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# AK LV - Risk Mangement for Life Insurance

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# Contents

# Chapter 1
# What is Risk Management

## 1.1 Introduction

### *1.1.1 Raison d'être of Risk Management*

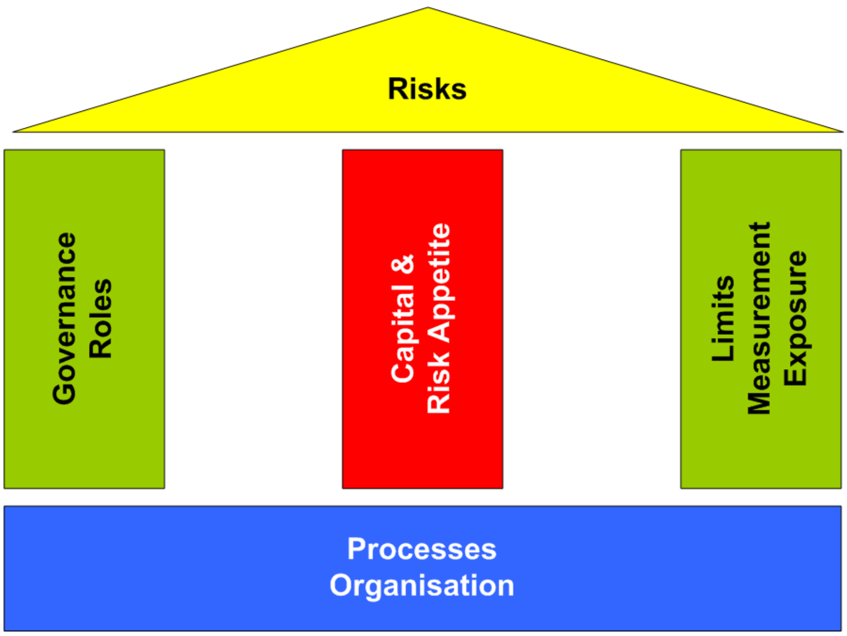



**Fig. 1.1** Overview

In order to understand the need for risk management it is necessary to look at the different building blocks of a holistic risk management. Figure 1.1 tries to decompose the risk management into its generic components. The overall aim is to manage the risks a company is facing. All employees in the company are expected to some larger or smaller extent to manage risks in order to limit a potentially adverse outcome and to generate profit and stability for the stakeholders of the company. The corresponding risk culture is of paramount importance. An open communication and a clear and unbiased view in respect of the risks are essential in order to become a professional risk-taker.

In order to control these risks it is first necessary to analyse and categorise the risks into its components. This decomposition is described in section 1.2.1.

After understanding the risks and their impact on the stakeholders, it is essential to understand what are the foundations and the pillars which allow us to operate in such a way that we manage risks in an optimal manner.

The foundations of each company are its organisation and its processes. Therefore it is necessary to define the relationship between these foundations and the risks. The second foundation of each organisation are its processes. Here the escalation processes (section **??**) are particularly important, since they define how to behave in the case a risk "gets out of control" and "needs to be fixed". The generic risk management process, in chapter **??**, helps to better analyse the risks in a consistent way and hence ensures better communication, understanding and analysis within the insurance company.

The three pillars which ensure that risks are taken in a conscious and value enhancing way are:

- Governance and roles,
- Capital and risk appetite, and
- Measurement, limits and exposure.

Each of the pillars has a particular purpose:

**Governance and Roles:** In order to ensure a "fit and proper" management it is essential to have adequate governance in place. A common understanding of the various parts within the organisation is of paramount importance. The corresponding definitions, that cover part of the roles and responsibilities are documented in section **??**. Together with the generic governance principles (section **??**) they form the corner stones of the company's governance structures. The governance structures are further detailed in section **??**.

**Capital and Risk Appetite:** Risk can be defined as a potential adverse outcome, and it can normally be measured in monetary terms. The capital resources available to the company serve as a buffer in order to limit the need for fresh capital to prevent bankruptcy. Hence it is of utmost importance to know the available resources (which can serve as buffer) and to ensure that the risk appetite is commensurate with the company's strategic aims (e.g. rating, capital level, etc.) and the limits imposed by its stakeholders (Board of Directors, regulators, etc.). This relationship is documented in chapters 2, 5 and 13.

**Measurement, Limits and Exposure:** The last pillar defines how to measure risk. This is particularly important in order to have reliable information for knowing the actual risk profile. To ensure that the company operates within its risk appetite, some of the risks are limited by a *limit system*. An example could be, that the company does not want to invest more than 10% of its assets in shares. How this is done, is documented in chapters 6 and 7.

Having all the before mentioned parts in place, means that the insurance company is a professional risk taker, which aims to outperform the market and its peers. This can *only* be achieved if everybody is responsible for risk management. The risk management function acts as a enabler and consolidator.

### 1.1.2 The role of Risk Management

The role of risk management can be summarised as follows:

- To ensure risk appetite is clearly articulated for all risk categories.
- To ensure the businesses operates within the established risk appetite through monitoring and controls.
- To ensure the level of capital held in the balance sheets is compatible with the risks taken.
- To ensure efficient capital structures operate within the business.
- To ensure compliance with risk policies.
- To ensure an efficient process is in place to identify emerging issues and risks.
- To help mitigate the risks which are outside the risk appetite.
- To define methods and processes to measure the available and required risk capital.

### 1.1.3 Three lines of defence in Risk Management

Finally, it needs to be stressed that risk management is not only carried out by the risk management function, but by the whole organisation.

The organisation can be split into the so called three lines of defence.

**First line of defence:** The *line management* as first line of defence is of paramount importance in risk management, because this function is essentially responsible for ensuring that all processes in the business adhere to the *risk management policies* and that the company operates within the *limits* as agreed upon by the Board of Directors and the executive.

**Second line of defence:** The *risk management function* lead by the Chief Risk Officer is the second line of defence. It has the duty to provide a reliable challenge to the first line of defence and it measures the necessary risk capitals and independently monitors the adherence to limits and appetite. In case of limit breaches it initiates together with the first line of defence mitigating actions. The *risk management function* is also responsible for the various risk committees and risk reporting.

**Third line of defence:** *Internal audit* is the third line of defence. Its main task in respect to risk management is to provide independent assurance to the Board of Directors and the senior executive that the risk management processes are adequately working within the first and second lines of defence.

## 1.2 Principles

The aim of this section is to define the generally applicable operating principles which are used within the company to ensure adequate and efficient risk management. These principles define on a high level the main risk categories and risk management principles and it is expected that the whole organisation adheres to them.

### *1.2.1 Risk Categories*

In order to have a systematic approach to measure, limit and to mitigate the risks the insurer is facing, a so called risk landscape has to be created. The main aim is to have a structured and uniform approach towards risks. Such a landscape normally takes the form of a tree where risks become more and more granular. The depth of the branches corresponds to the level of the model.

Each risk can be characterised by its impact (*severity*) and its probability (*frequency*). Furthermore we speak of *inherent risk* if we look at it before any dedicated controls or mitigating actions are put in place. We speak of a *residual risk* if we measure it taking into account the existence and effectiveness of controls.

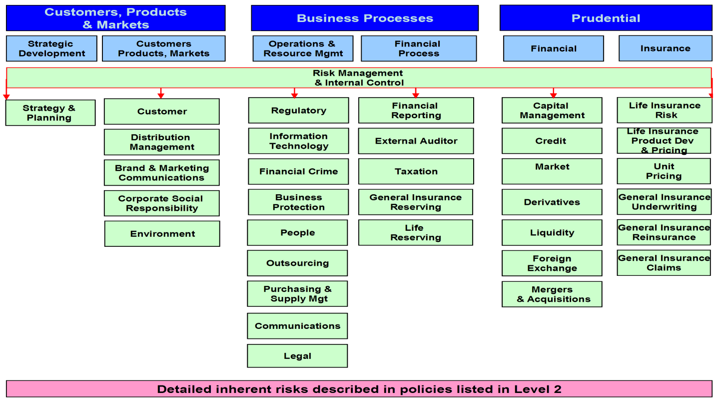



**Fig. 1.2** Risk Map

Distinguishing *inherent risks* and *residual risks* is necessary in order to know whether a certain control is efficient or sufficient in order to limit a risk to an adequate level. Obviously the full elimination of a risk by using a lot of mitigating actions might not be optimal in the sense that the corresponding costs for the mitigating actions could outweight the potential loss. Hence it is essential to have a commensurate risk appetite which takes this into consideration.

Risk is defined as the potential danger that an actual result will deviate (adversely) from the expected result. Risk is measured according to probabilities and the extent of negative deviations. Risk is defined as:

**The magnitude of a risk** expressed in terms of impact and probability before any dedicated controls or mitigating actions are put in place or assessed on the basis that the dedicated controls and mitigating actions in place fail.

**The impact and probability** of an *inherent risk* taking into account the existence and effectiveness of controls.

Risks are measured and assessed in financial terms, provided that this is both possible and appropriate. To weigh up and compare various risks, risk management ratios will be defined, providing consistent information on the probabilities and extent of negative deviations. Risks which cannot be directly quantified (especially operational and strategic risks) are also to be systematically recorded and represented in an appropriate form.

In order to identify, measure and limit certain risks, a systematic approach is needed. In a first step risks are categorised according to a risk map (figure 1.2). Examples of specific risks within the individual categories are:

| | |
|---|---|
| **Market risks** | ALM or gap risk, interest rate risk, equity risk, currency risk, real estate risk, commodity risk, etc. |
| **Liquidity risks** | Market liquidity risk, funding risk. |
| **Credit risks** | Counter-party risk, country risk, concentration risk, risk of rating changes, etc. |
| **Insurance risks** | Death, disability, longevity, illness, etc. |
| **Operational risks** | Distribution risk, financial crime, legal risk, reputation risk, business protection risk, HR risk, loss of expertise, etc. |
| **Strategic risks** | Risk of pursuing the wrong strategy or of being unable to implement the strategy (e.g., market access). |

## 1.3 Risk Management Process

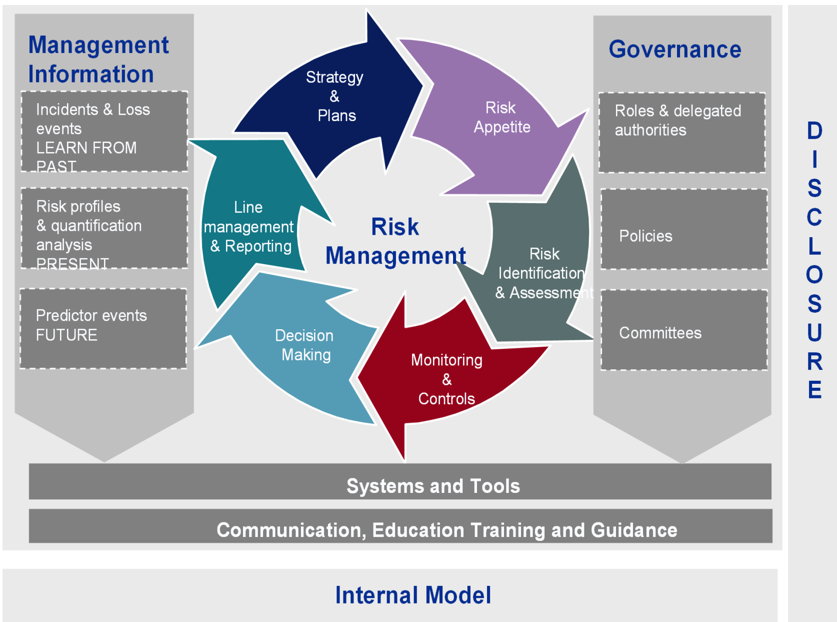



**Fig. 1.3** Risk Management Process

Figure 1.3 defines the generic risk management and controlling process:

**Strategy and plans:** This is the first step of this generic and cyclic process where, based on risk and reward, a strategy is determined in order to optimise return to shareholders on a risk adjusted basis. Implicit to this task is the high level risk measurement and capital consumption of a certain strategy. This part of the process is owned by the *risk owners*. Risk management information should be used to provide insight, inform the operational planning process and influence resource allocation including capital. Businesses must ensure that changes to their risk profile including control effectiveness are explicitly considered within strategy setting, business planning, objective setting and performance monitoring.

**Risk appetite:** Based on the plans and the high level risk and capital allocation, the risk appetite is defined and risk limits are set. This process is governed by the risk committee and the owner of this process step is the *risk owner*. Risk appetite statements and tolerances should be clearly defined and refreshed on a regular basis and as an integral part of the planning process. Risk appetite should be defined for a business as usual situation within an established business and also needs to be sufficiently flexible to deal with a variety of situations (e.g. rapid market expansion, managing significant change) and should support rather than constrain sensible risk taking to deliver business objectives.

**Risk Identification and Assessment:** As next step there is a detailed risk analysis comprising risk identification and assessment. This step, owned by the different *risk experts* and the *risk function* ensures that all risks are properly captured within the systems and processes of the company. Furthermore it ensures that material risks can be quantified adequately with high quality. All material inherent risks must be identified, assessed and recorded. Controls to mitigate each material inherent risk must be documented and assessed for their adequacy and effectiveness in risk mitigation in order to produce a residual risk assessment which is within appetite. The risk model must be used as the basis for considering all types of risk.

**Monitoring and Controls:** The agreed risk limits are entered into the models monitoring the risk and controls in order to ensure a timely detection of limit and control breaks. This part of the process is owned by the *risk function*.

**Decision Making:** During the year risks are taken according to the policies defined by the company. *Line management* is responsible for adherence to the risk policies and ensures the management control of them. By doing business *line management* ensures the embedding of the *risk management policies* and adherence to the limits granted. In case of limit breaks, the corresponding processes are initialised. During this task *line management* optimises the risk return profile and hereby generates value for the company and its shareholders. *Risk management* supports the *line management* by regular risk reporting and reports limit consumption and limit breaches to the *risk owners*.

**Line Management and Reporting:** This last step of the process ensures a proper feed-back over the cycle, by assessing the performance on a risk adjusted base. Risk adjusted returns and limit breaches are prepared by the *risk function* and are reported to the *line management* and the *risk owners*. This information serves as input to management remuneration and the strategy and planning process.

Risk is measured in two dimensions: frequency (likelihood) and severity (impact). The impact can be one of the following in decreasing order: catastrophic, critical, significant and important. Each level of impact corresponds to a monetary amount, which depends on the entity. The bigger the entity the higher the corresponding threshold. The probabilities in decreasing order of likelihood are: likely to happen, possible, remote and extremely remote. Based on the assessment of frequency vs severity the overall risk can be expressed in a more holistic way. The figure 1.4 shows a such an overview:

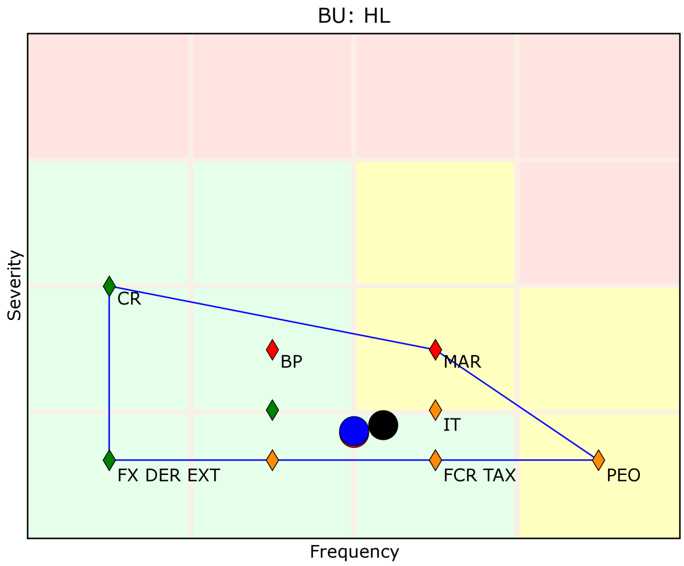



**Fig. 1.4** Frequency vs. Severity

In figure 1.4, the frequency of the emergence of a risk is plotted and each of the four categories has a specified probability. Similarly the potential impact is classified into one of four categories. Depending on frequency and severity each of the 16 squares is allocated to one of the three colour codings: green, amber and red. In a next step all policies are mapped against this grid and plotted. The label “PEO” refers for example to people risk, which has in this example a high probability of materialisation with a rather low impact. This policy has been evaluated as amber and hence the marker is amber. In the same sense one can see the market policy (“MAR”), the credit policy (“CR”) etc. In order to have an overall picture the circles represent the barycentre of the assessments, in black for the past reporting period, and in blue for the current. From this it can be seen that the over all risk moved slightly south–east, hence has reduced a little.

## 1.4 Risk Management policies and Risk Landscape

In order to define minimal standards on how different risks have to be treated and define minimal governance standard, insurance companies codify the corresponding rules and responsibilities in terms of risk management policies which cover risks, which belong together. The following list provides a quite complete list of risk management policies. Obviously there are different ways on how one can arrange these policies.

**Brand & Marketing Communications:** This risk management guideline describes the risk which is intrinsic in the brand and marketing communication process. Here the main aim is to safeguard the company's reputation and to ensure that the communication is aligned with the core values of the company.

**Business Protection:** Business protection aims to protect the orderly running of the business and is therefore concerned with things such as physical security, IT security, data recovery, business continuity in case of a damage of a property or in case of a pandemic etc. Hence the aim of the policy is to define the limits of acceptable risk with respect to these topics.

**Capital Management:** The aim of this guideline is to define the processes and the risk appetite the company has in respect to capital management, hence in respect of levels and quality of capital. Here also the process of raising capital, paying dividends and the risk appetite of becoming insolvent is anchored. One could for example state: "There is no risk appetite that the statutory capital level falls below 120%."

**Communications:** Communications cover both internal and external communication. Here it is defined how information is treated and who is allowed to communicate internally and externally. The corresponding risks are unhappy employees because of bad internal communication, or externally: reputational issues and communication leaks.

**Credit:** This is the financial credit risk, where credit migration, credit spread and default risk is addressed. Furthermore guidance is given in respect to concentration limits and the processes used in order to ensure the company operates within a given risk appetite. Hence some of the requirements limit financial risks and others aim to address operational (risk) issues.

**Customer:** One of the big reputational and regulatory risks of each insurance company is the relationship vis-a-vis the customers. Here it is important to define what "treating customers fairly" means and how the corresponding risk appetite is defined. In consequence governance rules are established in order that the company operates within these boundaries.

**Derivatives:** Since derivatives imply a much higher (operational and financial) risk than "normal" assets, it is important to define the corresponding governance processes in a stringent and efficient way. Hence this guideline addresses, besides the pure financial risk, also the important operational procedures and hence aims to limit also the operational risk.

**Distribution Management:** The distribution management policy aims to limit the risks which are induced by the insurer's distribution network. Here risk appetite and processes are set in respect to the quality of people acting as distributors, turnover of distribution managers, remuneration schemes etc.

**Environment:** Here the company states its risk appetite with respect to environmental issues, such as energy consumption etc.

**External Auditor:** As a consequence of Enron and Worldcom, the attitude vis-a-vis accounting has become much more stringent and most companies have no appetite to make accounting errors and a lot of them have also implemented quality standards such as SoX 404. This guideline defines the relationship towards the auditors and states which behaviours are not acceptable and which services may not be taken from the own external auditor.

**Financial Crime:** The financial crime policy states the required behaviour in respect to financial crime, such as fraud, money laundry etc. Most companies do not have the slightest appetite for financial crime and hence these guidelines are normally very prescriptive and restrictive.

**Financial Reporting:** The financial reporting guideline needs to viewed as a companion of the external auditor guideline with the aim to reduce errors and omissions with respect to financial reporting down to an acceptable (low) level.

**Foreign Exchange:** The FX guideline is also one of the financial risk guidelines and it has the same aim as all of these guidelines, namely that the company operates within a well defined risk appetite. In consequence the limit setting, monitoring and reporting processes are of utmost importance.

**GI Claims:** This guideline governs the GI claim processes and defines which measures have to be taken, to prevent fraud and to treat customers fairly. Obviously the claim settlement process for GI claims is of utmost importance, because there is a narrow margin between being too onerous and being too strict. As a consequence we speak here about operational risk, which has a direct financial impact.

**GI Reinsurance:** Since a lot of GI lines of business are heavily re-insured (say some 25% of the total GI premiums), it is important to have a clear guidance which level of risk is still acceptable and which risks need to reinsured. Besides the insurance risk (such as windstorm, earthquake, ...), it is important to recognise there is also credit risk involved, since reinsurers also might default. Hence a balanced reinsurance portfolio is important in order to avoid severe problems in case of a reinsurer default. In the reinsurance risk guideline the risk appetite is not only relevant quo lines of business but also quo counterparties.

**GI Reserving:** Looking at the balance sheet of a GI insurer it becomes obvious that a large part of the balance sheet consists of claim reserves. Hence it is important to have a clear risk appetite in order to ensure on one hand adequate reserves, which are on the other hand, not too onerous. Furthermore the GI reserving process involves, besides actuarial techniques, also considerable judgement. Hence in the light of financial reporting risk it is important to have rigid and robust processes in place.

**GI Underwriting:** The GI Underwriting guideline can be considered as a companion guideline to the GI claims guideline covering the underwriting process. A stringent process is needed in order to ensure an adequate portfolio quality. Let's assume for the moment that a company would attract all "bad" risks. In this case the company would obviously suffer because of an inadequate pricing. Hence also the GI pricing is anchored in this guideline.

**Information Technology:** Information technique per se is a vast topic and the corresponding intrinsic risks are big. This guideline steers the risk appetite in respect to IT risks, such as infrastructure, IT projects etc.

**Legal:** The legal risk policy speaks about the company's attitude in respect of legal issues, litigation etc. Here it is important to allocate the responsibilities and duties accordingly. This is in particular relevant when entering into a litigation or s settlement of a claim. The legal risk policy does not only cover the risks the corporate faces, but also risks which are consequences of disputed life and in particular GI claims.

**Life Insurance Product Development & Pricing:** As we will see in chapter 14 the product development and product pricing process for life insurance policies is a difficult one. As a consequence of the typically big volumes and long contract terms (20 years and more) and the fact that issues become costly quite easily. It is of paramount importance to have a clearly defined risk appetite in respect of product development and pricing, and corresponding robust governance processes. It is also important to recognise that besides the pure financial risks there are also significant operational risks which can materialise in ill-designed products.

**Life Insurance Risk:** This guideline covers the risk appetite of the pure technical insurance risks which are, for example, mortality, disability, surrender etc. In order to operate within a well defined risk appetite these technical risks are to be limited with a limit system.

**Life Reserving:** This is the companion guideline of the "GI Reserving".

**Liquidity:** Liquidity risk guideline governs the process to monitor liquidity and to ensure that the company has always enough liquidity to fulfil its obligations. This guideline is also one of the financial risk guidelines.

**Market:** From all the financial risk guidelines, this is the most important, covering the market risk of all financial assets (such as equities, bonds, hedge funds etc.) and the corresponding ALM risk if also taking the liabilities into consideration. In consequence governance, limit systems, escalation processes, risk mitigation and risk measurement play an important role in this guideline. Only if these building blocks are robust and accurate is it possible to operate in a well defined environment, taking risks in a conscious manner.

**Mergers & Acquisitions:** This guideline sets the risk appetite and standards for M&A processes. It is known that these processes are difficult and can lead to substantial problems if done in an inappropriate manner. Hence it is important to have a stringent guideline describing processes, governance arrangements and risk appetite.

**Outsourcing:** The Outsourcing guideline defines the risk appetite and the protocols to follow in case of outsourcing arrangements. Obviously it is the aim of such a guideline to limit the corresponding operational and counter-party risks.

**People:** For all financial institutions there are two main resources needed: capital and people (human capital). It is very important to clearly articulate the risk appetite in respect to people to ensure the attractiveness to key performers and to ensure an adequate turnover to get new talent on board.

**Purchasing & Supply Management:** See “Outsourcing”.

**Regulatory:** This guideline can be compared with the “External Auditor” guideline since it defines the risk appetite in respect to the different regulators of the company.

**Risk Management & Internal Control:** This guideline defines how risk management works in the corporate environment and covers many issues and questions of this book.

**Strategy & Planning:** Looking at the main processes of an insurance company, the strategy and planning process is particular, since it defines what the company will do in the following year. It is also known that ill-behaved strategies are one of the root causes for corporate failures. Hence it is important to also control this process and strategic planning in a environment with a well defined risk appetite.

**Taxation:** This is the companion guideline of “Regulatory” vis-a-vis the tax authorities.

Whereas the risk management policy view is efficient for managing the company another view is needed in order to decompose the risks in their generic risk factors. Assume, for example, credit risk. This risk factor influences more than one risk covered in one of the risk management polices, such as “Credit Risk”, “Reinsurance”, “Customer”, “Outsourcing” etc. Whereas the link is clear for “Credit Risk”, the relationship is not always as straight forward. The following table summarises these relationships:

| **Policy** | **Relationship** |
|---|---|
| Credit | Via the credit default and credit mitigation risk of bonds and mortgages. |
| Reinsurance | Via the counter-party credit risk of the reinsurance treaties and insurance linked securities. |
| Customer | The reputational issue if the customer suffers in case of a credit default independently on whether the insurance company bears the risk or not. |
| Outsourcing | Via the operational risk, which is induced by the default of an outsourcing partner. |

As a consequence it is necessary to decompose the risk-universe into its drives. This map is called *risk landscape*. Also here it is possible to have a coarser or finer view on the risk landscape. Figure 1.5 shows a quite high level risk landscape.

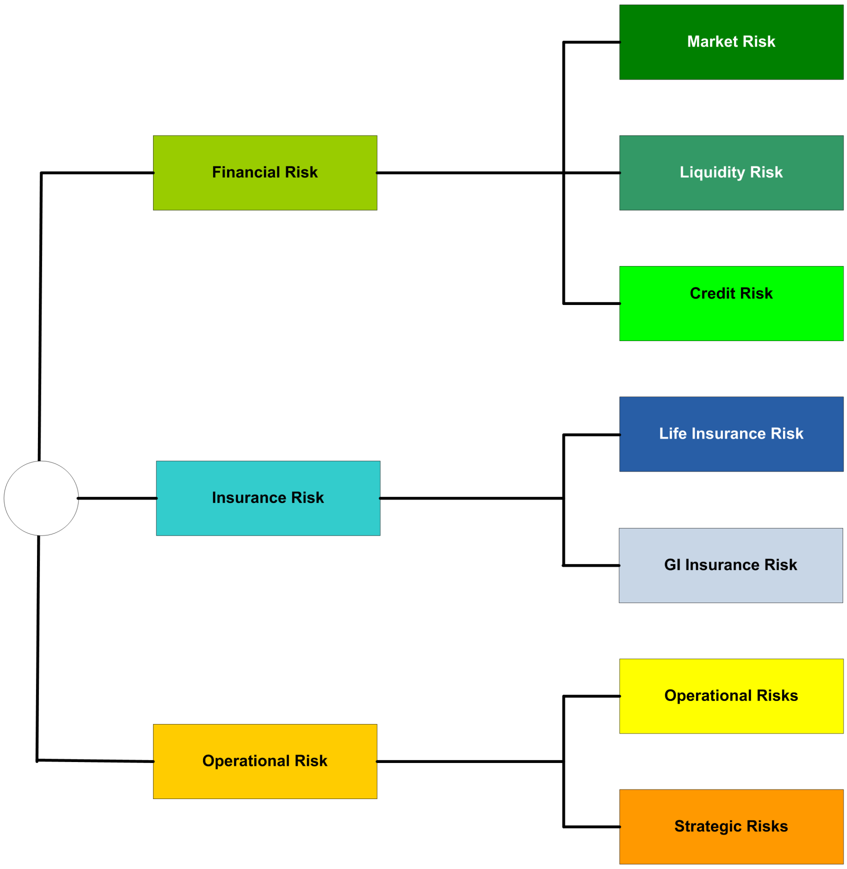



**Fig. 1.5** Risk Landscape

These risk factors form, from a mathematical point of view, the base for the risk capital calculations and represent a multi-dimensional random variable or stochastic process. All random fluctuation within a economic capital model are derived from these risk factors. The financial instrument sub-model of the Swiss solvency test uses for example about 80 different risk factors, which are modelled as a multi-dimensional normally distributed random variable ($X \sim \mathcal{N}(\mu, \Sigma)$).

# Chapter 2
# The role of the Balance Sheets and of Capital

## 2.1 The balance sheet of an Insurance Company

In order to assess the financial strength of an insurance company one often looks at its balance sheet. It is however not quite as easy as it sounds to look at the balance sheet of an insurance company, since the corresponding concepts are quite complex and because there exist different accounting standards. A further complexity is the fact that not all of the assets and liabilities are traded and their value is not directly observable.

## 2.2 Role of Valuation

In order to do physics it is essential to measure the different quantities accurately. Independently of the actual length of 1 meter the different laws and formulae are valid and correct. Therefore the meter serves mainly as an objective yardstick for comparison. In the world of economics the common measure is the face amount of the money, for example 1 EUR. Here the situation is however more difficult, in the sense that it is a priori not clear how to value complex financial instruments such as options, illiquid stocks, insurance policies etc. In order to be able to publish reliable financial statements and to undertake sensible risk management it is imperative to base on reliable valuation principles and methods. Without these methods neither financial accounting nor risk management make sense. The aim of this chapter is to give an introduction into economic valuation methods.

### *2.2.1 Valuation Methods*

For a given financial instrument or liability, a valuation can in principle be done based on market or book values. In case of book values the implicit aim is to prudently valuate the assets based on the purchase price. In case of stocks the corresponding principle results in the so called "lower cost or market" valuation, which means that the stock is in the books at the purchase price as long as the market price is not lower. Assume that a stock has been bought at EUR 100000 and has doubled its market price. In this case the book value would still be EUR 100000 and its market value EUR 200000. Correspondingly there is a revaluation reserve of EUR 100000 which is not accounted for in this type of balance sheet. In order to show these hidden values more transparently the so called market value accounting principles were introduced.

The market value of financial instruments can usually be determined looking at deep and liquid markets where these instruments are traded. In case of most stocks this is the case. There are however instruments, which are not regularly traded and here it is necessary to base the valuation on models. Typical instruments where models are required are for example:

| **Instrument** | **Method for valuation** |
|---|---|
| Illiquid Stocks | Usually last paid price |
| Synthetic Zero Coupon Bonds | By recursion based on bonds with coupons |
| Properties | By discounted cash flow method or expert judgement |
| Options and other derivatives | By mathematical methods such as Black-Scholes-Formula |
| Insurance Liabilities | Based on synthetic replicating portfolios |

The above table clearly shows the need for mathematical methods to approximate market values where such are not directly observable. One of the most useful theories is the arbitrage free pricing theory which will be explained in the next section.

### 2.2.2 Principle of no Arbitrage

The aim of this section is to give a high level overview about the so called arbitrage free pricing theory. If the reader wants to get a more mathematical representation of the corresponding topic, we refer to appendices D and E. For a general formal approach to abstract valuation we refer to appendix C.

In economics, arbitrage is the practise of taking advantage of a state of imbalance between two or more markets: a combination of matching deals are struck that capitalise upon the imbalance, the profit being the difference between the market prices. When used by academics, an arbitrage is a transaction that involves no negative value at any probabilistic or temporal state and a positive value in at least one state. A person who engages in arbitrage is called an arbitrageur. The term is mainly applied to trading in financial instruments, such as bonds, stocks, derivatives and currencies.

If the market prices do not allow for profitable arbitrage, the prices are said to constitute an arbitrage equilibrium or arbitrage free market. An arbitrage equilibrium is a precondition for a general economic equilibrium. The following example shows an arbitrage opportunity:

Suppose that the exchange rates (after taking out the exchange fees) in London are ï¿½5 = $10 = ï¿½1000 and the exchange rates in Tokyo are ï¿½1000 = ï¿½6 = $10. Converting $10 to ï¿½6 in Tokyo and converting that ï¿½6 into $12 in London, for a profit of $2, would be arbitrage. In reality, this "triangle arbitrage" is so simple that it almost never occurs.

The most important elements of the Arbitrage Free Pricing Theory are:

- Pricing systems,
- Arbitrage and
- Self-financing strategies.

We denote by $S_k(t)$ the price of the asset $k$ at time $t$, where $S_0(t)$ denotes usually the investment in cash. A portfolio at time $t$ is a vector $\phi_k(t)$ indicating the number of units of the corresponding asset hold at time $t$. The value of this portfolio at time $t$ equals

$$V(t) =< S(t), \phi(t) >= \sum_k S_k(t) \times \phi_k(t).$$

A self-financing trading strategy is a sequence of portfolios, which fulfils besides some additional mathematical requirements the following equation: $V(t^-) = V(t)$, which can be interpreted as the absence of injecting or withdrawing money during the changes of the portfolio. The trading strategy is called admissible if its value never falls below 0. The idea of arbitrage free pricing is to replicate a financial instrument such as a stock option by a corresponding self-financing trading strategy, which has exactly the same payout pattern as the financial instrument for (almost) all possible states of the financial market. As the strategy was self-financing the value of the instrument at time 0, needs to equal the value of the portfolio of the strategy at inception.

The arbitrage free pricing theory can today be considered as one of the cornerstones for pricing derivatives of financial instruments such as stock options, swaptions etc. From a mathematical point of view this theory is intrinsically linked to martingales - the prototype of a fair game. It can be shown that the absence of arbitrage implies the existence of a so called equivalent martingale measure $Q$. The price of the derivative is then the expected value of the discounted value of the instrument, not with respect to the original measure $P$, but with respect to the equivalent martingale measure $Q$. This can be interpreted that the value process under this new measure follows a fair game.

By using all the theoretical tools available for martingales it is possible to show a lot of nice features of these processes. The so called Itô-calculus allows the analytical and numerical treatment of such instruments.

In relation to the valuation it becomes obvious that options and other derivatives which have no deep and liquid market are priced and valuated based on these concepts. Furthermore they also play a significant role in the risk management of derivatives, because Itô-calculus allows the quantification of the changes in the price depending on the parameters resulting in the so called greeks. They represent the partial derivatives of the price and can be used to approximate the change in value by using a Taylor-approximation.

Another aspect of these tools is the possibility to simulate the price of financial instruments by Monte-Carlo-methods. Arbitrage free pricing theory and the need for equivalent martingale measures for pricing indicate the need to use the equivalent martingale measure for simulations - or equivalently to use so called deflators with respect to the original measure $P$. Deflators can be considered as a link between the two measures and are closely related to the concept of a Radon-Nikodym density $\frac{dQ}{dP}$. For further details we refer to appendix C.

A section about the Arbitrage Free Pricing Theory is certainly incomplete without mentioning the Black-Scholes Formula. The Black-Scholes model is a model of the evolving price of financial instruments, in particular stocks. The Black-Scholes formula is a mathematical formula for the theoretical value of European put and call stock options derived from the assumptions of the model. The formula was derived by Fischer Black and Myron Scholes and published in 1973. They built on earlier research by Edward Thorpe, Paul Samuelson, and Robert C. Merton. The fundamental insight of Black and Scholes is that the option is implicitly priced if the stock is traded. Merton and Scholes received the 1997 Nobel Prize in Economics for this and related work; Black was ineligible, having died in 1995.

### 2.2.3 Reconciliation of Balance Sheets

One of the main challenges with respect to economic balance sheets is the missing experience in doing so. Companies are much more used to producing their financial reports based on book value based principles where often virtual assets and liabilities and other "difficult animals" occur, such as:

- Deferred acquisition cost assets,
- Activated software assets,
- Deferred taxes,
- Equalisation reserves,
- Additional technical reserves for all types of insurance cover, etc.

In order to produce reliable economic balance sheets it is therefore advisable to start with an audited balance sheet of the company and to reconcile each position from book values to market values. In case of assets the reconciliation between book values and market values usually equals the revaluation reserves. But for some positions a reconciliation is difficult and therefore even more necessary. Just to mention one of the most difficult positions. What is the market value of a 100% consolidated subsidiary?

After having done the reconciliation it is far easier to explain a economic balance sheet to an audience understanding the traditional accounts. The reconciliation furthermore gives deep insights, where the company suffers small margins or has a lot of fat.

## 2.3 Bonds

In a typical insurance company most of its assets are bonds or bond like investments. A bond is a financial asset, where the investor pays at the time of buying a fixed amount. In the following years until the maturity of the bond, the investor gets a regular interest payment - a coupon. This payment is based on the nominal value of the bond. At the bond's maturity the investor gets the nominal value of the bond. Depending on the relationship of the relative interest rates the investor can buy the bond below the nominal value, at nominal value or above nominal value. Correspondingly the purchase is called below par, at par or above par.

For bonds there are many different possibilities and variations. Firstly there are so called perpetual bonds, which never mature. In actuarial terms they are perpetual annuities. Furthermore there are so called zero coupon bonds, where the interest rate of the coupon is 0%. Finally it is worth mentioning that there are also callable bonds, where the issuer can call the bond back before its maturity. So one could buy for example a bond which matures in 60 years from now, but which can be called after 10 years every 5 years. The idea behind a callable bond is to provide for a type of capital substitute for the issuer. So the issuer is not forced to refinance the bond in hard times. Normally the buyer of such bonds has to be compensated for this effect in case the bond is not called. A step up facility is such a method. In the above example one could for example get 5% for the first 10 years. After that, one could expect an uplift of 150 bp (eg 1.50%).

In today's environment, bonds are not issued anymore in paper form, but mostly only exist in a virtual form. In the past a bond consisted of a large piece of paper with attached small sections, the so called coupons. When the interest payment was due, these coupons were cut away and brought back to the bank. In exchange the bank payed the interest.

After understanding what a bond is we need to address the risks of a bond. There are two different types of risks, which are intrinsic to a bond, namely the interest rate risk and the credit risk.

As seen above the interest rate is the amount of money which the investor gets for borrowing his money. This price depends on the economical environment and can be higher or lower, depending on the moment in time. The so called *yield curves* describe the interest rate one could get at a certain point of time for borrowing the money for a certain term. In consequence yield curves depend per currency on two parameters, namely the time and the term of the bond. The following table shows the interest rates as at Jan 1st., 2008. It becomes obvious that the interest rates for example in CHF are lower than those in USD. Figure 2.1 shows the corresponding yield curves.

| 01.01.2008 | US | EURO | UK | JAPAN | AUS $ |
|---|---|---|---|---|---|
| 3 Month | 3.2745% | 3.9564% | 4.8201% | 0.5069% | 6.5797% |
| 6 Month | 3.4112% | 4.0640% | 4.8170% | 0.5103% | 6.5915% |
| 1 Year | 3.3624% | 4.0488% | 4.5518% | 0.6486% | 6.6866% |
| 2 Year | 3.0825% | 4.0549% | 4.3958% | 0.7062% | 6.9174% |
| 3 Year | 3.0936% | 4.0778% | 4.4141% | 0.7818% | 6.8486% |
| 4 Year | 3.3828% | 4.1624% | 4.4507% | 0.9138% | 6.7332% |
| 5 Year | 3.4584% | 4.1592% | 4.5251% | 1.0195% | 6.5823% |
| 7 Year | 3.8290% | 4.2489% | 4.5763% | 1.1847% | 6.3889% |
| 8 Year | 3.9680% | 4.2982% | 4.6010% | 1.2444% | 6.3642% |
| 9 Year | 4.0681% | 4.3348% | 4.6026% | 1.3916% | 6.3350% |
| 10 Year | 4.2713% | 4.3811% | 4.6145% | 1.5144% | 6.2876% |
| 15 Year | 4.5299% | 4.5805% | 4.5926% | 1.8450% | 6.2431% |
| 20 Year | 4.5214% | 4.6512% | 4.5123% | 2.1060% | – |
| 25 Year | 4.4676% | 4.6689% | 4.4309% | 2.2586% | – |
| 30 Year | 4.4601% | 4.6279% | 4.3449% | 2.3804% | – |

Please note that for some currencies (eg AUS) there exist no long dated bonds (eg for Australia there is no interest rate for 20 yrs) and hence the yield curve is not complete. This not an issue in respect of the bonds, but there is an issue in valuing long term liabilities such as life insurance contracts with a term longer than the longest dated bond.

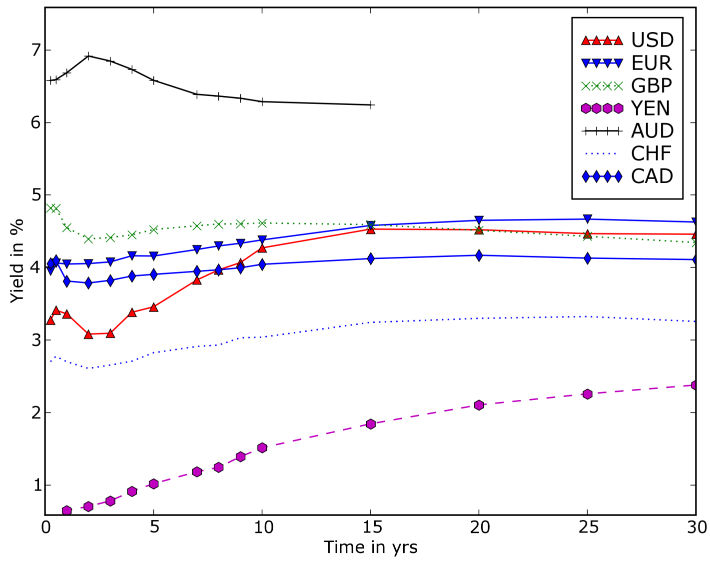



**Fig. 2.1** Yield curves as at 1.1.2008

In a next step it is interesting to have a look how the yield curves move over the time. In order to do this we want to have a look at the yield curves just about one year before:

| 29.12.2006 | US | EURO | UK | JAPAN | AUS $ |
|---|---|---|---|---|---|
| 3 Month | 4.9999% | 3.6216% | 5.2126% | 0.2996% | 6.2142% |
| 6 Month | 5.0386% | 3.7508% | 5.2517% | 0.4099% | 6.2337% |
| 1 Year | 4.9692% | 3.8736% | 5.2828% | 0.5956% | 6.2101% |
| 2 Year | 4.8293% | 3.8806% | 5.2632% | 0.7925% | 6.1771% |
| 3 Year | 4.7430% | 3.8841% | 5.1827% | 0.9505% | 6.1159% |
| 4 Year | 4.6693% | 3.9097% | 5.1239% | 1.1175% | 6.0616% |
| 5 Year | 4.6673% | 3.9180% | 5.0511% | 1.2481% | 6.0178% |
| 7 Year | 4.7125% | 3.9421% | 4.9372% | 1.4639% | 5.9519% |
| 8 Year | 4.7361% | 3.9497% | 4.8857% | 1.5418% | 5.9225% |
| 9 Year | 4.8751% | 3.9453% | 4.8241% | 1.6153% | 5.9001% |
| 10 Year | 4.7832% | 3.9591% | 4.7926% | 1.6916% | 5.8816% |
| 15 Year | 4.9564% | 4.0413% | 4.6623% | 1.9393% | 5.8335% |
| 20 Year | 4.9309% | 4.0914% | 4.4779% | 2.0721% | – |
| 25 Year | 4.8704% | 4.1142% | 4.3386% | 2.2340% | – |
| 30 Year | 4.8099% | 4.0749% | 4.2355% | 2.3115% | – |

In order to be able to compare the two sets of yield curves the table below provides a comparison between them.

| $\Delta$ | Time | US | EURO | UK | JAPAN | AUS $ |
|---|---|---|---|---|---|---|
| 3 Month | 2007 | 4.9999% | 3.6216% | 5.2126% | 0.2996% | 6.2142% |
| 3 Month | 2008 | 3.2745% | 3.9564% | 4.8201% | 0.5069% | 6.5797% |
| 3 Month | $\Delta$ | -1.7254 % | 0.3348 % | -0.3925 % | 0.2073 % | 0.3655% |
| 5 Year | 2007 | 3.4584% | 4.1592% | 4.5251% | 1.0195% | 6.5823% |
| 5 Year | 2008 | 4.6673% | 3.9180% | 5.0511% | 1.2481% | 6.0178% |
| 5 Year | $\Delta$ | 1.2089% | -0.2412 % | 0.5260% | 0.2286 % | -0.5645% |
| 15 Year | 2007 | 4.5299% | 4.5805% | 4.5926% | 1.8450% | 6.2431% |
| 15 Year | 2008 | 4.9564% | 4.0413% | 4.6623% | 1.9393% | 5.8335% |
| 15 Year | $\Delta$ | 0.4265% | -0.5392% | 0.0697% | 0.0943% | -0.4096% |
| 30 Year | 2007 | 4.4601% | 4.6279% | 4.3449% | 2.3804% | – |
| 30 Year | 2008 | 4.8099% | 4.0749% | 4.2355% | 2.3115% | – |
| 30 Year | $\Delta$ | 0.3498% | -0.5530% | -0.1094% | -0.0689% | – |

One can see for example that the interest rate has increased considerably over this year for the 5–year bond in USD.

But how does one determine the yield curves? This question is closely linked to the valuation of the bonds and hence we will speak in a first step about bond valuation. In order to do that we will for the moment denote by $\pi_t$ the price of a bond or cash flow stream at time $t$ and we denote by $y_t(n)$ the yield of a n-year zero coupon bond ($\mathcal{Z}_{(n)} = (\delta_{nk})_{k\in\mathbb{N}_0}$) at time $t$. An ordinary bond with a cash flow pattern $\mathcal{B} = (CF_k)_{k\in\mathbb{N}_0}$ has in this context the following value at time $t$:

$$\begin{aligned}\pi_t(\mathcal{B}) &= \sum_{k=0}^{\infty} CF_k \times \pi_t(\mathcal{Z}_{(k)}) \\ &= \sum_{k=0}^{\infty} CF_k \times (1+y_t(k))^{-k}.\end{aligned}$$

Now it is possible to determine the yield curve based on a set of ordinary bonds by means of a recursion. Another important concept are forward rates $f_t(n,m)$. The interpretation of $f_t(n,m)$ is the yearly interest rate which we would get from time $n$ to time $m$ ($n \le m$). We denote by:

$$f_t(n,m) = \left(\frac{\pi_t(\mathcal{Z}_{(n)})}{\pi_t(\mathcal{Z}_{(m)})}\right)^{\frac{1}{m-n}} - 1,$$

and remark that the following equation holds, by using a non-arbitrage argument (Barbel-strategy):

$$(1+y_t(n))^n = \prod_{k=0}^{n-1} (1 + f_t(k,k+1))\,.$$

After defining the yield curves, it is now possible to introduce different concepts for the valuation of bonds. There is on the one hand the statutory amortised cost valuation method. On the other hand there is the market value valuation for bonds. In order to understand the amortised cost method, we need to look at the relationship between the coupon of a bond and the current interest rate level. As seen above, a bond can be viewed as a vector of cash flows, and it looks normally as follows:

$$\mathcal{B} = (CF_k)_{k\in\mathbb{N}_0}\text{, with}$$
$$CF_k = \begin{cases} 0 & \text{if} \quad k = 0, \\ i & \text{if} \ 0 < k < n, \\ 1+i & \text{if} \quad k = n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

In the above example we consider a bond with a nominal interest rate $i$ and a term of $n$ years. We say that we purchase the bond

*below par* if $\pi_t(\mathcal{B}) < 1$,
*at par* if $\pi_t(\mathcal{B}) = 1$,
*above par* if $\pi_t(\mathcal{B}) > 1$.

If we apply an amortised cost method to value a bond, we fix the price $\pi_t(\mathcal{B}) < 1$ at purchase date and amortise it until maturity of the bond at time $n$ to the value 1, the nominal value of the bond. If we buy a bond above (below) par the corresponding amortisation leads to a yearly gain (loss), which is recognised in the profit and loss account. This implies in particular that the value of the bond does not change during its lifetime due to interest rate movements.

The other valuation method is based on the market value of the bond. Here the accounting value for the bond equals $\pi_t(\mathcal{B})$ and hence depends on the relative level of interest rates. It is worth noting that the value of a bond such as $\mathcal{B}$, decreases if interest rates increase and it increases in case of a reduction in interest rates.

In a next step we want to have a look at the following bond:

- Nominal value: EUR 100000,
- Term: 5 years,
- 4.0% interest rate,
- Bought at par and we assume a flat yield curve,

and we want to look at the value of its parts:

| t | Coupon | Principal | Total CF | PV 3% | PV 4% | PV 5% |
|---|---|---|---|---|---|---|
| 0 | | | 0 | – | – | – |
| 1 | 4000 | | 4000 | 3883.49 | 3846.15 | 3809.52 |
| 2 | 4000 | | 4000 | 3770.38 | 3698.22 | 3628.11 |
| 3 | 4000 | | 4000 | 3660.56 | 3555.98 | 3455.35 |
| 4 | 4000 | | 4000 | 3553.94 | 3419.21 | 3290.80 |
| 5 | 4000 | 100000 | 104000 | 89711.31 | 85480.41 | 81486.72 |
| **Total** | 20000 | 100000 | 120000 | **104579.70** | **100000** | **95670.52** |
| *Difference* | | | | *4579.70* | 0 | *-4329.47* |

It becomes obvious that the value of the bond changes some 4.5% per 1% shift in interest rate. In order to quantify this sensibility one normally uses the so called *Macaulay duration* $d(\mathcal{B})$. It is defined by

$$\begin{aligned} d(\mathcal{B}) &= \frac{\sum_{k=0}^{\infty} k \times CF_k \times \pi_t(\mathcal{Z}_{(k)})}{\sum_{k=0}^{\infty} CF_k \times \pi_t(\mathcal{Z}_{(k)})} \\ &= \frac{\sum_{k=0}^{\infty} k \times CF_k \times (1+y_t(k))^{-k}}{\sum_{k=0}^{\infty} CF_k \times (1+y_t(k))^{-k}}. \end{aligned}$$

In the concrete example above we have $d(\mathcal{B}) = 4.62$, which is remarkably close to the interest rate sensitivity above. This is not an accident, but rather the reason, why the concept of duration exists. It aims to estimate the interest rate sensitivity in case of a parallel shift of the yield curve. But how can we do this? In a first step we define a modified valuation based on a flat interest curve with an interest rate of $x$:

$$\tilde{\pi}_t(\mathcal{B})(x) = \sum_{k=0}^{\infty} CF_k \times (1+x)^{-k}.$$

Now using Taylor approximation (eg $\tilde{\pi}_t(\mathcal{B})(x+\Delta x) \approx \tilde{\pi}_t(\mathcal{B})(x)+\Delta x\times \frac{d}{dx}\tilde{\pi}_t(\mathcal{B})(x)$) we get in a first step:

$$\begin{aligned}\frac{d}{dx}\tilde{\pi}_t(\mathcal{B})(x) &= \sum_{k=0}^{\infty} CF_k \times (-k) \times (1+x)^{-k-1} \\ &= \frac{1}{1+x} \times \sum_{k=0}^{\infty} CF_k \times (-k) \times (1+x)^{-k}.\end{aligned}$$

If we now want a relative value for the change we correspondingly get

$$\frac{\frac{d}{dx}\tilde{\pi}_t(\mathcal{B})(x)}{\tilde{\pi}_t(\mathcal{B})(x)} = \frac{1}{\tilde{\pi}_t(\mathcal{B})(x)\times(1+x)} \times \sum_{k=0}^{\infty} CF_k \times (-k) \times (1+x)^{-k}.$$

This last expression is called *modified duration* $d^{mod}(\mathcal{B})$ and we have the

$$d^{mod}(\mathcal{B}) = \frac{d(\mathcal{B})}{1+x}.$$

Since the interest rate is normally small, the approximation above is quite accurate. As mentioned the Taylor approximation (or modified duration) is still more precise. In our case the modified duration amounts to $4.40$ which equals the average of $4.57$ and $4.23$ with an error of less than $0.01$.

Until now we have looked at counter-party risk free bonds. This means, that we have assumed that the payments are always paid and that the issuer of the bond can not default. In reality bonds can default and as a consequence there is the possibility that not all coupons and/or the principal are paid. Assume for a moment that we have the following survival probabilities:

| time | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| p(x) | 1.000 | 0.980 | 0.950 | 0.910 | 0.860 | 0.810 |

The above table assumes for example that in average 2% of the companies issuing a certain type of bond default in the first year. Obviously such a bond has less value than the one from an issuer which can not default. This latter bond is called risk free bond and it is normally assumed that its issuer is a government. There are two questions, which we want to answer in the sequel, namely how do we valuate such (defaultable) bonds and how can we assign a quality to such a bond, since the value of the bond obviousely depends on its default probabilities.

In a first step we want to have a look at the valuation of such a bond and we want to revisit the example from above. In this case we have:

| t | A<br>Total CF<br>nominal | B<br>Survival Prob | C<br>Total CF<br>risk adjusted | D<br>PV nom.<br>4.00% | E<br>PV risk adj.<br>4.00% | F<br>PV nom.<br>**8.37%** |
|---|---|---|---|---|---|---|
| 0 | 0 | 1.00 | 0 | – | – | – |
| 1 | 4000 | 0.98 | 3920 | 3846.15 | 3769.23 | 3690.83 |
| 2 | 4000 | 0.95 | 3800 | 3698.22 | 3513.31 | 3405.57 |
| 3 | 4000 | 0.91 | 3640 | 3555.98 | 3235.94 | 3142.35 |
| 4 | 4000 | 0.86 | 3440 | 3419.21 | 2940.52 | 2899.48 |
| 5 | 104000 | 0.81 | 84240 | 85480.41 | 69239.13 | 69559.90 |
| **Total** | 120000 | | 99040 | 100000 | **82698.15** | **82698.15** |

Column A in the above table denotes the expected cash flows for the bond in case we assume that it does not default. The corresponding survival probabilities are stated in column B and in consequence we get the expected cash flow payments including the probability to default in column C. Based on this calculation we get the value of the non-defaultable bond in column D and of the defaultable one in column E. Besides this direct calculation, one can also ask how much bigger interest rate is necessary, in order to get the same present value if we base our calculation on the nominal cash flows in column A. This results in an interest rate for discounting of 8.37%, which represents a risk premium of 437 bp. This is the way in which the market values defaultable bonds, by adding a risk premium on top of the risk free yield curve. Hence we get a yield curve for defaultable bonds including a *credit spread*. Figure 2.2 shows the development of the credit spread over time. It needs to be stressed that the increase of the credit spreads in this figure is only partially a consequence of higher default probabilities. The other effect is the fact that during the end of the year 2008 there was a liquidity and capital crunch. The absence of a liquid market can also lead to higher credit spreads, as observed in the crisis of 2008/09.

In a next step we want to understand how different bonds are assessed in terms of credit rating. In order to do this, bonds are evaluated by rating agencies, which put them in (homogeneous) classes which should have the same survival behaviour. To this end S&P classifies bonds between AAA, AA, A, BBB, BB, . . . C and D. Bonds with a higher credit quality than BBB are called investment grade and bonds classified as D are in default.

In order to model the credit rating process one normally uses *Markov chains* $(X_t)_{t \in \mathbb{R}^+}$ on a finite state space $S$ (see also appendix B). In case of the S&P rating one could define $S$ as

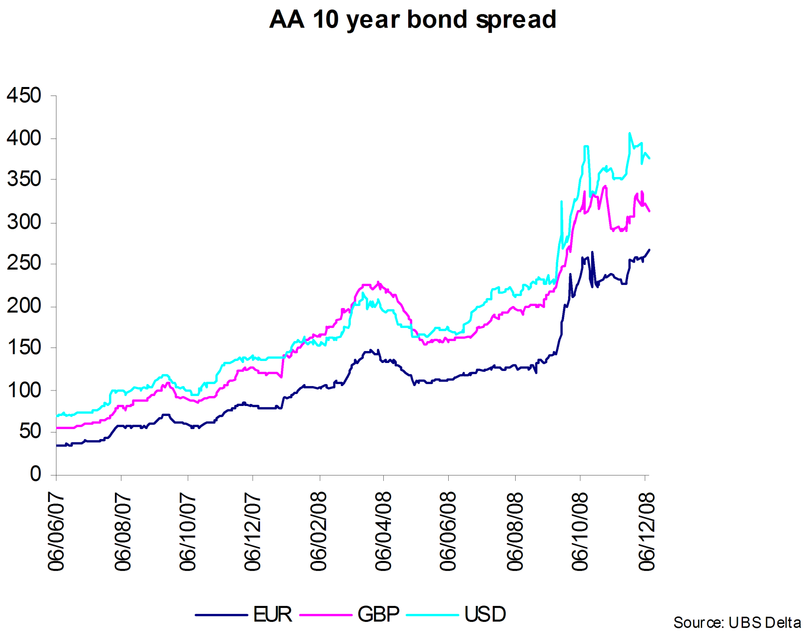



**Fig. 2.2** Credit spreads over time

$$S = \{AAA, AA, A, BBB, BB, B, C, D, NR\},$$

where $NR$ stands for not rated. For a Markov chain, one defines the transition probabilities $p_{ij}(s,t)$ as

$$\begin{aligned} p_{ij}(s,t) &= P[X_t = j \,|\, X_s = i], \text{ and} \\ P(s,t) &= (p_{ij}(s,t))_{(i,j)\in S\times S}\,. \end{aligned}$$

From the *Chapman-Kolmogorov equation* it is known that, we have the following relationship for $s < t < u$:

$$P(s,u) = P(s,t)\,P(t,u).$$

For credit risk it is normally assumed that $P(s, s+1)$ is constant ("time-homogeneous Markov chain"), and hence we can further simplify:

$$P(0,t) = P(0,1)^t.$$

After this formula we can now look at a concrete example of such a transition matrix ($P(0,1)$), remarking that the states "NR" and "D" have been merged:

| $j \rightsquigarrow$ <br> $i$ | AAA | AA | A | BBB | BB | B | C | D |
|---|---|---|---|---|---|---|---|---|
| AAA | 88.12% | 11.88% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| AA | 0.00% | 92.45% | 3.30% | 0.24% | 0.00% | 0.00% | 0.00% | 4.01% |
| A | 0.00% | 1.54% | 91.32% | 2.92% | 0.08% | 0.00% | 0.00% | 4.14% |
| BBB | 0.00% | 0.00% | 3.21% | 88.77% | 2.01% | 0.13% | 0.00% | 5.88% |
| BB | 0.10% | 0.00% | 0.10% | 4.66% | 80.55% | 5.17% | 0.20% | 9.22% |
| B | 0.00% | 0.00% | 0.00% | 0.33% | 7.22% | 72.54% | 3.50% | 16.41% |
| C | 0.00% | 0.00% | 0.72% | 0.00% | 3.6% | 18.71% | 50.36% | 26.62% |
| D | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 100.00% |

From this table one can for example read that within one year 11.88% of all AAA bonds are downgraded to AA. In a next step it is now possible to calculate for example $P(0,10)$:

| $j \rightsquigarrow$ <br> $i$ | AAA | AA | A | BBB | BB | B | C | D |
|---|---|---|---|---|---|---|---|---|
| AAA | 28.23% | 48.06% | 8.17% | 1.22% | 0.07% | 0.01% | 0.00% | 14.23% |
| AA | 0.00% | 46.81% | 15.88% | 3.07% | 0.22% | 0.03% | 0.00% | 33.98% |
| A | 0.00% | 7.34% | 43.42% | 11.91% | 1.15% | 0.2% | 0.01% | 35.96% |
| BBB | 0.03% | 1.06% | 12.98% | 33.54% | 5.13% | 1.19% | 0.1% | 45.97% |
| BB | 0.24% | 0.33% | 2.66% | 11.8% | 15.17% | 5.86% | 0.6% | 63.34% |
| B | 0.06% | 0.07% | 0.72% | 3.76% | 8.37% | 7.29% | 0.91% | 78.81% |
| C | 0.04% | 0.13% | 1.11% | 2.36% | 5.53% | 5.19% | 0.73% | 84.91% |
| D | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 100.00% |

From the above transition matrix from year 0 to year 10 it becomes obvious, that about 14% of the AAA have defaulted over this 10 year period. Furthermore it also becomes obvious, that the corresponding matrix has some flaws since there are no AA and A bonds which have after 10 years a AAA rating in contrast to some BB bonds. The reason for this is certainly the fact that the mitigation probabilities to AAA are very small and therefore not enough transitions have been observed over the observation time. In consequence the transition matrix contradicts expectations.

Finally it needs to be remarked that normally a change in rating implies a change in credit spread and hence implies a gain or a loss. In order to illustrate this, we use the same example from above and assume that the credit spread for a AA bond corresponds to 75 bp, the one for a A bond to 125 bp and the one for a BBB bond to 200 bp. In this case we have the following situation:

| t | Coupon | Principal | Total CF | PV AA 4.75% | PV A 5.25% | PV BBB 6.00% |
|---|---|---|---|---|---|---|
| 0 | | | 0 | – | – | – |
| 1 | 4000 | | 4000 | 3818.61 | 3800.47 | 3773.58 |
| 2 | 4000 | | 4000 | 3645.45 | 3610.90 | 3559.98 |
| 3 | 4000 | | 4000 | 3480.14 | 3430.78 | 3358.47 |
| 4 | 4000 | | 4000 | 3322.33 | 3259.65 | 3168.37 |
| 5 | 4000 | 100000 | 104000 | 82463.76 | 80523.53 | 77714.84 |
| **Total** | 20000 | 100000 | 120000 | **96730.32** | **94625.35** | **91575.27** |
| *Difference* | | | | *2104.97* | 0 | *-3050.07* |

Also here the duration approximation can be used to estimate the impact of a change in credit spread. If we want to estimate the impact of an increase of the credit spread by 75 bp (eg A $\rightsquigarrow$ BBB) we have $d^{mod}(\mathcal{B}) \times 0.75\% \times 94625.35 = 4.40 \times 0.75\% \times 94625.35 \approx 3122$, and also here, this approximation is of acceptable quality.

Finally we want to have a look at the valuation using a Markov model. The corresponding formulae can be found in appendix B. Above we have seen the transition probabilities and also the interest used for discounting is obvious. Hence we still need to define the corresponding benefits. Since the contractual terms are honoured in all cases where the bond is not in default, we have for all states in $S^* = \{AAA, AA, \ldots C\}$ the following:

$$a_{ij}^{\text{Post}}(t) = \begin{cases} 0 & \text{if} \quad k = 0 \\ i & \text{if} \ 0 < k < n \\ 1+i & \text{if} \quad k = n \\ 0 & \text{if} \quad k > n \end{cases} \qquad \forall (i,j) \in S^* \times S^*.$$

Now what happens for transitions $i \rightsquigarrow D$ with $i \in S^*$. In this case the investor gets back the value of the remaining cash flows $\times(1 - LGD)$, where $LGD$ denotes the *loss given default*. A loss given default of 65% means that you get back 35000 EUR back in case a bond with a value of 100000 EUR. For the example a simplified calculation of the remaining value of the bond was used, in the sense that the remaining value of the bond was taken undiscounted. In the concrete example below we have chosen a loss given default of 65 % and an interest rate of 3%.

| | AAA | AA | A | BBB | BB | B | C | D |
|---|---|---|---|---|---|---|---|---|
| 0 | 926111 | 819694 | 809241 | 755500 | 661762 | 575428 | 538721 | 0 |
| 1 | 936413 | 831033 | 821436 | 768823 | 674901 | 582776 | 541098 | 0 |
| 2 | 946761 | 843599 | 834953 | 784101 | 691281 | 593642 | 546108 | 0 |
| 3 | 956986 | 857477 | 849866 | 801507 | 711386 | 608958 | 554563 | 0 |
| 4 | 966877 | 872758 | 866253 | 821224 | 735744 | 629936 | 567639 | 0 |
| 5 | 976175 | 889540 | 884193 | 843447 | 764914 | 658154 | 587107 | 0 |
| 6 | 984565 | 907929 | 903766 | 868382 | 799460 | 695660 | 615791 | 0 |
| 7 | 991660 | 928040 | 925052 | 896252 | 839910 | 745080 | 658450 | 0 |
| 8 | 996993 | 949997 | 948133 | 927296 | 886697 | 809687 | 723574 | 0 |
| 9 | 1000000 | 973934 | 973090 | 961780 | 940069 | 893334 | 827069 | 0 |
| 10 | 1000000 | 1000000 | 1000000 | 1000000 | 1000000 | 1000000 | 1000000 | 0 |

The really interesting thing about the Markov model is the fact that you can not only calculate the expected values, but also higher moments and the complete probability distribution function (see [Kol10]).

### *2.3.1 Ranking of Bonds (seniority) and AT1 Bonds*

When a company defaults its debt is paid out in order of seniority in order to allow for an orderly wind down process. The following definition of seniority is taken from wikipedia, and is widely accepted: In finance, seniority refers to the order of repayment in the event of a sale or bankruptcy of the issuer. Seniority can refer to either debt or preferred stock. Senior debt must be repaid before subordinated (or junior) debt is repaid. Each security, either debt or equity, that a company issues has a specific seniority or ranking. Bonds that have the same seniority in a company's capital structure are described as being pari passu. Preferred stock is senior to common stock in a sale when preferred shareholders must receive back their preference, typically their original investment amount, before the common shareholders receive anything.

Bonds are classified by order of their priority in case of a default as follows in descending order of priority:

**Senior Secured Bonds:** Any security labeled "senior" in such a structure is one that takes primacy over any other companyâ€™s sources of capital. The most-senior securities holders will always be first to receive a payout from a companyâ€™s holdings in the event of default. Then would come those security-holders whose securities are deemed second-highest in seniority, and so forth until the assets used to pay off such debts run out.

**Senior Unsecured Bonds:** Senior unsecured corporate bonds are in most respects just like senior secured bonds with one significant difference: There is no specific collateral guaranteeing them. Other than that, such senior bondholders enjoy a privileged position in the event of default with respect to the payout order.

**Junior, Subordinated Bonds:** After the senior securities are paid out, the junior, unsecured debt will next be paid out from what assets remain. This is unsecured debt, meaning no collateral exists to guarantee at least a portion. Such unsecured bonds only have the issuerâ€™s good name and credit rating as security. Junior or subordinated bonds are named specifically for their position in the payout order: Their junior, or subordinate, status means they only are paid out after senior bonds, in the event of a default. In the financial industry such as in insurance and banks such types of bonds are also denoted as Tier 1 or Tier 2 bonds, since there count against the eligible capital of the respective financial institution. More precisely Tier 1 bonds are more junior than Tier 2 bonds and would be written down first in a case of default of the bak or insurance company. In order to allow for such loss-abosorbing capacity, typically Tier 1 bonds (also referred to "AT1" bonds are either very long dated or perpetual.

We note that the traditional belief is that even bonds with the lower seniority have a preference over the shareholder capital in case of a default, and hence philosophically one could consider equity as subordinated to all types of bonds and in consequence bonds are only written down in case of a default (ie state "D"). What does change for the various subordinated bonds is their loss given default. The more junior a bond the higher the loss given default. For example one could assume a loss given default for a senior secured bond to be closely to 0, whereas for subordinated loans close to 1. Hence according to the traditional view the probability of default is the same for all bonds issued by the same issuer, but the value of the bond post default is vastly different. In this context it is therefore acceptable to use the same transition matrices for all types of bonds for a given issuer.

On Sunday 19.3.2023 this view might have fundamentally changed as a consequence of the restructuring needed for Credit Suisse Group. As a consequence negociations between UBS, Credit Suisse ("CS"), the Swiss federal government, the Swiss National Bank and the Finma, the Swiss Financial Market Regulator, it was decided, that:

- The two banks are merged for an exchange of 22.48 CS shares into 1 UBS share, hence without a statutory default of CS;
- The Swiss Government and the Swiss Nation Bank provide for a variety of guarantees and liquidity facilities to ensure the survival of the combined entity;
- Emergency law is involved to make this deal possible, giving the Finma the possibility to write down the AT1 without an actual default; and
- Finma announced to the market on 16.3.2023 and on 23.3.2023, this notes in the amount of in excess of USD 16bn to written down immediately.

What this means (ignoring possible legal repercussions) is the fact that the traditional view on the pecking order of conditions of write-down of a bond has changed and that the model as per the former section using one transition matrix for all types of bonds has become inviable. In the following we want to have a look what this means for our model, and for the valuation of respective bonds.

Formally we need to split in a first instance the state $D$ into a state $D_e$ and a state $D_{reg}$, hence a economic default and a default triggered by the regulator. Moreover we remark that the absolute economic default probabilities will stay the same, but that there is an additional decrement. The probability for the additional decrement can only come from the "healthy" states $\{AAA, \ldots C\}$ and we note that $\sum_{k \in \mathcal{S}} p_{ik}(t, t+1) = 1$. Hence lets look at how this looks concretely based on a toy model (in the sense that we consider fewer states and respective transitions).

To be concrete we look at the following instrument:

- Nominal Value or the bond 300 (M CHF);
- Perpetual Maturity;
- Interest Rate $i = 0.005$ for discounting, coupon of the bond of 3.5%; and
- We assume that the Bond is BB rated.

In order to have a slightly more simplistic model we assume that a bond can be in the following states $\mathcal{S} = \{A, B, D\}$. In case of the allowance of a regulatory write down (ie "Regulatory Default") we use the following modified state space $tildemathcalS = \{A, B, D_e, D_{reg}\}$. In a next step we need to define the respective transition matrix, which is essentially based on the full transition matrix as per above and we chose in case of the regular model (all values in %):

$$p(1) = \begin{bmatrix} & A & B & D \\ A & 92.5 & 3.3 & 4.2 \\ B & 1.0 & 89.0 & 10.0 \\ D & 0.0 & 0.0 & 100.0 \end{bmatrix} \tag{2.1}$$

For the enhanced model we use the following transition matrix:

$$p(1) = \begin{bmatrix} & A & B & D_e & D_{reg} \\ A & 87.5 & 3.3 & 4.2 & 5.0 \\ B & 1.0 & 79.0 & 10.0 & 10.0 \\ D_e & 0.0 & 0.0 & 100.0 & 0.0 \\ D_{reg} & 0.0 & 0.0 & 0.0 & 100.0 \end{bmatrix} \tag{2.2}$$

Besides the transition matrices we need to define the loss-given-default, which we set to 80% for an economic and to 100% for a regulatory default.

Now we are able to calculate the present values of the bonds and we get the following results:

```
Model 1 - excluding regulatory default
```

```
  0 :      220.6520       153.5476          0.0000

 Model 2 - including regulatory default
  0 :      111.0362        72.1412          0.0000          0.0000
```

From the above calculation we can see that the bond value for a 'B' rated bond falls from c153 (M CHF) to c72 (M CHF), which also explains the material fall in observed AT 1 bonds after the declaration of the Swiss regulator to write such bonds off in the case of Credit Suisse. We note that those bonds contained regulatory triggers that such a thing could happen. It was however the belief that a regulatory write down could only happen if the company is in capital need and that in such a case the pecking order is maintained. We can also ask us: how much interest rate would the issuer have to offer to make good for the additional state of default. When doing the calculation on e finds out, that for the chosen parametrisation, the required interest rate would habe to move up from 3.5% to roughly 10%. An interesting side remark: given that these bonds can under regulatory default be worse of than equities, it can make sense that the respective return requirements are higher than the corresponding equity returns which are in the Swiss Market about 7-8% in average.

We note that we replicate the simpler model within the more complex model by using a one-parametric set of transition probabilities as follows:

$$p^{\alpha,\beta}(1) = \begin{bmatrix} & A & B & D_e & D_{reg} \\ A & 92.5-\alpha & 3.3 & 4.2 & \alpha \\ B & 1.0 & 89.0-\beta & 10.0 & \beta \\ D_e & 0.0 & 0.0 & 100.0 & 0.0 \\ D_{reg} & 0.0 & 0.0 & 0.0 & 100.0 \end{bmatrix}, \tag{2.3}$$

and we remark that for $\alpha = 0$ and $\beta = 0$ we get the prior and for $\alpha = 0.05$ and $\beta = 0.10$ the latter model. This now opens up for looking at the sensitivity of the value with respect to these parameters and finally how to potentially hedge such instruments against market value changes induced by changing $\alpha$ and $\beta$. Figure 2.3shows the respective sensitivity and we note that the steepness of the functions when moving from the original model away is high. This indicates that the risk of allowing this feature makes these bonds much more risky.

We will end this section by illustrating the difficulty to hedge these instruments by the respective equity of the underlying company. In case of Credit Suisse, the market capitalisation on the Friday before the forced merger with UBS was c8bn CHF and outstanding AT1 bonds had a value in the order of c10bn CHF. (nominal value c16bn CHF). On Monday the situation was that the company had a value of 3bn and the bonds nil. Hence the economic losses were 5bn CHF for equities and 10bn CHF bonds. Hence even if short selling all equity one could only recuperate half of the loss of the bonds, which essentially means that in this case a proxy hedge of the bonds via short selling the respective shares does not work in its entirety. Moreover

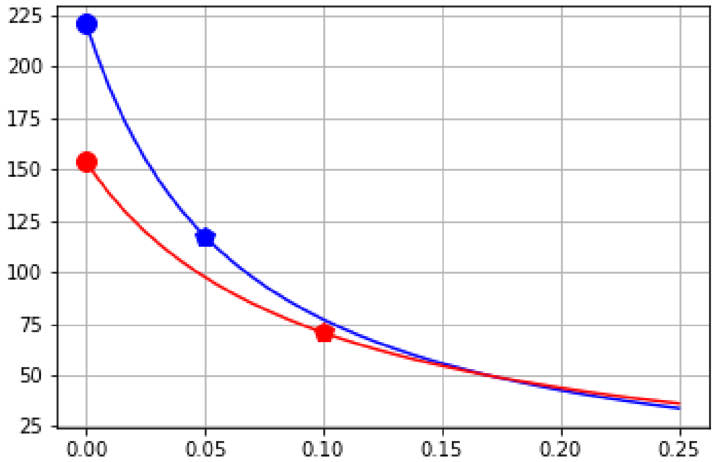



**Fig. 2.3** Senistivity of AT1 Bonds

the assumption of being able to short-sell all equities is quite heroic. This clearly underpins the statement above that this type of bond investments can be very risky.

## 2.4 Shares

In the previous section we have looked at bonds. Another important asset class are shares. Here one invests in a company and the share price reflects the value of the company. An example could be a share of an insurance company, of a utility company etc. A share has two economic aspects which need to be distinguished. On one hand there is the amount of dividends a company pays and on the other hand there is the value of the share.

As with bonds each share has a nominal value, for example CHF 50. At each moment in time, this share has a market value, which might for example be CHF 320. The dividend yield of 10% would result in a dividend payment of CHF 5. This means that the dividend payment in relation to its market value is 1.56%.

Shares can be valued according to book value and market value principles. Both of these valuations are conceptually easier than bonds. Assume that we have bought a share for a price of EUR 50 per unit and we have 10000 units. Furthermore assume that the unit price has changed upwards and we want to see how different valuation principles look like. The book value principle requests that the share is valued at purchase price, or lower if the current market price is below the purchase price ("lower book or market"). There are several variations to this principle and the concrete set up depends on the legislation of the country. There is a stronger form of the "lower book or market" principle, where the book value has to be written down, once the market value falls below the book value and there is no possibility to write it up again. However, there are countries where the minimum condition does not apply immediately and the value of the shares can be smoothed over time unless there is a permanent impairment.

On the other hand the market value valuation of a share is rather easy. One takes the last paid price of the instrument. This is however not always as easy as it might seem. If you have an illiquid share where there are only few transactions. In this case the last transaction might be quite old and the last share might have been traded some weeks ago. Besides, market prices of such shares move normally more erratic than the paid prices of more liquid shares, and hence more care is needed for the valuation of them.

Now lets have a look at the example mentioned above. The table below illustrates the values of the shares for different unit prices:

| Unit Price | Purchase Price | Number of Units | Market Value | Book Value |
|---|---|---|---|---|
| 40 | 50 | 10000 | 400000 | 400000 |
| 50 | 50 | 10000 | 500000 | 500000 |
| 60 | 50 | 10000 | 600000 | 500000 |

It is obvious that the book value moves like the market value once the share price has fallen below the purchase price. On the other hand it remains constant above the purchase price. The difference between market value and book value is called *revaluation reserve*. In case of a unit price 60, it is 100000. In the past some companies had considerable revaluation reserves in equities and also in properties (which are valued according book values in a similar way). If the company wanted to access a part of this reserve, it had to sell and potentially re-buy the instrument in order to *realise* the gain.

In a next step we want to look at the risks of a share. Normally one assumes that the share price (for one unit) $S_t$ moves according to a geometric Brownian motion. This implies that for any given time interval $\Delta t$ we have the following

$$\log\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim \mathcal{N}(\mu, \Delta t \times \sigma),$$

where $\mathcal{N}(\mu, \sigma)$ denotes the normal distribution with expected value $\mu$ and standard deviation $\sigma$. The parameter $\sigma$ is the standard deviation for a one year time horizon and it is called *volatility*. For typical equity indexes (eg a normalised basket of equities) the expected yield is in the area of 7 to 8%. The volatility depends on the market sentiment and ranges in normal markets between 15 to 25%. If there is a distressed market, the so called spot volatility can be well above 40%.

Based on the normal distribution assumption we can easily estimate the risk of a share. We know the probability distribution function for a random variable $X \sim \mathcal{N}(0, \sigma)$:

| x | $P[\frac{X}{\sigma} < x]$ |
|---|---|
| 1.000 | 84.134% |
| 1.281 | 90.000% |
| 1.644 | 95.000% |
| 2.326 | 99.000% |
| 2.575 | 99.500% |
| 3.090 | 99.900% |

Based on this, it is possible to form some rules of thumb. If we want to estimate the potential loss in a one in 200 year event we have to look at the 99.5% confidence level and we get 2.57. Assume now we want to estimate roughly the risk we are running for our 10000 shares at a market value of CHF 60 assuming a volatility of $\sigma = 18\%$. We get $10000 \times 2.57 \times 60 \times 0.18 = 277560$, which represents about 46% of the market value at this time. It is important to note that the rule of thumb uses an assumption, namely that the stock returns are log normally distributed. Since this is a model, it is only an approximation, and hence we might underestimate the true risk. Assume for sake of simplicity that we have observed a volatility of 18% over the last year. Our risk for a holding period of 1 year would be 277560. But a crisis might happen and the volatility could spike. In this case we would most likely underestimate the risk.

Before leaving shares we need to have a look at how volatility impacts the share price. Figure 2.4 shows a possible trajectory (share price development) over one year assuming three different levels of volatility 15%, 30% and 45%. We assume a drift $\mu = 7\%$ and a purchase price of CHF 500000.

## 2.5 Other Assets

This section will not go into the same detail as in the two previous section. Besides bonds and equities the most relevant asset classes an insurance company invests into are as follows:

**Cash** In order to be able to pay the claims due and because of regular premium income, insurance companies invest on a short time horizon into cash. This is a very risk adverse investment, where interest risk is normally virtually absent.

**Properties** In the past insurance companies have heavily invested in properties in order to safeguard the policy holder's assets. Properties are in a lot of regimes

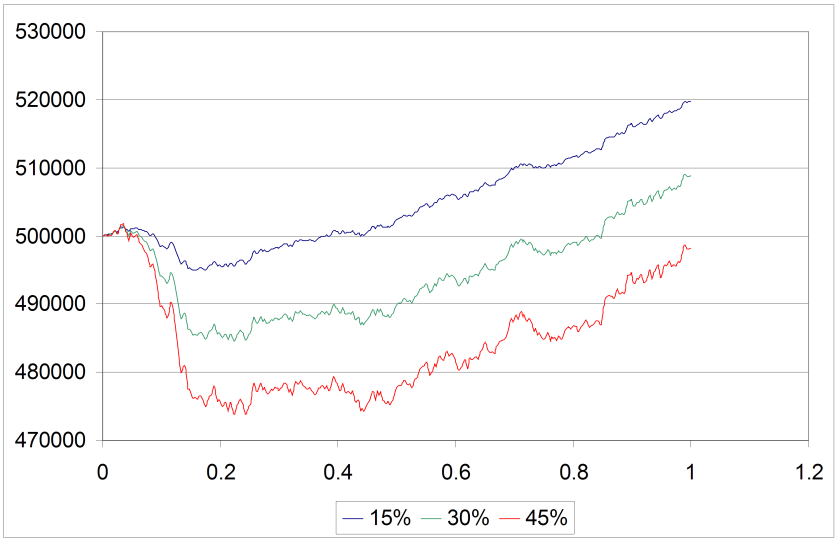



**Fig. 2.4** Possible Trajectories of Shares

accounted at depreciated purchase price. This means over time of 20 to 30 years the purchase price is amortised.

**Mortgages** A lot of people finance their properties by mortgages and in some countries it is a custom to buy an insurance policy which serves as collateral for the mortgage. As a consequence insurance companies have built up mortgage portfolios.

**Hedge Funds and Private Equities** These two asset classes of alternative assets are normally valued like a share. Please note however that from a risk point of view they may behave completely differently and hence it is very dangerous to use the same models.

**Commodities** Commodities such as oil, precious metals etc are not an often used asset class for insurance companies. In some countries they are not allowed for covering policyholder funds.

## 2.6 Insurance Liabilities

In this book it is not possible to describe and value all possible insurance liabilities and hence we should to focus on the most important life insurance liabilities. We can distinguish between insurance liabilities where the policyholder assumes all risk and consequently invests in funds (see also appendix D). Here the value is normally quite clear and so we can focus on life insurance forms with investment guarantees. In this case the majority of the investment risk is born by the insurance company. From a conceptual point of view life insurance cover behaves very similar to a bond. In principle one agrees some payments, which have to be weighted with the corresponding probabilities. In the following we want to introduce the corresponding concepts.

Insurance liabilities can be valued according to a book value or a market value principle. In the first case future cash flows are discounted using discount rates based on the technical interest rate $i$. In Europe this rate is determined in a prudent way and should according to the 3rd life insurance directive normally not exceeding 60% of the yield of governance bonds. So if we assume that governance bonds in EUR yields 4%, the maximal technical interest rate would be 2.4%. In reality the rule is interpreted in a somewhat more ingenious way and one looks for example at rolling averages of yields of government bonds. Based on the technical interest rate a payment of 1 due in one year is discounted with $v = \frac{1}{1+i}$. So here the book value approach yields to higher liabilities representing a prudent valuation approach.

In this section we will also focus on the market valuation on a best estimate basis. This is the first step to determine the market value of an insurance liability. We assume however that the insurance cash flows are certain. Since there is in reality a risk involved, it will be necessary to revisit the concept of market values for liabilities later (Section 4.3.1 and appendix C). We will see there how risk enters in the valuation and how we can use this knowledge for risk adjusted performance metrics.

### *2.6.1 Life Insurance Model*

In order to model a life insurance policy we consider a person aged $x$ and denote by $T$ the future life span and we remark that actually one would have to denote it $T(x)$ since it is dependent on the age $x$. The cumulative probability density function of $T$ is

$$G(t) = P[T \leq t], \tag{2.4}$$

and we assume that there exists a probability density function for T. Hence we can write:

$$g(t)dt = P[t < T < t + dt]. \tag{2.5}$$

In order to do life insurance mathematics it is useful the define the following standard quantities:

$$\begin{aligned}
{}_t q_x &:= G(t), && (2.6)\\
{}_t p_x &:= 1 - G(t), && (2.7)\\
{}_{s|t} q_x &:= P[s < T < s + t] && (2.8)\\
&= G(t+s) - G(s) = {}_{s+t} q_x - {}_s q_x, && (2.9)
\end{aligned}$$

$$\overset{\circ}{e}_x := \mathbb{E}[T(x)] = \int_0^\infty t g(t) dt \tag{2.10}$$

$$= \int_0^\infty (1 - G(t)) dt = \int_0^\infty {}_t p_x dt, \tag{2.11}$$

and we remark that $\overset{\circ}{e}_x$ is the expected future life span of a person aged $x$. We also remark that $q_x := {}_1q_x$ and $p_x := {}_1p_x$. Based on the above definitions we get the following equations:

$$\begin{aligned}
{}_tq_{x+s} &= G_{x+s}(t) && (2.12)\\
&= P[T(x+s) < t] && (2.13)\\
&= P[T(x) \leq s+t | T(x) > s] = \frac{G(s+t) - G(s)}{1 - G(s)}, && (2.14)\\
{}_tp_{x+s} &= P[T \geq s+t | T > s] = \frac{1 - G(s+t)}{1 - G(s)}, && (2.15)\\
{}_{s+t}p_x &= 1 - G(s+t) && (2.16)\\
&= (1 - G(s)) \frac{1 - G(s+t)}{1 - G(s)} = {}_sp_x \, {}_tp_{x+s}, && (2.17)\\
{}_{s|t}q_x &= G(s+t) - G(s) && (2.18)\\
&= (1 - G(s)) \frac{G(s+t) - G(s)}{1 - G(s)} = {}_sp_x \, {}_tq_{x+s}. && (2.19)
\end{aligned}$$

In order to calculate the quantities introduced above one normally uses mortality tables. Based on equation (2.17) we get the following

$$ {}_tp_x = \prod_{k=0}^{k<t} p_{x+k} = \prod_{k=0}^{k<t} (1 - q_{x+k}) \text{ for } t \in \mathbb{N}.$$

In order to simplify, one uses $K = \max\{k \in \mathbb{N}_0 : k \leq T\}$, the number of completely lived years before death.

### 2.6.2 *Capital Insurance*

Capital insurance is an insurance cover where there exists only one payment from the insurer during the contract and we distinguish between the following types of cover:

**Term Insurance and Whole Life Insurance:** In case of death a lump sum becomes due. The present value of this insurance type is denoted by $A^1_{x:\overline{n}|}$ if the

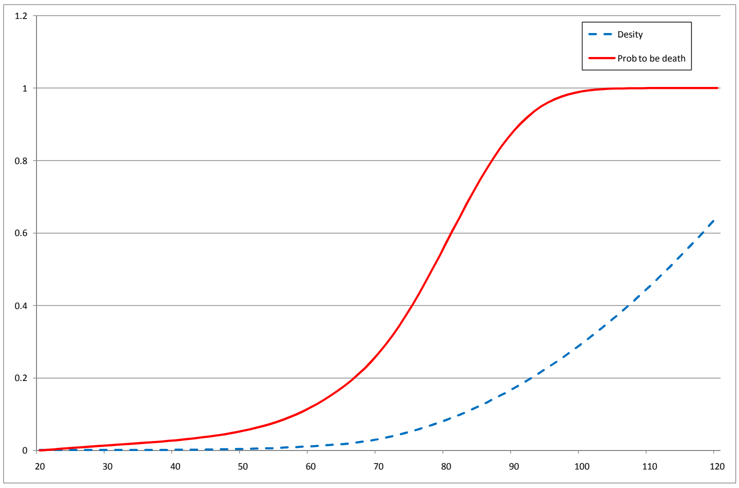



**Fig. 2.5** Probability density function for the future life span and hazard rate

cover is provided for $n$ years (eg a term insurance for a 45 year old person with a cover period of 10 years is denoted by $A^1_{45:\overline{10}|}$). A whole life insurance is an insurance where $n = \infty$ and we denote its present value by $A_x = A^1_{x:\overline{\infty}|}$.

**Pure Endowment:** In case a person reaches a certain age (eg 65) a lump sum becomes due. The present value of this type of insurance is denoted by $A_{x:\overset{1}{\overline{n}|}}$.

**Endowment:** Combination of the two types above, eg. if the person dies before the age 65 a lump sum becomes due at the moment of death, otherwise the person receives the lump sum at 65. The present value of this insurance is denoted by $A_{x:\overline{n}|}$. So the present value of the benefits to be paid for a 35 year old person with maturity at age 65 is denoted by $A_{35:\overline{30}|}$.

In order to value a life insurance policy we need to know its value. In the normal life insurance model one expects lump sums in case of death to become due at the end of the year.

### Whole Life Insurance

In case of death a payment of 1 is due, we have the following for the present value of the benefits as a random variable:

$$Z := v^{K+1}, \tag{2.20}$$

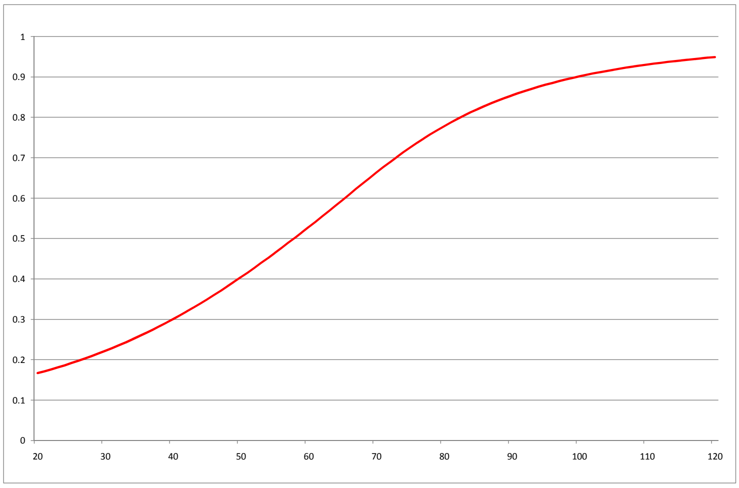



**Fig. 2.6** Value of the benefits of a whole life insurance

where $K = 0, 1, 2, \ldots$.. In case of a market consistent valuation $Z$ reads as follows:

$$Z := \pi(\mathcal{Z}_{(K+1)}), \tag{2.21}$$

$Z$ takes values $v$, $v^2$, $v^3, \ldots$ and $P[Z = v^{k+1}] = P[K = k] = {}_kp_x q_{x+k}$. Hence we get the following for the book value

$$A_x = \mathbb{E}\left[Z\right] = \mathbb{E}\left[v^{K+1}\right] = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k} \tag{2.22}$$

and

$$A_x = \mathbb{E}\left[Z\right] = \mathbb{E}\left[\pi(\mathcal{Z}_{(K+1)})\right] = \sum_{k=0}^{\infty} \pi(\mathcal{Z}_{(k+1)}) {}_kp_x q_{x+k} \tag{2.23}$$

for the market consistent valuation of the expected cash flows. In a next step we can calculate the variance of $Z$ as follows:

$$Var\left[Z\right] = \mathbb{E}\left[Z^2\right] - \mathbb{E}\left[Z\right]^2. \tag{2.24}$$

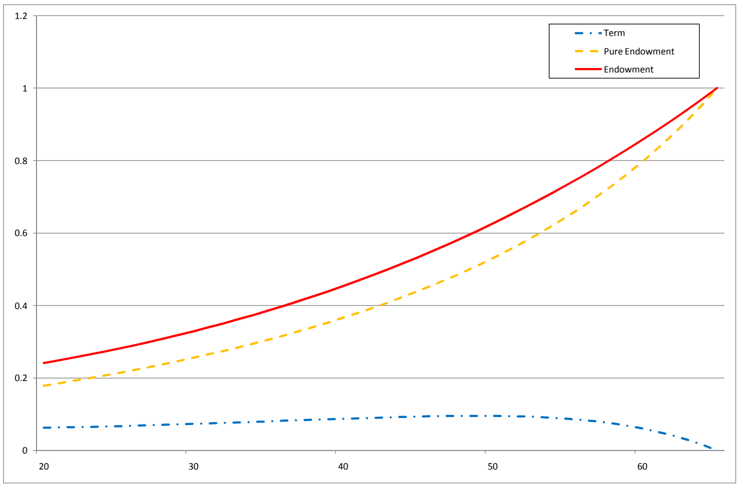



**Fig. 2.7** Value of a term insurance

### Term Insurance

The calculation is performed completely analogous: if the person dies within the contractual term (eg within $n$ years) a capital 1 becomes due. In consequence we get the following random variable for the present value of the insurance liability:

$$Z = \begin{cases} v^{K+1}, & \text{for } K = 0, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.25}$$

and hence we have the following for the book value valuation:

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x q_{x+k}.$$

For market values of the expected cash flows we have:

$$Z = \begin{cases} \pi(\mathcal{Z}_{(K+1)}), & \text{if } K = 0, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.26}$$

and hence

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} \pi(\mathcal{Z}_{(k+1)})\,{}_k p_x q_{x+k}.$$

### 2.6.3 Pure Endowment

The calculation is completely analogous. The only difference is the definition of the contractual payment stream.

$$Z = \begin{cases} 0, & \text{if } K = 0, 1, \ldots, n-1 \\ v^n, & \text{if } K = n, n+1, \ldots \end{cases} \tag{2.27}$$

and

$$\begin{aligned} A_{x:\overset{1}{\overline{n}|}} &= \sum_{k=0}^{\infty} Z(k) P[K = k] \\ &= \sum_{k=n}^{\infty} v^n P[K = k] = v^n P[K \geq n] \\ &= v^n(1 - P[K < n]) = v^n(1 - {}_n q_x) = v^n\,{}_n p_x. \end{aligned}$$

For the market value of the expected cash flows we have:

$$A_{x:\overset{1}{\overline{n}|}} = \mathcal{Z}_{(n)}\,{}_n p_x.$$

#### Endowment Insurance

Since an endowment is the sum of a term and a pure endowment insurance the arguments above apply mutatis mutandis and we get:

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}.$$

### 2.6.4 Annuities

As with capital insurance there exist a variety of different annuity covers and we need to focus on some particularly important ones:

**Immediate annuity:** This is an annuity where the insured person receives at the beginning of every year an annuity 1 until death. The present value of such an annuity is denoted by $\ddot{a}_x$.

**Deferred annuity:** Here the payment starts in the future, but otherwise it is the same as above. For the present value of the deferred annuity we use ${}_n\,\ddot{a}_x$, where $n$ stands for the number of years for which the annuity is deferred. So ${}_{30}\,\ddot{a}_{35}$ is a deferred annuity of a 35 year old person which is deferred by 30 years and hence starts at the age of 65.

**Temporary annuity:** This is the type of payment stream which is used to model a regular premium payment, starting immediately until death or when a certain term is reached.

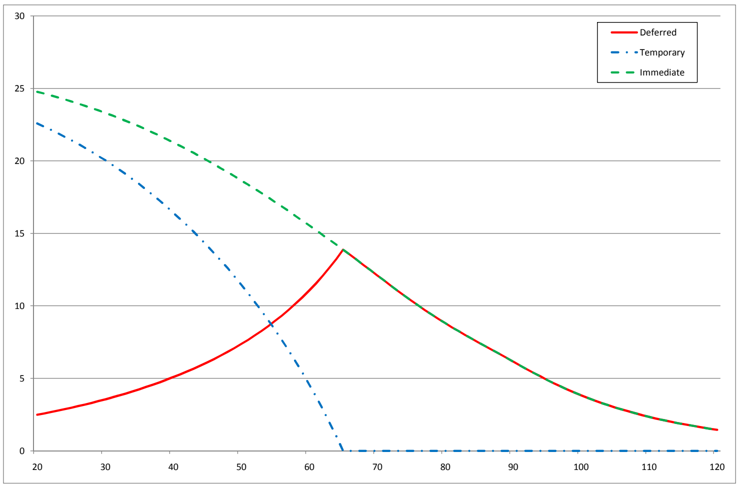



**Fig. 2.8** Value of different annuity covers

In order to evaluate the value of an annuity we need in a first step to define the corresponding present value as a random variable $Y$.

$$Y = 1 + v + v^2 + \ldots + v^K = \ddot{a}_{\overline{K+1|}} \tag{2.28}$$

and we know that $P[Y = \ddot{a}_{\overline{k+1|}}] = P[K = k] = {}_k p_x q_{x+k}$. Hence we can calculate the expected present value for the book valuation as follows:

$$\ddot{a}_x = \mathbb{E}\,[Y] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}}\, {}_k p_x q_{x+k}. \tag{2.29}$$

There is also a second possibility where we interpret an annuity as a portfolio of pure endowment policies and hence we can write:

$$Y = \sum_{k=0}^{\infty} v^k \chi_{\{K \geq k\}}. \tag{2.30}$$

Here the present value can be calculated as follows:

$$\ddot{a}_x = \mathbb{E}\left[Y\right] = \sum_{k=0}^{\infty} v^k P[K \geq k] = \sum_{k=0}^{\infty} v^k {}_k p_x.$$

We remark that for the market value of expected cash-flows we need to replace $v^k$ by $\mathcal{Z}_{(k)}$. In a next step it makes sense to indicate the relationship between capital insurance and annuities. We know the following relation:

$$Y = 1 + v + v^2 + \ldots + v^K = \frac{1 - v^{K+1}}{1 - v} = \frac{1 - v^{K+1}}{d},$$

which is valid as a random variable (please note that $d := \frac{1}{1-v}$). By applying the expected value operator we get the following useful relationship:

$$\begin{aligned} \ddot{a}_x = \mathbb{E}\left[Y\right] = \mathbb{E}\left[\frac{1-Z}{d}\right] &= \frac{1}{d} - \frac{\mathbb{E}\left[Z\right]}{d} \\ &= \frac{1 - A_x}{d}, \end{aligned}$$

or in terms of an actuary

$$1 = d\ddot{a}_x + A_x. \tag{2.31}$$

By means of the above equation we can also calculate the corresponding variances as follows:

$$Var\left[Y\right] = Var\left[\frac{1-Z}{d}\right] = \frac{1}{d^2} Var\left[Z\right]. \tag{2.32}$$

Please note that the above relationship is not true for the market consistent present value of expected cash-flows since we normally do not have a flat yield curve.

