}}{\Delta t}$$

$$= u_{i+1}^{n+1} - 2u_i^{n+1} - u_{i-1}^{n-1}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

3 unknowns in implicit scheme

Stencil



Need a set of coupled FD equations formed by writing fd formulas for all points. Transpose unknowns on left.

Key

$$\frac{\nu\Delta t}{\Delta x^2}u_{i-1}^{n+1} - \left(1+\frac{2\Delta t}{\Delta x^2}\right)u_i^{n+1} - \frac{\nu\Delta t}{\Delta x^2}u_{i+1}^{n+1} = -u_i^n$$

↳ linear system yielding a tridiagonal matrix

Implicit formulation which includes more than one unknown as an implicit method, $\boxed{\text{look up tridiagonal}}$

Crank - Nicholson method

→ Average of explicit and implicit schemes, for 1D diffusion

$$u_i^{n+1} = u_i^n + \frac{1}{2}\frac{\nu \Delta t}{\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \frac{1}{2}\frac{\nu \Delta t}{\Delta x^2}\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)$$

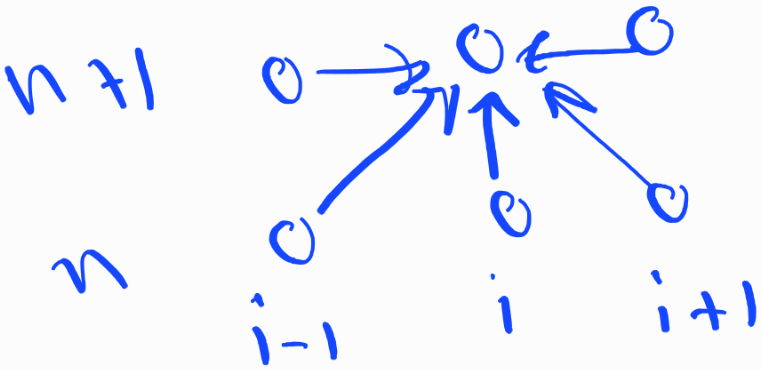
2nd order in time and space.
Implicit → tridiagonal system to solve

to solve:

We noted before that

$\frac{u_i^{n+1}-u_i^{n}}{\Delta t}$ can be a CD approx for a midpoint term $(n+\frac{1}{2})$.

Stencil



In terms of the grid points, we have a central difference representation of $\frac{\partial u}{\partial t}$ at A and the average difference at the same point A.

Lecture 6

To extend 1D to 2D

just apply algebra ... finite difference.

↳ Building a 2D grid defined by the path

$x_i = x_0 + i\Delta x, \quad y = y_0 + \Delta y \cdot j$

$u_{i,j} = u(x_i, y_j)$

Point $(i+1, j+1)$, TS in 2D

$$u_{i+1,j+1} = u_{ij} + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) u\Big|_{ij} + \frac{1}{2}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 u\Big|_{ij} + \ldots$$

1st order

FD at x

$$\left.\frac{\partial u}{\partial x}\right|_{ij} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$

$$\left.\frac{\partial u}{\partial y}\right|_{ij} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y} + O(\Delta x)$$

$$y_{j} \quad \partial y$$

CD

$$\left.\frac{\partial u}{\partial x}\right|_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$

2nd order
CD

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2}$$

2D linear convection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + c\,\frac{u_{ij}^{n} - u_{i-1,j}^{n}}{\Delta x} + c\,\frac{u_{ij}^{n} - u_{ij-1}^{n}}{\Delta y}$$

$$\Rightarrow u_{ij}^{n+1} = u_{ij}^{n} - \frac{c\Delta t}{\Delta x}\left(u_{ij}^{n} - u_{i-1,j}^{n}\right) - \frac{c\Delta t}{\Delta x}\left(u_{ij}^{n} - u_{ij-1}^{n}\right)$$

IC $u = 2$ $0.5 \le x \le 1$, $0.5 \le y \le 1$

$u = 1$ elsewhere

$u=1$ [illegible]

2D convection

$$u_t + u u_x + v u_y = 0$$
$$v_t + u v_x + v v_y = 0$$

#1

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + u_{ij}^{n} \frac{u_{ij}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{ij}^{n} \frac{u_{ij}^{n} - u_{ij-1}^{n}}{\Delta y} = 0$$

#2

$$\frac{v_{ij}^{n+1} - v_{ij}^{n}}{\Delta t} + v_{ij}^{n} \frac{v_{ij}^{n} - v_{i-1,j}^{n}}{\Delta x} + v_{ij}^{n} \frac{v_{ij}^{n} - v_{ij-1}^{n}}{\Delta y} = 0$$

#1

$$u_{ij}^{n+1} = u_{ij}^{n} - \Delta t \left(u_{ij} \frac{u_{ij}^{n} - u_{i-1j}^{n}}{\Delta x} + v_{ij}^{n} \frac{u_{ij}^{n} - u_{ij-1}^{n}}{\Delta y} \right)$$

#2

$$v_{ij}^{n+1} = v_{ij}^{n} - \Delta t \left(u_{ij} \frac{v_{ij}^{n} - v_{i-1j}^{n}}{\Delta x} + v_{ij} \frac{v_{ij}^{n} - v_{ij-1}^{n}}{\Delta y} \right)$$

See IC and boundary conditions.