### *2.6.5 Cash Flows and Valuation*

After this introduction to life insurance mathematics we want now to look at an example and we will focus on an immediate annuity for a 65 year old man. We assume that the annuity is paid in yearly instalments of EUR 12000. Furthermore we want to look at a valuation as of 29.12.2006 for the market values and a technical interest rate of $i = 2.5\%$ for the calculation of book values. So the annuity can be characterised as follows:

| **ANNUITY** | |
|---|---|
| Yearly payment - prenummerando | 12000 |
| Age | 65 |
| Currency | EUR |
| Valuation Date | 29.12.2006 |
| Technical Interest Rate | 2.5% |
| Mortality Table | Swiss Males |

Based on the above assumptions we get the following results:

| Age | ${}_t p_x$ | Annuity Nominal | Annuity Risk Ajd. | Price $i = 2.5\%$ | Price $\mathcal{Z}_{(t)}$ | Value Book | Value Market |
|---|---|---|---|---|---|---|---|
| 65 | 1.0000 | 12000 | 12000.00 | 1.0000 | 1.0000 | 12000.00 | 12000.00 |
| 66 | 0.9852 | 12000 | 11822.40 | 0.9756 | 0.9627 | 11534.04 | 11381.52 |
| 67 | 0.9692 | 12000 | 11630.87 | 0.9518 | 0.9266 | 11070.43 | 10778.13 |
| 68 | 0.9517 | 12000 | 11421.52 | 0.9285 | 0.8919 | 10606.01 | 10187.71 |
| 69 | 0.9331 | 12000 | 11197.65 | 0.9059 | 0.8577 | 10144.52 | 9605.12 |
| 70 | 0.9132 | 12000 | 10959.14 | 0.8838 | 0.8251 | 9686.29 | 9043.21 |
| 71 | 0.8918 | 12000 | 10702.70 | 0.8622 | 0.7935 | 9228.90 | 8492.71 |
| 72 | 0.8688 | 12000 | 10426.57 | 0.8412 | 0.7628 | 8771.51 | 7954.28 |
| 73 | 0.8441 | 12000 | 10129.93 | 0.8207 | 0.7335 | 8314.11 | 7430.54 |
| 74 | 0.8180 | 12000 | 9816.72 | 0.8007 | 0.7059 | 7860.52 | 6929.83 |
| 75 | 0.7900 | 12000 | 9481.08 | 0.7811 | 0.6782 | 7406.61 | 6430.32 |
| 76 | 0.7603 | 12000 | 9124.03 | 0.7621 | 0.6512 | 6953.83 | 5942.15 |
| 77 | 0.7288 | 12000 | 8746.38 | 0.7435 | 0.6251 | 6503.42 | 5468.02 |
| 78 | 0.6957 | 12000 | 8348.95 | 0.7254 | 0.5999 | 6056.49 | 5008.88 |
| 79 | 0.6610 | 12000 | 7932.33 | 0.7077 | 0.5755 | 5613.93 | 4565.42 |
| 80 | 0.6247 | 12000 | 7497.17 | 0.6904 | 0.5519 | 5176.53 | 4138.19 |
| 81 | 0.5870 | 12000 | 7044.19 | 0.6736 | 0.5297 | 4745.14 | 3731.38 |
| 82 | 0.5478 | 12000 | 6574.20 | 0.6571 | 0.5082 | 4320.53 | 3341.35 |
| 83 | 0.5073 | 12000 | 6088.43 | 0.6411 | 0.4875 | 3903.69 | 2968.54 |
| 84 | 0.4657 | 12000 | 5588.69 | 0.6255 | 0.4676 | 3495.88 | 2613.50 |
| 85 | 0.4231 | 12000 | 5077.49 | 0.6102 | 0.4484 | 3098.64 | 2276.94 |
| ... | | | | | | | |
| 90 | 0.2098 | 12000 | 2517.90 | 0.5393 | 0.3649 | 1358.13 | 918.94 |
| ... | | | | | | | |
| 100 | 0.0031 | 12000 | 37.95 | 0.4213 | 0.2471 | 15.99 | 9.37 |
| ... | | | | | | | |
| **Total** | | 552000.00 | 216736.59 | | | **170312.90** | **149811.51** |

The similarity of the above table with defaultable bonds is because of their intrinsic structural proximity. We can in particular observe that the reserve hold as book values is some 20000 EUR higher. One way to calculate the so called mathematical reserves $V_x$ is based on the approach outlined above. There is also another efficient way to recursively calculate the reserves, based on the so called Thiele's difference equation. To this end we denote by $(a_k)_{k \in \mathbb{N}_0}$ the annuity vector to be paid at age $k$ and by $(d_k)_{k \in \mathbb{N}_0}$ the death benefit at age $k$. Furthermore we denote by $v$ the one year discount rate, which is in the market value context the corresponding discount rate based on the forward rate for the corresponding time. Equipped with this notation we get the following backwards recursion:

$$V_x = a_x + q_x \times v \times b_x + p_x \times v \times V_{x+1}.$$

For a proof of the above recursion we refer to appendix B.

### 2.6.6 Primer on Life Insurance Risks

There are typically two different risks which impact the level of reserves. The level of the interest rate, just in the same spirit as with bonds. Hence we do not show the corresponding example here and remark that the duration of the annuity above is about 8.38. A parallel shift of the interest rate by 1% changes the corresponding reserve by about 8.4%.

The second risk is the one related to the mortality risk. The table below shows the sensitivity in respect to mortality and we assume that the stressed mortality with respect to a parameter $\alpha$ is defined as $q_x(\alpha) = \alpha \times q_x$. Based on this we get the following:

| Age<br>$\alpha$ | ${}_t p_x$<br>90% | ${}_t p_x$<br>100% | ${}_t p_x$<br>110% | MV<br>90% | MV<br>100% | MV<br>110% |
|---|---|---|---|---|---|---|
| 65 | 1.0000 | 1.0000 | 1.0000 | 12000.00 | 12000.00 | 12000.00 |
| 66 | 0.9866 | 0.9852 | 0.9837 | 11398.62 | 11381.52 | 11364.42 |
| 67 | 0.9722 | 0.9692 | 0.9661 | 10812.09 | 10778.13 | 10744.22 |
| 68 | 0.9565 | 0.9517 | 0.9470 | 10238.55 | 10187.71 | 10137.04 |
| 69 | 0.9396 | 0.9331 | 0.9266 | 9672.35 | 9605.12 | 9538.24 |
| 70 | 0.9216 | 0.9132 | 0.9049 | 9126.33 | 9043.21 | 8960.70 |
| 71 | 0.9022 | 0.8918 | 0.8816 | 8591.31 | 8492.71 | 8395.06 |
| 72 | 0.8812 | 0.8688 | 0.8566 | 8067.93 | 7954.28 | 7842.00 |
| 73 | 0.8587 | 0.8441 | 0.8298 | 7558.78 | 7430.54 | 7304.20 |
| 74 | 0.8348 | 0.8180 | 0.8015 | 7071.92 | 6929.83 | 6790.26 |
| 75 | 0.8091 | 0.7900 | 0.7714 | 6585.40 | 6430.32 | 6278.51 |
| ... | | | | | | |
| 80 | 0.6553 | 0.6247 | 0.5954 | 4340.88 | 4138.19 | 3944.25 |
| ... | | | | | | |
| 85 | 0.4621 | 0.4231 | 0.3871 | 2486.94 | 2276.94 | 2083.60 |
| ... | | | | | | |
| 90 | 0.2469 | 0.2098 | 0.1779 | 1081.75 | 918.94 | 779.31 |
| ... | | | | | | |
| 100 | 0.0063 | 0.0031 | 0.0015 | 18.71 | 9.37 | 4.51 |
| ... | | | | | | |
| **Total** | | | | **154327.11** | **149811.51** | **145675.21** |
| *Difference* | | | | *4515.60* | | *-4136.30* |

## 2.7 Shareholders Equity and Capital

After having discussed all the other assets and liabilities we will devote a short section to the shareholder equity or capital. As we have seen before there are different ways on how the different assets and liabilities are accounted. In principle shareholders equity is the difference between all assets and all liabilities other than the shareholder's equity. There might also be some adjustments for taxes not yet paid. The shareholder's equity represents the worth the shareholder has in the entity, and the purpose of the equity is twofold. It gives the shareholder the right to get a return on the capital he has invested. Moreover the shareholder equity or capital acts as a buffer in case of an adverse market development.

Since we want to illustrate this we will look first at a balance sheet based on book values and afterwards at the same balance sheet based on market values. In order to simplify we assume that we are living in a country with no taxes. Please note that we denote the mathematical reserves with MR. The insurance company we are looking at has the following balance sheet:

| Balance sheet | Book **A** | Book **L** | Market **A** | Market **L** | |
|---|---|---|---|---|---|
| Cash | 6200 | 47100 | 6200 | 48513 | MR |
| Bonds | 35700 | **2200** | 37842 | **3569** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 52082 | 52082 | |

From this balance sheet it becomes obvious that there are revaluation reserves in both bonds and shares and that the mathematical reserves are (not taking the interest rate effect into account) about 5% too high. For the example we assume a duration of the bonds of 6 and of the reserves of 8. This means that the company is suffering from an economical point of view in case of decreasing interest rates.

Another typical effect is the fact that the shareholder capital based on a book value approach is about 1500 less than if using an economic valuation. This means that the risk absorbing capacity of the company in nominal terms is higher if we look at a realistic valuation. We need to stress that the statutory accounts in continental Europe often use book value accounting. Hence the dividend paying capacity is based on the lower equity (eg 2200) and therefore it is of utmost importance to always keep also this number in mind. Since local insolvency laws are also based on the statutory shareholder equity, the company would be in deep trouble if this number reduces too much.

But now lets look at the following three scenarios:

- Drop in equity prices by 20%,
- Interest rate decrease by 1% and
- Interest rate increase by 1%.

In the first scenario we look at a 20% decrease in stock markets. It becomes obvious that the reduction in shareholder equity is proportionally smaller in the case of book value accounting, since the first hit is absorbed by the revaluation reserves which were present before the shock in the equities. Since the shock is bigger than the revaluation reserve, also in the book value accounting, the value of the shares had to be adjusted downwards in accordance with the "lower book or market" principle.

| Balance sheet | Book | Book | Market | Market | |
|---|---|---|---|---|---|
| *Shares -20%* | **A** | **L** | **A** | **L** | |
| Cash | 6200 | 47100 | 6200 | 48513 | MR |
| Bonds | 35700 | **1640** | 37842 | **2609** | SHE |
| Shares | 3840 | | 3840 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 48740 | 48740 | 51122 | 51122 | |

In the second scenario we see the impact of a reduction in interest rate levels by 1%. Due to the nature of the amortised cost method, the effect is not reflected in the accounts in the book value world. We also see a material deteriorisation in the marked to market balance sheet, as a consequence of the duration gap of 2. We can see that a further reduction of the interest rate levels could become dangerous for the insurance company. As remarked before book value accounting "is blind" with respect to the issue in case of low interest rate in its purest form. In order to compensate for it there are reserve adequacy tests where one tests, whether the earned interest rate yield is sufficient for financing the technical interest rate for the reserves. If this is not the case the reserves would have to grow commensurately, hereby reducing the statutory shareholder equity.

| Balance sheet | Book | Book | Market | Market | |
|---|---|---|---|---|---|
| *Int. -1%* | **A** | **L** | **A** | **L** | |
| Cash | 6200 | 47100 | 6200 | 52281 | MR |
| Bonds | 35700 | **2200** | 39984 | **1943** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 54224 | 54224 | |

The third scenario is a twin of the second one, just with the opposite direction of the interest rate shift. Here the available economic capital grows as a consequence of the duration gap. At this point it is worth mentioning that there are also other accounting standards, which are not symmetrical in the sense that book and market value principles are not applied in a consistent manner. IFRS for example can foresee market value principles for assets and book value principles for liabilities. We can see the corresponding consequences by regrouping the values accordingly. On doing so we acknowledge that under IFRS the shareholder equity reduces in case of an increase in interest rates for the simple reason, that the bonds lose value which is not compensated by a decrease in the policyholder mathematical reserves. Comparing IFRS accounting standards with economic principles also highlights one of the important paradoxes, namely that the two accounting standards contradict and that one can optimise both of them at the same time only in a limited manner.

| Balance sheet | Book | Book | Market | Market | |
|---|---|---|---|---|---|
| *Int. +1%* | **A** | **L** | **A** | **L** | |
| Cash | 6200 | 47100 | 6200 | 44745 | MR |
| Bonds | 35700 | **2200** | 35700 | **5195** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 49940 | 49940 | |

# Chapter 3

# Equity Derivatives and Unit-linked policies policies with Guarantees

## 3.1 Introduction

Up to now we have mostly considered models with deterministic interest rates or with an interest rate given by a Markov chain on a finite state space. This helped us to keep the calculations simple. In this chapter we have a look at variable annuities. We consider models for policies whose actual value depends on the performance of an underlying unit (usually a funds). Since we do not need at the beginning the entire complexity of a Markov model for the underlying demographic process, we will start using the simpler traditional approach. In case the reader is not aware of this we will formally introduce this below.

**Definition 1** *A classical life insurance model consists of a Markov model with a state space $S = \{\star, \dagger\}$, where $\star$ and $\dagger$ represent the states of being alive or death, respectively. We assume that the state $\dagger$ is absorbing (ie $p_{\dagger,\star} \equiv 0$. If we assume that the person is alive at age $x$ (ie $X_x = \star$), we can define the future life span $T_x$of a person aged $x$ as*

$$T_x = min\{\xi > x | X_\xi = \star\} - x,$$

*and we use the following (common) notations (for $\Delta_x > 0,\ t > 0$):*

$$\begin{aligned} {}_tq_{x+\Delta_x} &= P(T_x \leq t + \Delta_x | T_x > \Delta_x) \\ {}_tp_{x+\Delta_x} &= P(T_x > t + \Delta_x | T_x > \Delta_x) \\ \mu_{x+\Delta_x} &= \lim_{\Delta \to 0} \frac{{}_\Delta q_{x+\Delta_x}}{\Delta} \end{aligned}$$

**Remark 2** *In the same sense we can allow for a slightly refined insurance model allowing for lapse (or also called surrender). In this case the state space consists of $S = \{\star, \dagger\, \ddagger\}$, where $\ddagger$ represents the additional state of lapse. It is worth noting that also this state is absorbing. Furthermore, we have in this case (as also in the above case)*

$$\begin{aligned} {}_tp_x &= p_{\star,\star}(x, x+t) \\ &= \bar{p}_{\star,\star}(x, x+t). \end{aligned}$$

We begin with a look at policies whose value is tied to a bond or a funds. The payout of these so called “unit-linked policies” usually consists of a certain number of shares of a funds (the underlying unit) to the insured, in case of an occurrence of the insured event.

These policies have the characteristic feature that the level of the benefits (endowments or death benefits) are not deterministic, but random, depending on the underlying funds. A unit-linked policy is usually financed by a single premium. This type of financing is preferred due to the management of these policies. Note that this is in contrast to traditional policies. Moreover one has to note that the value at risk is constant for a traditional policy, but for a unit-linked policy it depends on the underlying funds.

To analyse unit-linked policies, we introduce the following notation:

$N(t)$ number of shares at time $t$,
$S(t)$ value of a share at time $t$.

We assume that $N(t)$ is deterministic. The relevant quantities for a life insurance in this setting and in the traditional setting are summarised in the following table:

| | traditional | (pure) unit-linked |
|---|---|---|
| death benefit | $C(t) = 1$ | $C(t) = S(t)$ |
| value (time 0) | $\pi_0(t) = \exp(-\delta t)$ | $\pi_0(t) = S(0)$ (assuming a "normal" economy) |
| single premium | $E\left[\int_0^T \pi_0(t) d(\chi_{T_x \leq t})\right]$ $= \int_0^T \exp(-\delta t)_t p_x \mu_{x+t} dt$ | $E\left[\int_0^T \pi_0(t) d(\chi_{T_x \leq t})\right]$ $= S(0) \int_0^T {}_t p_x \mu_{x+t} dt$ $= (1 - {}_T p_x) S(0)$ |

The above terms indicate that the financial risk taken by the insurer is smaller for a unit-linked product than for a traditional product with a fixed technical interest rate[1]. Furthermore, note that in the calculation of the single premium we implicitly assumed that the mean of the discounted (to time 0) value of the funds at time $t$ coincides with the value of the funds at time 0. This means, that we have to start with a discussion of the value or price of a funds.

Also the model did not include any guarantees. But generally one would like to add a guarantee (e.g. a refund guarantee for the paid in premiums) to the policy. In this case we have a unit linked insurance with a guarantee also known as variable annuity. For example, the guarantee could be of the form

$$G(t) = \int_0^t \bar{p}(s) ds,$$

where $\bar{p}(s)$ denotes the density of the premiums at time $s$. More generally one could be interested in a refund guarantee of the paid in premiums with an additional interest at a fixed rate:

$$G(t) = \int_0^t \exp(r(t-s)) \bar{p}(s) ds.$$

In these examples the payout function would be

$$C(t) = \max(S(t), G(t)).$$

[1] Actually, this is not true in general. Here we implicitly assumed that the capital market risks of a unit-linked policy are minimised by an appropriate trading strategy. For a classical insurance such a trading strategy replicates the cash flows by zero coupon bonds with the corresponding maturities.

Let us assume that the value of the funds is given by a stochastic process (with distribution $P$). What is, in this setting, the value of the discounted payment $C(t)$ at time $t$? A first guess might be

$$\pi_0\left(C(t)\right) = E^P\left[\max(S(t), G(t))\right].$$

But it is not that simple! If this would be the value of the payment, there would be the possibility to make a profit without risk (arbitrage). In order to prevent this possibility one has to change the measure $P$. In mathematical finance it is proved that an equivalent martingale measure exists, such that there is no arbitrage. Then in a "fair" market we have

$$\pi_0\left(C(t)\right) = E^Q\left[\max(S(t), G(t))\right],$$

where $Q$ is a measure equivalent to $P$ such that the discounted value of the underlying funds is a martingale.

Furthermore, in mathematical finance payments like $C(t)$ are called the payouts of an option. To determine the price of an option, one uses the arbitrage free pricing theory. A quick introduction to this theory will be given in the next section.

## 3.2 Pricing theory

In this section we look at modern financial mathematics. It is not our aim to give a comprehensive exposition with proofs of every detail, which would easily fill a whole book. We only want to give a brief survey which illustrates the theory. The reader interested in more details is referred to [Pli97], [HK79], [HP81] and [Duf92]. In the same sense we suggest [**?**] for further reading regarding stochastic integrals and stochastic differential equations.

In this context we clearly also have to mention the paper of Black and Scholes [BS73] with their famous formula for the pricing of options.

### *3.2.1 Definitions*

First we start with an example which illustrates the use of the pricing theory. The price of a share, modelled by a geometric Brownian motion $(S_t(\omega))$, might develop as shown in Figure 3.1.

A European call option for a certain share is the right to buy these shares at a fixed price $c$ (strike price) at a fixed time $T$. The value of this right at time $T$ is

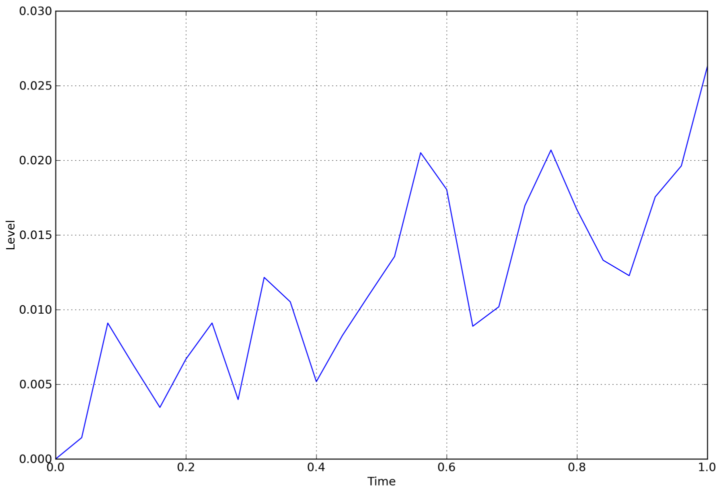



**Fig. 3.1** Movement of a share price

$$H = \max\left(S_T - c, 0\right).$$

Now a bank would like to know the value (i.e., the fair price) of this option at time $0$. As noted in the previous section, taking the expectation would systematically yield the wrong values. In many cases this would provide the possibility to make profits without risk. Thus there would be arbitrage opportunities.

To simplify the exposition we will consider the most simple models of the economy, i.e. finite models. In particular the time set will be discrete. The reader interested in the corresponding theorems for continuous time can for example find these in [HP81]. The ideas and concepts of the pricing theory are presented, as follows:

Let $(\Omega, \mathcal{A}, P)$ be a probability space where $\Omega$ is a finite set. Moreover, we assume that $P(\omega) > 0$ holds for all $\omega \in \Omega$.

We also fix a finite time $T$, as the time at which all trading is finished. The $\sigma$-algebra of the observable events at time $t$ is denoted by $\mathcal{F}_t$, and the shares are traded at the times $\{0, 1, 2, \ldots, T\}$. We note that $(\mathcal{F}_t)_{t \in T}$ is a filtration, ie $\mathcal{F}_t \subseteq \mathcal{F}_s$, for $t \leq s$.

We suppose that there are $k < \infty$ stochastic processes, which represent the prices of the shares $1, \ldots, k$, i.e.,

$$S = \{S_t, t = 0, 1, 2, \ldots, T\} \text{ with components } S^0, S^1, \ldots S^k.$$

As usual, we assume that each $S^j$ is adapted to $(\mathcal{F}_t)_t$. Here $S_t^j$ can be understood as the price of the $j$th share at time $t$. The fact that the price process has to be adapted reflects the necessity that one has to know at time $t$ the previous price of $S$. The share $S^0$ plays a special role. We suppose that $S_t^0 = (1+r)^t$, i.e., we have the possibility to make risk free investments which provide interest rate $r$. The risk free discount factor is defined by

$$\beta_t = \frac{1}{S_t^0}.$$

Next, we are going to define what is meant by a trading strategy.

**Definition 3** *A* trading strategy *is a previsible ($\phi_t \in \mathcal{F}_{t-1}$) process $\Phi = \{\phi_t, t = 1, 2, \ldots, T\}$ with components $\phi_t^k$.*

*We understand $\phi_t^k$ as the number of shares of type $k$ which we own during the time interval $[t-1, t[$. Therefore $\phi_t$ is called the portfolio at time $t-1$.*

**Notation 4** *Let $X, Y$ be vector valued stochastic processes. Then we use the notations:*

$$< X_s, Y_t > = X_s \cdot Y_t = \sum_{k=0}^{n} X_s^k \times Y_t^k,$$
$$\Delta X_t = X_t - X_{t-1}.$$

Next, we want to determine the value of the portfolio at time $t$:

| time | value of the portfolio |
|---|---|
| $t-1$ | $\phi_t \cdot S_{t-1}$ |
| $t^-$ | $\phi_t \cdot S_t$ |

Thus, the return in the interval $[t-1, t[$ is $\phi_t \cdot \Delta S_t$, and hence the total return is

$$G_t(\phi) = \sum_{\tau=1}^{t} \phi_\tau \cdot \Delta S_\tau.$$

We fix $G_0(\phi) = 0$, and $(G_t)_{t \geq 0}$ is called return process.

**Proposition 5** *$G$ is an adapted and real valued stochastic process.*

*Proof.* The proof is left as an exercise to the reader.

**Definition 6** *A trading strategy is* self financing, *if*

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \qquad \forall t = 1, 2, \ldots, T-1.$$

A self financing trading strategy is just a trading strategy where at no time further money is added to or deduced from the portfolio.

**Definition 7** *A trading strategy is* admissible*, if it is self financing and*

$$V_t(\phi) := \begin{cases} \phi_t \cdot S_t, \ \text{if } t = 1, 2, \ldots, T, \\ \phi_1 \cdot S_0, \ \text{if } t = 0 \end{cases}$$

*is non negative. (In other words, one is not allowed to become bankrupt.) The set of admissible trading strategies is denoted by $\Phi$.*

**Remark 8** *The idea of admissible trading strategies is to only consider portfolios which neither lead to bankruptcy nor allow an addition or deduction of money. This also indicates that the value of the trading strategies remains constant when the portfolio is rearranged. Thus a trading strategy, which generates the same cash flow as an option, can be used to determine the value of the option.*

**Definition 9** *A contingent claim is a positive random variable $X$. The set of all contingent claims is denoted by $\mathcal{X}$.*

*A random variable $X$ is attainable, if there exists an admissible trading strategy $\phi \in \Phi$ which replicates it, i.e.*

$$V_T(\phi) = X.$$

*In this case one says "$\phi$ replicates $X$".*

**Definition 10** *The price of an attainable contingent claim, which is replicated by $\phi$, is denoted by*

$$\pi = V_0(\phi)$$

*(We will see later, that this price is not necessarily unique. It coincides with the initial value of the portfolio.)*

### *3.2.2 Arbitrage*

We say, the model offers arbitrage opportunities, if there exists

$$\phi \in \Phi \text{ with } V_0(\phi) = 0 \text{ and } V_T(\phi) \text{ positive and } P[V_T(\phi) > 0] > 0,$$

i.e., money is generated out of nothing. If such a strategy exists, one can make a profit without taking any risks. One of the axioms of modern economy says that there are no arbitrage opportunities. This is fundamental for some important facts in the option pricing theory.

Now, we are going to define what is meant by a price system.

**Definition 11** *A mapping*

$$\pi : \qquad \mathcal{X} \to [0, \infty[, \qquad X \mapsto \pi(X)$$

*is called* price system *if and only if the following conditions hold:*

- $\pi(X) = 0 \iff X = 0,$
- $\pi$ *is linear.*

*A price system is* consistent*, if*

$$\pi(V_T(\phi)) = V_0(\phi) \qquad \text{for all } \phi \in \Phi.$$

*The set of all consistent price systems is denoted by $\Pi$, and $\mathbb{P}$ denotes the set*

$$\mathbb{P} = \{Q \text{ is a measure equivalent to } P, \text{ s.th. } \beta \times S \text{ is a martingale w.r.t. } Q\},$$

*where $\beta$ is the discount factor from time $t$ to $0$. The measures $\mu \in \mathbb{P}$ are called equivalent martingale measures.*

**Proposition 12** *There is a bijection between the consistent price systems $\pi \in \Pi$ and the measures $Q \in \mathbb{P}$. It is given by*

1. $\pi(X) = E^Q\left[\beta_T\, X\right]$.
2. $Q(A) = \pi(S_T^0\, \chi_A)$ *for all $A \in \mathcal{A}$.*

*Proof.* Let $Q \in \mathbb{P}$. We define $\pi(X) = E^Q\left[\beta_T\, X\right]$. Then $\pi$ is a price system, since $P$ is strictly positive on $\Omega$ and $Q$ is equivalent to $P$. Thus it remains to show, that $\pi$ is consistent. For $\phi \in \Phi$ we get

$$\begin{aligned}
\beta_T\, V_T(\phi) &= \beta_T\, \phi_T\, S_T + \sum_{i=1}^{T-1} (\phi_i - \phi_{i+1})\, \beta_i\, S_i \\
&= \beta_1\, \phi_1\, S_1 + \sum_{i=2}^{T} \phi_i\, \left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right),
\end{aligned}$$

where we used that $\phi$ is self financing. This yields

$$\begin{aligned}
\pi\left(V_T(\phi)\right) &= E^Q\left[\beta_T\, V_T(\phi)\right] \\
&= E^Q\left[\beta_1\, \phi_1\, S_1\right] + E^Q\left[\sum_{i=2}^{T} \phi_i\, \left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right)\right] \\
&= E^Q\left[\beta_1\, \phi_1\, S_1\right] + \sum_{i=2}^{T} E^Q\left[\phi_i\, E^Q\left[\left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right) | \mathcal{F}_{i-1}\right]\right] \\
&= \phi_1\, E^Q\left[\beta_1\, S_1\right]
\end{aligned}$$

$$= \phi_1 \, \beta_0 \, S_0,$$

since $\phi$ is previsible and $\beta \, S$ is a martingale with respect to $Q$.

Thus, $\pi$ is a consistent price system.

Now let $\pi \in \Pi$ be a consistent price system and $Q$ be defined as above. Then $Q(\omega) = \pi(S_t^0 \, \chi_{\{\omega\}}) > 0$ holds for all $\omega \in \Omega$, since $S_t^0 \, \chi_{\{\omega\}} \neq 0$. Moreover, we have $\pi(X) = 0 \iff X = 0$ and therefore $Q$ is absolutely continuous with respect to $P$.

In the next step, we are going to show that $Q$ is a probability measure. We define

$$\phi^0 = 1 \qquad \text{and} \qquad \phi^k = 0 \quad \forall k \neq 0.$$

Hence, by the consistency of $\pi$, we get

$$\begin{aligned} 1 &= V_0(\phi) \\ &= \pi(V_T(\phi)) \\ &= \pi(S_T^0 \cdot 1) \\ &= Q(\Omega). \end{aligned}$$

The prices of positive contingent claims are positive and $Q$ is additive. Therefore, Kolmogorov's axioms are satisfied, since $\Omega$ is finite. We have $Q(\omega) = \pi(S_T^0 \cdot \chi_{\{\omega\}})$ by definition. Hence, also

$$E[f] = \sum_\omega \pi(S_T^0 \cdot \chi_{\{\omega\}}) \cdot f(\omega) = \pi(S_T^0 \cdot \sum_\omega f(\omega)).$$

Thus, with $f = \beta_t \, X$, we have

$$E^Q[\beta_T \, X] = \pi(S_T^0 \cdot \beta_T \cdot X) = \pi(X).$$

Now we still have to show that $\beta_T \, S_T^k$ is a martingale for all $k$. Let $k$ be a coordinate and $\tau$ be a stopping time, and set

$$\begin{aligned} \phi_t^k &= \chi_{\{t \leq \tau\}}, \\ \phi_t^0 &= \left(S_\tau^k / S_\tau^0\right) \, \chi_{\{t > \tau\}}. \end{aligned}$$

(We keep the share $k$ up to time $\tau$, then it is sold and the money is used for a risk free investment.) It is easy to show, that the strategy $\phi$ is previsible and self financing. Finally, for an arbitrary stopping time $\tau$,

$$\begin{aligned} V_0(\phi) &= S_0^k, \\ V_T(\phi) &= \left(S_\tau^k / S_\tau^0\right) S_T^0 \end{aligned}$$

and

$$\begin{aligned} S_0^k &= \pi(S_T^0 \cdot \beta_\tau \cdot S_\tau^k) \\ &= E^Q\left[\beta_\tau \cdot S_\tau^k\right]. \end{aligned}$$

Thus $\beta_T\, S_T^k$ is a martingale with respect to $Q$.

Above we have proved one of the main theorems in the option pricing theory. Next, we will present further statements without proofs. They all can be found for example in [HP81].

**Theorem 13** *The following statements are equivalent*

1. *There is no arbitrage opportunity,*
2. $\mathbb{P} \neq \emptyset$,
3. $\Pi \neq \emptyset$.

**Lemma 1.** *Suppose there exists a self financing strategy $\phi \in \Phi$ such that*

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0,\ E[V_T(\phi)] > 0.$$

*Then there exists an arbitrage opportunity.*

**Example 14** *We are going to calculate the price of an option for a simple example. Consider a market with two shares $Z = (Z_1, Z_2)$ which are traded at the times $t = 0$, $t = 1$ and $t = 2$. Figure 3.2 shows the possible behaviour of these shares in form of a tree. To calculate the price of the option we suppose that all nine possibilities have the same probability.*

*We want to calculate the price of a complex option given by*

$$X = \{2Z_1(2) + Z_2(2) - [14 + 2\min\left(\min\left\{Z_1(t), Z_2(t)\right\}, 0 \leq t \leq 2\right)]\}^+ .$$

*First of all, we have to find an equivalent martingale measure. Thus we have to solve for the times $t = 0$ and $t = 1$ the following equations:*

$$\begin{aligned} 10 &= 11p + 11q + 8r, && \text{(martingale condition for } Z_1) \\ 10 &= 9p + 10q + 11r, && \text{(martingale condition for } Z_2) \\ 1 &= p + q + r. \end{aligned}$$

*The solution to these equations is $p = q = r = \frac{1}{3}$.*

*Here we can see explicitly which circumstances imply the existence and uniqueness of a martingale measure. In this example the martingale measure is, from a geometric point of view, defined as the intersection of three hyper-planes. Depending on their orientation, there is either one or there are many or there is none equivalent martingale measure.*

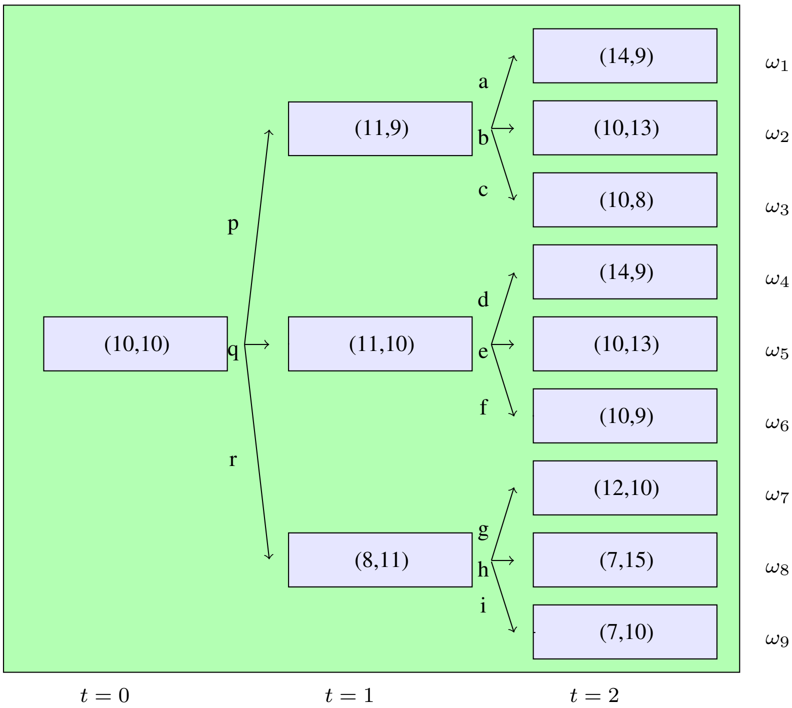



**Fig. 3.2** Example calculation of an option price

*Next, we can derive the xsequations for the times* $t = 1$ *and* $t = 2$*. These are*

$$\begin{aligned}
11 &= 14a + 10b + 10c,\\
9 &= 9a + 13b + 8c,\\
1 &= a + b + c,
\end{aligned}$$

$$\begin{aligned}
11 &= 14d + 10e + 10f,\\
10 &= 9d + 13e + 9f,\\
1 &= d + e + f,
\end{aligned}$$

$$\begin{aligned}
8 &= 12g + 7h + 7i,\\
11 &= 10g + 15h + 10i,\\
1 &= g + h + i,
\end{aligned}$$

*and they are solved by*

$$(a,b,c) = (\frac{1}{4}, \frac{3}{20}, \frac{3}{5})$$
$$(d,e,f) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$$
$$(g,h,i) = (\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$$

*Now we know the transition probabilities with respect to the martingale measure, which enables us to calculate the martingale measure $Q$ itself. The results of these calculations are summarised in the following table:*

| *state* | $X(\omega_i)$ | $Q(\omega_i)$ |
|---|---|---|
| $\omega_1$ | 5 | $1/12$ |
| $\omega_2$ | 1 | $1/20$ |
| $\omega_3$ | 0 | $1/5$ |
| $\omega_4$ | 5 | $1/12$ |
| $\omega_5$ | 0 | $1/12$ |
| $\omega_6$ | 0 | $1/6$ |
| $\omega_7$ | 4 | $1/15$ |
| $\omega_8$ | 1 | $1/15$ |
| $\omega_9$ | 0 | $1/5$ |

*Finally, we can calculate the price of the option as expectation with respect to $Q$. The result is $\frac{73}{60}$. This calculation can also be done recursively as per figure 3.3. Here one calculates the conditional expectations backwards. This in the end results in the same result – $\frac{73}{60}$. In a next step we want to determine the replicating portfolio from $t = 0 \rightsquigarrow t = 1$.*

*Assume that we hold at time $t = 0$ a portfolio of $\alpha$ cash and $\beta$ (resp. $\gamma$) units of security $Z_1$ (resp. $Z_2$). Then the following equation holds:*

$$\begin{pmatrix} 1 & 11 & 9 \\ 1 & 11 & 10 \\ 1 & 8 & 11 \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{5}{4} \\ 1 \end{pmatrix}$$

*Since the matrix is invertible the solution for the equation is unique:*

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{60} \begin{pmatrix} 143 \\ 2 \\ -9 \end{pmatrix}$$

*Hence we need to hold at time $t = 0$ cash in the amount of $143$, $2$ shares $Z_1$ and $-9$ shares $Z_2$. We note that the value of the replicating portfolio at time $t = 0$*

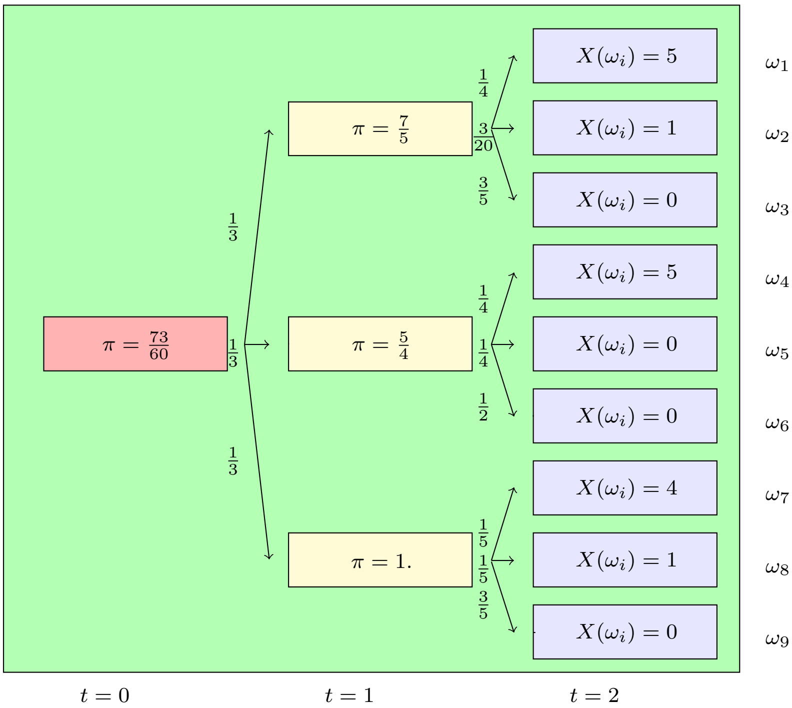



**Fig. 3.3** Calculation of price

$$\frac{143 + 2 \times 10 - 9 \times 10}{60} = \frac{73}{60}$$

*equals again the value of the option. This calculation can be performed for the entire tree resulting in the respective replicating portfolios (exercise). Figure 3.4 shows the corresponding replicating portfolios per state.*

### *3.2.3 Continuous time models*

For models in continuous time we restrict our exposition to the statements, the proofs can be found in the references previously mentioned. A major difference between the discrete and the continuous setting is that we are going to *assume* that $\mathbb{P} \neq \emptyset$ holds for the continuous time model.

We start with some basic definitions.

**Definition 15** • *A trading strategy $\phi$ is a locally bounded, previsible process.*

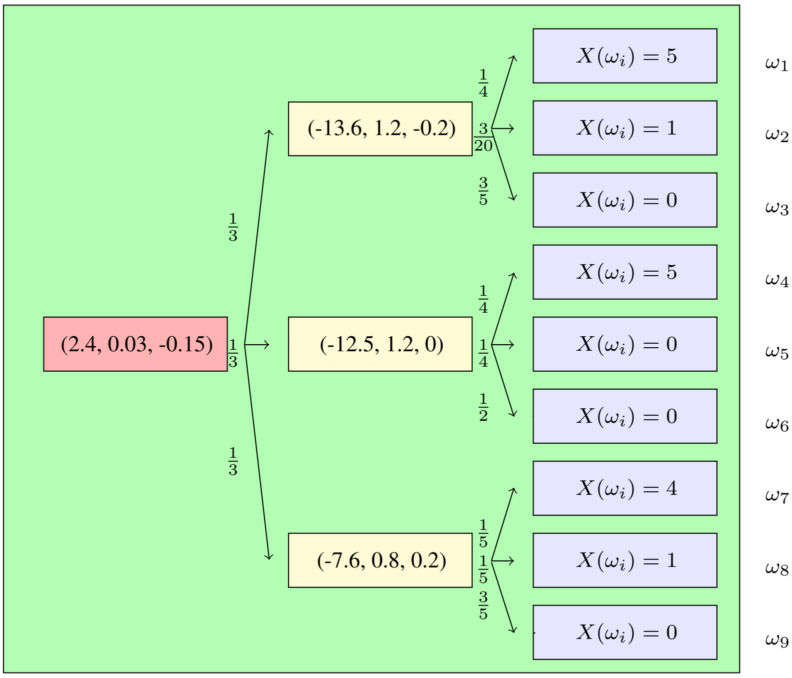



**Fig. 3.4** Calculation of replicating portfolios

- *The value process corresponding to a trading strategy $\phi$ is defined by*

$$V : \Pi \to \mathbb{R}, \phi \mapsto V(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{k} \phi_t^k \cdot S_t^k.$$

- *The return process $G$ is defined by*

$$G : \Pi \to \mathbb{R}, \phi \mapsto G(\phi) = \int_0^\tau \phi \, dS = \int_0^\tau \sum_{i=0}^{k} \phi^k dS^k.$$

- *$\phi$ is self financing, if $V_t(\phi) = V_0(\phi) + G_t(\phi)$.*
- *To define admissible trading strategies we use the notation:*

$$Z_t^i = \beta_t \cdot S_t^i, \qquad \textit{discounted value of share } i$$

$$G^*(\phi) = \int \sum_{i=1}^{k} \phi^i \, dZ^i, \qquad \textit{discounted return}$$

$$V^*(\phi) = \beta\, V(\phi) = \phi^0 + \sum_{i=1}^{k} \phi^i\, Z^i.$$

*A trading strategy is called admissible, if it has the following three properties:*

1. $V^*(\phi) \geq 0$,
2. $V^*(\phi) = V^*(\phi)_0 + G^*(\phi)$,
3. $V^*(\phi)$ *is a martingale with respect to* $Q$.

**Proposition 16** *1. The price of a contingent claim* $X$ *is given by* $\pi(X) = E^Q[\beta_T\, X]$.
*2. A contingent claim is attainable* $\iff$ $V^* = V_0^* + \int H dZ$ *for all* $H$.

**Definition 17** *The market is called complete, if every integrable contingent claim is attainable.*

Although this theory is very important, we only gave a brief sketch of the main ideas. It is therefore recommended that the reader extends his knowledge of financial mathematics by consulting the references.

## 3.3 The Black-Scholes Model and the Itô-Formula

As we have seen in the previous sections, we need an underlying economic model to calculate the price of an option. In principle one can use various different economic models. Exemplary, we are going to consider the most common model: geometric Brownian motion.

The following references are a good source for various aspects of the economic model: [Dot90], [Duf88], [Duf92], [CHB89], [Per94], [Pli97].

**Convention 18 (General conventions)** *For the remainder of this chapter we will use the following notations and conventions:*

- $T_x$ *denotes the future lifespan of an* $x$ *year old person.*
- *The* $\sigma$*-algebras generated by* $T_x$ *are denoted by* $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right)$.
- *We assume, that the values of the shares in the portfolio are given by standard Brownian motions* $W$. *(Compare with Figure 3.5.).*
- $\mathcal{G}_t$ *denotes the* $\sigma$*-algebra generated by* $W$ *augmented by the* $P$*-null sets.*

**Convention 19 (Independence of the financial variables)** • *We assume that* $\mathcal{G}_t$ *and* $\mathcal{H}_t$ *are stochastically independent. This means, that the financial variables are independent of the future lifespan.*


**Fig. 3.5** 5 Simulations of a Brownian motion

- $\mathcal{F}_t = \sigma(\mathcal{G}_t, \mathcal{H}_t)$ *denotes the* $\sigma$*-algebra generated by* $\mathcal{G}_t$ *and* $\mathcal{H}_t$.

Next step we look at specific semimartingales which are of particular interest for stochastic finance and variable annuities. For these the Itô-calculus is simpler. First we define the corresponding class of stochastic processes:

**Definition 20 (Money Market Account; Risk Free Bond)** *We assume that an investor can invest in a money market account* $B_t$ *which earns interest rate* $\delta$*, ie*

$$\begin{aligned} B_{\tau+\Delta\tau} &= \exp\left(\Delta\tau\, \delta\right)\, B_\tau, \text{ or equivalently} \\ dB_\tau &= \delta\, B_\tau\, d\tau. \end{aligned}$$

*We remark that this investment is risk free and that we do not assume any defaults.*

**Definition 21 (Forward Price of a Zero Coupon Bond)** *For* $t < T$*, we denote by* $B_0(t,T)$ *the expected price of a* $\mathcal{Z}_{(T)}$ *as at time* $0$ *, ie*

$$B_0(t,T) = E\left[\pi_t(\mathcal{Z}_{(T)}) \middle|\, \mathcal{F}_0\right].$$

**Proposition 22** *For* $t < T$ *the following formula holds:*

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

*In the context of the model of Definition 20 this leads to*

$$B(t,T) = \exp\left(-\delta\,(T-t)\right) \textit{ and },$$
$$dB(\tau,T) = \delta\, B(\tau,T)\, d\tau.$$

*Proof.* In an arbitrage free environment we can compare the following two strategies:

- Invest at $\tau = 0$ in one unit of $\mathcal{Z}_{(t)}$ and invest at $\tau = t$ the proceeds into $\mathcal{Z}_{(T)}$. At time $\tau = T$ we get the following proceeds:

$$\frac{1}{\pi_0(\mathcal{Z}_{(t)})}\,\frac{1}{\pi_t(\mathcal{Z}_{(T)})}.$$

- Alternatively we can invest the same value (ie $\pi_0(\mathcal{Z}_{(t)})$ into a $T$-year zero coupon bon ($\mathcal{Z}_{(T)}$), which will yield 1.

Since $\mathcal{Z}_{(t)}$ is risk free and because prices are linear, under the assumption of absence of arbitrage, this yields to

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

**Definition 23 (Itô process (1-dim))** *Let $(W_t)_{t\in\mathbb{R}^+}$ be a 1-dimensional Brownian motion on $(\Omega,\mathcal{A},P)$. A (1-dimensional) Itô process or stochastic integral is a stochastic process $(X_t)_{t\in\mathbb{R}^+}$ on $(\Omega,\mathcal{A},P)$ of the form:*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW_s,$$

*where $v \in \mathcal{W}_{\mathcal{H}}$ (see [**?**] chapter 4 for the precise meaning), such that*

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty \forall t \geq 0\right] = 1.$$

*We assume that $u$ is $\mathcal{H}_t$-adapted and*

$$P\left[\int_0^t |u(s,\omega)| ds < \infty \forall t \geq 0\right] = 1.$$

**Remark 24** *Note that writing*

$$dX = u(s)ds + v(s)dW_s$$

*is an abbreviation for:*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_o^t v(s,\omega)dW_s.$$

**Definition 25 (Share Price Process)** *We assume that the share price $S_t$ follows the following stochastic differential equation:*

$$dS_t = \eta \, S_t \, dt + \sigma \, S_t \, dW,$$

*where $\eta \in \mathbb{R}$ is the real world equity return drift term. $\sigma \geq 0$ represents the equity volatility and $(W_t)_{t \in \mathbb{R}_0^+}$ is a one-dimensional standard Brownian motion on $(\Omega, \mathcal{A}, P)$.*

**Remark 26** • *When considering*

$$\frac{dS}{S} = \eta \, dt + \sigma \, dW,$$

*we see that the Black-Scholes-Merton Model is based on a geometric Brownian motion, where we denote with $W$ the Wiener measure or Brownian motion.*

- *The equity drift term $\eta$ means that the value of the equity price would follow:*

$$S_{t+\Delta t} = S_t \, \exp\left(\eta \, \Delta t\right)$$

*in absence of volatility (ie for $\sigma = 0$). At the same time the risk free account value would yield*

$$B_{t+\Delta t} = B_t \, \exp\left(\delta \, \Delta t\right).$$

*In consequence the difference $\eta - \delta$ is called equity risk premium. It is typically positive and represents the additional return one can expect when investing in shares by assuming the additional risk induced by the volatility.*

- *The stochastic differential equation above is to be understood in sense of a stochastic integral (see appendix* **??***) and the Itô-calculus.*
- *We note that $(W_t)_{t \in \mathbb{R}_0^+}$ has P-a.e. continuous sample paths and is nowhere differentiable. As a consequence of this the stochastic differential equation is to be understood as an integral equation, ie*

$$dS_t = \eta \, S_t \, dt + \sigma \, S_t \, dW$$

*is a short form for*

$$S_t = S_0 + \int_0^t \eta \, S_\tau \, d\tau + \int_0^t \sigma \, S_\tau \, dW_\tau,$$

*where the integral is akin to a Riemann integral, eg one can define*

$$\int_0^t X_\tau dW_\tau = \lim_{n \to \infty} \sum_{j=0}^{n} X_{t_j^n} \left( W_{t_{j+1}^n} - W_{t_j^n} \right),$$

*with limit taken over arbitrary partitions of the interval [O,T], into n pieces*

$$0 = t_0^n < t_1^n < t_2^n \ldots < t_n^n = T \quad \textit{with}$$
$$\lim_{n\to\infty} \max\{|t_j^n - t_{j-1}^n| : j = 1, \ldots n\} = 0.$$

**Definition 27 (Black-Scholes-Merton model)** *This economic model consists of two investment options:*

$B(t) = \exp(\delta\, t)$ *risk free investment.*

$S(t) \; = S(0)\, \exp\left[\left(\eta - \frac{1}{2}\sigma^2\right)\, t \, + \, \sigma\, W(t)\right]$ *shares, modeled by a geometric Brownian motion (cf. Figure 3.6).*

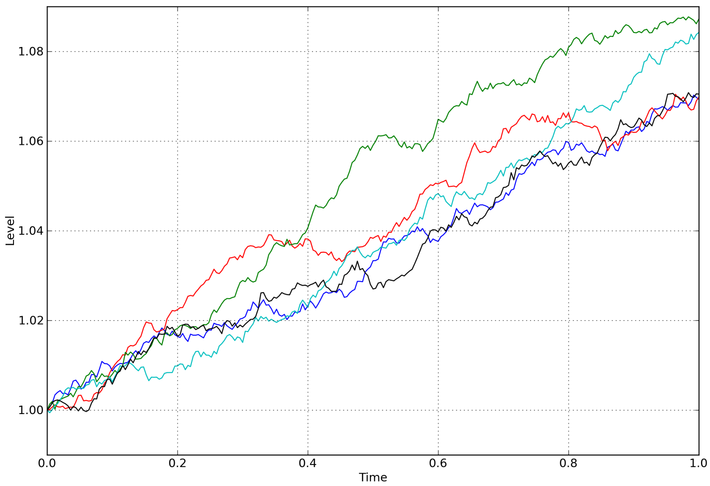



**Fig. 3.6** 5 Simulations of a geometric Brownian motion

$S$ is the solution to the following stochastic differential equation:

$$dS = \eta\, S\, dt \, + \, \sigma\, S\, dW.$$

**Exercise 28** *Prove that $S$ solves the stochastic differential equation given above.*

Next we will calculate the discounted values of $B$ and $S$:

$$
\begin{aligned}
B^*(t) &= \tfrac{B(t)}{B(t)} = 1, \\
S^*(t) &= \tfrac{S(t)}{B(t)} = S(0) \, \exp\left[\left(\eta - \delta - \tfrac{1}{2}\sigma^2\right) \, t \, + \, \sigma \, W(t)\right].
\end{aligned}
$$

Thus we have defined the investment options. To calculate the option prices we need to find an equivalent martingale measure.

Note that we interpret $dX$ in the sense of a stochastic integral, to which we can apply Itô's formula:

**Theorem 29 (Itô)** *Assume $X$ being an Itô process with*

$$dX = a \, dt + b \, dW$$

*and let $g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, (x,t) \mapsto g(x,t)$ be a function, where the following partial derivatives are continuous: $\frac{\partial}{\partial x} g$, $\frac{\partial^2}{\partial x^2} g$, and $\frac{\partial}{\partial t} g$. In this case $Y_t = g(X_t, t)$ is also an Itô process with the following stochastic differential equation:*

$$dY = (\frac{\partial g}{\partial x} \, a + \frac{\partial g}{\partial t} + \frac{1}{2} \, \frac{\partial^2 g}{\partial x^2} \, b^2) \, dt + \frac{\partial g}{\partial x} \, b \, dW$$

*Proof.* For the proof we refer to [**?**], or also [Pro90] or [IW81].

We can now apply Itô's lemma to the geometric Brownian motion, as follows:

**Proposition 30** *Let a stock price $S$ be modelled by a geometric Brownian motion*

$$dS = \eta \, S dt + \sigma \, S \, dW$$

*and let*

$$V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}, (s,t) \mapsto V(x,t)$$

*be a function fulfilling the regularity criteria of the function $g$ of the Itô lemma. In this case $V := V(S,t)$ is also an Itô process with the following stochastic differential equation:*

$$dV = (\frac{\partial V}{\partial s} \, \eta \, S + \frac{\partial V}{\partial t} + \frac{1}{2} \, \frac{\partial^2 V}{\partial S^2} \, \sigma^2 \, S^2) \, dt + \frac{\partial V}{\partial S} \, \sigma \, S \, dW.$$

*Proof.* This follows immediately from the Itô formula keeping in mind that

$$
\begin{aligned}
a &= \eta \, S, \text{and} \\
b &= \sigma \, S.
\end{aligned}
$$

In order to simulate a geometric Brownian motion, the following corollary is helpful:

**Proposition 31** *For a geometric Brownian motion*

$$dS = \eta \, S \, dt + \sigma \, S \, dW$$

*we define $Y = \log(S)$. Then we have the following:*

1. *$dY = (\eta - \frac{1}{2}\sigma^2)\, dt + \sigma \, dW$ is the unique solution to the above stochastic differential equation.*
2. *We can calculate $S_t$ by:*

$$S_t = S_0 \, \exp\left(\left(\eta - \frac{1}{2}\sigma^2\right) \times t + \sigma \, W_t\right)$$

3. *For a series of times $t_0 = 0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots$, and $(X_k)_{k\in\mathbb{N}}$ standard normally distributed independent variables we can simulate $(S(t_k))_{k\in\mathbb{N}}$ with $S(0) = 1$ by the following recursion $\forall n \in \mathbb{N}$:*

$$S(t_n) = S(t_{n-1}) \times \exp\left[(\eta - \frac{\sigma^2}{2}) \times (t_n - t_{n-1}) + \sqrt{t_n - t_{n-1}} \times \sigma \times X_n\right].$$

*Proof.* Define

$$dX_t = (\eta - \frac{1}{2}\sigma^2)\, dt + \sigma \, dW,$$

with $X_0 = 0$. We know that $\tilde{S}_t := g(X_t)$, with

$$g : \mathbb{R} \to \mathbb{R}, x \mapsto g(x) = \exp(x),$$

and that $g = \frac{d}{dx} g = \frac{d^2}{dx^2} g$. As a consequence of the Itô-formula we get

$$\begin{aligned}
d\tilde{S}_t &= dg(X_t) \\
&= g'(X_t)\,(\eta - \frac{1}{2}\sigma^2)\, dt + g'(X_t)\, \sigma \, dW_t \\
&\quad + \frac{1}{2} g''(X_t) \sigma^2 \, dt \\
&= g(X_t)\; (\eta \, dt + \sigma \, dW_t) \\
&= \tilde{S}_t \; (\eta \, dt + \sigma \, dW_t)\,.
\end{aligned}$$

Hence both $S$ and $\tilde{S}$ fulfill the same stochastic differential equation with identical boundary condition. The proposition follows as a consequence of a general result for stochastic differential equations an the Itô-calculus, stating the uniqueness if a stochastic differential equation if the coefficients of the stochastic differential equations are Lipschitz continuous.

**Proposition 32** *Under the natural filtration $\mathcal{F}$ induced by $(W_t)_{t\in\mathbb{R}_0^+}$ the following hold:*

1. *The Brownian motion is a martingale, and*
2. $M_t = \exp\left(\sigma\, W_t - \frac{1}{2}\sigma^2\, t\right)$ *is a martingale.*

*Proof.* Let $Z \sim \mathcal{N}(0,1)$ and let $u \leq t$. For the first equation we have

$$\begin{aligned} E^P\left[W_t|\mathcal{F}_u\right] &= E^P\left[W_t - W_u|\mathcal{F}_u\right] + E\left[W_u|\mathcal{F}_u\right] \\ &= E\left[(t-u)\, Z\right] + W_u \\ &= W_u. \end{aligned}$$

For the second equation we have:

$$\begin{aligned} E^P\left[M_t|\mathcal{F}_u\right] &= E^P\left[\exp\left(\sigma\, W_t - \frac{1}{2}\sigma^2\, t\right)|\mathcal{F}_u\right] \\ &= E^P\left[\exp\left(\sigma\left((W_t - W_u) + W_u\right) - \frac{1}{2}\sigma^2\left((t-u)+u\right)\right)|\mathcal{F}_u\right] \\ &= \exp\left(\sigma\, W_u - \frac{1}{2}\sigma^2\, u\right) \times E^P\left[\exp\left(\sigma\left(W_t - W_u\right) - \frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right] \\ &= M_u \times E^P\left[\exp\left(\sigma\left(W_t - W_u\right) - \frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right] \\ &= M_u \times E^P\left[\exp\left(\sigma\sqrt{t-u}\, Z - \frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right] \\ &= M_u \times E^P\left[\exp\left(\sigma\sqrt{t-u}\, Z - \frac{1}{2}\sigma^2\left(t-u\right)\right)\right] \\ &= M_u \end{aligned}$$

where we used the defining charcateristics of the Brownian motion and where $E^P\left[\exp\left\{\sigma\sqrt{t-u}\, Z\right\}\right] = \exp\left(\frac{1}{2}\,\sigma^2\left(t-u\right)\right)$ can be shown by elementary calculus (exercise).

**Corollary 1.** $(S_t)_{t \in \mathbb{R}_0^+}$ *follows a martingale under* $P$ *if and only if* $\eta = 0$.

**Remark 33** *We note that* $S_t$ *is a time continuous Markov-process under* $P$ *with respect to* $\mathcal{F}$ *and we have*

$$\begin{aligned} p_S(u,x,t,y) &:= P\left[S_t = y|S_u = s\right] \\ &= \frac{1}{\sqrt{2\pi(t-u)}\,\sigma\, y} \\ &\quad \times \exp\left\{-\frac{\left(\ln\left(\frac{y}{x}\right) - \eta\left(t-u\right) + \frac{1}{2}\sigma^2\left(t-u\right)\right)^2}{2\sigma^2\left(t-u\right)}\right\} \end{aligned}$$

**Definition 34 (Trading Strategy)** *A trading strategy in the Black-Scholes-Merton framework is a pair $\Phi = (\Phi^1, \Phi^2)$ of $\mathcal{F}$- previsible, progressively measurable stochastic processes. on $(\Omega, \mathcal{A}, P)$.*

**Definition 35** *A trading strategy $\Phi = (\Phi^1, \Phi^2)$ over $[0,T]$ is self-financing if the corresponding value process (also wealth process) $(V_t(\Phi))_{t\in[0,t]}$,*

$$V_t(\Phi) = \Phi_t^1\, S_t + \Phi_t^2\, B_t, (\forall t \in [0,T])$$

*satisfies the following condition:*

$$V_t(\Phi) = V_0(\Phi) + \int_0^t \Phi_\tau^1\, dS_\tau + \int_0^t \Phi_\tau^2 dB_\tau.$$

**Remark 36** • *As usual, we implicitly assume the existence of an integral if we write $\int X dW$. The following two conditions are sufficient that the integral in Definition 34 exists:*

$$P\left[\int_0^T (\Phi_\tau^1)^2 d\tau < \infty\right] = 1, \textit{ and}$$
$$P\left[\int_0^T |\Phi_\tau^2| d\tau < \infty\right] = 1.$$

- *Note that we will denote by $^*$ discounted quantities, with respect to the money market account, eg*
$$S_t^* = \frac{S_t}{B_t}.$$
- *Since $B_t = \exp(t\,\delta)\, B_0$, we get*
$$S_t^* = S_0^* \exp\left(\left(\eta - \delta - \frac{1}{2}\sigma^2\right) t + \sigma\, W_t\right),$$
*or equivalently*
$$dS_t^* = (\eta - \delta) S_t^*\, dt + \sigma S_t^* dW_t,$$
*with $S_0^* = S_0$.*

**Corollary 2.** *The discounted stock price process $(S_t^*)_{t\in[0,T]}$ is a martingale under $P$ if and only if $\eta = \delta$.*

**Definition 37 (Martingale Measure)** *A probability measure $Q$ on $(\Omega, \mathcal{A})$ equivalent to $P$ is called martingale measure for $S^*$ if $(S_t^*)_{t\in[0,T]}$ is a local martingale under $Q$.*

**Definition 38 (Spot Martingale Measure)** *A probability measure $P^*$ on $(\Omega, \mathcal{A}, P)$ on $(\Omega, \mathcal{A}, P)$ equivalent to $P$ is called spot martingale measure, if the discounted value process $V_t^*(\Phi) = \frac{V_t(\Phi)}{B_t}$ of any self-financing trading strategy is a local martingale under $P^*$.*

**Lemma 2.** *A probability measure is a spot martingale measure if and only if it is a martingale measure for the discounted stock price $S^*$.*

*Proof.* Let $\Phi$ be a self-financing strategy and denote $V^* := V^*(\Phi)$. Using Itô's product rule and the fact that $dB_t^{-1} = -\delta\, B_t^{-1}\, dt$ we the following:

$$\begin{aligned}
dV_t^* &= d(V_t\, B_t^{-1}) \\
&= V_t\, dB_t^{-1} + B_t^{-1}\, dV_t \\
&= \left(\Phi_t^1\, S_t + \Phi_t^2 B_t\right) dB_t^{-1} + B_t^{-1}\, \left(\Phi_t^1\, dS_t + \Phi_t^2 dB_t\right) \\
&= \Phi_t^1\, \left(B_t^{-1}\, dS_t + S_t\, dB_t^{-1}\right) \\
&= \Phi_t^1\, dS_t^*.
\end{aligned}$$

Hence we get

$$V_t^*(\Phi) = V_0^*(\Phi) + \int_0^t \Phi_\tau^1\, dS_\tau^*$$

Therefore the lemma follows

**Lemma 3.** *The unique martingale measure $Q$ for the discounted stock price process $S^*$ is given by the Radon-Nikodym density*

$$\begin{aligned}
\xi_t &= \frac{dQ}{dP} \\
&= \exp\left(-\frac{1}{2}\left(\frac{\eta-\delta}{\sigma}\right)^2 t - \frac{\eta-\delta}{\sigma}\, W(t)\right) \qquad \textit{for all } t \in [0,T].
\end{aligned}$$

*The discounted stock price $S^*$ satisfies under $Q$ the following:*

$$dS_t^* = \sigma\, S_t^*\, dW_t^*,$$

*and the continuous $\mathcal{F}$-adapted process $W^*$ is given by*

$$W_t^* = W_t - \frac{\delta-\eta}{\sigma}\, t\; \forall t \in [0,T].$$

- **Exercise 39** *Prove the following statements:*

*1.* $E[\xi_t] = 1,$

*2.* $Var[\xi_t] = \exp\left(\left(\frac{\eta-\delta}{\sigma}\right)^2 t\right) - 1,$

*3.* $\xi_t > 0$.

*(Hint:* $W(t) \sim \mathcal{N}(0,t)$.*)*

*Proof.* An application of Girsanov's theorem – a theorem in the theory of stochastic integration (e.g. [Pro90] Theorem 3.6.21) – shows that

$$\hat{W}_t = W(t) + \frac{\eta - \delta}{\sigma}\, t$$

is a Brownian motion with respect to $Q = \xi \cdot P$.

Naturally, after this transformation we want to prove that

$$S^*(t) = S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, \hat{W}(t)\right)$$

is a martingale with respect to $Q$. (Then the price of the option is given by its expectation with respect to $Q$.)

We have to show the equality

$$E^Q\left[S^*(u)|\mathcal{F}_t\right] = S^*(t)$$

for $t, u \in \mathbb{R}, u > t$. With the notation $u = t + \Delta t$, $W_u = W_t + \Delta W$ and $Z \sim \mathcal{N}(0,1)$ we have

$$\begin{aligned}
&E^Q\left[S^*(u)|\mathcal{F}_t\right] \\
&= E^Q\left[S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, \hat{W}(t) + (-\frac{1}{2}\sigma^2\, \Delta t + \sigma\, \Delta\hat{W})\right)|\mathcal{F}_t\right] \\
&= S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, W(t)\right)\, E^Q\left[\exp\left(-\frac{1}{2}\,\sigma^2 \Delta t + \sigma\sqrt{\Delta t} Z\right)|\mathcal{F}_t\right] \\
&= S^*(t).
\end{aligned}$$

Therefore the measure $Q$ is equivalent to $P$, and $S^*$ is a martingale with respect to $Q$. An economist would say, "it exists (at least) one consistent price system". For the uniqueness of $Q$ we refer to [**?**] lemma 3.1.3.

As a next step we want to understand the Black-Scholes-Merton partial differential equation from a theoretical aspect. The following theorem sheds light into this question. We note that this partial differential equation will be useful, when we will look at $\Delta$-hedging.

**Theorem 40 (Black-Scholes-Merton Differential Equation)** *Let $\delta$ be the risk-free interest rate, $\eta$ the equity drift rate and $\sigma$ the volatility of a asset $S_t$ in the Black-Scholes-Merton framework (eg the share price satisfying the following stochastic* *differential equation $dS = \eta\, S\, dt + \sigma\, S\, dW$). Suppose we have a contingent claim with expiry date $T$ and underlying asset $S_t$, with value function:*

$$v : \mathbb{R}^+ \times ]0, T[, \to \mathbb{R}, (x, t) \mapsto v(x, t),$$

*where $v \in C^{2,1}(\mathbb{R}^+ \times [0, T])$. Then the following partial differential equation holds:*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} = \delta v.$$

*Moreover the portfolio $\mathcal{P} = (V - \Delta \times S, \Delta)$ consisting $V - \Delta \times S$ units cash (valued at $1$) and $\Delta = \frac{\partial v}{\partial S}$ shares replicates the contingency claim.*

*Proof.* Since this theorem is very important we offer in a first step an *outline the proof*. The proof is an application of Itô's lemma. We have the following:

$$\begin{aligned} dS &= \eta\, S\, dt + \sigma\, S\, dW, \text{ and} \\ dv &= (\frac{\partial v}{\partial S}\, \eta\, S + \frac{\partial v}{\partial t} + \frac{1}{2}\, \frac{\partial^2 v}{\partial S^2}\, \sigma^2\, S^2)\, dt + \frac{\partial v}{\partial S}\, \sigma\, S\, dW \end{aligned}$$

We define $\Pi := -v + \Delta\, S$ and we see that for $\Delta = \frac{\partial v}{\partial S}$ the stochastic part of the SDE cancels out and we get:

$$d\Pi = \left\{ -\frac{\partial v}{\partial t} - \frac{1}{2}\, \frac{\partial^2 v}{\partial S^2}\, \sigma^2\, S^2 \right\}\, dt$$

On the other hand $\Pi$ can be interpreted as the value of an investment portfolio, shorting $-v$ of cash and buying $\Delta$ shares. One could also invest the respective value in cash, by means of absence of arbitrage. This needs to result in the same value and in the same increments, hence:

$$d\Pi = \delta\, \Pi\, dt$$

When now using the definition of $\Pi$ for the right hand side and using the above formula for the right hand side, this results in:

$$\left\{ -\frac{\partial v}{\partial t} - \frac{1}{2}\, \frac{\partial^2 v}{\partial S^2}\, \sigma^2\, S^2 \right\}\, dt = \delta\, \left\{ -v + \frac{\partial v}{\partial S}\, S \right\}\, dt.$$

In consequence we get the Black-Scholes-Merton differential equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} = \delta v$$

*Proof.* In a next step we offer a more formal proof. It follows essentially [**?**] theorem 3.1.1 where further details can be found. We will proof this theorem by directly determining the replicating strategy and note that the price $v_t$ of the contingency claim is a function $v_t = v(S_t, t)$. We may assume that the replicating strategy $\Phi$ has the form:

$$\Phi_t = (\Phi_t^{(1)}, \Phi_t^{(2)}) = (h(S_t,t), g(S_t,t)),$$

for $t \in [0,T]$ and $g,h : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ unknown functions. Note that $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ represent the amount of cash and the number for shares which we hold at time $t$. Since $\Phi$ is assumed self-financing, its wealth process $V(\Phi)$ given by

$$V_t(\Phi) = g(S_t,t)S_t + h(S_t,t)B_t = v(S_t,t), \tag{3.1}$$

needs to satisfy the following:

$$dV_t(\phi) = g(S_t,t)dS_t + h(S_t,t)dB_t. \tag{3.2}$$

Given that we operate in the Black-Scholes-Merton framework, we can conclude from (3.2) that

$$\begin{aligned} dV_t(\Phi) &= g(S_t,t)(\eta S_t dt + \sigma S_t dW_t) + h(S_t,t)\delta B_t dt \\ &= (\eta - \delta) S_t g(S_t,t)dt + \sigma S_t g(S_t,t)dW_t + \delta S_t g(S_t,t)dt + \delta h(S_t,t)B_t dt \\ &= (\eta - \delta) S_t g(S_t,t)dt + \sigma S_t g(S_t,t)dW_t + \delta S_t v(S_t,t)dt. \end{aligned} \tag{3.3}$$

The application of the Itô lemma to $v$ results in

$$dv(S_t,t) = \left( \frac{\partial v}{\partial t}(S_t,t) + \eta S_t \frac{\partial v}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t,t) dW_t.$$

Combining the above expression with (3.2), results in the following expression for the Itô differential of the process $Y_t = v(S_t,t) - V_t(\Phi)$:

$$\begin{aligned} dY_t = &\left( \frac{\partial v}{\partial t}(S_t,t) + \eta S_t \frac{\partial v}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t,t) dW_t \\ &+ (\delta - \eta) S_t g(S_t,t)dt - \sigma S_t g(S_t,t)dW_t - \delta v(S_t,t)dt \end{aligned}$$

On the other hand, in view (3.1), the process $Y$ vanishes identically (i.e. $Y \equiv 0$), and thus $dY_t = 0$. By virtue of the uniqueness of the canonical decomposition of continuous semimartingales, the diffusion term in the above decomposition of $Y$ vanishes. In our case, this means that we have, for every $t \in [0,T]$,

$$\int_0^t \sigma S_\tau \left( g(S_\tau,\tau) - \frac{\partial v}{\partial S}(S_\tau,\tau) \right) dW_\tau = 0 \quad P - a.s.$$

In view of the properties of the Itô integral (isometry used for the construction of the Itô integral), this is equivalent to:

$$\int_0^T S_\tau^2 \left( g(S_\tau, \tau) - \frac{\partial v}{\partial S}(S_\tau, \tau) \right)^2 d\tau = 0. \tag{3.4}$$

For (3.4) to hold, it is sufficient and necessary that $g$ satisfies

$$g(s,t) = \frac{\partial v}{\partial S}(s,t) \;\; \forall (s,t) \in \mathbb{R}^+ \times [0,T] \tag{3.5}$$

Strictly speaking, the equality above should hold $P \otimes \lambda-$almost surely, where $\lambda$ is the Lebesgue measure. Using (3.5), results in still another equation of $Y$:

$$Y_t = \int_0^t \left\{ \frac{\partial v}{\partial \tau}(S_\tau, \tau) + \frac{1}{2}\sigma^2 S_\tau^2 \frac{\partial^2 v}{\partial S^2}(S_\tau, \tau) + \delta S_\tau \frac{\partial v}{\partial S}(S_\tau, \tau) - \delta v(S_\tau, \tau) \right\} d\tau$$

It is thus apparent that $Y \equiv 0$ whenever $v$ satisfies the following partial differential equation, referred to as Black-Scholes PDE:

$$\frac{\partial v}{\partial t}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t, t) + \delta S_t \frac{\partial v}{\partial S}(S_t, t) - \delta v(S_t, t) = 0 \tag{3.6}$$

This terminates the first part of the proof. It thus remains to check that $\Phi$ is an admissible trading strategy.

If the contingent claim is attainable, it thus remains to check that the replicating strategy $\Phi$, given by the formula

$$\begin{aligned} \Phi^1 &= g(S,t) = \frac{\partial}{\partial S} v(S_t, t), \\ \Phi^2 &= h(S_,t) = B_t^{-1} \left( v(S_t, t) - g(S_t, t) S_t \right), \end{aligned}$$

is admissible. Let us first check that $\Phi$ is self-financing. We need to check that

$$dV_t(\Phi) = \Phi_t^1 \, dS_t + \Phi_t^2 \, dB_t.$$

Since $V_t(\Phi) = \Phi_t^1 \, S_t + \Phi_t^2 \, B_t = v(S_t, t)$, by applying Itô's formula, we get

$$dV_t(\Phi) = \frac{\partial}{\partial S} v(S_t, t) \, dS_t + \frac{1}{2}\sigma^2 S_t^2 \, \frac{\partial^2}{\partial S^2} v(S_t, t) \, dt + \frac{\partial}{\partial t} v(S_t, t) \, dt.$$

In view of (3.6), the last equality can also be given the following form

$$dV_t(\Phi) = \frac{\partial}{\partial S} v(S_t, t) \, dS_t + \delta \, v(S_t, t) \, dt - \delta \, S_t \frac{\partial}{\partial S} v(S_t, t) \, dt,$$

and thus

$$dV_t(\Phi) = \frac{\partial}{\partial S} v(S_t, t) dS_t + \delta v(S_t, t) dt - \delta S_t \frac{\partial}{\partial S} v(S_t, t) dt$$

$$
\begin{aligned}
&= \Phi_t^1\, dS_t + \delta\, B_t \frac{\frac{\partial}{\partial t} v(S_t, t) - \Phi_t^1\, S_t}{B_t} \\
&= \Phi_t^1\, dS_t + \Phi_t^2\, dB_t.
\end{aligned}
$$

This ends the verification of the self-financing property.

Note that the $\Delta$-portfolio is not permanently risk free, but only *instantaneously*. For the interested reader we suggest as further reading: [**?**] for the analytical basics such as integration theory, Banach and Hilbert spaces etc, and [Per94] and [**?**] for stochastic integration and stochastic differential equations. Finally we would also suggest [Hul97] and [**?**] as general valuable reference.

**Remark 41** *The proof of the above theorem is very helpful for understanding the concept of dynamical hedging. Assume a given contingency claim in the Black-Scholes framework which has been sold by an insurance company or a bank. Theorem 40 shows a way to calculate the value $V$ via solving the corresponding partial differential equation. More importantly it helps us also to understand the intrinsic risk to this contract in case of changes in equity prices ($S$). In many instances the bank or insurance company is not willing to take this risk on its balance sheet (eg letting fluctuate the shareholder equity) and tries to mitigate this risk by a suitable hedging strategy.*

*One classical hedging strategy is to buy at each point of time assets with the same partial derivative as $V$ with respect to $S$. Such a strategy is called $\delta$-hedging. Figure 3.7 shows this. Note that the $\delta$-hedging strategy has this name, because the first partial derivative of $V$ with respect of $S$ is called $\Delta$, similarly $\Gamma$ is the second partial derivative of $V$ with respect to $S$, ie. we have the following*

$$
\begin{aligned}
\Delta &= \frac{\partial}{\partial S} V \\
\Gamma &= \frac{\partial^2}{\partial S^2} V
\end{aligned}
$$

*Turning back to the proof of theorem 40, we see that for a pure $\delta$ strategy the value process of the hedge portfolio becomes deterministic, following*

$$
d\Pi = \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2}\, \frac{\partial^2 V}{\partial S^2}\, \sigma^2\, S^2 \right\} dt.
$$

*This however means that a pure $\delta$ strategy leads to a deterministic difference in price as a consequence of the $\Gamma$ term. This effect is known under the name $\gamma$-bleed. Figure 3.8 shows the effect of using a pure $\delta$-strategy vs using a $\delta - \gamma$ strategy for a simulated trajectory. The red line shows the true value of $V$, the blue one using a $delta$-hedge and the green one a $\delta - \gamma$-hedge.*

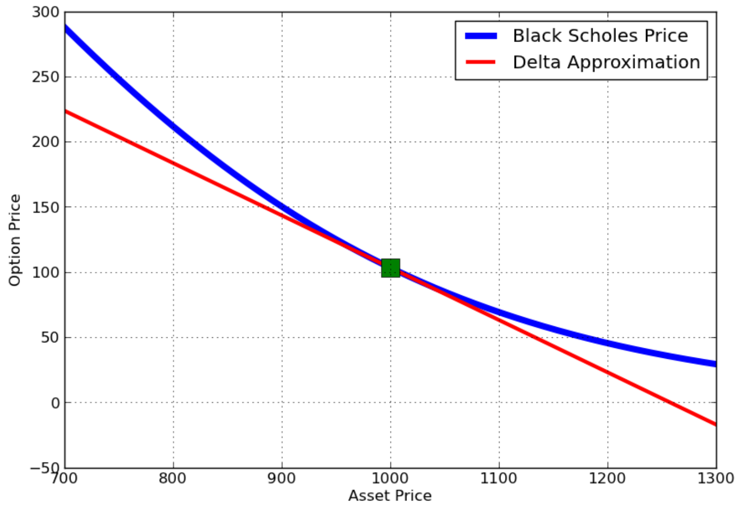



**Fig. 3.7** Delta Hedge

**Theorem 42** *Let the economic model defined above be given, i.e. it is defined by* $(\Omega, \mathcal{A}, P)$, $S$ *and* $B$. *Then at time* $t$ *the price of a policy with death benefit* $C(T)$ *is*

$$\pi_t(T) = E^Q \left[ \exp\left(-\delta(T-t)\right) C(T) | \mathcal{F}_t \right].$$

**Remark 43** *The main difference of this model in comparison to the classical model is that one has to calculate the expectation with respect to* $Q$ *and not with respect to* $P$. *Moreover one should note, that we have not proved the uniqueness of the price system.*

The following two formulas are an important consequence of the previous considerations.

**Proposition 44** *A single premium for a policy based on the economic model defined above is given by the following formulas.*

**Endowment policy:**

$$V(0) = E^Q \left[ \exp(-\delta T)\, C(T) \right] \cdot {}_T p_x.$$

**Term life insurance:**

$$V(0) = \int_0^T E^Q \left[ \exp(-\delta t)\, C(t) \right] {}_t p_x \, \mu(x+t) dt.$$

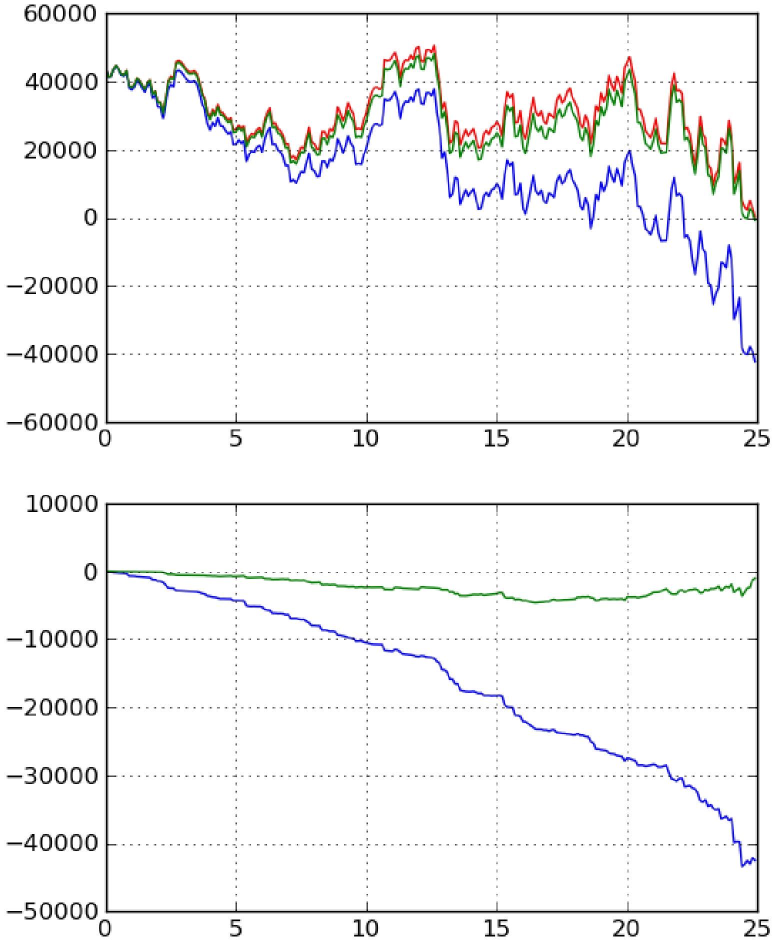



**Fig. 3.8** Delta Hedge

## 3.4 Calculation of single premiums

Up to now the calculations have been relatively simple, since we did not include any guarantees in our policy model. Next, we will consider a unit-linked policy with an additional guarantee. We recall some of the notations from the previous sections:

| | |
|---|---|
| $N(\tau)$ | Number of shares at time $\tau$, |
| $S(\tau)$ | value of a share at time $\tau$, |
| $G(\tau)$ | guaranteed benefits at time $\tau$, |
| $C(\tau) = \max\{N(\tau)S(\tau), G(\tau)\}$ | value of the insurance at time $\tau$. |

### 3.4.1 Pure endowment policy

**Proposition 45** *Let the Black-Scholes model be given. Then the single premium for a pure endowment policy with payout*

$$C(T) = \max\{N(T)S(T), G(T)\}$$

*is given by*

$${}_TG_x = {}_Tp_x \left[G(T)\, \exp(-\delta T)\Phi(-d_2^0(T)) + S(0)N(T)\Phi(d_1^0(T))\right],$$

*where*

$$\begin{aligned} \Phi(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{x^2}{2})dx, \\ d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s>t), \\ d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s>t). \end{aligned}$$

*Proof.* In the following we denote by $J^*$ the discounted value of a random variable $J$. The value of the pure endowment policy at time zero is $E^Q[C^*(T)]$. We set $Z = S^*(T)$. Then the following equations hold

$${}_TG_x = {}_Tp_x\, E^Q\left[\max\{N(T)Z, G^*(T)\}\right]$$

and

$$Z = S(0)\,\exp\left(-\frac{1}{2}\sigma^2 T + \sigma\hat{W}(T)\right) \qquad \text{where} \qquad \hat{W}(T) \sim \mathcal{N}(0,T).$$

Thus we get

$$\begin{aligned} {}_TG_x &= {}_Tp_x \int_{-\infty}^{\infty} \max\left[N(T)S(0)\exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi), G^*(T)\right] f(\xi)d\xi, \\ f(\xi) &= \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}\xi^2\right). \end{aligned}$$

Next we define $\bar{\xi} = \frac{1}{\sigma}\left[\ln\left(\frac{G^*(T)}{N(T)S(0)}\right) + \frac{1}{2}\sigma^2 T\right]$ and note that $\xi > \bar{\xi}$ implies $N(T)Z > G^*(T)$. Therefore, the single premium is given by

$$\begin{aligned}
{}_T G_x &= {}_T p_x \Bigg( G^*(T) \int_{-\infty}^{\bar{\xi}} f(\xi) d\xi \\
&\qquad + N(T)S(0) \int_{\bar{\xi}}^{\infty} \exp(-\frac{1}{2}\sigma^2 T + \sigma\, \xi) f(\xi) d\xi \Bigg) \\
&= {}_T p_x \Bigg( G^*(T) \int_{-\infty}^{\bar{\xi}} f(\xi) d\xi \\
&\qquad + N(T)S(0) \int_{\bar{\xi}}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp(-\frac{1}{2T}(\xi - \sigma T)^2 d\xi \Bigg).
\end{aligned}$$

This equation, with adapted notation, yields the statement of the theorem.

### *3.4.2 Term life insurance*

**Proposition 46** *Let the Black-Scholes model be given. Then the single premium for a term life insurance with death benefit*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

*is given by*

$$G^1_{x:T} = \int_0^T \left( G(t)\exp(-\delta t)\Phi(-d_2^0(t)) + S(0)N(t)\Phi(d_1^0(t)) \right) {}_t p_x \mu_{x+t}\, dt,$$

*where*

$$\begin{aligned}
\Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}$$

*for $s > t$.*

**Exercise 47** *Prove the previous theorem by the same methods which we used for the pure endowment policy.*

**Remark 48** *For the calculations above we have defined the guarantee for both temporary death benefit ("GMDB") and for the pure endowment ("GMAB") by*

$$C(T) = \max\{N(T)S(T), G(T)\}.$$

*This means that we have value before the total product, eg the value of the underlying funds plus the corresponding variable annuity guarantee. In practise the total value is often split into the value of the underlying funds ($N(T)S(T)$) and the value of the variable annuity guarantee or also called variable annuity rider. Most of the examples in the following will actually calculate the value of the variable annuity rider. Note that the split of the entire value of the variable annuity (including the value of the underlying funds) is simple as a consequence of the linearity of the expected value under the measure $Q$. Moreover, one can define the guaranteed part of the insurance benefit as follows:*

$$\begin{aligned} C^G(T) &= \max\{N(T)S(T), G(T)\} - N(T)S(T) \\ &= \max\{0, G(T) - N(T)S(T)\}. \end{aligned}$$

*In order to compare the value of the variable annuity rider on often compares it with the value of the underlying funds. Depending on lapse assumptions the value of the guarantee can often exceed 10% of the underlying funds value.*

## 3.5 Thiele's differential equation

Now we want to derive Thiele's differential equation. For this we need to determine premiums for the policies. We introduce the notation $\bar{p}(t)$ for the density of the premiums at time $t$. Then the equivalence principle yields the following two equations:

$${}_TG_x = \int_0^T \bar{p}(t) \exp(-\delta\, t)_t p_x dt$$

and

$$G^1_{x:T} = \int_0^T \bar{p}(t) \exp(-\delta\, t)_t p_x dt.$$

Also in this section the pure endowment policy and the term life insurance will be considered separately. The mathematical reserve for these policies is given by:

$$\text{Pure endowment: } V(t) = {}_{T-t}p_{x+t}\pi_t(T) - \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t} p_{x+t}\, d\xi.$$

$$\text{Life insurance: } V(t) = \int_t^T (\pi_t(\xi)\mu_{x+\xi} - \bar{p}(\xi)\exp(-\delta(\xi - t))) \times {}_{\xi-t}p_{x+t}\, d\xi,$$

where

$$\begin{aligned}\pi_t(s) &= G(s)\exp(-\delta(s-t))\Phi(-d_2^t(s)) \\ &\quad +N(s)S(t)\Phi(d_1^t(s)),\\ d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},\\ d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},\end{aligned}$$

for $s > t$.

**Remark 49** • *In the classical setting the reserves were deterministic, but here they depend on the underlying share $S$.*

- *Note that we are beyond the deterministic theory of differential equations. In particular we have to use Itô's formula, which takes the following form for the purely continuous case of a standard Brownian motion $W$:*

$$df(W) = f'\,dW + \frac{1}{2}f''\,ds.$$

For the policies defined above we have the following theorem.

**Theorem 50** *1. The differential equation for the price of a pure endowment policy is:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\,V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}.$$

*2. The differential equation for the price of a term life insurance is:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\,V(t) - C(t)\mu_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}.$$

Before we prove this theorem, we want to make some comments on the formulas.

**Remark 51** *1. One obtains Black-Scholes formula by setting $\mu_{x+t} = \bar{p}(t) = 0\,\forall t$.*

*2. The first terms in the differential equations in the theorem above coincide with the classical case (see section* **??***), i.e. the dependence of the values on the premiums, on the mortality and on the interest rate. Due to the shares in the model a further term appears: $-\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}$. It represents the fluctuations of the underlying shares.*

*Proof.* We have

$$\pi_t^*(T) = \exp(-\delta t)\pi_t(T).$$

Hence, by the definition of $V$, we get

$$V(t) = {}_{T-t}p_{x+t}\pi_t^*(T)\exp(\delta t) - \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}d\xi$$

and

$$\pi_t^*(T) = \Psi(t)\left[V(t) + \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}d\xi\right],$$

where

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

Now we can apply Itô's formula to the function $\pi_t^*(t, S)$, since $\pi_t^*$ is a function of $S$ and $t$. We get

$$\begin{aligned} dY_t &= U(t + dt, X_t + dX_t) - U(t, X_t) \\ &= \left(U_t dt + \frac{1}{2}U_{xx}b^2 dt\right) + U_x dX_t \\ &= \left(U_t + \frac{1}{2}U_{xx}b^2\right) dt + U_x\, b\, dB_t \end{aligned}$$

and

$$d\pi^* = \left(\frac{\partial \pi^*}{\partial t} + \frac{\partial \pi^*}{\partial S}\,a + \frac{1}{2}\,\frac{\partial^2 \pi^*}{\partial S^2}\,b^2\right) dt + \frac{\partial \pi^*}{\partial S} b\, d\hat{W}.$$

Furthermore we know that

$$dS = \delta S(t)dt + \sigma S(t)d\hat{W},$$

and thus we have $a = \delta S(t)$ and $b = \sigma S(t)$. In the next step we want to determine the two terms:

$$\begin{aligned} \frac{\partial \pi_t^*}{\partial S} &= \Psi(t)\,\frac{\partial V}{\partial S}, \\ \frac{\partial^2 \pi_t^*}{\partial S^2} &= \Psi(t)\,\frac{\partial^2 V}{\partial S^2}. \end{aligned}$$

To get $\frac{\partial \pi^*}{\partial t}$, we start with

$$\begin{aligned} \frac{\partial}{\partial t}{}_{\xi-t}p_{x+t} &= \mu_{x+t}\,{}_{\xi-t}p_{x+t}, \\ \frac{\partial}{\partial t}\Psi(t) &= \left(\frac{A}{B}\right)' = \frac{A'}{B} - \frac{A}{B^2}\,B' \\ &= -\left(\mu_{x+t} + \delta\right)\Psi(t). \end{aligned}$$

Now, with the formula from above we get

$$\begin{aligned}
\frac{\partial \pi^*}{\partial t} &= \frac{\partial \Psi}{\partial t}\left( V(t) + \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}dt \right) \\
&+ \Psi(t)\left( \frac{\partial V}{\partial t} + \frac{\partial}{\partial t}\int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}dt \right) \\
&= \Psi(t)\left( \frac{\partial V}{\partial t} - (\mu_{x+t} + \delta)\, V(t) - \bar{p}(t) \right),
\end{aligned}$$

where we applied the chain rule to

$$\frac{\partial}{\partial t}\int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}dt.$$

Thus we get

$$\begin{aligned}
\pi_s^*(T) = \pi_t^*(T) &+ \int_t^s \Psi(\xi)\frac{\partial V}{\partial S}\sigma S d\hat{W}(\xi) \\
&+ \int_t^s \Psi(\xi)\Bigg[ \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (\mu_{x+\xi} + \delta)V(\xi) \\
&\qquad\qquad + \frac{\partial V}{\partial t}(\xi) - \bar{p}(\xi)\Bigg] d\xi.
\end{aligned}$$

Now the drift term is equal to zero, since $\pi_\cdot^*(T)$ is a martingale. Therefore we finally get

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\, V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t)\frac{\partial V}{\partial S}.$$

**Exercise 52** *Prove the second part of the theorem above.*

**Remark 53** *Thiele's differential equation for pure endowments and term insurance (Theorem 50), has been stated assuming no lapses. In reality a lot of the corresponding GMDB and GMAB policies are lapse supported. Depending on the context the surrender value of the policy varies. If we assume that the surrender value in case of lapse is given by $f(x)$ for a function*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$$

*it is easy to show that the mathematical reserves for the model including lapsed with a lapse density $\mu^l(x)$ fulfill the following partial differential equations:*

1. *The differential equation for the price of a pure endowment policy is:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + \left(\mu_{x+t} + \mu_{x+t}^l + \delta\right) V(t) - f(x)\mu_{x+t}^l$$

$$-\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.$$

2. *The differential equation for the price of a term life insurance is:*

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} - f(x)\mu^l_{x+t} \\ &\quad -\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

*It is worth mentioning that in many cases we have $f \equiv 0$, meaning that the option value of the variable annuity guarantee is lost in case of a surrender.*

**Exercise 54** *Prove the modified partial differential equations as per Remark 53. Note that in this context one has to replace*

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

*by*

$$\Psi(t) = \frac{\exp(-\delta t)}{\bar{p}_{**}(x+t, x+T)}$$

*(see definition* **??***). By theorem* **??** *we can use the following equation:*

$$\bar{p}_{**}(s,t) = \exp\left( - \sum_{k \neq *} \int_s^t \mu_{*k}(\tau) d\tau \right). \tag{3.7}$$

**Remark 55** *Until now we have mainly considered insurance covers which pay a lump sum in case a person survives a certain number of years (endowment) or in case of death (term insurance). The same methodology shown above can be used to model unit-linked annuities (GMWB variable annuities). Since an annuity can be considered as a negative insurance premium, we have actually already included this case in Theorem 50 and in Remark 53. We will use the same notation as in Remark 53. Furthermore we denote by $\bar{r}(t)$ the annuity density being paid to the policyholder. Keeping in mind that $\bar{p}(t)$ in Theorem 50 needs to be replaced by $\bar{p}(t) - \bar{r}(t)$ we get the following Thiele partial differential equation for a GMWB / GMDB combined insurance product:*

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) - \bar{r}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} \\ &\quad - f(x)\mu^l_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

*By using a Taylor expansion we get the following movement of $V(t)$ during the interval $[t, t+\Delta t]$.*

$$
\begin{aligned}
V(t) + \bar{p}(t)\Delta t = \bar{r}(t)\,\Delta t + \sum_i \left(\mu^i_{x+t}\,C_i(t)\right)\,\Delta t \\
-\left(V(t) - S(t)\,\frac{\partial}{\partial S}V(t)\right)\,\delta\Delta t \\
-\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S^2}\Delta t \\
+\left(1 - \left(\sum_i \mu^i_{x+t}\right)\,\Delta t\right)\,V(t+\Delta t)
\end{aligned}
$$

*We will see in the following that the above roll forward of $V(t)$ is useful when introducing the concept of a hedge P&L.*

## 3.6 Example

In this section we consider a concrete example for which the corresponding calculations can be explicitly performed. We consider the following steps:

1. Description of the product,
2. Valuation of the product, and

The value of a variable annuity from a policyholder point of view consists of two parts:

1. The value of the underlying fund investment, and
2. the value of the variable annuity rider (eg GMDB, GMAB, GMIB and GM(L)WB).

From an insurer's point of view the fund investment is normally classified as a separate account and rather easy to value. The focus on valuation of variable annuities regards the actual variable annuity riders, on which we will focus. Figure 3.9 shows the different parts of a balance sheet of an insurance company which sells unit linked products with guarantees (aka "variable annuities").

In order to value such variable annuities there are different ways to determine the value of the underlying guarantee:

- Explicit formula or recursion (only for insurance valuation and very simple variable annuities),

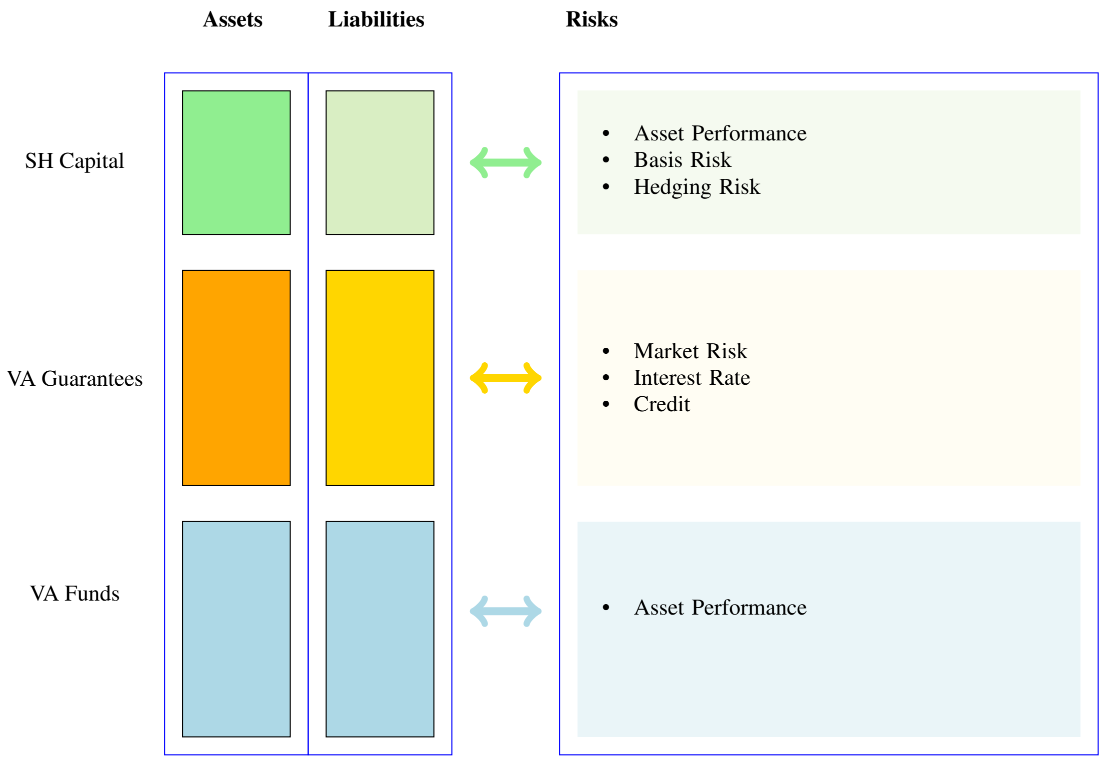



**Fig. 3.9** Balance sheet of an insurance company

- Solution of Black-Scholes-Merton differential equation / Thiele's differential equation (different methods including tree method),
- Monte Carlo Simulation (this is the approach most often used for variable annuities).

Monte Carlo is most commonly used for variable annuities since it is very versatile and can also cope with very complex option structures, such as ratchets. For this example, we will however limit ourselves to a product which can be valued explicitly. In order to keep things simple we look in a first step at a discrete time version of Theorems 45 and 46.

**Proposition 56** *Assume that the economy follows the Black-Scholes-Merton model and assume that deaths occur at time $K = \lfloor T \rfloor$. For*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

*we can calculate the single premiums as follows:*

$$
\begin{aligned}
{}_T G_x &= {}_T p_x \, \pi_0^*(T), \\
G^1_{x:T} &= \sum_{k=0}^{T-1} {}_k p_x \, q_{x+k} \, \pi_0^*(k),
\end{aligned}
$$

*where*

$$
\begin{aligned}
\Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}
$$

*for* $s > t$.

**Exercise 57** *Prove the previous theorem.*

### *3.6.1 Definition of the product*

In the following we consider a very simple variable annuity product consisting of a term assurance ("Guaranteed Minimal Death Benefit (GMDB)") and of a pure endowment policy ("Guaranteed Minimal Accumulation Benefit (GMAB)"). This is exactly the set up which we have considered in section 3.4.

- Variable annuity for a 30 year old man consisting of GMDB and GMAB for a term of 25 years.
- Variable annuity guarantee level according to a "doubler", eg
  - At maturity guarantees twice the initial fund value,
  - In case of death, at least double the value after 10 years.
- Single premium for both funds value and guarantee premium.

In order to illustrate things clearer we will base our calculations on the following decrement table ($l_{35} = 100000$, $l_{x+1} = l_x\,(1 - q_x)$, and $d_x = l_x\,q_x$):

| x | $l_x$ | $d_x$ |
|---|---|---|
| 35 | 100000.0 | 147.7 |
| 36 | 99852.3 | 157.5 |
| 37 | 99694.8 | 169.5 |
| 38 | 99525.4 | 183.8 |
| 39 | 99341.5 | 200.8 |
| 40 | 99140.8 | 220.3 |
| ... | | |
| 60 | 86622.9 | 1394.0 |

We note that one can calculate

$$
\begin{aligned}
{}_t p_x &= \frac{l_{x+t}}{l_x}, \text{ and,} \\
{}_t p_x \, q_{x+t} &= \frac{d_{x+t}}{l_x}.
\end{aligned}
$$

In a next step we want to do this example also allowing for lapses. For sake of simplicity, we have assumed a level lapse rate of 4% with an exception for year 10 where the assumed lapse rate is 12%. These uneven lapse rates are typical – after 10 years (or so, product dependent), the policyholder has the ability to lapse without surrender penalty, leading to higher lapses in this year.

The following table shows the decrement table including lapses:

| x | $l_x$ | $d_x$ | $l_x$ incl. lapse | $d_x$ incl. lapse |
|---|---|---|---|---|
| 35 | 100000.0 | 147.7 | 100000.0 | 147.7 |
| 36 | 99852.3 | 157.5 | 95852.3 | 151.2 |
| 37 | 99694.8 | 169.5 | 91860.7 | 156.2 |
| 38 | 99525.4 | 183.8 | 88016.8 | 162.6 |
| ... | | | | |
| 44 | 98114.7 | 326.8 | 67604.5 | 225.2 |
| 45 | 97787.9 | 360.9 | 60517.2 | 223.4 |
| 46 | 97426.9 | 398.4 | 57735.6 | 236.1 |
| ... | | | | |
| 60 | 86622.9 | 1394.0 | 23716.1 | 381.7 |

Note that also in this context we can calculate ${}_t p_x$ and ${}_t p_x \, q_{x+t}$ by means of $l_x$ and $d_x$. Based on the above table we see that the valuation of a variable annuities heavily depends on lapses and also that the guarantee gets cheaper when considering lapses, since less people profit from it. In consequence most variable annuity products are lapse supported.“Lapse supported” means that the pricing of the product relies on the fact that some of the policyholders will lapse early.

### 3.6.2 Valuation of the Product / Replicating of a Variable Annuity

Based on the example introduced before we now determine the value or price for this variable annuity guarantee. In order to do this, three steps are involved:

**Determine the number of people which benefit:** Since only a tiny percentage of the whole inforce dies within a given year, one only needs to provide the respective GMDB cover only to them. Similarly the GMAB cover is paid only to the people surviving the entire term of the policy. Hence we need to determine the respective percentages. This is done by means of life decrement tables, as introduced before.

**Calculate what these people receive:** We need to know what the respective policyholders are entitled to. Assume for example the people dying at age 40. They are entitled to get a GMDB at a certain level. Hence we need to determine the number of the corresponding units of guarantees. For our 40 year old dying person, this would be put options at a strike price.

**Calculate the value:** We know the valuation portfolio of guarantees representing the variable annuity guarantee (eg number of instruments and their characteristics). We now need to value them. For our example this is done via the Black-Scholes formula.

Note: only in this simple example we can easily distinguish between steps 2 and 3. Normally one performs 2 and 3 together using a Monte Carlo simulation. For our concrete example the table below provides the portfolio of guarantees at inception.

| Instrument | Strike | Amount |
|---|---|---|
| Fund | | 100000 |
| Put Fund value t=0 at | 100000 | 0.1 % |
| Put Fund value t=1 at | 107177 | 0.2 % |
| Put Fund value t=2 at | 114870 | 0.2 % |
| Put Fund value t=3 at | 123114 | 0.2 % |
| Put Fund value t=4 at | 131951 | 0.2 % |
| Put Fund value t=5 at | 141421 | 0.2 % |
| ... | | |
| Put Fund value t=9 at | 186607 | 0.3 % |
| Put Fund value t=10 at | 200000 | 0.4 % |
| ... | | |
| Put Fund value t=25 at | 200000 | 86.6 % |

Over time, as the policy matures more and more of these instruments are used to pay the guarantees of the corresponding period. Deviations from this modeled guarantee portfolio result in a profit or loss.

Based on the valuation portfolio we can now calculate the various metrics for the policy. In particular we can value the variable annuity guarantee of the policy by valuing each individual instrument:

| | Instrument | Strike | Amount %age | **Value** |
|---|---|---|---|---|
| 0 | Put Fund | 100000 | 0.1 % | 8.7 |
| 1 | Put Fund | 107177 | 0.2 % | 17.7 |
| 2 | Put Fund | 114870 | 0.2 % | 27.7 |
| 3 | Put Fund | 123114 | 0.2 % | 39.5 |
| 4 | Put Fund | 131951 | 0.2 % | 53.6 |
| 5 | Put Fund | 141421 | 0.2 % | 70.6 |
| ... | | | | |
| 9 | Put Fund | 186607 | 0.3 % | 146.0 |
| 10 | Put Fund | 200000 | 0.4 % | 181.8 |
| ... | | | | |
| 25 | Put Fund | 200000 | 86.6 % | 34789.4 |
| | **Total** | | | **40311.7** |

We need to look at the consequence of different market shocks at inception, such as lower equity prices, lower interest rates and higher volatility. Note that the valuation portfolio does not change and remains (in terms of respective types of guarantees and amounts) the same! Concretely we look at the following three shocks:

- Equity Drop by 10%,
- Interest Lower by 1%,
- Volatility up by 1%.

The following table summarises the corresponding results. We can observe the high dependency of the value on the market variables.

| | Instrument | Value Normal | Value Equity $-10\%$ | Value Interest $-1\%$ | Value Volatility $+1\%$ |
|---|---|---|---|---|---|
| 0 | Put Fund | 8.7 | 16.5 | 9.5 | 14.5 |
| 1 | Put Fund | 17.7 | 26.5 | 19.6 | 26.5 |
| 2 | Put Fund | 27.7 | 37.6 | 31.3 | 39.4 |
| 3 | Put Fund | 39.5 | 50.6 | 45.3 | 54.1 |
| 4 | Put Fund | 53.6 | 66.0 | 62.4 | 71.4 |
| 5 | Put Fund | 70.6 | 84.6 | 83.4 | 92.0 |
| ... | | | | | |
| 9 | Put Fund | 146.0 | 165.8 | 178.2 | 180.8 |
| 10 | Put Fund | 181.8 | 204.0 | 224.2 | 222.3 |
| ... | | | | | |
| 25 | Put Fund | 34789.4 | 38092.7 | 54782.4 | 50112.1 |
| | **Total** | **40311.7** | **44239.3** | **62445.2** | **57454.8** |

Finally we look how the value of the valuation portfolio changes over time. There are two effects which affect its value, namely that, over time, parts of it are used to finance the claims which have occurred in the past and also as a consequence of the market movement of the underlying fund. Figure 3.10 shows how the guarantee of the hedge liability moves over time:

- Upper figure shows movement in fund value. The lower figure shows the changing value of the underlying guarantee.

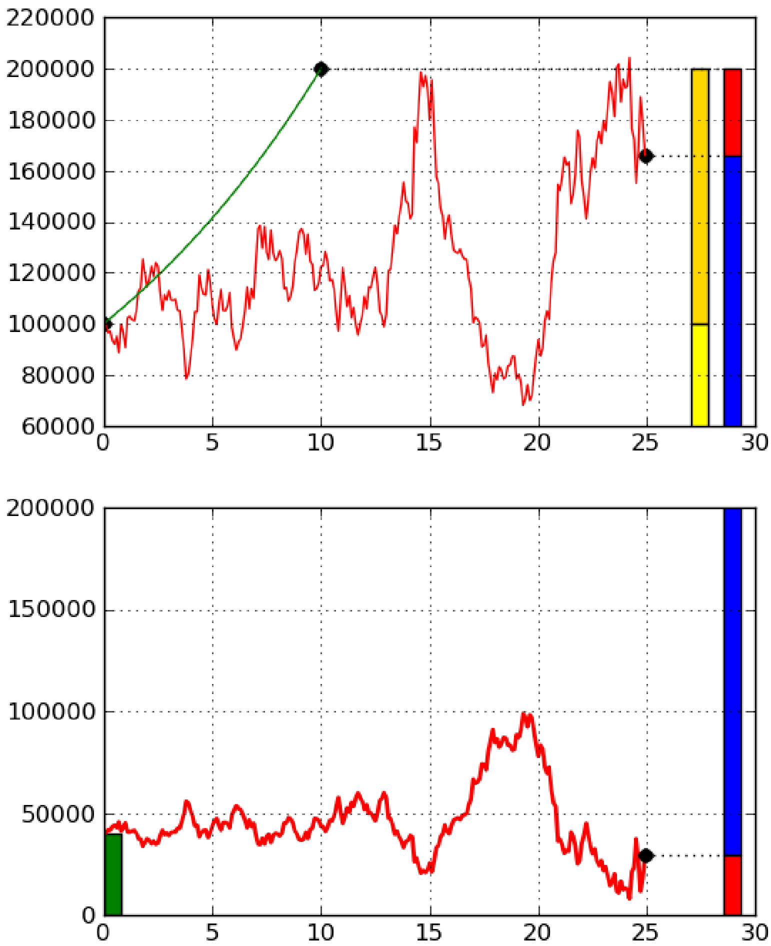



**Fig. 3.10** Value of Hedge Liability over time

- Note that at inception the value of the guarantee equals the value determined above (40311).
- Over time the valuation portfolio becomes smaller, since part of its instruments are used to finance the claims which have occurred in the past.
- Moreover the valuation portfolio changes its value as a consequence of changing fund levels, interest rates and volatilities.

- For this example interest rates and volatilities have been keep constant.
- It becomes obvious that the value of the guarantee increases each time fund value decreases and vice versa.

It is important to understand that none, except the most simple variable annuity guarantee structures can be calculated explicitly by the Black-Scholes formula. The examples in this text have been designed in such a manner that they still allow to use the Black-Scholes formula.

### 3.6.3 *Value of a variable annuity as a function of equity level*

Finally, the following table shows the dependency of the value ($\pi$) of the variable annuity guarantee as a function of the equity level and is called a trading grid. The lower the equity level, the more valuable the variable annuity guarantee.

| Equity Level | $\pi$ <br> $V(S)$ | $\delta$ <br> $\frac{\partial}{\partial S} V$ | $\gamma$ <br> $\frac{\partial^2}{\partial S^2} V$ | $\rho$ <br> $\frac{\partial}{\partial r} V$ | $\nu$ <br> $\frac{\partial}{\partial \sigma} V$ |
|---|---|---|---|---|---|
| -50 % | 65777 | -33818 | 39673 | -2341692 | 86714 |
| -40 % | 59397 | -36051 | 42961 | -2241519 | 111455 |
| -30 % | 53734 | -37319 | 44743 | -2136639 | 133559 |
| -20 % | 48710 | -37826 | 45431 | -2030460 | 152596 |
| -10 % | 44254 | -37744 | 45168 | -1925345 | 168470 |
| -5 % | 42219 | -37531 | 44705 | -1873714 | 175247 |
| 0 % | 40301 | -37230 | 44052 | -1822905 | 181284 |
| +5 % | 38494 | -36854 | 43243 | -1773034 | 186616 |
| +10 % | 36789 | -36419 | 42315 | -1724192 | 191283 |
| +20 % | 33662 | -35417 | 40235 | -1629845 | 198789 |
| +30 % | 30871 | -34295 | 38026 | -1540199 | 204141 |
| +40 % | 28372 | -33102 | 35808 | -1455379 | 207655 |
| +50 % | 26130 | -31872 | 33640 | -1375369 | 209618 |

# Chapter 4
# Accounting Principles

In this rather short chapter we want to have a deeper look at the different accounting principles which are normally used. As seen above, technically speaking an accounting principle on a balance sheet $(x)$ is a function which allocates to each asset and liability its value. Since there are different possibilities, it is not always easy to see the main differences between the different accounting standards. We want to have a look at the following accounting standards:

- Statutory accounting,
- IFRS and US GAAP accounting,
- Embedded value accounting,
- Economic balance sheet accounting.

In order to do this, we always need to look at the same balance sheet and we want to see what are the material differences between the different standards. It needs to be stressed, that it will be impossible to explain all the different aspects and hence this chapter cannot substitute the in depth study of the corresponding standards.

We will look at the following balance sheet, which we have introduced before:

| Balance sheet | Book **A** | Book **L** | Market **A** | Market **L** | |
|---|---|---|---|---|---|
| Cash | 6200 | 47100 | 6200 | 48513 | MR |
| Bonds | 35700 | **2200** | 37842 | **3569** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 52082 | 52082 | |

## 4.1 Statutory Accounting

Statutory accounting is, in most cases, one of the most prudent forms of accounting and the focus of the standards are smooth and continuous profits, as long as there are no market disruptions. Implicitly one assumes that the assets are held for a long period and bonds in particular are held until maturity. In consequence, bonds are valued according to the *amortised cost method* and shares are accounted for at the lower of book value and market value.

There are different ways how this “lower book or market” principle can be applied such as:

- Whether the book value needs to be written down and hence the asset stays afterwards at this lower price in the books or not,
- Whether there is only a need for a write-down if the decrease in asset value is permanent,
- Whether one can use other hidden values to offset the negative movement,
- Whether the impairment needs to take place at once or can be dispersed over some years.

One particular consequence of this set of accounting rules is the fact that there are normally unrealised capital gains and losses, which in this world only materialise if the asset is sold or the asset defaults.

## 4.2 IFRS and US GAAP Accounting

Since the different statutory accounting rules can vary considerably, US GAAP and IFRS accounting standards have been introduced with the aim to make balance sheets and income statements more comparable. As a consequence, the corresponding accounting standards, together with the guidance notes are very large, since the aim is to cover all possibilities. For a beginner it is not always easy to understand what is happening and why.

One can, in principle, try with each accounting standard to optimise the usability of both balance sheet and income statement. In the first case the focus is a most accurate representation within the balance sheet. In the second case the focus is in having profit and loss accounts where one tries to get a profit and loss statement, which allows best to judge the quality of the earnings of the company. This approach is also known as a deferral and matching approach. Both of the above mentioned accounting standards follow this philosophy. When looking at the DAC the deferral and matching approach will become more evident.

On the asset side there are normally different possible choices depending on the intention of the company. Whereas this might help the individual company to show their performance in the way they believe it is most suitable, these choices are also one of the root causes for the opacity of these standards (the other being the high intrinsic complexity).

These choices work as follows: Each asset is classified into a category and it is accounted for accordingly. In order to avoid accounting arbitrage there are limitations in respect to a change in the accounting category. For bonds the possible categories are shown in the table below:

| | B/S Treatment | P/L Treatment |
|---|---|---|
| Hold to Maturity | Amortised Cost | Amortised Cost |
| Available for Sales | Market Value | Amortised Cost |
| Trading | Market Value | Market Value |

This means that bonds are treated completely differently depending on the classification. For shares there also exists two different classifications, as for the ones above with the difference that obviously “Hold to Maturity” does not make a lot of sense. In the case of shares an impairment provision needs to be taken if there is a permanent impairment. Impairment of an asset is given if its carrying amount exceeds its recoverable amount. Since IFRS is very restrictive one usually finds accurate definitions of such terms, as:

The recoverable amounts for the following types of intangible assets should be measured annually whether or not there is any indication that it may be impaired. In some cases, the most recent detailed calculation of recoverable amounts made in a preceding period may be used in the impairment test for that asset in the current period:

- An intangible asset with an indefinite useful life.
- An intangible asset not yet available for use.
- Goodwill acquired in a business combination.

Depending on the valuation used for assets there may occur (gross) unrealised capital gains which step are broken down in different parts, such as latent taxes, etc. So summarising the situation can be quite difficult.

One particular asset is the so called DAC asset, which is a direct consequence of the deferral and matching principle. In order to understand this, one needs to understand how an insurance policy is sold. If an insurance policy is sold with a regular premium of 6000, one could, for example, expect that commissions of 10000 are paid to the distributor. This would result in an accounting loss even if no mathematical reserve would have to be set up. Hence the more the company sells, the worse its profit. Since it is expected that this initial loss is compensated in later times, there are several attempts to present an accounting standard which takes care of this. One of the possibilities is the embedded value method which we will describe in the following section. The other idea is to assume that the paid commission can be financed and amortised with the future gains. In consequence one creates in a first step an "intangible" asset called DAC ("Deferred Acquisition Costs") and one amortises it over time. Also here there is quite some discretionary, in respect to the following:

- How much of the acquisition costs are deferred?
- At the first application of the standards, which portfolios are considered going back and how?
- Which is the amortisation pattern which is used?[1]

After having looked at the assets we want next to look at the liabilities. There are, most importantly, the mathematical reserves, which are accounted at book values. This incongruence between assets and liabilities leads to an artificial balance sheet volatility. Furthermore it is known that the statutory reserves are based on a prudent approach resulting in hidden reserves as a consequence of this conservatism. Another important liability are latent taxes and deferred policyholder participations. The first effect is a consequence of having unrealised capital gains on the assets which are accounted according to market values. Once one sells them the unrealised capital gains would represent true gains before taxes. Since these gains are then taxed a corresponding liability is set up. The same is true for the policyholder participation. For a country such as Germany where 90 % of the gross profits have to be given back to the policyholder, it is clear that out of 1000 unrealised capital gains, the shareholder can, in normal circumstances, only expect 10% or equally 100 (before tax). As a consequence, the remaining 900 are deferred policy holder bonuses. Hence based on 1000 of unrealised capital gains we would have the following reparticipation assuming 34 % of tax:

| | |
|---:|---:|
| Deferred Policyholder Bonus | 900 |
| Deferred Tax | 34 |
| Part of Shareholder Equity | 66 |
| Unrealised Capital Gains | 1000 |

Please note that in reality the situation is still more complex since one would expect that a part of these gains are used for the accelerated amortisation of the DAC asset and other similar effects.

[1] Under US GAAP there are three different ways to amortise DAC, proportional to premium (FAS 60) and proportional to expected profits (FAS 97 and 120)).

## 4.3 Embedded Value and Economic Accounting

As we have seen before there is an intrinsic problem in respect to statutory accounting in the sense that companies writing profitable new business tend to show bad statutory returns as a consequence of the so called new business strain, the effect that the commissions paid are higher than the premium received. Obviously, statutory accounting misses some part in the value, namely the future gains. The aim of the embedded value accounting is to include future gains in the balance sheet. This is done by recognising the so called *present value of future profits (PVFP)*. Also embedded value accounting has changed over time since at the beginning it was based on statutory profits emerging over time. One realised that the original embedded value methodology was not clear enough and introduced the european embedded value (EEV). After recognising this later one is not *risk neutral* and that higher equity backing ratios (eg investments in equities) always leads to a higher value, one reconsidered this position with the introduction of a *market consistent embedded value (MCEV)*. Obviously, this topic could fill a whole book and hence the introduction must remain superficial. Depending on the parameters and interpretations chosen, a market consistent embedded value can be considered as an economic measure.

### *4.3.1 Economic Valuation / Market Consistent Embedded Value*

In contrast to the *traditional embedded value*, the economic valuation or the market consistent embedded value is based on modern valuation techniques such as arbitrage free pricing etc. The idea here is to base discounting on a risk free rate. Risk is considered by setting appropriate capital for the different points in time and by setting an adequate cost of capital. For a more rigid approach than the one presented in this section we refer to appendix C. There we will show how to calculate different replicating portfolios and will also present an abstract approach to ALM.

The most important additional insights which will be provided by Solvency II are *economic balance sheets*, in particular with respect to insurance liabilities. This means on the asset side that all unrealised capital gains and losses are taken into account in a transparent way. On the liability side the situation is somewhat different because there are no tradeable instruments which can be used to perfectly replicate the liabilities in order to determine their economic price. It is clear however, that this information is of the utmost importance for managing the risks and therefore usually a model approach is used to get a reasonable approximation of the market values for the insurance liabilities. First, one needs to calculate the expected present value of the future policyholder benefits, as seen before. On top of this amount one requires a so called *market value margin* ($MVM$). In order to calculate the expected present value of the future policyholder benefits, one needs to calculate the corresponding cash flows. If we assume that the expected cash flows are independent on asset returns, such as for guaranteed benefits (eg. annuities in payment), the expected present value of the cash flows $(CF_t)_{t\in\{0,1,2,\dots\}}$ can be calculated by

$$\mathbb{E}[PV] = \sum_{k=0}^{\infty} \pi_t(\mathcal{Z}_{(k)}) \times \mathbb{E}[CF_k],$$

where $\pi_t(\mathcal{Z}_{(k)})$ denotes the market price of a zero coupon bond with maturity $k$ at balance sheet date $t$, keeping in mind that we have assumed stochastic independence of the expected cash flows from the financial variables. If this independence is not the case, such as in products with discretionary bonus benefits or with GMDB products in relation to unit linked policies we need to apply a more general definition of replicating portfolios, such as the one presented in appendix C. Whereas this calculation is quite straight forward for P&C insurance, it requires some additional considerations for life portfolios, where it is typically based on a policy-by-policy calculation. In contrast to usual actuarial practise where mathematical reserves are based on the assumption that there are no lapses, it is key within a realistic valuation to also consider this effect. In doing so, the duration of liabilities usually reduces considerably. This clearly shows the importance to consider this effect.

On top on the expected present value, it is necessary to calculate the market value margin. In order to understand this we need to acknowledge that the $\mathbb{E}[PV]$ does not take into account that, even for guaranteed liability cash flows with no link to the capital markets, the cash flows can fluctuate over time for example as a consequence of a pandemic. As a consequence the insurance company need to carry a certain amount of risk capital to absorb such shocks and it will not be willing to assume the insurance liabilities just for $\mathbb{E}[PV]$. This can be shown mathematically (in appendix C) using the utility assumption of the investor and the Jensen-inequality. The cost of capital approach, which is also more formally introduced in appendix C aims to provide for a proxy of the market value of insurance liabilities.

Another motivation to use the cost of capital approach is based on the assumption that the originating life insurance company becomes insolvent and has to be wound up. In this case the market value margin, calculated based on the cost of capital approach incentives another company to assume the corresponding insurance liabilities, since it can produce a higher yield on the capital invested on the run-off portfolio assumed. Hence the cost of capital approach provides a mechanism to ensure that insurance portfolios are assumed once a life insurance company becomes insolvent (but not bankrupt).

The *cost of capital approach (CoC)* requires, that the risk capital ($RC_t$) is projected into the future. In a second step the $CoC$ equals the present value of the corresponding costs for the future periods:

$$MVM = CoC = \sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)}).$$

The parameter $\beta$ corresponds to the unit cost of capital and is usually in the order between 2% and 6%, for regulatory purposes. In case of a given hurdle rate $\gamma$ (eg $\gamma = 7\%$), $\beta$ can be calculated by the formula $\beta = \gamma -$ riskfree for the corresponding period, neglecting for the moment the effect of taxation. It is important to note that the CoC approach has two additional benefits: it can quite easily verify the corresponding results and it avoids double counting of capital.

Similarly one can calculate the internal rate of return by this approach. Assume that $\beta = \gamma -$ riskfree is constant (for example by introducing a constant spread over risk free), then the calculation becomes still easier:

$$IRR = \frac{\mathbb{E}[PV]}{\sum_{k=0}^{\infty} RC_k \times \pi_t(\mathcal{Z}_{(k)})} + \text{riskfree}.$$

In case of a yield curve which is not flat, the IRR (eg $\gamma$) can be calculated by the following formula:

$$IRR = \frac{\mathbb{E}[PV] + \sum_{k=0}^{\infty} i_k \times RC_k \times \pi_t(\mathcal{Z}_{(k)})}{\sum_{k=0}^{\infty} RC_k \times \pi_t(\mathcal{Z}_{(k)})},$$

where $i$ corresponds to the corresponding forward return on the capital, eg $i_k = \frac{\pi(\mathcal{Z}_{(k+1)})}{\pi(\mathcal{Z}_{(k)})} - 1$. It is obvious that in case of a flat yield curve the two formulae are equal.

Having stated the importance of basing the solvency regime on a reliable economic balance sheet, there is another important question relating to the market consistent valuation of liabilities. What is the value of the different policyholder options such as the possibility to surrender a policy or to take the capital or annuity in a pension scheme? It is clear that these implicit options can have a considerable value, but there are few reliable methods to value them which are generally accepted. Therefore, a pragmatic approach has to be taken. This means that only the most relevant policyholder options should be quantified. The most prominent example is the guaranteed unit linked insurance contract. Here the valuation of the corresponding put option on the fund is relatively easy to quantify based for example on the Black-Scholes formula and the corresponding risk management techniques (see also section C and example 125).

Finally we need to realise that in the real world there are additional constraints, which have an impact on the value of a portfolio or a product sold. The most relevant are listed below:

- Frictional costs and
- Taxes,

Frictional costs stem from the fact that the company needs to hold at a certain time the corresponding statutory reserves $V_t$ for an underlying block of business. Given the fact that the *best estimate liabilities* $\mathbb{E}[PV]$ may be inferior, the company needs to hold this additional amount, resulting in the above mentioned (pure) frictional capital costs:

$$FCC^* = \sum_{k=0}^{\infty} \beta \left\{ \max(0, V_k - \mathbb{E}[PV]_k) \right\} \times \pi_t(\mathcal{Z}_{(k)}),$$

where $\mathbb{E}[PV]_k$ denotes the expected present value of liabilities as seen at time $k$. Based on the fact that the risk capital also qualifies as capital to fill up missing reserves, the total frictional capital costs amount to:

$$FCC = \max(0, FCC^* - CoC).$$

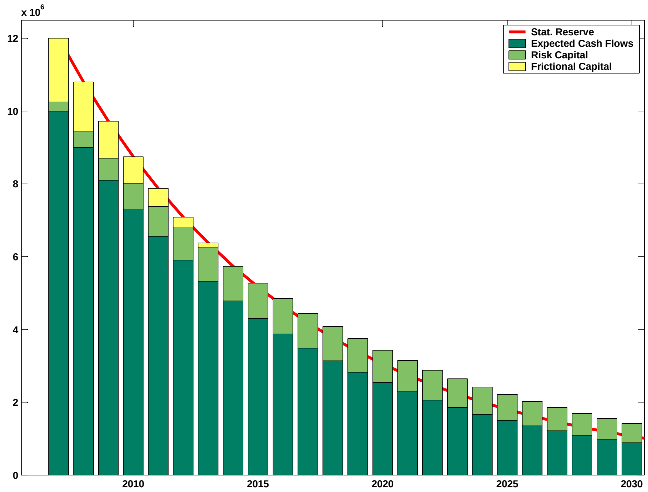
With respect to taxes, all values need to be considered after tax. Whether a certain tax applies and to what extent depends heavily on the country. In the simplest setting, pre-tax values can simply be multiplied by $(1 - \text{taxrate})$.

### *4.3.2 Valuation Methodology Revisited*

Risk management is based on a market consistent valuation of the insurance liabilities. The following effects have to be considered:

1. Expected present value of the cash flows using a risk free interest rate.
2. The market value margin (MVM) which compensates the buyer of the portfolio for the risk he assumes.
3. The present value of the frictional capital costs, which are essentially a consequence of higher external capital requirements than those on a pure economic basis. Typical examples are higher statutory reserves or solvency requirements.
4. Other cash flow streams which need to be valued, such as cash flow swaps induced by funds withheld, etc.

The following figure illustrates the different parts:




### Best Estimate Liability (BEL)

Firstly, one needs to calculate the expected present value of the future policyholder benefits. On top of this amount one requires a so called market value margin $MVM$. In order to calculate the expected present value of the future policyholder benefits, one needs to calculate the corresponding cash flows. The expected present value of the cash flows $(CF_k)_{k \in \{0,1,2,\dots\}}$ is then calculated by

$$\mathbb{E}[PV] = \sum_{k=0}^{\infty} \pi_t(\mathcal{Z}_{(k)}) \times \mathbb{E}[CF_k],$$

where $\pi_t(\mathcal{Z}_{(k)})$ denotes the market price of a zero coupon bond with maturity $k$ at balance sheet date $t$, assuming again that the cash flows are stochastically independent on the financial market variables. It has to be stressed that the consideration of lapses is key within a realistic valuation. Note that the calculation of cash flows including lapses can be done in a similar manner as in section 2.6. Moreover one can use the Markov chain life insurance model (see appendix B) by enlarging the state space, etc. by the state “lapse”.

### Market Value Margin/Cost of Capital

On top on the expected present value, it is necessary to calculate the market value margin. In a second step the $CoC$ equals the present value of the corresponding costs for the future periods:

$$
\begin{aligned}
MVM = CoC = & \sum_{k=0}^{\infty} \gamma \times RC_k \times \pi_t(\mathcal{Z}_{(k)}) \\
& - \sum_{k=0}^{\infty} \left( \frac{\pi_t(\mathcal{Z}_{(k-1)})}{\pi_t(\mathcal{Z}_{(k)})} - 1 \right) \times RC_k \times \pi_t(\mathcal{Z}_{(k)}) \\
& \quad \times (1 - \text{ Tax Rate}) \\
= & \sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)}).
\end{aligned}
$$

The parameter $\gamma$ corresponds to the unit cost of capital and is usually in the order between $2\%$ and $6\%$, for regulatory purposes. In case of a given hurdle rate $\gamma$ (eg $\gamma = 13\%$), $\beta$ can be calculated by the formula $\beta = \gamma -$ riskfree for the corresponding period, neglecting for the moment the effect of taxation.

### Positions of a Fair Value Valuation

| **Position** | **Amount in USD** | **Relative Amount** |
|---|---:|---:|
| Reserves in B/S | 753400 | 19.78% |
| Present Value Premium | 3055000 | 80.22% |
| Present Value Claims | -2945000 | -77.33% |
| PV Exp - Internal | -50400 | -1.32% |
| PV Exp - Overhead | -85730 | -2.25% |
| PV Exp - Commissions | -332400 | -8.72% |
| Subtotal | 394800 | 10.36% |
| Market Value Margin | -97070 | -2.54% |
| FCC | -33560 | -0.88% |
| Funds Withheld | 6281 | 0.16% |
| Tax | -127200 | -3.33% |
| **Total** | **143200** | **3.76%** |
| PV Profit | 284700 | |
| PV Capital | 2044000 | |
| RoRAC | | **13.93** % |

## 4.4 Formulae

$$
\begin{aligned}
\mathbb{E}[PV] &= \sum_{k=0}^{\infty} \pi_t(\mathcal{Z}_{(k)}) \times \mathbb{E}[CF_k], \\
CoC &= \sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)}), \\
FCC^* &= \sum_{k=0}^{\infty} \beta \left\{\max(0, V_k - \mathbb{E}[PV]_k)\right\} \times \pi_t(\mathcal{Z}_{(k)}), \\
FCC &= \max(0, FCC^* - CoC), \\
\text{Profit before Tax} &= \mathbb{E}[PV] - CoC - FCC, \\
\text{Profit after Tax} &= (1 - \text{taxrate}) \times \left\{\mathbb{E}[PV] - CoC - FCC\right\}, \\
\beta &= \gamma - \text{riskfree for the corresponding period.}
\end{aligned}
$$

Note again that we have assumed here that the insurance cash flows are independent on the capital market variables, which is the case for guaranteed benefits, but *not* for discretionary policyholder participation and unit linked policies with guarantees. For a more rigid approach we refer to appendix C.

## 4.5 Examples

### *4.5.1 Annuity*

We consider a real life annuity portfolio with a total face amount of about EUR 240 M p.a. In order to determine the expected cash flow and the replicating portfolio it is necessary to choose a mortality law for the description of the evolution of the mortality:

$$q_{x,t} = q_{x,t_0} \times \exp(-\lambda_x \times (t - t_0)).$$

Considering a $x$ year old person, the present value of the annuity in payment of 1 EUR is given by

$$\ddot{a}_x = \sum_k {}_kp_x \times v^k.$$

This indicates that the expected cash-flow at time $t$ equals ${}_tp_x$ and therefore the replicating portfolio corresponds to $\sum_t {}_tp_x \times \mathcal{Z}_{(t)}$, where $\mathcal{Z}_{(t)}$ represents an abstract basis for the corresponding zero coupon bonds. This policy has the following value at balance sheet date:

$$\mathbb{E}[PV] = \sum_k {}_kp_x \times \pi_t(\mathcal{Z}_{(k)}),$$

where $\pi_t(X)$ denotes the market price of the financial instrument $X$ at time $t$.

It is now necessary to calculate the market value margin. In order to do so, one needs to determine the relevant risk factors together with their probability functions. In case of the annuity portfolio we assume longevity as main risk factor and assume that the mortality for future years follows the following law:

$$q_{x,t} = q_{x,t_0} \times \exp(-\lambda_x(\omega) \times (t - t_0)).$$

In this case we model the risk by replacing $\lambda_x$ by $\lambda_x(\omega) = c(\omega) \times \lambda_x$. $c(\omega)$ corresponds to the relative change in mortality improvement in relation to the observed standard trend. In this case the present value of the loss equals the difference of the expected present values based on $\lambda_x$ and $\lambda_x(\omega)$ respectively. By integrating over $d\omega$ one gets the desired result for the present value of the risk capital. Multiplying by the unit CoC results in the desired result.

The analysis is based on a real life annuity portfolio with reserves summing up to EUR 2.7 bn and annuities in payment of ca. EUR 240 M. We use a 99.5% shortfall as risk measure for the calculation of the CoC. At this point in time it is worth to mention the fact that

$$\sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)}) = \beta \times \sum_{k=0}^{\infty} RC_k \times \pi_t(\mathcal{Z}_{(k)}).$$

This means that it is possible to model the present value of the risk capital directly, which is done for this example. Furthermore the function:

$$\mathbb{N} \to \mathbb{R}, n \mapsto \frac{1}{\pi_t(\mathcal{Z}_{(n)})} \sum_{k=n}^{\infty} RC_k \times \pi_t(\mathcal{Z}_{(k)})$$

defines the required risk capital for the different periods. Figure 4.1 illustrates the replicating portfolio, on the one hand side with $c = 100\%$, and on the other with $c = 130\%$, showing the longer duration and hence the higher present value in the latter case.

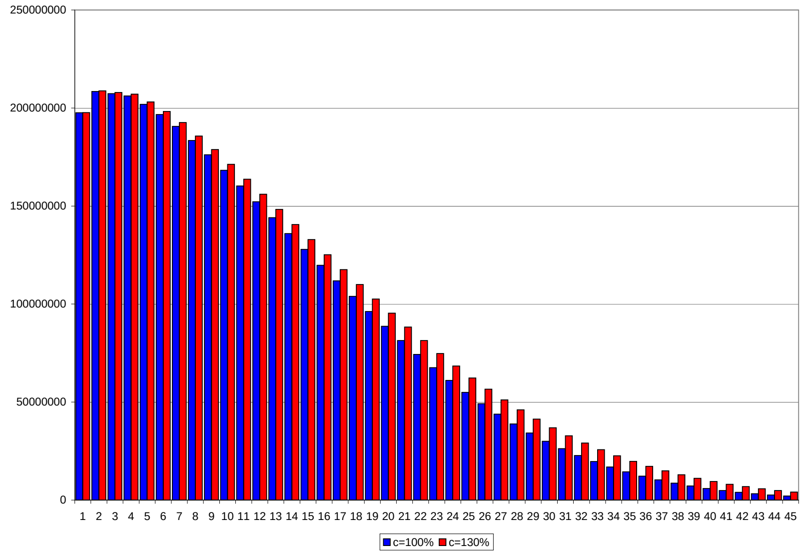



**Fig. 4.1** Replicating Portfolio

By using the replicating portfolios as given in figure 4.1, the development of the required capital corresponds to figure 4.2, using a somewhat simplified version for the capital.

As a next step these calculations have to be done for the different $c(\omega)$, and weighted with the corresponding probabilities. The following table illustrates this:

| $c(\omega)$ | $P[c(\omega)]$ | Loss($c(\omega)$) bn EUR | Contribution to TailVar bn EUR |
|---|---|---|---|
| 1.0 | | 0.000 | |
| 1.1 | | -0.032 | |
| 1.2 | | -0.065 | |
| 1.3 | | -0.099 | |
| 1.4 | | -0.133 | |
| 1.5 | | -0.169 | |
| 1.6 | | -0.205 | |
| 1.7 | | -0.242 | |
| 1.8 | 0.0005 | -0.280 | -0.028 |
| 1.9 | 0.0005 | -0.319 | -0.031 |
| 2.0 | 0.0004 | -0.359 | -0.028 |
| 2.5 | 0.0019 | -0.572 | -0.217 |
| 3.0 | 0.0017 | -0.801 | -0.272 |
| Total | **0.0050** | | **-0.578** |

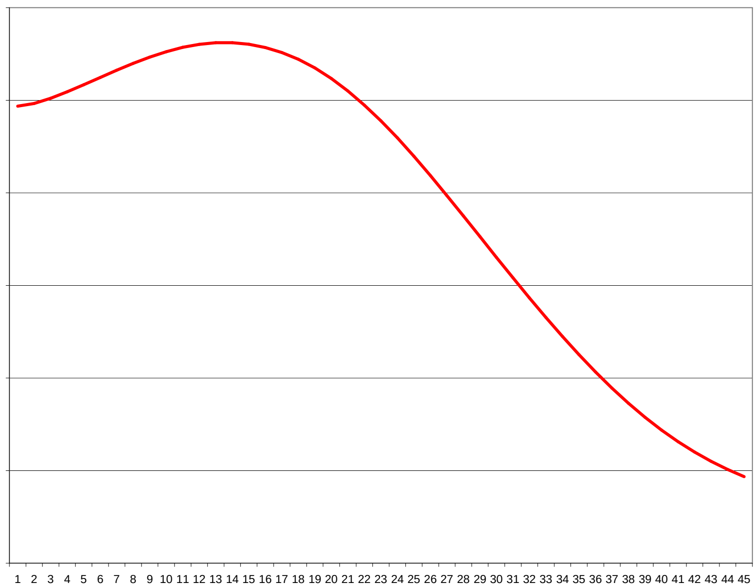



**Fig. 4.2** Required capital over time

This indicates that the present value of the risk capital, calculated with the 99.5% TailVar, amounts to 0.578 bn EUR. For more details in respect of TailVar see chapter 5. Assuming that the statutory reserves or the price the company pays corresponds to 2.934 bn EUR we get the following:

| | | bn EUR | % |
|---|---|---|---|
| + | Statutory reserve | 2.934 | 100.00% |
| − | $\mathbb{E}[PV]$ | 2.750 | 93.73% |
| − | CoC 13% | 0.075 | 2.56% |
| − | Tax 25% | 0.028 | 0.95% |
| = | **Profit Tax = 25%** | **0.080** | **2.72%** |

IRR ca. 30%

In this particular case the margins induced by the prudent mortality laws in the statutory reserves are partially offset by a low interest environment. It is however obvious that the statutory reserves carry about 4% of margin with respect to a market consistent valuation.

### 4.5.2 Capital Protection

Whereas we have considered in the first example an annuity portfolio, we now want to look at a life protection portfolio consisting of 100 $x = 30$ year old persons with a term of 30 years. The death benefit amounts (per policy) to 100000 EUR with a $\mathbb{E}[PV] = 14507$ EUR. In order to calculate the risk capital, we assume that the exogenous risk factors are types of pandemics, as follows:

| $\theta =$ Relative $q_x$-level | Return period for $\theta$ | $F_{q_x-\text{level}}(\theta)$ |
|---|---|---|
| 1.0 | 0 | 0.000 |
| 1.1 | 10 | 0.975 |
| 1.2 | 20 | 0.976 |
| 1.3 | 30 | 0.978 |
| 1.4 | 40 | 0.980 |
| 1.5 | 50 | 0.981 |
| 2.0 | 100 | 0.990 |
| 2.5 | 175 | 0.994 |
| 3.0 | 250 | 0.996 |
| 4.0 | 500 | 0.998 |
| 10.0 | 1100 | 0.999 |
| $20 + \epsilon$ | $\infty$ | 1.000 |

Based on this approach it is now possible to do a simulation by replacing the original $q_x$ by a new random $q_x(\omega)$ given by $F_{q_x-\text{level}}(\theta)$. The following table summarises the main results of this simulation, where SaR stands for "Sum at Risk". The sum at risk is the amount of money the insurer loses for a certain policy in case the insured person dies. Hence it equals the sum insured minus the corresponding mathematical reserves hold in the balance sheet for this policy.

| | | Relative | in% SaR |
|---|---|---|---|
| Expected Value at level 100% | 1407500 | | |
| Expected Addl Loss | 43100 | 3.1% | 0.43% |
| Sdtdev | 97700 | 6.9% | 0.97% |
| 2.5 $\sigma$ | 244400 | 17.4% | 2.44% |
| $F^{-1}(99\%)$ | 568800 | 40.4% | 5.68% |
| $F^{-1}(99.6\%)$ | 720500 | 51.2% | 7.20% |
| TVar(99%) | 697100 | 49.5% | 6.97% |

This indicates that given a hurdle rate of 13% and using the VaR with respect to a return period of 250 years, the required single premium for this contract can be calculated as follows, neglecting the impact of taxes:

| Item | | Amount |
|---|---|---|
| $\mathbb{E}[PV]$ | | 1450700 |
| CoC @ 13% | $720,500 \times 13\%$ | 93600 |
| **Total** | | 1544300 |

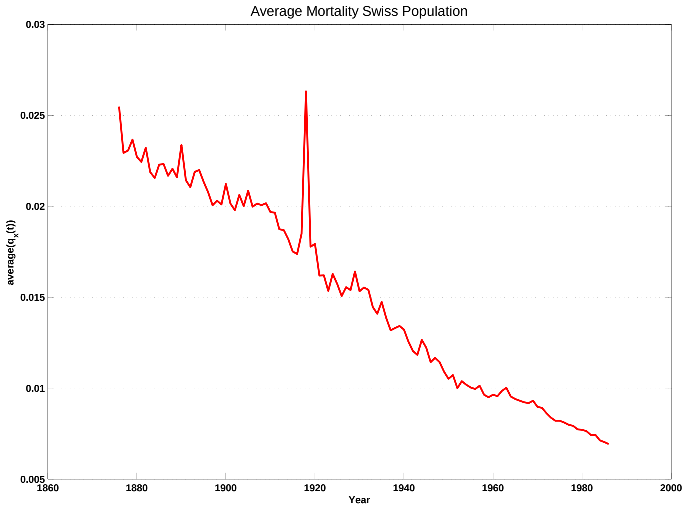



**Fig. 4.3** Development of Mortality over Time

The following figures illustrate the effect of the 1918' influenza pandemic ('Spanish Flu').

Figure 4.3 shows the change in average mortality over the years. It becomes obvious the average mortality for the year 1918, when the Spanish flu occurred equals roughly the one of 1860 and is much higher than the average mortality of neighbouring years. Figure 4.4 compares the mortality for the years 1908, 1918 and 1928. Interestingly the pandemic results in a much higher mortality for young people aged between 15 and 40. The older ages are relatively less affected.

Finally figure 4.5 shows the mortality per age-band and year. Again we see that the mortality for 30 year old people is considerably higher and equals the one of the 60 year old people. We observe at the same time that the main mortality improvement over this time span relates to the younger people and not to the very old.

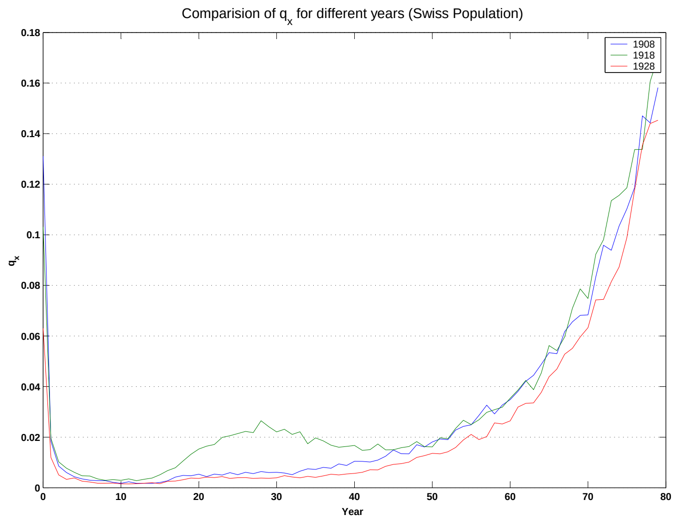



**Fig. 4.4** Comparison of different years

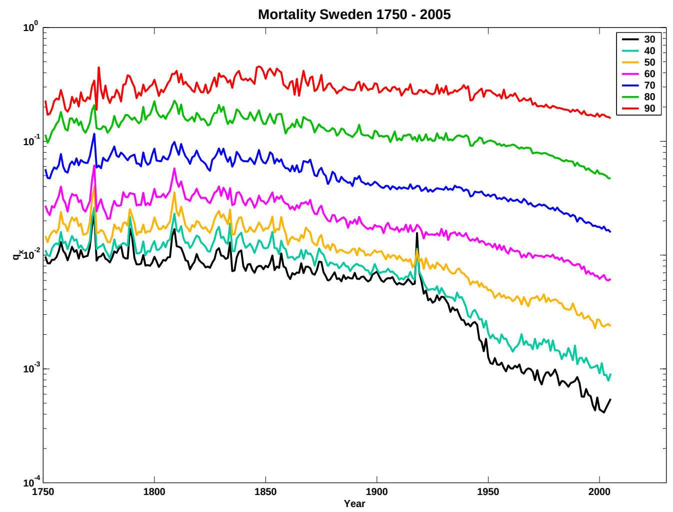



**Fig. 4.5** Mortality by Ages for Sweden 1751 - 2005

# Chapter 5
# Risk Appetite and Tolerance

## 5.1 Risk Capacity and Risk Appetite

Risk appetite is a term that is frequently used throughout the risk management community. Recent changes in global regulations that encompass security and risk and control implications, have raised the awareness around the concept of risk appetite, particularly among the management team.

Also, the financial crisis of autumn 2008 has shown the critical importance of considering the possible risks which may be faced by financial service companies. This crisis involved several cases in which financial services companies, including insurers, suffered highly damaging losses from risks which they had not fully or correctly evaluated, or perhaps even not been aware of. Many of the failings of companies can be attributed to the acceptance of excessive risks and the poor management of those risks or a lack of clarity around the level of compensation expected for risks taken. Also, in many cases, the link between risk and strategic planning or business decision making has been insufficient.

Risk capacity is defined by the available risk capital. From an economic viewpoint, this is defined as the adjusted difference between the market value of investments and the market value of liabilities (insurance liabilities and financial commitments). The risk appetite shall be commensurate with whichever risk the decision-makers (*risk owners*) are willing to assume. The risk appetite is measured in terms of the economic capital needed to cover a given risk exposure over a specified period of time and which must therefore be held in reserve. This capital must be sufficient with a high level of probability. Risk appetite must never exceed risk capacity.

Risk appetite, at the organisational level could be, in general terms, the amount of risk exposure or potential adverse impact from an event the organisation is willing to accept/retain.

With an increasing importance due to the latest events, risk appetite, tolerances, risk targets and limits are a critical element of prudent business management and an effective risk governance process and need to be more than a statement, but something you live every day by how decisions are made and companies are managed. The purpose of "defining risk appetite", whatever that may mean, is to control directly, or at least influence directly, how people make decisions on behalf of an organisation in the face of risk and uncertainty by specifying the importance of risk in some way. In establishing risk appetite, the picture of the whole strategy the risk management should follow is Fig. 5.1. In the concrete context it shows also the readiness of the organisation in implementing the respective steps, eg whether the step is implemented 25%, 50%, . . . 100%.

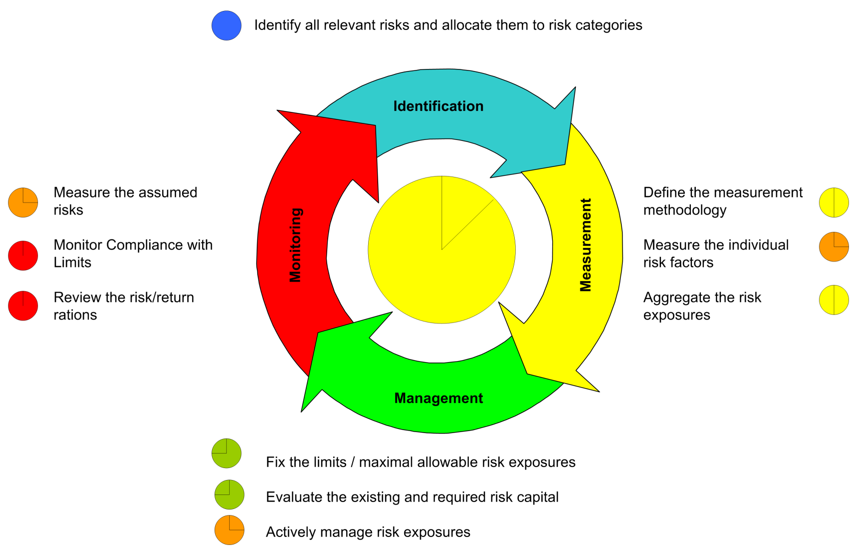



**Fig. 5.1** Risk Identification Process

Each one of the steps should be specified in the procedure at the organisational level, and needs to be analysed with a specific frequency to ensure the company is operating within the expectations of key stakeholders. Each step could broadly be defined as follows:

**Risk Capacity** Risk capacity could be defined as the maximum amount and type of risk a company is able to accept/retain in pursuit of its mission, vision, business objectives and value goals. It is directly related to an entity's capital and external stakeholder influences.

**Risk Appetite** Risk appetite, at the organisational level, is the amount of risk exposure, or potential adverse impact from an event, that the organisation is willing to accept/retain in pursuit of its mission, vision, business objectives and value goals.

The procedure for defining the risk appetite is called “limits system”, which analyses the total exposure to the different risks and identifies “exposure limits”. This means that for each risk it is possible to define the maximum threshold or exposure that the company is willing to accept and in a level above this limit the company needs to act to decrease the exposure.

The risk appetite, as we can see, is directly related to an entity’s risk capacity as well as its culture, desired level of risk, capability and business strategy.

Entities often consider risk appetite both qualitatively and quantitatively. It is often expressed in acceptable/unacceptable outcomes focused in the downside risk, such as:

- Rating target of AA.
- No activity that will impair the ability to continue as a going concern.
- Defined probability of ruin at a specific confidence level (as 90%).

Typically, the level of a risk will be measured by the likelihood of it occurring and the financial impact if it does. We can:

- Capture expert’s opinion of loss severities and frequencies.
- Calculating statistics for individual loss scenarios and the total losses an organisation could sustain as a result.

Most documents trying to define “risk appetite” say it is the amount of risk that “the organisation” is prepared to put up with. The idea is that it applies to the whole organisation. However, almost all practical applications of “risk appetite” involve multiple “appetites”, though these may be intended to reflect an overall “appetite”. “Risk appetites” in various forms are set out for the organisation as a whole, for sub-units of the organisation, for activities within an organisation, for types of risk, for individual risks, and even for liability products or asset portfolios. Sometimes a single “risk appetite” is laid down that is intended to be applied to all risks and sometimes each risk has its own individual appetite. All these are legitimate possibilities, though some only make sense for some kinds of decisions and there are some difficult issues to deal with in setting lower level appetites to reflect a higher level appetite.

**Risk Tolerance** Risk tolerance is typically a specific maximum applicable to each risk regarding the magnitude and the type of risks the organisation is willing to take in order to achieve its business strategy and objectives while operating within the broad risk appetite.

It should be set that the aggregation of risk tolerances ensures the organisation operates within the risk appetite.

Risk Tolerance can be expressed in terms of:

- Risk measures with a number (or category) that represents the quantity of risk, as the mathematically expected value of "impact", the total exposure, or VAR.
- Other variables that may be used are results that are not related to risk events. They may be solvency statements, results like profit or earnings, nominal measures as the amount of premiums written, or categories like high/medium/low applied to "impact" or to outcome levels.

Many constraints in "risk appetite statements" being written today do not mention risk at all. Instead, they are often rules on proxies for risk such as external conditions, activity levels, and results achieved. The reason for the use of this type of statement is that they are easy to analyse and reliable. For example, they may say that no more than 10% of investments will be in a certain currency, or that new offices will not be opened in countries with inflation above 4%, or that actual losses from operational incidents will not be more than 5% of revenues in any one month. All variables used should be carefully defined and their values, when used in decision making, should be made explicit, usually by being written down so that personal bias is harder to conceal. Historically, organisations typically consider risk appetite based on effects across the four general dimensions shown below by priority of use:

1. Statutory capital or level of surplus.
2. Credit ratings threshold.
3. Economic capital.
4. GAAP earnings.

**Risk Target or Range** The risk target is the optimal level of risk that the organisation desires to take to achieve its business strategy and objectives and to operate within its appetite/tolerance for risk. It is generally articulated as a range in the same units that risk tolerance is measured in.

The setting of risk targets should be based on the management's desired returns, the role of risk to achieve those returns (risk/return profile) and management's capability to manage each risk.

In setting specific risk targets, management aligns risk targets to ensure that it will meet both its strategic goals as well as operating within its risk appetite and tolerance.

**Risk Limits** The risk appetite is the amount of risk that it is willing to accept in pursuit of value. Risk appetite therefore reflects the desire to optimally exploit opportunities and minimise hazards to an acceptable level.

The actual implementation of the risk strategy is achieved through the fixing of permissible risk limits for the company as a whole. The corporate limits will be spread across the primary risk categories - separately for the markets, products, channels and functions. The risk committee delegates the limits to the next level of responsibility, documents the delegated limits and monitors compliance therewith. By doing this the risk committee ensures that it operates within the limits which have been granted by the group. In case of insufficient limits, the risk committee requests a higher capacity from group, if there is an appealing business case.

The risk committee shall be informed on a quarterly basis of the utilisation of limits by the company as a whole.

The risk limit is a threshold to control activities to ensure that variations from expected outcomes will be consistent with the risk target, but will not exceed the risk appetite/tolerance.

Limits are how the appetite/tolerance and the risk target are translated into practical constraints on business activity. For example, as mentioned in the risk appetite section above, each company has its own “exposure limits” for each of the risks, through the limits system.

Risk limits should be formulated so that on a probabilistic basis, considering utilisation, aggregation and correlation, they ensure that the organisation operates within its target range and does not exceed its aggregate risk tolerance.

The threshold can be set:

- By business unit.
- By individual risk exposure.
- Allowing for diversification.
- With controls and processes to maintain risk within risk appetite.

Once the risk appetite threshold has been breached, risk management treatments and business controls are implemented to bring the exposure level back to within the accepted range. However, we need to be aware that a threshold is a crude, all-or-nothing approach. No value for it is right in every situation and natural decision making can be disrupted by having to work with the threshold. For example, if an action results in a risk measure value that is above a limit then that action cannot be chosen. If the action results in a risk measure value that is below the limit then that action can be chosen. In other words, below the limit risk doesn’t have any importance, but above it risk is of overwhelming importance and rules out the action. Although limits put a ceiling on risk taking they do not ensure that risks are properly weighed against rewards. If risk limits were the only way that risk was weighed in decision making then there would be no difference in risk terms between an action that was just within the limit and another that was well within the limit. To promote good risk reward decisions some more progressive weighting of risk can be used, or at least management need to be aware of the level in which each risk operates and analyse each situation carefully before making a decision.

Additional potential problems in defining thresholds could be:

- Setting levels is very difficult and they can often seem rather arbitrary, leading to problems getting people to take them seriously.
- The decisions to which the rules apply may be unclear.
- Breaking an overall “risk appetite” into smaller parts is difficult. If each individual element is constrained so that, in total, the overall appetite is not exceeded (in some sense) then the freedom to act given to individual elements has to be cut down. Many strategies that would be possible if risk was constrained only at the top level are blocked by having multiple lower level constraints. In effect, the breakdown into subsidiary risks and into activity levels can create an ever tightening straight jacket.

And also notice that risk appetite, tolerance and limits are not static. They must be updated with changes in strategy, the environment and market expectations. Ultimately, they should be a key element in driving risk taking and in turn in performance measurement.

## 5.2 Limit Systems

Limit systems are a way to express risk appetite. Normally risk appetite statements are given on a more global level and need to be broken down in a second step to actual limit systems which are more granular. As with risk appetite there are several possibilities on how to design a limit system and hence the example below is for illustration purpose only.

In this first section an overview is given in respect to the relative size of the underlying insurance company. Hence these are no limits but serve for comparing the corresponding risks.

| In M EUR | Metric | BU |
|---|---|---|
| Local Cur | | EUR |
| Total B/S | IFRS | 53000 |
| SH Equity | IFRS | 1700 |

**1 Risk Capital Limits**

| In M EUR | Metric | BU |
|---|---|---|
| Risk Capital Limit | | |
| Market (ALM) Risk Cap | ICA | 800 |
| Credit Risk Capital | Group Method | 200 |

In the above table the risk capital limit refers to the required capital which the company is willing to put at risk, as defined in chapter 12. The ALM risk capital and the credit risk capital refer to the required capital for the corresponding financial risk (see chapter 6).

### 2 Market Risk Limits

| In M EUR | Metric | BU |
|---|---|---|
| Exposure Limits FX in% | | |
| Local Currency | MV | 100% |
| CHF | MV | 15% |
| EUR | MV | 100% |
| GBP | MV | 20% |
| USD | MV | 15% |
| Other FX | MV | 5% |
| Total Equities | MV | 500 |
| Max. Single Stock Position | MV | 50 |
| Maximum interest rate sensitivity per 10 bps | Against guaranteed CF, scaled | 300 |

All of the above quantities aim to steer the financial risk taking. One tries for example to limit the FX risk or also the maximal amount which can be invested in a single counter-party.

**3. Credit Risk Limits**

| In M EUR | Metric | BU |
|---|---|---|
| Exposure Limits Credit | | |
| Local Government | Nominal Value | unlimited |
| AAA Rating | Nominal Value | 400 |
| AA Rating | Nominal Value | 200 |
| A Rating | Nominal Value | 100 |
| BBB Rating | Nominal Value | 50 |
| Below BBB and NR | Nominal Value | 25 |

**4. Insurance Limits**

| In M EUR | Metric | BU |
|---|---|---|
| PV of Premium per Contract | EUR | 15 |
| Annuities | PV Annuities | 5 |
| Mortality | Sum at Risk | 2 |
| Disability | 10 x annuity or lump sum | 2 |
| Stop Loss | Max Loss | 15 |

The above table aims to limit some risks in relation to life insurance as outlined in chapter 7.

**5. Other-Operational Limits**

| In M EUR | Metric | BU |
|---|---|---|
| Operational Limits | in M EUR | 5 |
| Capital Expenditure | in M EUR | 2 |
| Revenue Expenditure | in M EUR | 2 |
| Claims Settlement Authority | in M EUR | 4 |
| Bad debt write off p.a. | in M EUR | 1 |
| Asset Dispose | in M EUR | 2 |
| Reinsurance Commutation | in M EUR | 4 |
| Letters of Credit | in M EUR | 4 |

## 5.3 Hedging Strategies and Response Strategies

We have seen in this chapter how risk appetite can be defined and we also know that there are the following principles for taking risks:

- Risk is rather limited than eliminated as responsible risk taking contributes to value creation. The approved risk appetite governs the level of risk an insurance company is willing to accept. Any risks outside of appetite will be proactively managed in a timely manner.

- Risks are only accepted where the required organisational capability, expertise and infrastructure to manage the risks are in place. In addition sufficient risk based capital buffers to withstand risks materialising even under extreme stressed conditions are required.
- In accepting risk we strive for capital efficiency and profitable growth.

The above means in particular that there is a need to reduce risks and have response strategies in place in order to bring risks outside risk appetite or risks violating some limits back into risk appetite.

The corresponding strategies for the reduction of financial risks are called *hedging strategies*. In the wider context also including all other types of risks one names them *response strategies*. The aim of this short section is to provide an overview what this could mean. For the (financial) assessment of the strategies we refer in particular to chapters 6 and 13. The objective of such strategies is to reduce the risk to an acceptable (agreed) level and to optimise the risk adjusted performance. In order to do that the different strategies are analysed and compared in order to choose an optimal one. An example can be found in section 14.4.

In the following we will show what a hedging or response strategy could mean for different risks an insurance company is facing:

**Equity Risk:** If an insurance company faces a too high equity exposure which could threaten it, there are different possible response strategies:

1. Do nothing (I will mention this only once ... ).
2. Selling equities: This can be a lengthy process since a well diversified equity portfolio consists of may different equities. Furthermore not all equities are liquid enough to change the risk portfolio fast enough.
3. Using an overlay strategy. One possible choice is to sell futures in order to reduce the equity exposure. The selling of index futures can be performed very fast and hence it is possible to rapidly reduce the equity exposure. As with all proxy hedges there remains a *basis risk*. This means that the derisked equity portfolio (eg equities plus short future) will behave differently that the correspondingly reduced equity portfolio, since the actual equity portfolio may not fit the index chosen. Hence it is important in applying such hedges to see how close the hedge is to the actual portfolio and how big the deviation could be in an distressed environment.
4. Using derivative structure, such as buying an (index) put, or puts on individual equities. When using a put option on an index one also faces here a certain basis risk. Furthermore put options on individual equities tend to be quite costly, in particular if the underlying share is illiquid. The difference to the strategy using futures is that one pays for the options up front. For futures this is not the case. On the other hand futures require regular *margin calls*. This mechanism limits the counter-party exposure of the two parties engaging

in the future contract. Hence in case of short futures the insurance company has to pay cash to its counter-party (eg bank), when the stock market rises. There are two things to consider. On one hand is the basis risk. On the other hand the insurance company might be forced to sell equities in a rising market to pay the margin calls. For derivative structures it is worth to remark that there is also the possibility to do a self-financing derivative strategy. Hence one sells the upside (by selling a call) and uses the proceeds to buy the put. Such strategies could lead to the situation, where one acquires a downside protection at -15 % for the price of limiting its upside to +10 %.

**Credit Risk:** The question here is how to reduce credit risk; the possible choices are very similar to the one stated above with respect to equities. One can either sell the corresponding bonds directly or one can look for derivative structures (CDS) which mitigate the risk. The typical approach for a lot of insurance companies is to assume a very limited amount of credit risk and the tendency is then to sell the titles which are considered to carry excessive risk.

**Interest Rate Risk:** For interest rate risk we refer to chapter 6 and remark that the migration of interest rate risk can be performed either by directly selling and buying bonds, or by using *swaps* or *swaptions*. A Swap allows transforming the duration of a bond and it can be considered as an exchange of two different cash flow streams. Hence one can "change" a 3-year bond into a 10-year bond. A Swaption is an option which allows you to enter into a Swap contract at a predefined price.

**Insurance Risk:** The mitigation of insurance risk is done mainly via reinsurance contracts, where the insurer cedes a part of his risk to a reinsurer. There are different ways of doing this either by a *quota share*, where a certain percentage of the original risk is ceded to the reinsurer. A *non proportional treaty* is a reinsurance treaty where the cession is not linear (as with a quota share). A non proportional treaty would qualify in the context of financial risk as an option. A *stop loss treaty* is an example of such a non proportional contract. Here the reinsurer starts to pay if a certain threshold of the total loss is exceeded.

## 5.4 Introduction: Use of Capital

Economic capital is one of the cornerstones of risk management. It has, roughly speaking, the same purpose as a meter stick for an engineer: It serves to measure and compare different risks and to limit them. Hence the following tasks are performed using economic capital:

- Limit and control risks,
- Allocation of capital to different markets and different functions and lines of business,

- Measure risk adjusted profitability,
- For regulatory purposes and to define the risk appetite.

In order to understand the methodology outlined in this document we need to understand how such a model works and which are the generic steps to define it. In order to produce an economic capital model the following steps need to be performed. It should be stressed that this is not an easy task and that there exist models which can be used out of the box, such as the Swiss solvency test, JP Morgan Risk Metrics and others:

The following steps, which are explained in the sequel, are needed in order to calculate the required risk capital

- Definition of the risk factors,
- Definition of a probability density functions per risk factor,
- Definition of a valuation methodology,
- Definition of the joint distribution of all risk factors – diversification,
- Definition of risk measures,
- Definition of stress scenarios.

### 5.4.1 Definition of the Risk Factors considered

The risk factors define a hierarchy of disjointed risks, which are modelled separately. Not all the risks will be quantified at the beginning. Normally, risk factors can be thought as a tree. Figure 1.5 shows the risk landscape, which can be used for a risk management framework.

In order to have a sufficiently granular risk map for measuring financial risk, the map as shown in figure 1.5 needs to be enhanced.

### 5.4.2 Probability Density Functions per Risk Factor

The probability density function for each risk factor describes mathematically its behaviour. In other words: We define these functions by how likely or unlikely a certain event is. As a first step this is done risk factor by risk factor. Next, the different risk factors are bound together. Then the question is whether the events are more likely to occur together or not. Is it, for example, more or less likely that the temperature drops in case of a thunder storm?

The modelling of the different probability density functions and the interaction of the risk factors is based on observable market data, e.g. one looks how the different risk factors have behaved in the past and assumes that this relationship is also valid for the future. Obviously this is a bold assumption and it is therefore important to acknowledge the limitations of *each* model. This however has to be put in relation to the use of a model in general. Assume you have a small lamp (model) in a dark night. Obviously this lamp cannot replace the sun (reality). Nevertheless nobody would go out and leave the lamp at home, only because it is not as bright as the sun ....

We have spoken before about linking different risk factors together in order to be able to observe whether two risk factors interact stronger or weaker. Technically speaking the benefit gained from two risk factors which level themselves out is called diversification or diversification benefit. By choosing a clever asset allocation one can "save" capital by relaying on this effect. *However* there is also a dark side, since it is known that diversification is normally collapsing in case of extreme market movements. Statistically there is a diversification benefit between equities and credit risk in normal circumstances. In September and October 2008 this diversification disappeared and equity and credit markets were highly correlated, resulting in higher losses than in a normal market.

In the above section we have seen that economic models shed some light on the risk characteristics of an insurance portfolio, but that it is dangerous to solely rely on this number. One practical method to know what might happen in extreme conditions is to use stress scenarios, which reflect such extreme states of the economy. With their use it is also possible to do "what if" calculations and regulators often expect companies to use them.

### 5.4.3 Valuation Methodology

After the definition of the various risks which are considered in the economic capital model in order to determine the required risk capital, it is necessary to know how much capital is available. This available risk capital serves as a buffer in order to absorb shocks which are induced by an adverse market movement. To determine shareholder capital and available risk capital there are several possibilities. We know statutory equity, IFRS equity, embedded value and realistic balance sheet equity. The choice of the type of accounting system to be used to determine the available resources is closely linked to the intrinsic methodology applied. Since economic capital models are usually based on economic principles, the normal approach is to use a so called economic or realistic balance sheet.

We assume here the familiarity with the IFRS accounting standards - at least on a very high level. This standard is - roughly speaking characterised by:

- Assets valued at market values,
- Liabilities valued at book (statutory) values,
- In order to partially compensate for this discrepancy, additional elements are used in IFRS such as DAC, shadow adjustments and goodwill.

As seen above IFRS is a hybrid accounting system somewhere between a traditional statutory balance sheet and a fully economic balance sheet. As such this balance sheet normally serves as a basis to determine the economic / realistic balance sheet used for economic capital purposes.

### 5.4.4 Risk Measures

Using a risk measure allows us to assign an amount of capital to the corresponding probability distribution if a ruin event occurs.

Without going into details there are two commonly used risk measures. One is the *value at risk* for a given confidence level and the other is the *Tail VaR* or *expected shortfall*. Both measures relate to a certain probability of occurrence. One normally speaks about a 99.6% VaR or a 99% TailVaR. This means that we consider events which happen on average every 250 years (eg $99.6\% = 1 - \frac{1}{250}$) or once in one hundred years (eg $99\% = 1 - \frac{1}{100}$) respectively. Since the occurrence of an event is linked to the time which elapses it is important to recognise the fact that a *VaR* or *TailVaR* is linked to a time span. In banking one uses, for example, a *daily VaR*, which means one looks at events which occur with a given probability within the next day. For insurance a yearly consideration is normal, hence we look a *99.6% yearly VaR*. In consequence these events can also be considered to have a 250 year return period. (Note that this concept is mainly used for natural perils in reinsurance.)

In order to better understand the two concepts – VaR and TailVaR – we want to show this based on an example and we assume that we have observed 1000 different observations of losses and we have sorted them according to their loss. The worst 12 losses were:

| Number | Loss (in M EUR) |
|---|---|
| 1000 | 210 |
| 999 | 175 |
| 998 | 150 |
| 997 | 145 |
| 996 | 140 |
| 995 | 130 |
| 994 | 125 |
| 993 | 120 |
| 992 | 115 |
| 991 | 112 |
| 990 | 110 |
| 989 | 105 |

We want now to determine the 99.6% VaR and the 99% Tail VaR. For the 99.6% VaR we look at the loss at position 997 (eg 4 events out of 1000) and find that the $VaR = 145$ M EUR. For the Tail VaR we have to look at the average loss, once we know that the loss is within the 1% worst. Hence we have to look at all losses between number 991 and 1000 and have to take the average. Hence we have:

$$\begin{aligned} TailVar &= \frac{210 + 175 + 150 + 145 + 140 + 130 + 125 + 120 + 115 + 112}{10} \\ &= 142.2. \end{aligned}$$

Since one can also plot the probability distribution, one can also locate the VaR and TailVaR within this graphic. Figure 5.2 show this relationship.

**Stress Scenarios**

We know that each economic capital model has limitations, in particular in case of extreme states of the market. This is also the case because there are not enough observations available to calibrate the model. Hence the use of a model which has been calibrated around the mean is dangerous, since the effective risk might be underestimated. In order to compensate for this effect it is common to use stress scenarios. Stress scenarios can be thought as instantaneous changes of the economy according to some predefined risks.

## 5.5 Risk Measures

Using a risk measure allows us to assign an amount of capital to the corresponding probability distribution if a ruin event occurs (a catastrophic scenario occurs when

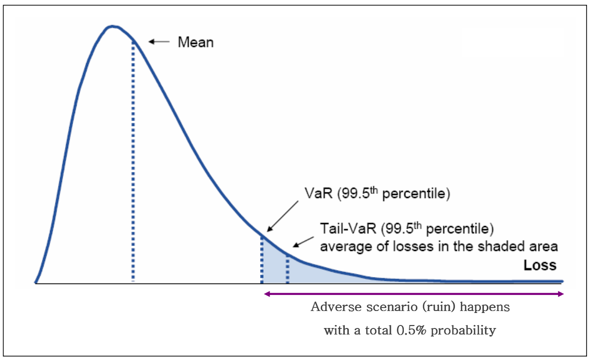



**Fig. 5.2** Comparison VaR and TailVaR

the amount of admissible assets is lower than the total amount of technical provisions). Thus, a risk measure is what will allow calculation of the solvency margin and which will serve as a benchmark for the standard formula and the various internal models that companies may develop.

The chosen risk measure is a function $\rho : X \longmapsto \rho(X) \in \mathbb{R}$ where $X$ is a real random variable which represents the possible loss for an insurer,

- $X > 0 \Longrightarrow$ loss,
- $X < 0 \Longrightarrow$ profit.

In the concrete set-up the possible loss $X$ of an insurer is the difference in shareholder capital over a given time interval. A risk measure $\rho$ is considered as "coherent" if it satisfies the four following axioms:

**Monotonicity:** $\rho(X) \leq \rho(Y)$ whenever $X \leq Y$
If the risk of a portfolio $Y$ is greater than that of a portfolio X then the capital needed will be greater too.

**Positive homogeneity:** $\forall \alpha > 0 : \rho(\alpha \times X) = \alpha \times \rho(X)$
If each risk is multiplied by a factor $\alpha$ then the risk measure will also be.

**Translational invariance:** $\forall \alpha \in \mathbb{R}$: $\rho(X + \alpha) = \alpha + \rho(X)$

**Sub-additivity:** $\rho(X+Y) \leq \rho(X) + \rho(Y)$
Risk measure for two portfolios is lower than the sum of risk measures for these two portfolios. It represents the advantage of diversification effect.

### 5.5.1 *Value at Risk (VaR)*

This measure is much used in banks and insurance companies.

Definition: Given some confidence level $\alpha \in [0,1]$ the Value at Risk of a portfolio is the smallest number $x$ such that the probability that the loss $X$ does not exceed $x$ is larger than $(1-\alpha)$. It represents the maximal potential loss accepted.

$$\mathrm{VaR(X)} = -\inf \{x \,| P[X \leq x] > (1-\alpha)\} \tag{5.1}$$

where X is a real random variable which represents the possible loss for an insurer.

In fact it is defined as a quantile $\alpha$ level: $VaR^{\alpha}(X) = Q^{\alpha}(X) = F_X^{-1}(1-\alpha)$ where $F$ is the distribution function of the random value $X$ continuous and purely increasing such as $F_X(x) = P[X \leq x] = 1 - \alpha$.

The drawback of this measure is that VaR doesn't give any information about the tail of the distribution. Moreover it is not an "coherent"measure, indeed VaR is monotone, positively homogeneous and translationally invariant but not sub-additive (aggregating many risks can increase the risk and it is not really conservative).

Thus Artzner has developed the Tail Value at Risk as a risk measure which is more convenient than VaR especially in reinsurance since tail risks are covered. TVaR requires a less simulations to be estimated and to reaches stability faster than VaR.

### 5.5.2 *Tail Value at Risk (TVaR)*

Definition: Tail Value at Risk (TVaR) is the expected value of the loss in those cases where it exceeds the predefined confidence level. This measure is equal to the average loss a company will suffer in case of (extreme) situations where losses exceed a predefined threshold. Contrary to VaR, TVaR is "coherent"and considers the shape of the tail of the distribution.

$$\mathrm{TVaR}^{\alpha} = \frac{1}{1-\alpha} \int_{\alpha}^{1} VaR(X;\xi) d\xi \tag{5.2}$$

where X is a real random variable which represents the possible loss for an insurer.

We can define similar risk measures such as Conditional Tail Expectation (CTE), Expected Shortfall (ES) or Expected Tail Loss.

- Conditional Tail Expectation (CTE)
  $\text{CTE}^{\alpha}(X) = \mathbb{E}[X|X > VaR^{\alpha}(X)]$
  $\hookrightarrow$ CTE represents the mean value over VaR and satisfies the property of sub-additivity only in the case of a continuous distribution.
- Conditional Value at Risk (CVaR)
  $\text{CVaR}^{\alpha}(X) = \mathbb{E}[X - VaR^{\alpha}(X)|X > VaR^{\alpha}(X)]$
  $\text{CVaR}^{\alpha}(X) = \mathbb{E}[X|X > VaR^{\alpha}(X)] - VaR^{\alpha}(X)$
  $\text{CVaR}^{\alpha}(X) = CTE^{\alpha}(X) - VaR^{\alpha}(X)$
  $\hookrightarrow$ CVaR represents a weighted average between the value at risk and losses exceeding the value at risk
- Expected Shortfall (ES)
  $\text{ES}^{\alpha}(X) = \mathbb{E}[X|X \geq VaR^{\alpha}(X)]$
  $\hookrightarrow$ Expected shortfall is the conditional expectation of loss given that the loss is beyond the VaR level.

As we can see, in the graph above, a 99.5% TVaR gives the average of the highest 0.5% of losses. For this reason TVaR will be higher than the VaR estimate for the same percentile.
According to the CEIOPS[1], experience suggests that, on average, a 99% confidence level with a TailVaR risk measure may roughly be equivalent to a 99.6% confidence level with a VaR risk measure.

Afterwards, these risk measures will be used to determine our catastrophic scenario and consequently, the necessary amount of RAC.

### 5.5.3 *Relationship between Value at Risk and Expected Shortfall*

We know that

$$\begin{aligned}
\text{VaR}^{\alpha}(\text{X}) &= -\inf\left\{x \,|P[X \leq x] > (1-\alpha)\right\}, \\
ES^{\alpha}(X) &= \mathbb{E}[X|X \geq VaR^{\alpha}(X)],
\end{aligned}$$

and we define

[1] www.ceiops.eu

$$e^{\alpha}(X) = \mathbb{E}[X - \mathrm{VaR}^{\alpha}(\mathrm{X}) \,|\, X \geq VaR^{\alpha}(X)]$$

the excess loss over the value at risk[2], and remark that $ES^{\alpha}(X) - VaR^{\alpha}(X) = e^{\alpha}(X)$. Hence this excess loss over VaR measures how much a possible loss "eats" in average more of the available capital in case a $\alpha$–quantile event occurs. We remark that the $e^{\alpha}(X)$ is dependent on the distribution function of $X$. The example of section 5.4.4 show this additional "eating" of capital.

Hence the understanding of $e^{\alpha}(X)$ is essential when choosing a risk measure or defining risk appetite. In this section we want to have a quick look at the respective advantages and disadvantages of the two risk measures.

The main advantage of the VaR is that it is very well understood and widely used. It can be argued that this is the correct over all measure from a shareholders point of view if he wants to define how likely it is to lose all of its capital. If the idea behind measuring risk is however to consider a going concern and to limit the downside, the value at risk is dangerous since an additional amount of capital $e^{\alpha}(X)$ is needed to survive an $\alpha$–quantile event.

It is therefore important to use the expected shortfall measure for the definition of risk appetite statements and for considerations where a going concern in an $\alpha$–quantile event is envisaged. This is particularly true in case one wants to break down the total required risk capital in smaller pieces to define the risk appetite at a more granular level. A concrete example is a risk appetite statement of “We want to be able to continue operating normally and adhering to the one in 200 VaR after a one in 10 year event.” In this example we need to hold an excess capital over the one in 200 Var of the 10 % expected shortfall (and *not* VaR) over the 99.5 % VaR, assuming the independence of the two events.

We finally remark that we obviously can stick to the VaR if we keep these facts in mind and have a view on the possible different outcomes by using VaR. From a theoretical point of view coherent risk measures have obviously its advantages when tying to analyse them.

Assume for the moment, that we use VaR as a risk measure, for example because the regulatory regime requests it and that in consequence the insurance company bases its target capital and its risk appetite on the $\alpha$–quantile with respect to VaR. In case such a default occurs, in average, the amount $e^{\alpha}(X)$ is needed to run off the company in an orderly fashion. In consequence the average cost of default can be calculated as $\alpha \times e^{\alpha}(X)$.

We finally remark that the mean excess function for a random variable $X$ is given by

[2] It is important to remark, that $e^{\alpha}(X)$ has an interesting relationship to reinsurance, if we assume that $VaR^{\alpha}(X)$ is the retention of a reinsurance treaty, with no upper limit. In this case we can interpret $e^{\alpha}(X)$, as the expected loss of the reinsurer in case the retention limit is triggered.

$$e_X(x) = \mathbb{E}[X - x \,|\, X > x] = \frac{\int_x^\infty \{1 - F_X(\xi)\} d\xi}{1 - F_X(x)},$$

and that we have $e^\alpha(X) = e_X(VaR^\alpha(X))$.

For the convenience of the reader we have listed some mean excess functions for different probability distributions in appendix A.3.

Before entering into a new topic, we want to have a look how VaR and TVaR concretely compare, assuming a standard $\mathcal{N}(0,1)$ normal distribution. We start at a confidence level $\alpha$ and calculate then VaR and TVaR. Moreover we also want to calculate the equivalent confidence level $\tilde{\alpha}$ with respect to VaR for a given TVaR at level $\alpha$. The table below summarises the results and we remark that these numbers are heavily dependent on the distribution function chosen. For probability distribution functions with heavier tails, the difference between VaR and TVaR increases, and hence it is important to devote enough time and thought when choosing a suitable family of probability distributions for capital models.

| Confidence Level | VaR | $e(x)$ | TVaR | Equivalent VaR Level |
|---|---|---|---|---|
| 0.84134 | 1.00000 | 0.52513 | 1.52513 | 0.93638 |
| 0.90000 | 1.28155 | 0.47343 | 1.75498 | 0.96036 |
| 0.95000 | 1.64485 | 0.41785 | 2.06271 | 0.98043 |
| 0.99000 | 2.32634 | 0.33886 | 2.66521 | 0.99615 |
| 0.99500 | 2.57582 | 0.31611 | 2.89194 | 0.99808 |
| 0.99900 | 3.09023 | 0.27685 | 3.36709 | 0.99962 |

### *5.5.4 What is a Stress Scenarios mathematically*

A stress scenario, can be considered a stochastic process, associated with a point (Dirac) measure. For stock markets the drop of 30% can be considered as a stress scenario. At the beginning such a stress scenario is not linked with a probability. In order to use this methodology for determining a risk capital one needs to have a view on its probability.

The stress scenario is used to calculate the loss linked to it.

# Chapter 6

# Financial Risks and their Modelling

The aim of this chapter is to educate the readers, in order that they understand the basics of financial risk management and so that they can interpret the numbers within this report. For the underlying abstract valuation concept we refer to appendix C.

## 6.1 The Model underlying Financial Risks

In order to develop a model for managing and measuring financial risks we have a look at the balance sheet, which have seen earlier in this book:

| Balance sheet | Book **A** | Book **L** | Market **A** | Market **A** | |
|---|---|---|---|---|---|
| Cash | 6200 | 47100 | 6200 | 48513 | MR |
| Bonds | 35700 | **2200** | 37842 | **3569** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 52082 | 52082 | |

It is clear that we need to decouple the valuation $\pi_t$ from the underlying asset. So formally the balance sheet consists of assets $(\mathcal{A}_i)_{i \in S_A}$ and Liabilities $(\mathcal{L}_i)_{i \in S_L}$ and we assume that both index sets $S_A$ and $S_L$ are finite. Now assume we have 1000 shares from HSBC. We could say that these 1000 shares are "one" asset. On the other hand we could model the same holding as holding 1000 pieces of the asset "1 HSBC share". Therefore we denote by $(\alpha_i)_{i \in S_A}$ and $(\lambda_i)_{i \in S_L}$ the number of units which we own at the certain point of time. Furthermore we want to separate the shareholder equity from the liabilities and we denote it $\mathcal{E}$.

If we write $\alpha_1 \mathcal{A}_1$ we assume that we are holding $\alpha_1$ units of the asset $\mathcal{A}_1$. Hence our portfolio is an abstract finite dimensional linear vector space $\mathcal{Y} = \text{span}\{(\mathcal{A}_i)_{i \in S_A}, (\mathcal{L}_i)_{i \in S_L}, \mathcal{E}\}$. In this context our balance sheet is a point $x = \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \in \mathcal{Y}$.

As seen before some assets and liabilities can be further decomposed in simpler assets and liabilities and hence we can find a suitable basis for the vector space $\mathcal{Y} = \text{span}\{e_1, \ldots, e_n\}$, where $(e_k)_{k \in \mathbb{N}_n}$ is its basis, and we remark that we can also write our balance sheet as $x = \sum_{k \in \mathbb{N}_n} \gamma_k \, e_k$.

The idea to introduce $\mathcal{Y}$ is to have a normalised vector space. Assume for example that we hold some ordinary bonds. In this case we would use as $e_k = \mathcal{Z}_{(k)}$, the corresponding zero coupon bonds, etc.

We finally remark that the balance sheet $x \in \mathcal{Y}$ actually represents a random cash flow vector, and hence we strictly have $x_t$ or $X_t(\omega) \in \mathcal{X}$ if we assume that the changes of the portfolio follow a stochastic process (cf. appendix D). For measuring the risk of the actual balance sheet it is normally sufficient to assume that $y \in \mathcal{Y}$ does not change.

Next we need to look at the second part, namely the valuation $\pi_t$, and we remark that:

- The valuation is dependent on time.
- We assume that the valuation is a linear functional $\pi_t : \mathcal{Y} \to \mathbb{R}$ which allocates to each asset its value (see also appendix C).
- A liability $\mathcal{L}$ is characterised by $\pi(\mathcal{L}) \leq 0$. In the same sense and asset has a positive value. As a consequence an $x \in \mathcal{Y}$ can in principle be both an asset or a liability, depending on the economic environment and also depending on the valuation functional.

After having defined the different parts we need to have a closer look at what equity or capital ($\mathcal{E}$) means. In the context of the balance sheet we observe that the sum of the value of all assets equals the sum of the value of all liabilities (neglecting the sign). Hence we have the following:

$$\begin{aligned} x &= \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i + \mathcal{E} \in \mathcal{X}, \text{ and} \\ \pi(x) &= \pi \left( \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i + \mathcal{E} \right) \\ &= 0, \text{ and hence} \\ SHE = \pi(\mathcal{E}) &= -\pi \left( \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \right). \end{aligned}$$

This means that we can always calculate the value of the shareholders' equity if we know the value of all other assets and liabilities.

Finally we want to show how to tackle the stochastic valuation functional $\pi_t$. Since we live in a linear vector space $\mathcal{Y}$ with a basis $(e_k)_{k\in\mathbb{N}_n}$, it is sufficient to define the price $\pi_t(e_k)$. The idea is to decouple the operator from the economy and the corresponding set up is to define the state of the economy by a stochastic process $(R_t)_{t\in\mathbb{R}} \in \mathbb{R}^m$. You could think that one of the components could be inflation, another could be the level of the 10 year interest rate, etc. In this setup we can define:

$$\pi_t(e_k) = f_k(R_t),$$

where $f_k : \mathbb{R}^m \to \mathbb{R}$ is a sufficiently regular function. If we assume for example that $R_t[10]$ is the interest rate for the 10 year bond, then we have (depending on our definition of $\pi$)

$$\pi_t(\mathcal{Z}_{(10)}) = (1 + R_t[10])^{-10}.$$

The idea of financial risk management is to assess and control the change of the value of the shareholder equity, e.g. the profit and loss induced by this change. If we assume for the moment that the time $t$ is denoted in years, one is normally interested in the following quantity:

$$PL_T = (\pi_T(\mathcal{E}) - \pi_0(\mathcal{E}))\,.$$

The loss which we encounter within the time interval $[0, T]$. Banks normally look at one week, eg $T = 1/52$, Solvency II looks at $T = 1$. One measures the risk, as indicated before based on the random variable $PL_T$.

Here again is a more formal environment: In order to assess the financial risk of an insurance company the following steps are needed.

1. Define the valuation methodology $\pi_t$,
2. Define (note this is a big model assumption) which stochastic process $R_t$ models the economy,
3. Define the universe of all assets and liabilities $\mathcal{Y}$,
4. Define and calculate the functions $(f_k)_{k\in\mathbb{N}_n}$,
5. Analyse the possible balance sheets $x \in \mathcal{Y}$ and decompose each $\mathcal{A}_i$ and $\mathcal{L}_i$ into the basis $(e_k)_{k\in\mathbb{N}_n}$,
6. Define the risk measure to be used such as VaR, etc.,
7. Implement the model.

The implementation of the above steps in its purest form is very complex and therefore one normally has to make approximations.

## 6.2 Approximations

A common approximation starts with the simplification of the function $f_k$, by using a *Taylor approximation*. Since we are interested in

$$\begin{aligned} PL_T &= \pi_T(\mathcal{E}) - \pi_0(\mathcal{E}) \\ &= [\pi_T - \pi_0] \circ \left( \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \right), \end{aligned}$$

we use the following first order Taylor approximation

$$\begin{aligned} \pi_T(e_k) - \pi_0(e_k) &= f_k(R_T) - f_k(R_0) \\ &\approx \nabla f_k(x)\|_{x=R_0} \times \Delta(R). \end{aligned}$$

If we apply this formula to all assets and liabilities we get a model where the gains and losses are linear in the risk factors $R$. If there is a balance sheet $x = \sum_{k \in \mathbb{N}_n} \gamma_k \, e_k$ we can obviously sum over the different $e_k$ and we get the following approximation:

$$\begin{aligned} \pi_T(x) - \pi_0(x) &= \sum_{k \in \mathbb{N}_n} \gamma_k \times (f_k(R_T) - f_k(R_0)) \\ &\approx \delta^T \times \Delta(R), \end{aligned}$$

where

$$\delta = \sum_{k \in \mathbb{N}_n} \gamma_k \times \nabla f_k(x)\|_{x=R_0},$$

and where we denote with $x^T$ the transposed of a matrix or vector.

Another simplification is to use a stochastic process, which is analytically easy to tackle. Both the risk metrics method and also the Swiss solvency test use a multidimensional normal distribution for $Z = \Delta R$.

Hence we have

$$Z \sim \mathcal{N}(0, \Sigma),$$

where we $\Sigma$ denotes the *covariance matrix*. One can express this matrix by the standard deviation vector $s$ for each of the risk factors and the correlation matrix $\rho$. In a first step we define the matrix $S = (v_i \times \delta_{ij})_{i,j}$. Furthermore we need to know that if $X_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$ is a multidimensional normal distribution and $A$ and $b$ are a matrix and a vector, respectively, we then know that $X_2 := A \times X_1 + b \sim \mathcal{N}(\mu_1 + b, A \times \Sigma \times A^T)$. Using this formula we finally get the following relationship:

$$\Sigma = S \times \rho \times S,$$

keeping in mind that $S = S^T$.

If we use the two approximations, the calculation of the VaR at a level $\alpha$ (eg 99.5%) can be calculated as follows. In a first step we denote by

$$\zeta = F^{-1}_{\mathcal{N}(0,1)}(\alpha),$$

and we get in consequence:

$$\begin{aligned} VaR_{PL}(\alpha) &= F^{-1}_{\mathcal{N}(0,1)}(\alpha), \\ &= \zeta \times \sqrt{(s \times \delta)\rho(s \times \delta)^T}. \end{aligned}$$

Hence the value at risk can easily be calculated using some simple matrix multiplications. The example which follows is based on these approximations.

At this point it is important to remark that every model has flaws and hence it is of utmost importance to understand the *limitations* of a model. The risk to choose a “wrong” or “inaccurate” model is called *model risk*. Here it important how a model is constructed. Figure 9.1 aims to show this. In principle there are the reality (left hand side of the figure) which one tries to model in order to answer “difficult” questions which can not be answered directly. In order to do that one creates a model (right hand side of the figure) and one should be able to answer the corresponding questions within the model. Next one translates the results back to reality and “hopes” that the diagram is commutative. From this point of view the model risk is the missing “approximate” commutativity of the model. As a corollary one needs to acknowledge that each model is suited and best adapted for a certain purpose and that it is dangerous to use the model outside that.

Another interesting aspect with respect to model risk is the fact that one can, from time to time, observe difficult and lengthy discussions between experts on which

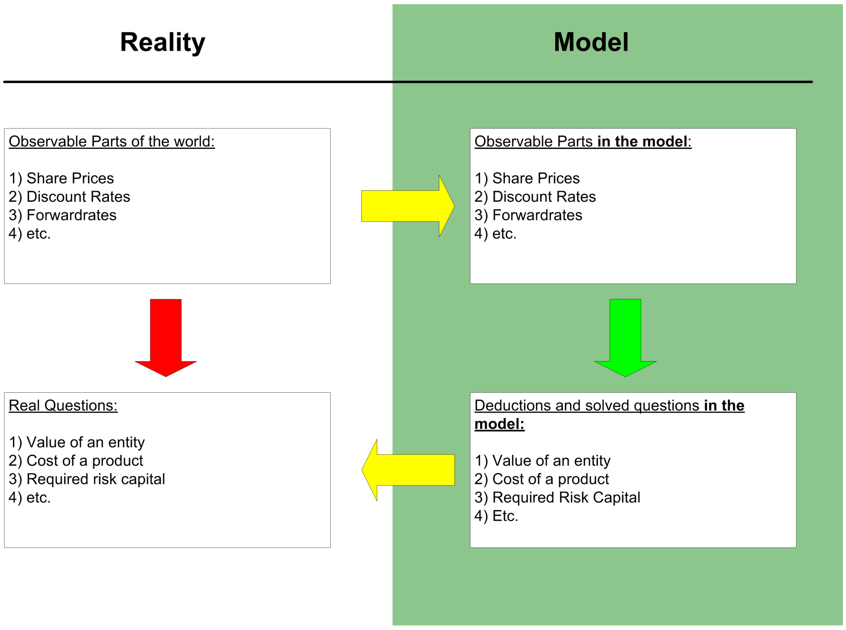



**Fig. 6.1** Models and Model Risk

model is better. Such discussions can stem from the fact that these people do not distinguish between reality and the model and hence these discussions can end up in religion like beliefs.