2D Diffusion

$$u_t = \nu(u_{xx} + u_{yy})$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \nu\left(\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{\Delta x^2}\right)$$

$$+ \nu\left(\frac{v_{i+1,j}^{n} - 2u_{ij}^{n} + v_{i-1,j}^{n}}{\Delta y^2}\right)$$

$= 1$

$$u_{ij}^{n+1} = u_{ij}^{n} + \Delta t\left(\nu\left(\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{\Delta x^2}\right)\right.$$

$$\left. + \nu\left(\frac{v_{i+1,j}^{n} - 2u_{ij}^{n} + v_{i-1,j}^{n}}{\Delta y^2}\right)\right)$$

8) 2D Burgers

1. $u_t + uu_x + vu_y = \underline{\nu}(u_{xx} + u_{yy})$

2. $v_t + uv_x + vv_y = \underline{\nu}(v_{xx} + v_{yy})$

D iscretization

1. $$\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t}+u_{ij}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{ij}\frac{u_{ij}^{n}-u_{ij-1}^{n}}{\Delta y}$$
$$=\underline{v}\left(\frac{u_{i+1j}^{n}-2u_{ij}^{n}-u_{i-1j}^{n}}{\Delta x^2}+\frac{u_{i,j+1}^{n}-2u_{ij}^{n}+u_{ij}^{n}}{\Delta y^2}\right)$$

2. $$\frac{v_{ij}^{n+1}-v_{ij}^{n}}{\Delta t}+u_{ij}^{n}\frac{v_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{ij}\frac{v_{ij}^{n}-v_{ij-1}^{n}}{\Delta y}$$
$$=\underline{v}\left(\frac{v_{i+1j}-2v_{ij}-v_{i-1j}^{n}}{\Delta x^2}+\frac{v_{ij+1}^{n}-2u_{j}+v_{ij}^{n}}{\Delta y^2}\right)$$

Lecture 11

NS: $^{1}\nabla\vec{u}=0$, $^{2}\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{u}$

① Mass conservation for ②, but there is no obvious way to compute velocity and pressure. When compressible we have this.

In 2D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Diff top equation by $\frac{\partial}{\partial x}$, bottom by $\frac{\partial}{\partial y}$ and add

$$\hookrightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y}$$
$$= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \nu\frac{\partial}{\partial x}(\nabla^2 u)$$

$$\hookrightarrow \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial y^2}$$
$$= \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \nu\frac{\partial}{\partial y}(\nabla^2 v)$$

Add them element wise, so the RHS terms

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$+ \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 u}{\partial xy} + v\frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \text{ but } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \boxed{\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2}$$

RHS

$$\boxed{-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)} + v\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] *$$

$$* \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

9 While equation

Poisson equation

$$-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\left(\frac{\partial u}{\partial x}\right)^2+2\,\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\left(\frac{\partial v}{\partial y}\right)^2$$

Roughly equation of following type

$\nabla^2 p = f \Rightarrow$ Poisson equation for pressure, ensures ~~that~~ continuity is satisfied. **Discretising**

Second order central diff

$$\frac{\partial \vec{u}}{\partial t}+\vec{u}\,\nabla\vec{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2 u$$

discretise in time: forward euler in time, explicit.

$$\vec{u}^{\,n+1}=\vec{u}^{\,n}+\Delta t\left\{-\vec{u}^{\,n}\cdot\nabla\vec{u}^{\,n}-\frac{1}{\rho}\nabla p^n+\nu\nabla^2\vec{u}^{\,n}\right\}$$

$$\nabla\cdot\vec{u}^{\,n+1}=\nabla\cdot\vec{u}^{\,n}+\Delta t\left\{-\nabla\left(\vec{u}^{\,n}\cdot\nabla\vec{u}^{\,n}\right)\right.$$

$$-\frac{1}{\rho}\nabla^2 p^{n+1} + \nu\nabla^2(\nabla\cdot\vec{u}^n)\Big)$$

In the numerical scheme we want $\nabla\cdot\vec{u}^{n+1} = 0$. But we may have $\nabla\cdot\vec{u}^n \neq 0$. (fractional step?)

Poisson equation for p at time $n+1$

$$\nabla^2 p^{n+1} = \rho\frac{\nabla\cdot\vec{u}^n}{\Delta t} + \Big\{-\rho\nabla\cdot(\vec{u}^n\cdot\nabla\vec{u}^n) + \mu\nabla^2(\nabla\cdot\vec{u}^n)\Big)$$

Thus if velocity obtained from Navier-Stokes on an intermediate step is $u^{n+0.5}$ and $\nabla\cdot\vec{u}^{n+\frac{1}{2}} \neq 0$

↳ Need to calculate a pressure p^{n+1} such that continuity holds

Lecture 12

① Laplace equation $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=0$

Discretize second order centre difference

$$\frac{p^n_{i+1,j}+2p^n_{i,j}+p^n_{i-1,j}}{\Delta x^2}+\frac{p^n_{i,j+1}-2p^n_{i,j}+p^n_{i,j-1}}{\Delta y^2}=0$$