In the same sense the results of every model depend on the parameters chosen. The risk of inaccurate model parameters is called *parameter risk*. An easy example is the equity volatility, which is for example used for the Black-Scholes model. The value of the corresponding options is heavily dependent of the volatility chosen. As remarked before the volatility for equity market indexes is normally in the region of say 17 %. In case of market disruptions this parameter can spike up to 30 % and above. Hence it is crucial to exactly know how the model behaves with respect to different parameters.

Finally it is worth noting that the distinguishing between model and parameter risk is not always clear.

## 6.3 Concrete Implementation

For the concrete implementation of a risk model for financial risk there are, in principle, the following three different approaches:

1. *Analytical approach*, such as the one used in the Swiss solvency test: Here the required risk capital is calculated based on a closed formula. The advantage here is the fast calculation times because this approach is only feasible for a limited class of model.
2. *Model based simulation* (aka Monte Carlo approach): One can, in principle, use whichever model is deemed to be adequate and one simulates the corresponding random variables. Here one can also use sophisticated methods to link variables together such as the copula method. This method is very flexible - for the price of having normally longer running times, since one requires normally a sizable amount of simulations in order to determine the tail probabilities with an adequate accuracy. Assume for example that we are interested in the 99.5% VaR. In this case we have only 500 simulations which are beyond this level for a sample of 100000.
3. *Historical Simulation*: In this case one uses past observed financial data to predict the future. The big advantage is the fact that we do not need to assume which is the correct distribution. In this class of methods we can either run through the past time series or one can use the boot-strapping method. The problem with this method is the fact that there are only quite short time series (say 50 years) for the underlying financial data. Since one is normally interested in rare events such as the one in a 200 year event one needs to amend this method correspondingly. Furthermore, one needs also to remark that the behaviour of some financial variables has changed considerably over the past 50 years, such as foreign exchange rates, which were fixed until the 1970s and are now floating.

As seen above there are different methods on which we can implement financial risk management. In this section we will have a closer look at the multi-normal model, as used in the analytic part of the Swiss solvency test. First we need to look at the risk factors used and then to calculate the risk capital for the balance sheet introduced above, based on a simplified model.

The Swiss solvency test uses the following risk factors:

- Zero coupon prices for CHF, EUR, GBP and USD, for 13 time buckets,
- Interest rate volatility,
- Credit Spreads for four different rating categories,
- Four different currencies vs CHF: EUR, GBP, USD and YEN,
- Seven equity indexes,
- Equity volatility,
- Real estate, hedge funds and private equity indexes,

each of which is modelled as a normal distribution. Before making a concrete example we want to have a look on how big the different quantities are. Since there are 81 risk factors, this would result in a 81x81 covariance matrix. In consequence we will have a look at a part of it. Firstly we want to look at the corresponding standard deviations (as of 31/12/08).

| Risk Factor $RF_i$ | | Quantity | $\sigma(RF_i)$ |
|---|---|---|---|
| EUR | 1 | bps | 61.82 |
| EUR | 2 | bps | 72.08 |
| EUR | 3 | bps | 73.00 |
| EUR | 4 | bps | 73.12 |
| EUR | 5 | bps | 83.53 |
| EUR | 6 | bps | 70.43 |
| EUR | 7 | bps | 68.09 |
| EUR | 8 | bps | 65.93 |
| EUR | 9 | bps | 64.88 |
| EUR | 10 | bps | 63.54 |
| EUR | 15 | bps | 58.91 |
| EUR | 20 | bps | 60.94 |
| EUR | 30 | bps | 59.95 |
| EURO STOXX | | in% | 18.78 |
| Credit | AAA | bps | 11.08 |
| Credit | AA | bps | 12.00 |
| Credit | A | bps | 23.80 |
| Credit | BBB | bps | 52.60 |

From the above table we see that the volatility for equities was about 19% and the standard deviation for spread risk increases if the credit quality deteriorates. In a next step we want to have a look at the (simplified) correlation matrix:

| $\rho_{i,j}$ | EUR 5 | EUR 10 | EUR 20 | STOXX | AA | BBB |
|---|---|---|---|---|---|---|
| EUR 5 | 1.00 | 0.89 | 0.66 | 0.36 | -0.14 | -0.23 |
| EUR 10 | | 1.00 | 0.73 | 0.32 | -0.17 | -0.21 |
| EUR 20 | | | 1.00 | 0.16 | -0.09 | -0.09 |
| STOXX | | | | 1.00 | -0.45 | -0.50 |
| AA | | | | | 1.00 | 0.61 |
| BBB | | | | | | 1.00 |

**No picture yet**

**Fig. 6.2** Correlation Matrix

Looking at figure 6.2 we see clearly how the different risk factors are situated in the matrix. One sees the four times 13 risk factors relating to interest rates as first block, which is highly correlated between themselves and slightly less between different currencies. Afterwards one sees the correlation with the credit block, followed by the equity–like investments etc.

From the above table it becomes obvious that the credit spreads have a quite high negative correlation with stock market prices. This means that credit spreads increase normally at the same time when equity markets fall. One can observe that increasing stock market prices normally imply increasing interest rates. These two remarks are for example valid for the credit market crisis in 2008. Here we observed decreasing stock market indexes, reduced interest rate levels and increased credit spreads.

In order to make a concrete example based on the above data we need to base the calculation on a balance sheet and we assume:

| **Item** | **EUR** | **Term** | **Rating** |
|---|---|---|---|
| MR | -10000 | 10 | Government |
| Bond 1 | 5000 | 5 | BBB |
| Bond 2 | 4000 | 20 | AA |
| STOXX | 2000 | | |
| Capital | 1000 | | |

So as a first step we need to calculate the sensitivities regarding our risk factor vector EUR 5,EUR 10, EUR 20, STOXX, AA, BBB. We assume for the sake of simplicity each of the bonds and the mathematical reserve (MR) is zero coupon bonds with the corresponding term. In this case the duration equals the term, as one can easily verify. Since the volatility for interest rates and credit spread movement is stated in bps, we also need to calculate the sensitivity of the corresponding values per bp.

Since the MR is considered as a $\mathcal{Z}_{(10)}$ there is only a sensitivity with respect to the EUR 10 year risk factor and an upward movement of 1% reduces the reserve by 10%, so from a capital point of view we have an entry of $+1000$. For 1 bp we hence have $+10$ and the sensitivity factor for this liability reads as $\delta_{x_1} = (0, 10, 0, 0, 0, 0)$. In the same sense we can calculate Bond 1 ($\delta_{x_2}$) and Bond 2 ($\delta_{x_3}$), remarking that both of them are sensitive also with respect to credit spreads we get

$$\begin{aligned} \delta_{x_2} &= (-2.5, 0, 0, 0, 0, -2.5), \\ \delta_{x_3} &= (0, 0, -8, 0, -8, 0). \end{aligned}$$

For the share we calculate the sensitivity for an increase of 1% and hence we get:

$$\delta_{x_4} = (0, 0, 0, 20, 0, 0),$$

and therefore we get for the total sensitivity:

$$\begin{aligned} \delta_{Tot} &= \sum_{k=1}^{4} \delta_{x_k} \\ &= (-2.5, +10, -8, +20, -8, -2.5). \end{aligned}$$

In a next step we need to calculate:

$$
\begin{aligned}
s \times \delta_{Tot} &= (83.53 \times -2.5, \dots, 52.61 \times -2.5) \\
&= (-208.83, 635.42, -487.52, 375.70, -96.03, -131.51).
\end{aligned}
$$

Now we can calculate the standard deviation of the capital, considered as a random variable by:

$$
\begin{aligned}
\sigma &= \sqrt{(s \times \delta) \rho (s \times \delta)^T} \\
&= 546.6\text{M EUR}.
\end{aligned}
$$

As a consequence of this the VaR for the 99.5% corresponds to $VaR_{99.5\%} = 2.57 \times 546.6 = 1404.7$ M EUR. If one further decomposes the VaR, one could look at pure interest rate VaR. In this case one would look at the corresponding $\delta$:

$$
s \times \delta_{Interest} = (-208.83, 635.42, -487.52, 0, 0, 0),
$$

and we would get in the same way $VaR_{99.5\%}^{\text{Interest}} = 2.57 \times 487.9 = 1253.9$. This is the way how one determines which parts of the balance sheet contribute most to the risk. In the concrete example we get (in M EUR):

| **Item** | **Std Deviation** | **99.5% VaR** |
|---|---|---|
| Bonds | 487.9 | 1253.9 |
| Equities | 375.7 | 965.5 |
| Credit | 204.8 | 526.3 |
| Simple Sum | 1068.4 | 2745.7 |
| Diversification | -521.8 | -1341.0 |
| **Total** | 546.6 | **1404.7** |

Finally, we want to have a look at the accuracy of the linear approximation, which we have used. For shares there is nothing to do, since the change in value of the asset is linear. Therefore we want to look at the accuracy of the approximation for bonds. For simplicity we assume that the yield curve is flat at $i = 4\%$. In this context we have:

$$
\pi_t(\mathcal{Z}_{(n)})[i + \Delta_i] - \pi_t(\mathcal{Z}_{(n)})[i] = (1 + i + \Delta_i)^{-n} - (1 + i)^{-n}.
$$

We remark that the volatility of bonds is about 60 bps, therefore looking at the 99.5% (which is about $2.57 \times \sigma$) implies, looking at the precision of the approximation, at a shift of c 150 bps.

| **Term 20 yrs** | True Change | Delta Approx. | Error |
|---|---|---|---|
| -300 | 0.3631 | 0.2738 | -24.5% |
| -200 | 0.2165 | 0.1825 | -15.7% |
| -150 | 0.1538 | 0.1369 | -11.0% |
| -100 | 0.0972 | 0.0912 | -6.1% |
| -50 | 0.0461 | 0.0456 | -1.1% |
| 50 | -0.0417 | -0.0456 | 9.3% |
| 100 | -0.0794 | -0.0912 | 14.8% |
| 150 | -0.1136 | -0.1369 | 20.4% |
| 200 | -0.1445 | -0.1825 | 26.2% |
| 300 | -0.1979 | -0.2738 | 38.3% |

Consequently, we see that this approximation has some non negligible errors, which can be mitigated by adjusting the duration accordingly. Another source of such non-linearities are options, where a standard model is inadequate. Hence we want to have a look at a possible solution. In order to do that we need to go back to first principles, which define the model *before* approximations. In our case we assume that the risk factors are following a multi-normal distribution for $Z = \Delta R$. This means that for some of our assets $(\mathcal{A}_i)_{i \in S_A}$ or liabilities $(\mathcal{L}_i)_{i \in S_L}$ the linear approximation is inadequate. The method which we want to show here works in general and can either be applied to one or more of the underlying assets and liabilities. It works for example for plain-vanilla stock options, which can be valued using the :

The price for a *put*-option with payout $C(T, P) = \max(K - S; 0)$ at time $t$ and strike price $K$ and equity price $S$ is given by:

$$\begin{aligned}
P &= K \times e^{-r \times T} \times \Phi(-d_2) - S_0 \times \Phi(-d_1), \\
d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2) \times T}{\sigma \times \sqrt{T}}, \\
d_2 &= d_1 - \sigma \times \sqrt{T}, \\
\Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(\frac{-\zeta^2}{2}) d\zeta.
\end{aligned}$$

The risk factors which enter into the calculation of the price $\pi(\mathcal{A}) := P$ are the following:

- Share price $S_t$,
- Volatility of the share price $\sigma$, and
- The interest rate for the corresponding term $r$.

It needs to be stressed that the price of such an option is clearly not linear in the risk factors.

In order to assess the corresponding risk, one can for example use a partial or full simulation approach. In the first case the whole distribution is simulated with a sufficiently big sample and the effective change in capital is evaluated and recorded in order to determine the value of the chosen risk measure such as the VaR or TVaR. One can also use a partial simulation approach remarking that only nonlinear instruments need to be simulated. Here the question is how to "marry" the simulated and the analytical parts. One approach is to use control variables.

In order to describe this approach let's assume that we have one asset $\mathcal{A}$, which is not linearly dependent on the risk factor and assume for sake of simplicity that we have the following:

$$\begin{aligned}\tilde{\pi}_t(x) - \pi_0(x) &= \delta^T \times \Delta(R), \text{ and}\\ \pi_t(x) - \pi_0(x) &= f(\Delta(R)).\end{aligned}$$

In the above we denote with $\tilde{\pi}_t(x)$ and $\pi_t(x)$, the approximated and the "true" change in value, respectively. The function $f$ denotes the "true" change in value for a given $\Delta(R)$. So in order to do a partial simulation approach one needs to do the following:

1. Simulate $n$ (say 50000) times the random variable $R$, resulting in a series of $(r_k)_{k=1,2,\dots n}$.
2. Calculate the analytic value of the risk measure $C_a^l$ for the linear approximation.
3. Calculate the simulated value of the risk measure $C_r^l$ for the linear approximation, using the series $(r_k)_{k=1,2,\dots n}$.
4. Calculate the simulated value of the risk measure $C_r^f$ for the "true" value, using the series $(r_k)_{k=1,2,\dots n}$.

As a consequence of the weak law of big numbers we have $C_a^l = \lim_{n\to\infty} C_r^l(n)$. Hence using the difference $C_a^l - C_r^l$ as a correction to $C_r^f$ normally improves the quality of the approximation.

The table below show an example for the accuracy of the linear approximation in case of a plain vanilla put option. At time $t = 0$ we assume a stock price of 1000 and we consider a strike for the put option at 900. For this example a sample size of 50000 has been chosen and analysed for the first 500, the first 1000 samples etc.

| Sample size | Linear Value | Linear Error | BS - Price Value | BS - Price Error |
|---|---|---|---|---|
| 500 | 97680 | +0.7% | 159238 | +1.5% |
| 1000 | 97911 | +0.9% | 160029 | +2.0% |
| 2000 | 97908 | +0.9% | 160017 | +2.0% |
| 5000 | 96412 | -0.6% | 154571 | -1.4% |
| 10000 | 97911 | +0.9% | 160029 | +2.0% |
| 20000 | 96672 | -0.4% | 157896 | +0.7% |
| 50000 | 97038 | ref | 156831 | ref |

What can be seen from the above example is that the linear model converges much faster and in this case the value of the put-option to the company holding it is *underestimated*.

## 6.4 Interpreting the Results

This section provides a reporting template which can be used for financial risk management. This template risk report is subdivided into the following parts:

**Summary:** The aim of this section is to provide a concise summary. In order to get a high level view on the duration gap between assets and liabilities, the corresponding durations are calculated. Furthermore we see the impact of an increase in interest rates of 10 bps and an increase of 1% in equity prices, separately for assets and liabilities only and combined. After these deterministic measures we see some important key measures in terms of VaR, for both a one in ten year (1:10) and a one in 250 year (1:250) event. Here we look at combinations of risk factors. Namely we look separately at equities, bonds, surrenders and the total. This total VaR needs to be compared with the available capital (Market Value). Finally, also the Tail VaR or Expected Shortfall (ES) is shown. The figure underneath shows the required capital for different return periods (separately for assets and for the total). The two red balls represent the VaR in a 1:10 and a 1:250 year event and these numbers reconcile to the table.

**Decomposition of VaR:** In order to better understand where capital is consumed the total VaR is further decomposed into its components. It is possible to see which parts of the assets and liabilities consume the majority of the capital. In the concrete example we can see that most of the total required capital of 3216 M EUR is consumed by credit risk (2042 M EUR). Furthermore it becomes obvious that the pure ALM risk (in terms of interest rates) is quite small with 764 M EUR. Finally we see that equity risk and hedge funds account still with 764 M EUR and 464 M EUR respectively. At the bottom of the page we see that the surrender risk amounts to 736 M EUR.

**Individual Capital Assessment (ICA):** In order to be able to compare the model with the regulatory standard ICA model the corresponding results have been included in this section. It needs to be stressed that the ICA model covers more risk factors, but is not as granular for the market risk factors as the own model.

**Scenarios:** In this section some scenarios are shown and in particular, how the company balance sheet would look after such an event. Section 6.4.2 shows the main characteristics of the scenarios used. The balance sheet items for each of the scenarios follow a typical IFRS balance sheet. The figures underneath the table show the change in shareholders equity and the decomposition of the balance sheet post stress respectively.

**Stress Tests:** The section stress tests is thought to represent some additional scenarios as described above. The only difference is the fact that here the scenarios are shown in a summarised version and are based on group requirements. The scenarios currently used have been defined by FSA[1] and are quite self-explanatory.

**Limits:** This section aims to show the limits currently in place to limit the ALM risk. The table shows the limits currently in place: The target which is limited, the threshold, the current level and the headroom. The program has been built up in such a way that every number which is checked against a limit is either printed in green (e.g. within limit) or red (e.g. limit breach).

### *6.4.1 Notation*

In this section the main elements in respect of notation are documented.

[1] Financial Services Authority in the UK.

| | |
|---|---|
| VaR | Value at risk, eg the Loss which occurs according to a certain probability. In the analysis a 99.6% VaR is used. This means that the loss represents a loss which in the long run is expected to occur every 250 years. It needs to be noted at this point of time, that analytical models tend to underestimate such losses since the risk factors have been modelled as normally distributed. |
| 1:10 | This symbol also relates to a VaR, in this case corresponding to the 90th quantile, e.g. once every 10 years. |
| Duration | The modified duration which represents the risk intrinsic to a bond portfolio |
| Sensi Bonds (+10 bps) | The change in value of a bond portfolio if the yield curve is shifted by 10 bps (= 0.1%). |
| ES 99% | The expected shortfall in a 1 in 100 year event is defined of the average loss looking at all events occurring less than once in 100 years. This measure is more sensitive in the tails than the VaR. |
| Intangibles | The intangibles in the balance sheet (eg goodwill etc.) In case of an impairment of participation the model reduces the intangibles in a first step. |
| MR | mathematical reserves for traditional business. They are moving in line with the interest rates. |
| UDS | Undistributed surplus. |
| Tax | Taxes and deferred tax assets and liabilities are not modelled. |
| SHE | Market Value Shareholder funds. This corresponds in principle to the corresponding MCEV. |
| $\Delta$ SHE | Change in SHE in case of a certain scenario. |

- The figure *Distribution of Losses* shows the probability density function of the losses. The two red circles represent the VaR 1:10 and in respect of the 99.5% quantile.
- The figure *Cash Flow Profile* shows the inflows (red: bond payments and yellow: premiums) v.s. the outflows (blue: expected claims).
- The figure *Diversification* shows the diversification effect in relation to the main asset risks.
- The figure *Decomposition of required Capital* and *Credit Risk by Rating* show which risk and which credit risk absorbs most of the required capital.

### 6.4.2 Scenarios

In the following section the different used scenarios are defined in some greater detail:

| | Credit | Yen | Depr. | FSA | Hard Land. | Depr. ii |
|---|---|---|---|---|---|---|
| i-rate 2 yrs (bps) | 0 | -399 | -399 | 50 | 90 | -220 |
| i-rate 7 yrs (bps) | 0 | -316 | -316 | 50 | 90 | -220 |
| i-rate 10 yrs (bps) | 0 | -323 | -323 | 50 | 90 | -220 |
| i-rate 25 yrs (bps) | 0 | -303 | -303 | 50 | 90 | -220 |
| Shares (%) | -18 | -18 | -65 | -34 | 9 | -32 |
| Properties (%) | -5 | -5 | -55 | -19 | -28 | -36 |
| AA Credit Spread (bps) | 103 | 0 | 103 | 50 | 40 | 110 |

### *6.4.3 What can and what cannot be done with this*

As indicated above, a model is not reality and hence it is of utmost importance to recognise the limitations of such a model. In this section we try to show some of the limitations of the model currently used. One of the possible risks of this model is that it is overly simplistic.

From a high level point of view the main shortcomings of the model are:

**The model is linear:** The different risk factors enter linearly into the calculation of the loss. Therefore for options, the corresponding delta equivalent is used. In a next step such effects should be captured better.

**The model uses a standard multi-normal distribution.**

**Management actions:** Management actions are not taken into account.

**Insurance and operational risks:** The model purely focuses on ALM risk.

**Dynamic Lapses:** Dynamic lapses are also an area where the model used needs refinement.

## 6.5 Reporting Example

### *6.5.1 Summary*

**Assessment and key Figures**

- Globally the company has currently not enough risk capital, from a purely economic perspective, to run the corresponding ALM risks, since the margins have become tighter due to the losses in the equity and corporate bond portfolio and widening of the credit spread.
- The statutory reserve set up for GMDB at the year-end (31.12.2008) was 130m EUR whereas the more economic vision used in the MCEV calculation produced a value of 162m EUR. During the first quarter the statutory reserves increased to c230m EUR, affecting the IGD Solvency adversely by c82m EUR.

**Data Quality**

- Private Equities are currently within the category hedge funds
- The current analysis is still in draft form and is based on data as of 31.12.2008.

| **Item** | **Assets** | **Liabilities** | **Total** |
|---|---:|---:|---:|
| Duration | 5.81 | 5.12 | |
| Sensi Bonds (+10bps) | -177.87 | 144.98 | -32.88 |
| Sensi Equities (+1%) | 14.90 | -0.21 | 14.69 |
| 1:10 Bonds | 1505.30 | 1174.40 | 352.94 |
| 1:10 Equities | 370.32 | 15.68 | 380.18 |
| 1:10 Total | 1809.60 | 983.14 | 1600.50 |
| VaR Bonds | 3025.60 | 2360.40 | 709.39 |
| VaR Sx | – | 736.69 | 736.69 |
| VaR Equities | 744.32 | 31.51 | 764.13 |
| VaR Total | 3637.10 | 1976.00 | 3216.90 |
| ES 99% | | | 3098.60 |
| **Market Value** | – | 45343.00 | **2780.00** |

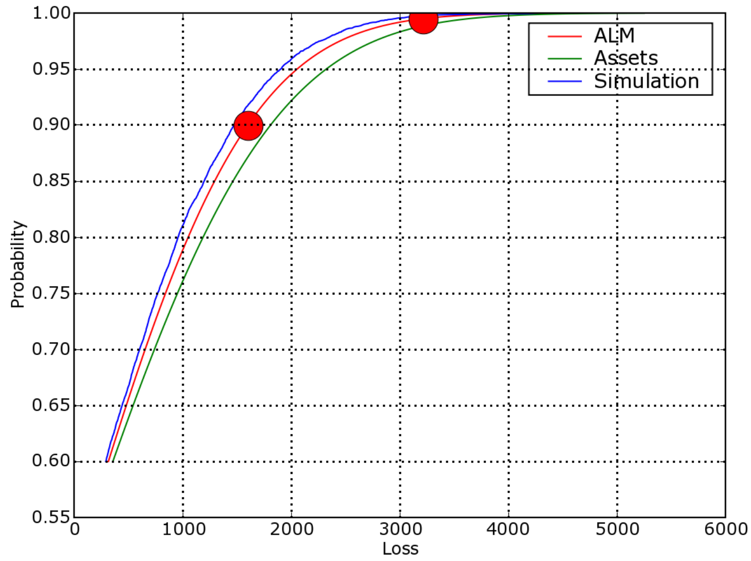



**Fig. 6.3** Distribution of Losses

## 6.5.2 Decomposition of VaR

| Assessment and key Figures | Data Quality |
|---|---|
| • The total required risk capital amounts to 3.2 bn EUR (pre Tax) and to c 2.1 bn EUR (post Tax), compared with a available risk capital of c 2.7 bn EUR (after Tax). The 2.1 bn EUR compare to 2.5 bn EUR for the YE2008 ICA calculation. The biggest difference is the fact the ALM Capital does not take into account risk other than market and surrender risks. The biggest additional contribution is the expense risk capital of c 0.6 bn EUR. Adding this to the ALM Capital the two numbers get closer with a difference below 0.1 bn EUR. Overall, both metrics result in similar numbers.<br>• The ALM mismatch consumes about $\frac{1}{2}$ and the equities et al exposure ca. $\frac{1}{2}$ of the total risk capital. This indicates the company has a rather high risk in equities, private equities, properties and hedge funds. In particular, the capital needed for alternative investments is almost 20% of the total available risk capital.<br>• Most of the ALM mismatch stems from the long duration liabilities which are not matched with corresponding assets. | • Replicating the portfolio for one major product line under review<br>• Available risk capital not yet calculated and the current figure is based on an estimation.<br>• GMDB exposure reflected via $\delta$–equivalent for equities and volatility via $\theta$. Interest rate sensitivity not yet reflected. |

| Item | Assets | Liabilities | Total |
|---|---|---|---|
| **Market Value** | – | 45343.00 | **2780.00** |
| Bonds EUR <3 | 48.26 | 52.35 | 11.78 |
| Bonds EUR 3-7 | 616.54 | 454.94 | 162.66 |
| Bonds EUR 8-12 | 1446.40 | 810.34 | 636.14 |
| Bonds EUR 13-24 | 1007.60 | 925.96 | 109.24 |
| Bonds EUR >25 | 26.64 | 263.45 | 236.81 |
| *Bonds EUR Total* | 3025.60 | 2360.40 | 709.39 |
| *Div. Ben.* | -119.88 | -146.66 | -447.24 |
| Bonds GBP <3 | – | – | – |
| Bonds GBP 3-7 | – | – | – |
| Bonds GBP 8-12 | – | – | – |
| Bonds GBP 13-24 | – | – | – |
| Bonds GBP >25 | – | – | – |
| *Bonds GBP Total* | – | – | – |
| *Div. Ben.* | – | – | – |
| Bonds USD <3 | – | – | – |
| Bonds USD 3-7 | – | – | – |
| Bonds USD 8-12 | – | – | – |
| Bonds USD 13-24 | – | – | – |
| Bonds USD >25 | – | – | – |
| *Bonds USD Total* | – | – | – |
| *Div. Ben.* | – | – | – |
| Bonds CHF <3 | – | – | – |
| Bonds CHF 3-7 | – | – | – |
| Bonds CHF 8-12 | – | – | – |
| Bonds CHF 13-24 | – | – | – |
| Bonds CHF >25 | – | – | – |
| *Bonds CHF Total* | – | – | – |
| *Div Ben.* | – | – | – |
| **All Bonds** | **3025.60** | **2360.40** | **709.39** |
| *Div. Ben.* | – | – | – |
| Credit Risk | 2042.20 | – | 2042.20 |
| Shares MSCIEMU | 744.32 | – | 744.32 |
| Shares MSCICHF | – | – | – |
| Shares MSCIUK | – | – | – |
| Shares MSCIUS | – | – | – |
| **All Shares** | **744.32** | **31.51** | **764.13** |
| *Div. Ben.* | – | – | *-11.70* |
| FX GBP | – | – | – |
| FX USD | – | – | – |
| FX GBP | – | – | – |
| **FX Total** | **–** | **–** | **–** |
| *Div. Ben.* | – | – | – |
| Real Estate | 131.20 | 9.45 | 121.75 |
| Alternatives | 464.46 | – | 464.46 |
| Participations | 191.90 | – | 191.90 |
| **Total** | **3637.10** | **1976.00** | **3216.90** |
| *Div. Ben.* | *-920.34* | *-425.30* | *965.29* |
| Surrenders | – | 736.69 | 736.69 |

### 6.5.3 Figures

| **Assessment and key Figures** | **Data Quality** |
|---|---|
| • The duration of the bonds with 5.6 years is considerably shorter than the ones of the liabilities with 10.6 years. In part this is due to the special characteristics of a particular insurance portfolio and corresponding analysis are under way.<br>• From a liquidity point of view the company has considerable amounts of bonds maturing within 1 year and 2 years leading to an excess liquidity of ca 2bn EUR and 1bn EUR respectively.<br>• The table relating the shift in asset value for a shift in credit spreads shows clearly the high credit quality of the underlying assets corresponding to a average rating of slightly above AA | • Replicating portfolio for particular product line under review<br>• Derivatives not yet reflected in analysis |

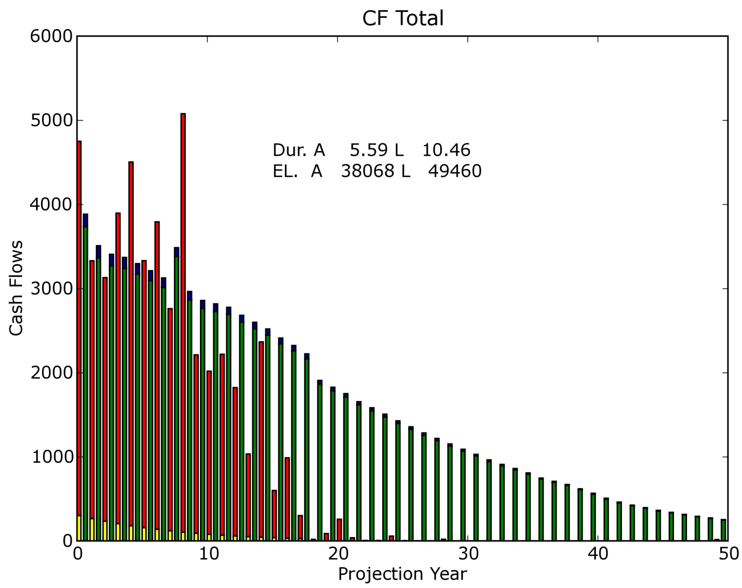



**Fig. 6.4** Cash Flow Profile

| **Credit Quality** | **+10 bps Spread** | **Percentile** | **$\Delta$ Profit** |
|---|---|---|---|
| EURO AAA | -89.13 | 1% | 2905.30 |
| EURO AA | -23.76 | 5% | 2054.20 |
| EURO A | -42.69 | 10% | 1600.50 |
| EURO BBB | -24.16 | 33% | 538.04 |
| USD AAA | – | 66% | -537.70 |
| USD AA | – | 90% | -1600.50 |
| USD A | – | 95% | -2054.20 |
| USD BBB | – | 99% | -2905.30 |
| **Total** | **-179.76** | 99.5% | -3216.90 |

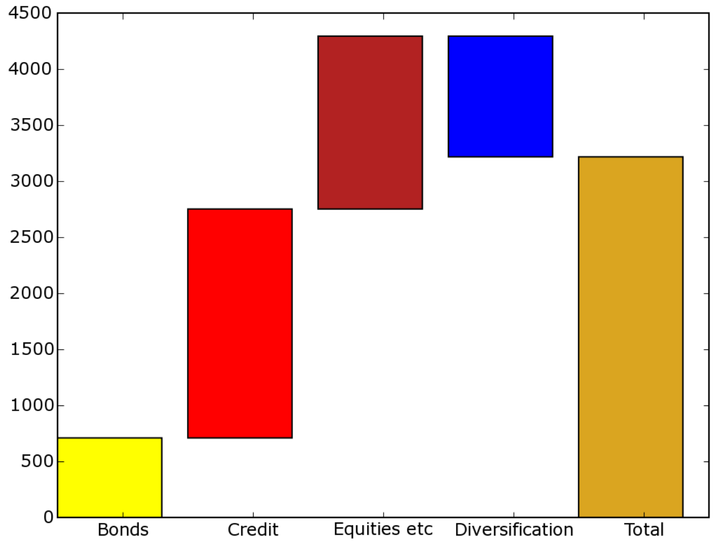



**Fig. 6.5** Diversification

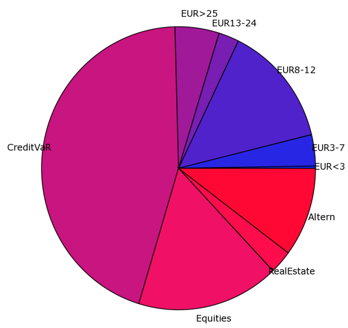



**Fig. 6.6** Required Capital

## *6.5.4 Scenarios*

| **Assessment and key Figures** | **Data Quality** |
|---|---|
| • The main three scenarios consist of a widening of credit spreads by a further 50%, a falling of the interest rates to YEN levels and a global severe depression. | • The current analysis is work in progress and is based on data as of 30.12.08 |

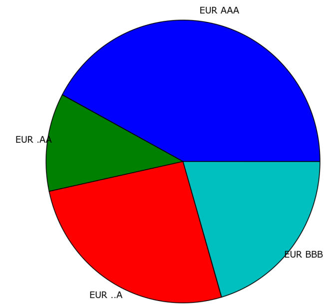



**Fig. 6.7** Credit Risk by Rating

| Item | Start | Y to D | Credit | YEN | Depression |
|---|---|---|---|---|---|
| Cash | 6221.10 | 6221.10 | 6221.10 | 6187.00 | 6187.00 |
| Bonds | 35734.00 | 35734.00 | 33730.00 | 41567.00 | 39564.00 |
| Shares | 4468.30 | 4207.50 | 4207.50 | 4207.50 | 3499.50 |
| Properties | 1137.10 | 1081.70 | 1081.70 | 1081.70 | 528.15 |
| Hedge Funds | 595.70 | 521.24 | 521.24 | 521.24 | 59.57 |
| Private Equity | 64.00 | 56.00 | 12.80 | 56.00 | 6.40 |
| Loans | 1452.10 | 1452.10 | 1452.10 | 1452.10 | 1452.10 |
| Unit Linked Assets | 14330.00 | 14330.00 | 14330.00 | 14330.00 | 14330.00 |
| Other Assets | 3612.00 | 3612.00 | 3612.00 | 3612.00 | 3612.00 |
| Intangibles | 204.90 | 152.75 | 167.65 | 152.75 | -63.30 |
| MR | 47066.00 | 47066.00 | 47326.00 | 51773.00 | 51773.00 |
| Unit Linked Liabilities | 14330.00 | 14330.00 | 14330.00 | 14330.00 | 14330.00 |
| UDS | – | – | – | – | 37.60 |
| Debt | – | – | – | – | – |
| Deferred Tax | 127.70 | 127.70 | 127.70 | 127.70 | 127.70 |
| Other Lia | 3801.90 | 3797.90 | 3797.90 | 3797.90 | 3758.00 |
| SHE | 2493.00 | 2046.20 | -245.49 | 3139.00 | -851.01 |
| $\Delta$ SHE | | **-446.82** | **-2738.50** | **645.97** | **-3344.00** |

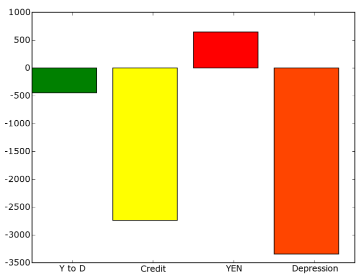



**Fig. 6.8** $\Delta$ SHE for the scenarios

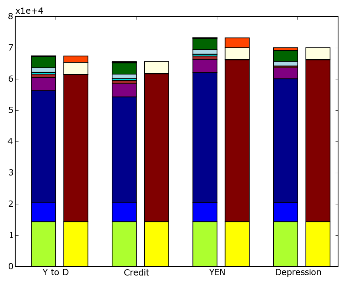



**Fig. 6.9** BS for Scenarios

**Year-to-date:** The interest rates decreased by c 200 bps at the short and 30 bps at the long end. At the same time credit spreads widened between 90 bps (AAA) and 380 bps (BBB). Stock markets reduced by 35+%.
**Credit Scenario:** An additional spread widening of 50%.
**YEN Scenario:** Based on the current scenario, interest rates have been lowered to levels where the Yen was at the its deepest level
**Depression:** YEN interest rates, a 40% credit spread widening and a cumulative reduction of 65% for shares, PE, HF and properties.

### *6.5.5 ICA Capital*

**Assessment and key Figures**

- The falls in available economic capital and the increase in capital requirements are largely driven by:
  Falls in equity markets and increases in credit risks have lead to minimum investment guarantees “biting”, with a direct burn-through impact on shareholder assets.
  Saving products experienced significant erosion in value of future profits (VIF), with asset returns over the year close to minimum guarantees.
  Falls in the equity market meant unit linked contracts experienced an increase in Guaranteed Minimum Death Benefit (GMDB) risk.
- Changes in YE2008 stress methodology, in particular, the “softening” of equity and credit spread tests were key to keeping the funds from going into deficit on an economic basis.
- The company has completed an equity de-risking initiative, leading to a further fall in capital requirement for equity risk.
- The company believes a significant part of the credit spread widening is linked to liquidity premium and for some products, creates an artificial and unnecessarily high capital requirement.
- A separate exercise will need to be performed to quantify the reputational risk associated to structured products that have been sold with the underlying guarantees provided by third parties. The default risk is borne by the client but a reputational risk would remain with the company (total reserves 4,844m). This is not part of the YE08 SSTEC requirements, but will be investigated given the potentially material impact.

| in M EUR | YE 2007 | YE 2008 | 1Q2009 | 4Q2009 |
|---|---|---|---|---|
| Available Economic Capital | 3657 | 2780 | | |
| Reg. Capital Required | 2083 | 2508 | | |
| Cover | 176% | 111% | | |
| Diversification Benefit | 39% | 41% | | |
| MV Assets | 59390 | 56390 | | |
| MV Liabilities | 54674 | 52558 | | |

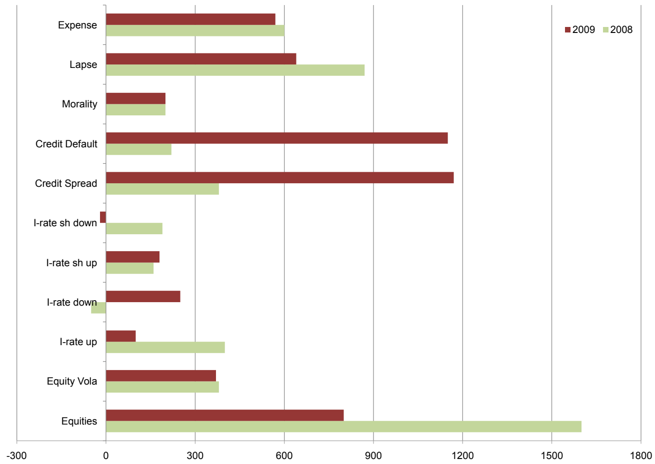



**Fig. 6.10** Required ICA Capital by Risk

## 6.5.6 Stress Tests

| Nr | Name | Equity | △ Assets | △ Lia | △ Equity |
|---|---|---|---|---|---|
| | B/S | 2493.00 | | | |
| | Y to D | 2046.20 | -450.81 | -3.98 | -446.82 |
| 1 | Equities -10% | 1763.00 | -603.96 | 126.01 | -729.97 |
| 2 | Equities -20% | 1479.90 | -757.11 | 256.01 | -1013.10 |
| 3 | Equities -40% | 1043.60 | -1063.40 | 386.01 | -1449.40 |
| 4 | Equity Vola + 10% | 2044.10 | -450.81 | -1.88 | -448.92 |
| 5 | Property -7.5% | 1908.00 | -529.70 | 55.33 | -585.03 |
| 6 | Property -15% | 1769.80 | -608.58 | 114.65 | -446.82 |
| 7 | Property -30% | 1558.40 | -766.35 | 168.28 | -934.64 |
| 8 | I rates -50 bps | 2210.60 | 438.52 | 720.94 | -282.42 |
| 9 | I rates -100 bps | 2375.00 | 1327.80 | 1445.90 | -118.01 |
| 10 | I rates -200 bps | 2703.80 | 3106.50 | 2895.70 | 210.79 |
| 11 | I rates Twist: long down | 1988.70 | 834.64 | 1339.00 | -504.31 |
| 12 | I rates Twist: long up | 2102.20 | -1750.40 | -1359.60 | -390.84 |
| 13 | Cred spread +50 bps | 1147.40 | -1349.60 | -3.98 | -1345.60 |
| 14 | Cred spread +100 bps | 248.60 | -2248.40 | -3.98 | -2244.40 |
| 15 | Cred spread +200 bps | -1549.00 | -4046.00 | -3.98 | -4042.00 |
| 16 | FSA | 530.28 | -2703.00 | -740.28 | -1962.70 |
| 17 | FSA Hard Landing | 1287.10 | -2533.00 | -1327.00 | -1205.90 |
| 18 | FSA Depression 2010 | 206.99 | 875.04 | 3161.10 | -2286.00 |

## 6.5.7 Limits

In this section the various limits are checked:

| Limit | Threshold | Actual | Headroom |
|---|---|---|---|
| VaR Equity Asset (20%) ◇ | 409.24 | 744.32 | **-335.08** |
| Total VaR (80%) ◇ | 1636.90 | 3216.90 | **-1580.00** |
| Base Point Sensitivity | 87.98 | -32.88 | 55.10 |
| Alternatives VaR (10%) ◇ | 204.62 | 464.46 | **-259.84** |
| Credit VaR (20%) ◇ | 409.24 | 2042.20 | **-1633.00** |
| Credit Scenario SHE $> 0$ | 0.00 | 3139.00 | 3139.00 |
| Yen Scenario SHE $> 0$ | 0.00 | -245.49 | 245.49 |
| Combined 2 Scenario SHE $> 0$ | 0.00 | 1287.10 | 1287.10 |
| Depression Scenario SHE $> 0$ | 0.00 | -851.01 | 851.01 |
| Properties VaR (10%) | 204.62 | 131.20 | 73.42 |

## 6.6 Summary Reporting Example

Figure 6.11 provides an example of a summary on a page for the financial risk a company is facing. The aim is to be concise and also action oriented. Hence the table envisages the following entries:

**Name** The name of the risk is indicated together with a measure for its size, such as the amount of assets affected, a risk measure etc.

**Risk Category** The risk category aims to indicate which type of risk is described, such as credit risk, liquidity risk, . . .

**Risk** The risk is described in a concise manner in order that a knowledgeable third party can understand, what the risk is.

**Actions** This one is the most important column, since the mitigation actions performed and planned are described. This helps to see the development with respect to the corresponding risk

**Remarks** Here additional information needed to better understand the issue is documented.

The principle for the writing of such reportings must be *relevant, concise and action oriented*.

| Name | Risk Category | Risk | Actions | Remarks |
|---|---|---|---|---|
| **Credit Defaults** | Credit Risk | The company had and has substantial credit exposure, in particular towards the banking sector and might suffer credit defaults | • Reduction in holdings in Italian and Greek government bonds<br>• Reduction (€ 100 M) of counterparty exposure to X<br>• Regular exposure control and reporting | The counterparty exposure vis-à-vis our strategic partners is a particular concern due to its strategic rationale. |
| **Hedge Funds** | Market Risk | An unclear asset investment rationale and risk controlling was found at the beginning of this year. As a consequence the company faces a variety of risks – exposure € c500M | • Decision to exit hedge funds<br>• So far € 70M have been sold | Due to the introduction of gates and side pockets within the hedge funds the reduction of hedge fund investments will take some time |
| **Equity and GMDB** | Market Risk | As a consequence of the credit crises in 2008, the market value of equities dropped considerably together with a spike in equity-volatility. As a consequence the company was and still is vunerable against equity market movements. | • Reduction of € >3bn so far<br>• Hedging of Equity exposure with derivatives<br>• Mitigation actions with respect to GMDB exposure examined<br>• Continuous monitoring | • Equity exposure (€ c900M) protected with put option strategy maturing Mar 2010<br>• GMDB reserve € 160M exposed towards volatility increase |
| **Interest Rate** | Market Risk | Since many of the bond portfolios covering insurance liabilities are not perfectly matched, the company is vunerable against interest rate movements. Most liability portfolios are of a shorter duration than the assets. | • Cash flow analysis performed for major blocks of business. As a consequence the corresponding risk for life portfolios is adequate<br>• GI portfolios were short duration in liabilities and a reduction in asset duration is under way | Regular ALM risk controlling to be built up |
| **Lapse Risk** | Insurance Risk | As a consequence of the financial crisis, which now also effects the "real" economy, a lot of our policyholders lapsed their policies, with a potentially adverse impact on our franchise value | • Actions are under way to safeguard the franchise value of the company<br>• The distribution channels have been asked to report back on the steps undertaken so far<br>• Regular reporting of lapse experience | A side effect of the financial crisis is the fact that some of our clients switch their unit linked policies into traditional ones, which are less profitable |

**Fig. 6.11** Financial Risk Reporting

# Chapter 7
# Insurance Risks

## 7.1 Method for Allocation of Capital

The aim of this chapter is to introduce a very concrete risk capital model for life insurance risks and should help to understand the approach which needs to be taken and the respective necessary steps. It is important to understand that there are many other risks that need to be analysed and modelled, such as all GI risks and for life insurance also disability and in particular the lapse risk.

### 7.1.1 Steps required

The following steps are needed in order to calculate the required risk capital for life risk

- Definition of the risk factors,
- Definition of a probability density functions per risk factor,
- Definition of a valuation methodology,
- Definition of the joint distribution of all risk factors – diversification,
- Definition of risk measures,
- Definition of the concept of stress scenarios.

### 7.1.2 Probability Density Functions per Risk Factor

Each of the risk factors needs to be described in terms of stochastic processes and corresponding probability density function.

### 7.1.3 Diversification

After the definition of the individual probability density function, it is necessary to define the joint distribution of all individual random variables. First, we will use a very simple model and assume that the present values of the corresponding losses are linked by a covariance matrix. Whilst not being the most elaborate method, this approach is pragmatic enough to capture the most relevant interactions.

## 7.2 Stochastic Models used

### 7.2.1 Mortality

Historical statistics show that mortality has generally been improving for many decades. However, in contrast to the longevity, the major risk for mortality, from an insurer's point of view, is a pandemic-like event which could cause an exceptional mortality shock.

A pandemic is an epidemic that spreads worldwide or at least across a large region. A worldwide pandemic is recognised to be "virulent and contagious with high rates of illness and deaths, as well as significant social and economic disruption."

According to an article written by the *British Columbia Pandemic Influenza Advisory Committee (BCPIAC)*, several medical experts think that the threat of a severe "influenza" pandemic is to be feared.
Indeed, not only have past events shown that influenza pandemic strikes about three times a century (e.g. the Spanish Flu (1918 - 1919), the Asian Flu (1957 - 1958) and the Hong Kong Flu (1968 - 1969)), but all the factors contributing to the risk are in place.

In light of this, it is necessary to develop a model integrating a shock to evaluate the amount of capital to hold. As for the longevity model, we will take the future estimation of mortality rate ($q_{x,t}$) as an input.

### Model description

The following model for the mortality process assumes a mortality shock. The best estimate mortality is given by $q_{x,t}$. The shock modelled as a percentage of $q_{x,t}$ is simply given by a Bernoulli random variable ($\mathcal{B}(\gamma)$, with $\gamma$ the frequency of the shock) multiplied by a log-normal random variable (severity of the shock) :

$$Q^i_{x,t}(\omega) = q_{x,t} \times [1 + I_t(\omega) \times R_t(\omega)] + \epsilon^i_{x,t}(\omega).$$

With,

$$\begin{aligned}
(I_t)_{t\in\mathbb{N}} &\sim \mathcal{B}(\gamma) && \text{i.i.d.,}\\
(R_t) &\sim \text{Severity distribution, eg log-normal,}\\
R_t &= \exp(Y),\\
Y &\sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}),\\
\epsilon^i_{x,t} &\sim F && \text{i.i.d. with } \mathbb{E}(F) = 0,\\
(R, I, \epsilon) & \quad \text{Independent.}
\end{aligned}$$

We will first concentrate our efforts on determining the parameters of the $(R_t)_{t\in\mathbb{N}}$ and $(I_t)_{t\in\mathbb{N}}$ distribution, the aim of which is to simulate the non diversifying risk of mortality. The $(\epsilon)^i_{x,t}$ vector since it represents the diversifying risk.

### Determination of $\gamma$, $\tilde{\mu}$, $\tilde{\sigma}$

We use the probability of death from the Human Mortality Database (HMD)[1] to fit the parameters needed. We used data from different countries: Sweden, England and France, in order to obtain greater information on the stability of the parameters.

To illustrate the data we used, here are the evolution of the probability of death at age 40 for the 3 countries, see figures 7.1, 7.2 and 7.3.

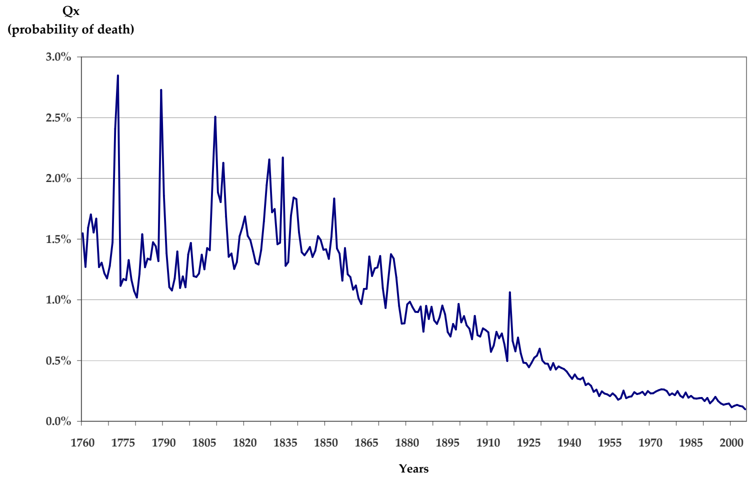



**Fig. 7.1** Evolution of death rate at age 40 for males from Sweden

### Estimation of shocks in past data

As a first step, we tried to identify the shocks in mortality rate and express it as a percentage of the 10 years moving average $q_x$. Concretely, in total, 6 series of historical death rate (one for males and one for females for each of the following countries: France, England and Sweden) were at our disposal in a period of time $[T_1, T_2]$ (see values in table 7.1):

$(\hat{q}_{x,t})_{t \in [T_1;T_2]; x \in [0;110]}$

Let us define, for $t \in [T_1 + 5; T_2 - 5]$ the following kernel estimator:

[1] www.mortality.org

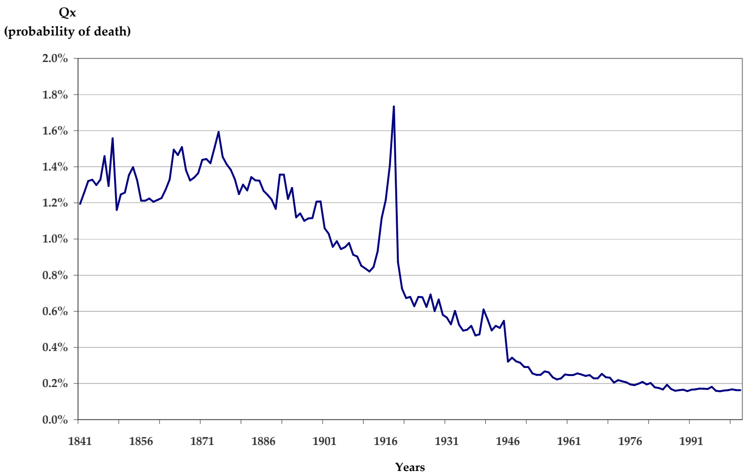



**Fig. 7.2** Evolution of death rate at age 40 for males from England

**Table 7.1** Value of $T_1$ and $T_2$ by country

| Country | $T_1$ | $T_2$ |
|---|---|---|
| France | 1899 | 2004 |
| England | 1841 | 2003 |
| Sweden | 1751 | 2005 |

$$\tilde{q}_{x,t} = \frac{\sum_{i=-5}^{5} \hat{q}_{x,t+i}}{11}$$

$$\text{Shock}_{x,t} = \frac{\hat{q}_{x,t}}{\tilde{q}_{x,t}} - 1$$

In a next step we study the series of $\text{Shock}_{x,t}$ in order to find in past data which percentage of the average $q_x$ should correspond to a shock in mortality rate which happens once in 250 years (99.6% percentile). Results per age band are shown in table 7.2.

These results show that if we look at the past data, the 1/250 catastrophic scenario should be based on a shock in mortality about 70% of the $q_x$ for all combined ages, and about 150% for the specific age band $[20; 40]$.

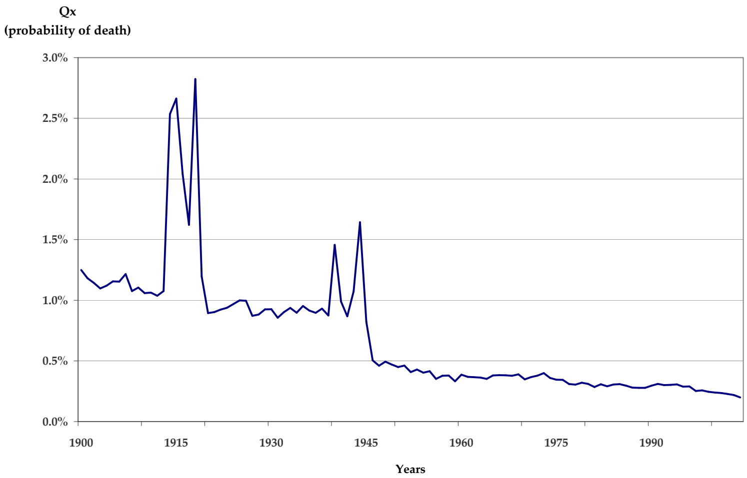



**Fig. 7.3** Evolution of death rate at age 40 for males from France

**Table 7.2** Historical mortality shocks (99.6% percentile of the $\mathrm{Shock}_{x,t}$ series by age band)

| Age band | France M | France F | England M | England F | Sweden M | Sweden F |
|---|---|---|---|---|---|---|
| 0-20 | 53% | 55% | 53% | 35% | 51% | 49% |
| 20-40 | 154% | 135% | 162% | 51% | 104% | 87% |
| 40-60 | 37% | 34% | 21% | 19% | 61% | 54% |
| 60-80 | 15% | 18% | 15% | 15% | 39% | 31% |
| Total | 68% | 63% | 62% | 27% | 60% | 53% |

## Determination of $\gamma$

We keep studying the series of $\mathrm{Shock}_{x,t}$. We want to obtain the $\gamma$ parameter. In table 7.3 we summarise frequencies of $\mathrm{Shock}_{x,t}$ which are greater than 20%.

**Table 7.3** Historical mortality shocks frequency series by age band

| Age band | France M | France F | England M | England F | Sweden M | Sweden F |
|---|---|---|---|---|---|---|
| 0-20 | 7.3% | 4.2% | 3.9% | 1.3% | 7.8% | 7.8% |
| 20-40 | 8.3% | 4.2% | 5.2% | 2% | 8.3% | 6.8% |
| 40-60 | 3.1% | 2.1% | 1.3% | 0.7% | 5.7% | 5.7% |
| 60-80 | 0% | 0% | 0% | 0% | 2.6% | 3.6% |
| Total | 7.3% | 4.2% | 3.3% | 0.7% | 5.2% | 4.2% |

We can observe that for the age band 60-80 shocks frequencies are null for England and France. So we will skip this age band for the following part of the study.

For the risk capital in relation to mortality risk we define $\gamma^{\text{MORT}} := \gamma$.

#### Relationship between $\mathrm{Shock}_{x,t}$, $I_t$ and $R_t$

Let us assume that:

$$\mathrm{Shock}_{x,t} \sim I_t \times R_t.$$

We want to establish a link between the percentile from $\mathrm{Shock}_{x,t}$ and the one from $R_t$. If $\phi$ is the 99.6% percentile from $\mathrm{Shock}_{x,t}$, then we have :

$$P[\mathrm{Shock}_{x,t} \geq \phi] = 0.004,$$
$$P[I_t = 1, R_t \geq \phi] = 0.004.$$

Then, because $I_t$ and $R_t$ are independent,

$$P[R_t \geq \phi] = \frac{0.004}{P[I_t = 1]}. \tag{7.1}$$

Then $\phi$ is the $1 - \frac{0.004}{P[I_t=1]}$ percentile from $R_t$.

#### Determination of $\tilde{\mu}$, $\tilde{\sigma}$

Let us introduce $F[\tilde{\mu}, \tilde{\sigma}]$ the cumulative density function of a log-normal distribution with parameters $\tilde{\mu}$ and $\tilde{\sigma}$:

Given relationship 7.1, we will be able to fit parameters $\tilde{\mu}, \tilde{\sigma}$ with the historical data. We define 2 sets of conditions to fit $\tilde{\mu}$ and $\tilde{\sigma}$ :

$$F[\tilde{\mu}, \tilde{\sigma}]^{-1}(1 - \frac{0.004}{\gamma}) = \mathrm{Shock}_{x,t}(99.6\%) \text{ and}$$
$$\exp\left(\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}\right) = AV(\mathrm{Shock}_{x,t} | \mathrm{Shock}_{x,t} > 20\%).$$

Where $\mathrm{Shock}_{x,t}(99.6\%)$ is the 99.6% percentile of the $\mathrm{Shock}_{x,t}$ historical series, and $AV(\mathrm{Shock}_{x,t}|\mathrm{Shock}_{x,t} > 20\%)$ is the average of the $\mathrm{Shock}_{x,t}$ historical series where $\mathrm{Shock}_{x,t} > 20\%$ (see table 7.4).

**Table 7.4** Historical mortality shocks average where the shock is greater than 20%

| Age band | France M | France F | England M | England F | Sweden M | Sweden F |
|---|---|---|---|---|---|---|
| 0-20 | 38% | 42% | 38% | 37% | 32% | 34% |
| 20-40 | 95% | 86% | 87% | 46% | 48% | 40% |
| 40-60 | 30% | 32% | 21% | 22% | 35% | 37% |
| Total | 46% | 42% | 47% | 42% | 39% | 39% |

We obtain the following set of parameters:

**Table 7.5** Values for $\tilde{\mu}$

| Age band | France M | France F | England M | England F | Sweden M | Sweden F |
|---|---|---|---|---|---|---|
| 0-20 | -5.41 | -3.74 | -3.46 | -1.64 | -5.49 | -5.64 |
| 20-40 | -4.59 | -2.59 | -2.84 | -1.94 | -4.59 | -4.05 |
| 40-60 | -3.36 | -2.57 | -2.08 | -1.57 | -4.22 | -4.54 |
| Total | -5.08 | -3.41 | -2.88 | -1.1 | -4.12 | -3.69 |

**Table 7.6** Values for $\tilde{\sigma}$

| Age band | France M | France F | England M | England F | Sweden M | Sweden F |
|---|---|---|---|---|---|---|
| 0-20 | 2.98 | 2.4 | 2.23 | 1.13 | 2.95 | 3.01 |
| 20-40 | 3.01 | 2.2 | 2.32 | 1.51 | 2.77 | 2.49 |
| 40-60 | 2.07 | 1.69 | 1.03 | 0.28 | 2.52 | 2.65 |
| Total | 2.93 | 2.25 | 2.06 | 0.68 | 2.52 | 2.34 |

At this stage, it is difficult to have a strong idea of which parameter we should use except that since most of our portfolio comes from UK, we should choose parameters for UK and for the age band concerned.

For Mortality we denote by $(\mu^{\text{MORT}}, \sigma^{\text{MORT}}) = (\tilde{\mu}, \tilde{\sigma})$.

### 7.2.2 Longevity

As with mortality, there are many different possibilities to model the longevity risk. We want to show one particular model in order to understand the different steps needed. For an overview of other models and approaches we refer to [PDHO09] and sources therein.

We use the following model for longevity. Assume the best estimate mortality is given by $q_{x,t}$. Then we use the following model for the mortality process:

$$Q^i_{x,t}(\omega) = q_{x,t} \times C_t(\omega) + \epsilon^i_{x,t}(\omega).$$

With,

$$\begin{aligned}
C_t &= \exp(X_t) \times C_{t-1}, \\
(X_j)_{j\in\mathbb{N}} &\sim \mathcal{N}(\mu,\sigma) && \text{i.i.d.,} \\
\epsilon^i_{x,t} &\sim F && \text{i.i.d. with } \mathbb{E}(F) = 0, \\
(C,\epsilon) &\quad \text{Independent,} \\
C_0 &= 1,
\end{aligned}$$

where $x$ represents the age of the insured, t the period in time we focus on.

This model aims to describe how longevity behaves around the projected expected values. We will concentrate our efforts on determining the parameters of the $(C_t)_{t\in\mathbb{N}}$ distribution the aim of which is to simulate the non diversifying risk of longevity. We will not work on the $(\epsilon)^i_{x,t}$ vector since it represents the diversifying risk.

This approach allows a best estimate mortality projection given by the common assumption:

$$q_{x,t} = q_{x,t_0} \times \exp\left(-\lambda_x \times \{t - t_0\}\right).$$

#### Comparison with the Lee-Carter Model

In 1992 Lee and Carter [LC92] developed an new method to forecast mortality and so to obtain prospective mortality tables. This stochastic methodology suggested a log-bilinear form for the central death rate $\mu_x(t)$ for age x at time t . It consists in decomposing the age-specific mortality in two components:

- A set of age-specific constants $a_x$, $b_x$,
- A time-varying index of mortality $\kappa_t$.

The model has the following form:

$$\ln(\mu_x(t)) = a_x + b_x \times \kappa_t + \epsilon_{x,t}.$$

Since this admits several solutions, restrictions (on the parameters) must be added to obtain an identifiable model:

$$\sum_{x_{min}}^{x_{max}} \beta_x = 1,$$
$$\sum_{t_{min}}^{t_{max}} \kappa_t = 0.$$

The signification of the parameters is described below:

$a_x$ represents the average level of the $\ln(\mu_x(t))$ surface over time. Therefore, $\exp(a_x)$ is the general shape of mortality at age x over time. $\kappa_t$ is an indicator of mortality evolution over time. It describes the change in overall mortality over time. $b_x$ indicates the sensitivity of rates to the mortality evolution over time, for a given age x; so, mortality evolution at age x depends on the index of mortality $\kappa_t$.

To build prospective mortality tables, it is necessary to

- Estimate the parameters of the Lee-Carter Model.
  An optimal solution can be found with the method of least squares and is given by a singular value decomposition (SVD) (for the determination of parameters $\kappa_t$ and $b_x$).
- Model the time-varying parameter $\kappa_t$ describing mortality evolution. In fact it is necessary to specify the process shape of $\kappa_t$ ($AR(p)$, $MA(q)$, $ARMA(p,q)$, ...) in order to make forecasts and thus, to extrapolate the trend.
  In this way it is possible to obtain predictions of the probabilities of death and interval forecasts by doing simulations.

**Choice of the model**

Since there is no generally accepted model, we chose this one that could be considered as a simplified version of Lee-Carter as we can see if we express the 2 models in the same way :

$$\ln(\mu_x(t)) = [a_x + b_x \times \kappa_t] + \epsilon_{x,t} \quad \text{(Lee-Carter model)},$$
$$\ln(Q_{x,t}) = \ln(q_{x,t}) + \sum_{i=0}^{t} X_i \quad \text{(our model)}.$$

Indeed, both use a log-normal distribution to estimate the deviation around a global tendency. But we have also some major differences. Lee-Carter has a stochastic predictive part $(a_x + b_x \times \kappa_t)$ whereas our model separates this prediction work and assumes it is given by the $(q_{x,t})$. The variance of the stochastic part of our model $(\sum_{i=0}^{i=t} X_i)$ is growing with time.

### Model parameters

Since the model assumes that our best estimate mortality is given by $q_{x,t}$, we need the following condition:

$$E\left[\exp\left(X_t\right)\right] = 1, \tag{7.2}$$

since $\exp\left(X_t\right)$ follows a log-normal distribution. This implies:

$$E\left[\exp\left(X_t\right)\right] = \exp\left(\mu + \frac{\sigma^2}{2}\right). \tag{7.3}$$

Equations (7.2) and (7.3) yield to:

$$\mu = -\frac{\sigma^2}{2}.$$

So the only parameter for our model is given by $\sigma^{\text{LONG}} := \sigma$.

## 7.3 Concrete Example: An Annuity Portfolio

Annuity portfolios are normally re-insured in the form of a quota share or in form of a mortality swap. In both cases the premiums paid are compared with the annuities paid out. For an individual policy$k$ the payout is given by the following formula:

$$R_k(\omega, t) = \chi_{\{T_k \geq t\}} \times R_k,$$

where $T_k(\omega)$ denotes the (random) future life span of policy $k$ and $R_k$ the annuity to be paid per annum. As a consequence the price (BEL) of this annuity corresponds to

$$V_k(t) = \sum_{t=0}^{\infty} {}_t p_x \times R_k \times \pi(\mathcal{Z}_{(t)}),$$

denoting by $\pi(\mathcal{Z}_{(t)})$ the price of the $t$-year zero coupon bond at valuation date. The problem with the above mentioned cover is the fact that both the cover length and the potential loss for the reinsurer is unlimited, leading to relatively high prices. In contrast, the typical non-life reinsurance covers are limited in time and in maximal amounts. The aim is to analyse the applicability of the corresponding methods in mortality swaps.

The analogy of the instruments which we will describe in the financial world are most likely swaptions to the extent that there is a real swap in certain situations and that there is a limit in time and amount.

In order to illustrate this type of product, we will have a look at the different steps. The main ingredient is a clear definition of the risk covered and the corresponding cash flows. We remark that figure 7.3 shows the corresponding results by putting always 5 years together. Hence bucket number 1 corresponds to years 1 to 5, bucket number 2 to the years 6 to 10, etc.

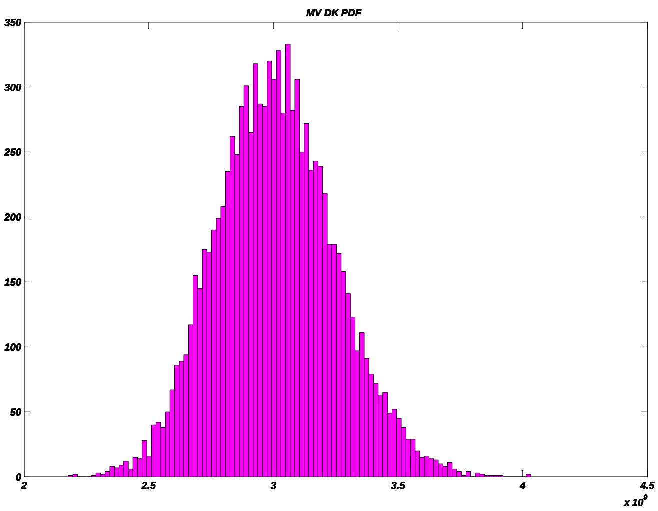



**Fig. 7.4** Distribution of the present value of the future cash flows

**no picture yet**

**Fig. 7.5** Sample of CF Trajectories

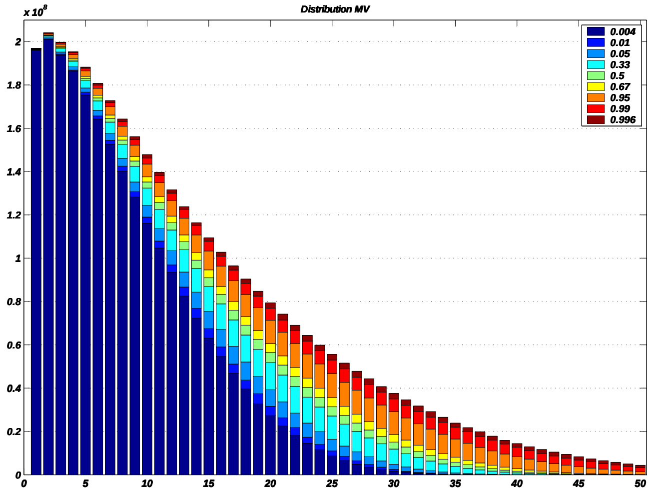



**Fig. 7.6** Probability Density Function of the Present Values

## 7.3.1 Formulae

In principle, the usual calculations for expected values of cash flows $CF_t$ apply, with exception that one needs to replace them by the correspondingly adjusted ones. Assume, for example, a layer between $\alpha$ and $\beta$ within the time interval $[t_1, t_2]$. In this case we have

$$CF_t^* = \begin{cases} 0 & \text{if } t \notin [t_1, t_2], \\ \min(\beta - \alpha, \max(CF_t - \alpha, 0)) & \text{else.} \end{cases}$$

Normally one would agree that both $\alpha$ and $\beta$ are defined relative to the expected value of the corresponding cash flows. For example $\alpha_t = 110\% \times \mathbb{E}[CF_t]$ and $\beta_t = 120\% \times \mathbb{E}[CF_t]$ The present values and premium need to be calculated by using a standard approach.

In the following we will use the Swiss mortality table ERM/F 2000 for the calculations. The mortality law for ERM/F 2000 can be defined as follows, with $C(t, \omega) = 1 \, \forall t$:

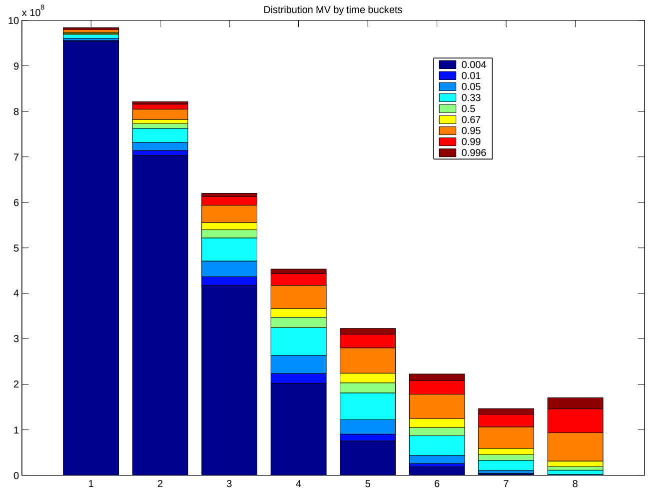



**Fig. 7.7** Market Values by Time Buckets

$$q_{x,t,\omega} = q_{x,t_0} \times \exp\{\lambda_x \times (t - t_0)\} \times C(t, \omega).$$

The model applied for these calculations uses a non-trivial $C$: We denote by $(X_t)_{t \in \{0,1,2,...\}}$ iid $\mathcal{N}(\mu, \sigma)$ normal distributed random variables, and define

$$C(t) = \exp(\sum_{k=0}^{t} X_k).$$

Note that this model is very similar to the well known Lee Carter model, which is given by:

$$\begin{aligned} \log(q_{x,t}) &= a_x + b_x\, k_t + \epsilon_{x,t}, \\ k_{t+1} &= k_t + \mu + \sigma\, X_t, \\ (X_t)_{t \in \mathbb{N}} &\sim \mathcal{N}(0, 1). \end{aligned}$$

Obviously the two models interrelate by

$$\begin{aligned} q_{x,t_0} &\mapsto a_x, \\ \lambda_x &\mapsto b_x. \end{aligned}$$

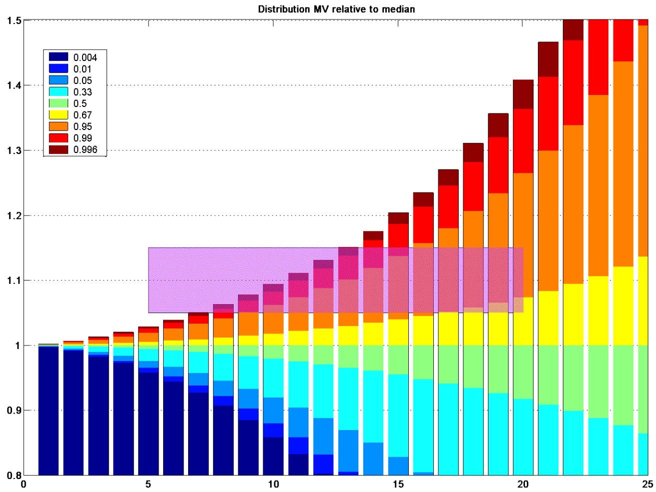



**Fig. 7.8** Example of a covered layer

The main difference between the two models is the different parameters for the fluctuation of the trend (relative to the model introduced above):

| | |
|---|---|
| Model used | $X_k \sim \mathcal{N}(0, \sigma)$ |
| Lee Carter | $X_k \sim \mathcal{N}(\lambda_x \times \mu, \lambda_x \times \sigma)$ |

### *7.3.2 Application to Insurance Linked Securities*

The mortality swap is intrinsically linked to insurance linked securities (ILS) and therefore this section describes some of the above mentioned features. Firstly it needs to be noted that investors have the following criteria to invest in certain investments. The investments should satisfy as many of the following characteristics as possible:

- High return in relation to the corresponding risk,
- Clearly defined risks and the possibility to quantify the risk,
- Liquid secondary market in order to offset the risk in adverse situations,
- Short binding period, in case there is no liquid secondary market.

In case of longevity ILS the risk is clearly defined but difficult to quantify. The recent introduction of well defined indexes relating to these risks and the companion documents and methods (cf "JPMorgan LifeMetrics") help to close this gap. Until now there exists no liquid secondary market and therefore the binding period of the risk needs to be analysed further. The easiest thing would be to offer longevity ILS only for a limited period of time to the investor. This is clearly feasible as defined above, but has the drawback that neither the most relevant risk for the pension funds and primary life companies is considered. Furthermore, interestingly, non-correlated, investment opportunities would be left out. The idea is therefore to slice a given mortality portfolio in analogy to CDS constructions in bits which have different risk characteristics, ranking from Bond investments with a equivalent counter-party risk of "AAA" down to junk bond and an equity part. In contrast to the CDS the differentiation would be done over time. As the longevity risk increases with time, the risk related to the annuities which will be paid out in the near future corresponds to bonds with a higher rating and less return. The further away the annuity is paid, the higher risk and return for the corresponding bond. Figure 7.9 shows this.

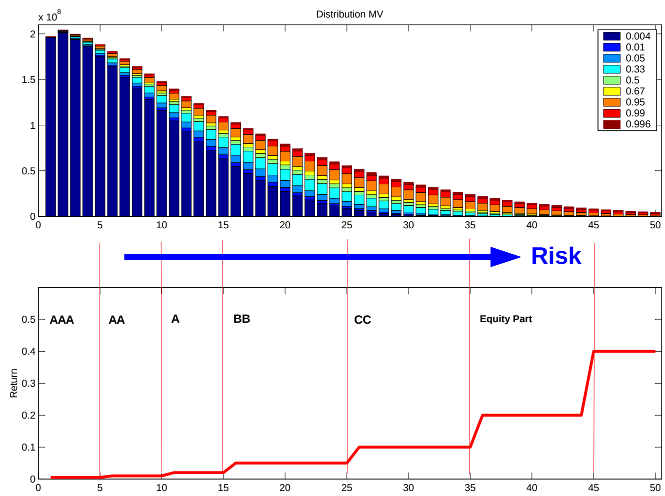



**Fig. 7.9** CDS structuring for longevity ILS

Moreover the following table shows how the differences match to different investor characteristics:

| **Time Bucket** | **Rating idea** | **Investor Characteristics** |
|---|---|---|
| $[0,5[$ | AAA | Corporate Bonds |
| $[6,10[$ | AA | Corporate Bonds / Reinsurance |
| $[11,15[$ | A | Reinsurance |
| $[15,25[$ | BB | Reinsurance / Hedge Funds |
| $[25,35[$ | CC | Hedge Funds |
| $[35,45[$ | Equity Part | Hedge Funds / Private Equity |
| $[46,\infty[$ | Equity Part | Private Equity |

After this abstract structuring there is a mechanism to calculate the profit and loss for each slice and time needs to be calculated. The profit and loss for a certain period is given by

$$L_t(\omega) = \{\, {}_tp_x - \chi_{T>t}\} \times \pi(\mathcal{Z}_{(t)}).$$

Therefore the calculation of the profit/loss in a certain point in time can be calculated if either $q_{x,t}$ or ${}_tp_{x_0,t_0}$ is given. In case of a slice the corresponding profits and losses are calculated by integrating over the exposure, eg the loss within time $[t_1,t_2]$ is given by:

$$L_{[t_1,t_2[}(\omega) = \sum_{\tau=t_1, x\in\mathbb{N}}^{t_2-1} R_{\tau,x} \times \{\, {}_\tau p_x - \chi_{T_x>\tau}\} \times \pi(\mathcal{Z}_{(\tau)}),$$

where $R_{x,t}$ denotes the annuity (notional amount) to be paid for the persons aged $x$ at inception of the ILS at time $\tau$.

We consider now the annuity part at time $n$ with an expected annuity of $A_n$ and a risk capital at that time, for example with respect to the 99.6 % Var of $\eta_n$ of the expected annuity. For simplicity we assume $A_n = 1$. For this treaty the investor wants a return of $\kappa_n$ on the capital invested. By $A_n(\omega)$ we actual annuity paid out at time $n$. We have the following relations, denoting with $C_n$ the corresponding risk capital.

$$\begin{aligned} A_n &= \mathbb{E}[A_n(\omega)] \\ C_n &= F^{-1}_{A_n}(99.6\%) \end{aligned}$$

From the investor's point of view we have the following cash flows:

| Time | Amount |
|---|---|
| $t = 0$ | $-C_n + \xi$ |
| $t = n$ | $[C_n - A_n(\omega)]^+ + C_n \times \kappa_n$ |

where $\xi$ denotes the risk premium for the investor. Because of equilibrium we have the following equation:

$$C_n - \xi = \pi(\mathcal{Z}_{(n)}) \times \left\{ \int [C_n - A_n(\omega)]^+ d\omega + C_n \times \kappa_n \right\}.$$

### 7.3.3 *Statistics*

| | |
|---|---|
| Table used | ERM/F 2000 second order for $q_x$ and trend |
| Date of Valuation | 1.1.2006 |
| Model for Morality law | Mortality improvements following cumulative log-normal path |
| $\mu$ | $-\frac{\sigma^2}{2}$ |
| $\sigma$ | 10 % |
| Discount factors | Yield curve in Euro as of 29.12.2006 |
| Annuities paid out p.a. | EUR 244.18 M |
| Proxy for stat. MR 3.5 % | EUR 3,164.16 M |

```
MV DK Mean     3011.22 M    Stddev      239.47 M (7.95 %)
MV DK -.---- Min         2176.70 M    (  72.29 % of mean)
MV DK 0.0040 Quantile    2407.04 M    (  79.94 % of mean)
MV DK 0.0100 Quantile    2485.36 M    (  82.54 % of mean)
MV DK 0.0500 Quantile    2632.77 M    (  87.43 % of mean)
MV DK 0.3300 Quantile    2898.58 M    (  96.26 % of mean)
MV DK 0.5000 Quantile    3005.98 M    (  99.83 % of mean)
MV DK 0.6700 Quantile    3109.00 M    ( 103.25 % of mean)
MV DK 0.9500 Quantile    3420.31 M    ( 113.59 % of mean)
MV DK 0.9900 Quantile    3599.37 M    ( 119.53 % of mean)
MV DK 0.9960 Quantile    3691.62 M    ( 122.60 % of mean)
MV DK -.---- Max         4032.62 M    ( 133.92 % of mean)

Bucket [ 1, 5] DK Mean      970.83 M    Stddev        5.78 M (0.60 %)
Bucket [ 1, 5] DK -.---- Min          946.91 M    (  97.54 % of mean)
Bucket [ 1, 5] DK 0.0040 Quantile     953.44 M    (  98.21 % of mean)
Bucket [ 1, 5] DK 0.0100 Quantile     955.80 M    (  98.45 % of mean)
Bucket [ 1, 5] DK 0.0500 Quantile     960.67 M    (  98.95 % of mean)
Bucket [ 1, 5] DK 0.3300 Quantile     968.64 M    (  99.77 % of mean)
Bucket [ 1, 5] DK 0.5000 Quantile     971.19 M    ( 100.04 % of mean)
Bucket [ 1, 5] DK 0.6700 Quantile     973.63 M    ( 100.29 % of mean)
Bucket [ 1, 5] DK 0.9500 Quantile     979.67 M    ( 100.91 % of mean)
Bucket [ 1, 5] DK 0.9900 Quantile     982.73 M    ( 101.23 % of mean)
Bucket [ 1, 5] DK 0.9960 Quantile     984.00 M    ( 101.36 % of mean)
Bucket [ 1, 5] DK -.---- Max          989.74 M    ( 101.95 % of mean)

Bucket [ 6,10] DK Mean      771.25 M    Stddev       22.21 M (2.88 %)
Bucket [ 6,10] DK -.---- Min          662.71 M    (  85.93 % of mean)
Bucket [ 6,10] DK 0.0040 Quantile     703.04 M    (  91.16 % of mean)
Bucket [ 6,10] DK 0.0100 Quantile     713.85 M    (  92.56 % of mean)
Bucket [ 6,10] DK 0.0500 Quantile     731.78 M    (  94.88 % of mean)
Bucket [ 6,10] DK 0.3300 Quantile     762.71 M    (  98.89 % of mean)
Bucket [ 6,10] DK 0.5000 Quantile     773.09 M    ( 100.24 % of mean)
Bucket [ 6,10] DK 0.6700 Quantile     782.07 M    ( 101.40 % of mean)
Bucket [ 6,10] DK 0.9500 Quantile     804.71 M    ( 104.34 % of mean)
```

```
Bucket [ 6,10] DK 0.9900 Quantile     816.04 M    ( 105.81 % of mean)
Bucket [ 6,10] DK 0.9960 Quantile     821.40 M    ( 106.50 % of mean)
Bucket [ 6,10] DK -.---- Max          843.75 M    ( 109.40 % of mean)

Bucket [11,15] DK Mean      536.56 M    Stddev      37.94 M (7.07 %)
Bucket [11,15] DK -.---- Min          351.99 M    (  65.60 % of mean)
Bucket [11,15] DK 0.0040 Quantile     417.68 M    (  77.84 % of mean)
Bucket [11,15] DK 0.0100 Quantile     436.71 M    (  81.39 % of mean)
Bucket [11,15] DK 0.0500 Quantile     471.03 M    (  87.79 % of mean)
Bucket [11,15] DK 0.3300 Quantile     521.64 M    (  97.22 % of mean)
Bucket [11,15] DK 0.5000 Quantile     539.72 M    ( 100.59 % of mean)
Bucket [11,15] DK 0.6700 Quantile     555.49 M    ( 103.53 % of mean)
Bucket [11,15] DK 0.9500 Quantile     593.68 M    ( 110.65 % of mean)
Bucket [11,15] DK 0.9900 Quantile     613.08 M    ( 114.26 % of mean)
Bucket [11,15] DK 0.9960 Quantile     619.86 M    ( 115.52 % of mean)
Bucket [11,15] DK -.---- Max          657.53 M    ( 122.54 % of mean)

Bucket [16,20] DK Mean      344.11 M    Stddev      47.36 M(13.76 %)
Bucket [16,20] DK -.---- Min          135.87 M    (  39.49 % of mean)
Bucket [16,20] DK 0.0040 Quantile     202.06 M    (  58.72 % of mean)
Bucket [16,20] DK 0.0100 Quantile     223.62 M    (  64.98 % of mean)
Bucket [16,20] DK 0.0500 Quantile     263.32 M    (  76.52 % of mean)
Bucket [16,20] DK 0.3300 Quantile     324.41 M    (  94.28 % of mean)
Bucket [16,20] DK 0.5000 Quantile     347.04 M    ( 100.85 % of mean)
Bucket [16,20] DK 0.6700 Quantile     366.76 M    ( 106.58 % of mean)
Bucket [16,20] DK 0.9500 Quantile     417.42 M    ( 121.31 % of mean)
Bucket [16,20] DK 0.9900 Quantile     443.26 M    ( 128.82 % of mean)
Bucket [16,20] DK 0.9960 Quantile     453.25 M    ( 131.72 % of mean)
Bucket [16,20] DK -.---- Max          493.63 M    ( 143.45 % of mean)

Bucket [21,25] DK Mean      202.24 M    Stddev      47.95 M(23.71 %)
Bucket [21,25] DK -.---- Min           32.43 M    (  16.04 % of mean)
Bucket [21,25] DK 0.0040 Quantile      75.37 M    (  37.27 % of mean)
Bucket [21,25] DK 0.0100 Quantile      90.32 M    (  44.66 % of mean)
Bucket [21,25] DK 0.0500 Quantile     122.05 M    (  60.35 % of mean)
Bucket [21,25] DK 0.3300 Quantile     181.05 M    (  89.52 % of mean)
Bucket [21,25] DK 0.5000 Quantile     203.22 M    ( 100.49 % of mean)
Bucket [21,25] DK 0.6700 Quantile     224.45 M    ( 110.98 % of mean)
Bucket [21,25] DK 0.9500 Quantile     279.91 M    ( 138.41 % of mean)
Bucket [21,25] DK 0.9900 Quantile     310.11 M    ( 153.34 % of mean)
Bucket [21,25] DK 0.9960 Quantile     322.97 M    ( 159.70 % of mean)
Bucket [21,25] DK -.---- Max          372.89 M    ( 184.38 % of mean)

Bucket [26,30] DK Mean      106.99 M    Stddev      40.99 M(38.32 %)
Bucket [26,30] DK -.---- Min            5.61 M    (   5.24 % of mean)
Bucket [26,30] DK 0.0040 Quantile      18.68 M    (  17.46 % of mean)
Bucket [26,30] DK 0.0100 Quantile      25.57 M    (  23.90 % of mean)
Bucket [26,30] DK 0.0500 Quantile      43.53 M    (  40.69 % of mean)
Bucket [26,30] DK 0.3300 Quantile      86.64 M    (  80.99 % of mean)
Bucket [26,30] DK 0.5000 Quantile     104.68 M    (  97.85 % of mean)
Bucket [26,30] DK 0.6700 Quantile     124.07 M    ( 115.96 % of mean)
Bucket [26,30] DK 0.9500 Quantile     178.23 M    ( 166.59 % of mean)
Bucket [26,30] DK 0.9900 Quantile     208.37 M    ( 194.77 % of mean)
Bucket [26,30] DK 0.9960 Quantile     222.52 M    ( 207.99 % of mean)
Bucket [26,30] DK -.---- Max          278.46 M    ( 260.27 % of mean)

Bucket [31,35] DK Mean       49.80 M    Stddev      29.55 M(59.33 %)
Bucket [31,35] DK -.---- Min            0.65 M    (   1.30 % of mean)
Bucket [31,35] DK 0.0040 Quantile       2.66 M    (   5.34 % of mean)
Bucket [31,35] DK 0.0100 Quantile       4.20 M    (   8.43 % of mean)
Bucket [31,35] DK 0.0500 Quantile      10.34 M    (  20.76 % of mean)
Bucket [31,35] DK 0.3300 Quantile      32.84 M    (  65.94 % of mean)
Bucket [31,35] DK 0.5000 Quantile      45.17 M    (  90.70 % of mean)
Bucket [31,35] DK 0.6700 Quantile      59.07 M    ( 118.63 % of mean)
Bucket [31,35] DK 0.9500 Quantile     106.17 M    ( 213.20 % of mean)
Bucket [31,35] DK 0.9900 Quantile     134.08 M    ( 269.24 % of mean)
Bucket [31,35] DK 0.9960 Quantile     146.43 M    ( 294.05 % of mean)
```

```
Bucket [31,35] DK -.---- Max          198.37 M    ( 398.34 % of mean)

Bucket [36,50] DK Mean       29.45 M    Stddev      31.12 M(105.69 %)
Bucket [36,50] DK -.---- Min            0.11 M    (   0.36 % of mean)
Bucket [36,50] DK 0.0040 Quantile       0.30 M    (   1.01 % of mean)
Bucket [36,50] DK 0.0100 Quantile       0.50 M    (   1.69 % of mean)
Bucket [36,50] DK 0.0500 Quantile       1.67 M    (   5.67 % of mean)
Bucket [36,50] DK 0.3300 Quantile      11.08 M    (  37.62 % of mean)
Bucket [36,50] DK 0.5000 Quantile      18.93 M    (  64.29 % of mean)
Bucket [36,50] DK 0.6700 Quantile      31.22 M    ( 106.02 % of mean)
Bucket [36,50] DK 0.9500 Quantile      93.36 M    ( 317.04 % of mean)
Bucket [36,50] DK 0.9900 Quantile     145.94 M    ( 495.60 % of mean)
Bucket [36,50] DK 0.9960 Quantile     170.37 M    ( 578.55 % of mean)
Bucket [36,50] DK -.---- Max          273.39 M    ( 928.41 % of mean)
```

## 7.3.4 Allocation of Risk Capital in proportion to Premium

Based on the simulation approach it is possible to allocate a risk capital per year in proportion of the annuity paid out, as follows. We know from the simulation the 99.6% quantile as a percentage of the best estimate annuity paid out

| $t$ | $\eta_t$ | $t$ | $\eta_t$ |
|---|---|---|---|
| 1 | 100.18 % | 26 | 186.87 % |
| 2 | 100.67 % | 27 | 197.50 % |
| 3 | 101.30 % | 28 | 210.19 % |
| 4 | 102.06 % | 29 | 222.65 % |
| 5 | 102.95 % | 30 | 238.05 % |
| 6 | 104.03 % | 31 | 255.49 % |
| 7 | 105.20 % | 32 | 276.07 % |
| 8 | 106.59 % | 33 | 299.82 % |
| 9 | 108.13 % | 34 | 324.58 % |
| 10 | 109.82 % | 35 | 349.16 % |
| 11 | 111.60 % | 36 | 383.51 % |
| 12 | 113.62 % | 37 | 422.22 % |
| 13 | 115.77 % | 38 | 462.09 % |
| 14 | 118.26 % | 39 | 503.20 % |
| 15 | 121.21 % | 40 | 560.08 % |
| 16 | 124.48 % | 41 | 619.86 % |
| 17 | 128.09 % | 42 | 693.29 % |
| 18 | 132.12 % | 43 | 770.48 % |
| 19 | 136.80 % | 44 | 859.48 % |
| 20 | 142.08 % | 45 | 945.10 % |
| 21 | 147.77 % | 46 | 1033.18 % |
| 22 | 153.63 % | 47 | 1149.51 % |
| 23 | 160.64 % | 48 | 1277.50 % |
| 24 | 168.35 % | 49 | 1382.71 % |
| 25 | 176.94 % | 50 | 1506.23 % |

Assume that for the portfolio considered the best estimate annuities are given by $(R_t)_{t\in\mathbb{N}}$. In this case the present value of the necessary risk capital can be calculated by

$$C_0 = \sum_{\tau=0}^{\infty} R_\tau \; (\eta_\tau - 1) \; \pi(\mathcal{Z}_{(\tau)}) \text{ and,}$$

$$C_t = \frac{1}{\pi(\mathcal{Z}_{(t)})} \sum_{\tau=t}^{\infty} R_\tau \; (\eta_\tau - 1) \; \pi(\mathcal{Z}_{(\tau)}).$$

At this point there is still a need for a better explanation for the capital needed at a certain point in time, especially for annuities:

1. The determination of the capital $C_0$ can be put in a relation to the annuities paid, in the sense that the present value of the difference equals the risk capital. A re-

lated question is now whether there is an additional need for risk capital during the projection for $t = 1$ etc. This is not necessary as the following calculation shows: The economic risk at a given time $t$ for an annuity 1 at inception is equivalent to the difference of the relevant payments:

$$L_t(\omega) = \{ \, {}_tp_x - \chi_{T>t}\} \times \pi(\mathcal{Z}_{(t)}).$$

Therefore the risk capital to be allocated at this time is the corresponding possible annuity in an adverse scenario. The sum over the different points in time results in the above mentioned $C_0$. Therefore the actual risk capital allocated at a certain point in time equals $\Gamma(L_t)$, the corresponding risk measure for the possible excess annuity to be paid.

2. The second question, which needs some explanation relates to the required capital $C_1$, but not as seen from time $t = 0$, but if we are at time $t = 1$ and know the development until then. In this situation the corresponding mortality processes have moved forward by one year and therefore, we need to take the *conditional* risk measure at time $t = 1$, as a formula:

$$\begin{aligned}\text{Risk Capital at time } 1 &= \Gamma\left[\frac{1}{\mathcal{Z}_{(1)}} \sum_{t=1}^{\infty} L_k \,\middle|\, \mathcal{F}_1\right] \\ &= \mathbb{E}\left[\frac{1}{\mathcal{Z}_{(1)}} \sum_{t=1}^{\infty} L_k \,\middle|\, \mathcal{F}_1, L > F_\alpha^{-1}(L)\right].\end{aligned}$$

in case of the 99 % shortfall as risk measure. This relates assuming some homogenity assumptions in the following formula in therms of $\eta$:

$$\begin{aligned}C_0 &= \sum_{\tau=0}^{\infty} R_\tau \; (\eta_\tau - 1) \; \pi(\mathcal{Z}_{(\tau)}) \text{ and,} \\ \tilde{C}_t &= \frac{1}{\pi(\mathcal{Z}_{(t)})} \sum_{\tau=t}^{\infty} R_\tau \; (\eta_{\tau-t} - 1) \; \pi(\mathcal{Z}_{(\tau)}).\end{aligned}$$

### *7.3.5 Reporting Templates*

This section provides an easy reporting template in relation to insurance risks. The concrete example is quite detailed and was intended for use in a life reinsurance company as its core business. For a primary insurance company the reporting template might be too detailed and one would expect other dimensions which matter more such as the lapses per distribution channel. As one can see the numbers in the template are purely virtual. Most of the risk categories are self explaining with the exception of ILA, which stands for impaired life annuities, eg annuities sold to people with a short remaining life time, be it for their old age, or be it for the fact that they are ill.

*a) By Risk Factor*

| | Var(1:75) | Var(1:250) | TVar(1:100) | Limit (1:250) |
|---|---|---|---|---|
| Mortality | 25.0 | 50.0 | 60.0 | 250.0 |
| Longevity | 25.0 | 50.0 | 60.0 | 200.0 |
| ILA | 25.0 | 50.0 | 60.0 | |
| Disability | 25.0 | 50.0 | 60.0 | |
| Lapses | 25.0 | 50.0 | 60.0 | |
| Other | 25.0 | 50.0 | 60.0 | |
| Simple Sum | 150.0 | 300.0 | 360.0 | |
| Diversification | -50.0 | -100.0 | -120.0 | |
| Total | 100.0 | 200.0 | 240.0 | 500.0 |

*b) By Lines Of Business:*

| | Var(1:75) | Var(1:250) | TVar(1:100) | Limit (1:250) |
|---|---|---|---|---|
| TCI | 25.0 | 50.0 | 60.0 | |
| GMDB | 25.0 | 50.0 | 60.0 | 150.0 |
| Financing | 25.0 | 50.0 | 60.0 | |
| Longevity /MS | 25.0 | 50.0 | 60.0 | |
| ILA | 25.0 | 50.0 | 60.0 | |
| Other | .... | .... | .... | |
| Simple Sum | 150.0 | 300.0 | 360.0 | |
| Diversification | -50.0 | -100.0 | -120.0 | |
| Total | 100.0 | 200.0 | 240.0 | |

*c) Stress Scenarios:*

| Scenario | SC1 | SC2 | SC3 |
|---|---|---|---|
| TCI | 25.0 | 50.0 | 60.0 |
| GMDB | 25.0 | 50.0 | 60.0 |
| Financing | 25.0 | 50.0 | 60.0 |
| Longevity /MS | 25.0 | 50.0 | 60.0 |
| ILA | 25.0 | 50.0 | 60.0 |
| Other | .... | .... | .... |
| Total | 100.0 | 200.0 | 240.0 |

**SC1:** Pandemic as seen in 1918
**SC2:** Global increase in life span by 3 yrs at age 65
**SC3:** Run of the bank due to economic distress

# Chapter 8
# Policyholder Behaviour

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We will see that allowing for lapses reduces the people receiving the variable annuity benefits. The price for the variable annuity guarantee for our sample policy (see section 3.6) reduces, when including lapses, from 40311.70 down to 12222.80. This big difference raises the question, what happens if policyholders become more efficient and lapse less. To understand this it is important to know that the insurer will price this guarantee possibly at 15000, to have some safety and profit margin. However the 15000 could not cover the guarantees in case no lapses occur. Such effects have led to losses in some variable annuity portfolios. To illustrate this, assume that the best estimate lapses ("BE") indicated above (eg 4% for all years, except for year 10 where lapses are 12%) were inaccurate and need to be revalued to 8% at year 10 and 2% thereafter ("New BE"). The following table shows the value of the valuation portfolio as at time 2. We note that maturity is now in 23 years.

| | Instrument | Value Normal | Value BE | Value New BE | Value P&L |
|---|---|---|---|---|---|
| 1 | Put $t = 1$ | 25.6 | 23.5 | 23.5 | – |
| 2 | Put $t = 2$ | 39.7 | 35.1 | 35.1 | – |
| 3 | Put $t = 3$ | 55.7 | 47.3 | 47.3 | – |
| ... | | | | | |
| 7 | Put $t = 7$ | 156.7 | 112.6 | 112.6 | – |
| 8 | Put $t = 8$ | 195.4 | 134.6 | 134.6 | – |
| 9 | Put $t = 9$ | 241.2 | 149.3 | 149.3 | 0.0 |
| ... | | | | | |
| 23 | Put $t = 23$ | 36986.0 | 10138.3 | 14802.2 | -4663.9 |
| | **Total** | 42844.5 | 12973.2 | 18006.3 | **-5033.1** |

In this example the loss of the updated lapse assumptions exceeds the assumed profit and safety margin. We wish to remark that a lot of companies have more refined lapse assumptions, in the sense that they make them dynamic depending on the value of the underlying guarantee. The higher the fund value the more likely the policyholder is going to lapse the insurance policy. In such instances the best estimate lapses could be, for example 50% of normal lapse levels. In case of low fund values the dynamic lapses try to model the reduced lapse levels and hence the policyholder behaviour. There is a high incentive for the policyholder to lapse in times of good fund performance and to stick in times of adverse fund development.