Transpose p_{ij}^n

$$p_{ij}=\frac{\Delta y^2\left(p^n_{i+1,j}+p^n_{i-1,j}\right)+\Delta x^2\left(p^n_{i,j+1}+p^n_{i,j-1}\right)}{2(\Delta x^2+\Delta y^2)}$$

I.C $\Rightarrow$ $p=0$ everywhere in $(0,2)\times(0,1)$

BC $\Rightarrow$ $p=0$ at $x=0$

$p=y$ at $x=2$

$\frac{\partial p}{\partial y}=0$ at $y=0,1$

Analytical solution

$$p(x,y)=\frac{x}{4}-4\sum_{\substack{n=1\\ \text{odd}}}\frac{\sinh(n\pi x)\cos(n\pi y)}{(n\pi)^2\sinh(2\pi n)}$$

and

① 5 point stencil, typical for Diffusion. diffusion is isotropy

⇒ have to be central difference to match physics

② 2nd order CD widely used scheme for Laplace operator.

⑧ iterative method for steady state, so we need to iterate.

(pseudo Jacobi method).

⑩ Poisson equation $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$

$$\frac{p^n_{i+1,j} - 2p^n_{ij} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{(\Delta y)^2}$$

$$= b^n_{ij}$$

$$\Rightarrow p^n_{ij} = \Big((p^n_{i+1,j} + p^n_{i-1,j}) \Delta y^2 + (p^n_{i,j+1} + p_{i,j-1}) \Delta x^2$$

$$-b_{ij}^{n}\,\Delta x^2 \Delta y^2 \Big)$$

$$\overline{2(\Delta x^2 + \Delta y^2)}$$

IC $p = 0 \ \forall \ x, y \in (0,2) \times (0,1)$

BC $=$ $p = 0$ @ $x = 0, 2$ and $y = 0, 1$

$b_{ij} = 100$ @ $i = \frac{nx}{4}$, $j = \frac{ny}{4}$

$b_{ij} = -100$ @ $i = \frac{3}{4} nx$, $j = \frac{3}{4} ny$

$b_{ij} = 0$ else.

vi. Navier Stokes Cavity. (Basically Burgers)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)$$

$$\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} \quad \frac{\partial x}{} \right)$$

$$+\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Discretized

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{ij}^{n}\frac{u_{ij}^{n}-u_{i-1,j}^{n}}{\delta x}+v_{ij}^{n}\frac{u_{ij}^{n}-u_{i,j-1}^{n}}{\Delta y}$$

$$=-\frac{1}{\rho}\frac{p_{i+1,j}^{n}-p_{i-1,j}^{n}}{2\Delta x}+\nu\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^{2}}\right.$$

$$\left.+\frac{u_{i,j+1}^{n}-2u_{ij}+u_{i,j-1}}{\Delta y^{2}}\right)$$

$$\frac{p_{i+1,j}^{n}-2p_{ij}^{n}+p_{i-1,j}^{n}}{\Delta x^{2}}+\frac{p_{i,j+1}^{n}-2p_{ij}+2p_{i,j-1}}{\Delta y^{2}}$$

$$=-\rho\left[\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2\Delta x}\right)^{2}+2\left(\frac{u_{ij+1}-u_{ij-1}}{\Delta y}\right)\left(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2\Delta x}\right)\right.$$

$$\left.+\left(\frac{v_{i,j+1}^{n}-v_{ij-1}}{2\Delta y}\right)^{2}\right]+$$

$$\rho\frac{1}{\;}\left(u_{i+1,j}^{n}-u_{i\ldots}^{n}\quad u^{n}\quad u^{n}\right)$$

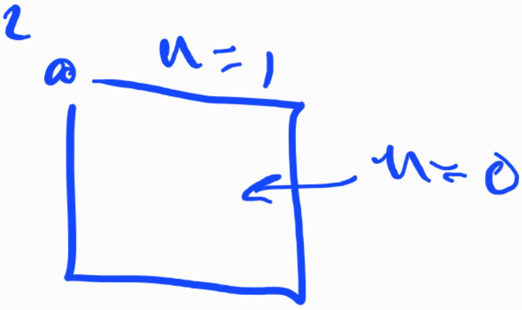
$$\frac{1}{\Delta t}\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right)$$

Transpos $u^{n+1}_{i,j}, v^{n+1}_{ij}, p^{n}_{ij}$

IC $u,v,p=0$

BC $u=1$ at $y=2$, $u,v=0$ $x=0,2$

$y=0$



$p=0$ at $y=2$

$\frac{\partial p}{\partial y}=0$ at $y=0$

$\frac{\partial p}{\partial x}=0$ at $x=0,2$,

(Basically Poisson and 2D Burger's.)

(12) same as 11 but F is in equation for u

IC $u,v,p=0$ everywhere

BC $\quad u, v, p$ periodic at $0, L$

$u, v = 0$ @ $y = 0, 2$

$\frac{\partial p}{\partial y} = 0$ @ $y = 0, 2$