From a theoretical background the inclusion of the policyholder behaviour makes things much more complicated. We recall convention 19, where we assumed that $\mathcal{G}_t$ and $\mathcal{H}_t$ are stochastically independent. This means, that the financial variables are independent of the future lifespan and from lapses. Whereas we can defend the independence of mortality from the equity market prices, this is not quite the case if we take lapses into consideration. To model policyholder behaviour better, it is in a first step necessary to understand what a policyholder can do with his policy over time. This will be the topic of the following section.

## 8.1 What can a policyholder do during the lifetime of his policy

In this section we look at the actions a policyholder can take with respect to his policy. From an insurer's point of view this list indicates some of the policyholder behaviour risks:

**Change Asset Allocation:** The policyholder can change his asset allocation and invest in different assets, which are more or less risky.

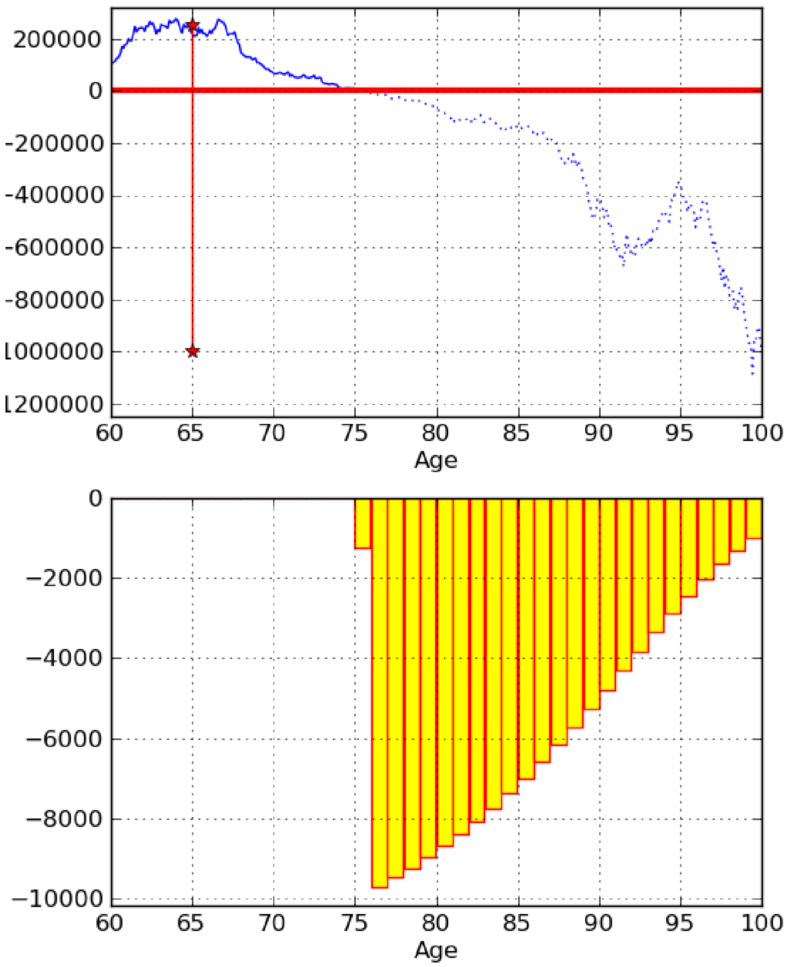


**Fig. 8.1** Example of GMWB contingency claims one trajectory

**Top up investment:** The policyholder can invest an additional amount in the underlying fund. This can change the guarantees:

**Lapse:** policyholder can end the policy.

**Start withdrawing:** The policyholder can start to withdraw money from the fund.

**Change amount of withdrawal:** Within a given period, the policyholder can decide to withdraw more or less.

**Partial Surrender:** Withdrawing more than regularly allowed.

**Sell Policy:** He can sell the policy to a third party to monetize the value of the policy.

Policyholder behaviour is a risk that needs to be considered, in particular for the product design. Here one needs to avoid product designs, which promote crystallisation of losses for many policyholders at the same time.

There are different ways how policyholder behaviour can be considered. One school of thought is to assume that the policyholder behaves efficient, in the same way the price of an option is determined. In reality one can observe that policyholders do not behave fully efficient and therefore implicitly have another utility function which determines their behaviour (see for example [**?**]). Ultimately the different ways to consider policyholder behaviour boils down to the way the respective models are defined. In the following we will try to incorporate observed policyholder behaviour in our models.

One term which important in the way policyholder behaviour is modelled is the in-the-moneyness ("itm"). The "itm" is defined as one minus the ratio between the fund value "FV" and the corresponding remaining guarantee, hence $itm = 1 - \frac{FV}{Guarantee}$ (as proxy for the guarantee one could for example use $\alpha\, \ddot{a}_x$). The higher this ratio, the higher the value of the underlying guarantee and the lower the likelihood that the policyholder lapses.

With respect to withdrawing (GMWB policies) the policyholder has usually two additional choices to make. He has to decide at which point in time he will start to consume his funds (aka withdraw) and also the amount. A lot of GMWB policies have step-up ("ratchets") features - ie the benefit base ($B$) increases at certain points in time ($\tau_1, \tau_2, \ldots, \tau_N$) if the at this time the fund value ($FV$) exceeds the benefit base. Since the maximum withdrawal ($R_t$) amount is usually a percentage of the benefit base it is normally beneficial for the policyholder to wait withdrawing since this way he can expect an overall higher level of guaranteed withdrawals. On the other hand, this effect can be materially be muted in case of equities having fallen considerably. In this case the policyholder would be better advised to start withdrawing earlier. For the above case we have the following formulae:

$$B_0 = FV_0$$

$$
\begin{aligned}
dB_t &= \sum_{k=1}^{N} \delta_{\tau_k,t} \, (FV_t - B_t)^+ \\
R_t &= \beta_t \times B_t
\end{aligned}
$$

Though far from exhaustive, we discuss some of the main considerations and drivers for policyholders of the four options in the table below:

| Policyholder Option | Description | Pros | Cons | Drivers |
|---|---|---|---|---|
| **Leave the account untouched for another year** | Account remains invested. Guarantee can be utilised on next anniversary. Some benefits may become more onerous. | Opportunity to benefit from potentially better returns to come as remain invested; Potentially higher income can be drawn in future | Increases reliance on other sources of income | Guarantee is not itm; Expectation of strong equity market performance; Income not needed; Deferred retirement; Lack of interesting alternative investments. |
| **Utilise the guarantee and draw an income** | Policyholder draws income in the form of a withdrawal (GMWB) | Income level is secured; GMWB: Future income withdrawal can be deferred. | GMIB: Income is fixed based on benefits base; GMIB: Control of principle is lost; GMWB: Level of income is restricted to the contractually agreed maximum.; GMWB: Higher withdrawals can come with a surrender charge. | Guarantee is itm; Lack of income from other sources; Immediate annuity rates are competitive – low interest rates. |
| **Surrender the account value for cash** | Policyholders can surrender the account value at the end of the waiting period for cash to be reinvested or consumed as wished | Flexibility; Control of principle. | Variable annuity features not utilised; Lack of income; Exposure to reinvestment and inflation risk. | Guarantee is not itm; Bird in the hand theory; Interesting alternative investment opportunities. |
| **Surrender the account value for cash and purchase an immediate annuity** | Policyholders can surrender the account value at the end of the waiting period for cash to be reinvested in an immediate annuity | Income level is secured; May offer more attractive income. | Variable annuity features not used; Income is fixed based on account value and prevailing IA rates; Control of principle is lost. | Guarantee is not itm; Prevailing immediate annuity rates are attractive – high interest rates. |

## 8.2 Dynamic Policyholder Lapses for GMWB

In this example we want to look at a Guaranteed Minimum Withdrawal benefit for life, where we model the lapse behaviour dependent from the underlying equity market returns. We will look at how to model lapse rates which depend on the moneyness of the underlying guarantee. A GMWB is the variable annuity analogon of an immediate or deferred annuity. For this type of insurance we want see how the independence assumption (convention 19) interacts with the calculations: Hence we look at the following two different cases:

1. Assuming only mortality as decrement (where we assume that convention 19 holds), and
2. Including as a second effect the dynamic lapse behaviour.

In a first step we want to have a look at the stochastic differential equation which governs the dynamics of the fund value $(FV_t)_{t\in\mathbb{R}^+}$. Hence we assume as before that the equity value follows

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dW_t,$$

with $(W_t)_{t\in\mathbb{R}^+}$ the Brownian motion, $r$ the interest rate density and $\sigma$ the equity volatility. Assume also that $(A_t)_{t\in\mathbb{R}^+}$ and $(P_t)_{t\in\mathbb{R}^+}$ represent the cumulative annuity paid out and the cumulative guarantee fee respectively. In this case the fund value $FV_t$ is governed by the following equation:

$$dFV_t = r\, FV_t dt + \sigma\, FV_t dW + dA_t + dP_t.$$

We note that the product is the defined via the concrete choices of $A$ and $P$. We could for example envisage a product where the guarantee fee is constant multiple $\alpha$ of the fund value and where the annuity is a constant $\beta$ as long as the person is alive. In this case we would have:

$$\begin{aligned} dP_t &= -\alpha\, FV_t\, dt \text{ and} \\ dA_t &= -\beta\, I_\star(t)\, dt, \end{aligned}$$

where $I_\star(t)$ refers to definition 76.

A GMWB is a unit linked annuity for a person of age $x$. We use the following notations, assuming a time discrete model (eg $t \in \mathbb{N}_0$):

| Variable | Meaning |
|---|---|
| $S_t$ | Denotes the equity price process introduced in chapter D. |
| $X_t$ | Process with values $\star, \dagger, \ddagger$ to reflect the state of the policy, whereby $\star$ stands for alive, policy in force, $\dagger$ stands for death and $\ddagger$ stands for surrender. The set up is much the same as for the Markov chain models which we encountered before, however depending of the use of convention 19, the stochastic process is assumed to be Markovian conditional $S_t$. |
| $FV_t$ | The fund value at time $t$; we assume that $FV_0$ represents the fund value at inception of the policy. |
| $R_t$ | The annuity which is withdrawn at time $t$ in case the policy is in state $\star$. Please note that there are a variety of possibilities for $R$ in practice. The easiest one is to assume that $R_t = \alpha \times FV_0$, which means that a constant proportion of the funds can be withdrawn per period. In general $R_t$ will be more complex (in particular if taking ratcheting options into account), and will depend on $S_t$. |
| $C_t$ | The contingent claim in relation to the GMWB variable annuity at time $t$. The value of the GMWB option is then the expected value of the sum of the present values of $(C_t)_{t \in \mathbb{N}_0}$ under the risk neutral measure $Q$. |

We use the same notation as in chapter D. This means in particular that:

- The $\sigma$-algebras generated by $X_t$ are denoted by $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right)$.
- We assume, that the values of the shares in the portfolio are given by standard Brownian motions $W$. (Compare with Figure 3.5.).
- $\mathcal{G}_t$ denotes the $\sigma$-algebra generated by $W$ augmented by the $P$-null sets.
- $\mathcal{F}_t$ denotes the sigma algebra generated by $\mathcal{G}_t$ and $\mathcal{H}_t$.

This means that we assume, as in chapter D, that $(S_t)_{t \in \mathbb{N}_0}$ is adapted with respect to $(\mathcal{G}_t)_{t \in \mathbb{N}_0}$. We also assume that the annuity to be paid out (conditionally that $X_t = \star$) is previsible with respect to $(S_t)_{t \in \mathbb{N}_0}$, which can be interpreted in the sense that $R_n$ is determined by the sigma algebra $\mathcal{G}_{n-1}$ hence at the beginning of the respective period.

In order to make things simple we use a flat risk free interest rate of $r$ per time step and define $v = \frac{1}{1+r}$. This will help to make formulae simpler. The inclusion of an interest rate term structure does not pose additional complexities (except the formulae get more convoluted). Please note that we assume that the annuity $R_t$ is deducted and paid out at the beginning of the time interval $[t, t+1[$. Since we are looking at an annuity financed by a single premium, one would normally consider $R_0 = 0$. In the same sense we can model deferred annuities by setting $R_k = 0$ for $k < m$, where $m$ denotes the deferral period. We assume that in case of a lapse the policyholder can get hold of the residual fund value $FV_t$ but has no right to claim back a residual value of the GBWB contingency claim.

Before diving into the two distinct cases, lets look at the mechanics and interaction of the different processes. We first note that the annuity paid out at time $t$ has the form

$$I_\star(t) \times R_t,$$

where $I_\star(t)$ refers to definition 76. Hence we have the following recursion for the fund:

$$FV_{t+1} = \max\left\{0, \frac{S_{t+1}}{S_t} \times (FV_t - I_\star(t) \times R_t)\right\}.$$

Based on this we can now determine the contingency claim at time $t$ by

$$\begin{aligned} C_t &= \max\{0, (R_t - FV_t) \times I_\star(t)\} \\ &= \max\{0, (R_t - FV_t)\} \times I_\star(t)\}. \end{aligned}$$

As a next step we need to calculate the price of the GMWB guarantee by taking the expected present values under the risk neutral measure $Q$ of the contingency claims $(C_t)_{t\in\mathbb{N}_0}$. Hence we have the following:

$$\begin{aligned} \pi(GMWB) &= E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \max\{0, (R_k - FV_k)\} \times I_\star(k)\right] \\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[\max\{0, (R_k - FV_k)\} \times I_\star(k)\right] \\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[E^Q\left[\max\{0, (R_k - FV_k)\} \times I_\star(k)|\mathcal{G}_k\right]\right] \\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[\max\{0, (R_k - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right]. \end{aligned}$$

We note that the last equality holds be construction of $FV$ and $R$ and the definition of the sigma algebra $\mathcal{G}$. At this point things become interesting. Until now we did not assume stochastic independence between $\mathcal{G}$ and $\mathcal{H}$, as we did in chapter D. If we assume stochastic independence we can take $E^Q\left[I_\star(k)|\mathcal{G}_k\right]$ out of the outer expectation and get:

$$\begin{aligned} \pi(GMWB) &= E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \max\{0, (R_k - FV_k)\} \times I_\star(k)\right] \\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[\max\{0, (R_k - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k \in \mathbb{N}_0} v^k \, E^Q \left[\max\{0, (R_k - FV_k)\}\right] \times E^Q \left[I_\star(k)\right] \\
&= \sum_{k \in \mathbb{N}_0} v^k \, {}_k p_x \, E^Q \left[\max\{0, (R_k - FV_k)\}\right].
\end{aligned}
$$

This is exactly the same formula we reached in chapter D. In this case it is possible to first calculate the value of all the contingency claims assuming the person lives for ever and then weigh the respective values with the respective survival probabilities. This makes the calculation simpler. In practice one can not assume this stochastic independence, particularly when taking lapses in consideration. It is known that the lapse rates reduce more when the variable annuity guarantees are in the money. Normally one models this conditional lapse behaviour to determine $E^Q \left[I_\star(k) | \mathcal{G}_k\right]$ via the so called “in-the-moneyness”. Here one compares the ratio between the guaranteed value of the variable annuity and the underlying funds. The lower the funds value the more the guarantee is “in-the-money” and consequently a lower lapse rate. Conversely if the fund value is above the nominal guarantee value the policy is called “out-of-the-money” resulting in higher lapse rates.

We end this example with some figures which show the mechanics intrinsic to GMWB’s. Figure 8.1 illustrates the interaction between the fund value ($FV_k$ in red) and the corresponding expected contingency claims (in yellow) for one trajectory. One can see that the contingency payments decrease after the payment in proportion to the respective decrement table. In order to value the entire contingency claim one needs to look at the corresponding risk-neutral expected contingency claim cash flows (measure $Q$). Figure 8.2 shows the respective expected cash flows. For the example they have been simulated.

Please note that example 8.2 focuses on the GMWB part of a typical US variable annuity only. Normally this type of product is sold together with a GMDB part. The questions regarding the dynamic policyholder behaviour remain the same. It makes sense however to look at the following example which considers both parts together. In contrast to example 8.2, we will be slightly more concrete in the definition of the respective benefits and we will show the corresponding formulae.

## 8.3 Dynamic Policyholder Lapses for GMWB/GMDB Policy

We assume that this product is sold to $x$ year old man, which pays a single premium of $V_0$. In order to make things a little bit easier, we assume that all payments take place annually, and that the ratchet feature is also annual. This will allow us to use $t \in \mathbb{N}_0$ as time elapsed in years. The product has the following features:

- The fund management fees amount to $\beta = 1.5\%$ part of the fund value at the beginning of each period,

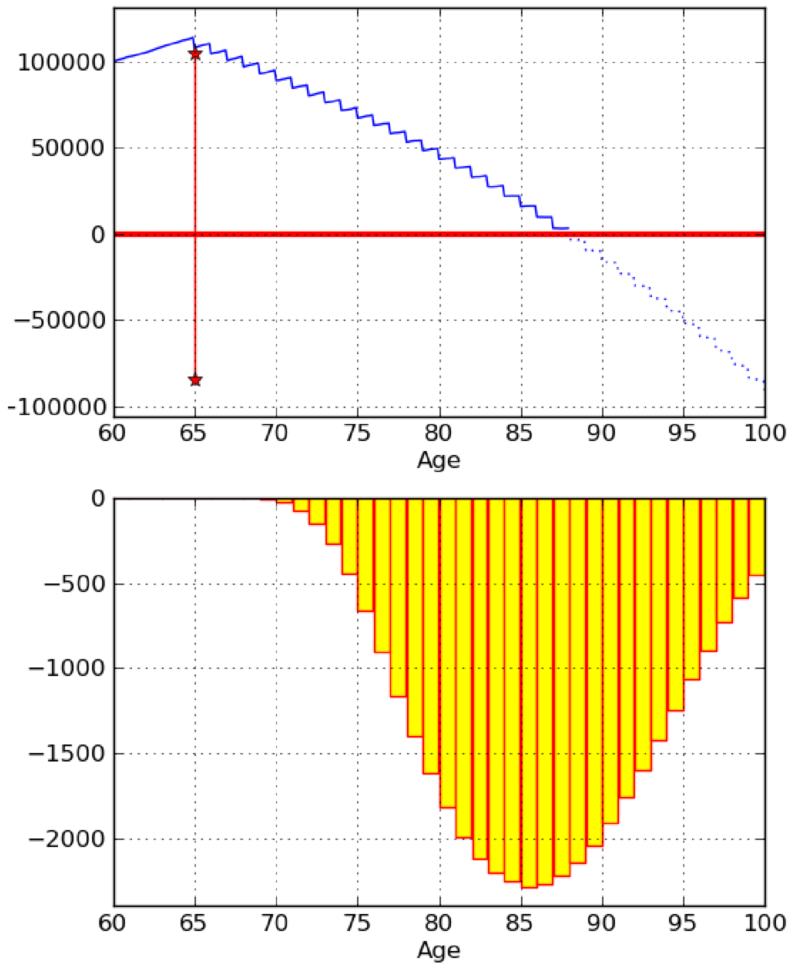


**Fig. 8.2** Example of GMWB contingency claims - expected value under $Q$

 

- The guarantee fee (for financing the variable annuity benefits) amounts to $\alpha = 0.9\%$ times the nominal value of the funds $\tilde{FV}_k$ to be charged at the beginning of the period.
- The annuity to be withdrawn equals $\gamma = 5\%$ of the nominal fund value $\tilde{FV}_k$ staring at $t = 1$ until the death of the policyholder.
- The GMDB benefits amount to the maximum of the nominal fund value and $FV_0$ and is paid in case of death before age $85$.
- The nominal fund value is ratcheted up by taking the maximum of the fund value of the previous year and the fund value at the end of the period before deduction of the charges mentioned before.
- The charges are deducted in the order of the list above.

We denote by $\pi(x)$ for $x \in \{$ "GMDB", "GMWB", "GP", "Total" $\}$ the risk neutral prices of the GMDB, GMWB and Guarantee Premium's respectively. "Total" represents the sum of the first three components. By $N_{\star\dagger}(k)$ we denote the event of the policyholder dying during period $k$, ie

$$N_{\star\dagger}(k) = I_\star(k-1)\, I_\dagger(k)$$

We will use all the notation of example 8.2. In case we have the following recursions

$$\begin{aligned}
FV_{t+1} &= \max\left\{0, \frac{S_{t+1}}{S_t} \times \Big(FV_t - I_\star(t) \times (R_t + \tilde{FV}_t \times (\alpha+\beta)\Big)\right\} \text{ and} \\
\tilde{FV}_{t+1} &= \max\left\{\tilde{FV}_t, \frac{S_{t+1}}{S_t} \times FV_t\right\}.
\end{aligned}$$

Based in the above definitions, we can now calculate $R_t = \gamma\, \tilde{FV}_t$. For the different contingency claims we define by $C_k^i$ for $i \in \{$ "GMDB", "GMWB", "GP", "Total" $\}$ the respective cash flows. We have the following relationships:

$$\begin{aligned}
C_t^{GMWB} &= \max\left\{0, \Big(R_t - (FV_t - \tilde{FV}_t \times (\alpha+\beta)\Big)\right\} \times I_\star(t), \\
C_t^{GMDB} &= \max\left\{0, \tilde{FV}_t - FV_t\right\} \times N_{\star\dagger}(t), \\
C_t^{GP} &= -\min\left\{FV_t, \tilde{FV}_t \times \alpha\right\} \times I_\star(t)
\end{aligned}$$

and $C_k^{total}$ being the sum of the three other ones. Now the standard argument as above applies also to this more general situation:

$$\pi(i) = E^Q \left[ \sum_{k \in \mathbb{N}_0} v^k \, C_k^i \right],$$

for $i \in \{$ "GMDB", "GMWB", "GP", "Total" $\}$. We refine the formula for each of the three basis types as follows:

$$\pi(GMWB) = \sum_{k \in \mathbb{N}_0} v^k \, E^Q \left[ \max\{0, (R_k - FV_k)\} \times E^Q \left[ I_\star(k) | \mathcal{G}_k \right] \right],$$

$$\pi(GP) = - \sum_{k \in \mathbb{N}_0} v^k \, E^Q \left[ \min \left\{ FV_k, \tilde{FV}_k \times \alpha \right\} \times E^Q \left[ I_\star(k) | \mathcal{G}_k \right] \right], \text{ and}$$

$$\pi(GMDB) = \sum_{k \in \mathbb{N}_0} v^k \, E^Q \left[ \max \left\{ 0, \tilde{FV}_k - FV_k \right\} \times E^Q \left[ I_\star(k-1) | \mathcal{G}_k \right] \right] \times q_{x+k},$$

where we have assumed for the GMDB that mortality is independent from capital markets. We finally remark that the last formula for the GMDB benefit is equivalent to the same formula in chapter D, if we assume stochastic independence. In this case we get:

$$\pi(GMDB) = \sum_{k \in \mathbb{N}_0} {}_kp_x \times q_{x+k} \, v^k \, E^Q \left[ \max \left\{ 0, \tilde{FV}_k - FV_k \right\} \right].$$

**Exercise 58** *Calculate the present value of each of the contingency claims for example 8.3 by simulation.*

## 8.4 Model for dynamic lapses

As we have seen, the lapses - ie the conditional probabilities for $X$ to move from $\star$ to $\ddagger$ are dependent on $\mathcal{G}_t$. Normally this is modeled in practice by the definition of the lapse rates via the "in-the-moneyness" (itm), which is defined as $itm = 1 - \frac{FundValue}{ValueofGuarantee}$. This means that the higher the guarantee relative to the funds value, the higher the "in-the-moneyness". A policy which is deep in the money (eg $itm \approx 100\%$) means that the guarantee is very valuable and in turn less people lapse. Figure 8.3 shows the lapse level in function of duration ($d$) of the policy for different levels of $itm$, and we can observe the above mentioned relationship. A higher "in-the-moneyness" results in a higher persistency of the policyholders, eg more policyholder will stay up to older ages and in consequence the variable annuity guarantee becomes more valuable. Figure 8.5 shows this effect. It is important to understand how this dynamical lapse assumption is determined. In principle one

 

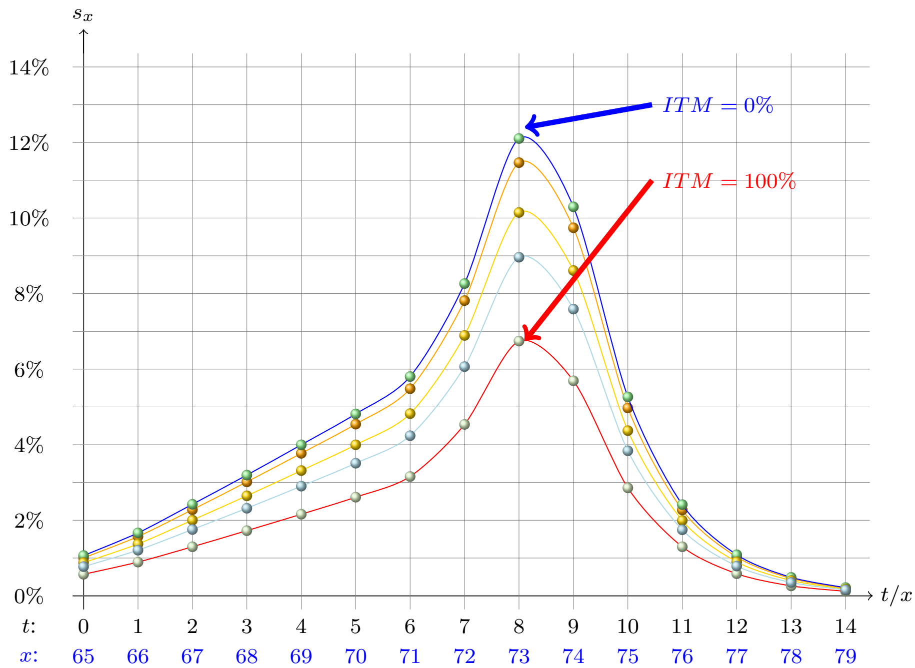


**Fig. 8.3** Dynamic Lapses

has to start with observed policy data and one has to estimate the lapse probability for a period based on a set of explanatory variables. The aim is to find a dynamic lapse function which depends on these explanatory variables. To fit these one can choose a standard statistical methodology. The examples presented here are based on a realistic model portfolio and the underlying analysis has been performed using general additive models ("GAM") using a so called "logit" link:

$$\text{logit}(x) = \ln\left\{\frac{x}{1-x}\right\}, \text{ for } x \in ]0,1[.$$

This means that the logit transformation maps the parameters for a binomial distribution into a linear space, where we can determine the underlying probabilities. In the context of the dynamic lapse function above we have used a "GAM" model, where each of the dependent variables was used via a cubic spline. Hence we have the following relationship:

$$\text{logit}(s_x) = c + s_1(x) + s_2(d) + s_3(itm),$$

 

where $c \in \mathbb{R}$ denotes the intercept, and where $s_i$, $i \in \{1, 2, 3\}$ are the three cubic splines representing the impact of age ("$x$"), duration ("$d$") and "in-the-moneyness" ("$itm$"). Based on the intercept and the three splines it is then possible to calculate the yearly lapse rate ("$sx$") by application of the inverse logit function:

$$\text{logit}^{-1}(x) = \frac{e^x}{1 + e^x}.$$

Instead of showing the three splines, we separately show in figure 8.4 the impact of changes in age and duration. We have set the 100% level at age 73. This is the age where most lapses occur, as a consequence of tax rules in the country where these variable annuities are offered[1].

Besides the actual model it is also important to discuss possible explanatory variables. The following table lists some possible explanatory variables together with a reason why this could be relevant. The underlying statistical analysis normally shows, that some of the discussed variables are more or less relevant for the concrete context. This has to do with the fact that the dynamic lapse behaviour is dependent on the tax environment and also on the general structure of the underlying products. Hence it is *dangerous* to just take some assumption which is readily available without checking its suitability for the concrete application!

| Quantity | Description |
|---|---|
| itm | The "in-the-moneyness" drives the lapse behaviour considerably, hence is present in most dynamic lapse assumptions for variable annuities. |
| Duration | Lapse behaviour mainly depends on the duration of a contract, since in a lot of countries tax benefits emerge only if the money is for a certain minimal time in an insurance policy. |
| Age | Evidence suggests that lapses are normally dependent on age, tending to decrease for old ages. |
| Gender | Males and females often show different lapse behaviour. |
| Product | Lapse rates are normally quite dependent on the underlying product, also because different products have different surrender charges and tax treatments. |
| Interest Rate | In particular for GMWB one would expect that lapse rates decrease in a low interest environment because the underlying variable annuity guarantees get more costly. |
| Volatility | Implied volatility (such as the VIX index) tracks the implied volatility. Again economics would suggest that policyholders lapse less in a high volatility environment as a consequence of the increased value of the underlying guarantees. |

[1] Note that the lapse rates shown do *not* represent a concrete existing portfolio. Hence they must not be taken to value a concrete variable annuity portfolio

 

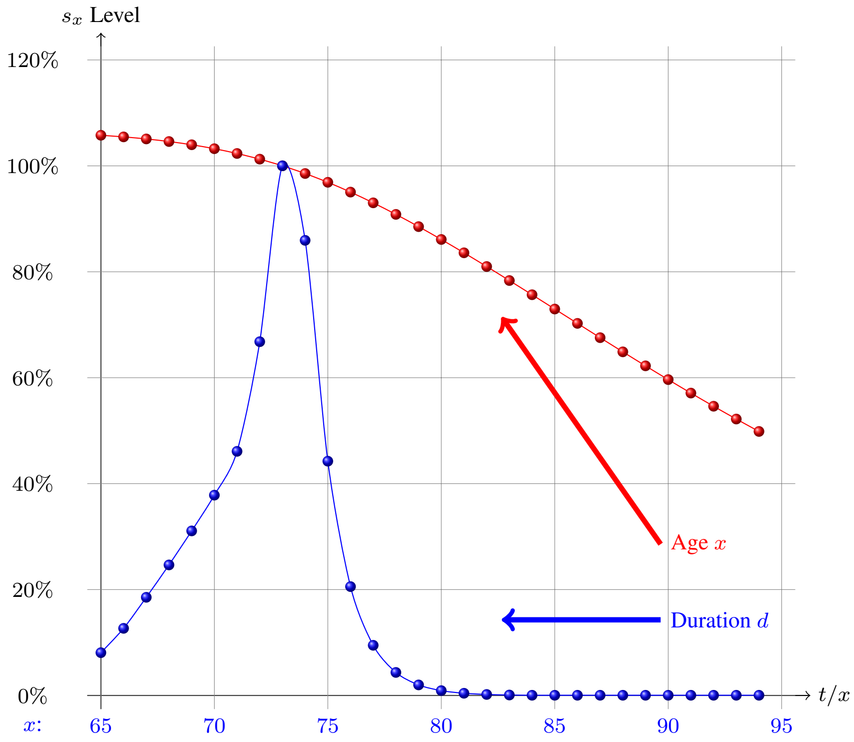


**Fig. 8.4** Impact of age and duration on lapses

The above list does not aim to be comprehensive and there are other factors which should be considered. What is quite clear is the underlying analysis is quite complex and it is often advisable to use a statistical software packet such as "R" to do the corresponding analyses. We end this example by remarking that based on the dynamic lapse assumptions from period to period, it is possible to construct the corresponding decrements ($E^Q\left[I_\star(k)|\mathcal{G}_k\right]$). Since the decrements are actually path dependent, it is rather difficult to plot them. Assuming a constant $itm$ level it is however possible to calculate the decrements. Figure 8.5 shows such an example for a policy with GMWB for a $x = 65$ year old man. We remark the underlying GAM model depends on the following explanatory variables: duration, age and "in-the-moneyness".

Before moving to utilisation it is worth noting that the inclusion of dynamic lapses versus flat lapse assumptions means that deeply in-the-money policies become more expensive and that in turn a higher amount of hedging is needed. Figure 8.6 shows this effect. We conclude that the inclusion of dynamic lapse assumption is essential in modeling variable annuities.

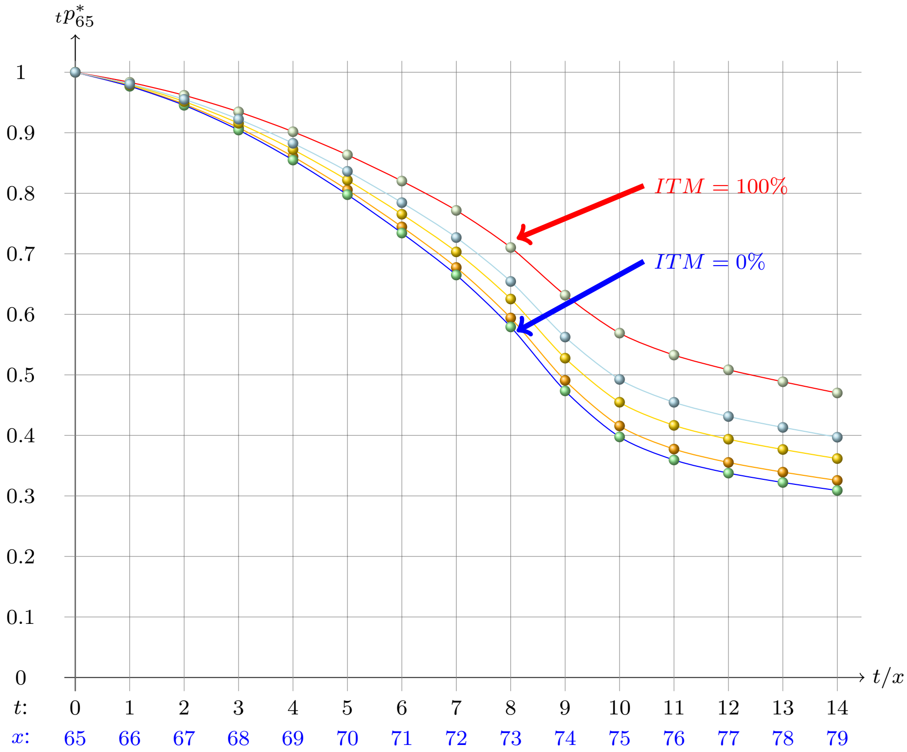


**Fig. 8.5** Decrement using Dynamic Lapses

## 8.5 Modelling of Utilisation for GMWB

In example 8.2 we have seen the importance of modeling the lapses dynamically. In reality variable annuities are more complex than the example would suggest. In this section we want to focus on a product where utilisation depends on policyholder behaviour. In example 8.2, we used the following set up for $R$:

$$I_\star(t) \times R_t,$$

with the assumption that $R_t$ depends on the financial market only (ie. $\mathcal{G}$ previsible). This covers all the cases where the annuity is deterministic dependent on the capital market and where the start of the annuity payment is already known at inception. In a lot of American GMWB's the policyholder has the option to start withdrawing at a point in time which suits them and they also have the right to use only a part of the maximal withdrawal benefit. If we denote with $\psi_k \in [0, 1]$ the ratio which the Policyholder chooses to with withdraw, the above formula has to be modified to

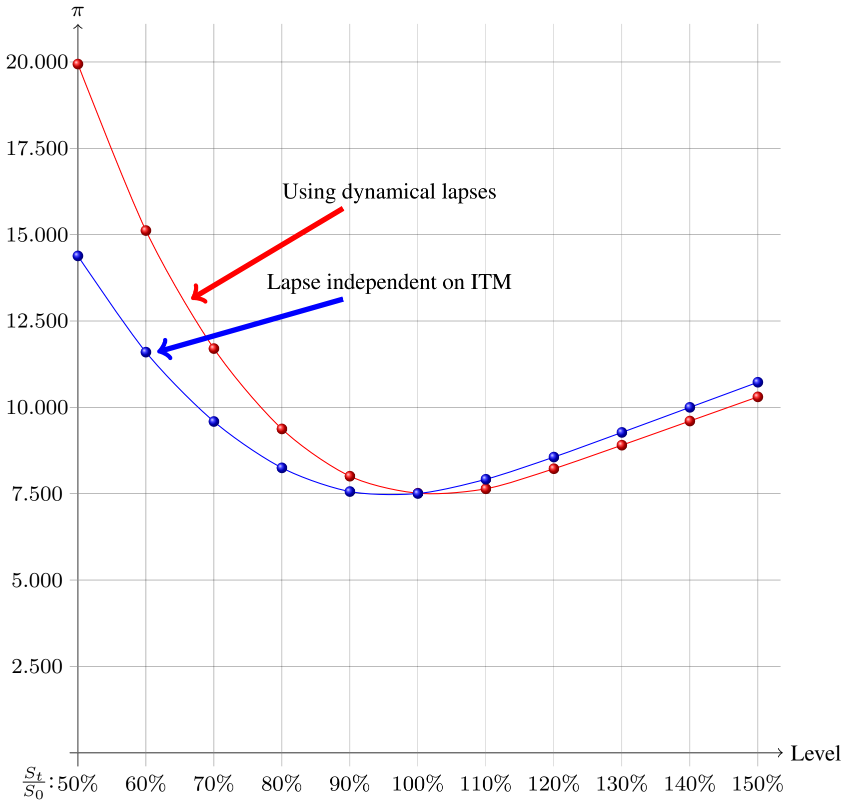


**Fig. 8.6** Comparison of trading grid with and without dynamic lapses

$$I_\star(t) \times \psi_t \times R_t.$$

The process $(\psi_k)_{k \in \mathbb{N}_0}$ is called the utilisation process. Again, this process is normally modeled in a very similar manner as the dynamic lapse assumption by modeling it with a suitable set of explanatory variables. This time however the process may be more complex. It can for example be seen that once the policyholder starts to withdraw the likelihood that he reduces utilisation is quite low and in turn $\psi_{t-1}$ is a good predictor for $\psi_t$. One possibility to model the process $\psi$ is to assume stochastic independence of utilisation process from capital market variables (represented by $(\mathcal{G}_t)_{t \in \mathbb{N}_0}$). In this case one could choose a Markov process (see section **??**) using a discretised version of the observed utilisation. Denote with $S = \{0\%, 25\%, 50\%, 75\%, 100\%\}$ the underlying state space, and denote $\psi_t$ the Markov chain on $S$. We note that $t \in \mathbb{N}_0$ denotes the actual policy year and identifies it with the respective ages. If we assume that the current policyholder's age is $x_0$

and that the policyholder withdraws currently $i \in S$, it is then possible to calculate the probability $P_{ij}(0, t0)$ to withdraw $j \in S$ at age $x_0 + t$ by the standard approach using the Chapman-Kolmogorov equation, eg

$$P(s, u) = P(s, t)\, P(t, u),$$

for $s \leq t \leq u$. It is worth noting that it makes sense to use a time-inhomogeneous Markov chain, because it can be expected that older people normally behave in average differently from younger policy holders. Reconsidering example 8.2, we see that we need to calculate the following quantity:

$$\begin{aligned}\pi(GMWB|\psi_o = i) &= E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \, \max\{0, (R_k \times \psi_k - FV_k)\} \times I_\star(k)|\psi_o = i\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[E^Q\left[\max\{0, (R_k \times \psi_k - FV_k)\} \times I_\star(k)|\mathcal{G}_k\right]|\psi_o = i\right].\end{aligned}$$

Since we have assumed stochastic independence of $\psi$ from the capital market variables, we need to first calculate

$$\begin{aligned}E^Q\left[\psi_t|\mathcal{G}_t\right] &= E^Q\left[\psi_t\right]\\ &= \sum_{j\in S} j \times p_{ij}(0, t).\end{aligned}$$

Moreover if we assume that $\psi_t$ and $I_\star(t)$ are independent, we can calculate $\pi(GMWB)$ as follows:

$$\pi(GMWB|\psi_o = i) = E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \, \max\{0, (R_k \times \psi_k - FV_k)\} \times I_\star(k)\big|\psi_o = i\right]$$

$$\begin{aligned}&= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[E^Q\left[\max\{0, (R_k \times \psi_k - FV_k)\} \times I_\star(k)|\mathcal{G}_k\right]|\psi_o = i\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q\left[\max\{0, (R_k \times E[\psi_k|\psi_o = i] - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k \, E^Q[\max\{0, (R_k \times \{\sum_{j\in S} j \times p_{ij}(0, t)\} - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]].\end{aligned}$$

Finally we look at an abstract example of the above concept. We assume for the sake of simplicity a time-homogeneous Markov chain and we consider a $x = 65$ year old man. To model the transition matrix $P(1)$ we assume the following:

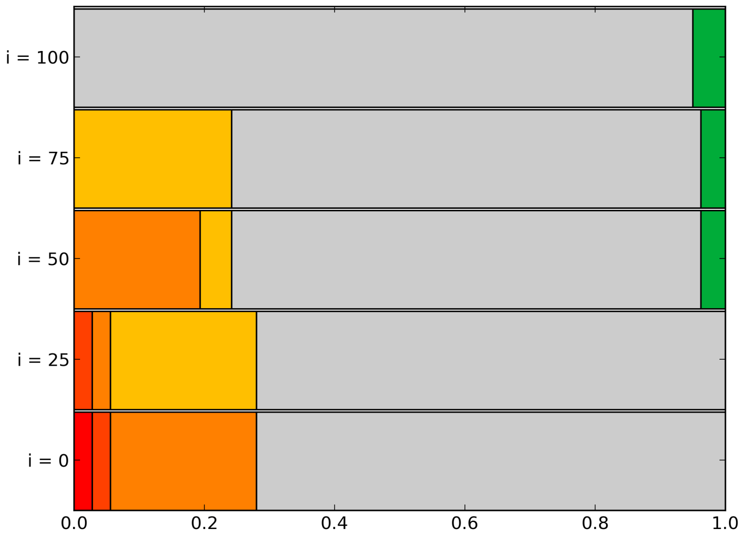


**Fig. 8.7** Decomposition of $P(1)$ into its parts

1. After 5 years 80% of the policyholders withdraw.
2. For each year the percentage of people not taking higher withdrawals and withdrawing one step less (eg 50% instead of 75%) is 5%, the reminder (95%) of the policyholders not withdrawing more remains at the same level.
3. After starting to withdraw we assume that after 5 years, 75% withdraw more.

Note that the above set of assumptions do not allow determining $P(1)$. Which other assumptions have been made (excise), in order for the assumptions result in the following one year transition matrix $P(1)$? Figure 8.7 shows this matrix in graphical form. It shows for each state $i \in S$ the decomposition of the probabilities $\{p_{ij}(1) | j \in S\}$. We know that $\sum_{j \in S} p_{ij}(1) = 1$ and we use reddish colors for transitions with $j < i$ (ie higher utilisation in the next period), gray for the probability $p_{ii}(1)$ and greenish colors for $j > i$ (ie lower utilisation in the next period).

| | 100% | 75 % | 50% | 25% | 0% |
|---|---|---|---|---|---|
| 100% | 0.95 | 0.05 | – | – | – |
| 75% | $1-\alpha$ | $0.95\alpha$ | $0.05\alpha$ | – | – |
| 50% | $0.8\,(1-\alpha)$ | $0.2\,(1-\alpha)$ | $0.95\alpha$ | $0.05\alpha$ | – |
| 25% | $0.1\,(1-\beta)$ | $0.1\,(1-\beta)$ | $0.8\,(1-\beta)$ | $\beta$ | – |
| 0% | $0.1\,(1-\beta)$ | $0.1\,(1-\beta)$ | $0.8\,(1-\beta)$ | – | $\beta$ |

with

$$\begin{aligned}
\alpha &= \sqrt[5]{1-75\%}\\
&\approx 0.758\\
\beta &= \sqrt[5]{1-80\%}\\
&\approx 0.724
\end{aligned}$$

Figure 8.8 shows the utilisation for the Markov model specified above. As in reality, we can observe in figure 8.8 that the utility increases for a closed book of business. For example we have assumed that the Markov chain is time homogeneous and we would like to remark, that in reality an accelerated increase of the utilisation for older ages would occur.

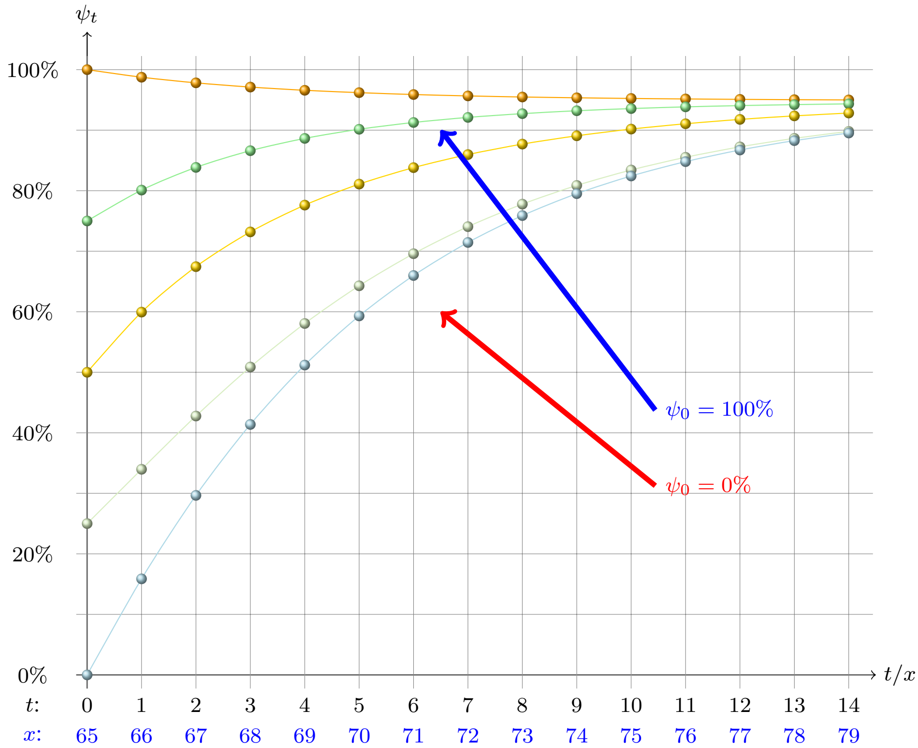


**Fig. 8.8** GMWB utilisation over time

We end this section with a concrete example of the Markov chain model, which we have introduced before.

**Example 59** *We consider the following variable annuity product consisting of a GMDB / GMWB combination:*

**Product definition:** *We assume a GMWB product where the benefit base is defined via a ratchet of the funds value, in the sense that at certain periods the benefit base can ratchet up to the funds value. Concretely we consider a product where the ratcheting is either quarterly, half-yearly or yearly. To this end we introduce the following notation:*

$$\begin{aligned}
FV(t) &\quad \textit{denotes fund value at time } t \textit{ (before withdrawal),}\\
GV(t) &\quad \textit{denotes benefit base (guaranteed) value at time } t\textit{,}\\
R(t) &\quad \textit{annuity paid at time } t\\
\psi(t, t+\Delta t) &\quad \textit{denotes the fund performance from time } t \textit{ to } t+\Delta t\textit{, and}\\
\mathfrak{T} \subset \mathbb{R}^+ &\quad \textit{denotes the set of times at which a ratchet takes place.}
\end{aligned}$$

*In this set up we have the following (assuming for sake of simplicity that $\mathfrak{T} \subset \{k \times \Delta t | k \in \mathbb{N}_0\}$ and also that annuity payments only take place at direct times $k\,\Delta t$ for some $k \in \mathbb{N}_0$):*

$$\begin{aligned}
FV(0) &= EE > 0\\
GV(0) &= FV(0)\\
FV((k+1)\Delta t) &= (FV(k\,\Delta t) - R(k\,\Delta t))\,\psi(k\,\Delta t, (k+1)\,\Delta t)\\
GV((k+1)\Delta t) &= \begin{cases} \max\left(GV(k\Delta t), FV((k+1)\Delta t) - R((k+1)\,\Delta t)\right) & \textit{if } (k+1)\,\Delta t \in \mathfrak{T},\\ GV(k\Delta t) & \textit{else.}\end{cases}
\end{aligned}$$

*The death benefit is defined as the maximum of the current funds value and the difference between the current $GV(t)$ and the annuities paid out until this point in time (ie $\sum_{k \in \mathbb{N}_0}^{k\,\Delta t \leq t} R(k\,\Delta t)$) until age 85. Afterwards there is no death benefit.*

**Annuity Definition:** *The annuity can be withdrawn at times $\mathfrak{S} \subset \{k \times \Delta t | k \in \mathbb{N}_0\}$ and it amounts at time $t \in \mathfrak{S}$ to $\rho(\xi_0) \times GV(t)$, where $\xi_0$ is the first time $\xi_0 \in \mathfrak{S}$ where the person can withdraw. The person is allowed to withdraw less than this amount in line with the model as defined before.*

**Fund dynamics:** *The fund performance follows a geometric Brownian Motion with a risk free interest rate of 2.5% and a volatility of 16.5%.*

**Policy:** *We assume a 65 year old policyholder which invests 100'000 \$.*

*Based on this product two different values for the guarantee have been calculated, namely the guarantee based on the best estimation parameters for the Markov chain $X_t$ and with respect to $X_t$ but assuming that the policyholder will never reduce the utilisation. The following table shows the respective result for the best estimate utilisation:*

| **GMDB** | | | |
|---|---|---|---|
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *2855.3* | *3432.0* | *4572.3* |
| *Half-yearly* | *2650.9* | *3193.0* | *4257.1* |
| *Yearly* | *2450.5* | *2904.1* | *3888.6* |
| **GMWB** | | | |
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *4439.0* | *4321.6* | *4392.1* |
| *Half-yearly* | *4219.0* | *4107.4* | *4182.7* |
| *Yearly* | *3966.0* | *3744.4* | *3820.7* |
| **Total** | | | |
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *7294.3* | *7753.6* | *8964.5* |
| *Half-yearly* | *6869.9* | *7300.5* | *8439.9* |
| *Yearly* | *6416.5* | *6648.5* | *7709.3* |

*If we assume in a next step a more conservative utilisation assumption, where the policyholder does not reduce its consumption we get the following results:*

| **GMDB** | | | |
|---|---|---|---|
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *2256.0* | *2874.7* | *4196.2* |
| *Half-yearly* | *2101.9* | *2663.6* | *3908.4* |
| *Yearly* | *1928.9* | *2428.5* | *3563.6* |
| **GMWB** | | | |
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *8378.6* | *8043.2* | *8069.0* |
| *Half-yearly* | *7989.1* | *7637.0* | *7623.3* |
| *Yearly* | *7470.0* | *7065.1* | *7037.9* |
| **Total** | | | |
| *Start state* | *100%* | *50%* | *0%* |
| *Ratchet* | | | |
| *Quarterly* | *10634.6* | *10918.0* | *12265.3* |
| *Half-yearly* | *10091.0* | *10300.7* | *11531.8* |
| *Yearly* | *9398.9* | *9493.7* | *10601.5* |

 

*We see for both Markov chain models that the value of the variable annuity guarantee depends on the initial state. Figure 8.9 shows an example of the respective cash flows for the best estimate version. Finally figure 8.10 shows the response function with respect to both models. The x-axis shows the relative equity level together with the values of the GMWB (red) and GMDB (blue) cover for the best estimate cash flows. The black line represents the total value of the variable annuity under the maximum utility model. It can be clearly seen that this second assumption increases the price of the variable annuity and also the hedging costs. The latter one can be seen by comparing the relative changes in value for the two models for example for a -40% downwards stress (at eg going down from* 1.0 *to* 0.6*).*

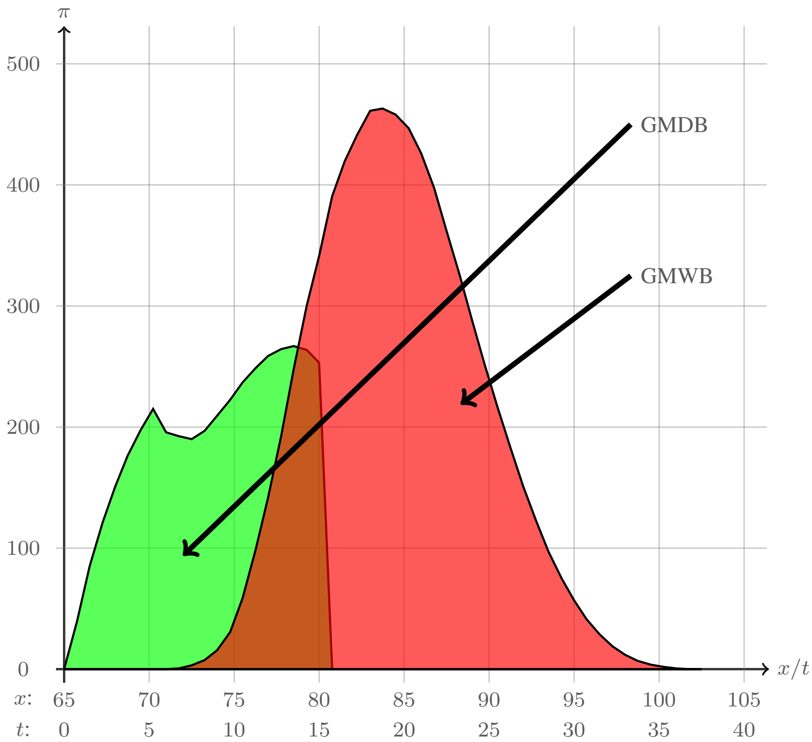


**Fig. 8.9** Risk neutral cash flows for quarterly ratchet and $\psi_0 = 50\%$ and $x = 65$

Based on this last example we can now look at the different response functions of the various types of ratchets and states. Figure 8.12 compares them for a 60 year old policyholder. We see that for a downward stress (eg equity levels $< 1$), the different types of ratchets start to convert to the same value. This is due to the reduced likelihood of a speedy recovery to allow the policy to ratchet up again. For the upward stress we can see that all the nine different combinations yield to different liability

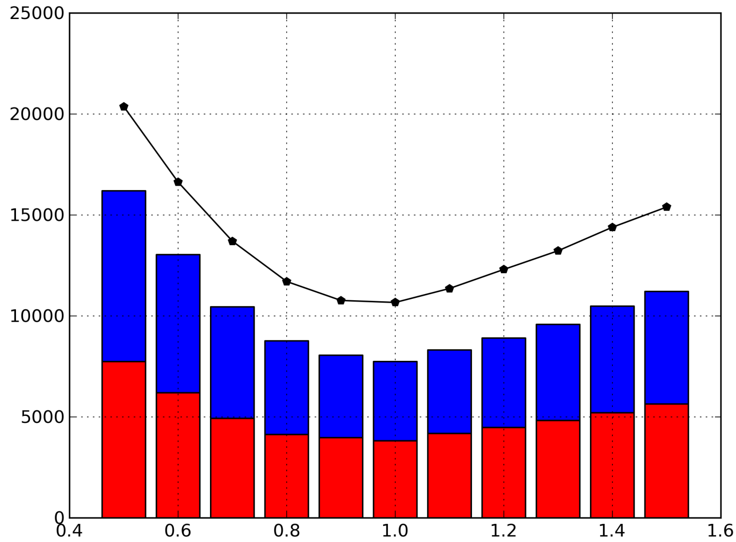


**Fig. 8.10** Trading Grid for quarterly ratchet and starting state 50%

values (and also to different hedging requirements). It is also worth pointing out that there is considerable dependency on the starting state of the Markov model – the value of the variable annuity guarantee varies depending on whether the policyholder starts to immediately withdraw his funds. We see that the policyholder who starts withdrawing immediately has the highest hedging requirement for material downwards stresses. At the same time the value of the variable annuity guarantee is the smallest (with respect to the other cases) for a market which has a strong positive performance. These results are in line with the comments of the table in section 8.1.

We note it is quite simple to combine the effects of dynamic lapsing and utilisation. The simpler possibility is to assume that these two effects are indecent and to calculate the hedge liability this way. It is also possible to estimate (and then simulate) a conditional Markov chain for utilisation and lapse, conditional to explanatory variables such as product type, in-the-moneyness etc. If choosing this approach, one needs to consider the state ‡ (lapse) explicitly as part of the Markov chain.

**Example 60** *We want to end this section by providing a link between the deterministic function $\psi_t$ and the class of Markov models introduced previously. Such a link may be useful in interpreting deterministic utilisation assumptions $(\psi_t)_{t\in\mathbb{N}_0^+}$. Such assumptions can be based either on expert judgment or observed utilisation pattern. In order to illustrate this link we use a Markov model with two states, namely "N" for the inforce which is not utilising at a certain age and "U" for the inforce utilising. We assume that each policyholder either utilises at 100% or not at all, noting*

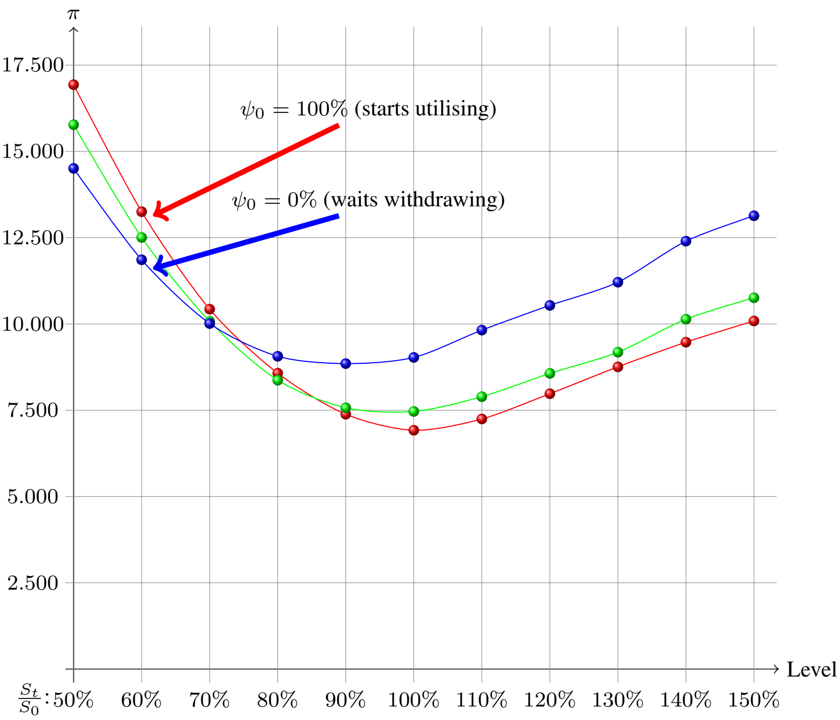


**Fig. 8.11** Comparing Trading Grids for 65 year old Policyholder (different states)

*that these 4 model assumptions can easily be generalised. Furthermore we assume that we have given an utilisation pattern $(\psi_t)_{t\in\mathbb{N}_0^+}$ and the task is to determine the Markov probabilities intrinsic to these assumptions. Assume for example:*

| $x$ | $\psi_x$ | $\psi_{x+1}$ | $\psi_{x+2}$ | $\psi_{x+3}$ | $\psi_{x+4}$ |
|---|---|---|---|---|---|
| *65* | *0.2500* | *0.3354* | *0.4207* | *0.5061* | *0.5914* |
| *70* | *0.7518* | *0.7663* | *0.7663* | *0.7663* | *0.7663* |
| *75* | *0.7663* | *0.7663* | *0.7963* | *0.7963* | *0.7963* |
| *80* | *0.7963* | *0.7963* | *0.7963* | *0.7963* | *0.7963* |
| *85* | *0.7963* | *0.7963* | *0.7963* | *0.7963* | *0.7963* |
| *90* | *0.7963* | *0.7963* | *0.7963* | *0.7963* | *0.7963* |

*Hence the task is to determine the initial distribution of the states and the respective transition probabilities. In order to do this we denote with $w_0(x)$ and $w_1(x)$ the probability distribution of the Markov chain at age $x$. Furthermore we denote with $r(x)$ and $q(x)$ the probabilities at age $x$ to not utilising. Hence we have:*

$$p_{00}(x) = r(x),$$

 

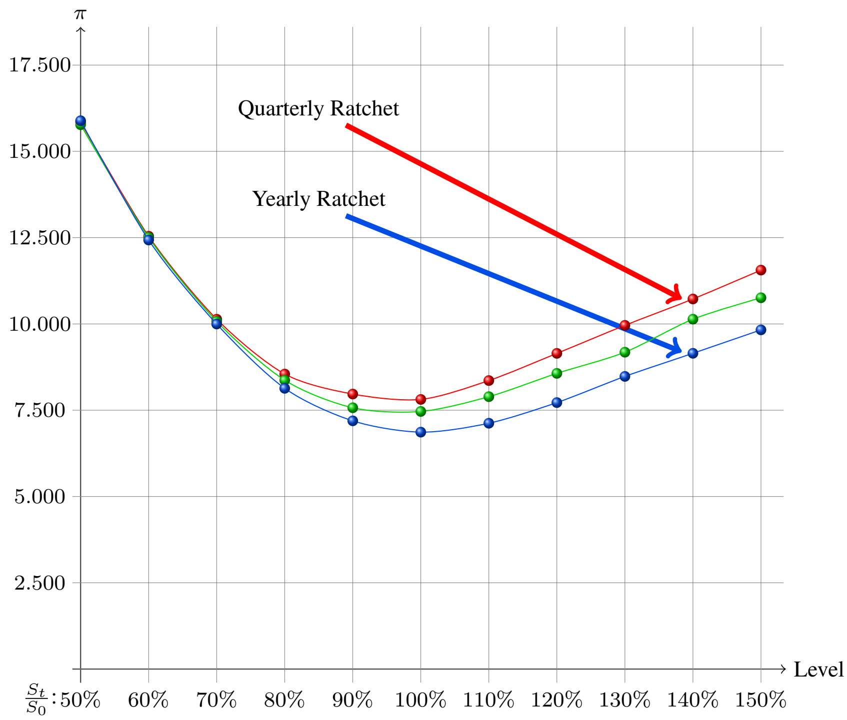


**Fig. 8.12** Comparing Trading Grids for 65 year old Policyholder (different ratchets)

$$p_{01}(x) = 1 - r(x),$$
$$p_{10}(x) = q(x), \textit{ and}$$
$$p_{11}(x) = 1 - q(x).$$

*We also denote by*

$$P(x) = (p_{ij}(x))_{(i,j)\in S\times S}$$

*the corresponding transition matrix. By definition of the transitions matrix we know that*

$$(w_0(x+1), w_1(x+1)) = (w_0(x), w_1(x)) \cdot P(x),$$

*or in coordinate form*

$$w_0(x+1) = w_0(x) \times p_{00}(x) + w_1(x) \times p_{10}(x), \text{ and}$$
$$w_1(x+1) = w_0(x) \times p_{01}(x) + w_1(x) \times p_{11}(x).$$

*We see that there are two unknown variables for each age, namely $r(x)$ and $q(x)$ and one boundary condition $\psi_x$. Therefore the equation system will be under-determined and we therefore need an additional assumption. Based on the estimation of the transition matrices and the raw inforce data, it is reasonable to assume $q(x) = 5\% \, \forall x$. With this additional assumption we can now solve the problem, by recognising that $\psi$ can be expressed by $\mathbf{w}$ via*

$$\begin{aligned}\psi_x &= 0 \times w_0(x) + 1 \times w_1(x)\\ &= w_1(x)\end{aligned}$$

*Therefore we immediately get the boundary condition (for age 65)*

$$w_0(65) = 0.75, \text{ and}$$
$$w_1(65) = 0.25.$$

*By means of the recursion we can now solve for $r(65)$ again considering the needed boundary condition for $\mathbf{w}(66)$:*

$$\begin{aligned}0.6646 = w_0(66) &= 0.75 \times r(65) + 0.25 \times q(65),\\ 0.3354 = w_1(66) &= 0.75 \times (1 - r(65)) + 0.25 \times (1 - q(65)), \text{ and}\\ q(65) &= 0.05.\end{aligned}$$

*This results in $r(65) \approx 87\%$. Figure 8.13 shows the values for $1 - r(x)$ and we can observe two marked spikes in the utilisation uptake probability $(1 - r(x))$ at ages of about 70 and 75, which stem from the tax treatment and the structure of the underlying product. The following table shows the numerical values when solving for $r$, $q$ and $\mathbf{w}$ respectively:*

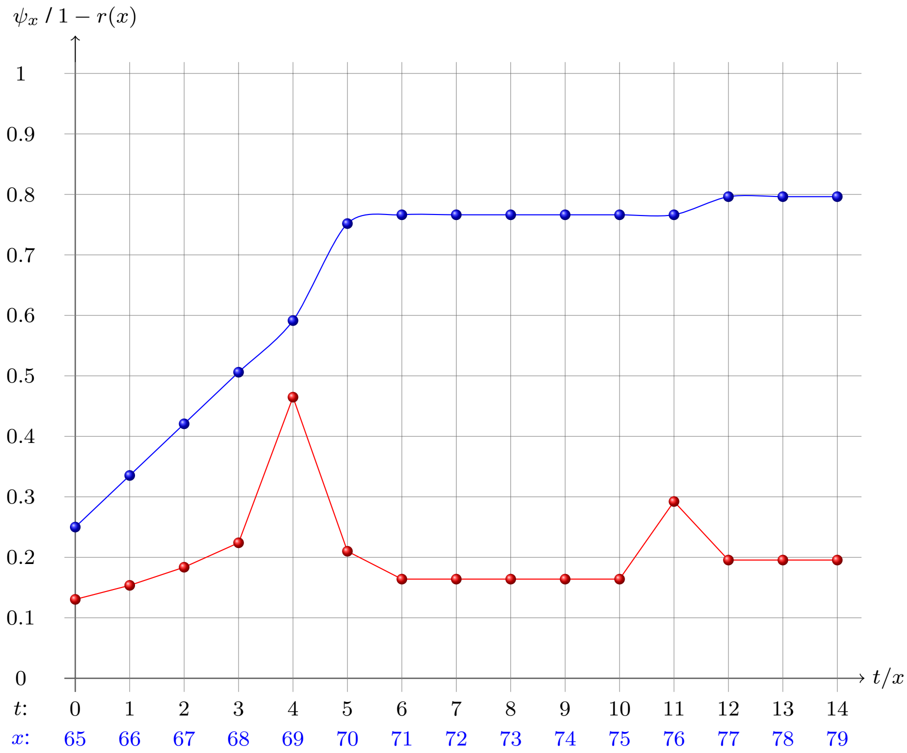


**Fig. 8.13** Implicit Markov Transition Probability $r(x)$

| $x$ | $\psi_x$ | $q(x)$ | $r(x)$ | $w_0(x)$ | $w(1)$ |
|---|---|---|---|---|---|
| *65* | **0.2500** | *0.0500* | *0.8695* | *0.7500* | **0.2500** |
| *66* | *0.3353* | *0.0500* | *0.8463* | *0.6646* | *0.3353* |
| *67* | *0.4207* | *0.0500* | *0.8163* | *0.5793* | *0.4207* |
| *68* | *0.5060* | *0.0500* | *0.7759* | *0.4939* | *0.5060* |
| *69* | *0.5914* | *0.0500* | *0.5351* | *0.4086* | *0.5914* |
| *70* | *0.7517* | *0.0500* | *0.7899* | *0.2482* | *0.7517* |
| *71* | *0.7663* | *0.0500* | *0.8360* | *0.2337* | *0.7663* |
| *80* | *0.7963* | *0.0500* | *0.8045* | *0.2037* | *0.7963* |
| *90* | *0.7963* | *0.0500* | *0.8045* | *0.2037* | *0.7963* |

 

## 8.6 Other types of policyholder behaviour

Besides lapse and utilisation there are some more types of policyholder behaviour which we mention mainly for the sake of completeness. It is important to understand policyholder behaviour in the context of the product, since policyholder behaviour impact is intrinsically linked with the way the product works. From this point of view the treatment in this section is necessarily limited. We also note that the techniques that we have seen previously can also be applied in such a context as "mutatis-mutandis", eg one needs to change the things that need to be changed.

The following lists some types of policyholder behaviour and the corresponding impact:

**Demographic anti-selection:** Under demographic anti-selection we understand that people buy certain products because they have a better understanding of their health status than an insurance company. Assume you have a GMWB benefit for life, where people can elect to withdraw their money or to leave it with the company. If a person perceives himself particularly healthy he will most likely opt for a GMWB benefit for life, because he is of the opinion that he will get this benefit cheaply. On the other hand a rather ill person will buy a GMDB cover in order to protect their inheritance. This will lead to an anti-selection, which is also present in most traditional products. In order to avoid this arbitrage opportunity it is essential to design the products in such a way that the financial benefit of such an arbitrage remains small. Moreover for death benefits it is normal that a material increase in the level of death benefits trigger a medical examination of the policyholder. A good example of a benefit design which is rather immune for arbitrage is a guaranteed annuity in payment. Assume a 65 year old policyholder has purchased an immediate payout annuity which is guaranteed for 15 years. One year later he decides to repurchase (lapse) the policy. The canonical method to avoid arbitrage is to pay back the guaranteed part of the annuity (eg after a year the 14 year guaranteed part of the annuity). After this partial lapse the policyholder will still receive a 14 year deferred annuity. Hence if he survives the remaining 14 years of the guarantee, he will receive the respective annuity payments. This product design is rather immune against anti-selection when ill people try to lapse the policy.

**Change in asset allocation:** Some variable annuity writers offer the possibility to switch between investments funds. We have seen that the price of the variable annuity guarantee is a function of the underlying volatility ($\sigma$) of the fund. If the guarantee fee for the variable annuity does not depend on the actual asset allocation of the policyholders funds, the insurance company has to bear the corresponding policyholder behaviour risk. Concretely, the values of the variable riders will increase if the policyholder chooses more risky funds. Again this type of risk is not existent for all types of variable annuities and it can be (partially) migrated by limiting the funds' choice or by charging the effective economic price of the rider in function of the concrete asset allocation.

**Top up's:** In many variable annuity products the policyholder has the ability to pay in additional funds at his digression and he can potentially benefit from the pricing level at inception of the policy, also in case the economic environment has changed. Assume for example that the pricing level has risen since inception of the policy due to a higher level of volatility and lower interest rates. In this case it would be economically beneficial for the policyholder to invest additional money since he would get the guarantee cheaper.

**Partial Withdrawals:** Partial withdrawals are the converse of top-ups. Here also it might be economical for the policyholder to partially withdraw money at certain times. Again this question is best addressed (as with top-ups) by an adequate product design.

**Premium holidays:** Typically, variable annuities are single premium products. In case they are financed by a regular premium, the policyholder also has the possibility to take premium holidays.

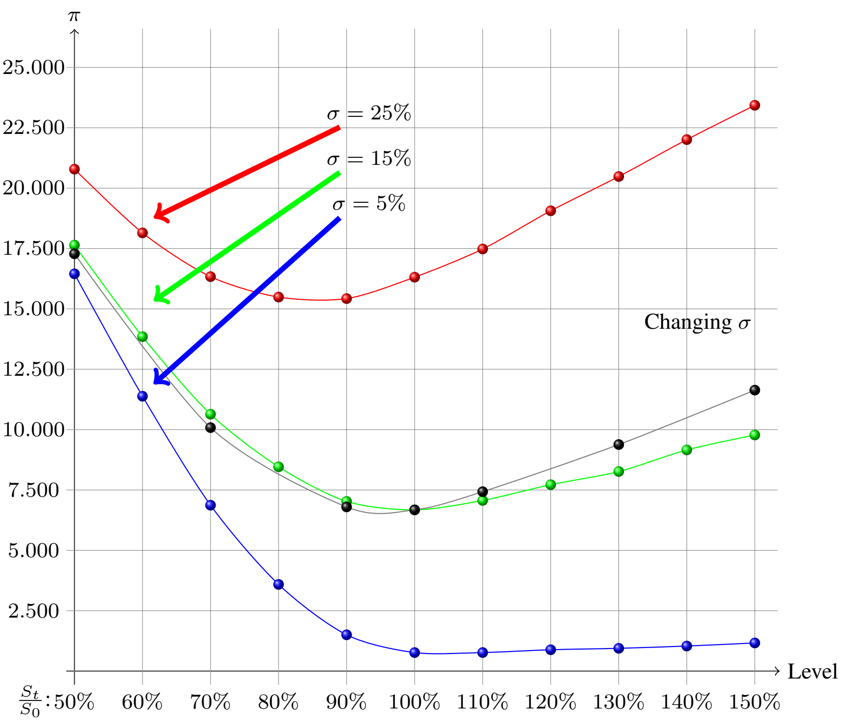


**Fig. 8.14** Response function for a 65 year old with $\psi_0 = 50\%$

In order to better understand the effect of the strategic asset allocation of the policyholder we look in a first step at the response function of the hedge liability with respect to example 59. Figure 8.14 shows the dependency of the value of the hedge liability with respect to varying fund volatility $\sigma$. We see that higher volatility means a higher value of the variable annuity and also the shape changes. It becomes obvious that a lower volatility means in general a steeper response function for a equal amount of equity level shift. This has however to be put into context, since a material reduction in equity values (eg -50%) is much less likely with a lower volatility. In turn the recovery in a low volatility environment would be commensurately more rare.

In respect to policyholder behaviour the asset allocation piece depends crucially on the approach the policyholder is taking. In principle he can take an active or a passive approach. When taking a active approach the policyholder would actively re-balance his investments in order to maintain the desired asset allocation. Adopting a passive approach would mean that he does not re-balance the his assets. This is best explained with an example.

Assume that there are two investment at disposition of the policyholder: cash (with $\sigma = 0\%$) and equities (with $\sigma = 20\%$). Also we assume that at time $T = 0$ the policyholder invests 75% of his assets in equities, the remainder in cash. This leads to a volatility of his investments of $\sigma = \frac{3}{4} \times 20\% = 15\%$.

Adopting an active approach the equity portion will stay at $75\%$ and in consequence $\sigma = 15\%$. The following table shows how the volatility will change for an immediate change in equity values, assuming a passive approach. We see that now the volatility can not be assumed to be constant.

| | Bond Value | Equity Value | Volatility |
|---|---|---|---|
| -50 % | 0.250 | 0.375 | 12.0% |
| -30 % | 0.250 | 0.525 | 13.6% |
| -10 % | 0.250 | 0.675 | 14.6% |
| – | 0.250 | 0.750 | 15.0% |
| 10 % | 0.250 | 0.825 | 15.4% |
| 30 % | 0.250 | 0.975 | 15.9% |
| 50 % | 0.250 | 1.125 | 16.4% |

Figure 8.14 shows the consequence of adopting a passive re-balancing approach in terms of the value of the hedge liability. We can observe that a passive re-balancing approach necessitates a higher amount of protection for the ratchets than using an active re-balancing. As a consequence of this there are variable annuity products in the market where the underlying funds is actively re-balanced in order to have lower hedging costs.

## 8.7 Summary

In this section we want to summarise the formalism we have learned in this section to model policyholder behaviour and to put it in context to the formulation of the respective processes in terms of stochastic integrals. We will consider the following two effects for policyholder behaviour: lapse and utilisation. As before we will assume the Black-Scholes-Merton framework for the share-price process and we denote with $(W_t)_{t \in \mathbb{R}^+}$ the standard Brownian motion. The demographic and policyholder behaviour process is based on a Markov Chain (eventually conditional to the moneyness as we have seen in section 8.4. In this context we use the following states:

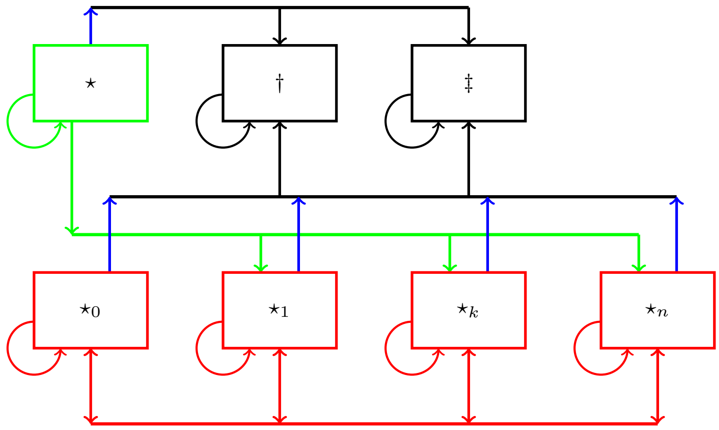


**Fig. 8.15** Modelling Policyholder Behaviour

**Alive – Never Utilised:** This is the canonical starting state and we denote it by $\star$.

**Alive – Utilising:** We denote this states with $\star_0$ to $\star_n$ with corresponding utilisation $\psi_t(j)$. We note that we have introduced this model in section 8.5 and that the state $\star_0$ is considered not utilising after having utilised beforehand. For convenience reasons we denote the set of all utilising states with $S = \{\star_k \,|\, k = 0, 1, \ldots, n\}$.

**Death:** We denote this state $\dagger$.

**Lapse:** We denote this state $\ddagger$.

**State Space:** The state space of the entire demographic model is denoted $\tilde{S} = S \cup \{\star, \dagger, \ddagger\}$. Figure 8.15 shows the state-space diagram for this model.

We note that the model above can be chosen more generic that in the prior section by making all transition intensities $\mu_{ij}$ conditional to the moneyness. Until now we have used this technique only for lapsing eg for the intensities $\mu_{i\ddagger}$, $i \in \tilde{S} \setminus \{\ddagger\}$. It is worth pointing out that such a generalised model heuristically makes sense, since policyholder may tend to withdraw more if the funds starts to be depleted and in consequence the policy becomes more "in-the-money". We note that the main complexity of such a model is the need for a complex estimation of the respective intensities as a function of moneyness (eg $\mu_{ij}(itm)$).

**Fund return:** The fund return is given in the Black-Scholes-Merton context via

$$dF = \eta\, F\, dt + \sigma\, F\, dW.$$

**Benefit Base:** The benefit base $B$ fulfils the following

$$B_0 = f(F_0).$$

This means that the start benefit base depends on the initial fund value. The propagation of the Benefit base depends on the underlying guarantee. There are various possibilities such as:

$dB = \bar{\eta}\, B\, dt$ Geometric Increase, or
$dB = \bar{\eta}\, dt$ Linear Increase, or
$$dB = \sum_{k=1}^{N} \delta_{\cdot,t_n}((F_{t_n} - F_{t_{n-1}}) - (A_{t_n} - A_{t_{n-1}}) - (P_{t_n} - P_{t_{n-1}}))^+,$$

where we have a series of ratchets at times $\{t_1, t_2, \ldots t_N\}$ in the latest case.

**Guarantee Premium:** The guarantee premium $P_0 = 0$ and

$$dP = \frac{P}{F}\, dF + \bar{\xi}\, \chi_{\{F-A-P>0\}} B\, dt.$$

**Withdrawals ($A$):** There are two withdrawal processes. The first one in order to get the right guarantee premium. Since we do not deduct the annuity paid out from the fund when this happens, but as a sum (integral), we have to allow the total amount of annuities paid out to grow in line with fund return:

$$dA = \frac{A}{F}\, dF + \left( \sum_{j \in S} I_j\, \psi_t(j) \right) B\, dt.$$

Since the death benefit (below) is defined as the reminder of the benefit base we need also keep track of the sum of annuities paid out:

$$dAS = \left( \sum_{j \in S} I_j\, \psi_t(j) \right) B\, dt.$$

Since the death

**GMWB Guarantee Process ($G$):**

$$G = -(F - A - P)^+.$$

**Death Benefit ($DB$), GMDB Guarantee:** We limit ourselves to a very death benefit, the return of the Benefit base in case of death.

$$dDB = \sum_{i \in S \setminus \{\dagger, \ddagger\}} \big((B - AS)^+ - (F - A - P)^+\big)^+ \times dN_{i\dagger}.$$

**Residual Guarantee Premium / Value:**

$$\pi_t(G + DB) = E^Q[e^{\delta\, t} \times \int_t^\infty e^{-\delta\, \tau}(dG_\tau + dDB_\tau)|\mathcal{F}_t].$$

We note that $-\pi_t(G + DB)$ is the economic value of this policy from the company's point of view. In case of using the equivalance pricicple, one would have the boundary condition

$$\pi_0(G + DB) = 0.$$

We would like to finally note that the above quantities are normally calculated by using a simulation approach.

**Exercise 61** *1. Use the formalism above to model a GMDB/GMWB product in discrete time.*

2. *Use the formulae to calculate the value of this product by means of simulation.*

# Chapter 9
# Model Risks

Ë˜

## 9.1 Ways to assess the risk intrinsic to Variable Annuities

In order to assess the risks intrinsic in Variable annuities one normally follows the following approach:

- Collect the different risks which could affect a variable annuity product adversely. The respective risks are highly dependent on context and in particular on the product features and the regulatory environment one is operating in. These risks will be collected in a so called risk register. At this stage one does not normally attempt to quantify these risks.
- In a second step one determines the dimensions which are relevant in the given context and in particular the metrics which will be employed to measure the respective risks. Common metrics are the regulatory statutory capital and also various types of economic capital measures such as the ones shown in the previous chapters. During this step one needs to decide which level of risk is acceptable by, for example, defining what statutory insolvency means. The definition of these types of risk appetite will be ultimately helpful to determine whether a company has enough capital or whether management actions need to be taken (such as changing the product design, the hedging or also raising capital).
- After having defined the relevant risk factors and the metrics to be considered, one needs to try to estimate the quantum of risk each risk factor contributes. Ie how much additional capital would be needed if equity levels fall by 40%? In order to do this, a number of stress scenarios are applied to each of the quantities which have been defined in the prior step. This activity is called stress testing. On distinguishes between single factor stresses where only one quantity is stressed and combined stresses where several factors are stressed at once. Note that a stress does per se have no frequency how often it occurs. Hence we can only answer the question what would, for example, happen if equity levels fall by 40% but not how often such an event occurs.
- Then one needs to builds a capital model which can also answer the question how often a certain stress occurs and how much capital is needed to survive for example a one in 25 years event.
- Finally one tries to answer the question how certain risks can be mitigated, for example by hedging out a part of the equity risk. For variable annuities in particular, this risk mitigation by means of hedging is of paramount importance.

## 9.2 The Model underlying Financial Risks

In order to develop a model for managing and measuring financial risks we need to look at the balance sheet, which we have seen earlier in this book (see also section D and figure 3.9):

| Balance sheet | Book **A** | Book **L** | Market **A** | Market **A** | |
|---|---|---|---|---|---|
| Cash | 6200 | 47100 | 6200 | 48513 | MR |
| Bonds | 35700 | **2200** | 37842 | **3569** | SHE |
| Shares | 4400 | | 4800 | | |
| Properties | 1100 | | 1300 | | |
| Loans | 1400 | | 1400 | | |
| Alternatives | 500 | | 540 | | |
| Total | 49300 | 49300 | 52082 | 52082 | |

It is clear that we need to decouple the valuation $\pi_t$ from the underlying asset. So formally the balance sheet consists of assets $(\mathcal{A}_i)_{i \in S_A}$ and Liabilities $(\mathcal{L}_i)_{i \in S_L}$ and we assume that both index sets $S_A$ and $S_L$ are finite. Now assume we have 1000 shares from HSBC. We could say that these 1000 shares are "one" asset. On the other hand we could model the same holding as holding 1000 pieces of the asset "1 HSBC share". Therefore we denote by $(\alpha_i)_{i \in S_A}$ and $(\lambda_i)_{i \in S_L}$ the number of units which we own at the certain point of time. Furthermore we want to separate the shareholder equity from the liabilities and we denote it $\mathcal{E}$.

If we write $\alpha_1 \, \mathcal{A}_1$ we assume that we are holding $\alpha_1$ units of the asset $\mathcal{A}_1$. Hence our portfolio is an abstract finite dimensional linear vector space $\mathcal{Y} = \text{span}\{(\mathcal{A}_i)_{i \in S_A}, (\mathcal{L}_i)_{i \in S_L}, \mathcal{E}\}$. In this context our balance sheet is a point $x = \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \in \mathcal{Y}$.

As seen before some assets and liabilities can be further decomposed in simpler assets and liabilities and hence we can find a suitable basis for the vector space $\mathcal{Y} = \text{span}\{e_1, \ldots, e_n\}$, where $(e_k)_{k \in \mathbb{N}_n}$ is its basis, and we remark that we can also write our balance sheet as $x = \sum_{k \in \mathbb{N}_n} \gamma_k \, e_k$.

The idea to introduce $\mathcal{Y}$ is to have a normalised vector space. Assume for example that we hold some ordinary bonds. In this case we would use as $e_k = \mathcal{Z}_{(k)}$, the corresponding zero coupon bonds, etc.

We finally remark that the balance sheet $x \in \mathcal{Y}$ actually represents a random cash flow vector, and hence we strictly have $x_t$ or $X_t(\omega) \in \mathcal{X}$ if we assume that the changes of the portfolio follow a stochastic process (cf. appendix D). For measuring the risk of the actual balance sheet it is normally sufficient to assume that $y \in \mathcal{Y}$ does not change.

Next we need to look at the second part, namely the valuation $\pi_t$, and we remark that:

- The valuation is dependent on time.
- We assume that the valuation is a linear functional $\pi_t : \mathcal{Y} \to \mathbb{R}$ which allocates to each asset its value (see also appendix C).
- A liability $\mathcal{L}$ is characterised by $\pi(\mathcal{L}) \leq 0$. In the same sense and asset has a positive value. As a consequence an $x \in \mathcal{Y}$ can in principle be both an asset or

a liability, depending on the economic environment and also depending on the valuation functional.

After having defined the different parts we need to have a closer look at what equity or capital ($\mathcal{E}$) means. In the context of the balance sheet we observe that the sum of the value of all assets equals the sum of the value of all liabilities (neglecting the sign). Hence we have the following:

$$\begin{aligned}
x &= \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i + \mathcal{E} \in \mathcal{X}, \text{ and} \\
\pi(x) &= \pi \left( \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i + \mathcal{E} \right) \\
&= 0, \text{ and hence} \\
SHE = \pi(\mathcal{E}) &= -\pi \left( \sum_{i \in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \right).
\end{aligned}$$

This means that we can always calculate the value of the shareholders' equity if we know the value of all other assets and liabilities.

Finally we want to show how to tackle the stochastic valuation functional $\pi_t$. Since we live in a linear vector space $\mathcal{Y}$ with a basis $(e_k)_{k \in \mathbb{N}_n}$, it is sufficient to define the price $\pi_t(e_k)$. The idea is to decouple the operator from the economy and the corresponding set up is to define the state of the economy by a stochastic process $(R_t)_{t \in \mathbb{R}} \in \mathbb{R}^m$. You could think that one of the components could be inflation, another could be the level of the 10 year interest rate, etc. In this setup we can define:

$$\pi_t(e_k) = f_k(R_t),$$

where $f_k : \mathbb{R}^m \to \mathbb{R}$ is a sufficiently regular function. If we assume for example that $R_t[10]$ is the interest rate for the 10 year bond, then we have (depending on our definition of $\pi$)

$$\pi_t(\mathcal{Z}_{(10)}) = (1 + R_t[10])^{-10}.$$

The idea of financial risk management is to assess and control the change of the value of the shareholder equity, e.g. the profit and loss induced by this change. If we assume for the moment that the time $t$ is denoted in years, one is normally interested in the following quantity:

$$PL_T = (\pi_T(\mathcal{E}) - \pi_0(\mathcal{E})).$$

 

The loss which we encounter within the time interval $[0, T]$. Banks normally look at one week, eg $T = 1/52$, Solvency II looks at $T = 1$. One measures the risk, as indicated before based on the random variable $PL_T$.

Here again is a more formal environment: In order to assess the financial risk of an insurance company the following steps are needed.

1. Define the valuation methodology $\pi_t$,
2. Define (note this is a big model assumption) which stochastic process $R_t$ models the economy,
3. Define the universe of all assets and liabilities $\mathcal{Y}$,
4. Define and calculate the functions $(f_k)_{k\in\mathbb{N}_n}$,
5. Analyse the possible balance sheets $x \in \mathcal{Y}$ and decompose each $\mathcal{A}_i$ and $\mathcal{L}_i$ into the basis $(e_k)_{k\in\mathbb{N}_n}$,
6. Define the risk measure to be used such as VaR, etc.,
7. Implement the model.

The implementation of the above steps in its purest form is very complex and therefore one normally has to make approximations.

## 9.3 Approximations

A common approximation starts with the simplification of the function $f_k$, by using a *Taylor approximation*. Since we are interested in

$$\begin{aligned} PL_T &= \pi_T(\mathcal{E}) - \pi_0(\mathcal{E}) \\ &= [\pi_T - \pi_0] \circ \left( \sum_{i\in S_A} \alpha_i \mathcal{A}_i + \sum_{i \in S_L} \lambda_i \mathcal{L}_i \right), \end{aligned}$$

we use the following first order Taylor approximation

$$\begin{aligned} \pi_T(e_k) - \pi_0(e_k) &= f_k(R_T) - f_k(R_0) \\ &\approx \nabla f_k(x)\|_{x=R_0} \times \mathit{\Delta}(R). \end{aligned}$$

If we apply this formula to all assets and liabilities we get a model where the gains and losses are linear in the risk factors $R$. If there is a balance sheet $x = \sum_{k\in\mathbb{N}_n} \gamma_k \, e_k$ we can obviously sum over the different $e_k$ and we get the following approximation:

 

$$\begin{aligned}\pi_T(x) - \pi_0(x) &= \sum_{k \in \mathbb{N}_n} \gamma_k \times (f_k(R_T) - f_k(R_0)) \\ &\approx \delta^T \times \Delta(R),\end{aligned}$$

where

$$\delta = \sum_{k \in \mathbb{N}_n} \gamma_k \times \nabla f_k(x)\|_{x=R_0},$$

and where we denote with $x^T$ the transposed of a matrix or vector.

Another simplification is to use a stochastic process, which is analytically easy to tackle. Both the risk metrics method and also the Swiss solvency test use a multi-dimensional normal distribution for $Z = \Delta R$.

Hence we have

$$Z \sim \mathcal{N}(0, \Sigma),$$

where we $\Sigma$ denotes the *covariance matrix*. One can express this matrix by the standard deviation vector $s$ for each of the risk factors and the correlation matrix $\rho$. In a first step we define the matrix $S = (v_i \times \delta_{ij})_{i,j}$. Furthermore we need to know that if $X_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$ is a multidimensional normal distribution and $A$ and $b$ are a matrix and a vector, respectively, we then know that $X_2 := A \times X_1 + b \sim \mathcal{N}(\mu_1 + b, A \times \Sigma \times A^T)$. Using this formula we finally get the following relationship:

$$\Sigma = S \times \rho \times S,$$

keeping in mind that $S = S^T$.

If we use the two approximations, the calculation of the VaR at a level $\alpha$ (eg 99.5%) can be calculated as follows. In a first step we denote by

$$\zeta = F^{-1}_{\mathcal{N}(0,1)}(\alpha),$$

and we get in consequence:

$$\begin{aligned}VaR_{PL}(\alpha) &= F^{-1}_{\mathcal{N}(0,1)}(\alpha), \\ &= \zeta \times \sqrt{(s \times \delta)\rho(s \times \delta)^T}.\end{aligned}$$

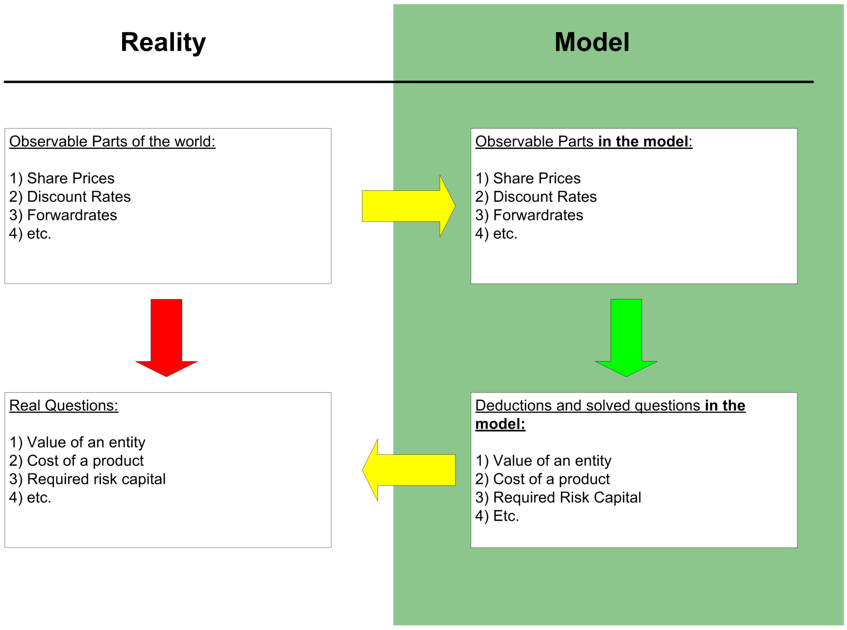


**Fig. 9.1** Models and Model Risk

Hence the value at risk can easily be calculated using some simple matrix multiplications. The example which follows is based on these approximations.

At this point it is important to remark that every model has flaws and hence it is of utmost importance to understand the *limitations* of a model. The risk to choose a “wrong” or “inaccurate” model is called *model risk*. Here it important how a model is constructed. Figure 9.1 aims to show this. In principle there are the reality (left hand side of the figure) which one tries to model in order to answer “difficult” questions which can not be answered directly. In order to do that one creates a model (right hand side of the figure) and one should be able to answer the corresponding questions within the model. Next one translates the results back to reality and “hopes” that the diagram is commutative. From this point of view the model risk is the missing “approximate” commutativity of the model. As a corollary one needs to acknowledge that each model is suited and best adapted for a certain purpose and that it is dangerous to use the model outside that.

Another interesting aspect with respect to model risk is the fact that one can, from time to time, observe difficult and lengthy discussions between experts on which model is better. Such discussions can stem from the fact that these people do not distinguish between reality and the model and hence these discussions can end up in religion like beliefs.

In the same sense the results of every model depend on the parameters chosen. The risk of inaccurate model parameters is called *parameter risk*. An easy example is the equity volatility, which is for example used for the Black-Scholes model. The value of the corresponding options is heavily dependent of the volatility chosen. As remarked before the volatility for equity market indexes is normally in the region of say 17 %. In case of market disruptions this parameter can spike up to 30 % and above. Hence it is crucial to exactly know how the model behaves with respect to different parameters.

Finally it is worth noting that the distinguishing between model and parameter risk is not always clear.

## 9.4 Market Risks for Variable Annuities

In this section we want to apply the theoretical concepts of the prior sections for Variable Annuities. It is important to define the risk factors that have a material impact on the value of a variable annuity. As a consequence of the valuation of variable annuities (section D) we know that the main market risks for variable annuities comprise the following:

- Equity price risk including basis risk,
- Equity volatility risk,
- Interest Rate risk.

Based on these risks a typical insurance company will try to (partially) hedge out some liability risks, by means of derivatives, such as equity stock options, swaps and swaptions. By doing that we are facing all the risks which will affect these instruments. Moreover they will also face risks, which have to do with the liquidity of the derivative markets, the credit counterparty risk to the sellers of the respective derivatives and more generally to risks intrinsic in the operation of the company's hedging programme. Hence we need to complement the above list of risk factors by the following:

- Liquidity Risk,
- Credit Risk,
- Hedging Risk.

Since most of the above risks can be modeled in a very canonical form; we refer to the corresponding literature such as [**?**] or [Hul97] and focus on the ones which are more specific for Variable Annuities:

- Liquidity Risk, and
- Hedging Risk.

For liquidity risk it is important to understand that the insurance company is exposed to two sorts of liquidity risk, namely the liquidity of the derivatives market in a first instance and the liquidity of the insurance company in the narrower sense for posting collateral. It is important to understand that market closures are risky for the insurance company, since during such times the insurance company can not re-balance its hedge and it therefore partially exposed to risk it actually wanted to hedge away. Insurers which run a pure delta hedging strategy are most exposed to such risks, because the intrinsic way such a strategy works is by updating the corresponding Greeks on a very regular basis, particular during times of high market volatility. This risk can be partially mitigated by complementing the hedging strategy by some static options which longer time to expiry. In this case the company is exposed only to the extent such derivatives mature and can not be replaced.

The most complex part of the capital model for a variable annuity writer is the way the hedging strategy is implemented in the models. The actual problem is founded in the fact that most hedging strategies give the ALM function some digression how they want to operate their hedging strategy (eg both in terms of how much risk needs to be hedged away and the instruments which are used to do so). Hence such models rely on a lot of model assumptions which aim to replicate the actual hedging strategy to the best possible level.

## 9.5 Hedging Risk and Basis Risk

In this section we aim to explain to shortly explain hedging and basis risk intrinsic to variable annuities. We will look at hedging in further detail in section **??** and limit ourselves in this section to explain the rationale and need to hedge variable annuities and the companion concept of basis risk. We have seen in section 3.6.3 the trading grid of a variable annuity:

| Equity Level | $\pi$ <br> $V(S)$ | $\delta$ <br> $\frac{\partial}{\partial S} V$ | $\gamma$ <br> $\frac{\partial^2}{\partial S^2} V$ | $\rho$ <br> $\frac{\partial}{\partial r} V$ | $\nu$ <br> $\frac{\partial}{\partial \sigma} V$ |
|---|---|---|---|---|---|
| -50 % | 65777 | -33818 | 39673 | -2341692 | 86714 |
| -40 % | 59397 | -36051 | 42961 | -2241519 | 111455 |
| -30 % | 53734 | -37319 | 44743 | -2136639 | 133559 |
| -20 % | 48710 | -37826 | 45431 | -2030460 | 152596 |
| -10 % | 44254 | -37744 | 45168 | -1925345 | 168470 |
| -5 % | 42219 | -37531 | 44705 | -1873714 | 175247 |
| 0 % | 40301 | -37230 | 44052 | -1822905 | 181284 |
| +5 % | 38494 | -36854 | 43243 | -1773034 | 186616 |
| +10 % | 36789 | -36419 | 42315 | -1724192 | 191283 |
| +20 % | 33662 | -35417 | 40235 | -1629845 | 198789 |
| +30 % | 30871 | -34295 | 38026 | -1540199 | 204141 |
| +40 % | 28372 | -33102 | 35808 | -1455379 | 207655 |
| +50 % | 26130 | -31872 | 33640 | -1375369 | 209618 |

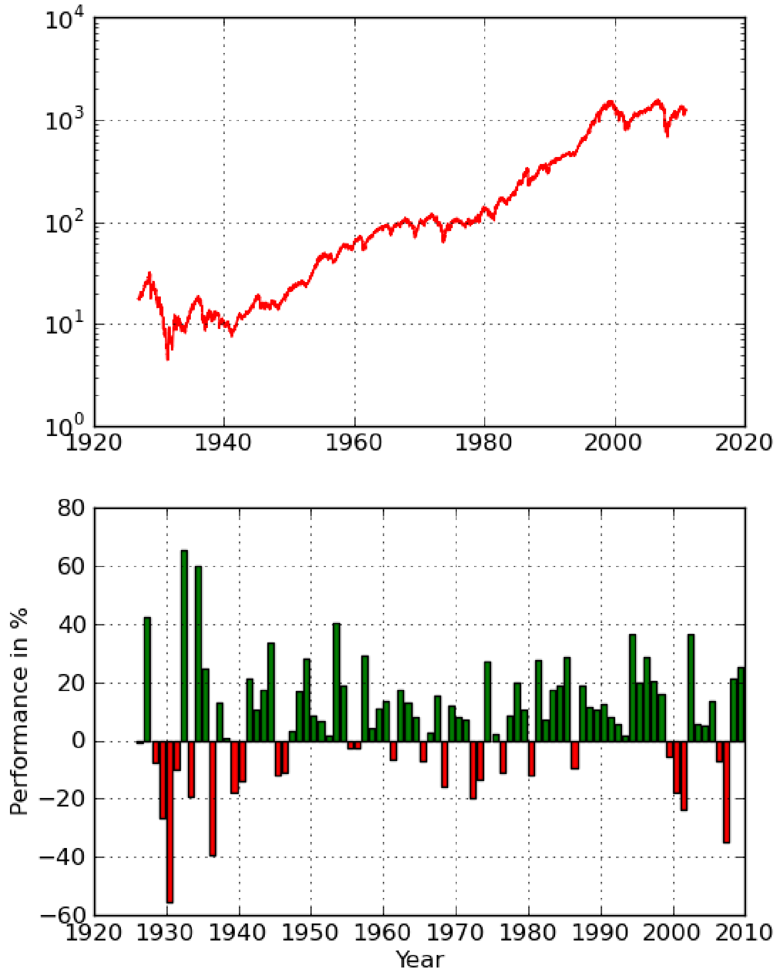


**Fig. 9.2** Equity performance over time (S&P 500 index)

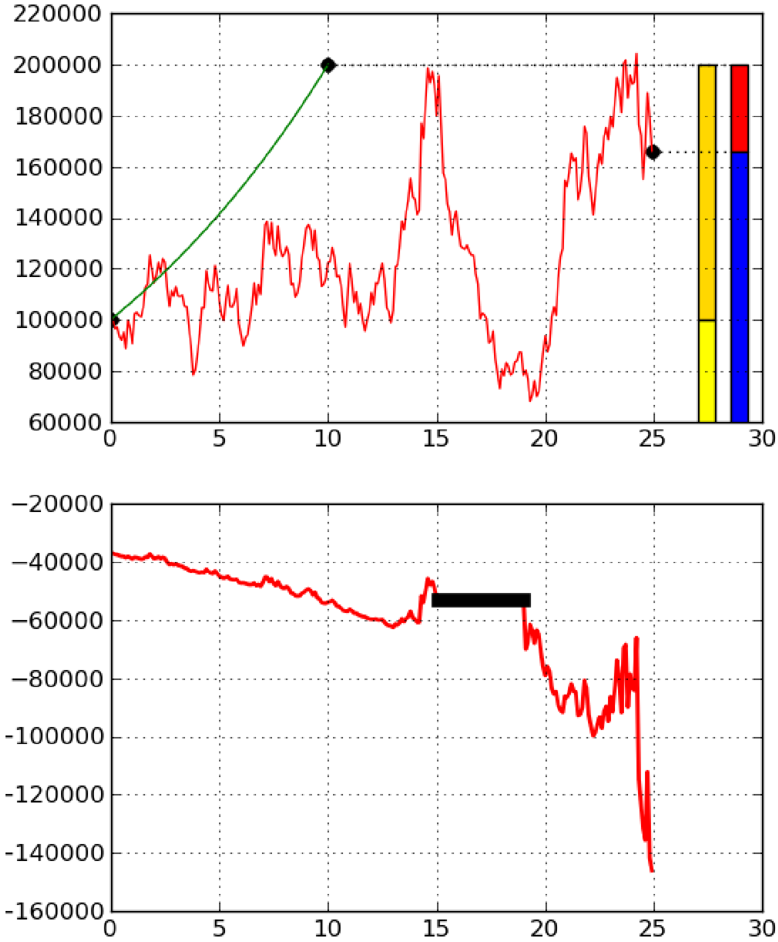


**Fig. 9.3** Effect of absence of Derivative Market

It becomes obvious that the value of the variable annuity increases (decreases) from c40'000 to c65'000 (c26'000) in case equity markets decrease by 50% (resp. increase by the corresponding amount). In relative terms the value of the variable annuity increases by 50% for such a shock. Market shocks of 50% and more are quite rare events but they do occur such as 1929, 1974/75, 2001/2002 and hence can have a detrimental effect on the company's balance sheet. Figure 9.2 shows the performance of the S&P 500 index since 1920. Ultimately the insurance company can go bankrupt during such market events and both shareholders and policyholder can lose material amounts of money. In order to mitigate this risk most insurance companies implement a hedging strategy which has the aim to offset a part of the corresponding market risk. A hedging strategy, which would fully remove the instantaneous equity risk would for example consist of financial assets which fully offset the respective liability movement. For such a strategy the value of the assets for a 40% equity fall would have to increase by $59397 - 40301 = 19096$.

We note that event though the above hedging strategy would fully immunise the company at the current time $t$, this normally not anymore the case for $t + \Delta t$, and therefore one needs to constantly rebalance the hedging assets in order to maintain the desired level of protection. The intrinsic reason why it is quite rare to have hedges which need to be rebalanced over time is a consequence of the absence of a sufficiently large universe of hedging assets to replicate the variable annuity liabilities over time. Derivative markets usually lack liquidity (eg the availability to purchase derivatives in an efficient market) for longer dated options.

Figure 9.4 shows how a dynamic hedging strategy actually works. At time $t = 0$ the company decides on the hedging strategy to be applied and rebalances its hedges accordingly. After a certain time (eg one day or one week in practice) the markets have moved forwards and both the values of variable annuity liabilities and hedging assets have moved. If the hedging was not perfect this results in a hedging gain or loss.

As a consequence of this effects (which you could call time congruent hedging), there are two material risks to be considered in respect to such *dynamic hedging strategies*:

**Hedging Risk:** By its nature running a hedging operation is an operationally complex process, which can fail at times. As a consequence of this the insurance company can suffer material losses. Operational failures comprise for example key man risk (eg the dependency on some few knowledgeable people), rough traders etc. In order to mitigate this risk it is necessary to have adequate processes with robust controls and also sufficient hedging knowledge (both quo quality of the people doing this and quo quantity in order nor to be overly reliant on some few key individuals).

**Absence of Market:** One implicit assumption of being able to execute a dynamic hedging strategy is the requirement that there is always a sufficiently deep and liquid derivative market, which allows the company to rebalance its hedging assets. Locking back in time there have been instances where derivative markets

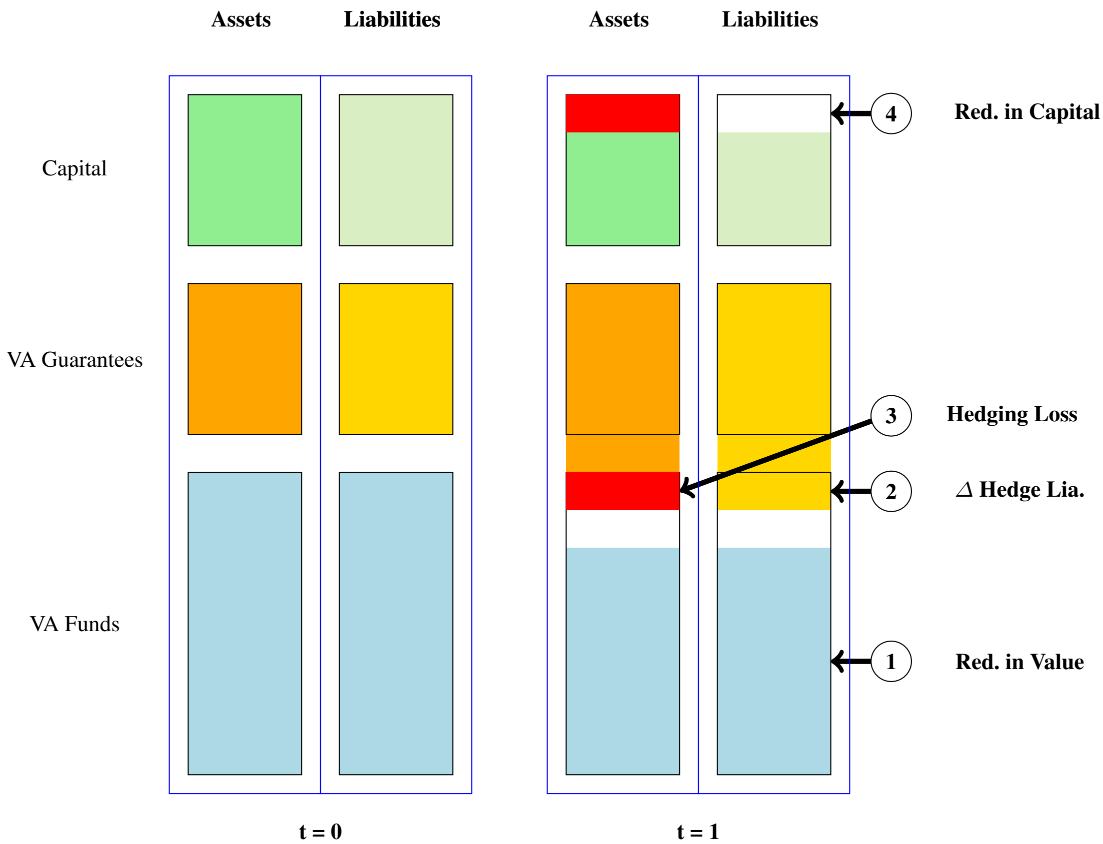


**Fig. 9.4** Hedging Strategy

were closed, such as at the day of the assassination of president Kennedy or also the week after the terror attacks which took place on the 11.9.2001. As a consequence of this it is wise to ensure that not all derivative assets mature at the same time and are adequately spread over time.

Figure 9.3 shows the effect of an absent market to the hedging process. We see that as a consequence of the market being closed the company suffers a corresponding hedging loss.

Besides the hedging risk there is a second particular risk - basis risk, which we want to address section. In the past we have always assumed that the funds is represented by an investment fund with certain characteristics. Since there are normally no derivatives on fund level, the insurance company has to hedge the fund returns by means of derivatives which relate for example to an index (such as the S&P 500 equity index). Since fund and index are not fully correlated the performance of the funds and the index are likely to differ over time and in consequence the value of the hedge liability will. This effect (eg the different risk characteristics of the underlying fund and the hedging indices) is called basis risk. The following exercise illustrates this:

**Exercise 62** *Model the fund and the corresponding hedge index with two geometric Brownian motions $X_1$ and $X_2$ respectively. Assume that both have the same drift term $\eta$ and volatility $\sigma$ and that the two underlying Brownian motions are correlated via a correlation coefficient $\rho$. By $Y$ we denote the relative performance of the funds with respect to the index, eg*

$$Y = \frac{X_1}{X_2}$$

1. *What is the distribution of $Y$ at time $t = 1$?*
2. *What is the adverse 1:10 performance of the funds vis-a-vis the index after one year?*
3. *What would be the approximate increase to the hedge as a consequence to this? (Take the trading grid above and assume that the $\Delta = \frac{\partial}{\partial S} V$ is constant over the year and use the Taylor approximation.*

We will see that we in section **??** how basis risk is treated in the hedge P&L.

## 9.6 Policyholder Behaviour Risk

In this section we want to analyse policyholder behaviour risk and develop some possible capital models. To this end we need in a first step to better understand what types of risks we are looking at:

**Lapse level risk:** As we have seen before, expected lapse behaviour is typically modeled for variable annuities, because in some countries the majority of policyholders will lapse their policy before being able to completely benefit from it features. Therefore lowering the lapse rate is typically detrimental to value creation of these products. In consequence a reduced lapse level is typically risky.

**Lapse trend risk:** Besides the immediate fall in lapse rate there is the risk that the lapse rates are not constant over time and the might expose a trend. As per the point above, the main risk is typically a downwards trend.

**Moneyness function lapse risk:** A particular feature of variable annuities is the fact that the guarantees become more valuable after an equity crash or in instances of a prolonged period with low equity returns. As a consequence of this policyholders will typically lapse less in such economic environments and the respective dynamic lapses are modeled via the moneyness function. This might mean that the lapse level is reduced by 50 % for policies which are deep in the money. Given that there are not a lot of observations of such policies there is considerable uncertainty about the shape of the moneyness function. In particular a steeper moneyness function (eg reducing lapse levels by 75% for policies which are deep in the money) results in a more adverse outcome for the insurance company.

**Utilisation level risk:** For GMWB and GMLB policies the policyholder can often decide when to start withdrawing and the change the corresponding amount within given boundaries. In consequence there is a risk that the average amount of withdrawal changes. For most of these types of products higher withdrawal rates are adverse for the company.

**Utilisation trend risk:** In the same sense as for lapses one can distinguish between level utilisation risk and utilisation trend risk.

It is worth pointing out that the above list provides an indicative illustration what changes lapse behaviour means. It is of paramount importance to analyse the effects on adverse policyholder behaviour in the context of a given product, as the following example illustrates:

**Example 63** *In the following we will consider two different types of products, both of them for life withdrawal variable annuities ("GLWB"), the difference being the product definition:*

1. *Product "D" has a so called* doubler benefit*. This means that the benefit level (benefit base) will double after 10 years provided the policyholder did not start withdrawing until then. In case the policyholder decides to withdraw the corresponding double benefit is void. In consequence an efficient policyholder would wait the 10 years before starting to withdraw.*
2. *Product "R" is a yearly ratchet which is indifferent whether the policyholder starts withdrawing or not. Note that there is an effect of starting to withdraw earlier, since this means that the fund value is consumed earlier, reducing the likelihood of a ratchet and reducing also the potential uplift.*

*We will consider that products "R" and "D" are sold to a 65 year old policyholder, who pay 1'000 single premium into the investment funds which is considered to have fees of 1.75% and a volatility of $\sigma = 16.5\%$. We also assume that the policyholder starts in a non-withdrawing state and consider also a dynamic lapse behaviour as per section 8.4. Together with the GMLB the product foresees also a GMDB rider, where the death benefit is equal to the $(GV - FV)^+$, where $FV$ and $GV$ denote the funds value and the benefit base (guarantee value) respectively.*

*For this given product, we will calculate the value of the guarantees for best estimate utilisation rates and for utilisation rates which are stressed as per figure 9.5. We note*

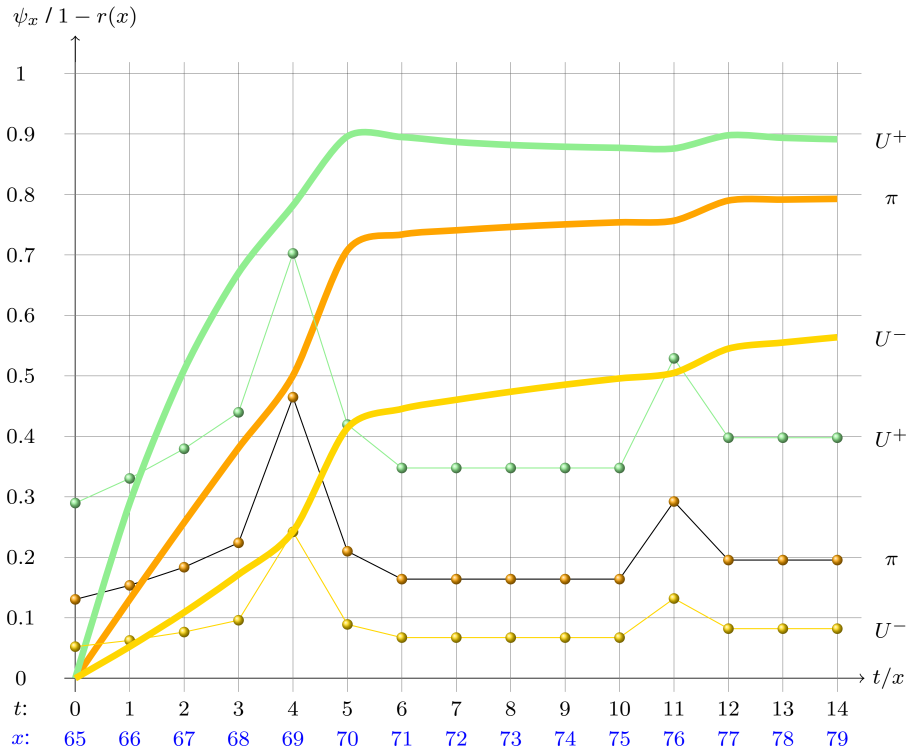


**Fig. 9.5** Implicit Markov Transition Probability $r(x)$

*that the stressed incidence rates and expected utilisation patterns are represented by the dashed lines. We use the notation of example 60.*

*Looking at figure 9.5 we can already see that the number of policyholder violating the doubler benefit is materially different depending on the utilisation pattern used. Ignoring for the moment that $w_0(x)$ consists also from people which stopped withdrawing, we can see that over 90% of the policyholders would have violated the doubler benefit for the case where a higher take up rate is expected. On the other hand the respective figure is around 50% for the lower utilisation take-up pattern. This means that in this context the doubler could become materially more expensive. It is worth nothing that neither of the two utilisation stresses would however pick up the most adverse shape in respect to the doubler bonus. This one would consist of a materially lower take up rate for the first 10 years followed by a materially increased tack up rate in the later years.*

*In a next step we want compare the response functions for the respective products. We will look at the sum of the hedge liability as a function of equity levels (on the horizontal axis) and utilisation stress (on the vertical axis). We will denote by $U^-$*

*and $U^+$ the value of the hedge liability with respect of an utilisation take up downwards and up-wards stress respectively. The central value of the un-stressed hedge liability will be denoted by $\pi$ and it will comprise both the GMWB and the GMDB part.*

**Response function product "D"**

*Sensitivity*

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $U^-$ | 419.3 | 352.9 | 292.8 | 240.6 | 196.5 | 159.6 | 129.2 | 105.4 | 86.3 | 71.0 | 58.5 |
| $\pi$ | 357.7 | 299.1 | 246.6 | 201.3 | 163.3 | 132.0 | 106.3 | 86.4 | 70.5 | 57.8 | 47.6 |
| $U^+$ | 354.0 | 294.0 | 240.4 | 194.7 | 156.4 | 125.4 | 100.8 | 81.9 | 66.6 | 54.5 | 44.8 |

**Response function product "R"**

*Sensitivity*

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $U^-$ | 418.8 | 354.3 | 298.8 | 256.4 | 229.8 | 220.0 | 222.4 | 234.9 | 251.1 | 269.6 | 288.9 |
| $\pi$ | 347.6 | 291.5 | 244.1 | 208.3 | 187.4 | 179.7 | 182.1 | 192.0 | 205.7 | 220.7 | 236.4 |
| $U^+$ | 345.5 | 287.9 | 239.3 | 203.8 | 182.7 | 176.0 | 179.5 | 189.9 | 203.5 | 218.5 | 234.0 |

*From the above response functions we see the clearly different product structures, in particular that the downside protection in the case of the doubler benefit is pretty independent whether there are material equity downwards shocks. On the other hand we see the different behaviour in respect to the utilisation stress. In a first step we look how the hedge liability changes when changing utilisation take-up assumptions:*

*Sensitivity*

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| D $U^-$ | 61.6 | 53.8 | 46.2 | 39.3 | 33.2 | 27.6 | 22.9 | 19 | 15.8 | 13.2 | 10.9 |
| D $U^+$ | -3.7 | -5.1 | -6.2 | -6.6 | -6.9 | -6.6 | -5.5 | -4.5 | -3.9 | -3.3 | -2.8 |
| R $U^-$ | 71.2 | 62.8 | 54.7 | 48.1 | 42.4 | 40.3 | 40.3 | 42.9 | 45.4 | 48.9 | 52.5 |
| R $U^+$ | -2.1 | -3.6 | -4.8 | -4.5 | -4.7 | -3.7 | -2.6 | -2.1 | -2.2 | -2.2 | -2.4 |

*We see that not only the change of the hedge liability differs materially depending on the utilisation take up stress applied, but also its shape. The change in shape means that a policyholder behaviour change can mean that the ex ante defined hedging strategy is not optimal and could give rise to a risk which materialised if the equity markets and the policyholder behaviour changes at the same time. Note that we will in a further example what it means if the moneyness function changes.*

*The following table shows the hedge requirements for a $\pm 30\%$ equity shock:*

| | D0.7 | D1.3 | R0.7 | R1.3 |
|---|---|---|---|---|
| $U^-$ | 133.2 | -73.3 | 78.8 | 31.1 |
| $\pi$ | 114.6 | -61.5 | 64.4 | 26 |
| $U^+$ | 115.0 | -58.8 | 63.3 | 27.5 |

In a next step we want to have a look at another aspect of policyholder behaviour and we will look at our prior example again. This time we will look how changing the moneyness function impact both hedging liability and the respective response function. We note that we have already looked at the most extreme change to the moneyness function in Figure 8.6 where we compared the response function once including the moneyness adjustment and once ignoring it. Now we want to look at the impact of an even stronger moneyness function.

**Example 64** *Figure 9.6 compares the approximate effect of the moneyness spline as a function of in-the-moneyness. It shows the different in-the-moneyness function which we will consider and we can observe that a higher leverage to the moneyness function will lead to higher lapses for at-the-moneys policies and to lower lapses for policies which are deep in the money. On the other hand removing the moneyness effect results in equal lapses for all moneyness levels. We will have a look how the hedge liability changes when steepening or flattening the moneyness function.*

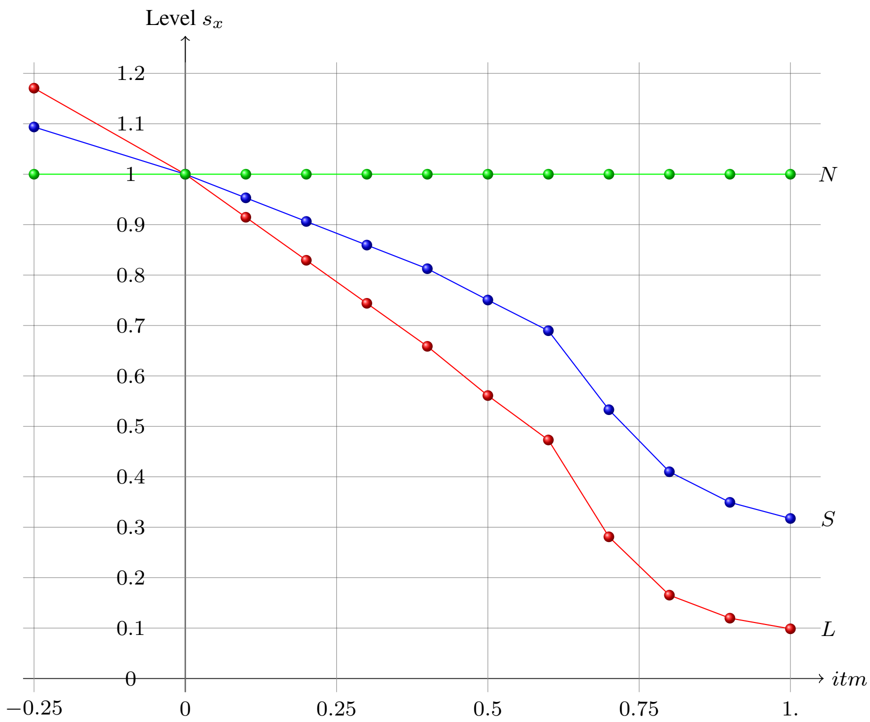


**Fig. 9.6** Moneyness functions considered

*For this example we will consider the ratchet product from example 63, where the variable annuity is financed via a guarantee premium equal to 1% of the funds* *value per annum. Hence early lapses in case of high fund value will result in a corresponding loss of guarantee fees. On the other hand the present value of guarantee fees will be lower in case of policies which are deep in the money. We will consider a policy which is still in deferral period, eg not withdrawing funds. The following table shows the net hedge liability for the three moneyness functions considered: "S" standard moneyness function, "L" levered moneyness function and "N" nil moneyness function:*

*Sensitivity*

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 241.1 | 179.3 | 127.9 | 88.4 | 61.9 | 46.5 | 40.0 | 39.2 | 41.4 | 44.5 | 48.3 |
| L | 303.2 | 232.5 | 172.3 | 125.9 | 95.4 | 78.8 | 72.2 | 73.2 | 77.7 | 83.6 | 90.1 |
| N | 127.1 | 85.7 | 51.5 | 24.6 | 5.5 | -7.1 | -14.0 | -17.6 | -19.5 | -20.8 | -21.7 |
| $L-S$ | 62.1 | 53.1 | 44.4 | 37.5 | 33.5 | 32.3 | 32.2 | 33.9 | 36.3 | 39.0 | 41.8 |
| $N-S$ | -113.9 | -93.6 | -76.4 | -63.7 | -56.4 | -53.7 | -54.1 | -56.9 | -60.9 | -65.4 | -70.0 |

*It becomes obvious that the levered moneyness function results in both a higher hedge liability (in absolute terms) and also in a steeper response function. Similarly considering no moneyness function results in a flatter response function and a lower hedge liability. Hence the mis-estimation or not considering the moneyness function is potentially costly. The reason for the above effect is due to the fact that for a steeper moneyness function less policyholders stick to the company and hence the company looses guarantee fees is markets rise. Conversely more policyholders stay with the company in case of adverse stock market performance and in turn the variable annuity becomes more costly. The following table shows the hedge requirement for the different moneyness function for different equity levels.*

*Hedge requirement*

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 194.5 | 132.8 | 81.3 | 41.8 | 15.3 | – | -6.4 | -7.2 | -5.1 | -1.9 | 1.7 |
| L | 224.3 | 153.6 | 93.5 | 47.0 | 16.5 | – | -6.5 | -5.6 | -1.1 | 4.7 | 11.2 |
| N | 134.3 | 92.8 | 58.6 | 31.8 | 12.6 | – | -6.9 | -10.5 | -12.4 | -13.6 | -14.6 |

### 9.6.1 Capital Models for Variable Annuity Lapse Rates

In this section we will have a look how to model lapse capital. We have seen before that lapse has a material impact, both quo absolute lapse level and also in respect to the moneyness function. In order to address this question we consider following figure shows a possible choice of a dynamic lapse rate, as a function of duration ($d$), age ($x$) and in-the-moneyness ($itm$):

and we assume in the following that

$$f(\mathbf{p}) = sx(d, x, itm) \tag{9.1}$$

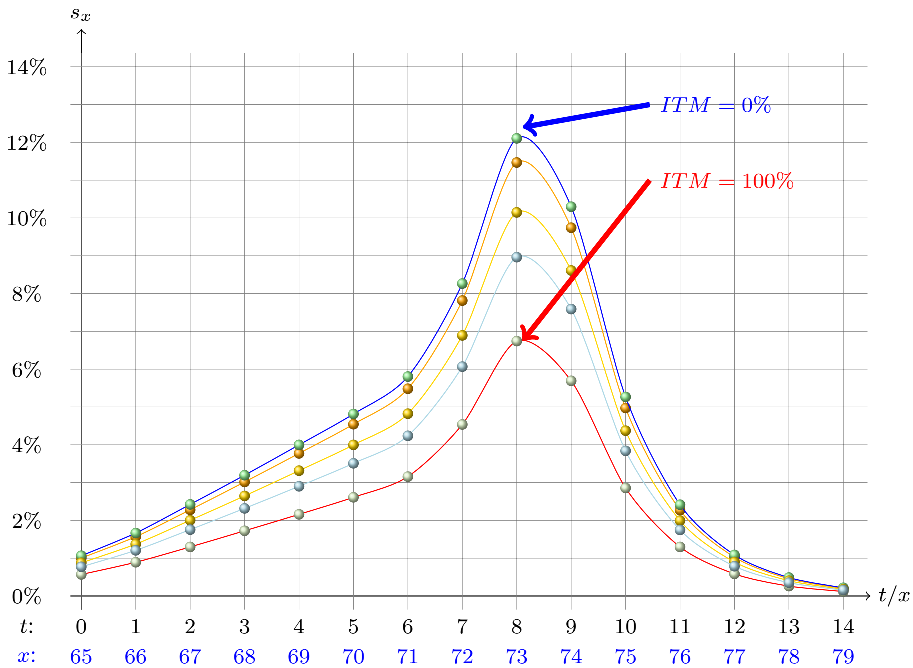


**Fig. 9.7** Dynamic Lapses

represents the best estimate lapse rate as a function of explanatory variables $\mathbf{p}$. We will in the following look at some possible models for the lapse rate. We will call the resulting random variables for model $i$ as

$$f_i(\mathbf{p})[\omega] = f_i(\mathbf{p}), \tag{9.2}$$

and we take the liberty to omit $[\omega]$

## *9.6.2 Lapse Model 1*

The idea of model 1 is to use the same approach one normally uses for longevity trend risk. In this case one assumes a random innovation in mortality improvement. In the same sense one can model an innovation in lapses.

$$f_1(\mathbf{p})[\omega] = f(\mathbf{p}) \times Z_t, \tag{9.3}$$

where $(Z_t)_{t\in\mathbb{N}_0}$ is a geometric Brownian Motion with drift term $\mu$ and volatility (per time) of $\sigma$. In this context we interpret $t$ as the time parameter at which we will use the respective lapse rates. In order to ensure that

$$E[f_i(\mathbf{p})] = f_i(\mathbf{p}), \tag{9.4}$$

we need choose $\mu = -\frac{\sigma^2}{2}$.

Please note that given the high level of lapses at the time of peak lapses, Model 1 (and also the following Models) need to reformulated in the logit space. We will do this for model 1 and leave the corresponding formalism for the reminding models to the reader. To this end we the following bijective mapping from $]0,1[$ to $\mathbb{R}$:

$$\psi(x) := \ln\left(\frac{x}{1-x}\right) \tag{9.5}$$

$$\psi^{-1}(x) = \frac{\exp(x)}{1+\exp(x)} \tag{9.6}$$

In this context the model reads now

$$\tilde{f}_1(\mathbf{p})[\omega] = \psi^{-1}(\psi(f(\mathbf{p})) + \tilde{Z}_t), \tag{9.7}$$

where now $(\tilde{Z}_t)_{t\in\mathbb{N}_0}$ is a (ordinary) Brownian Motion with suitable parameters.

### *9.6.3 Lapse Model 2*

Model 2 is a variation of model 1, in the sense that one assumes that a certain percentage $\beta$ of lapses is given and stable. In this case we get:

$$f_2(\mathbf{p})[\omega] = f(\mathbf{p}) \times (\beta + (1-\beta) \times Z_t, ) \tag{9.8}$$

### 9.6.4 Lapse Model 3

Model 3 is a variant of Model 1 assuming there is a minimal surrender floor. To this end we define

$$\mathbf{q} = (\mathbf{p}, t) \tag{9.9}$$

and minimal surrender floor $g(\mathbf{q})$ which depends on the parameter vector $\mathbf{p}$ and the time $t$. Now we can define the model as follows:

$$f_3(\mathbf{p})[\omega] = \max\left(f_i(\mathbf{p}, g(\mathbf{p}, t))\right), \tag{9.10}$$

where $i = 1, 2$

### 9.6.5 Lapse Model 4

One could generalise the above models by assuming a geometric Brownian Motion for each possible $\mathbf{p}$. This would be most likely to complex. There is however a case to be made to choose three different geometric Brownian motions, as a function of duration, namely before, at and after peek lapse.

We have seen that there are different possibilities how to model lapse risk capital. The choice of the concrete model and the respective calibration is both a function of available data and the generic best estimate lapses. If we consider a typical American variable annuity one can see that the lapse levels in the first few years are very low, followed by a peak lapse year. Peak lapse year is a consequence of the typical product structures which allow lapsing the policy at this point in time for no penalty. After the peak lapse year policyholders have typically an age of 70+ years and the remaining variable annuity GMWB policies start withdrawing.

**Example 65** *In order illustrate Model 1, Figures 9.8 and 9.9, show the distribution of the risk neutral cash flows for a GMDB and GMWB product (without guarantee fees) as per example 63, product "R". We have chosen a volatility of lapse levels of $\sigma = 20\%$ p.a. Figure 9.8 shows the distribution of cash flows per year for the quantiles*

$$\{0.01, 0.05, 0.10, 0.20, 0.50, 0.80, 0.90, 0.95, 0.99\}$$

*and we see that the main lapse behaviour risk is after policyholders start withdrawing after 10 and more years. Figure 9.9 shows the so called critical scenario for this lapse model. The critical scenario for a given confidence level and capital model is the simulation which results in the corresponding shock in the capital model. Since*

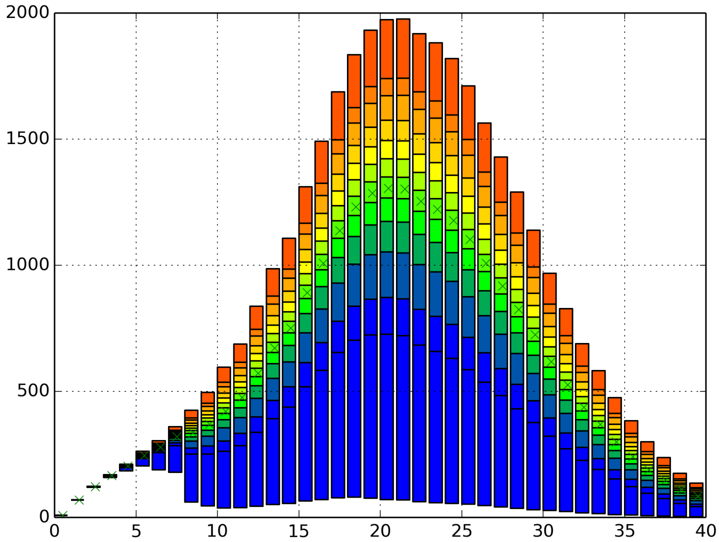


**Fig. 9.8** Cash Flow distribution for Lapse capital model

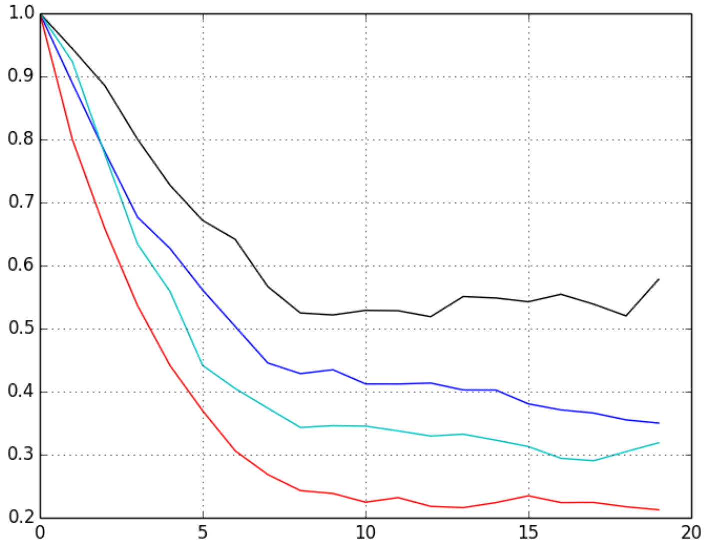


**Fig. 9.9** Tail lapse reduction patterns

*the pure critical scenario is very unstable, one normally averages the neighbouring scenarios. For the example out of 10'000 simulation the* $\pm 20$ *neighbouring scenarios have been considered for the confidence levels* $\{0.80, 0.90, 0.95, 0.99\}$. *We can see that model together with the chosen calibration leads to material downwards adjustments in the respective lapse levels and the required capital varies between 2% and 4%. We note that the corresponding capital numbers and are highly sensitive to concrete product offering and the way the guarantee is financed.*

### *9.6.6 Capital Models for Variable Annuity Utilisation and Lapse Rates*

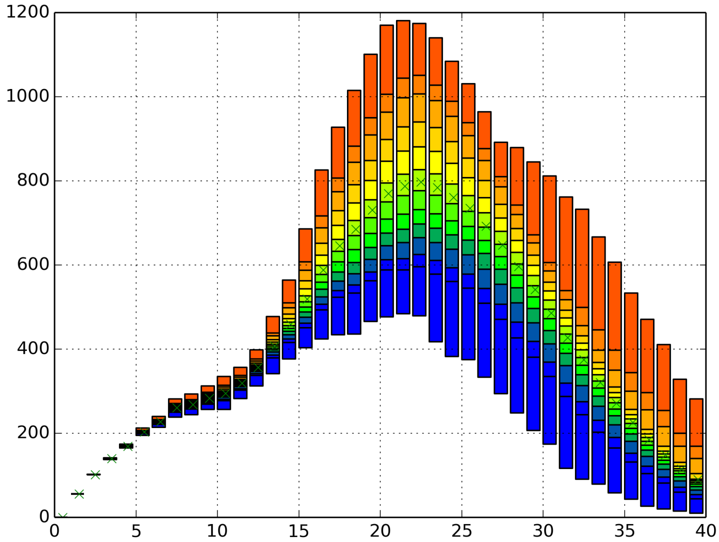


**Fig. 9.10** Cash Flow distribution for Utilisation capital model

In this section we discuss some possible utilisation models based on the Markov model for utilisation. Assume we have a state space $S$ representing the discretised utilisation for a GMWB policy such as $S = \{0, 0.25, 0.5, 0.75, 1.0, 1.25\}$. For simplicity we assume that the governing Markov chain is (time-) homogeneous in discrete time. Hence the matrix $P := P(0,1)$ determines the Markov Chain $X$ completely.

The idea of the models which we will introduce, will be based on the idea that for all $t > 0$ the Markov chain is conditional to some process $Y$, and in turn we can calculate

$$P(0,n|(Y_k)_{k\in\mathbb{N}}) = \prod_{k=0}^{n-1} P(k,k+1|(Y_k)_{k\in\mathbb{N}}).$$

This means that we first take one $Y$ in order to then calculate the conditional transition matrix given $Y$. It is worth mentioning that we can combine utilisation and lapse in one model by enlarging the state space by the state lapse (‡). This means that for the above example we would use $S = \{0, 0.25, 0.5, 0.75, 1.0, 1.25, \ddagger\}$. Also we could enlarge the model further to include the entire demographic process by also adding the state of death resulting in the following state space for the above example

$$S = \{0, 0.25, 0.5, 0.75, 1.0, 1.25, \ddagger, \dagger\}.$$

In all of the above models the matrix $P(k,k+1|(Y_k)_{k\in\mathbb{N}})$ needs to be defined. By using a bijective transformation function

$$\psi : [0,1] \to \bar{\mathbb{R}}, x \mapsto \Psi(x)$$

we can map a transition matrix in a matrix of the same dimensions in $\bar{\mathbb{R}}^{n\times n}$. One example for $\Psi$ is the logit function, eg

$$\psi : [0,1] \to \bar{\mathbb{R}}, x \mapsto \Psi(x) = \ln\left(\frac{x}{1-x}\right)$$

In order to define the transformation $\Psi$ from $M_n([0,1])$ to $M_n(\bar{\mathbb{R}})$, we need to transform the matrix into its cumulative analogon $\tilde{P}$ via the bijection

$$\Phi : M_n([0,1]) \to M_n([0,1]), (x_{i,j})_{i=1,\dots n, j=1,\dots n} \mapsto (\Phi(x)_{i,j})_{i=1,\dots n, j=1,\dots n},$$

with

$$\Phi(x)_{i,j} = \sum_{k=1}^{j} x_{i,k}.$$

We note that $\Phi^{-1} \circ \Psi^{-1} \circ \Psi \circ \Phi = 1$. The idea in defining $P(k, k+1|(Y_k)_{k\in\mathbb{N}})$, is to use the identity before and modify it by shifting the values for each line of the matrix. In order to make things still rather simple we define for each $n \in \mathbb{N}$ a function $f_n(Y_1, \dots Y_n)$ which takes values in $\mathbb{R}$ and we define

$$P(n, n+1|Y) = \Phi^{-1} \circ \Psi^{-1}(\Psi \circ \Phi\left(P(0,1) + (f_n(Y)_{i,j})_{i=1,\dots n, j=1,\dots n}\right).$$

What we have done is to shift all entries of the matrix by a amount $f_n(Y)$. One can simply generalise this type of model by using a different shift amount per line of the matrix. Also one can introduce a different shift per column, but has to me more careful in order to preserve monotonicity of the entries of the matrix $\tilde{P}$. This latter type of model is useful when combining utilisation and lapse stresses, assuming that the relative quanta are different.

In a next step we want to formulate one concrete model withing the entire family of models defined above.

To this end we define a family $(Z_k)_{k\in\mathbb{N}}$ of $\mathcal{N}(0,1)$ iid random variables, and we define the respective random walk as

$$\begin{aligned} Y_1 &= 0, \text{ and} \\ Y_{n+1} &= Y_n + \sigma \times Z_n, \end{aligned}$$

and $f_n(Y) = Y_n$.

**Example 66** *We want to end this section by showing how much policyholder behaviour depends on the underlying product and the actual status of the policyholder. In order to do this we will revisit our doubler bonus product from example 63. The main characteristic of this product is the doubler bonus, which means that the benefit base is doubled if the policyholder postpones withdrawal for at least 10 years. If he chooses to start withdrawing earlier, the corresponding guarantee is void and he loses the ability to double the benefit base. In consequence the hedge liability is materially different if the policyholder is already withdrawing (eg $\psi_{x_0} \neq 0\%$) or not.*

*As a consequence of this also the direction of adverse utilisation is different. In case the policyholder is already withdrawing, a higher utilisation is adverse. On the other hand in case of a policyholder not withdrawing, the most adverse withdrawal patters is a lower utilisation at the beginning followed by a higher utilisation later. Figures 9.11 and 9.12 show the respective critical scenarios on the level of the* *underlying Brownian motion process ("starves") for not utilising at the beginning and utilising at 100% respectively.*

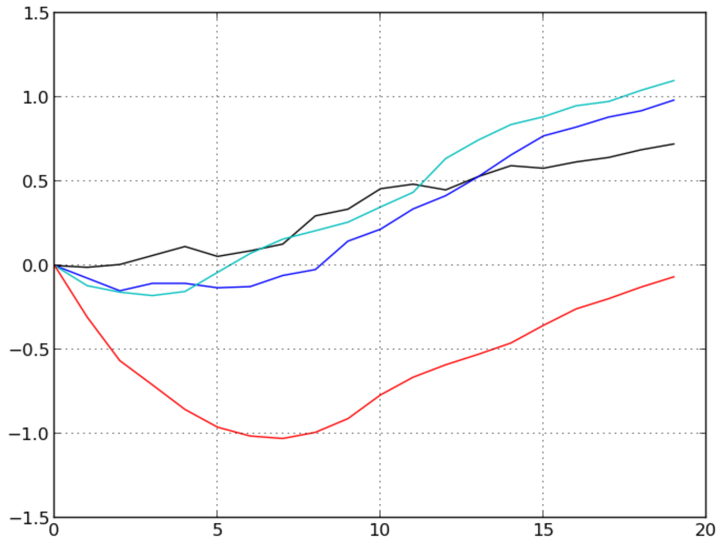


**Fig. 9.11** Critical Scenario Utilisation $\psi_{x_0} = 0\%$

*In order to better translate the respective starves into utilisation patterns figure 9.13 shows the expected utilisation for both stressed and unstressed conditions. This figure clearly evidences the dependency of the stress on the product and the state of the individual policyholder.*

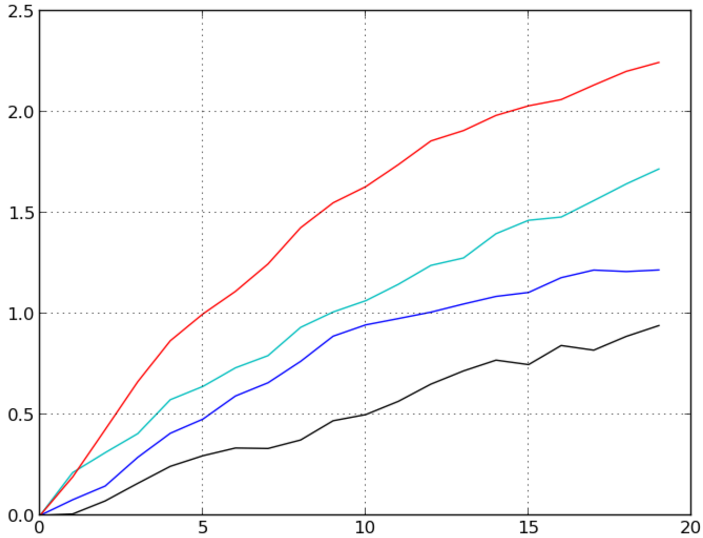


**Fig. 9.12** Critical Scenario Utilisation $\psi_{x_0} = 100\%$

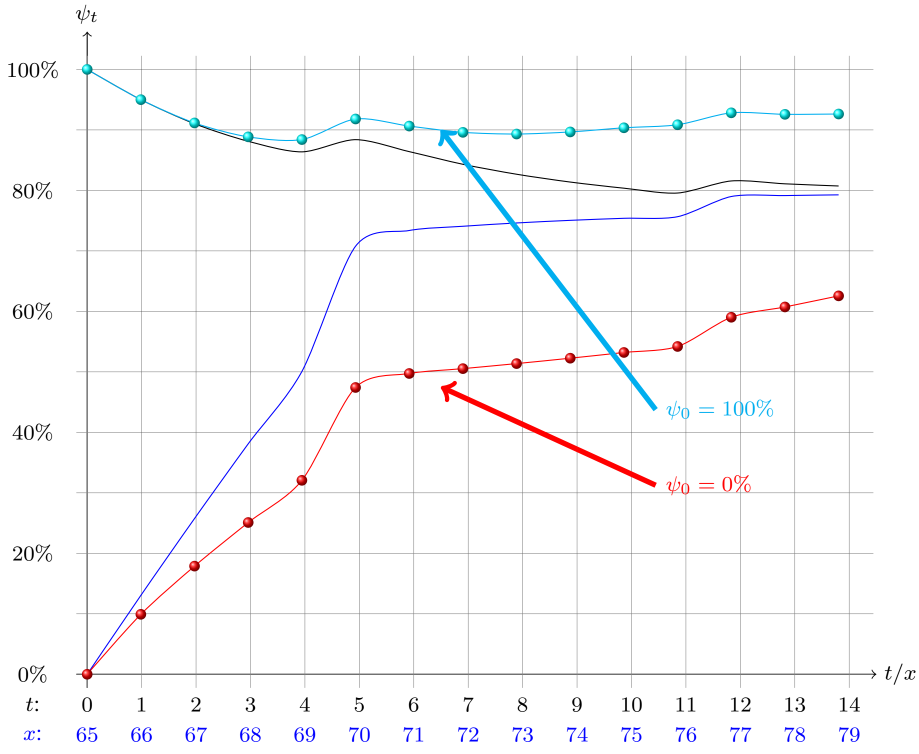


**Fig. 9.13** Expected Utilisation for $\psi_{x_0} = 0\%$ and $\psi_{x_0} = 100\%$

# Chapter 10
# Risk Mitigation Strategies

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## 10.1 Introduction

In this section we look at hedging a variable annuity and how this can be achieved. From an abstract point of view the value of the variable annuity depends on several factors. The aim of hedging is to reduce the risk for the insurance company in case of market movements and market shocks.

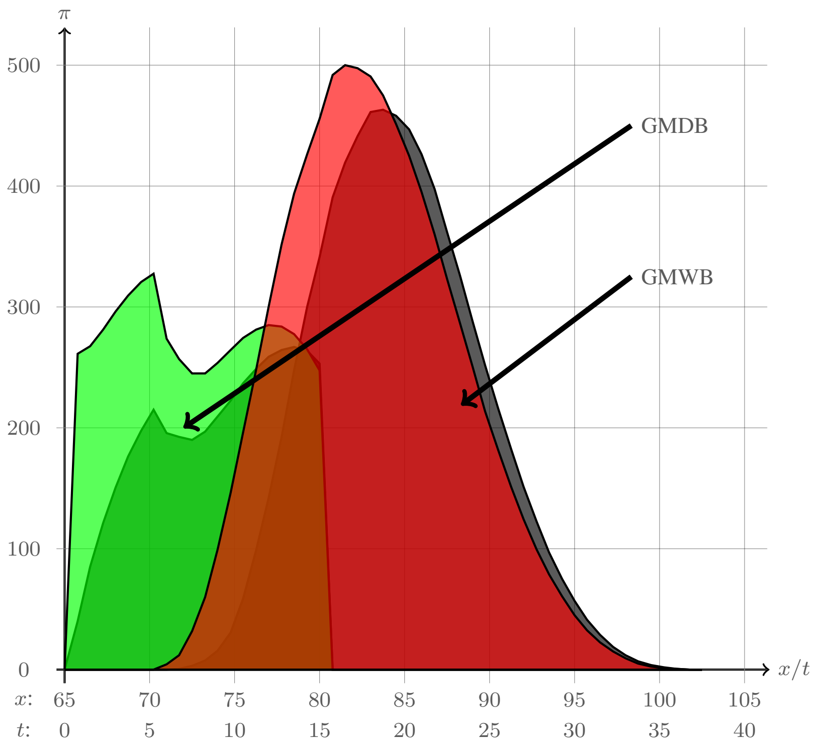


**Fig. 10.1** Cash-flows under stress (-30%) with $\psi_0 = 0.5$ and $x = 65$

Let's look again at example 59 and analyze what happens to the expected cash flows under the risk neutral measure $Q$ if we allow for an immediate $30\%$ equity fall. Figure 10.1 illustrates this. One can see the original (ie unstressed cash flows) together with the cash flows under stressed conditions (in green and red) versus the unstressed cash flows as per figure 8.9. The value of the product under the neutral position amounts to 7'166 (3'090 for GMDB and 4'076 for GMWB). Under the stress the corresponding option value increases by c2'200 to 9'422 (4'485 for GMDB and 4'936 for GMWB). This becomes apparent when considering figure 10.1. We see that the GMDB cash-flows in particular spike at the early years. Also the GMWB cash-flows increase in size and the maximum is reached earlier.

This clearly shows there is a need to off-set the corresponding potential loss of c2'200 by using a corresponding hedging strategy. The following table shows how the option prices vary with changing equity stresses:

Sensitivity

| | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| DB | 6'570 | 5'463 | 4'485 | 3'676 | 3'189 | **3'090** | 3'249 | 3'545 | 3'845 | 4'189 | 4'522 |
| WB | 7'343 | 5'969 | 4'936 | 4'352 | 4'104 | **4'076** | 4'299 | 4'823 | 5'224 | 5'542 | 5'871 |
| Total | 13'913 | 11'432 | 9'422 | 8'028 | 7'294 | **7'166** | 7'549 | 8'369 | 9'069 | 9'732 | 10'394 |

We see that the value of the variable annuity increases both for falling equity prices (as a consequence of the guarantee being further in the money) and also for increasing equity levels (as a consequence of the respective ratchets). The idea of hedging is to reduce the impact of changing equity levels by having a mitigation strategy in place. There are various possible objectives for such hedging strategies in terms of:

- the general underlying hedging strategy,
- the instruments to be used, and
- the cost for the respective hedging strategy.

In order to define a hedging strategy, one has to find a suitable set of assets which replicate the liabilities as best as possible in order to then define a concrete hedge. It is key to define the meaning of a hedging strategy and the respective metrics to be considered. A hedging strategy needs to consider and establish objectives in respect of:

**Economic Risks:** Which economic risks are hedged and to what extent?

**Financial Statement Risks:** How important are the risks regarding the publicly stated accounts and to what extent is there a need to hedge them?

**Regulatory Capital Risks:** What are the regulatory capital risks and to which extent need they to be hedged?

Therefore a hedging strategy needs to establish objectives for the different dimensions and define a corresponding risk appetite.

## 10.2 Aim of Hedging and general Principles

The simplest way to hedge a variable annuity is to use a so-called "$\Delta$-hedge"[1] or linear approximation of the value of the variable annuity liability. Let's look at this in some greater detail. Firstly one has to determine the assets needed to linearly approximate the current variable annuity. The amount of shares is given by $\Delta$. We invest $\Delta$ amount in shares and the difference to the value of the option in cash. This is the current hedge portfolio. At the next valuation date we do the following:

[1] The delta ($\Delta$) is the change in value of a derivative when the underlying asset price changes. In the context of a VA, on calculates in a first step the $\Delta$ of the variable annuity rider. The hedge seeks to offset this change or $\Delta$ by choosing appropriate assets/derivatives with a similar $\Delta$.

1. Revalue Option – the value has changed because of a different market environment,
2. Calculate the profit and loss over the period, and
3. Adjust the hedge portfolio in the same way as described above (“re-balancing hedge portfolio”).

### *10.2.1 Example of Hedge Portfolio “Doubler at” $t = 5$*

**Assume the following:**

| Time $t$ | Equity Index $S_t$ | Value Hedge Liability $\pi$ | Delta $\Delta$ | Interest $r$ |
|---|---|---|---|---|
| 5.0 | 1.07 | 43851.6 | -45037.2. | 2.5 % |
| 5.1 | 1.02 | 46193.3 | N/R | N/R |

**Hedge Portfolio:** Short Equities: -45037.2, Cash: $43851.6 + 45037.2 = 88888.8.$
**Value Hedge Portfolio** $t = 5.1$**:**

$$\underbrace{1.025^{\frac{1}{10}} * 88888.8}_{\text{Performance Cash}} + \underbrace{(-45027.2) * \frac{1.02}{1.07}}_{\text{Performance Equities}} = 45399.4.$$

**Hedge P&L:** $= 45399. - 46193.3 = -794.3.$

**Now re-balance the hedge portfolio.**

Depending on the market movement the approximation is more or less effective:

- The value of the option is approximated linearly with a $\Delta$-approximation.
- The error is small in case of minor equity movements and is
- Bigger in case of more material equity movements.
- Considering also the curvature in hedging is called a $\Delta$–$\Gamma$–hedge.

To price and hedge these guarantees the concepts of arbitrage free pricing theory (“Black-Scholes”) together with the Itô-Calculus are used to immunise and hedge the liabilities as much as possible with a dynamic hedging strategy. The value of the guarantees depend on financial market variables such as equity market, volatility and interest rates. They also depend on mortality and the various parameters regarding the policyholder behaviour. Since one aims to hedge the market variables, the values of the liabilities are represented in terms of an (Taylor) approximation and the relevant variables such as equity returns, interest rate movement and volatility of the funds in sense of the Itô-calculus:

$$\Delta f(S,t,r,\sigma) = \underbrace{\frac{\partial f}{\partial S}}_{Delta} \Delta S + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial S^2}}_{Gamma} (\Delta S)^2 + \underbrace{\frac{\partial f}{\partial t}}_{Theta} \Delta t + \underbrace{\frac{\partial f}{\partial r}}_{Rho} \Delta r + \underbrace{\frac{\partial f}{\partial \sigma}}_{Vega} \Delta \sigma + \dots$$

This means that the changes in values in each of the underlying quantities is approximated. These approximations of the liability values (and also assets) are called "greeks". Using assets with the same underlying value and greeks means that the change in economic equity moves in parallel, thereby reducing the risk up to higher order errors of both assets and liabilities. The concept of dynamic hedging foresees the updating of the underlying liability values and greeks and the corresponding re-balancing of the assets in continuous time in order to result in an optimal hedge. In reality the recalculation of the liability values and the re-balancing is done on a less frequent basis, resulting in a tracking or hedging error. There is also a basis risk, when individual assets behave differently from the index chosen for hedging. The following example shows how this concept works in practice: Assume that our portfolio has the following characteristics:

### 10.2.2 Example with several greeks (1)

| Quantity | M $ | Meaning |
|---|---|---|
| Value of Guarantee $\pi$ | Net: 4526 | The economic value of the liability. |
| $\Delta$ (1%) | $-85$ | The "Delta" is the sensitivity of the value of the liability with respect to the change of the underlying equity index. If the equity market would fall by say 2%, the value of the guarantee would increase by approximately $-2 \times -85 = 170$ M \$. |
| $\rho$ (1bp) | -23.6 | The "Rho" is the sensitivity of the insurance liability with respect to interest rate movements. Assume for example that the interest rates increase by 0.2%. In this case the value of the guarantee decreases by approximately $20 \times -23.6 = -472$ M \$. |
| $\nu$ (1%) | 240 | The "Vega" is the sensitivity with respect to the volatility of the equity index. If volatility in market increases from say 18% to 20%, this means that the value of the guarantee would increase by $2 \times 240 = 480$ M \$. |

As indicated above a dynamic hedging programme aims to reduce/minimise the changes of the (economic) value of the guarantee by buying assets with offsetting characteristics. In order to do so the company, in a first instance decides which of the "greeks" it wants to hedge. The following table illustrates the different types of hedge programmes:

Based on the above example we look how a hedge would evolve and what the corresponding hedge error could be. We assume for example (as above) a fall in equities of 2%, an increase in interest rates of 0.2% and increase in volatility of 2%. Then we get the following:

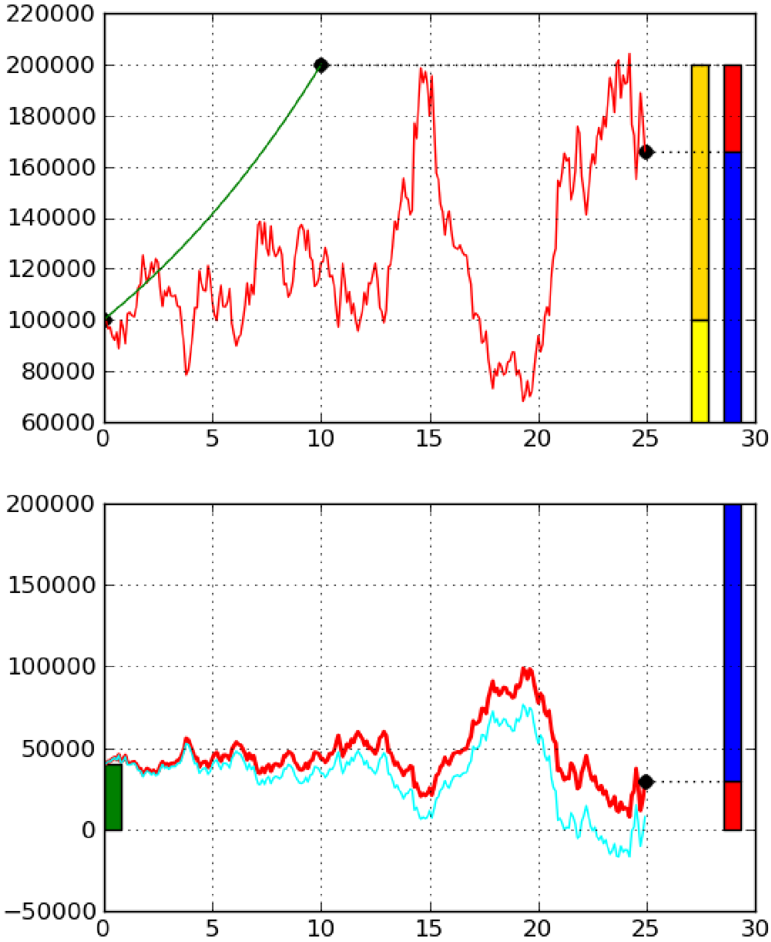


**Fig. 10.2** $\Delta$–hedge vs exact value of the hedge liability

 

### 10.2.3 *Example with several greeks (2)*

| Quantity | Liabilities Sensitivity | Assets Sensitivity | Liabilities Value | Assets Value | Net Change in Equity |
|---|---|---|---|---|---|
| Value of Guarantee/Assets BoP $\pi$ | | | 4526 | 0 | – |
| $\Delta$ (Change -2%) | $-85$ | $-53$ | +170 | +106 | -64 |
| $\rho$ (Change +20bp) | -23.6 | -0.3 | -472 | -6 | +464 |
| $\nu$ (Change +2%) | 240 | 19 | +480 | + 38 | -442 |
| Value of Guarantee/Assets EoP | | | 4704 | 138 | **-40** |

This means that the hedge loss would equal to $(4526-4704)+(138-0) = -40$. In absence of a hedging the corresponding loss would equal to $(4526-4704) = -181$ and hence hedging allowed to reduce the loss in this example by 78% ($= 100\% \times (1 - \frac{40}{181})$).

We see that we need to somehow calculate the value of these liabilities together with their corresponding greeks in order to set up a dynamic hedging strategy. For simple types of liabilities the underlying calculations can be performed by closed form formulae. In the concrete set up, the liabilities are more complex and hence the valuation and the calculation of the greeks are done via simulation (“Monte Carlo") and depend on the Economic Scenario Generator used. The expected values are estimated in the simulation context by taking suitable averages and the partial derivatives are estimated with a difference quotient with a suitable small shock.

## 10.3 Hedge Strategies

To better understand the different hedging strategies, it is in a first step important to remember the fundamental principle of replicating a contingency claim. For this we refer back to section 3.3, where we have seen the Black-Scholes-Merton differential equation and Theorem 40 which also states that a contingency claim can be replicated perfectly by a suitable dynamic hedging strategy $\Phi = (\Phi^1, \Phi^2)$ (see definition 34). Given that that the amount of shares we hold equals $\Delta = \frac{\partial v}{\partial S}$, the partial derivative of the price of the contingency claim, such a hedging stratgey is called a *replicating* $\Delta$-strategy. Refinements of these dynamic replicating strategies are

**Trivial Hedge:** Nothing is hedged and the insurance company keeps the entire risk;

**$\Delta$-hedge:** Only the equity is hedged, no interest rate hedge. A $\Delta$–$\Gamma$–hedge is a variant of this, where equities are hedged more accurately than with a pure $\Delta$–hedge;

**$\Delta$–$\rho$–hedge:** Interest rates and equities are hedged; and

**3 greeks hedge:** $\Delta$, $\rho$ and the equity volatility $\nu$ is hedged.

It is intrinsic to all such strategies that the contingency claim is replicated by a continuous re-balancing of the underlying hedging portfolios and in the theoretical set up Theorem 40 ensures perfect replication. There is however a practical problem with these strategies, which becomes very apparent, when looking at the days after 11.9.2001. We remember that on the that day there was a terrorist attack on New York which resulted in the collapse of the twin towers and the New York Stock exchange was closed for several days. The following table shows the S&P 500 index for these days:

| Date | S&P 500 | Change to next trading day |
|---|---|---|
| 10.9.2001 | 1092.54 | -11.6 % |
| 11.9.2001 | closed | n/a |
| 12.9.2001 | closed | n/a |
| 13.9.2001 | closed | n/a |
| 14.9.2001 | closed | n/a |
| 15.9.2001 | closed | n/a |
| 16.9.2001 | closed | n/a |
| 17.9.2001 | 1038.77 | -3.40 % |
| 18.9.2001 | 1032.74 | -1.98 % |
| 19.9.2001 | 1016.10 | -0.89 % |

It becomes obvious that as a consequence of the closed market, a continuous rebalancing was not possible, resulting in a corresponding non-nil hedge P&L. We will in the following look at the effect of material daily index movements in further detail. We remark again that there were several trading days in the autumn 2008 where we observed daily losses of 2 % or more. Even bigger equity market swings have been observed beginning May 2010, after fears of state bankruptcies in the Euro zone. On Thursday 6.5.2010 the NYSE (Dow Jones Industrial Average) fell temporarily over 9 %, because of such fears and automated trading. The same day Procter and Gamble lost temporarily more than 35 % of its value. On Monday 10.5.2010 the Euro Stoxx index performed 10.35 % within one day, after the announcement of a EUR 750 bn bail-out plan. Assuming a volatility of 20 % and log-normally distributed equity-market returns, this represents a $9.8 \times \sigma$–event. Such an event has a return period of $5.9 \times 10^{17}$ years. This number is considerably bigger than the age of the universe of $1.375 \times 10^{10}$ years and hence it is obvious that the log-normally distributed model is not correct in the tails. Figure 10.3 shows how the S&P 500 moved between 1.8.2008 and 1.4.2010.

To see the effects of such market movements, mean for a dynamic replicating strategy we will have a look at the following trading grid, which has been normed to 1M USD as at staring date (1.8.2008).


**Fig. 10.3** S&P 500 performance during 2008 financial crisis

| Change in Index | Date | Date |
|---|---|---|
| | 09.03.09 | 10.03.09 |
| Index | -46 % | -43% (+c6%) |
| | | |
| -50% | 3'418'116 | 3'433'834 |
| **-45%** | **3'066'963** | **3'082'383** |
| -40% | 2'743'110 | 2'758'206 |
| -35% | 2'444'987 | 2'459'706 |
| -30% | 2'171'849 | 2'186'127 |
| -25% | 1'923'926 | 1'937'682 |
| -20% | 1'702'022 | 1'715'171 |
| -15% | 1'507'336 | 1'519'804 |
| -10% | 1'341'427 | 1'353'103 |
| -5% | 1'206'280 | 1'217'046 |
| 0% | 1'102'989 | 1'112'803 |
| 5% | 1'027'233 | 1'036'330 |
| 10% | 972'206 | 980'768 |
| 15% | 933'233 | 941'424 |
| 20% | 906'734 | 914'666 |
| 25% | 890'216 | 897'973 |
| 30% | 881'652 | 889'293 |
| 35% | 879'287 | 886'878 |
| 40% | 881'766 | 889'348 |
| 45% | 888'049 | 895'669 |
| 50% | 897'369 | 905'055 |

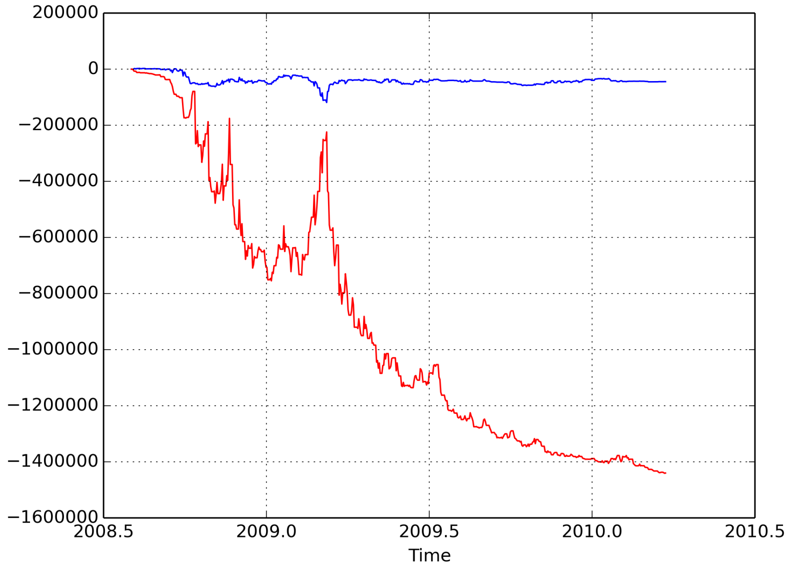


**Fig. 10.4** Performance of a dynamic $\Delta$-hedging strategy during the 2008 financial crisis

Figure 10.4 shows how the underlying dynamic $\Delta$-hedging strategy (red) has performed over this period and we see that the risk mitigation is moderate. To better understand the dynamics, it is important to understand, that we have assumed to rebalance the corresponding replicating portfolio once a day. Let us look of what would have happened between the 8.3.2009 and 9.3.2009. The following table summarises the relevant facts:

```
S&P Date                  09.03.09         10.03.09
Trading Grid Date         09.03.09         10.03.09
Index (100%=1.1.08)       -46%             -43%
alpha (weight grid)       0.735930         0.419412
alpha*(weight delta gr)   0.235930         0.919412

Liabilities
-----------
Price current            3'159'692        2'946'419
Price next day           2'931'135
Delta current            6'894'246        6'521'056

Assets
------
Short futures           -6'894'246
```

```
Cash                       10'053'939

Profit and Loss (day)
---------------------

Change in Liabilities                      228'557
Mark on Futures                           -438'909

                                         ---------
P&L w/o interest                          -210'352
                                         =========
```

We can see from this example that the loss of the dynamic $\Delta$–hedging strategy is a consequence of the changing $\Delta$ in function of the underlying equity levels. To improve the quality of the hedge there are various possibilities. In the context of a dynamic replicating strategy, one could in a first instance decrease the times between rebalancing the dynamic asset portfolio. The problem is that this does not help in case the equity market being closed. The other canonical approach to mitigate the underlying risk is to use a higher order Taylor approximation of the trading grid:

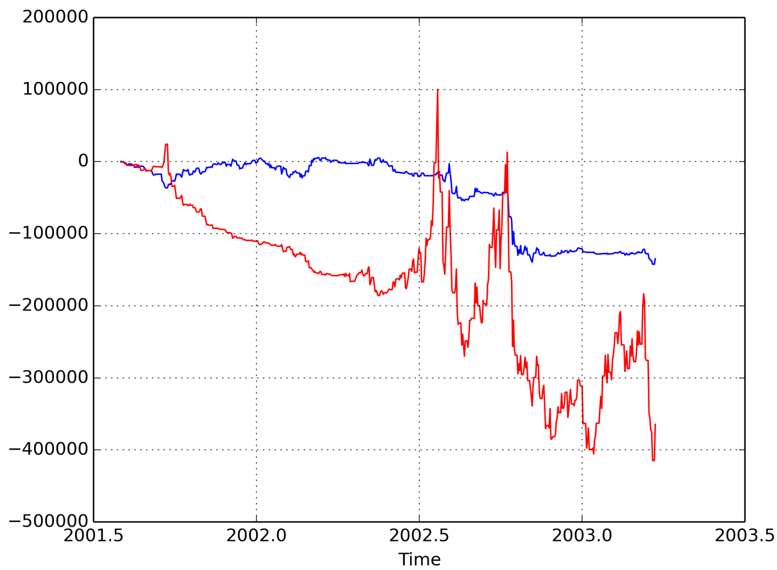


**Fig. 10.5** Performance of a dynamic $\Delta$-hedging strategy during the 2001 financial crisis

$$\Delta V(S,t,r,\sigma) = \underbrace{\frac{\partial V}{\partial S}}_{Delta} \Delta S + \frac{1}{2} \underbrace{\frac{\partial^2 V}{\partial S^2}}_{Gamma} (\Delta S)^2 + \underbrace{\frac{\partial V}{\partial t}}_{Theta} \Delta t$$
$$+ \underbrace{\frac{\partial V}{\partial r}}_{Rho} \Delta r + \underbrace{\frac{\partial V}{\partial \sigma}}_{Vega} \Delta \sigma + \dots$$

Another solution to the same problem is to complement the dynamic hedging strategy by static hedges. Hence the idea is to buy an asset (hedging) portfolio which approximates the liability trading grid. Such a approximation does not need to be perfect, since the reminder can still be dynamically hedged. Such a hedging strategy is called semi-static replicating hedging strategy or also replicating macro-hedging strategy. Hence the overarching aim is still to replicate the variable annuity liabilities by a replicating $\Phi_L = (\Phi_L^1, \Phi_L^2)$. In case static hedges $(\mathcal{A}_i)_{i=1,\dots n}$ have been chosen to approximate the insurance liabilities, we know that each $\mathcal{A}_i$ can be generated by $\Phi_i$. In consequence the dynamic part of the hedging strategy can be calculated as

$$\Phi_D = \Phi_L - \sum_{i=1}^{n} \Phi_i$$

since

$$\Phi_L = \Phi_D + \sum_{i=1}^{n} \Phi_i.$$

In principle one can still define a more general macro-hedging strategy by not requiring a perfect replication, but instead requiring that the underlying P&L remains within certain levels for a given time period. The blue line of figure 10.4 shows how the corresponding macro-hedging strategy would have performed and we note that this type of strategy is in this case much more resilient versus tail events. Figure 10.5 compares a pure daily rebalanced $\Delta$–strategy with the above mentioned macro-hedging strategy.

In summary we can distinuish between the following hedging strategies:

**Trivial Hedge:** Nothing is hedged and the insurance company keeps the entire risk.

**Dynamic Replicating Hedging Strategy:** $\Delta$-hedge, $\Delta$–$\Gamma$–hedge; $\Delta$–$\rho$–hedge: interest rates and equities are hedged; 3 greeks hedge: $\Delta$, $\rho$ and the equity volatility $\nu$ is hedged.

**Semi-dynamic hedging strategy:** This is a variant of the dynamic hedging strategy, where the dynamic strategy is complemented by static hedging assets.

 

**Macro Hedge:** The aim here is to rather hedge the tail (or big movement) risks, potentially however trading-off protection against the accuracy of the hedge for smaller magnitude movements.

## 10.4 What can and cannot be hedged

The following table provides a summary of the effects where hedging is more or less difficult:

### *10.4.1 Hedging different effects*

Easier to hedge

- Short dated options,
- Equity prices,
- Short term volatility,
- Interest rates.

More difficult to hedge

- Long dated options,
- Long term volatility,
- Interest rates at the long end,
- Policyholder behaviour (lapses, ...),
- Basis risk.

## 10.5 Delta Hedging and Tracking Error

In this section we elaborate on our example, by looking at the performance of a $\Delta$–hedging programme over the lifetime of the policy. In particular we will compare a more frequent re-balancing of the hedge portfolio with a less frequent one, to see that the quality of the hedge programme improves in the former case.

The following table shows the hedge P&L over time and we observe that the respective profits and losses are particularly sizable in case of bigger market moves.

 

| Time | Price Guarantee | Cumulative $\Delta$-Hedge | PnL Period |
|---|---|---|---|
| 0 | 40311.7 | 40311.7 | – |
| 1 | 45395.2 | 45600.2 | 205.0 |
| 2 | 37483.6 | 37941.5 | 252.8 |
| 3 | 39139.7 | 40128.8 | 531.2 |
| 4 | 51622.0 | 52597.4 | -13.7 |
| ... | | | |
| 20 | 77885.0 | 86199.2 | -2103.3 |
| 21 | 37311.6 | 43522.2 | -2103.7 |
| 22 | 35134.3 | 44409.0 | 3064.2 |
| 23 | 21765.1 | 34420.6 | 3380.7 |
| 24 | 13406.0 | 33654.4 | 7593.0 |
| 25 | 29278.1 | 57477.4 | 7950.9 |

Below we show year 17, showing the individual profit and loss entries. It becomes apparent that the profits and losses are small in case of minor market movements and big in case of more material market movements.

### *10.5.1 Delta PnL Detail Year 17*

| Time | Equity Level | Price Guarantee | Cumulative $\Delta$-Hedge | PnL Period |
|---|---|---|---|---|
| 17.0 | 1.00 | 66612.0 | 76981.1 | – |
| 17.1 | 1.03 | 64609.6 | 75706.8 | 728.1 |
| 17.2 | 1.02 | 65325.8 | 76540.5 | 117.5 |
| 17.3 | 1.00 | 66921.1 | 78238.0 | 102.2 |
| 17.4 | 0.91 | 74192.7 | 85176.7 | -332.9 |
| 17.5 | 0.92 | 73924.3 | 84996.9 | 88.6 |
| 17.6 | 0.96 | 71250.2 | 82333.9 | 11.2 |
| 17.7 | 0.85 | 80215.3 | 90657.0 | -642.1 |
| 17.8 | 0.79 | 85469.8 | 95705.1 | -206.4 |
| 17.9 | 0.73 | 90905.4 | 100872.4 | -268.3 |
| 18.0 | 0.81 | 84692.8 | 94159.9 | -499.9 |

In a next step we want to reconsider example from section 3.6. In this section we have considered a simple GMAB and GMDB example, where the benefits over the term of 10 years double. The beauty of this example is the fact that it is rather easy to determine the respective valuation portfolio, which consists of European put options only. This allowed us the calculate the value of the variable annuity guarantee for each time by the use of the Black-Scholes formula. Figure 3.10 has illustrated this based on one sample trajectory. Now we want to look at the same trajectory and look how well the $\Delta$-hedge performs vis-a-vis the explicit formula. We note that we expect to areas which lead to an imperfect hedge, namely the absence of the $\Gamma$ part of the hedge (cf section 41) and the fact that we hedge in the model based on a time step of $\Delta t = 0.1$ years. Figure 10.2 illustrates the performance of the $\Delta$-hedge for this trajectory ($\Delta$-hedge approximation in cyan). We can improve the $\Delta$-hedging by using a higher order approximation, eg a $\Delta - \Gamma$–hedging programme. Figure 3.8, which he have already seen shows the difference between these two types of hedges for the same trajectory and product as shown in figure 10.2.

## 10.6 Proxy Hedging

We have seen above, that some effects such as mortality are not easy to hedge directly with capital market instruments. The aim of proxy hedging is to hedge an a priory unhedgeable risk by something more fuzzy. Assume you want to protect yourself against the severe mortality as it occurred during the "Spanish Flu" in 1918 where mortality increased by 1%. One way is to put together a (short) equity portfolio which offsets the losses in the case of such an extreme event. Since equities have prima facie nothing to do with mortality this is called a proxy hedge. In our concrete case we could, for example, short the shares of transport and health enterprises, since they would most likely be most affected by such an event.

With respect to variable annuities one could consider a proxy hedge in particular with respect to lapses and utilisation, since ample data and respective statistical tools are available.

## 10.7 Comparison of different hedging strategies

In this section we will compare and contrast some different hedging strategies. The base for determining the hedge assets is the trading grid. We use the following trading grid (see section 3.6.3) for which we will determine the respective hedge assets for the following hedging strategies:

1. Trivial hedging strategy
2. $\Delta$-hedge
3. $\Delta - \Gamma$-hedge
4. Macro/tail hedge

| Equity Level | $\pi$<br>$V(S)$ | $\Delta$<br>$\frac{\partial}{\partial S} V$ | $\Gamma$<br>$\frac{\partial^2}{\partial S^2} V$ | $\rho$<br>$\frac{\partial}{\partial r} V$ | $\nu$<br>$\frac{\partial}{\partial \sigma} V$ |
|---|---|---|---|---|---|
| -30 % | 8442.9 | n/a | n/a | n/a | n/a |
| -20 % | 5931.9 | n/a | n/a | n/a | n/a |
| -10 % | 4051.7 | n/a | n/a | n/a | n/a |
| 0 % | 2784.5 | -9925.5 | -45.31 | 325170 | 47226 |
| +10 % | 2004.0 | n/a | n/a | n/a | n/a |
| +20 % | 1531.3 | n/a | n/a | n/a | n/a |
| +30 % | 1232.2 | n/a | n/a | n/a | n/a |

As we have seen before a hedging strategy has to fulfill are variety of requirements and has to be seen in the context of risk appetite and regulatory capital. In remark 41 we have explained that a $\Delta - \Gamma$, which is updated in continuous time, can eliminate all market risk and that in consequence the variable annuity liabilities can be perfectly replicated. There are however some impediments to this:

- There is no continuous trading, hence there remains always some residual risk. More importantly stock markets are normally closed outside business hours and there may be considerable changes in equity prices between the closing of the market and the opening next morning.
- The whole concept is based on the principal assumptions made withing the Black-Scholes-Merton model, namely deep, friction less markets and the equity prices following a geometric Brownian motion. While some of these assumptions are at least approximately fulfilled, there are always transaction costs.

As a consequence of these difficulties a variety of hedging strategies has been developed to overcome some of the problems intrinsic to the dynamic $\Delta - \Gamma$ hedging strategy. When considering a a $\Delta - \Gamma$-hedge it is obvious that the protection of the balance sheet takes place at the body of the risk neutral probability distribution given current prices. Hence such a strategy performs normally best in normal markets with only minor changes in equity prices from day to day. If one want to protect the tail, one rather would concentrate on the entries of the trading grid at the more extreme end (such as for a equity price fall of 30%). Hence from an abstract point of view one wants to solve a minimisation problem. Eg given values

$$x_k = f_k(S_k, r_k, \sigma_k, \ldots), \text{ for k } = 1 \ldots n,$$

and weights $(w_k)_{k=1\ldots n}$, $w_k > 0$ one seeks hedge assets $(\mathcal{A}_j)_{j=1\ldots m}$ with corresponding response values

$$y_k(\mathcal{A}_j) = g_k(\mathcal{A}_j, S_k, r_k, \sigma_k, \ldots) \text{ for } k = 1 \ldots n, \text{and for } j = 1 \ldots m,$$

such that

$$\sum_{k=1}^{n} w_k \left( x_k - \sum_{j=1}^{m} y_k(\mathcal{A}_j) \right)^2 \stackrel{!}{=} min.$$

In a concrete set up $f_k$ could for example represent the $\Delta$ at current market levels or also the pro-forma loss in case of an immediate equity fall by 20% (ie. $\pi(0.8 \times S(0)) - \pi(S(0))$). Hence one picks some of the entries of the trading grid with corresponding weights, and seeks for the correct amount of hedge assets such that the differences between variable annuity liabilities and hedge assets becomes minimal. Such an optimisation can also consider a maximal amount of spend to be used for buying hedge assets. In the following we will use the above trading grid to explain this concept in further detail. The minimisation problem was in the concrete set up solved using a numerical method for minimisation.

In order to chose adequate hedge assets one needs to give the optimiser an adequate set of candidate assets. We consider the following hedge assets:

| Asset | Remarks |
|---|---|
| $\mathcal{B}$ | This is an investment in cash (ie."bank"). In consequence all Greeks $(\Delta, \Gamma, \ldots)$ are zero. |
| $\mathcal{F}$ | Futures. At current market level the value of the future $\pi(\mathcal{F}) = 0$ with $\Delta = 1$ and $\Gamma = 0$. |
| $\mathcal{P}(c, n)$ | European put option with strike price $c$ and term $n$. |
| $\mathcal{C}(c, n)$ | European call option with strike price $c$ and term $n$. |

### *10.7.1 Trivial hedge*

The connotation of a "Trivial Hedge" is a synonym for not hedging at all and assuming full absorption capacity of the market value changes by the shareholder equity. It is obvious from the above example that this is a very risky strategy and during the 2008' financial crises some fallout has been seen for some of the companies utilising a trivial hedging strategy. In the best case the lost a sizable amount of money and had to close their variable annuity portfolio for new business.

### *10.7.2 Delta - hedge*

The $\Delta$ hedge is the most simple of all possible non trivial hedging strategies. In the set-up of above there is just one function $f$ which is the $\Delta(S(0))$. Most commonly one uses futures as corresponding hedge assets. The $\Delta$ of a future is 1, with $\Gamma = 0$. Hence we have the following set up:

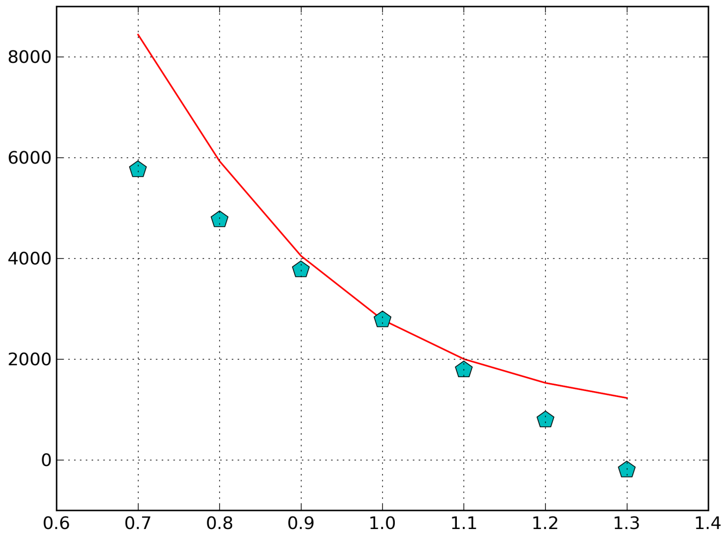


**Fig. 10.6** Adjusted Comparison

| | $\frac{S_t}{S_0}$ | $\Delta r$ | $\Delta\nu$ | Type | Value | $w_k$ |
|---|---|---|---|---|---|---|
| 1 | 0.70 | 0.00 | 0.00 | $\pi$ | 8’442.90 | 0.00 |
| 2 | 0.80 | 0.00 | 0.00 | $\pi$ | 5’931.90 | 0.00 |
| 3 | 0.90 | 0.00 | 0.00 | $\pi$ | 4’051.70 | 0.00 |
| 4 | 1.00 | 0.00 | 0.00 | $\Delta$ | -9’925.50 | 1.00 |
| 5 | 1.00 | 0.00 | 0.00 | $\Gamma$ | -45.31 | 0.00 |
| 6 | 1.00 | 0.00 | 0.00 | $\pi$ | 2’784.50 | 10’000.00 |
| 7 | 1.00 | 0.00 | 0.00 | $\rho$ | 325’170.00 | 0.00 |
| 8 | 1.00 | 0.00 | 0.00 | $\nu$ | 47’226.00 | 0.00 |
| 9 | 1.10 | 0.00 | 0.00 | $\pi$ | 2’004.00 | 0.00 |
| 10 | 1.20 | 0.00 | 0.00 | $\pi$ | 1’531.30 | 0.00 |
| 11 | 1.30 | 0.00 | 0.00 | $\pi$ | 1’232.20 | 0.00 |

As explained above the candidate assets consist of $\mathcal{T} = \{\mathcal{B}, \mathcal{F}\}$. The optimiser determines the following hedge portfolio:

| | Type | $n$ | $S_t$ | Strike | Number |
|---|---|---|---|---|---|
| 1 | $\mathcal{B}$ | 0.00 | – | – | 0.00 |
| 2 | $\mathcal{F}$ | – | 1’841.10 | 1’841.10 | -5.39 |

This results in the following response function (trading grid for the hedge assets):

| | $\pi$ | $\Delta$ | $\Gamma$ | $\rho$ | $\nu$ |
|---|---|---|---|---|---|
| 1.30 | -2'975.70 | -12'903.00 | 0.00 | 0.00 | 0.00 |
| 1.20 | -1'983.10 | -11'911.00 | 0.00 | 0.00 | 0.00 |
| 1.10 | -990.55 | -10'918.00 | 0.00 | 0.00 | 0.00 |
| 1.00 | 2.00 | -9'925.50 | 0.00 | 0.00 | 0.00 |
| 0.90 | 994.55 | -8'933.00 | 0.00 | 0.00 | 0.00 |
| 0.80 | 1'987.10 | -7'940.40 | 0.00 | 0.00 | 0.00 |
| 0.70 | 2'979.70 | -6'947.90 | 0.00 | 0.00 | 0.00 |

The following pictures (10.6 and 10.7) show how variable annuities and hedge assets move for various equity level movements. One can clearly see that the futures do not have any convexity (ie. $\Gamma = 0$) and that in consequence the hedge is more effective around the current market levels, than a more extreme levels.

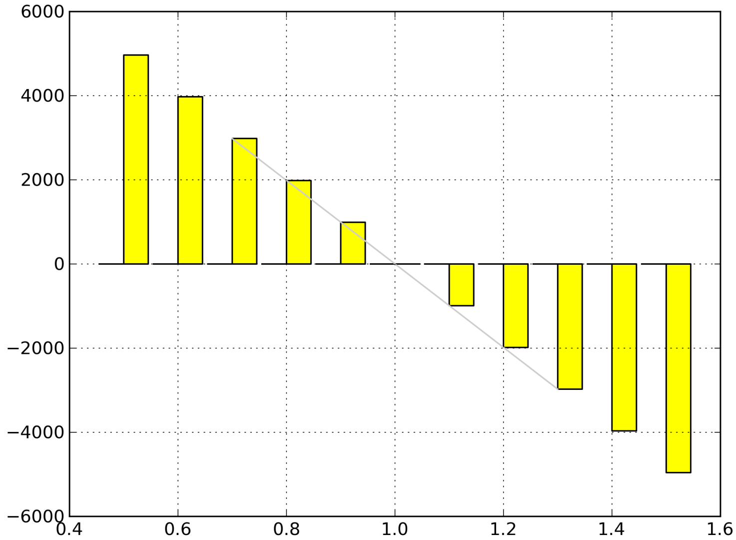


**Fig. 10.7** Decomposition P&L

### 10.7.3 Delta - Gamma - hedge

For a $\Delta - \Gamma$-hedge the approach is similar, namely we try to approximate both $\Delta$ and $\Gamma$ at current market levels. Since futures do not have any convexity we still need to add an option to the candidate assets such that the optimiser can optimise for $\Gamma$. Hence we have the following set up:

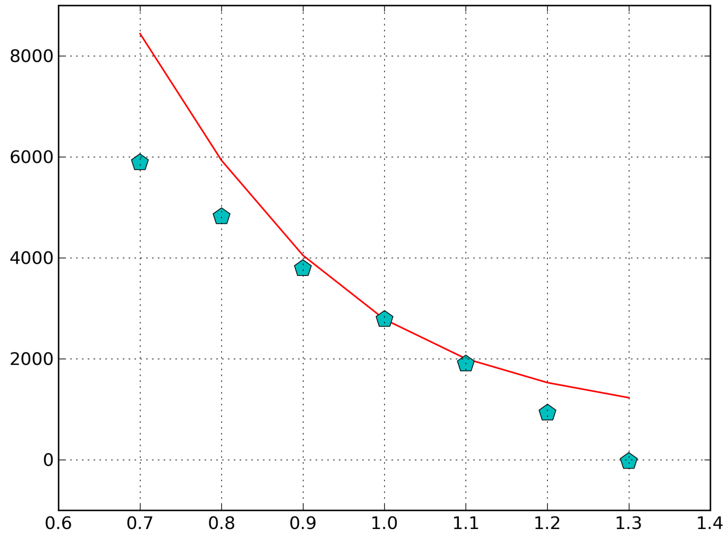


**Fig. 10.8** Adjusted Comparison

| | $\frac{S_t}{S_0}$ | $\Delta r$ | $\Delta \nu$ | Type | Value | $w_k$ |
|---|---|---|---|---|---|---|
| 1 | 0.70 | 0.00 | 0.00 | $\pi$ | 8'442.90 | 0.00 |
| 2 | 0.80 | 0.00 | 0.00 | $\pi$ | 5'931.90 | 0.00 |
| 3 | 0.90 | 0.00 | 0.00 | $\pi$ | 4'051.70 | 0.00 |
| 4 | 1.00 | 0.00 | 0.00 | $\Delta$ | -9'925.50 | 1.00 |
| 5 | 1.00 | 0.00 | 0.00 | $\Gamma$ | -45.31 | 400.00 |
| 6 | 1.00 | 0.00 | 0.00 | $\pi$ | 2'784.50 | 10'000.00 |
| 7 | 1.00 | 0.00 | 0.00 | $\rho$ | 325'170.00 | 0.00 |
| 8 | 1.00 | 0.00 | 0.00 | $\nu$ | 47'226.00 | 0.00 |
| 9 | 1.10 | 0.00 | 0.00 | $\pi$ | 2'004.00 | 0.00 |
| 10 | 1.20 | 0.00 | 0.00 | $\pi$ | 1'531.30 | 0.00 |
| 11 | 1.30 | 0.00 | 0.00 | $\pi$ | 1'232.20 | 0.00 |

In this case we consider the following test assets consisting of cash, futures and one option:

$$\mathcal{T} = \{\mathcal{B}, \mathcal{F}, \mathcal{P}_1(c \in [0.7, 0.9], t \in [1, 2])\}.$$

The optimiser determines the following hedge portfolio:

| | Type | $t$ | $S_t$ | Strike | Number |
|---|---|---|---|---|---|
| 1 | $\mathcal{B}$ | 0.00 | – | – | 5.00 |
| 2 | $\mathcal{F}$ | – | 1’841.10 | 1’841.10 | -5.13 |
| 3 | $\mathcal{P}_1$ | 2.00 | 1’841.10 | 1’657.00 | 1.07 |

This results in the following response function (trading grid for the hedge assets):

| | $\pi$ | $\Delta$ | $\Gamma$ | $\rho$ | $\nu$ |
|---|---|---|---|---|---|
| 1.30 | -2’810.30 | -12’421.00 | 0.40 | -315.35 | 398.60 |
| 1.20 | -1’851.50 | -11’555.00 | 0.60 | -500.57 | 552.51 |
| 1.10 | -883.09 | -10’722.00 | 0.87 | -771.61 | 722.66 |
| 1.00 | 99.97 | -9’925.50 | 1.15 | -1’145.10 | 874.49 |
| 0.90 | 1’104.40 | -9’158.20 | 1.40 | -1’619.40 | 952.11 |
| 0.80 | 2’137.40 | -8’394.20 | 1.48 | -2’157.40 | 895.51 |
| 0.70 | 3’204.00 | -7’587.90 | 1.29 | -2’677.90 | 684.21 |

The following pictures (10.8 and 10.9) show how variable annuities and hedge assets move for various equity level movements. In comparison to the pure $\Delta$-hedge assets and liabilities are moving more in parallel and the some of the tail risk has been removed.

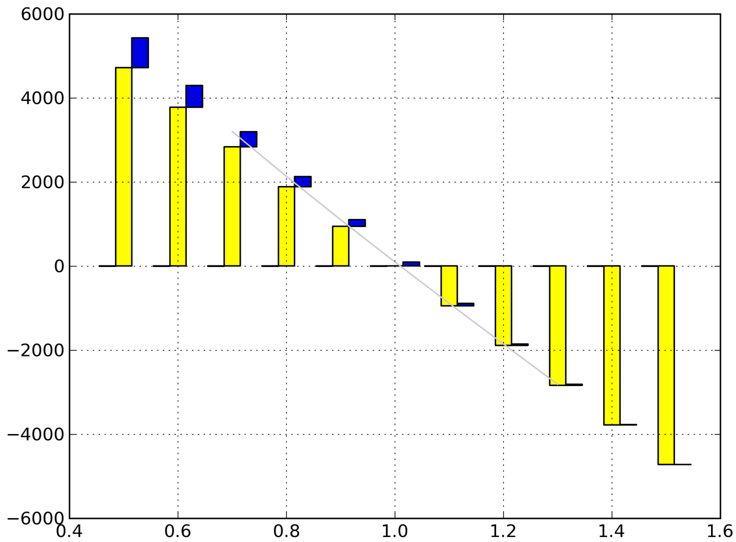


**Fig. 10.9** Decomposition P&L

### 10.7.4 Tail-hedge

We have seen that both the $\Delta$ and the $\Delta - \Gamma$ hedging strategy focus on the body of the distribution. Another approach is to try to protect the balance sheet for more extreme stresses. Hence one tries to immunise for a wider range of market stresses such as for stresses up to $\pm 30\%$. In consequence more options are needed as candidate assets. The following table shows how we choose the weights in this case:

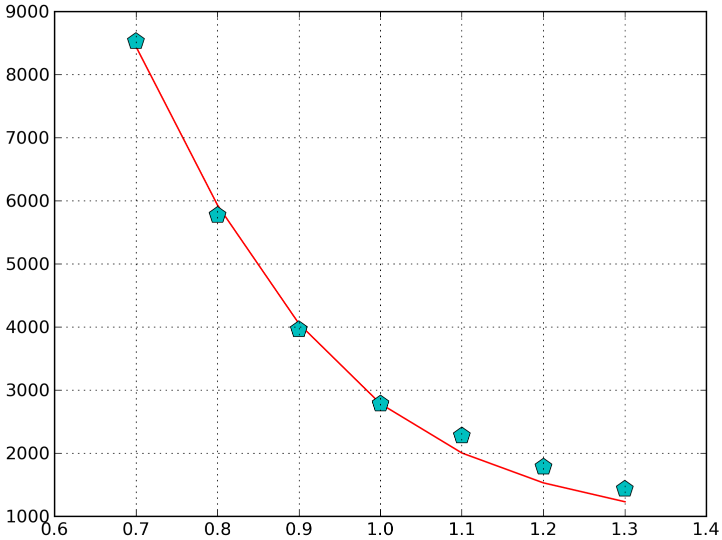


**Fig. 10.10** Adjusted Comparison

| | $\frac{S_t}{S_0}$ | $\Delta r$ | $\Delta \nu$ | Type | Value | $w_k$ |
|---|---|---|---|---|---|---|
| 1 | 0.70 | 0.00 | 0.00 | $\pi$ | 8’442.90 | 1.00 |
| 2 | 0.80 | 0.00 | 0.00 | $\pi$ | 5’931.90 | 1.00 |
| 3 | 0.90 | 0.00 | 0.00 | $\pi$ | 4’051.70 | 1.00 |
| 4 | 1.00 | 0.00 | 0.00 | $\Delta$ | 9’925.50 | 0.00 |
| 5 | 1.00 | 0.00 | 0.00 | $\Gamma$ | 45.31 | 0.00 |
| 6 | 1.00 | 0.00 | 0.00 | $\pi$ | 2’784.50 | 10’000.00 |
| 7 | 1.00 | 0.00 | 0.00 | $\rho$ | 325’170.00 | 0.00 |
| 8 | 1.00 | 0.00 | 0.00 | $\nu$ | 47’226.00 | 0.00 |
| 9 | 1.10 | 0.00 | 0.00 | $\pi$ | 2’004.00 | 1.00 |
| 10 | 1.20 | 0.00 | 0.00 | $\pi$ | 1’531.30 | 1.00 |
| 11 | 1.30 | 0.00 | 0.00 | $\pi$ | 1’232.20 | 1.00 |

In this example we also consider a rich portfolio of test assets. The following table shows the test assets which we use ($\mathcal{T}$):

 

| Asset | Freedom |
|---|---|
| $\mathcal{P}_1$ | $t \in [4,5], n \in [0.5,0.6], \pi(\bullet) \in [0,200]$ |
| $\mathcal{P}_2$ | $t \in [9,10], n \in [0.5,0.6], \pi(\bullet) \in [0,200]$ |
| $\mathcal{P}_3$ | $t \in [1,2], n \in [0.5,0.6], \pi(\bullet) \in [0,200]$ |
| $\mathcal{P}_4$ | $t \in [9,10], n \in [0.8,0.9], \pi(\bullet) \in [0,200]$ |
| $\mathcal{P}_5$ | $t \in [1,2], n \in [0.8,0.9], \pi(\bullet) \in [0,200]$ |
| $\mathcal{P}_6$ | $t \in [4,5], n \in [0.8,0.9], \pi(\bullet) \in [0,200]$ |
| $\mathcal{C}_1$ | $t \in [9,9.2], n \in [1.2,1.4], \pi(\bullet) \in [-2,2]$ |
| $\mathcal{C}_2$ | $t \in [4,4.2], n \in [1.2,1.4], \pi(\bullet) \in [-2,2]$ |
| $\mathcal{C}_3$ | $t \in [1,1.2], n \in [1.2,1.4], \pi(\bullet) \in [-2,2]$ |
| $\mathcal{B}$ | |
| $\mathcal{F}$ | |
| Total Value | $\sum_i \pi(\mathcal{A}_i) \leq 250$ |

For this minimisation problem the optimiser found the following result:

| | Type | $t$ | $S_t$ | Strike | Number |
|---|---|---|---|---|---|
| 1 | $\mathcal{B}$ | 0.00 | – | – | 75.55 |
| 2 | $\mathcal{C}$ | 1.19 | 1'841.10 | 2'419.00 | 0.66 |
| 3 | $\mathcal{C}$ | 4.01 | 1'841.10 | 2'211.90 | -1.48 |
| 4 | $\mathcal{C}$ | 9.12 | 1'841.10 | 2'209.40 | -1.59 |
| 5 | $\mathcal{F}$ | – | 1'841.10 | 1'841.10 | 1.05 |
| 6 | $\mathcal{P}$ | 1.00 | 1'841.10 | 920.57 | 64.37 |
| 7 | $\mathcal{P}$ | 1.38 | 1'841.10 | 1'657.00 | 15.24 |
| 8 | $\mathcal{P}$ | 4.29 | 1'841.10 | 1'657.00 | 0.00 |
| 9 | $\mathcal{P}$ | 4.36 | 1'841.10 | 921.19 | 47.81 |
| 10 | $\mathcal{P}$ | 9.00 | 1'841.10 | 1'472.90 | 0.01 |
| 11 | $\mathcal{P}$ | 9.00 | 1'841.10 | 1'051.50 | 0.00 |

We get the following response function for the hedge assets:

| | $\pi$ | $\Delta$ | $\Gamma$ | $\rho$ | $\nu$ |
|---|---|---|---|---|---|
| 1.30 | -1'349.50 | -3'927.20 | 5.27 | -24'718.00 | 857.25 |
| 1.20 | -1'000.10 | -4'894.80 | 9.31 | -25'599.00 | 3'803.20 |
| 1.10 | -504.29 | -6'625.00 | 15.17 | -28'053.00 | 7'884.70 |
| 1.00 | 250.00 | -9'335.50 | 22.67 | -32'875.00 | 12'892.00 |
| 0.90 | 1'426.00 | -13'087.00 | 30.89 | -40'904.00 | 18'276.00 |
| 0.80 | 3'239.40 | -17'762.00 | 39.98 | -53'010.00 | 23'848.00 |
| 0.70 | 5'996.00 | -23'731.00 | 57.61 | -70'853.00 | 31'398.00 |

The following pictures (10.10 and 10.11) show how variable annuities and hedge assets move for various equity level movements. We see that this tail hedging strategy protects the balance sheet best in case of material adverse developments.

**Example 67** *In a next step we will apply this technique to example 59.. In this case we have the following variable annuity response function*

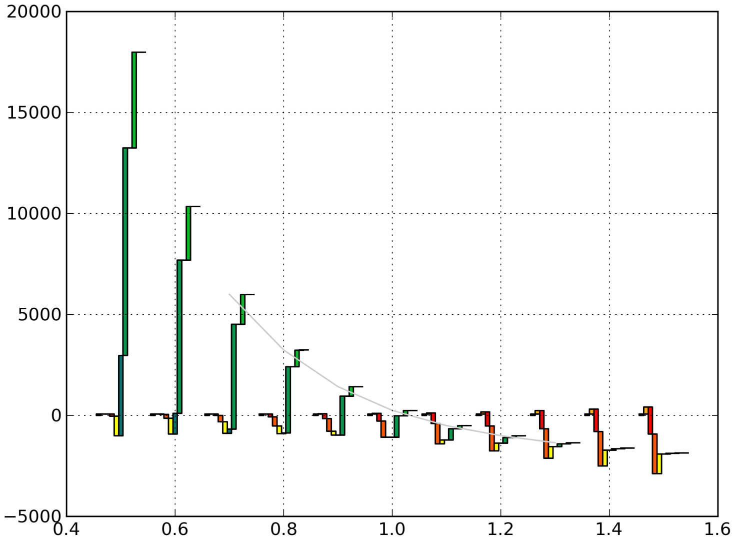


**Fig. 10.11** Decomposition P&L

| | $\frac{S_t}{S_0}$ | $\Delta r$ | $\Delta \nu$ | *Type* | *Value* | $w_k$ |
|---|---|---|---|---|---|---|
| *1* | *0.50* | *0.00* | *0.00* | $\pi$ | *15'837.00* | *1.00* |
| *2* | *0.60* | *0.00* | *0.00* | $\pi$ | *12'532.00* | *1.00* |
| *3* | *0.70* | *0.00* | *0.00* | $\pi$ | *10'129.00* | *1.00* |
| *4* | *0.80* | *0.00* | *0.00* | $\pi$ | *8'541.80* | *1.00* |
| *5* | *0.90* | *0.00* | *0.00* | $\pi$ | *7'962.10* | *1.00* |
| *6* | *1.00* | *0.00* | *0.00* | $\pi$ | *7'806.50* | *10'000.00* |
| *7* | *1.10* | *0.00* | *0.00* | $\pi$ | *8'353.20* | *1.00* |
| *8* | *1.20* | *0.00* | *0.00* | $\pi$ | *9'140.40* | *1.00* |
| *9* | *1.30* | *0.00* | *0.00* | $\pi$ | *9'952.20* | *1.00* |
| *10* | *1.40* | *0.00* | *0.00* | $\pi$ | *10'715.00* | *1.00* |
| *11* | *1.50* | *0.00* | *0.00* | $\pi$ | *11'551.00* | *1.00* |

*As before we first define a rich portfolio of test assets. The following table shows the test assets which we use ($\mathcal{T}$):*

| *Asset* | *Freedom* |
|---|---|
| $\mathcal{P}_1$ | $t \in [4,5], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{P}_2$ | $t \in [9,10], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{P}_3$ | $t \in [1,2], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{P}_4$ | $t \in [9,10], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{P}_5$ | $t \in [1,2], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{P}_6$ | $t \in [4,5], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$ |
| $\mathcal{C}_1$ | $t \in [9,9.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$ |
| $\mathcal{C}_2$ | $t \in [4,4.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$ |
| $\mathcal{C}_3$ | $t \in [1,1.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$ |
| $\mathcal{B}$ | |
| $\mathcal{F}$ | |
| *Total Value* | $\sum_i \pi(\mathcal{A}_i) \leq 250$ |

*For this minimisation problem the optimiser found the following result:*

| | *Type* | $t$ | $S_t$ | *Strike* | *Number* |
|---|---|---|---|---|---|
| *1* | $\mathcal{B}$ | *0.00* | *None* | *None* | *64.27* |
| *2* | $\mathcal{C}$ | *1.20* | *1'841.10* | *2'238.70* | *-2.00* |
| *3* | $\mathcal{C}$ | *4.20* | *1'841.10* | *2'209.40* | *-1.99* |
| *4* | $\mathcal{C}$ | *9.20* | *1'841.10* | *2'221.50* | *-1.99* |
| *5* | $\mathcal{F}$ | *None* | *1'841.10* | *1'841.10* | *9.61* |
| *6* | $\mathcal{P}$ | *1.00* | *1'841.10* | *1'657.00* | *6.99* |
| *7* | $\mathcal{P}$ | *1.00* | *1'841.10* | *920.57* | *66.28* |
| *8* | $\mathcal{P}$ | *4.00* | *1'841.10* | *1'634.50* | *9.92* |
| *9* | $\mathcal{P}$ | *5.00* | *1'841.10* | *920.57* | *38.77* |
| *10* | $\mathcal{P}$ | *9.00* | *1'841.10* | *1'472.90* | *0.00* |
| *11* | $\mathcal{P}$ | *9.00* | *1'841.10* | *1'071.80* | *0.00* |

*In a first step we calculate the (asset) trading grid for the assets which we have found:*

| | $\pi$ | $\Delta$ | $\Gamma$ | $\rho$ | $\nu$ |
|---|---|---|---|---|---|
| *1.30* | *2'148.00* | *9'541.30* | *0.92* | *-43'607.00* | *1'562.20* |
| *1.20* | *1'426.30* | *8'427.60* | *3.64* | *-44'763.00* | *4'525.00* |
| *1.10* | *764.27* | *6'680.90* | *8.53* | *-47'550.00* | *8'755.20* |
| *1.00* | *249.97* | *3'970.40* | *15.64* | *-52'845.00* | *14'077.00* |
| *0.90* | *28.52* | *108.40* | *23.99* | *-61'586.00* | *19'850.00* |
| *0.80* | *303.40* | *-4'860.70* | *33.40* | *-74'685.00* | *25'691.00* |
| *0.70* | *1'370.80* | *-11'370.00* | *51.88* | *-93'739.00* | *33'203.00* |

**Exercise 68** *Calculate the trading grid for example 59 and identify a hedging portfolio alongside the example 67.*

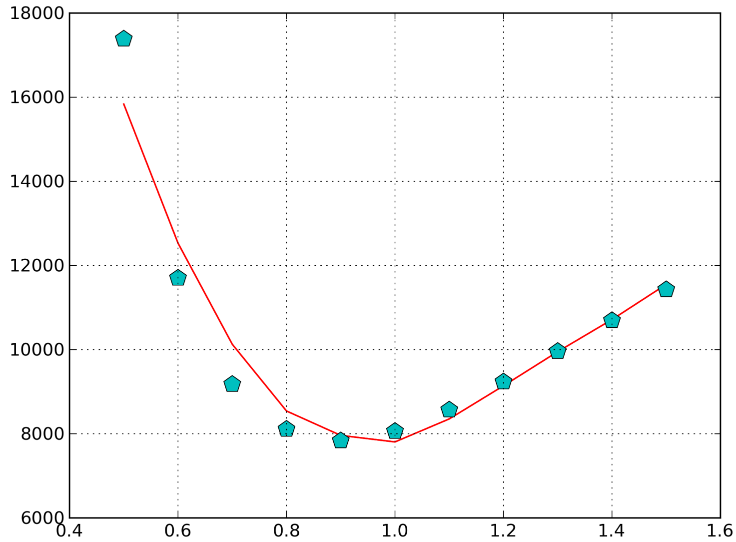


**Fig. 10.12** Adjusted Comparison

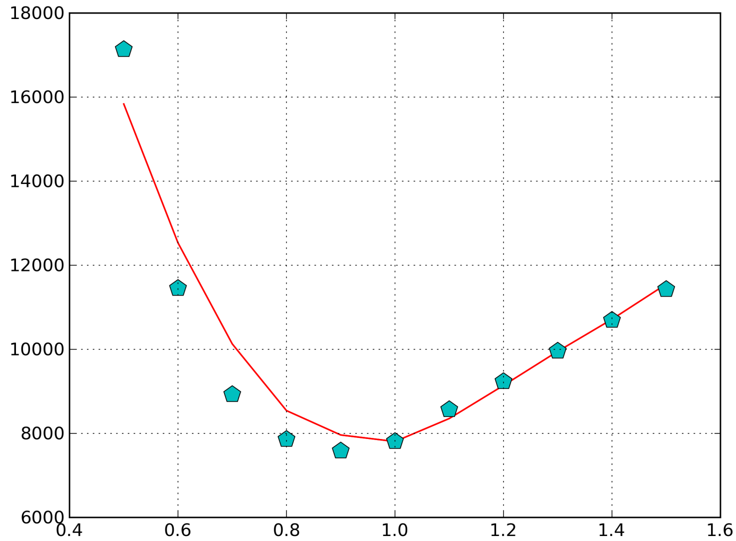


**Fig. 10.13** Adjusted Comparison

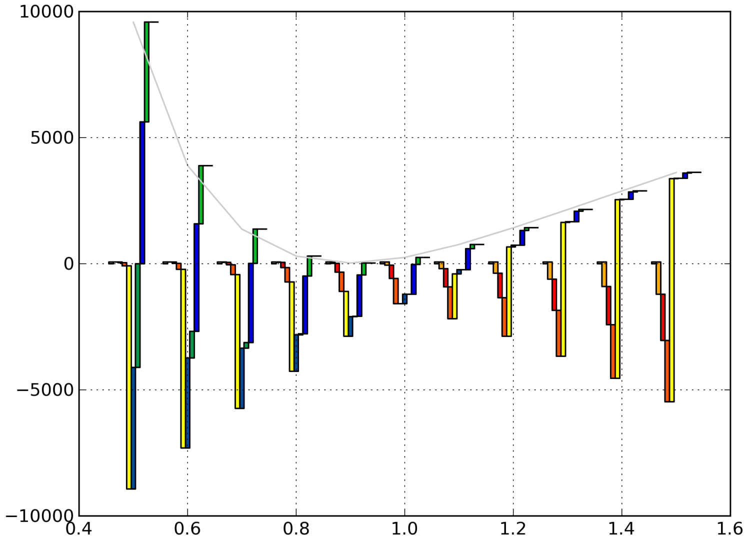


**Fig. 10.14** Decomposition P&L

## 10.8 Risk Mitigation Strategies within Funds

For most of the traditional VA's the hedging of the fund values takes place outside the fund, as described above. There is the possibility to mitigate this risk within the fund, by applying respective fund management strategies. The advantage of such strategies is that one can implicitly hedge some of the more complex and costly risks, such as the long term volatility risk. Doing so one introduces additional risks, which relate to the way this dynamic hedging strategies within the fund work. The downside of such "self-hedging funds" is their opacity for the client and the fact that the fund performance does normally not match an index. In the following we will have a look at two typical strategies of self-hedging funds:

- Target volatility funds, and
- Mandatory asset transfers.

## 10.9 Target Volatility Funds

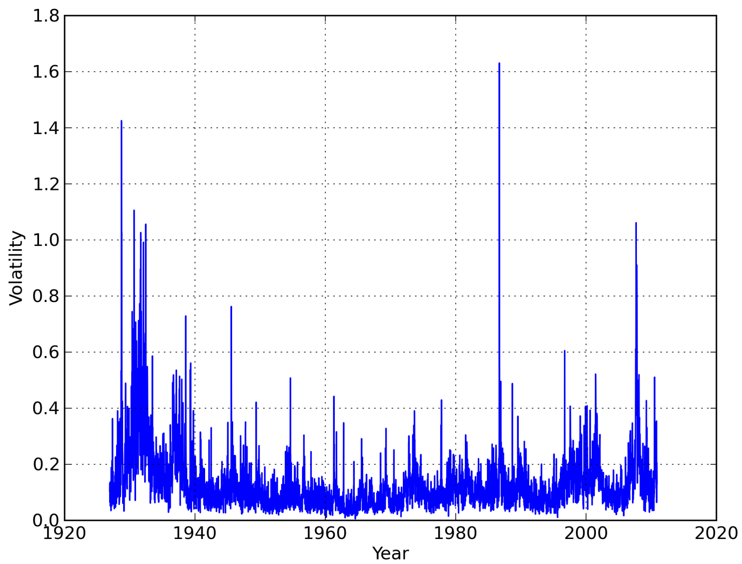


**Fig. 10.15** Realised Volatility of S&P 500 over time

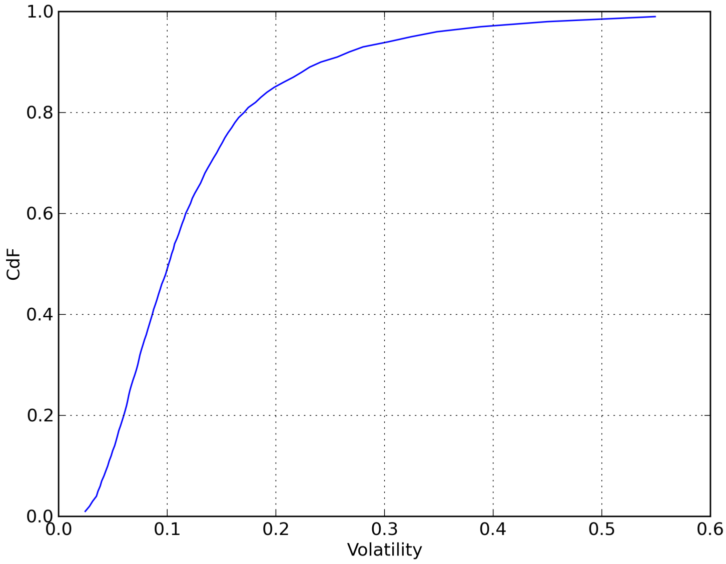


**Fig. 10.16** Probability Distribution of Realised Volatility of S&P 500

In order to do $\Delta$-hedging the underlying volatility assumption is crucial. A higher effective volatility generally results in hedging losses and buying options to hedge the variable annuity liabilities become more expensive. This effects are particularly important in case of a paradigm shift to a generally higher level of volatility. The idea of a target volatility fund is to dynamically manage the equity backing ration of the funds in such a way that it results it the required target volatility. Hence in cases of a low volatility environment, the equity backing ratio is increased and decreased again if volatility levels start to rise.

In terms of a model we assume that we have two assets, namely a bank account $(B_t)$ and shares $(S_t)$. We assume that the volatility is as follows:

$$\sigma(B_t) = 0 \text{ and}$$
$$\sigma(S_t) = f(t) \times \sigma$$

for a suitable function $f(t)$ which depends on the time. When holding a portfolio with $\alpha$ shares and $1-\alpha$ cash, we can calculate the total volatility as follows:

$$\sigma((1-\alpha)\,B_t + \alpha\,S_t) = \alpha\,f(t).$$

Hence for a target volatility of $\tilde{\sigma}$ one needs to choose $\alpha$ as $\alpha = \frac{\tilde{\sigma}}{f(t)\,\sigma}$.

While this approach seems rather easy there are some complexities hidden in it. The first one is the way one finds $f(t)$. Normally $f$ is based on a suitable estimator based on the effective past fund's return. Here it is important to choose an updating volatility estimator which is able to update fast enough times where there are volatility spikes, since otherwise there will be hedging losses. The second complexity is a consequence of possible limits with respect to the funds composition. Hence one might need to leverage (eg $\alpha > 1$) in case of a very low volatility environment. This would necessitate, that the fund can do this leverage and borrow cash. Moreover in case of very high volatility, the equity backing ratio can fall to very low levels. Also this needs to be possible in the funds.

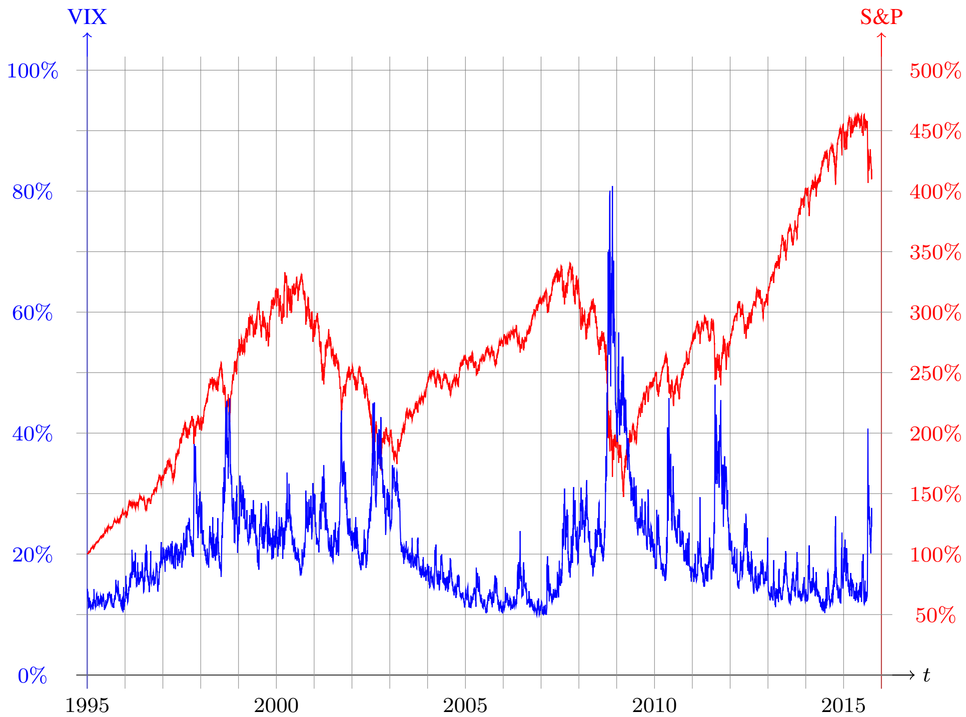


**Fig. 10.17** Implied Volatility ("VIX") and S&P level

The figure 10.17 shows the movement of the implied volatility during 2010 and 2011. This figure shows besides the spot observed volatility, also the average volatility over the last 20 and 200 trading days. Here the effect of a volatility estimator based on the past becomes apparent. In case of rapidly rising volatility levels (eg August 2011), the estimator underestimates the effective volatility. Hence during this time period the fund has a too high equity backing ratio. On the other hand after


**Fig. 10.18** Performance of volatility controlled funds

a fall of volatility (eg October 2011), the estimator shows a too high volatility, and in consequence the fund has a too low equity backing ratio. Hence the choice of this estimator is very important to construct target volatility funds. We want this section with an example showing the performance of volatility controlled funds over the period from 1995 to 2015. We consider an S&P funds, which has a target volatility of $\sigma_T = 20.5\%$, which almost matches the average volatility of VIX over this time period. There are two way how the target volatility can be interpeted, namely:

1. The funds is allowed to be levered and invest more than 100% in the S&P index, and
2. The fund can only reduce the equity holding.

In both cases we estimate the current volatility at time $t$ as

$$\hat{\sigma}(t) = \frac{1}{60} \sum_{k=-59}^{0} VIX(t+k),$$

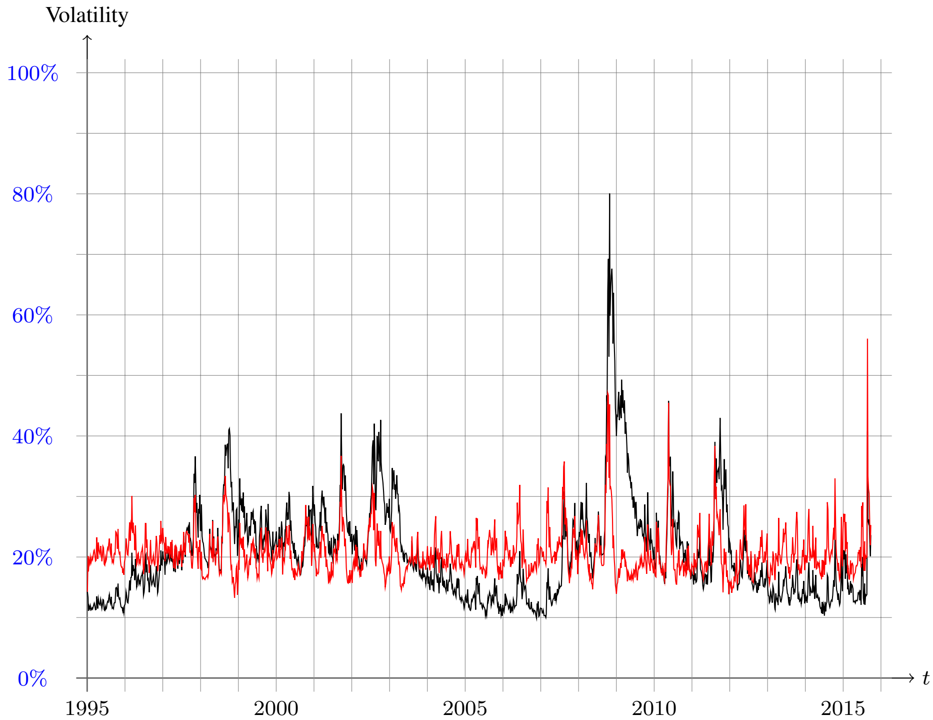


**Fig. 10.19** Volatility of volatility controlled funds

which means that we take the average volatility over the past 60 trading days (eg over c. 3 months). For the two interpretations above we define the equity backing ratio $\alpha(t)$ (eg the percentage the funds invests in equities) as follows:

**No leverage:**

$$\alpha(t) = \max\{1, \frac{\sigma_T}{\hat{\sigma}(t)}\}.$$

**Leverage possible:**

$$\alpha(t) = \frac{\sigma_T}{\hat{\sigma}(t)}.$$

Figure 10.18 shows the respective performance of the two volatility controlled funds together with the performance of the S&P index. It becomes obvious that the funds, where no leverage is allowed shows a worse performance over the considered time period. On the other hand it is worth mentioning that the equity backing ratio varies over this period from c40% in 2009 to 190% in 2007. Finally Figure 10.19 shows the implied volatility of the funds for each point in time.

## 10.10 CPPI type Funds

The idea of the mandatory asset transfer mechanism is to derisk the funds in case a guarantee gets into the money. Hence this approach is very similar to CPPI products, where the equity backing ratio depends on the ratio on the “in-the-moneyness”. The more the guarantee is in the money, the lower the equity backing ratio. Hence the investments of the policyholder are shifted in to cash to protect them from downside movements. The issue with this approach is the fact that once all the money is switched into cash, there is no way back again into more risky assets. Hence the state “cash” is “absorbing”. This absorption can take also in case the equity market dip is only very short.

Hence this type of funds are effective in managing the downside risk, but case of a bad outcome, the policyholder sits on a cash investment. The way such as CPPI product works, is based on a similar concept as the “in-the-moneyness” concept. We consider a GMWV with a funds value $FV$ and a currently guaranteed withdrawal amount of $R$. The quantity which determines the mandatory asset derisking is based on the following function:

$$f(FV, x, i) = \frac{FV}{R \times \ddot{a}_x(i)},$$

where $\ddot{a}_x(i)$ denotes the present value factor of an immediate annuity for a x year aged person, using a technical interest rate of $i$. The function $f$ measures the over-/under-coverage of the actual funds value to cover an immediate annuity. The higher this ration, the less the guarantee is *economically* in the money. Below a certain threshold the mandatory asset transfer is triggered. Figure 10.20 shows the level at which a such a mandatory asset transfer could be triggered for different interest rate levels. The lower the interest rate level, the earlier the trigger. Figure 10.21 shows the impact of this CCPI mechanism for a certain asset performance trajectory. The bars in gray are financed by the actual funds value of the VA. The green bars represent the parts of the GMLB guarantee which is paid by the rider, in both cases, ie with and without the CPPI mechanism. The red bars represent the part of the guarantee which is mitigated by the application of the CPPI mechanism, hence they form only part of the guarantee in case of absence of the CPPI mechanism. We see that in this example about 4 annuities can still be paid by the funds of the policyholder when applying the CPPI approach.

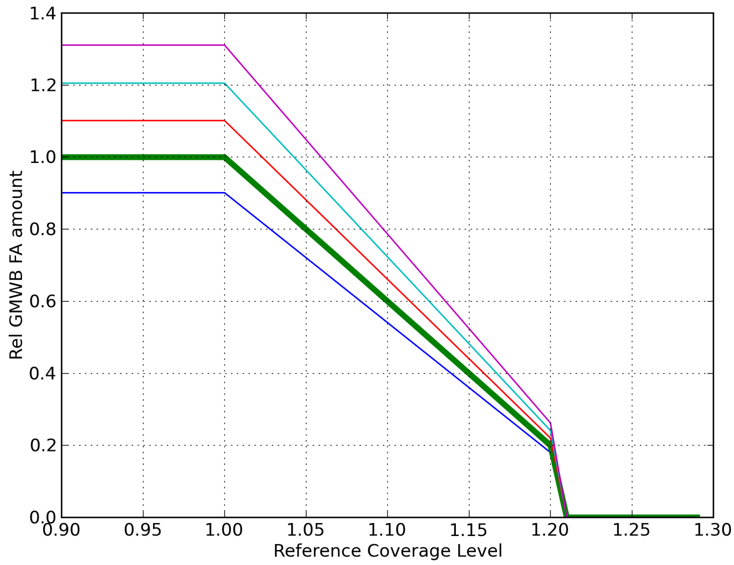


**Fig. 10.20** Level of mandatory asset allocation shift

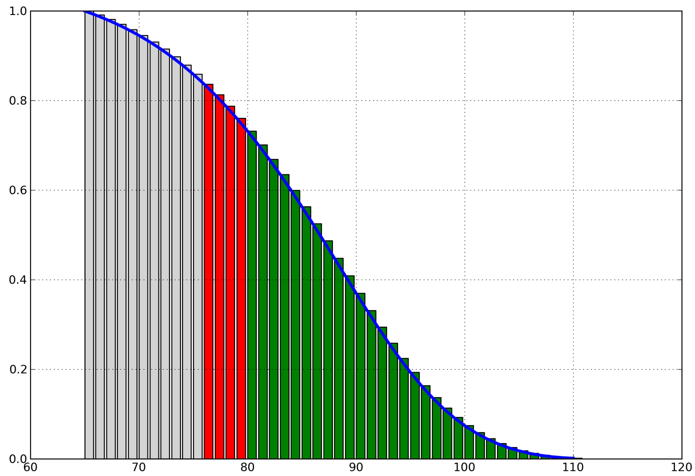


**Fig. 10.21** Impact of a CPPI mechanism

 

# Chapter 11

# Strategic Risk Management

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# Chapter 12

# Capital Models and Integrated Risk Management

## 12.1 Introduction

In this chapter we want to see how the different pieces of the capital models flow together in order to get an integrated capital model, covering all the different risk categories. In a lot of cases the capital models for insurance companies have been designed along the following categories:

- Financial and ALM risk,
- Life insurance risk,
- General insurance risk,
- Operational risk.

The reason for building these distinct risk modules was that there were people focusing on ALM issues, such as life risk etc. Hence it was a consequence of the relative skill set of the people and of the relative importance of the risks. Sometimes some of the risks were merely modelled as a consequence of a regulatory requirement, such as operational risks. The methods used for the different risk categories are often different and ultimately there is the question on how to link the sub-modules together. This is the same question as linking the individual risk factors within each risk module together.

From a holistic point of view it is important that the company can cover the required capital stemming from all risk types with the available risk capital.

## 12.2 Bringing the Puzzle together

From a technical point of view we are in a situation where we have a set of risk categories $(\mathcal{R}_\kappa)_{\kappa\in\mathcal{K}}$ and for each risk category $\kappa \in \mathcal{K}$ we have a corresponding loss function $X_\kappa$ with a probability density function $F_{X_\kappa}(t)$. So what we actually know per $X_\kappa$ is its marginal distribution if we consider $(X_\kappa)_{\kappa\in\mathcal{K}}$ as a multidimensional random variable.

One possibility is to link the $(\mathcal{R}_\kappa)_{\kappa\in\mathcal{K}}$ together with *copulas*. In order to understand this concept, we need to look at two random variables $X$ and $Y$ with cumulative distribution functions $F_X$ and $F_Y$ respectively. Furthermore we remark that in this case both of the following random variables $\tilde{X}$ and $\tilde{Y}$ are uniformly $[0,1]$–distributed:

$$\tilde{X} = F_X(X),$$
$$\tilde{Y} = F_Y(Y).$$

We remark that if $(X,Y)$ are dependent, this holds true also for $(\tilde{X},\tilde{Y})$. A copula is hence a function which transforms the random variables $(\tilde{X},\tilde{Y})$. More formally: A copula is a multivariate (n-dimensional) joint distribution on $[0,1]^n$, such that every marginal distribution is uniform on the interval $[0,1]$. A function

$$C : [0,1]^n \to [0,1], (x_1,\dots x_n) \mapsto C(x_1,\dots x_n)$$

is an n–dimensional copula if the following hold:

1. $C(u) = 0$ for $u = (x_1,\dots x_n) \in [0,1]^n$ if one of the $x_k = 0$,
2. $C(u) = x_j$ for $u = (x_1,\dots x_n) \in [0,1]^n$ if $x_i = 1 \forall i \neq j$,
3. $C$ is increasing for all hyperrectagles $R \subset [0,1]^n$.

Sklar's theorem states the following for the bivariate case: For $H(x,y)$ a bivariate cumulative probability distribution function with marginal cumulative probability functions $F_X(x) := H(x,\infty)$ and $F_Y(y) := H(\infty,y)$, there exists a copula with

$$H(x,y) = C(F_X(x), F_Y(y)).$$

Hence copulas are a means to link individual random variables together and we want to have a look at the most important classes of copulas:

**Gaussian copula:** For $\rho \in \mathbb{R}$, we denote by

$$C_\rho(x,y) := \Phi_\rho(\Phi^{-1}(x), \Phi^{-1}(y)), \text{ with}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(\frac{\zeta^2}{2}) d\zeta,$$
$$\Phi_\rho(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right]\right).$$

**Archimedian copula:**

$$H(x_1, \dots x_n) = \Psi^{-1}\left(\sum_{j=1}^{n} \Psi(F_{x_j}(x_j))\right).$$

$\Psi$ is known as generator function. If some conditions for $\Psi$ are fulfilled, the resulting $C$ is a copula.

**Product copula:** The product copula is a special kind of an Archimedian copula with $\Psi(x) = -\ln(x)$. This copula is also known as independent copula, since we get $H(x,y) = F_X(x) \times F_Y(y)$.

After having determined the relationship between the different loss functions $(X_\kappa)_{\kappa \in \mathcal{K}}$ for the different risk categories $(\mathcal{R}_\kappa)_{\kappa \in \mathcal{K}}$, it is possible to calculate the corresponding n-dimensional cumulative probability density function $F_{(X_1, \dots X_n)}$. The distribution function for the total loss $L = \sum_{i=1}^{n} X_i$ can hence be calculated.

## 12.3 Diversification

Figure 12.1 provides an example for the total capital consumed by an insurance entity. It becomes obvious in this example that consumed capital is not uniform over the risk categories:

| | Risk Category | | Required Capital |
|---|---|---|---|
| (i) | Financial Risk | $X_1$ | 7680 |
| (ii) | Operational Risk | $X_2$ | 900 |
| (iii) | Regulatory | $X_3$ | 450 |
| | Simple Sum | | 9030 |
| | Diversification | | -3010 |
| (iv) | Required Capital | $\sum_{i=1}^{3} X_i$ | 6020 |

Assume for the moment that we are using a 99.5 % VaR as a risk measure. In this context the numbers above are determined in the following manner:

$$7680 = F_{X_1}^{-1}(0.995),$$

$$\begin{aligned} 900 &= F_{X_2}^{-1}(0.995), \\ 450 &= F_{X_3}^{-1}(0.995), \\ 6020 &= F_{\sum_{i=1}^{3} X_i}^{-1}(0.995). \end{aligned}$$

This means in particular that the diversification effect $((iv) - (i) - (ii) - (iii))$ is a pure consequence on how we group the different risk factors. It is dependent on the way we construct our risk models and how we link them together. It represents the amount of capital which is released by linking the different risks together. Now looking again at the numbers, one might wonder why the diversification is so high. The solution in the concrete example is actually rather easy, since the numbers such as 7860 already represent simple sums. Hence actually (assuming that we have the 13 risk factors mentioned in the table) we have

$$3010 = \sum_{i=1}^{13} F_{X_i}^{-1}(0.995) - F_{\sum_{i=1}^{13} X_i}^{-1}(0.995),$$

which highlights the risk of the concept of diversification. In the table on page 153 you see that diversification can be calculated at different levels of the aggregation.

| Financial Risks | | Operational Risks | | Legal/Regulatory/Compliance Risks | |
|---|---|---|---|---|---|
| **Risk** | **in M €** | **Risk** | **in M €** | **Risk** | **in M €** |
| Credit Risk | 2660 | ALM and Investment Processes | c 300 | Legislative Changes | c 400 |
| Alternative Investments | 520 | Reputational Risks | c 150 | Quantum / Genesis Regulatory Risk | c 25 |
| Equity and GMDB Risk | 2040 | Quantum Leap Programme | c 70 | Solvency II Risk | c 25 |
| Interest Rate Risk | 470 | Switch at known prices of Unit Lined Funds | c 250 | | |
| Lapse Risk | 1990 | Underperformance of distribution partners | c 130 | | |
| **Subtotal** | **7680** | **Subtotal** | **c 900** | **Subtotal** | **c 450** |
| | **c 85%** | | **c 10 %** | | **c 5%** |

| | |
|---|---|
| Total economic capital | 9030 |
| Diversification effect | -3010 |
| **Total Required Economic Capital** | **6020** |

**Fig. 12.1** Required Economical Capital Reporting

As with all other relevant risk capital measurements, it makes sense to define a risk appetite with corresponding limits for the total capital consumed by the insurance company.

# Chapter 13
# Risk adjusted performance Metrics


## 13.1 Introduction

When doing business there are different metrics of great importance, such as

- Statutory profit, since this is the base in order to determine and pay dividends.
- IFRS Profit, since this is one of the most regarded measures, which allows to compare different insurance entities.

- MCEV earnings and the corresponding value of new business, since this allows to determine an proxy for the economic value of the insurance entity and its ability to write profitable new business.
- Required and available risk capital, allowing to steer and measure the risk the company is running in order to achieve its strategic ambitions.

Most of the above mentioned measures are well known and hence we limit ourselves to remark that in some of the above mentioned performance metrics, there is considerable judgement needed for the set-up and the calibration of the underlying models. Furthermore it is worthwhile to remark that for all the metrics one can distinguish between a value and a performance metric. In case of the IFRS, the absolute profit is the value metric and the profit per shareholder equity (eg return on equity) is the performance metric. In the same sense the value of new business measures the value creation as a consequence of writing new business, and the new business margin (eg the value of new business per present value of new premiums).

## 13.2 Performance and Value Metrics

Looking at the above metrics it becomes obvious that all of the above metrics are quite complex and some of them such as the market consistent embedded value extremely difficult to understand. Hence the aim of this section is to introduce a value and a performance metric which is easily understandable and which allows to compare different lines of business, channels or companies. We will introduce a cash flow based performance metric and remark that a suitable interpretation of the market consistent embedded value will be consistent with this view.

We have seen in chapter 4 that the market consistent value of the insurance liabilities is based on cash flows and we will show here that this concept is suited to build a fully blown performance management system.

For convenience purposes the formulae already shown in chapter 4, assuming that the cash-flows considered as random variables are stochastically independent of the financial variables:

$$\begin{aligned}
\mathbb{E}[PV] &= \sum_{k=0}^{\infty} \pi_t(\mathcal{Z}_{(k)}) \times \mathbb{E}[CF_k], \\
MVM = CoC &= \sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)}), \\
IRR &= \frac{\mathbb{E}[PV] + \sum_{k=0}^{\infty} i \times RC_k \times \pi_t(\mathcal{Z}_{(k)})}{\sum_{k=0}^{\infty} RC_k \times \pi_t(\mathcal{Z}_{(k)})},
\end{aligned}$$

$$FCC^* = \sum_{k=0}^{\infty} \beta \left\{ \max(0, V_k - \mathbb{E}[PV]_k) \right\} \times \pi_t(\mathcal{Z}_{(k)}),$$
$$FCC = \max(0, FCC^* - CoC).$$

### 13.2.1 IBNR Reserves

IBNR is usually calculated using a formula based on actual claim experience for prior years, adjusted for current trends and other factors. IBNR should be calculated for all lines of business in all countries. Companies should determine the IBNR reserve either by relying on past experience modified for current conditions or by determining the actual claims reported up to some point in time, such as 30 days after the balance sheet date, and estimating the claims yet to be reported beyond that date. Because death claims are usually reported quickly, the adequacy of the current year IBNR reserve can generally by determined by developing prior year's IBNR and by comparing this development to the year's reserve and giving consideration to various factors such as premiums in force. IBNR for disability business usually reflects a combination of historical claim experience, reasonable future expectations and the actual waiting periods for the in-force block.

The valuation of IBNR reserves must be according to the corresponding standards. In case of a traditional embedded value there is usually no value attributed to IBNR reserves, which themselves qualify as technical reserves. In case of a market consistent embedded value, IBNR Reserves are the present values of the expected future cash flow streams in relation to the IBNR cash flows using risk free discount rates.

### 13.2.2 Financial Options

In order to reflect financial options in the value of business in force, such as GMDB's etc, calculations to assess their value need to be performed based on a risk neutral method such as arbitrage free pricing or also Black-Scholes.

This valuation can either be done by explicit formulae (such as in the Black-Scholes context), or can also be based on a general risk neutral valuation method (such as deflators, martingale methods, Monte Carlo simulations, etc.)

The value of options needs to be shown separately and the parameters for its calculation are based on observable data at balance sheet date (for example in relation to the risk free rate, the volatility etc.). These parameters can either be estimated directly by analysing the underlying assets or also by using implicit methods (such as the calculation of the implicit volatility given the price of stock options.)

### 13.2.3 Frictional Capital Costs

Finally we need to realise that in the real world there are additional constraints, which have an impact on the value of a portfolio or a product sold. The most relevant are listed below:

- Frictional costs and
- Taxes.

Frictional costs stem from the fact, that the company needs to hold at a certain time the corresponding statutory reserves $V_t$ for an underlying block of business. Additional frictional costs are induced by solvency requirements which are higher the economical risk capital. Given the fact that the best estimates liabilities $\mathbb{E}[PV]$ may be inferior, the company needs to hold this additional amount, resulting in the above mentioned (pure) frictional capital costs:

$$FCC^* = \sum_{t=0}^{\infty} \beta \left\{ \max(0, V_t - \mathbb{E}[PV]_t) \right\} \times \pi(\mathcal{Z}_{(t)}),$$

where $\mathbb{E}[PV]_t$ denotes the expected present value of the future liabilities as seen at time $t$. Based on the fact that the risk capital also qualifies as capital to fill up missing reserves, the total frictional capital costs amount to:

$$FCC = \sum_{t=0}^{\infty} \beta_2 \left\{ \max(0, V_t - \mathbb{E}[PV]_t - RC_t) \right\} \times \pi(\mathcal{Z}_{(t)}).$$

In a further step we will distinguish between the sort of frictional capital we need:

- Frictional capital which has been financed by the policyholder: This is the difference between the best estimate liabilities ($\mathbb{E}[PV]_t$) and the carrying amount of reserves in the company's balance sheet. This part of capital is similar to capital provided by letters of credits by bank for a relatively small cost and will therefore be charged less ($\beta_3$).
- Frictional capital which needs to be financed by the shareholder: This is the remaining part of the difference as indicated by the formula above and the WACC (weighted average cost of capital) of the company ($\beta_2$) will be charged.

Technically the above formula reads now as follows:

$$FCC = \sum_{t=0}^{\infty} [\beta_2 \{\max(0, W_t(1)\} + \beta_3 \{\max(0, W_t(2)\}] \times \pi(\mathcal{Z}_{(t)}),$$

where $W_t(1)$ and $W_t(2)$ denote the above mentioned differences.

As the market consistent valuation is based on a full balance sheet approach the market consistent value of insurance liabilities is to be calculated before tax, as taxes are taken care of in the calculation of the market consistent equity.

### 13.2.4 Duration of Projection

Projections should be sufficiently long duration to capture all important financial events in the life of a policy. The projections are subject to a minimum projection period of 40 years (or policy duration if less).

### 13.2.5 Formulae

$$\mathbb{E}[PV] = \sum_{t=0}^{\infty} \pi(\mathcal{Z}_{(t)}) \times \mathbb{E}[CF_t], \tag{13.1}$$

$$\mathbb{E}[PV]_t = \frac{1}{\pi(\mathcal{Z}_{(t)})} \sum_{k=t}^{\infty} \pi(\mathcal{Z}_{(k)}) \times \mathbb{E}[CF_k \mid \mathcal{F}_t], \tag{13.2}$$

$$CoC = \sum_{t=0}^{\infty} \beta_1 \times RC_t \times \pi(\mathcal{Z}_{(t)}), \tag{13.3}$$

$$FCC^* = \sum_{t=0}^{\infty} \beta_2 \{\max(0, V_t - \mathbb{E}[PV]_t)\} \times \pi(\mathcal{Z}_{(t)}), \tag{13.4}$$

$$FCC = \sum_{t=0}^{\infty} \beta_2 \{\max(0, V_t - \mathbb{E}[PV]_t - RC_t)\} \times \pi(\mathcal{Z}_{(t)}), \tag{13.5}$$

$$\text{MV of Ins. Lia.} = \mathbb{E}[PV] - CoC - FCC, \tag{13.6}$$

$$\gamma = (\beta + i) \times (1 - \text{ Tax rate}), \tag{13.7}$$

where $i$ denotes the risk free interest rate for the corresponding period.

### 13.2.6 Example

In order to show how the different pieces work together we have chosen an annuity portfolio for a valuation as of 31.12.2006. We have the following main characteristics:

| Item | Amount in EUR |
|---|---|
| Balance Sheet Reserve | 341723220 |
| of which from in-force | 272912264 |
| of which from New Business | 25610375 |
| of which IBNR for late reporting | 45303338 |
| Annuities to be paid out per Year | 16371791 |

There are principally two effects for which the reserve has to be adjusted. In the concrete example – a reinsurance company – the last settling of annuities payed went back to September 2005. Therefore 16 months of annuity payments are outstanding, leading to the IBNR Reserve of EUR 45.3 M. The data for the projection is as of 30.6.2006. Therefore we need on the one hand roll forward the projection to the valuation date. On the other hand the new production for the 6 missing months needs to be modelled, resulting in an increase of reserve of EUR 25.6 M.

In order to make a market consistent valuation we need to take the following effects into consideration and we have chosen the following parameters:

| Item | Parameter |
|---|---|
| Risk capital | 15 % of $\mathbb{E}[PV]$ |
| Unit CoC - riskfree | 10 % |
| FCC | Difference between $V_t$ and $\mathbb{E}[PV]_t$ |

Figure 13.1 shows the development of the different cash flows. One now needs to calculate the different parts, namely the market value margin (CoC), the change in value due to funds withheld and also the frictional capital costs. The funds withheld can economically be considered as a loan of the reinsurer to the insurer with a valuation corresponding to a cash flow swap; eg reinsurer pays forward interest and receives fixed interest as agreed in the contract. In this particular case this process is in favour of the reinsurer. Figures 13.2 shows the three different pieces.

The above mentioned calculation results in the following results:

| Item | Amount in EUR |
|---|---|
| $+\ \mathbb{E}[FV]$ | 230158544 |
| $-$ FundsWithheld | -8212193 |
| $-\ MVM$ | 29581548 |
| $FCC^*$ | 20249339 |
| $-\ FCC$ | 181688 |
| $= \mathbb{E}[FV] +$ FuWi $+ CoC + FCC$ | **251709588** |
| $\Delta$ Difference to MR | +21202676 |
| $+$ Gross up for New Business | 9.3 % |
| $=$ Gross Eco Value of Lia. | **275330282** |

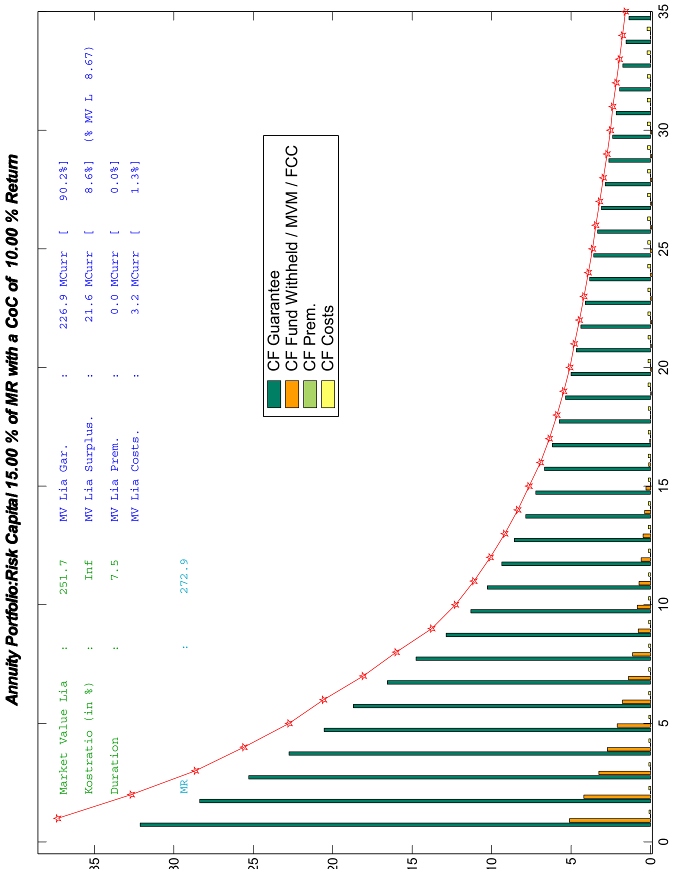



**Fig. 13.1** Cash Flows

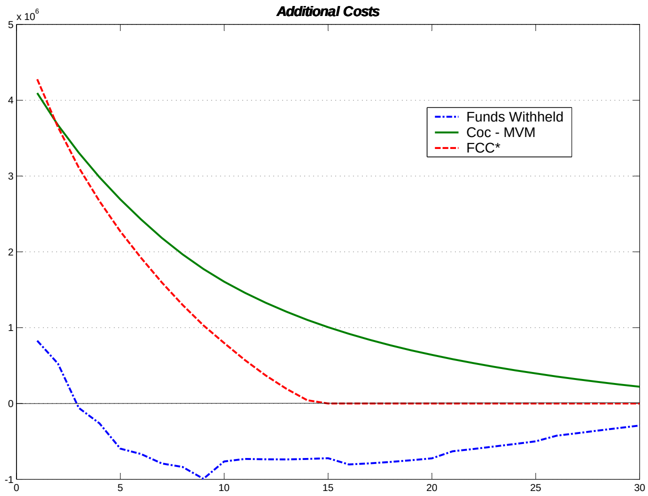



**Fig. 13.2** Details

## 13.3 Examples

This section provides a concrete reporting example in order to see the different parts. The first table shows an overview of all the different product portfolios for an insurance entity, the second one the corresponding details. In order to understand the notation, the table below provides an explanation:

**Reserves in B/S:** These are the statutory reserves in the balance. In this direct method they serve also as a proxy for the amount of assets covering the liabilities and hence one part of the value attributed to the corresponding product

**PV Premium:** The present value of future premiums is the second contributor to value. In the context of market consistent valuation these future premiums are weighted according to persistency and discounted by risk free discount rates.

**PV Claims:** Here the expected claims are indicated. All different types of claims such as surrender and maturity benefits, but also claims in case of death, etc are subsumed here. It would be possible to be more granular in this position.

**PV Exp - Internal:** The next three positions relate to expenses. They are split into the different pieces in order to allow a variety of break downs, looking for example at a marginal cost base. Please note that the expenses are important since the company can influence them better than most of the other parts. In this position the present value of future internal costs in relation to the product are displayed.

**PV Exp - Overhead:** In this position the present value of future overhead costs in relation to the product are displayed.

**PV Exp - Commissions:** In this position the present value of future commissions in relation to the product are displayed. In case of a value of new business this position carries all the commissions which are paid for the product.

**Market Value Margin:** The MVM is the amount described above. eg $MVM = \sum_{k=0}^{\infty} \beta \times RC_k \times \pi_t(\mathcal{Z}_{(k)})$.

**FCC:** Frictional costs of capital as mentioned above.

**Funds Withheld:** This is the swap arrangement which is often used in a reinsurance treaty, where the cedent pays fixed and receives floating.

**Tax:** In this position the taxes are deducted.

**Total:** This line, the sum of the above, represents the total value inherent in the corresponding product from an economic point of view.

**PV Profit:** Market consistent present value of profits not allowing for risk capital.

**PV Capital:** Present value of risk capital for the corresponding line of business.

**RoRAC (in %):** That's the Return on risk adjusted capital.

**VIF:** This is the expected present value of the profits after tax, using a risk discount rate.

**Lock-in:** This is the so called lock in effect, eg the opportunity loss as a consequence that shareholder capital is immobilised as a consequence of regulatory capital requirements.

**PVFP:** Sum of the two above.

| **Value at B/S Date** | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| M EUR | **Prod 1** | **P2** | **P3** | **P4** | **P5** | **P6** | **P7** | **P8** | **P9** | **P10** | **Total** |
| Reserves in B/S | 1.2 | 0.0 | 79.0 | 214.5 | 8.0 | 1.6 | 29.8 | 448.8 | 12.1 | -41.9 | 753.4 |
| PV Premium | 0.6 | 0.0 | 0.5 | 738.3 | 0.0 | 34.3 | 285.8 | 0.0 | 432.9 | 1563.0 | 3055.0 |
| PV Claims | -0.0 | -0.0 | -66.8 | -678.6 | -3.6 | -2.9 | -18.4 | -387.8 | -436.3 | -1351.0 | -2945.0 |
| PV Exp - Internal | -0.1 | 0.0 | -0.0 | -14.7 | -0.0 | -0.1 | -9.7 | -2.9 | -0.1 | -22.3 | -50.4 |
| PV Exp - Overhead | -0.2 | 0.0 | -0.0 | -29.5 | -0.1 | -0.2 | -14.5 | -4.0 | -0.2 | -36.6 | -85.7 |
| PV Exp - Commissions | -0.0 | 0.0 | -0.0 | -171.0 | 0.0 | -31.8 | -39.9 | -33.1 | -2.3 | -53.8 | -332.4 |
| Market Value Margin | -0.0 | 0.0 | -0.0 | -14.2 | -0.1 | -1.1 | -17.9 | -4.7 | -10.0 | -48.6 | -97.0 |
| FCC | -0.0 | -0.0 | -0.0 | -0.6 | -0.0 | -2.1 | -7.9 | -19.7 | -0.0 | -3.0 | -33.5 |
| Funds Withheld | -0.0 | -0.0 | -0.3 | -2.6 | 0.0 | -1.3 | 0.0 | 10.6 | 0.0 | 0.0 | 6.2 |
| Tax | -0.4 | -0.0 | -4.1 | -17.5 | -1.4 | 0.0 | -79.4 | -10.8 | -1.3 | -12.0 | -127.2 |
| **Total** | **0.9** | **0.0** | **8.0** | **23.7** | **2.4** | **-3.7** | **127.6** | **-3.9** | **-5.4** | **-6.5** | **143.2** |
| **Profitability** | | | | | | | | | | | |
| PV Profit | 1.0 | 0.0 | 8.0 | 48.7 | 2.6 | -3.7 | 151.8 | -14.0 | 12.6 | 77.6 | 284.7 |
| PV Capital | 0.7 | -0.0 | 0.7 | 300.9 | 3.4 | 23.4 | 378.0 | 99.3 | 212.0 | 1025.0 | 2044.0 |
| RoRAC (in %) | 22.1 | 15.7 | 9.2 | 8.8 | 14.8 | -19.5 | 27 | -25.4 | 5.5 | 5.6 | 8.5 |
| **Classical EV** | | | | | | | | | | | |
| VIF | 1.1 | 0.0 | 8.6 | 37.7 | 2.6 | 0.9 | 131.3 | 13.0 | 5.0 | 33.0 | 233.5 |
| Lock-in | -0.0 | 0.0 | -0.0 | -8.6 | -0.0 | -0.6 | -10.0 | -2.8 | -6.2 | -28.4 | -56.9 |
| PVFP | **1.0** | **0.0** | **8.6** | **29.1** | **2.5** | **0.3** | **121.2** | **10.2** | **-1.2** | **4.5** | **176.5** |
| **Sensitivities** | | | | | | | | | | | |
| Expenses | -0.0 | 0.0 | -0.0 | -3.0 | -0.0 | -0.0 | -1.6 | -0.4 | -0.0 | -4.5 | -9.7 |
| Claims | -0.0 | -0.0 | -4.4 | -46.0 | -0.2 | -0.2 | -1.2 | -25.4 | -34.0 | -103.0 | -214.7 |
| Capital | -0.0 | 0.0 | -0.0 | -1.9 | -0.0 | -0.1 | -2.3 | -0.6 | -1.5 | -7.1 | -13.7 |
| Tax | -0.0 | -0.0 | -0.4 | -1.7 | -0.1 | 0.0 | -7.9 | -1.0 | -0.1 | -1.2 | -12.7 |
| **Profits p.a.** | | | | | | | | | | | |
| 1990 – 2006 | 0.9 | 0.0 | 0.0 | 1.4 | 0.0 | -0.0 | 0.7 | -0.1 | 0.0 | -0.1 | 2.9 |
| 2007 | 4.4 | 0.3 | 1.1 | 9.4 | 0.5 | 1.8 | 9.7 | -20.6 | 0.1 | 1.6 | 8.6 |
| 2008 | 1.1 | 0.0 | 9.4 | 11.6 | 0.9 | -0.2 | 20.5 | -2.4 | 0.5 | 11.3 | 52.9 |
| 2009 | 0.0 | 0.0 | 0.0 | 10.4 | 0.8 | 0.4 | 19.7 | 3.0 | 0.5 | 9.4 | 44.6 |
| 2010 | 0.0 | 0.0 | 0.0 | 7.2 | 0.6 | 0.3 | 18.4 | 2.8 | 0.5 | 5.2 | 35.3 |
| 2011 | 0.0 | 0.0 | 0.0 | 4.7 | 0.1 | -0.4 | 15.9 | 2.6 | 0.5 | 2.9 | 26.5 |
| 2012 – 2016 | 0.0 | 0.0 | 0.0 | 2.7 | 0.1 | 0.1 | 13.6 | 2.0 | 0.5 | 1.1 | 20.5 |
| 2017 – 2021 | 0.0 | 0.0 | 0.0 | 0.4 | 0.0 | 0.1 | 9.5 | 1.0 | 0.6 | 0.3 | 12.0 |
| 2022 – 2031 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 5.5 | 0.3 | 0.2 | 0.1 | 6.2 |
| 2032 – 2107 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.4 | 0.0 | -0.1 | 0.0 | 0.2 |
| **Total** | 1.1 | 0 | 8.6 | 37.7 | 2.6 | 0.9 | 131.3 | 13 | 5 | 33 | 233.5 |
| **Capital p.a.** | | | | | | | | | | | |
| 1990 – 2006 | 4.3 | 0.6 | 4.5 | 14.8 | 0.7 | 1.2 | 2.7 | 2.9 | 11.2 | 6.8 | 50.1 |
| 2007 | 10.9 | 0.0 | 14.1 | 65.7 | 4.8 | 3.6 | 40.9 | 15.0 | 33.3 | 78.9 | 267.7 |
| 2008 | 0.7 | 0.0 | 0.7 | 71.0 | 2.3 | 0.8 | 33.7 | 15.1 | 31.1 | 105.2 | 261.1 |
| … | | | | | | | | | | | |
| **Total** | 0 | 0 | 0 | 8.6 | 0 | 0.6 | 10 | 2.8 | 6.2 | 28.4 | 56.9 |

**Valuation at B/S date**

| Position | Amount in EUR | Relative Amount |
|---|---|---|
| Reserves in B/S | 753400000 | 19.78 % |
| Present Value Premium | 3055000000 | 80.22 % |
| Present Value Claims | -2945000000 | -77.33 % |
| PV Exp - Internal | -50400000 | -1.32 % |
| PV Exp - Overhead | -85730000 | -2.25 % |
| PV Exp - Commissions | -332400000 | -8.72 % |
| Subtotal | 394800000 | 10.36 % |
| Market Value Margin | -97070000 | -2.54 % |
| FCC | -33560000 | -0.88 % |
| Funds Withheld | 6281000 | 0.16 % |
| Tax | -127200000 | -3.33 % |
| **Total** | **143200000** | **3.76 %** |
| PV Profit | 284700000 | |
| PV Capital | 2044000000 | |
| RoRAC | | **13.93** % |

**Decomposition of Profit**

| Time | $\Phi$ P/L | $\Phi$ Capital | RoRaC w/o FCC |
|---|---|---|---|
| 1990 – 2006 | 2971000 | 50190000 | 5.92 % |
| 2007 | 8633000 | 267700000 | 3.22 % |
| 2008 | 52920000 | 261100000 | 20.27 % |
| 2009 | 44640000 | 237000000 | 18.84 % |
| 2010 | 35370000 | 221800000 | 15.95 % |
| 2011 | 26580000 | 203100000 | 13.09 % |
| 2012 – 2016 | 20540000 | 166500000 | 12.34 % |
| 2017 – 2021 | 12050000 | 99450000 | 12.12 % |
| 2022 – 2031 | 6277000 | 43480000 | 14.44 % |
| 2032 – 2107 | 279600 | 2526000 | 11.07 % |
| **PVFP** | 233500000 | 56990000 | 409.80 % |

The above tables can also be shown in a graphical form, such as in figure 13.3. Here the 100 % mark represents the mathematical reserves plus the present value of future premium. Hence a product is profitable if the sum is below 100 %.

## 13.4 Capital Allocation Process

Having a performance metric as the one described above and a process such as the one in section **??** it is now possible to do a concrete capital allocation process. In order to bring the two things together we have a second look at the process itself. Figure 13.4 provides an example which we want to analyse closer.

In order to do that we need to understand the different parts. The quantifies to be analysed are listed in the table below. It needs to be stressed that the framework is very flexible in terms what capital and return actually mean. In a lot of circumstances, capital is a synonym for a VaR (at a one in 200 year event) or for a TVaR (at

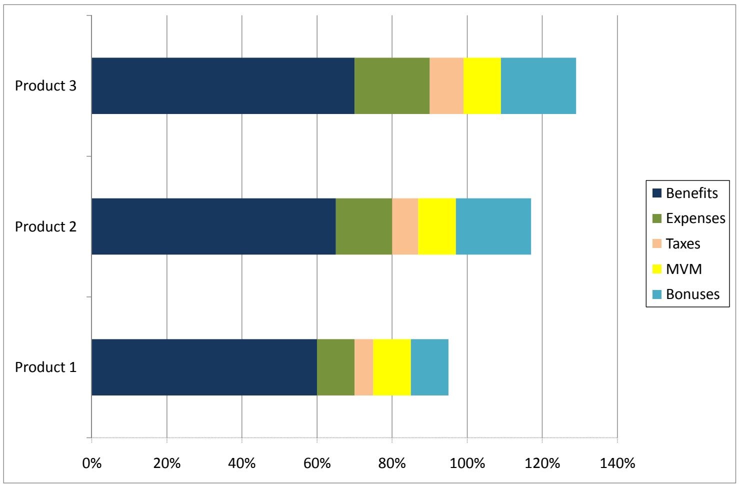



**Fig. 13.3** Example Profitability of three products

a one in 100 year event). In some instances it also makes sense to look at two or more capital measures together (say at a VaR in a 1 in 10 and a one in 200 year event). Such considerations are reasonable if one wants to either look at different metrics at the same time or if there are differentiated risk appetite statements for different confidence levels. Also return can be interpreted in different ways and stands in an economic context mainly for the IRR measure introduced before.

| | |
|---|---|
| Available Capital | This is the maximal capital which can be put at risk for the underlying period. It is normally the available economic capital. |
| Type of Opportunity | These are the different types of business opportunities, which absorb capital in order to generate shareholder value. |
| Required Capital | This is capital needed for the corresponding business opportunity. |
| Break-even return | This is the minimal return required in order to create shareholder value. These numbers are different, since there may also enter strategic and other considerations, which are not captured in the capital model. |
| Offered return | The expected return offered by the corresponding business opportunity. |
| Capital allocated | In this next step capital is allocated to each business opportunity. |

| | |
|---|---|
| Hurdle rate for bonus | At the same time the minimal required return is defined in order to incentivice the management accordingly. |
| Capital limit | Once the capital is allocated, the corresponding numbers become limits and capital is managed in such a way that it is optimally allocated and used, hereby *not* violating capital limits. |
| Capital used | At every point of time the capital used is compared with its capital limit in order to prevent limit breaches and in order to initiate corrective actions. |
| Effective Return | At the end of the cycle the realised return is calculated and compared with the agreed hurdle rates in order to determine the value creation for the shareholders and to compensate management accordingly. |

In order to offer some alternatives for capital and return, below some possible choices:

| Metric | Capital | Return |
|---|---|---|
| Economic | VaR or TVar | IRR as mentioned above |
| Accounting Profit | IFRS Shareholder equity | Return on equity |
| Risk adj Accounting Profit | VaR of IFRS SHE | Return on equity |
| Dividends | Free Surplus | Free surplus generated |

Finally some remarks to this process:

- The aim of the capital allocation process is twofold. On the one hand one wants to optimise return on capital in order to optimise shareholder returns on a risk adjusted base. On the other hand one wants to limit the risk by using a diversified opportunity portfolio and by agreeing capital limits.
- Normally each business opportunity comes with its capital needs and with its expected returns. In a lot of instances these models are rather crude. Therefore it is essential to robustly challenge the models and their assumptions in order to increase the probability of successful shareholder value creation.
- It is important to fix the management remuneration based on the agreed metric at the time of planning in order to avoid a principal - agent problem. It is essential (while being trivial) to remark that people taking the risk should not be able to determine the exogenous parameters which are used for remuneration purposes in order to avoid self-fulfilling promises. It is key that the yard-stick used for measuring remuneration is reliable.
- In the same sense it is important to remark that a mechanical capital allocation can not replace an expert judgement, in particular in respect to strategic developments and initiates, since such opportunities are very difficult to assess. There are

**1) Planning:**
- Opportunities:

**Available Capital: 1'500**

| Type | Capital needed | Break-even return | Return offered |
|---|---|---|---|
| Reinsurance | 100 | 13.0 % | 15.0 % |
| Strat Project | 150 | 11.0 % | 12.0 % |
| GI | 500 | 11.5 % | 13.0 % |
| Life Trad | 800 | 9.0 % | 9.0 % |
| Life UL | 400 | 9.0 % | 11.0 % |
| **Blended Total** | **1'950** | **10.0 %** | **10.9 %** |

**2. Capital allocation:**

| Type | Capital allocated | Return (plan) | Hurdle for Bonus |
|---|---|---|---|
| Reinsurance | 100 | 15.0 % | 14.0 % |
| Strat Project | 50 | 12.0 % | 12.0 % |
| GI | 350 | 13.0 % | 11.0 % |
| Life Trad | 700 | 9.0 % | 8.5 % |
| Life UL | 300 | 11.0 % | 10.5 % |
| **Blended Total** | **1'500** | **10.8 %** | **10.0 %** |

**3) Risk Taking:**
- Independent Challenge
- Monitoring of Limits
- Redeployment of capital

| Type | Capital limit | Capital used | Limit Headroom |
|---|---|---|---|
| Reinsurance | 100 | 99 | |
| **Strat Project** | **50** | **70** | |
| **GI** | **350** | **360** | |
| Life Trad | 700 | 650 | |
| Life UL | 300 | 290 | |
| **Blended Total** | **1'500** | **1469** | |

**4) Review:**
- Monitoring and evaluation of risk adjusted performance and capital generation
- Calculation and definition of remuneration

| Type | Capital used | Return effective | Level Bonus |
|---|---|---|---|
| Reinsurance | 99 | 15.5 % | 14.0 % |
| Strat Project | 65 | 12.0 % | 12.0 % |
| GI | 350 | 10.5 % | 11.0 % |
| Life Trad | 640 | 8.7 % | 8.5 % |
| Life UL | 300 | 12.0 % | 10.5 % |
| **Blended Total** | **1'454** | **10.4 %** | **10.0 %** |

**Fig. 13.4** Example Capital Allocation Process

many exogenous factors which also need to be considered outside the traditional models.

# Chapter 14

# Products and their Risks


The aim of this chapter is to look at some concrete product offerings that went wrong. Looking at the main risk on a large scale for an insurance company we have certainly the relationship between assets and liabilities, the set up of new large scale IT projects (aka insurance administration systems) and products. All of these risks manifest differently. The ALM question is certainly the one which most determines whether a company can survive after a corresponding event. Hence it is characterised by a high impact but also by a continuous evolution and hence one can mitigate it by setting up corresponding processes to govern and limit this risk. The IT risk is a typical project risk which normally has its roots in a too big appetite for systems which can do everything. Also this type of risk can, in principle, be managed in a canonical way by the application of the corresponding change and project management processes. If we finally look at the product risk we face a very different animal. Product risk is for most products relatively small since there are a lot of similar product designs, which are well known and for which one, in principle, knows very well the corresponding risks. The crystallisation of a product risk is hence a rather rare event. But on the other hand there may be huge impacts. Therefore one needs to be very vigilant when introducing new or adjusting existing products. This chapter aims to show some of the pitfalls to avoid in the form of real case studies. The reader is invited to think whether he would have fallen into the corresponding pit.

## 14.1 Nuptualite

The first product we want to have a look at is an endowment policy which is sold for young children and mainly provides them with a savings contract, which matures when they reach an age of somewhere between 20 and 25. So far this product is plain vanilla and the risk for the insurance company is modest. The real treat is a small rider which we will see in a second. So the characteristics of the main policy are as follows (for our example):

| | |
|---|---|
| Entry-age | 0 |
| Age at maturity | 25 |
| Maturity and Death Benefit | 100000 |
| Financed by regular premium payment | |
| Technical interest rate | 3 % |

For this rider the following two additional options were sold, respectively given for free:

**Premium Holiday:** After the first premium payment there is a possibility for a premium holiday. If premium were not paid, the premium holiday starts and the insurance cover is adjusted correspondingly. It is possible to pay the outstanding premiums later if the accrued interest is paid. This option was offered for free.

**Nuptualite:** The idea was that the child would get the maturity benefit before maturity if he/she got married before this moment. The calculation was based on the most accurate statistical information of the relevant country and premiums were calculated accordingly.

If we have a look at the above cover we would have the following yearly premiums:

| | |
|---|---|
| Main Policy | 2780.50 p.a. |
| Nuputalite Raider | 19.80 p.a. |
| Total Premium | 2800.30 p.a. |

This product was sold in the late 80's and early 90's by a mid-sized company with a shareholder's equity of 200+ M. The sales volume for this product was very high - so high that management were worried and stopped selling it. The loss of this product line was higher than the above mentioned shareholder's equity and the company only survived because it had a wealthy caring parent ...

But what had happened? Obviously something did not really work out. In order to analyse the situation a little closer, let's look at the following table:

| Question | Answer |
|---|---|
| Actuarial model correct? | Yes |
| Statistical basis reliable | For the country and the population it was designed: yes? |
| Insurability criteria fulfilled? | NO. One of the main criteria is based on the fact that the insured person cannot decide himself whether he is eligible to get a benefit or not and that in consequence the occurrence of paying benefits is random. Obviously the time when getting married can be influenced. |
| Statistical base relevant for the population? | NO. This type of product was largely sold to an ethical group, which usually get married at the age of 17. As a consequence the statistical basis was not adequate. |

Now we know what went wrong: an ill-designed product was sold to a population, where the statistical basis was not adequate.

In a next step we want to look how this product works. To this end we assume the following statistical base, where $h_x$ the probability to get married at age $x$ for the population of the country considered and where $\tilde{h}_x$ denotes the probability to get married for the ethical group mainly buying this sort of policy:

| | $q_x$ | $p_x$ | $h_x$ | $\tilde{h}_x$ |
|---|---|---|---|---|
| 0 | 0.00200 | 0.99800 | 0.00000 | 0.00000 |
| 5 | 0.00200 | 0.99800 | 0.00000 | 0.00000 |
| 10 | 0.00200 | 0.99800 | 0.00000 | 0.00000 |
| 17 | 0.00200 | 0.99800 | 0.00100 | **0.60000** |
| 18 | 0.00200 | 0.99800 | 0.00152 | 0.00152 |
| 19 | 0.00200 | 0.99800 | 0.00374 | 0.00374 |
| 20 | 0.00200 | 0.99800 | 0.00733 | 0.00733 |
| 21 | 0.00200 | 0.99800 | 0.00123 | 0.00123 |
| 22 | 0.00200 | 0.99800 | 0.01742 | 0.01742 |
| 23 | 0.00200 | 0.99800 | 0.02349 | 0.02349 |
| 24 | 0.00200 | 0.99800 | 0.03084 | 0.03084 |
| 25 | 0.00200 | 0.99800 | 0.03953 | 0.03953 |

We observe that the "normal" probability to get married is about 1-2 % starting at the age of ca 18. On the other hand we have assumed that for this particular community we have a marriage probability of 60 % at the age of 17. This is naturally a simplification in order to better understand the problematic. Based on the above assumption we get the following prices for the insurance offering:

| (A) | (B) | (C) W/o NUP | (D) W NUP | (E) Antisel. | (F) Arbitrage |
|---|---|---|---|---|---|
| PV Benefit | Death | 0.03410 | 0.03398 | 0.02984 | 0.00000 |
| | Survival | 0.45428 | 0.41613 | 0.16611 | 0.00000 |
| | Nupt | 0.00000 | 0.04004 | 0.35640 | 0.58739 |
| **Total** | | **0.48839** | **0.49016** | **0.55236** | **0.58739** |
| PV Prem | 1 | 17.56504 | 17.50421 | 15.36879 | 14.16611 |
| Premium | | 2780.50 | 2800.28 | 3594.07 | 4146.47 |
| PV | Ben | | | | 100000.00 |
| $x = 18$ | Prem | | | | 79384.28 |
| Delta Prem | | | 19.78 | 813.56 | |
| **Loss** | | | 177.20 | 6396.86 | **20615.71** |

In the table above column (C) describes the main policy, (D) the one including nuptualite, based on the observed statistics of the country, (E) ditto with the expected behaviour of the specific community. This table tells us for example that the additional premium for the raider costs some 20 per annum. Column (F) is the one to look at. Here the economic effect is indicated if we consider that a lot of policies have used the premium holidays and pay the remaining premium only when they know that the child is going to marry soon. It becomes obvious that in this case the loss per policy amounts to c 20000 per policy. In the concrete set up the in-force portfolio consisted out of ca 12000 policies having therefore in the model the cumulative loss of c 240 M.

## 14.2 Index Linked Products and other Contractual Issues

Some three years ago I would have been quite dogmatic in respect of measuring (in terms of capital) operational risk. The following example shows how operational risk can materialise. In the early 21st century structured products for insurance companies were very popular in Italy. In order to better understand this product let's look at the corresponding product characteristics, which I have put in the table below:

| | |
|---|---|
| Term of contract | 10 years |
| Single Premium | 100000 EUR |
| Benefit at maturity | 90 % of Yield of STOXX 50 index, minimally 2 % p.a. |

Obviously, one could also offer this product including a mortality cover, but that is not relevant for this example. The concrete numbers for this example are also irrelevant, since for this structured product, the insurer went to an investment bank (Lehmans for example) and gave them say 95000 EUR and the investment bank replicated the guarantee, by issuing a Lehman structured bond, which would pay according to the sometimes complex derivatives structure. We will see in section 14.3 what can go wrong if one tries to replicate these derivatives.

In order to be protected, the insurance contracts were written in a way stating the counter-party risk is explicitly born by the policyholder . . .

And now the unexpected happens - the bank defaults. Meanwhile the insurer has issued some 200 M EUR of Lehman structured bonds which trade, for arguments sake, at 4% and have therefore a value of 8 M EUR. So at first glance one could be of the opinion that there is no issue, because people have been advised correctly and are aware of the corresponding risks. What happened in reality in Italy was the following:

1. The lawyers of the company confirm that the insurance company has no legal obligation.
2. It is confirmed that the clients have been correctly advised.
3. The regulator confirms the legal position and mentions that he would be happy if the companies could take some "customer care" action, - eg voluntary payments to help the clients which suffered the loss.
4. Other companies start to compensate the clients for the losses and there is a reputational issue and hence the whole market seeks ways to make good the corresponding loss.
5. The regulator issues a new regulation, which foresees that such products can in the future only be offered if the insurer provides the guarantee.

So at the end, the loss ended up in the balance sheet of the insurer. Most customer care action requests the policyholder to inject additional money, the contract term was prolonged from say 5 to 10 years (because of the interest effect) and the insurer and the distributors (mostly banks in the case of Italy) injected the remaining funds. At the end a considerable part of the loss was taken by the distributor and the insurer, and hence the corresponding loss was an operational loss (reputational category) which was triggered by a credit event.

Now it is necessary to formulate some learnings:

- It is necessary not only to think in legal terms but always to keep the reputational consequences in mind when you believe that a liability has shifted to the customer.
- One must not underestimate the influence of the regulator even though he might not issue legal binding orders.

- The investment strategy chosen by the insurer on behalf of the customer is not acceptable. One would expect that a professional investor (or an adviser) would not put all eggs in one basket. The insurance company would have had concentration limits in place for the funds on its balance sheet.

It is important to recognise that the ex-ante finding of such issues, as the ones mentioned above, is anything but trivial, because one needs always to think what happens if the impossible happens (eg Lehman default). Finally it is worth mentioning that the design of structured products is undergoing considerable change since the issue is at the very end a design issue.

## 14.3 Variable Annuities

There are several types of performance guarantees for unit linked policies and one may often choose them a la carte, with higher risk charges for guarantees that are riskier for the insurance companies. The first type is comprised of guaranteed minimum death benefits (GMDB), which can be received only if the owner of the contract dies.

GMDBs come in various flavors, in order of increasing risk to the insurance company:

- Return of premium (a guarantee that you will not have a negative return),
- Roll-up of premium at a particular rate (a guarantee that you will achieve a minimum rate of return, greater than 0),
- Maximum anniversary value (looks back at account value on the anniversaries, and guarantees you will get at least as much as the highest values upon death),
- Greater of maximum anniversary value or particular roll-up.

Unlike death benefits, which the contract holder generally can't time, living benefits pose significant risk for insurance companies as contract holders will likely exercise these benefits when they are worth the most. Annuities with guaranteed living benefits (GLBs) tend to have high fees commensurate with the additional risks underwritten by the issuing insurer.

Some GLB examples, in no particular order:

- Guaranteed Minimum Income Benefit (GMIB, a guarantee that one will get a minimum income stream upon annuitisation at a particular point in the future)
- Guaranteed Minimum Accumulation Benefit (GMAB, a guarantee that the account value will be for a certain amount at a certain point in the future)

- Guaranteed Minimum Withdrawal Benefit (GMWB, a guarantee similar to the income benefit, but one that doesn't require annuitising)
- Guaranteed-for-life Income Benefit (a guarantee similar to a withdrawal benefit, where withdrawals begin and continue until cash value becomes zero, withdrawals stop when cash value is zero and then annuitisation occurs on the guaranteed benefit amount for a payment amount that is not determined until annuitisation date.)

In order to value this guarantee, one needs to rely on option pricing techniques such as the Black-Scholes formula. The price for a *put*-option with payout $C(T, P) = \max(K - S; 0)$ at time $t$ and strike price $K$ and equity price $S$ is given by:

$$\begin{aligned}
P &= K \times e^{-r \times T} \times \Phi(-d_2) - S_0 \times \Phi(-d_1), \\
d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2) \times T}{\sigma \times \sqrt{T}}, \\
d_2 &= d_1 - \sigma \times \sqrt{T}, \\
\Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(\frac{-\zeta^2}{2}) d\zeta.
\end{aligned}$$

The reader should be reminded that the formula is based on the efficient market hypothesis which requests that the following holds:

- Deep and friction-less market, and
- Absence of arbitrage.

In order to understand how these options are synthetically "constructed" one needs to understand the concept of a replicating portfolio. Hence one holds at every point in time a portfolio $P_t$ with the aim that this portfolio matches at time $T$ just the payout of the option mentioned above. In order to construct such portfolios one usually uses the "greeks". These greek letters represent the sensitivity of an option in case of a change of the underlying economic parameters such as equity price, interest rate levels, etc. We have the following relationships:

$$\begin{aligned}
\Delta_P &= \frac{\partial P}{\partial S} \\
&= \Phi(d_1), \\
\Gamma &= \frac{\partial^2 P}{\partial S^2} \\
&= \frac{\Phi'(d_1)}{S \times \sigma \times \sqrt{T}}, \\
\Lambda &= \frac{\partial P}{\partial \sigma} \\
&= S \times \Phi'(d_1) \times \sqrt{T - t},
\end{aligned}$$

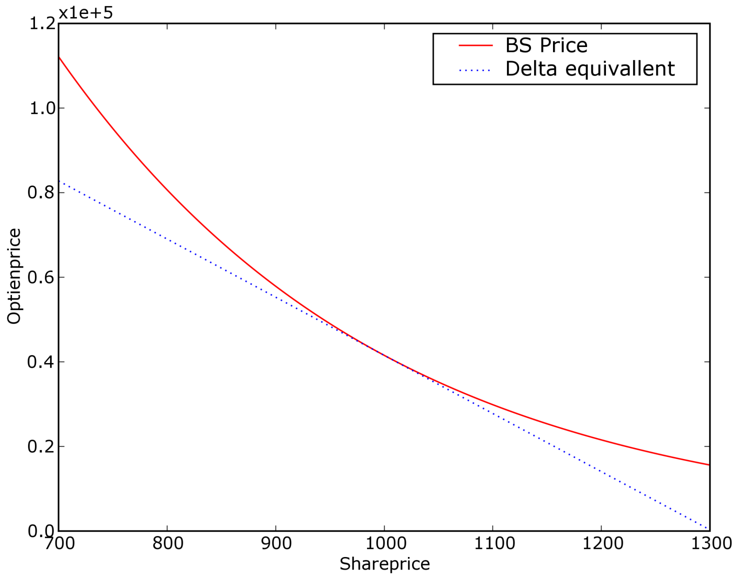



**Fig. 14.1** $\delta$–Hedging

$$\begin{aligned} P_P &= \frac{\partial P}{\partial r} \\ &= -(T-t) \times K \times e^{-r\times(T-t)} \times \Phi(-d_2). \end{aligned}$$

Based on the above partial derivatives, it is now possible to define different hedging strategies, one of them being a “delta-hedge”. The idea is to define at a point in time $t$ a portfolio $P_t$ consisting of cash and shares, which have the same value and for which the partial derivative with respect to equity price $S$ is the same. Hence we look for a Taylor approximation of order 1 in the variable $S$. Figure 14.1 shows such a delta hedge. For the concrete example we have the following put option:

| | |
|---|---|
| Interest | $r = 3.0\%$ |
| Term | 10 years |
| Equity Price | $S_0 = 1000$ |
| Strike | $K = 900$ |
| Volatility | $\sigma = 15\%$ |
| Number of Shares | 1000 |
| Value of Put | $P = 41535.7$ |
| Delta | $\Delta_P = -137472$ |

What becomes obvious is the fact that the hedge is quite good if the stock market does not move too far away during the time between the updating of the replicating portfolio, for example updating the hedge portfolio once a day.

It is interesting to see what starts to happen if there are days with high volatility and market disruption. There are, in principle, two effects of stock market movements which inhibit a perfect hedging of the underlying guarantees. This is the change of stock price overnight. You may observe that the last paid price of a share is 15.5 and that the sentiment overnight has changed and the first paid price is 15.0. The other effect is a high inter-day volatility of the underlying asset. Assume for arguments sake that we consider a company with the following portfolio:

| | |
|---|---|
| Spot price beginning of day | $K = 1000$ |
| Strike | $K = 950$ |
| Volatility for $\delta$–hedge | $\sigma = 15\%$ |
| Number of Index Baskets | 1000000 |
| Value of the portfolio | $P = 1000000000$ USD |
| Value of Put | $P = 52198940$ USD |
| Delta | $\Delta_P = -164095996$ USD |

Now let's see what happens if we have set up our hedge portfolio at the beginning of the day at an index of 1000 and when the index is at 970 at the end of the day. Obviously this example is fictional. But one could see such extreme market value movements more than once in the autumn of 2008. Before doing the calculation let's see how rare this event actually is with an underlying volatility of $\sigma = 15\%$. Based on the Brownian motion assumption we know that the variance increases linearly in time and hence we know that the log-returns for one day have a $\sigma_{day} = \frac{15}{\sqrt{365}} = 0.79\%$. The probability is that we have a day return of -3 % or less which amounts to 0.000066. The following table shows the corresponding numbers for other moves:

| $\alpha$ | $P[X \leq \alpha]$ |
|---|---|
| -3.0 % | 0.000066 |
| -2.5 % | 0.000725 |
| -2.0 % | 0.005427 |
| -1.5 % | 0.028034 |
| -1.0 % | 0.101391 |
| -0.5 % | 0.262116 |

And we remark again that there were several trading days in the autumn 2008 where we observed daily losses of 2 % or more. Even bigger equity market swings have been observed beginning May 2010, after fears of state bankruptcies in the Euro zone. On Thursday 6.5.2010 the NYSE (Dow Jones Industrial Average) fell temporarily over 9 %, as a consequence of such fears and automated trading. The same day Procter and Gamble lost temporarily more than 35 % of its value. On Monday 10.5.2010 the Euro Stoxx index performed 10.35 % within one day, after the announcement of a EUR 750 bn bail-out plan. Assuming a volatility of 20 % and log-normally distributed equity-market returns, this represents a $9.8 \times \sigma$–event. Such an event has a return period of $5.9 \times 10^{17}$ years. This number is considerably bigger than the age of the universe of $1.375 \times 10^{10}$ years and hence it is obvious that the log-normally distributed model is not correct in the tails. From a risk management point of view it becomes obvious that all capital models are prone to *model risk*. Hence it is of utmost importance to test the model in respect to its reliability. This can be done by back-testing and also by the application of statistical tests. In the same sense it is important to understand that all estimates are prone to *parameter risk*, e.g. the risk that a "wrong" parameter is chosen. Concluding, it is important to understand the behaviour of a model with respect to changed parameters.

Now let's have a look at what happens in one single such day. Our hedge portfolio consists of cash of 1.164 bn USD and we are short in the stock market index with an amount of 0.164 bn USD. We also know that the option value amounts to 52.2 M USD. Now let's look at the end of the day.

| in M USD | Value at 1000 | Value at 970 |
|---|---|---|
| Bonds | 1164 | 1164 |
| Shares | -164 | -159 |
| Options (index impact) | -52 | -57 |
| Options (interest - 25bps) | | -6 |
| Options ($\sigma$ to 20 %) | | -4 |
| Total | 948 | 938 |

As you see from the table above we have assumed that the volatility spiked up to 20 % and that the interest rates reduced by 25 bps. Together, all these effects have an adverse impact of 10 M USD which represents 1 % of the funds value. The impact in reality was far bigger than that and some sizable insurers have closed their corresponding portfolios for new business.

But now let's ask ourselves what went wrong. Actually, the underlying model worked in theory but in reality the situation was somewhat different. The whole arbitrage free pricing theory is based on a few relevant assumptions: frictionless, deep and arbitrage free market. The calculation can be interpreted as a market which is not deep and liquid enough, resulting in losses in the corresponding hedge portfolios. The true issue is that there was an over reliance on models and people did not ask what would happen if the model breaks down. Now I do not want to say that models are useless, but rather that we need to be always sure which are the corresponding limitations and what happens if these are not fulfilled.

The other conclusion is that derivatives can be very useful. On the other hand they are dangerous if not managed and analysed carefully.

## 14.4 Investment Guarantees and Bonus Rates

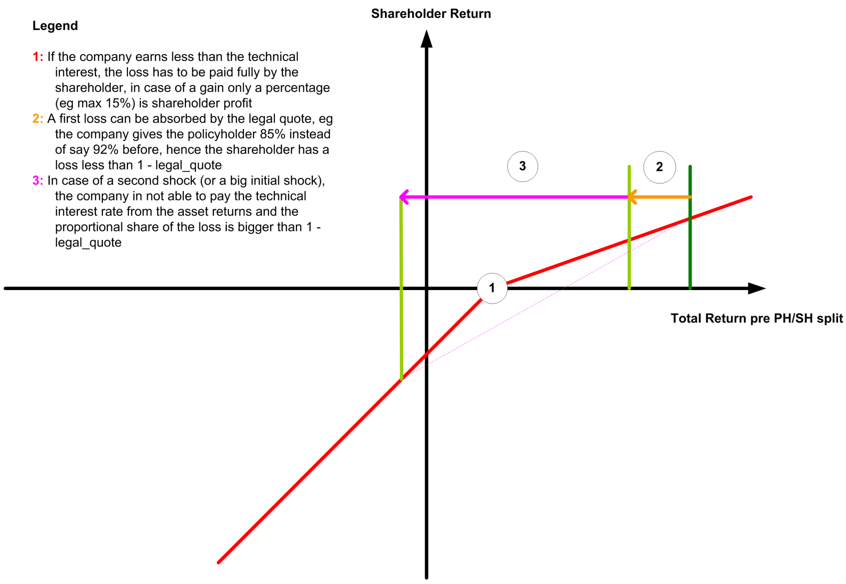



**Fig. 14.2** Policyholder – shareholder split

One reason for large issues with insurance products is that sometimes the existence and the level of interest guarantees the absence of a reasonable ALM. In order to understand the corresponding issues we look at Swiss pension schemes. Without going into detail, you can save money during the time you are working and the money saved is converted into an annuity and a corresponding widows pension at a fixed rate. This rate has for a long time been fixed at 7.2 % and is now going to decrease over an extended period. There are also other issues which we will exclude for the moment and we assume the following[1].

| | |
|---|---|
| Interest during deferral period | $i = 2.0\%$ |
| Current age of the insured | 45 |
| Current age of the partner | $\Delta_{XY} = -1$ |
| Conversion rate at 65 | 7.2 % including widow pension |
| Contribution rate | 15 % |
| Pensionable salary | 100000 |
| Single Premium at age 45 | 300000 |
| Valuation date | 29.12.2006 |
| Profit share mechanism | Reserves for longevity provision can be deducted. Shareholder can claim 10 % of the remaining profit. |

[1] We also remark that the example does not fully reflect the Swiss legislation.

We will consider two separate cases. In the first case we assume that the person is a man, in the other we assume that this benefit is offered to a woman. Tables 14.1 and 14.2 clearly shows that there is a sizable difference in future life span between the man and the woman.

**Table 14.1** Expected future life span for Swiss men

| Age | 1881-88 | 1921-30 | 1939-44 | 1958-63 | 1978-83 | 1988-93 | 1998-03 |
|---|---|---|---|---|---|---|---|
| 1 | 51.8 | 61.3 | 64.8 | 69.4 | 72.1 | 73.8 | 76.6 |
| 20 | 39.6 | 45.2 | 47.9 | 51.5 | 53.8 | 55.3 | 58.0 |
| 40 | 25.1 | 28.3 | 30.4 | 32.8 | 35.1 | 36.8 | 39.0 |
| 60 | 12.4 | 13.8 | 14.8 | 16.2 | 17.9 | 19.3 | 21.1 |
| 75 | 5.6 | 6.2 | 6.6 | 7.5 | 8.5 | 9.2 | 10.3 |

**Table 14.2** Expected future life span for Swiss women

| Age | 1881-88 | 1921-30 | 1939-44 | 1958-63 | 1978-83 | 1988-93 | 1998-03 |
|---|---|---|---|---|---|---|---|
| 1 | 52.8 | 63.8 | 68.5 | 74.5 | 78.6 | 80.5 | 82.2 |
| 20 | 41.0 | 47.6 | 51.3 | 56.2 | 60.1 | 61.8 | 63.4 |
| 40 | 26.7 | 30.9 | 33.4 | 37.0 | 40.7 | 42.5 | 43.8 |
| 60 | 12.7 | 15.1 | 16.7 | 19.2 | 22.4 | 24.0 | 25.2 |
| 75 | 5.7 | 6.7 | 7.4 | 8.6 | 10.7 | 11.9 | 12.8 |

First we look at the cash flow stream which is induced by the above contact. In order to do that we need to be aware that the conversion of capital in an annuity at age 65 is optional and in particular only the people reaching the age 65 have this option. Because of that we have the following cash flow pattern:

| Age | Saving amount | Cash Flow | Discount | MR |
|---|---|---|---|---|
| 45 | 15000 | -15000 | 1.00000 | 55609 |
| 55 | 187251 | -15000 | 0.78271 | 243309 |
| 60 | 290703 | -15000 | 0.68753 | 358147 |
| 64 | 383169 | -15000 | 0.62467 | 457903 |
| 65 | **392749** | 28277 | 0.61004 | **484246** |
| 66 | 0 | 27990 | 0.59656 | 466274 |
| 70 | 0 | 26626 | 0.54677 | 392517 |
| 75 | 0 | 24441 | 0.47350 | 312295 |
| 80 | 0 | 21605 | 0.41803 | 227625 |
| 85 | 0 | 17953 | 0.36905 | 148613 |
| 90 | 0 | 13183 | 0.32582 | 81052 |
| 100 | 0 | 2621 | 0.25395 | 7763 |

From the above table we see that the savings amount at age 65 equals 392749 CHF and that we need 484246 CHF for paying the liabilities assuming a risk free investment return. Hence c. 23 % of funds are missing at age 65 and the present value of the loss at age 45 equals 55609 CHF which equals 3.7 times the yearly contribution. For women the situation is worse because they live longer. Figure 14.3 shows the development of the available reserves compared with the necessary reserves.

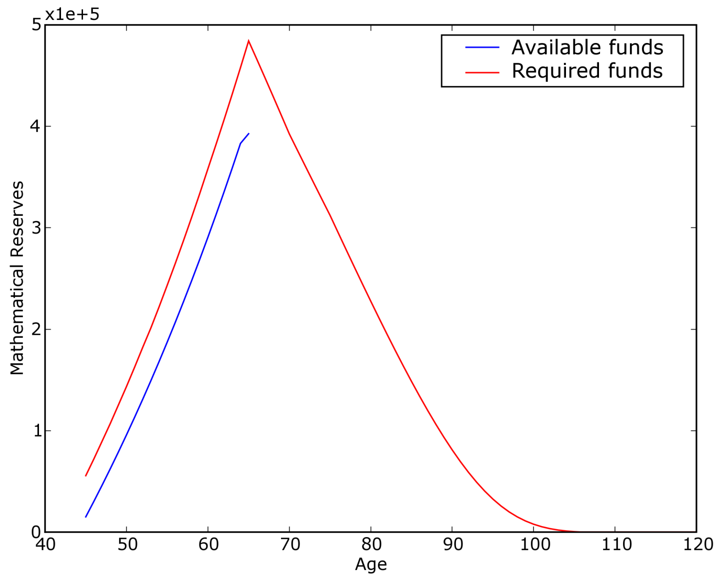



**Fig. 14.3** Development of mathematical reserves

But what does this loss now mean for the business. There are three ways how one can look at this:

- Do not do such business,
- Take the loss up front,
- Invest in a asset allocation where one can in average achieve the goal.

It was at this point things started to go wrong. Obviously the big players in the group life scheme market with assets under management above 20bn CHF did not want to cease this business and it was also not acceptable to show such big losses for each new contract and so the companies started to take more investment risk. We need now to know what investment return would be needed to fulfil the given obligations. By backward solving we find that we need to get an investment yield of 3.47% [2]. Hence we have the following situation, assuming that shares yield 400 bps more than risk free:

[2] Please note that the situation at that time was even worse, since the companies needed to credit not only 2 % but rather 4 % to the savings account during the accumulation phase.

| | |
|---|---|
| Required yield | 3.5 % |
| Risk free yield | 2.5 % |
| Required uplift | 1.0 % |
| Required equity backing ratio | 25.0 % |

If you now go back to the balance sheets of the Swiss life insurers at the beginning of this century, you will find that they were investing heavily in equities with equity backing ratios of 25 % and more. So this investment strategy worked well, because for most of the latter years of the 20th century, equities had a good return. In order to understand what happened in the year 2001, let's look at the corresponding balance sheets. To this end we look at a mid-sized insurer with assets of 20 bn CHF and insurance liabilities of 18 bn CHF and assume that it has an equity backing ratio of 20 %.

The expected yield of this company and its balance sheet looks as follows

| | SAA | in CHF | Yield | | Return |
|---|---|---|---|---|---|
| Shares | 20% | 7% | $4000000 \cdot 7\%$ | = | 280000 |
| Bonds | 60% | 4% | $12000000 \cdot 4\%$ | = | 480000 |
| Properties | 10% | 5% | $2000000 \cdot 5\%$ | = | 100000 |
| Mortgages | 10% | 4% | $2000000 \cdot 4\%$ | = | 80000 |
| | | | Total | | 940000 |
| Math. Res. | | 3.5% | $-18000000 \cdot 3.5\%$ | = | $-630000$ |
| | | | Total | | 310000 |

Hence the insurance has an average return to both shareholders and policyholders of 310 M CHF.

In the year 2001, the equity index fell by 21 %. What has happened to the insurers income statement?

| | SAA | in CHF | Yield | | Return |
|---|---|---|---|---|---|
| Shares | 20% | -21% | $4000000 \cdot (-21\%)$ | = | $-840000$ |
| Bonds | 60% | 4% | $12000000 \cdot 4\%$ | = | 480000 |
| Properties | 10% | 5% | $2000000 \cdot 5\%$ | = | 100000 |
| Mortgages | 10% | 4% | $2000000 \cdot 4\%$ | = | 80000 |
| | | | Total | | $-180000$ |
| Math. Res. | | 3.5% | $-18000000 \cdot 3.5\%$ | = | $-630000$ |
| | | | Total | | $-810000$ |

We can see that while this investment strategy worked for some time, the loss in this one year was so big, that the company lost almost half of its shareholder equity capital. This is also the reason why there were very big insurance companies in Switzerland which had to go to the capital markets in the years 2002 and 2003 to raise capital. There is another pitfall which one needs to be aware of if one takes excessive investment risk. In a lot of countries such as Germany, France but also in the UK there is a policyholder – shareholder split in respect to gross profits. Assume for example, that we have a *legal quote* such as 85 %. Assume for the moment that the gross profit before policyholder – shareholder split and before tax amounts to 1000 M. In this case there is a legal requirement to allocate 850 M to the policyholder and the shareholders get a pre-tax profit of 150 M. Then assume that we have a gross loss before shareholder – policyholder split of -500 M. In this case the shareholder takes the whole loss, since the minimal investment return for the policyholder is guaranteed. In consequence we get a shareholder – policyholder split as indicated in figure 14.2.

The following table shows a comparison between two different investment strategies, assuming a legal quote of 85 % and a tax-rate of 0 %. We assume the following:

| | |
|---|---|
| Mathematical Reserve | 1000000000 EUR |
| Technical interest | 3.0 % |
| Yield of a bond investment | 4.0 % |
| Expected yield shares | 7.0 % |
| Volatility of shares | 18.0 % |
| Strategy 1 | 100 % invested in bonds |
| Strategy 2 | 25 % invested in shares , 75 % in bonds. |

For strategy 1 we know that we have a gross profit of 10 M EUR and hence the shareholder (SH) gets 1.5 M EUR and the policyholder (PH) 8.5 M EUR. For strategy 2, the situation is more complex and we need to look at the corresponding probability distribution:

| Return Shares | Probability Return | Portfolio Gross | P/L $\sum$ | P/L SH | P/L PH |
|---|---|---|---|---|---|
| -40 % | 0.00086 | -7.00 % | -100000000 | -100000000 | 0 |
| -35 % | 0.00169 | -5.75 % | -87500000 | -87500000 | 0 |
| -30 % | 0.00426 | -4.50 % | -75000000 | -75000000 | 0 |
| -25 % | 0.00962 | -3.25 % | -62500000 | -62500000 | 0 |
| -20 % | 0.01948 | -2.00 % | -50000000 | -50000000 | 0 |
| -15 % | 0.03530 | -0.75 % | -37500000 | -37500000 | 0 |
| -10 % | 0.05730 | 0.50 % | -25000000 | -25000000 | 0 |
| -5 % | 0.08331 | 1.75 % | -12500000 | -12500000 | 0 |
| 0 % | 0.10851 | 3.00 % | 0 | 0 | 0 |
| 5 % | 0.12659 | 4.25 % | 12500000 | 1875000 | 10625000 |
| 10 % | 0.13229 | 5.50 % | 25000000 | 3750000 | 21250000 |
| 15 % | 0.12383 | 6.75 % | 37500000 | 5625000 | 31875000 |
| 20 % | 0.10383 | 8.00 % | 50000000 | 7500000 | 42500000 |
| 25 % | 0.07799 | 9.25 % | 62500000 | 9375000 | 53125000 |
| 30 % | 0.05247 | 10.50 % | 75000000 | 11250000 | 63750000 |
| 35 % | 0.03162 | 11.75 % | 87500000 | 13125000 | 74375000 |
| 40 % | 0.03097 | 13.00 % | 100000000 | 15000000 | 85000000 |
| Expected Value | 1.00000 | 5.34 % | 23472093 | -1518018 | 24990112 |

It becomes obvious that this second investment strategy is much worse for the shareholder since he makes, on average, a loss. As a consequence one needs to be very clear about bonus sharing mechanisms when determining the target asset allocation.

## 14.5 Longevity and the Ability to Forecast

In this last section I would like to elaborate further on the longevity issue, which I started to introduce in section 14.4. There the conversion rate of 7.2 % was stated, but it was not clear whether it was due to interest guarantees or because people are living longer. Whereas we put the focus on investment guarantees in section 14.4, we want to focus here on the longevity aspect of the issue.

We have seen in tables 14.1 and 14.2 that the future life span of men and women is still increasing at a high pace. The task of the actuary is to develop tables which forecast this (relatively stable) trend in order to avoid future losses. In order to check this need to have a look at the results by comparing the corresponding mortality tables. I use the Swiss tables but I want to stress that I have not encountered yet a single country where I could not observe the same effect:

| $\ddot{a}_{65}(i = 3.5)\%$ | men | $\Delta$ men | women | $\Delta$ women |
|---|---|---|---|---|
| ERM/F 70 | 12.491 | 3.958 | 13.923 | 3.820 |
| ERM/F 80 | 13.199 | 3.250 | 14.789 | 2.954 |
| ERM/F 90 | 14.387 | 2.062 | 16.221 | 1.522 |
| ERM/F 00 @ 2005 | 16.450 | | 17.744 | |

The table above compares the single premiums to be paid for an immediate annuity of 1 at age 65. We see that the price for this cover has increased by 3.958 when switching from the table ERM 70 to the table ERM 80.

The table above overstates the situation (because some of the people to whom the older products were sold have already died) but it shows the right direction. So assume that our insurance company has the following portfolio of people (men) who are aged 65 and assume that we expect them to live according to the most recent tables

| Tariff generation | MR reserve Original base M CHF | MR reserve ERM/F 2000 M CHF | Difference M CHF |
|---|---|---|---|
| ERM/F 70 | 400 | 526 | 126 |
| ERM/F 80 | 800 | 997 | 197 |
| ERM/F 90 | 2000 | 2286 | 286 |
| ERM/F 00 @ 2005 | 400 | 400 | 0 |
| Total | 3600 | 4210 | 610 |

From the above table it becomes obvious that the wrong mortality estimate is quite costly and amounts to CHF 610 M. The question why this happens so consistently cannot be answered easily, but there are some reasons listed below:

- In the past the analysis–tools were not as developed as now.
- There was a disbelief that the existing trend in an increased lifespan would persist in the future.
- Applying lower mortality rates to the in-force book is very expensive and hence one was reluctant to apply tables with stronger trends. Furthermore, there was a fear that the annuity product could not be sold anymore because it would become too expensive.

## 14.6 Long Term Care

In the previous section we looked at longevity risk and we want now to focus on long term care business. In order to do this, we need to understand the corresponding cover and how to value it. Afterwards we want to have a look at the risks of this cover.

Assume you are a healthy person living at home, able to feed yourself, to wash yourself et cetera. Hence you are able to perform the essential daily living activities (DLA) without help. Once you get older this may not be possible anymore and you are threatened to go into care, which you may not want. You would rather have home help. The long term care (LTC) cover aims to protect you from this, by paying for long term care support. How does this work in practise?

First the insurer defines the main daily living activities which you should be able to perform yourself and an amount which is paid if the person is not able to perform these anymore. Technically speaking we have, for example, 8 DLA which are monitored and you can perform between 0 and 8 of them. One could have a cover where you do not receive anything if your ability is 6 and above and gradually increase for the fewer DLAs you can perform yourself. In the concrete example the respective states are numbered from 1 to 6, where 1 indicates that everything can be autonomously and 6 represents the fact that we need help for all daily living activities. Formally the states are called $S = \{\dagger, 1, 2, 2a, 3, 3a, 4, 5, 6\}$. Assume that the benefits are given by the following table:

| Number of DLA | Amount payable pa. |
|---|---|
| $DLA \geq 7$ (S1, S2) | No benefit, only premium payment |
| $DLA = 6$ (S2a, S3) | 6000 |
| $DLA \leq 5$ | 12000 |

In order to price and value this cover we need to use a Markov chain model (see appendix B):

- The Markov model consists of the following states $S = \{\dagger, 1, 2, \ldots 8\}$, where $\dagger$ stands for the state of being dead.

- In a next step one needs to define the corresponding transition probabilities $p_{ij}(t, t+1)$, for $(i,j) \in S \times S$.
- For a market consistent valuation the discount rates follow risk free curves as seen before.
- Finally the table above has to be translated in payment functions $a_{ij}^{\text{Post}}(t)$ and $a_i^{\text{Pre}}(t)$. In order not to be overly complicated we assume that the benefits defined above are paid at the beginning of the year. Furthermore we denote with $P$ the premium and we assume that this is only paid in states S1 and S2.

Based on the above we have the following:

$$
\begin{aligned}
a_{ij}^{\text{Post}}(t) &= 0, \\
a_i^{\text{Pre}}(t) &= \begin{cases} -P & \text{if} \quad i \in \{S1, S2\}, \\ 6000 & \text{if} \quad i \in \{S2a, S3\}, \\ 12000 & \text{else.} \end{cases}
\end{aligned}
$$

In order to determine the premium and the mathematical reserves we use the recursion (107) in appendix B and assume that the person has currently an age $x = 65$. We want to have a closer look at the following questions:

1. What is the price and the mathematical reserves?
2. What happens if we consider an increase in life span and we assume that the remaining probabilities are reduced proportionally?
3. What does it mean if people become older and the time they are healthy remains constant?

Using the elements defined above and Thiele's difference equation, we can calculate the premium $P$ for the two states S1 and S2, where we see an obvious difference in the present value of a premium 1. Figure 14.4 shows this effect. A person buying this cover at age 75 would have to pay about 1000 if he is in state S1 and about 4300 if he is in state S2. This difference shows clearly the risk the company is assuming, as a person who is not able to perform 1 DLA has a materially higher risk. This also explains why the underwriting of this type of policy is of utmost importance. You can imagine what would happen if a person is assumed in state S1 soon becomes unable to perform his daily living activities. Figure 14.4 also shows the relative size of the premiums between states S1 and S2, and we see that there is at age 65 a factor of about 4.5 between the two.

With the same calculation, we can also determine the present value of future cash flows, as shown in Figure 14.5. In order to have a comparison, the mathematical reserves have been scaled relative to state S1. Note that state S4 is the one which is the most expensive and that both states S5 and S6 are cheaper. This is because people in these two states have a higher probability of death and therefore, the time

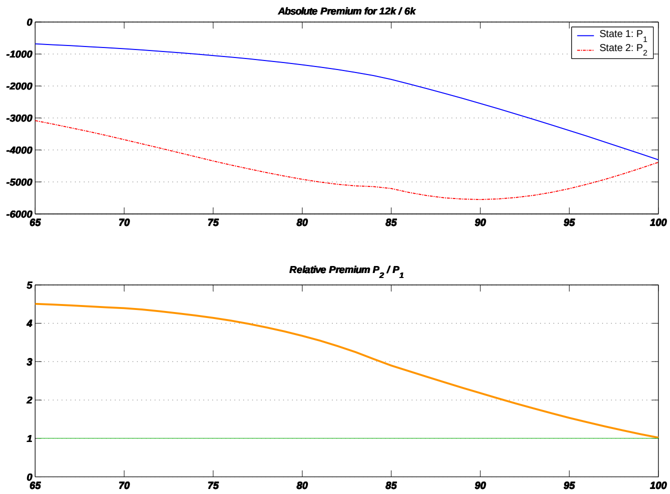



**Fig. 14.4** Level Premiums for LTC cover

the insurer has to pay is shorter. We will see, that the future development of mortality could impact this. Finally figure 14.6 shows the distribution of the losses. It also shows the dependency of the corresponding state. Such a calculation can either be performed by recursion, or as in the concrete case by a simulation. From this figure we can for example see that the probability of never receiving a benefit for a person starting in state S1 is about 34%. In the same sense we see that the death probability is higher in state S4 than in state S3.

Next we want to look what happens, if we assume that the mortality reduces faster as shown in Figure 14.7. We see that this improvement has a considerable impact. More concretely two versions have been calculated, one (variant 1) where the people remain healthy and stay longer in state S1. In the other, the reduction in mortality goes in parallel with an increased time where the people are not anymore able to perform the different DLAs. Obviously this has a material impact, which needs to be considered when constructing and pricing this type of product. We finally see in figure 14.8 the way the reduction in mortality leads to higher claims. In respect to variant 1 we see that the main additional cash flows are a consequence of living longer, starting at about age 80. We also see that for variant 2 the higher losses start soon after age 70.

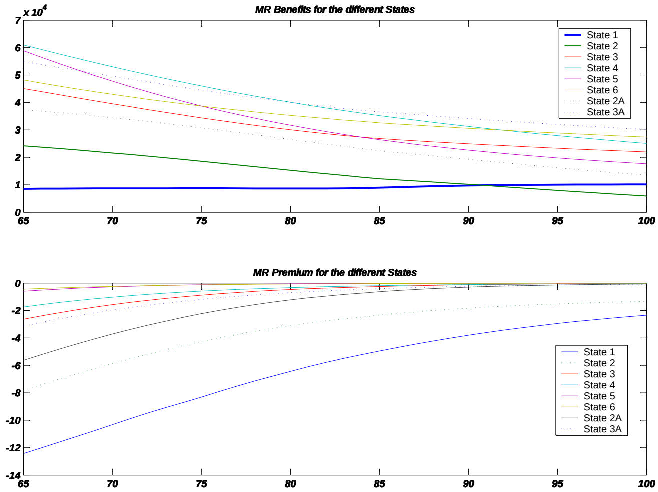



**Fig. 14.5** Relative Mathematical Reserves for LTC cover

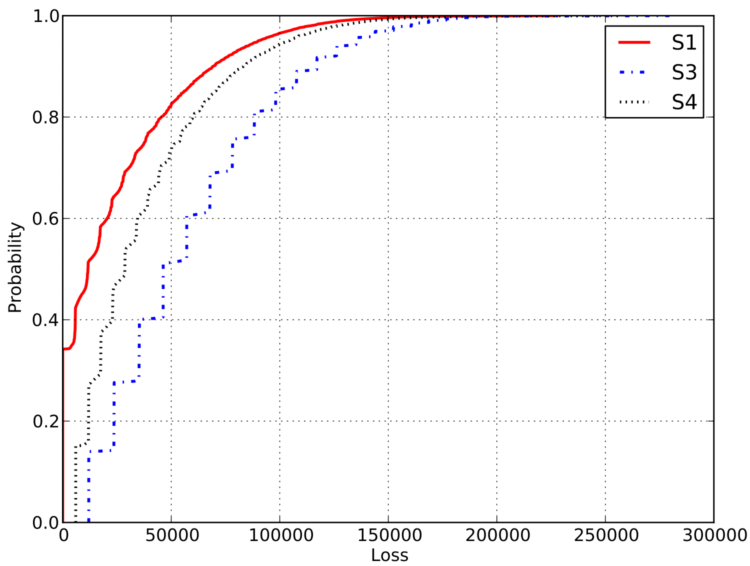



**Fig. 14.6** Distribution of Mathematical Reserves for LTC cover

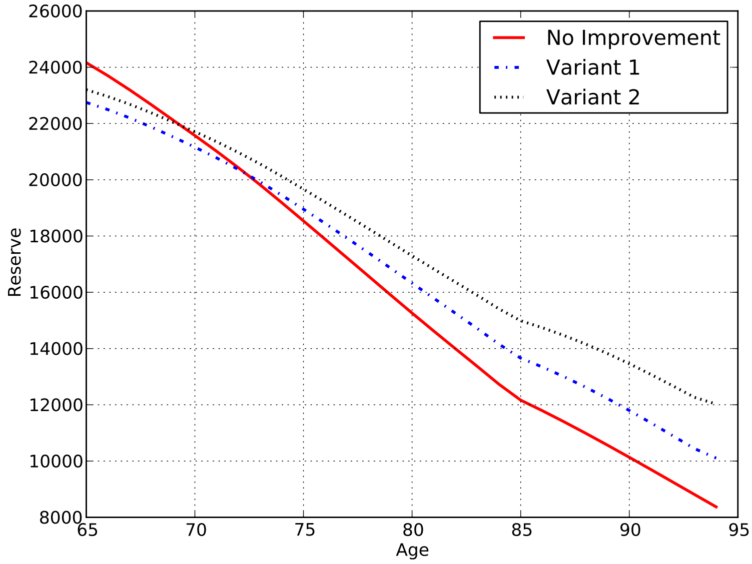



**Fig. 14.7** LTC Mathematical reserves when reducing mortality

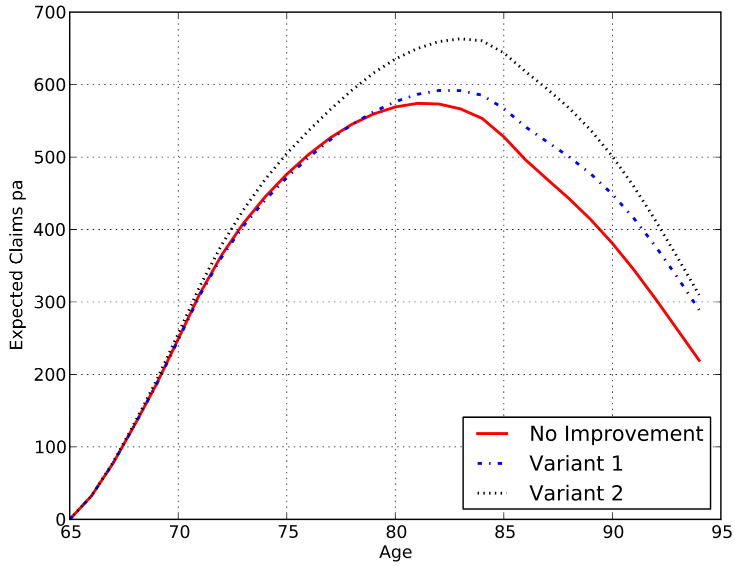



**Fig. 14.8** LTC expected losses when reducing mortality

## 14.7 Imperfect Cash Flows Matching

We have seen in section 2.3 that there are no prices for long dates bonds in some currencies. Moreover even if there are prices for these bonds, there might be only a limited market for long dated bonds, as a consequence of states not wanting to issue long term bonds. A typical example is the CHF, where the market is liquid only up to durations of about 15 years. As a consequence insurance companies and pension funds are not able to match their guaranteed cash flows with corresponding bonds.

In this section we want to have a closer look at this question and the corresponding risks. The best way to understand this risk is to look at concrete examples:

- A portfolio of annuities in payment,
- A portfolio of deferred annuities,
- A portfolio of endowment policies.

In all three cases we assume that the benefits are denominated in CHF and we furthermore assume there is only a liquid market for CHF bonds until year 15 and hence the best thing to do is to use investments according to this. In order to valuate what could happen we look at the following scenarios:

1. Yield curve and investment opportunities as seen today,
2. At time 15 there is a flat yield of 0%, 1%, 2 % and 3 % respectively.

In order to be able to better describe this problem, we denote with $(CF_k)_{k\in\mathbb{N}}$ the vector of expected cash flows and for the moment we neglect the fact that these cash flows are actually random and can depend on the market environment. For the analysis we assume that the company invests as follows in $\sum_{k\in\mathbb{B}} \alpha_k \, \mathcal{Z}_{(k)} \in \mathcal{X}$:

$$\alpha_k = \begin{cases} CF_k & \text{if } k < 14, \\ \sum_{k\geq 15} CF_k & \text{else.} \end{cases}$$

This means that the company actually tries to invest as long as possible. We furthermore assume that the company follows a passive investment strategy and reinvests the excess assets in $\mathcal{Z}_{(15)}$ at time 15 according the investment condition at this time. We need to remark that the chosen investment strategy is obviously a simplification and that reality is more complex. It however exposes the risk the company is facing, when not being able to invest in the corresponding bonds. In order to calculate the corresponding risk we follow a rather easy approach by adjusting the yield curve after year 15. We remember that the prices of zero coupon bonds and corresponding yields have the following relationship:

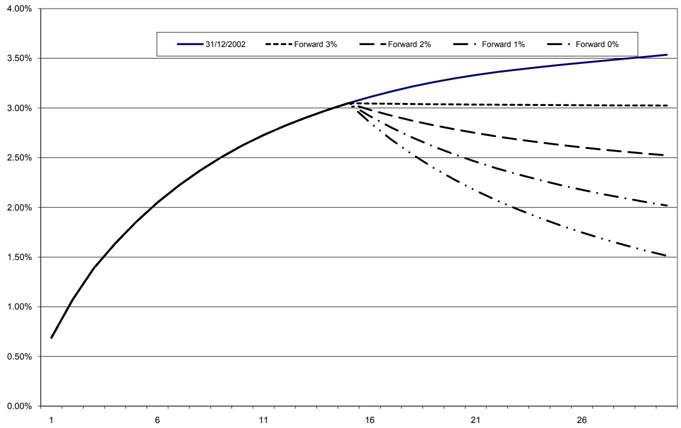



**Fig. 14.9** Modified yield curves

$$\begin{aligned}\pi_t(\mathcal{B}) &= \sum_{k=0}^{\infty} CF_k \times \pi_t(\mathcal{Z}_{(k)}) \\ &= \sum_{k=0}^{\infty} CF_k \times (1 + y_t(k))^{-k}.\end{aligned}$$

Moreover the forward rates can in this case be calculated by

$$f_t(n,m) = \left(\frac{\pi_t(\mathcal{Z}_{(n)})}{\pi_t(\mathcal{Z}_{(m)})}\right)^{\frac{1}{m-n}} - 1,$$

and hence the following equation holds:

$$(1 + y_t(n))^n = \prod_{k=0}^{n-1} (1 + f_t(k, k+1)).$$

At this stage it is now easy to “construct” suitable yield curves representing the scenarios above by setting:

$$f_t(n, n+1) = \theta,$$

for all $n \geq 15$, where $\theta$ represents the interest rate going forwards, according to the scenario, after year 15. Figure 14.9 shows the corresponding yield curves. It is obvious that the the overall yield reduces considerably in particular when using 0 % as forward rate. It is worth noting that the case of 0% is the worst case, since we could in this case hold the cash after time 15 until it is used, assuming that both the bonds and the cash is risk-free.

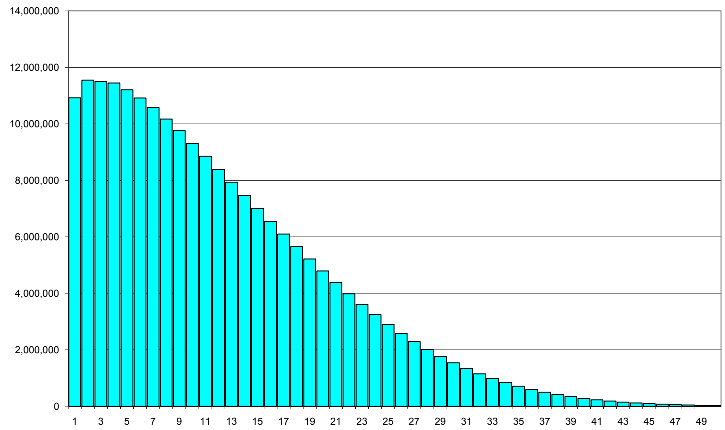



**Fig. 14.10** Cash Flow Pattern of Annuities in Payment

Now we need to have a look at the concrete portfolios. We denote the annuity portfolio in payment as (A), the deferred annuity portfolio as (B) and the endowment portfolio as (C)

| in M CHF | Portfolio A | Portfolio B | Portfolio C |
|---|---|---|---|
| Benefit | 13.6 p.a. | 108.1 p.a | 4211.1 |
| Statutory Reserves | 162.7 | 841.5 | 2474.4 |
| Premiums | – | – | 20.4 |
| Duration | 8.9 | 25.3 | 11.0 |
| Figure for Cash Flows | Fig. 14.10 | Fig. 14.11 | Fig. 14.12 |

Next we can calculate the corresponding amounts at risk as seen from today by using the different yield curves. The following table summarises this:

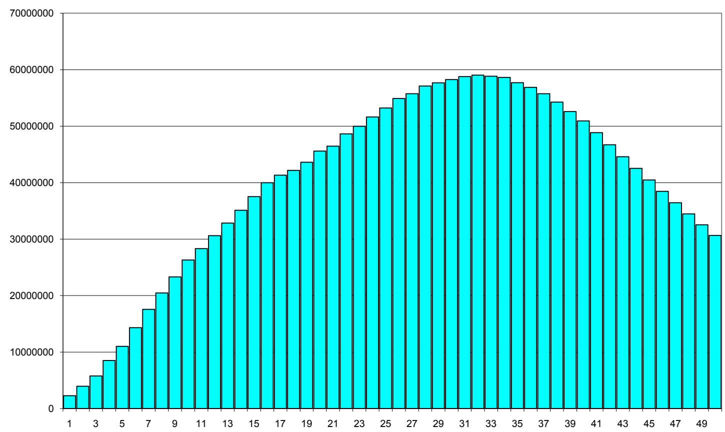



**Fig. 14.11** Cash Flow Pattern of Deferred Annuities

| in M CHF | Portfolio A | Portfolio B | Portfolio C | Total |
|---|---|---|---|---|
| Statutory Reserves 3.5 % | 162.7 | 841.5 | 2474.3 | 3478.6 |
| Value using yield as at 31.12.2002 | 159.4 | 880.0 | 2617.3 | 3656.9 |
| Forward 3% | 161.5 | 994.4 | 2678.9 | 3834.8 |
| Forward 2% | 163.7 | 1135.6 | 2746.0 | 4045.4 |
| Forward 1% | 166.3 | 1322.3 | 2821.5 | 4310.2 |
| Forward 0% | 169.2 | 1575.5 | 2906.7 | 4651.5 |
| **Coverage in %** | | | | |
| Value using yield as at 31.12.2002 | 102.0 % | 95.6 % | 94.5 % | 95.1 % |
| Forward 3% | 100.7 % | 84.6 % | 92.3 % | 90.7 % |
| Forward 2% | 99.3 % | 74.1 % | 90.1 % | 85.9 % |
| Forward 1% | 97.8 % | 63.6 % | 87.6 % | 80.7 % |
| Forward 0% | 96.1 % | 53.4 % | 85.1 % | 74.7 % |
| **Coverage absolute** | | | | |
| Value using yield as at 31.12.2002 | 3.2 | -38.5 | -143.0 | -178.3 |
| Forward 3% | 1.2 | -152.9 | -204.5 | -356.2 |
| Forward 2% | -1.0 | -294.0 | -271.7 | -566.8 |
| Forward 1% | -3.6 | -480.8 | -347.2 | -831.6 |
| Forward 0% | -6.5 | -734.0 | -432.3 | -1172.9 |

Looking at the example above one sees that the reinvestment risk, as a consequence of missing long duration assets, can be extremely dangerous for a life insurance company. We also see that deferred annuities in particular are a treat since the have a very long duration. Considering the current yields in CHF we would most likely have to look at a scenario between 1% and 2%. In this case we see that the company

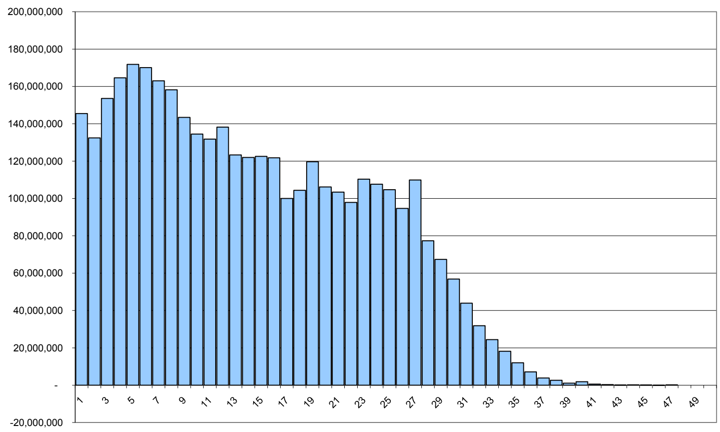



**Fig. 14.12** Cash Flow Pattern of Endowment Policies

needs to strengthen their total reserves of c3.5 bn CHF by approximately 17 % (c0.7 bn CHF). This clearly shows the magnitude of this risk and the need for an adequate ALM and risk management.

Another interesting aspect which can be analysed with the above portfolios is the possible impact of the EU gender directive, which postulates the equal treatment of men and women. This would concretely mean that one needs to have to use the same pricing for men and women. The real risk would be a retrospective application of the directive. The table below shows the corresponding impact, which is material for all types of annuities as a consequence of the different future life expectancy.

| in M CHF | Portfolio A | Portfolio B | Portfolio C | Total |
|---|---|---|---|---|
| Men as men | 137.2 | 663.7 | 1749.0 | 2549.9 |
| Women as women | 25.4 | 177.8 | 725.3 | 928.6 |
| Total | 162.7 | 841.5 | 2474.3 | 3478.6 |
| All as men | 157.7 | 804.2 | 2483.5 | 3445.5 |
| All as women | 197.1 | 1037.5 | 2451.1 | 3685.8 |
| Impact | 4.9 | 37.3 | 23.2 | 207.2 |
| **Relative Impact** | **3.0 %** | **4.4 %** | **0.9 %** | **5.9 %** |

# Chapter 15
# Emerging Risks

The aim of this chapter is to have a look at emerging risks. First this concept needs to be explained and also why it is important. In principle answering this question is at the centre of risk management.

## 15.1 What are Emerging Risks and why are they important?

Emerging risk are the ones which are not yet very obvious. Consequently they are not easy to detect. The reason for writing this chapter is based on a person chal-

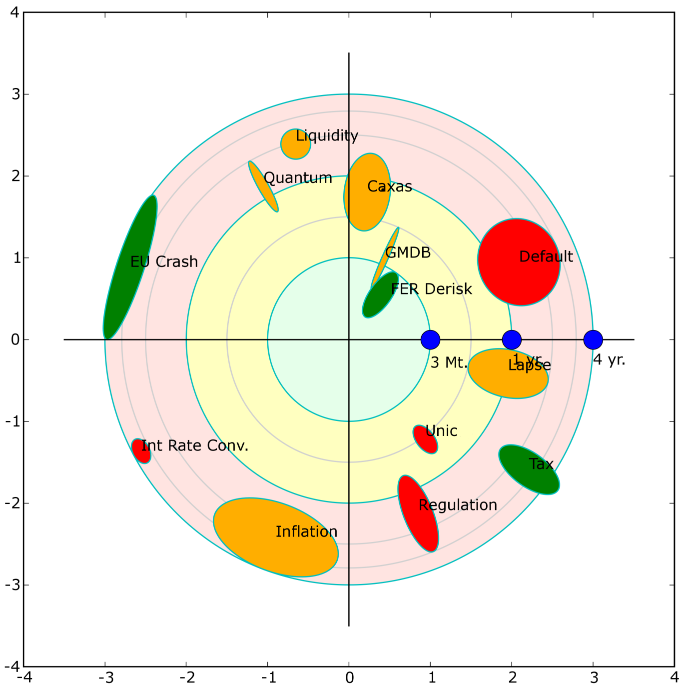



**Fig. 15.1** Example of a Risk Radar

lenging me by asking, how risk management can create value. Assume for example some big banks would have seen the consequence of the 2008 financial meltdown early *and* would have reacted accordingly. In this case many losses could have been prevented and the banks would not have lost a lot of their intrinsic value and credibility.

In this sense emerging risks are the ones which are believed to emerge not in the very near future, but rather a little bit later. A typical emerging risk could be global warming. So it is first important to detect these emerging risks and to act accordingly. Both of these tasks are not as easy as they may seem. For the detection of the risks, corresponding methods need to be applied and as a second step it is important to convince the relevant people that these risks are not only issues on the paper, but that there is a likelihood that they materialise over time.

When thinking about emerging risks, I imagined to sit in a valley with high mountains around me. Obviously (if there is no fog), I can see all the mountains and can identify the risks and perils. What I can not do is to look on the other side of the mountain. When imagining to cross over the mountains the perils of the part which I can not see are considered emerging risks. Obviously I would not try to cross a mountain if there is danger (such as a dragon) on the other side and would choose a less risky route, even though the alternative could be steeper in the short run. Based on this trivial idea, I tried to show the risks in the form of a radar (lets call it a *risk radar*). In contrast to a typical radar, where as the radius is the distance, I have chosen time as radius, and in consequence the task is to detect emerging risks, which may occur in say 1 year's time.

This method can not only be applied for regular risk management but this method is particularly useful for projects and change programmes. I have used this method and graphical support in one of my former jobs and it worked really well, because it helped to stimulate a discussion and also to visualise the risks.

## 15.2 What Process is needed for Emerging Risks?

Here I want to explain the above mentioned risk radar in more detail. Figure 15.1 shows a concrete example. The picture is characterised by bubbles of different size and colour and each ellipse is characterised by the following attributes:

**Likelihood that risk materialises:** This likelihood is expressed as the colour of the bubble. Red means high likelihood and green a relative low likelihood.

**Time when the risk is expected to start:** This may be in 1 year and that's where the ellipse starts.

**Time how long the risk may persist:** That's the diameter of the ellipse towards the centre. Hence the ellipse ends when the risk is "over".

**Expected monetary impact if risk materialises:** The areas of the bubbles have been calibrated in order that the areas represent the relative severity

Finally it has to be remarked that the figures use a logarithmic scale for the time and that the best way to produce such graphics is to use a small program doing the job for you, since messing around with the ellipses is very time consuming.

Obviously having the figure is not yet sufficient to detect emerging risks. The best way to detect them is an honest and open discussion. For the change programme I mentioned and where I already had a lot of insights, I have taken some time in a silent environment and put together the zoo of emerging risks and produced the graphics, which I then discusses with people working in risk and on the programme at different levels. This helped to improve the content of the risk radar.

The nice thing when using a program is also that you can project in the future and produce the same graphic in a year and look what may hit then. What was really astonishing was the fact that this method provided a good predictor of the upcoming risks and it was possible to avoid some of them in an early stage, hereby helping the change programme considerably.

It is important to note that a risk radar should not be produced once, but it needs to be embedded in a process. In the concrete set up the risk radar is updated every 3 months and the output is discussed, in order to migrate the risks.

Finally I would like to say some words in relation to the required skill set for detecting emerging risks successfully. Obviously models are not really of great help, since this process aims to detect the risk which are somehow hidden and not as easily detectable. Hence the following characteristics are important:

- Good and holistic understanding what is happening for example in a change programme,
- The ability and honesty to analyse what has gone wrong, and what could go wrong also as a consequence of inadequate skills,
- The ability to carefully listen to the programme managers and in particular also to the people working on the project,
- The ability to abstract from the day to day frustrations and fears of the people,
- The creativity to think what could go wrong, and experience,
- The will and ability to these exercises as processes.

# Chapter 16

# Regulatory view on Risk Management: Solvency II

## 16.1 Introduction

Solvency regulation in the EU is under reform. The Solvency II ([ECSO2]) project will introduce a new solvency regime which will be characterised by an integrated risk approach better taking into account the risks an insurer is facing than the current solvency regime. Securities for these risks will have to be held in the form of solvency capital. There is however a difference between the risk management for an insurance entity from the company's point of view and from a regulator's point of view. Whereas the assessment of risks and the calculation of the available and required risk capital *should* follow a strict market consistent approach with no hidden buffers, the regulatory approach focuses mainly on the security for the policyholders. The main aim of the insurance entity is to optimise its risk adjusted returns and it has therefore no incentive to under- or overestimate its capital requirements. On the other hand the regulator puts a bigger emphasis on security and the return on capital is rather a secondary point of view.

Furthermore we need to acknowledge that there is a principal – agent problem from a systemic point of view, since the principal (the policy setters and the general public) aims to have an efficient insurance market with competitive products at reasonable prices. This implies that the capital requirements should not be too onerous. The regulators protects the policyholders' interests and aim a capital requirement at the upper end of the reasonable range, since then they can sleep better. I do not want to state that Solvency II is not reasonable but I just want to say that there is a risk. This issue can be seen when following the discussions between policy setters, regulators and the insurance industry. There is also a principle – agent problem between policy setters on the one hand and the insurance industry, mutatis mutandis.

But what is the ideal outcome from a principal's point of view? In my view the accurate best estimate assessment of available and required capital, for the reason that both a too high capital level with implicit margins and also a too low capital level, is dangerous. For the second case this is obvious. For the first one it is a little bit trickier: since there are in this case implicit margins, one might feel in a secure region, despite the fact that one is not anymore, for the simple reason of not knowing the extent of the implicit margins.

For the insurance companies in Europe, the wave of deregulation in the 1990s brought more freedom - as well as more independent responsibility. This affected the insurance companies, their shareholders and the supervisory bodies. The companies sought to utilise the opportunities offered by deregulation and booming stock markets in order to expand internationally and to enter into more risky investments. At the same time, risk management was often neglected and companies made themselves increasingly dependent on capital gains. This trend became particularly marked among life insurers, who often made huge promises: they promised high surpluses which could only be achieved by assuming a high degree of risk. The turnaround came in 2001, when the climate for the insurance companies deteriorated dramatically - due to the events of 11 September, and to massive stock market losses. As a result, the largest Swiss insurance companies had to contend with very high losses and needed to rebuild their capital base. Outside Switzerland, some insurers even went bankrupt. Another example is Equitable Life, one of the oldest life insurance companies on the European continent. Here the problem were 'quasi' guarantees (say 8% return guarantee including policyholder bonus payments) which have been offered to the policyholders in an interest environment which was then very high (eg GBP interest rate at over 12%). As the company did not valuate these liabilities in a market consistent way, Equitable Life was not able to anticipate its problems in a timely manner and the company was forced to close down their new business (see also section 14.4).

What had happened? It was not the economic principle of diversification that had failed. In fact, what was lacking in the companies was appropriate risk management. On top of this, the instruments of supervision were often not applied with sufficient consistency, nor were they suitable to provide adequate measurements of companies' risks. The yardstick of capital - based on the old solvency regime - was not capable of measuring the asset-liability risk. In other words, the risk that the asset side of the balance sheet (investments) might behave differently than the liabilities side (technical obligations) could not be assessed correctly. The result was that assets were used to enter into risks that were out of any proportion to the insurance portfolio on the liabilities side.

This shows the need for the European regulators to adjust their tools and methods in order to be able to keep up with dramatic increase in complexity in the financial sector. It is however of utmost importance to acknowledge the economic fundamentals of all insurance undertakings: the law of large numbers or the diversification effect. It is from this point of view key that new regulation accepts diversification on all levels: between individual risks, between regions and also between legal entities (group diversification). It is clear that diversification goes hand in hand with capital fungibility and also with mutual trust between the regulators of the different legal entities in a group. Assuming a reduction of the diversification benefit would ultimately lead to either higher costs for the policyholder or to a deterioration of the risk adjusted profitability for the owners of the insurance company. The latter clearly leads to withdrawal of capital from the insurance sector which by itself leads to a reduction in available capacity. Hence it is key for Solvency II to accept the diversification benefit and ensure the corresponding capital fungibility.

## 16.2 What is Solvency II?

Currently Solvency II follows a one fits all approach in the sense that one tries to have one big risk based solvency framework which should be applied to all the different insurance entities in the same manner. This however does not reflect the relative importance of the different types of insurers. Figure 16.1 aims to explain this. Looking from a policy holder's standpoint one needs to distinguish at least between life, non-life and reinsurance. From the policyholder's point of view life insurance serves to protect him from the risk of suffering famine in case of ageing - it protects his (or his descendants) individual economic wealth after retirement. Clearly this aspect is one of the most important from the individual's (and also from the social state's) point of view. Therefore the security of the corresponding insurance undertaking needs to be highest. This is even more important under the point of view that these types of contracts consist normally out of very long tailed liabilities involving major financial resources. The pension benefit for a person after retirement is usually the biggest asset of him. In order to protect this wealth one needs to consider

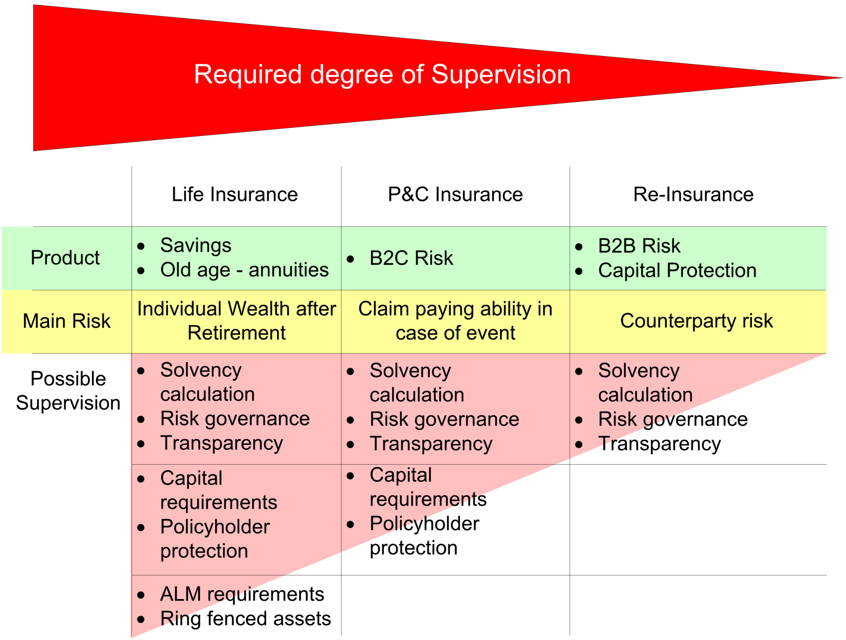



**Fig. 16.1** Supervision Intensity

risk based solvency requirements, ALM requirements and possibly also ring fenced assets to protect the policyholder in case of the default of the insurance company.

P&C insurance has another aim. Here the typical example is the car which had an accident or the house which burned down. The main risk from the policyholder's point of view is the claims paying ability and is correspondingly of less importance for the individual compared with the life covers. Therefore the regulation should be lighter. One could for example only require solvency requirements but no ALM requirements. Moving to reinsurance one has to remark that we are here in a B2B environment where there should not be an explicit retail customer protection. It is however important that there is enough transparency in order that the buyer of such products can access the financial stability of the counter-party (eg reinsurer). Therefore the *counter-party risk* represents the major risk category and correspondingly the regulation should be lighter. Looking to capital markets one can ask the rhetoric question whether junk bonds should be disallowed? The answer is clear: an investor can outperform with respect to his risk return profile by adding junk bonds to his investments. In analogy insurance regulation should also reflect this idea and one could think of a solvency regime which requires only transparency here but does not explicitly require a certain capital level. Even more philosophically one could ask the question whether the state should at all prescribe the level of security in insurance entities - nobody would ever prescribe a private asset portfolio's minimal credit quality.

Now what is the consequence of these ideas? Regulation should not fall in the pit to apply one approach to all possible insurance undertakings. Looking at the proposed *standard formulae* it becomes clear that they were created with the aim to cover all possible risks for every possible insurance company. It is very likely that such an approach neither qualifies as transparent nor as efficient. Furthermore it will be very difficult to develop compatible internal models. It would be much better to start with principles a risk based solvency model should fulfil and in a second step to develop suitable simplifications. This is exactly the way how the *Swiss solvency test* was developed; the first joint working meeting between the industry and the Swiss supervising authorities centred about principal questions. For example should such a model be book-value or market-value based? How should insurance liabilities be modelled - is there a need for the valuation of policyholder options etc. This methodical approach was also recognised by the CEA, the European insurance association and it allows the development of both internal models and standard simplifications in a consistent way. A last general comment: as such models start to become rather complex at a very early stage it is very important to be pragmatic and simple. In the following section we will dig a little bit deeper in some of the relevant areas.

## 16.3 Economic Balance Sheet and Prudence

The most important additional insight which will be provided by Solvency II are *economic balance sheets* as introduced in chapter 2. The "quasi" guaranteed annuities with a 8% return guarantee indicated above show clearly the need for an economic balance sheet. This means on the asset side that all unrealised capital gains and losses are taken into account in a transparent way. On the liability side the situation is somewhat different, because no tradable instruments exist which can be used to perfectly replicate the liabilities in order to determine their economic price. It is however clear that exactly this information is of utmost importance for managing the risks and one therefore usually uses a model approach to get a reasonable approximation of the market values for the insurance liabilities. In first a step one needs to calculate the expected present value of the future policyholder benefits. On top of this amount one requires a *market value margin* (MVM) or also called *prudence* in a regulatory environment. For details re refer to chapter 2.

In the cost of capital approach (CoC) the required risk capital is projected into the future. In a second step the CoC equals the present value of the corresponding costs for future periods. The parameter corresponds to the unit cost of capital and is usually in the order of between 2% and 6%. Figure 16.2 shows corresponding calculations performed by different Swiss insurers during the field test 2005[1]. These results show that even in the low interest environment for the CHF there are significant margins in the P&C reserves and also to a minor extent in the life reserves. Finally it is im-

[1] The results of the SST field test can be found under http://www.finma.ch.

portant to remark that the CoC approach has two additional benefits: one can quite easily verify the corresponding results and it avoids double counting of capital.

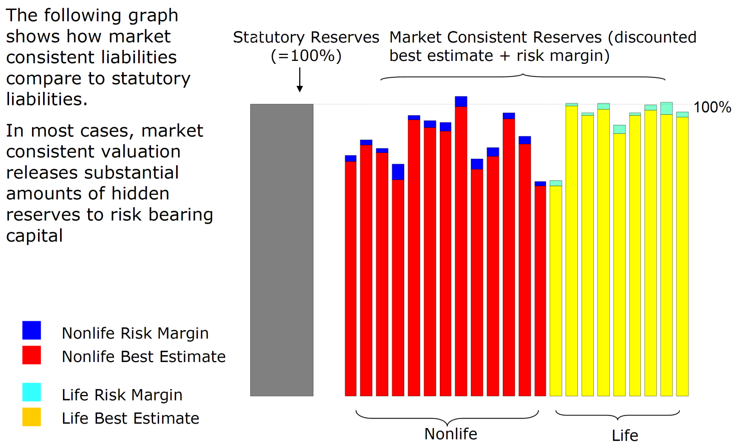



**Fig. 16.2** Market Value Margin for the SST

Having stated the importance of basing the Solvency regime on a reliable economic balance sheet, there is another important question relating to the market consistent valuation of liabilities. What is the value of the different policyholder options such as the possibility to surrender a policy or to take capital or annuity in a pension scheme? It is clear that these implicit options can have a considerable value, but there are few reliable methods to value them, which are generally accepted. Therefore a pragmatic approach has to be taken. This means that only the most relevant policyholder options should be quantified. The most prominent example is for the guaranteed unit linked insurance contract. Here the valuation of the corresponding put option on the fund is relatively easy to quantify based for example on the Black-Scholes formula and the corresponding risk management techniques.

## 16.4 Risk Modelling and Internal Models

After the calculation of the economic balance sheet and the *available capital* it is necessary to define a risk measure and to choose adequate models for the different risks. Within an insurance undertaking, the main risks are ALM risks, including credit risk and liability risk induced by the insurance contracts. The risk model serves to quantify these risks in order to monitor and steer them adequately. Also it is key to keep the model as simple as possible based on predefined principles. Otherwise the model becomes very opaque and model risk increases significantly. Allocating the total risk of an insurance company a typical outcome could be as follows (according to the quantitative impact studies for Solvency II – QIS 4):

| Risk | Life Insurer | P&C Insurer |
|---|---|---|
| Insurance Risk | 44% | 75% |
| ALM Risk | 72% | 41% |
| Credit Risk | 4% | 7% |
| Diversification | -20% | -23% |
| Total | 100 % | 100 % |

This table clearly shows that the ALM risk is the most important risk for a life insurer and needs to be modelled very accurately. The insurance risk is very relevant for a typical P&C insurer. Interestingly however is also the relative importance of the ALM risk for the P&C insurer. Looking at the origin of many internal models it can be observed that they have been designed based on this assumption: Replicating portfolio plus a standard ALM model. Only in a second step the pure insurance risk was included. This is also reasonable within Solvency II. Special consideration however needs be taken with respect to bonus reserves and legal quotes. Here it is key to identify the amount of the bonus reserves which can serve as a risk buffer and allow the company to take more risks. Therefore Solvency II is also an opportunity to discuss legal quote regimes in order to make them more efficient resulting in a higher performance for policyholders and shareholders (see also section 14.4). An example is a legal quote where the full bonus reserve can serve as a risk buffer in case the company suffers a loss. Solvency II foresees the application of *internal models* as an alternative to the *standard approach*. Here it is important to recognise the fears around this topic. On the one hand supervisors fear opacity and regulatory arbitrage . On the other hand small undertakings fear to be put into disadvantage with respect of capital requirements. With respect to the first topic it is necessary as mentioned before to base Solvency II on underlying design principles which need to be valid for internal models and also for the standard approach. With respect to the possible capital disadvantage of undertakings using the standard approach there is only one possibility to avoid this: the standard approach must be powerful enough to be close to the reality. Therefore a *standard formula* is likely to be dangerous in contrast to the example the Swiss solvency test approach.

## 16.5 Good Regulation

In order to understand the need for regulation let's go back in time and think about the roots of insurance. In ancient Rome poor people could not afford their funerals. Therefore they agreed to help each other in the case of death in order to finance the costly funeral ceremonies. This is diversification (raison d'être of insurance).

But did they need regulation and supervisors? No, because the whole was based on trust. Now the insurance industry has become a global play and there is a need for an efficient regulation which does not destroy the underlying principle of diversification. But what does this mean?

- We do not need a lot of regulation but need relevant one.
- Transparency is not the art of producing telephone books full of information, but rather concise and relevant information for transparency.
- Beware of the principal agent problem of regulators.
- It is key that the new regulation is developed in a coordinated effort between regulators and industry:

    Only by this does Regulation becomes relevant and applicable;

    Is accepted by all parties;

    Can enhance the value creation of the sector.

As a mathematician I like Axioms and hence I tried to summarise some relevant Axioms for good regulation:

1. It must be anticipatory – No formulae but principles.
2. It must be nimble – Defined solvency Axioms, pragmatic adaptations.
3. It must cultivate dependable relationships with regulators – Active dialogue between the stakeholders during the design and implementation.
4. It must be capable of implementing strategies to accomplish corporate goals – No monolithic solutions
5. It must be able to manage a crisis to minimise negative impacts and reputational harm – Try to prevent them by requiring the people to think about risk

## 16.6 Swiss Solvency Test

The Swiss solvency test is based on clear design principles, which can also help do design a consistent internal risk model:

1. All assets and liabilities are valued market consistently.
2. Risks considered are market, credit and insurance risks.
3. Risk-bearing capital is defined as the difference of the market consistent value of assets less the market consistent value of liabilities, plus the risk margin (eg. market value margin).

4. Target capital (eg. required capital) is defined as the sum of the expected shortfall of change of risk-bearing capital within one year at the 99% confidence level plus the risk margin.
5. Under the SST, an insurer's capital adequacy is defined if its target capital is less than its risk bearing capital (eg the available capital > required capital).
6. The scope of SST is legal entity and group / conglomerate level domiciled in Switzerland.
7. Scenarios (see also chapter 6) defined by the regulator as well as company specific scenarios have to be evaluated and, if relevant, aggregated within the target capital calculation
8. All relevant probabilistic states have to be modelled probabilistically.
9. Partial and full internal models can and should be used.
10. The internal model has to be integrated into the core processes within the company.
11. SST report to supervisor such that a knowledgeable 3rd party can understand the results.
12. Disclosure of methodology of internal model such that a knowledgeable 3rd party can get a reasonably good impression on methodology and design decisions
13. Senior management is responsible for adherence to principles.

Most of the of the design principles above are self explanatory, and so I would like to point only out a few things:

**Operational risks** have been excluded in the capital calculations in order to focus in a first step on the financial risks. This does not mean that they are not important, but the exclusion allowed the insurance entities to focus on the financial risks and economic balance sheets in order to be able to deliver the required results in a relatively short time.

**Risk Margin:** The expression risk margin is used as a synonym for the market value margin. The inclusion of the market value margin in the required capital instead of considering it as a part of the market value of insurance liabilities is different to Solvency II. This particular choice has been made, since the concept of a MVM was not yet generally accepted. From a conceptual point of view the inclusion of the MVM in the market value of insurance liabilities is a more sensible choice.

**Standard Model for ALM:** The standard model for ALM risk for the Swiss solvency test follows the approach shown in chapter 6. In addition to the analytical model there are additional stress scenarios which need to be defined and evaluated. In contrast to the material shown in chapter 6, the Swiss solvency test performs an additional step, in the sense that the total required capital is calculated based on the results of the analytical model and the outcomes of the different stress scenarios. In order to do this a discrete probability is attached to each stress scenario, considering it as a Dirac (point) measure. (Eg the scenario occurs with a certain probability and in all other cases the incremental loss is $0$). In a next step one considers the $n+1$ random variables $X_0, X_1, \ldots X_n$, where $X_0$ is the analytical model and the other $(X_k)_{k \in \mathbb{N}_n}$ denote the stress scenarios and forms $X = \sum_{k=0}^{n} X_k$ the total loss. The distribution of the total loss $X$ is calculated by a standard convolution technique[2] for independent random variables.

**Own Scenarios:** There is a requirement within the Swiss solvency test not only to use the standard stress scenarios, but also the need to define entity specific scenarios, which could occur and threaten the insurance company. The rational behind this is the idea that the SST should not become a pure compliance exercise and that the individual companies should think about their specific risks. Finally it is important to remark that the introduction of scenarios had not only the purpose to use them within the capital models of the different insurance companies, but the outcome of these scenarios allows the regulator to also assess the systemic risk for the whole insurance market within a country, since the standardised scenarios can easily be aggregated.

**Core processes:** The required use of the SST in the insurance companies' core processes should ensure that the economic capital model is used for making business decisions. In Solvency II this concept is known as *use test*.

The table below summarises the scenarios to be used for the Swiss solvency test:

- Longevity: mortalities fall suddenly and stay low.
- Disability: disability inception rates spike.
- Insufficient P&C reserves.
- Accident: Accident of a tourist group
- Severe Incident plus panic in a (sport–)stadium.
- Hail Storm.
- Industry Incident (eg Seveso / Bhopal type); besides financial and business interruption loss also casualties.
- Pandemic scenario (1914 Spanish Flu).
- Financial Distress (Run on the bank/insurance company, eg combined distressed asset values and a high liquidity demand).
- Default of the company's biggest reinsurer.
- Terrorist attack (aka 09.11.2001).
- "Own Scenarios" $(4\times)$.
- Equity markets drop $60\%$.

[2] See http://www.finma.ch.

- Real estate crash combined with increase in interest rates.
- Stock market crash (1987).
- Nikkei crash (1990).
- European currency crisis (1992).
- US interest rate crisis (1994).
- LTCM (1998).
- Stock market crash (2000/2001).
- Global deflation.
- Financial crisis 2008.
- Spike in lapse rates.
- Global inflation scenario.

## 16.7 Solvency II Standard Model

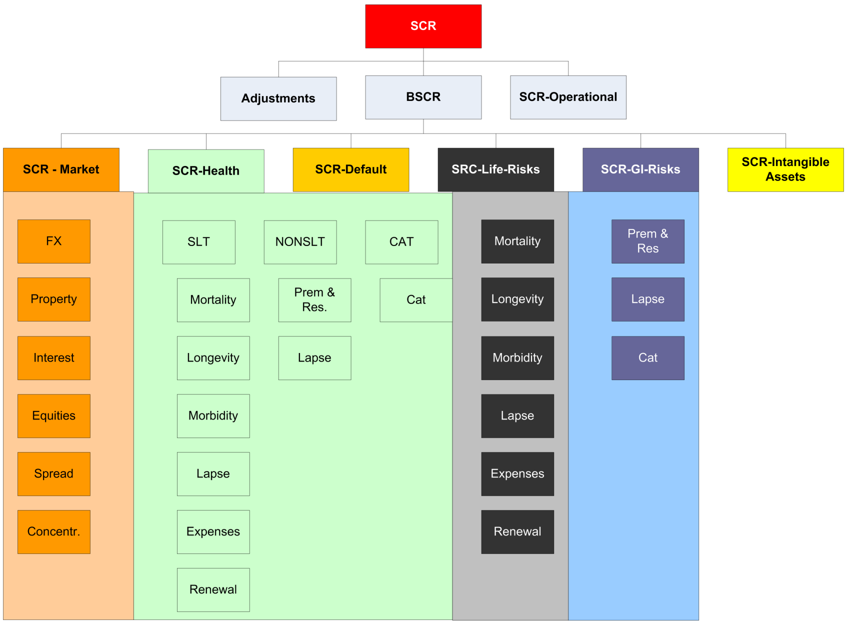



**Fig. 16.3** Solvency II Standard Model

It is import to recognise that the Solvency II standard formula is still under development and that this section can not replace the relevant regulation for Solvency II[3]. This section is based on the DRAFT Technical specifications for QIS 5, which were issued in April 2010. Hence it is possible that parts of the framework are still going to change.

In order to define a solvency frame work the following steps need to be performed:

**Definition of Capital and Valuation:** Define what is capital – eg market value of liabilities minus market value of capital. This question relates to the topics treated in chapters 2 and 4. In technical specification this is treated in section 1.

**Time Horizon:** The time horizon over which risk capital needs to cover risks. For Solvency II this is one year.

**Risks to Quantify:** In this step the risk map is defined, see for example chapter 6 for financial risks. Figure 16.3 defines the risk landscape taken for Solvency II.

**Definition of Ruin:** Solvency II defines ruin, if the market value of assets falls below the market value of liabilities.

**Risk Measure:** For Solvency II the 99.5 % VaR is taken as risk measure.

**Operational Implementation (Standard Model):** The standard model is described below.

## *16.7.1 Structure of the Model*

In a first step we need to understand the structure of the model (Figure 16.3): On the highest level we have the following building blocks:

| **Module** | **Description** |
|---|---|
| `SCR-mkt` | Capital charge for market risk |
| `SCR-def` | Capital charge for counter-party default risk |
| `SCR-life` | Capital charge for life underwriting risk |
| `SCR-nl` | Capital charge for non-life underwriting risk |
| `SCR-health` | Capital charge for health underwriting risk |
| `SCR-Intangibles` | Capital charge for intangible assets risk |

These building blocks are linked together via a mixed correlation approach and we have:

$$\text{Basic SCR} = \sqrt{\sum_{i,j \neq \text{Intang.}} \rho_{i,j} \times SCR_i \times SCR_j} + \text{SCR-Intangibles}$$

[3] www.ceiops.eu

All the above building blocks are defined based on more granular risk factors and algorithms to link them together. In the following we will have a closer look at some of the submodules. The correlation coefficients have been chosen as follows:

| | Market | Default | Life | Health | Non-Life |
|---|---|---|---|---|---|
| Market | 1.00 | 0.25 | 0.25 | 0.25 | 0.25 |
| Default | 0.25 | 1.00 | 0.25 | 0.25 | 0.50 |
| Life | 0.25 | 0.25 | 1.00 | 0.25 | 0.00 |
| Health | 0.25 | 0.25 | 0.25 | 1.00 | 0.00 |
| Non-life | 0.25 | 0.50 | 0.00 | 0.00 | 1.00 |

### *16.7.2 Market Submodule*

As seen above the market submodule consists itself out of the following submodules, for which the capital requirement $SCR$ is calculated. The respective SCR's are calculated by means of stress scenarios. For each submodule there are two scenarios to be evaluated, an upside movement ($\uparrow$) and a downside movement ($\downarrow$). We denote by $\mathcal{G} = \{\uparrow, \downarrow\}$ and we formally calculate for each $\kappa \in \mathcal{G}$ the following:

- Interest rate risk ($SCR_{irate}(\kappa)$),
- Spread risk ($SCR_{spread}(\kappa)$),
- Concentration risk ($SCR_{Co}(\kappa)$),
- Equity risk ($SCR_{Eq}(\kappa)$),
- Property risk ($SCR_{Prop}(\kappa)$),
- FX risk ($SCR_{FX}(\kappa)$).

All of the above then result into the SCR for the total market risk ($SCR_{mkt}$) with the following formula:

$$SCR_{mkt} = \max\left(SCR_{mkt}(\uparrow), SCR_{mkt}(\downarrow)\right),$$
$$SCR_{mkt}(\kappa) = \sqrt{\sum_{i,j} \rho_{i,j}(\kappa) \times SCR_i(\kappa) \times SCR_j(\kappa)},$$

with $\rho(\uparrow)$ given as follows:

| $\rho(\uparrow)$ | Interest | Equity | Property | Spread | Currency | Concentration |
|---|---|---|---|---|---|---|
| Interest | 1 | | | | | |
| Equity | 0 | 1 | | | | |
| Property | 0 | 0.75 | 1 | | | |
| Spread | 0 | 0.75 | 0.5 | 1 | | |
| Currency | 0.25 | 0.25 | 0.25 | 0.25 | 1 | |
| Concentration | 0 | 0 | 0 | 0 | 0 | 1 |

$\rho(\downarrow)$ is slightly different:

| $\rho(\downarrow)$ | Interest | Equity | Property | Spread | Currency | Concentration |
|---|---|---|---|---|---|---|
| Interest | 1 | | | | | |
| Equity | 0.5 | 1 | | | | |
| Property | 0.5 | 0.75 | 1 | | | |
| Spread | 0.5 | 0.75 | 0.5 | 1 | | |
| Currency | 0.25 | 0.25 | 0.25 | 0.25 | 1 | |
| Concentration | 0 | 0 | 0 | 0 | 0 | 1 |

For the interest rate model a stress scenario has to be applied with an upward and a downward stress of interest rates as follows:

| **Maturity in years** | **Relative change up ($\uparrow$)** | **Relative change down($\downarrow$)** |
|---|---|---|
| 0.25 | 70% | -75% |
| 0.50 | 70% | -75% |
| 1 | 70% | -75% |
| 2 | 70% | -65% |
| 3 | 64% | -56% |
| 4 | 59% | -50% |
| 5 | 55% | -46% |
| 6 | 52% | -42% |
| 7 | 49% | -39% |
| 8 | 47% | -36% |
| 9 | 44% | -33% |
| 10 | 42% | -31% |
| 15 | 33% | -27% |
| 20 | 26% | -29% |
| 25 | 26% | -30% |
| 30 | 25% | -30% |

Based on the above relative shifts (eg $i_{\text{after Stress}} = i_{\text{before Stress}} \times (1 +$ relative amount)) all values are recalculated, in the same sense as shown in chapter 6.

For the other risk factors the approach is quite similar and hence we want finally to have a look on how the credit spreads movement looks like. For further details we refer to the relevant literature [ECSO2].

The credit risk factors using the duration approximation mentioned in chapter 6 are defined as follows. Please note that the duration to be taken for the approximation has a floor and a cap.

| Rating | ↑ | ↓ | Floor | Cap |
|---|---|---|---|---|
| AAA | 1.0 % | -0.4 % | 1 | $\infty$ |
| AA | 1.5 % | -1.0 % | 1 | $\infty$ |
| A | 2.6 % | -1.7 % | 1 | $\infty$ |
| BBB | 4.5 % | -3.0 % | 1 | 7 |
| BB | 8.4 % | -6.3 % | 1 | 5 |
| $\leq$ B | 16.2 % | -8.6 % | 1 | 3.5 |
| Unrated | 5.0 % | -3.3 % | 1 | 7 |

Finally we want to have a look at the equity sub-model. Also here a stress scenario approach is applied where it is assumed that the equities fall by 30 % for global equity indices and 40 % for all other indices respectively. The corresponding results are aggregated using a correlation matrix. It needs to be stressed that it obviously can not replace the in depth study of the material in [ECSO2].

Please note that Solvency II foresees that insurance companies can use partial or full *internal models*. Figure 16.4 shows how such a modified internal model could look like. Obviously such internal models need to be approved by the relevant regulator.

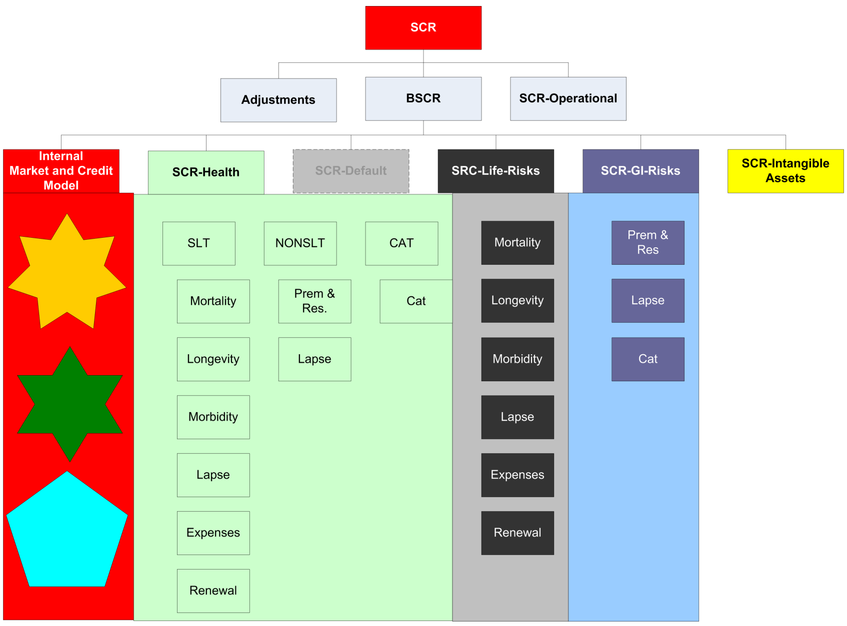



Fig. 16.4 Solvency II Partial Internal Model

## 16.8 Quo Vadis?

It is clear that Solvency II is a great opportunity for the European insurance industry, for example with respect to product innovation. However there is still a lot to do in order to get Solvency II up and running. It is important to base the whole system on clearly understandable principles. Only by that double counting of capital and capital inefficiencies can be avoided. With respect to valuation it is of utmost importance to base the calculation on an economic view with a transparent method for the market value margin such as the cost of capital approach. The valuation needs to be done on such a level of granularity that companies can calculate their replicating portfolios, in order to do a proper risk management with respect to the ALM risk. Finally the risk management models should be designed in a transparent manner in order that internal models follow logically from the standard principles. This ensures that the main risks are taken into consideration and that the standard approach needs not to be overloaded by additional security measures. Doing that will allow the companies to use their capital more efficiently leading to higher returns for shareholders and for policyholders. Furthermore such a model would also allow the companies to understand their risks better and to do better with respect to ALM. In order to achieve this challenging goal all stakeholders need to engage in an open and constructive discussion. This will ultimately lead to a better result and to a better mutual understanding between the insurance industry and the regulators.

# Appendix A
# Stochastic Processes

## A.1 Definitions

In this section we will introduce the definitions which we will use throughout this book and we assume that the reader is familiar with elementary calculus, measure theory and probability theory.

**Definition 69 (Sets)** *In the following we denote with*

$$
\begin{aligned}
\mathbb{N} \quad &= \textit{the set of all natural numbers including } 0, \\
\mathbb{N}_+ \quad &= \{x \in \mathbb{N} : x > 0\}, \\
\mathbb{R} \quad &= \textit{the set of real numbers}, \\
\mathbb{R}_+ \quad &= \{x \in \mathbb{R} : x \geq 0\}.
\end{aligned}
$$

*Furthermore we use the following notation for intervals. For $a, b \in \mathbb{R}, a < b$ we denote*

$$
\begin{aligned}
[a, b] \;&:= \;\{t \in \mathbb{R} \,:\, a \leq t \leq b\}, \\
]a, b] \;&:= \;\{t \in \mathbb{R} \,:\, a < t \leq b\}, \\
]a, b[ \;&:= \;\{t \in \mathbb{R} \,:\, a < t < b\}, \\
[a, b[\; k &:= \{t \in \mathbb{R} \,:\, a \leq t < b\}.
\end{aligned}
$$

**Definition 70 (Characteristic Function)** *For a set $A \subset \Omega$ we denote $\chi_A : \Omega \to \mathbb{R}, \omega \mapsto \chi_A(\omega)$ the characteristic function, where*

$$
\chi_A(\omega) := \begin{cases} 1, & \textit{if } \omega \in A, \\ 0, & \textit{if } \omega \notin A. \end{cases}
$$

*With $\delta_{ij}$ we denote the Kronecker-Delta, which is equal to $1$, if $i = j$ and $0$ otherwise.*

**Definition 71** *For a function*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$$

*we define the limit from below and from above, if they exist, as follows:*

$$f(x^-) := \lim_{\xi \uparrow x} f(\xi),$$
$$f(x^+) := \lim_{\xi \downarrow x} f(\xi).$$

**Definition 72** *A real valued function $f : \mathbb{R} \to \mathbb{R}$ is of order $o(t)$, if*

$$\lim_{t \to 0} \frac{f(t)}{t} = 0.$$

*In this case we write $f(t) = o(t)$.*

**Definition 73 (Functions with Bounded Variation)** *Let $I \subset \mathbb{R}$ be a finite interval. The total variation for a function $f$*

$$f : I \to \mathbb{C}, t \mapsto f(t)$$

*with respect to the interval $I$ is given by*

$$V(f, I) = \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)|,$$

*where the supremum is taken over all decompositions of the interval $I$, with*

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n.$$

*The function $f$ has a* bounded variation *on $I$, if $V(f, I)$ is finite.*

Properties of functions with bounded variation can be found in [DS57].

It is important to know that functions with bounded variation are both an algebra and a lattice. Hence for $f, g$ functions with bounded variation and $\alpha \in \mathbb{R}$ the following functions have also a bounded variation: $\alpha f + g$, $f \times g$, $\min(0, f)$ and $\max(0, f)$.

**Definition 74 (Probability spaces, stochastic processes)** *By $(\Omega, \mathcal{A}, P)$ we denote always a probability space which fulfils the axioms of Kolmogorov.*

*Let $(S, \mathcal{S})$ be a measurable space (eg $S$ a set and $\mathcal{S}$ a $\sigma$-algebra over $S$) and $T$ a set. We denote by $\mathcal{R} = \sigma(\mathbb{R})$ the $\sigma$-algebra over the Borel set of the real numbers.*

*A family $\{X_t : t \in T\}$ of random variables*

$$X_t : (\Omega, \mathcal{A}, P) \to (S, \mathcal{S}), \omega \mapsto X_t(\omega)$$

*is a* stochastic process *over $(\Omega, \mathcal{A}, P)$ with* State space $S$.

*For every $\omega \in \Omega$ we define by*

$$X_\cdot(\omega) : T \to S, t \mapsto X_t(\omega)$$

*the corresponding* trajectory. *We assume that these trajectories are right continuous and that the left side limit exists.*

**Definition 75 (Expected Value)** *For a random variable $X$ on $(\Omega, \mathcal{A}, P)$ and $\mathcal{B} \subset \mathcal{A}$ a $\sigma$-algebra we denote:*

- $\mathbb{E}[X]$ *the* expected value *of the random variable $X$,*
- $Var[X]$ *the* variance *of the random variable $X$,*
- $\mathbb{E}[X|\mathcal{B}]$ *the* conditional expected value *of $X$ with respect to $\mathcal{B}$.*

**Definition 76** *For a stochastic process $(X_t)_{t \in T}$ on $(\Omega, \mathcal{A}, P)$ with values in a countable set $S$ and $i \in S$ we define*

$$I_j(t)(\omega) = \begin{cases} 1, \text{ falls } X_t(\omega) = j, \\ 0, \text{ falls } X_t(\omega) \neq j \end{cases}$$

*the indicator functions with respect to the stochastic process $(X_t)_{t \in T}$ at time t.*

*Analogously we define for $j, k \in S$ with*

$$N_{jk}(t)(\omega) = \# \{\tau \in ]0, t[ : X_{\tau^-} = j \text{ and } X_\tau = k\}$$

*the number of jumps from $j$ to $k$ during the time interval $]0, t[$.*

**Definition 77 (Normal distribution)** *A random variable $X$ on $(\mathbb{R}, \sigma(\mathbb{R}))$ with probability density function*

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}$$

*is called* normally distributed *with expected value $\mu$ and variance $\sigma^2$. We denote $X \sim \mathcal{N}(\mu, \sigma^2)$.*

Examples of stochastic processes are:

**Example 78 (Brownian motion)** *An example of a non-trivial stochastic process is the Brownian motion. This process $X = (X_t)_{t \geq 0}$ in continuous time ($T = \mathbb{R}_+$) with state space $S = \mathbb{R}$ is used for describing many natural phenomena.*

*The Brownian motion can be characterised by the following properties:*

1. $X_0 = 0$ *almost surely.*
2. $X$ *has independent increments: For all* $0 \leq t_1 < t_2 < \ldots < t_n$ *and all* $n \in \mathbb{N}$ *we know that the random variables:* $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent.*
3. $X$ *has stationary increments.*
4. $X_t \sim \mathcal{N}(0, t)$.

*One can show that $X$ has almost surely continuous paths, which are nowhere differentiable.*

**Example 79 (Poisson process)** *The Poisson process $N = (N_t)_{t \geq 0}$ is a* count process *with state space $\mathbb{N}$, which is used for example for the modelling of number of claims within an insurance company. The time homogeneous poison process can be characterised by the following properties:*

1. $N_0 = 0$ *almost surely.*
2. $N$ *has independent, stationary increments.*
3. *For all* $t > 0$ *and all* $k \in \mathbb{N}$ *the following property holds:* $P[N_t = k] = \exp(-\lambda\, t)\, \frac{(\lambda\, t)^k}{k!}$.

## A.2 Markov Chains with Countable State Space

In the following we denote by $S$ a countable set.

**Definition 80** *Let $(X_t)_{t \in T}$ be a stochastic process over $(\Omega, \mathcal{A}, P)$ with state space $S$ and $T \subset \mathbb{R}$. The process $X$ is called a Markov chain, if for all*

$$n \geq 1, t_1 < t_2 < \ldots < t_{n+1} \in T, i_1, i_2, \ldots, i_{n+1} \in S$$

*with*

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] > 0$$

*the following equation holds:*

$$P[X_{t_{n+1}} = i_{n+1} | X_{t_k} = i_k \forall k \leq n] = P[X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n]. \quad \text{(A.1)}$$

**Definition 81** *Let $(X_t)_{t \in T}$ be a stochastic process over $(\Omega, \mathcal{A}, P)$. In this case we denote*

$$p_{ij}(s,t) := P[X_t = j \,|\, X_s = i], \qquad \text{where } s \leq t \text{ and } i, j \in S,$$

*the conditional probability, to change from time $s$ to $t$ from state $i$ to state $j$.*

The Chapman-Kolmogorov-theorem states the relationship of $P(s,t)$, $P(t,u)$ and $P(s,u)$ for $s \leq t \leq u$:

**Theorem 82 (Chapman-Kolmogorov-equation)** *Let $(X_t)_{t \in T}$ be a Markov chain and let $s \leq t \leq u \in T$, $i, k \in S$ with $P\left[X_s = i\right] > 0$. Then we have the following equations:*

$$p_{ik}(s,u) = \sum_{j \in S} p_{ij}(s,t)\, p_{jk}(t,u), \tag{A.2}$$

$$P(s,u) = P(s,t) \times P(t,u). \tag{A.3}$$

*Hence we can calculate $P(s,u)$ for $s \leq t \leq u \in T$ by matrix multiplication of $P(s,t)$ and $P(t,u)$.*

*Proof.* For $t = s$ or $t = u$ the equation is obviously true and hence we can assume that $s < t < u$. We denote by:

$$\begin{aligned} S^* &= \{ j \in S : P\left[X_t = j \,|\, X_s = i\right] \neq 0 \} \\ &= \{ j \in S : P\left[X_t = j\,,\, X_s = i\right] \neq 0 \}\,. \end{aligned}$$

The Chapman-Kolmogorov-equation can be proved by the use of the following equations:

$$\begin{aligned} p_{ik}(s,u) &= P\left[X_u = k \,|\, X_s = i\right] \\ &= \sum_{j \in S^*} P\left[X_u = k,\, X_t = j \,|\, X_s = i\right] \\ &= \sum_{j \in S^*} P\left[X_t = j \,|\, X_s = i\right] \times P\left[X_u = k \,|\, X_s = i,\, X_t = j\right] \\ &= \sum_{j \in S^*} p_{ij}(s,t) \times p_{jk}(t,u) \\ &= \sum_{j \in S} p_{ij}(s,t) \times p_{jk}(t,u), \end{aligned}$$

where we have used the Markov property.

**Definition 83 (Transition matrix)** *A family $p_{ij}(s,t)$ is called transition matrix if the following four conditions are fulfilled:*

*1.* $p_{ij}(s,t) \geq 0.$

*2.* $\sum_{j \in S} p_{ij}(s,t) = 1.$

*3.* $p_{ij}(s,s) = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases} \quad \text{if } P[X_s = i] > 0.$

*4.* $p_{ik}(s,u) = \sum_{j \in S} p_{ij}(s,t)\, p_{jk}(t,u)$ *for* $s \leq t \leq u$ *and* $P\,[X_s = i] > 0.$

**Proposition 84** *For a Markov chain* $(X_t)_{t \in T}$ *is* $p_{ij}(s,t)$ *is a transition matrix.*

*Proof.* This is a direct consequence of the Chapman-Kolmogrov-Theorem (Thm. 82).

**Definition 85** *A Markov chain* $(X_t)_{t \in T}$ *is called* time homogeneous*, if for all* $s, t \in \mathbb{R}, h > 0$ *and* $i, j \in S$ *with* $P[X_s = i] > 0$ *and* $P[X_t = i] > 0$ *the following homogeneity condition is fulfilled:*

$$P[X_{s+h} = j \,|\, X_s = i] = P[X_{t+h} = j \,|\, X_t = i].$$

*In this case we write:*

$$p_{ij}(h) := p_{ij}(s, s+h),$$
$$P(h) := P(s, s+h).$$

## A.3 Mean Excess Function

The mean excess function for a random variable $X$ is given by

$$e_X(x) = \mathbb{E}[X - x \,|\, X > x] = \frac{\int_x^\infty \{1 - F_X(\xi)\} d\xi}{1 - F_X(x)},$$

and that we have $e^\alpha(X) = e_X(VaR^\alpha(X))$.

For the convenience of the reader we have listed below some mean excess functions for different probability distributions:

**Normal Distribution:** 1. Probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$

2. Cumulative probability density function for standard normal distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(\frac{-\xi^2}{2}) d\xi.$$

3. Mean: $\mathbb{E}[X] = \mu$.
4. Variance: $Var[X] = \sigma^2$.
5. Mean excess function: For the standard normal distribution we have the following:

$$\begin{aligned} e(x) &= \frac{\int_x^{\infty} (\xi - x) \frac{1}{\sqrt{2\pi}} \exp(-\frac{\xi^2}{2}) d\xi}{1 - \Phi(x)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\exp(-\frac{x^2}{2})}{1 - \Phi(x)} - x. \end{aligned}$$

We remark that based on the $e(x)$ for the standard normal distribution we can easily calculate $e(x)$ for every normal distribution. In particular we can calculate the difference between VaR and TVaR for a normal distribution with standard deviation $\sigma$ by $\sigma \times (\frac{1}{\sqrt{2\pi}} \frac{\exp(-\frac{x^2}{2})}{1-\Phi(x)} - x)$.

**Log–Normal Distribution:** 1. Probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma\, x} \exp(-\frac{(\log(x) - \mu)^2}{2\sigma^2}), \text{ for } x > 0.$$

2. Mean: $\mathbb{E}[X] = \exp(\mu + \frac{\sigma^2}{2})$.
3. Variance: $Var[X] = (\exp(\sigma^2) - 1) \exp(2\mu + \sigma^2)$.
4. Mean excess function:

$$e_X(x) = \frac{\exp(\mu + \frac{\sigma^2}{2}(1 - \Phi(x)(\frac{\log(x)-\mu-\sigma^2}{\sigma}))}{1 - \Phi\left(\frac{\log(x)-\mu}{\sigma}\right)} - x.$$

**Exponential Distribution:** 1. Probability density function: $f_X(x) = \beta \exp(-\beta\, x)$, for $x > 0$.

2. Mean: $\mathbb{E}[X] = \frac{1}{\beta}$.
3. Variance: $Var[X] = \frac{1}{\beta^2}$.
4. Mean excess function: $e_X(x) = \frac{1}{\beta}$.

**Pareto Distribution:** 1. Probability density function: $f_X(x) = \frac{\alpha\, \lambda^\alpha}{(\lambda+x)^{\alpha+1}}$, for $x > 0$.

2. Mean: $\mathbb{E}[X] = \frac{\lambda}{\alpha-1}$. Note exists only if $\alpha > 1$.
3. Variance: $Var[X] = \frac{\alpha\, \lambda^2}{(\alpha-1)^2(\alpha-2)}$. Note exists only if $\alpha > 2$.

4. Mean excess function: $e_X(x) = \frac{\lambda + x}{\alpha - 1}$.

**Gamma Distribution:** 1. Probability density function: $F_X(x) = \beta(\beta\, x)^{\alpha-1} \frac{e^{-\beta\, x}}{\Gamma(\alpha)}$, for $x > 0$.

2. Mean: $\mathbb{E}[X] = \frac{\alpha}{\beta}$.
3. Variance: $Var[X] = \frac{\alpha}{\beta^2}$.
4. Mean excess function:

$$\begin{aligned} e_X(x) &= \frac{\alpha}{\beta} \times \frac{1 - F_X(x, \alpha+1, \beta)}{1 - F_X(x, \alpha+1, \beta)} \\ &= \frac{1}{\beta}(1 + o(1)), \end{aligned}$$

where $F_X(x, \alpha, \beta)$ denotes the cumulative probability density function of the gamma distribution with parameters $\alpha$ and $\beta$.

## A.4 Deterministic Cash Flow Streams

**Definition 86 (Payout function)** *$A$* deterministic payout function *$A$ is a function*

$$A : T \to \mathbb{R}, t \mapsto A(t),$$

*with the following properties, with $T \subset \mathbb{R}$:*

1. *$A$ is right continuous,*
2. *$A$ has bounded variation.*

We interpret $A(t)$ as the amount of money which has been paid until time $t$. One can show the following properties for functions with bounded variation [DS57]:

1. A function of bounded variation $A$ can be extended to a measure on $\sigma(\mathbb{R})$, which we denote also with $A$. This measure is called *Stieltjes measure*.
2. For a function $A$ on $\mathbb{R}$ with bounded variation there exist two positive, increasing functions with bounded variation and disjoint support such that $A = B - C$. We interpret $B$ inflow of money and $C$ as outflow of money.

**Definition 87 (Decomposition of measures)** *Let $f$ be a function of bounded variation and denote with $A$ the corresponding Stieltjes measure. In this case we define:*

$$\mu_f := A.$$

*Since this decomposition into $A = B - C$ is unique (with $B$ and $C$ positive and with disjoint support), we define*

$$A^+ := B,$$
$$A^- := C.$$

In order to define the mathematical reserves we need to introduce the discounting functions.

$$v(t) = \exp(-\int_0^t \delta(\tau)\, d\tau)$$

Now we can define the present value of a cash flow as follows

**Definition 88 (Value of a cash flow)** *Let $A$ be a deterministic cash flows and $t \in \mathbb{R}$. In this case we define:*

1. *The value of a cash flow $A$ at time $t$ is defined by:*

$$V(t, A) := \frac{1}{v(t)} \int_0^\infty v(\tau)\, dA(\tau).$$

2. *The value of the future cash flow is given by*

$$V^+(t, A) := V(t, A \times \chi_{]t,\infty]}).$$

## A.5 Random Cash Flows

**Definition 89 (Random cash flow)** *A* random cash flow *is a stochastic process $(X_t)_{t \in T}$, for which almost all paths are of finite variation.*

Consider now a stochastic process $A$ with bounded variation and $\omega \in \Omega$ with $t \mapsto A_t(\omega)$ right continuous and increasing. For a bounded Borel function $f$ we can now define the integral $\int f(\tau) d\mu_{A.(\omega)}(\tau)$ define. Analogously it is possible to define the integral $\int f(\tau, \omega) d\mu_{A.(\omega)}(\tau)$ P-a.e. for a bounded function $F_t = f(t, \omega)$, which is measurable with respect to the product sigma algebra. This construction can be extended to stochastic processes with bounded variation by using the decomposition of functions with bounded variation into its positive and negative parts.

**Definition 90** *For a stochastic process $(A_t)_{t \in T}$ on $(\Omega, \mathcal{A}, P)$ with bounded variation and a product measurable, bounded function $F : \mathbb{R} \times \Omega \to \mathbb{R}$ we define as described above:*

$$(F \cdot A)_t(\omega) = \int_0^t F(\tau, \omega) dA_\tau(\omega) = \int_0^t F\, dA,$$

*and we write*

$$d(F \cdot A) = F\, dA.$$

Now we can define the present value of a cash flow as follows

**Definition 91 (Value of a random cash flow)** *Let $A$ be a stochastic cash flows and $t \in \mathbb{R}$. In this case we define:*

1. *The expected value of a cash flow $A$ at time $t$ is defined by:*

$$V(t, A) := \mathbb{E}[\frac{1}{v(t)} \int_0^\infty v(\tau)\, dA(\tau)].$$

2. *The value of the future cash flow is given by*

$$V^+(t, A) := \mathbb{E}[V(t, A \times \chi_{]t,\infty]})].$$

# Appendix B

# Application of the Markov model to Life Insurance

## B.1 Traditional Rating of Life Contracts

Before starting with the Markov model, I would like to summarise how traditional calculations using commutation functions are performed. Usually one starts with the probabilities of death and then calculates a decrement table starting with, say, 100000 persons at age 20.

After that one, has to calculate the different commutation functions, which I assume everybody knows by heart. These numbers depend on the persons alive and on the technical interest rate i. Only when you have done this it is (in the classical framework) possible to calculate the necessary premiums. In the following we will look a little bit closer at the calculation of a single premium for an annuity. To do this we need the following commutation functions:

$D_x = v \times l_x$where $l_x$ denotes the number of persons alive at age $x$.
$C_x = v \times (l_{x+1} - l_x)$

Having this formalism it is well known that

$$\ddot{a}_x = \frac{N_x}{D_x}$$

From this example is easily seen that almost all premiums can be calculated by summation and multiplication of commutation functions. Such an approach has its advantages in an environment where calculations have to be performed by hand, or where computers are expensive. Calculation becomes messy if benefits are considered with guarantees or with refunds.

The Markov model here presented offers rating of life contracts without using commutation functions. It starts with calculation of the reserves and uses the involved probabilities directly. In order to see such a calculation let's review the above-mentioned example: We will use ${}_np_x$ to denote the probability of a person aged exactly $x$ surviving for $n$ years.

$$\begin{aligned}\ddot{a}_x &= \sum_{j=0}^{\infty} {}_jp_x \times v^j \\ &= 1 + p_x \times \ddot{a}_{x+1}\end{aligned}$$

The above formula gives us a recursion for the mathematical reserves of the contract. Hence one can calculate the necessary single premiums just by recursion. In order to do this, we need an initial condition, which is in our case $V_\omega = 0$.

The interpretation of the formula is easy: The necessary reserve at age $x$ consists of two parts:

1. The annuity payment, and
2. The necessary reserve at age x+1. (These reserves must naturally be discounted.)

It should be pointed out that the calculation does not need any of the commutation functions; only $p_x$ and the discount factor v are used. As a consequence this method does not produce the overheads of traditional methods.

In the following paragraphs the discrete time, discrete state Markov model is introduced and solutions of some concrete problems are offered.

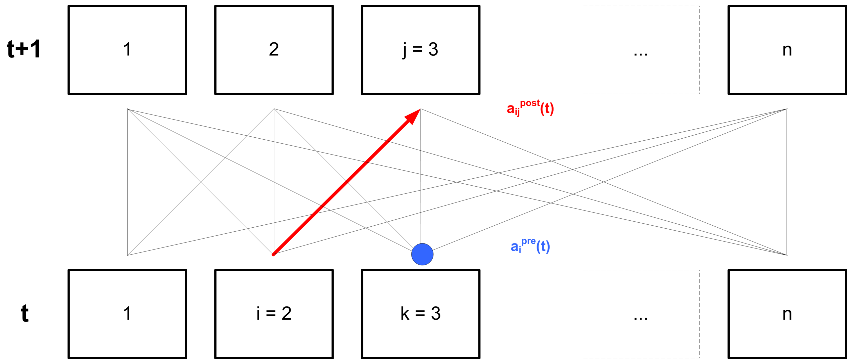
At this point, it is necessary to stress the fact that the following frame work can be used, with some modifications, in an environment with stochastic interest. But as we are limited in space and time we have to restrict ourselves to deterministic constant discount rates.

## B.2 Life Insurance considered as Random Cash flows

The starting point of the Markov model is a set of states, which correspond to the different possible conditions of the insured persons. In life insurance the set of states usually consists of alive, dead. The set of states will be denoted by $S$.

The second point which originates from the life contract has to do with the so-called contractual functions which depend on the states and the time. Hence the structure of a generalised life contract can be thought of:

Contractual situation between time $t$ and time $t+1$




From the above diagram it can be seen that a finite number of states is considered, and that for

each transition $i \to j$ two different sums are paid, namely $a_{ij}^{\text{Post}}(t)$ at the end of the considered time interval and $a_i^{\text{Pre}}(t)$ at the beginning of it. It is clear that the value of the payment stream $a_{ij}^{\text{Post}}(t)$ has to be discounted by v in order to be compatible with $a_i^{\text{Pre}}(t)$. Probably it is worth remarking that the use of the two payment streams $a_i^{\text{Pre}}(t)$ and $a_{ij}^{\text{Post}}(t)$ eases the solution of things like payments during the year and the distinction between lump

sums (generally payable at the end of the period) and annuities (at the beginning). Finally it must be said that premiums payable to the insurer can (not must (!)) be considered as benefits with the opposite sign.

Until now we have defined the sums which are payable if a certain insured event occurs. Now there has to be a probability law in order to rate the different transitions. In the following we denote by $p_{ij}(t, t+1)$ the probability of transition at time t from state $i \to j$ . Hence in the language of the above diagram there is one transition probability assigned to each line between two states.

So summarising a Markov life insurance model consists of the following:

| | |
|---|---|
| $S$ | A finite state space (set). |
| $((p_{ij}(t))_{(i,j)\in S^2})_{t\in(1,2,\ldots\omega)}$ | The transition probabilities describing the Markov chain $X_t$ on $S$. |
| $((a_i^{\text{Pre}}(t))_{i\in S})_{t\in(1,2,\ldots\omega)}$ | The prenumerando benefits relating, paying at the beginning of the corresponding period. |
| $((a_{ij}^{\text{Post}}(t))_{(i,j)\in S^2})_{t\in(1,2,\ldots\omega)}$ | The postnumerando benefits relating, paying at the end of the corresponding period, if a transition $i \to j$ happens. |
| $((v_i(t))_{i\in S})_{t\in(1,2,\ldots\omega)}$ | The yearly discount rate from $[t, t+1[$. We have $v_t = \sum_{j\in S} I_j(t)\, v_i(t)$. |

## B.3 Reserves, Recursion and Premiums

One of the most important quantities in actuarial science is the prospective reserve, as the insurer must have this amount of money for each policy. Therefore the concept of the prospective reserve is known to all actuaries. It is defined to be the present value of the future cash flow $A$ given the information at present. Formally we write

$$V_j^+(t, A) := \mathbb{E}[V(t, A \times \chi_{]t,\infty]}) \,|\, X_t = j],$$

(where $j$ denotes the state at time t). This notation tells us, that the reserve depends heavily on the state of the policy.

In the context of the above we have

$$\begin{aligned}
\Delta A(t) &= \sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t), \\
A(t) &= \sum_{k \leq t} \Delta A(k), \\
\Delta V(t, A) &= v(t) \Delta A(t), \\
&= v(t) \left[ \sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t) \right], \\
v(t) &= \prod_{\tau \leq t} \left[ \sum_{j \in S} I_j(\tau) \times v_j(\tau) \right].
\end{aligned}$$

The direct calculation of the necessary reserves for the different states is not too easy if you consider a general time continuous Markov model. An advantage of this model is the existence of a powerful backwards recursion. The following formula (Thiele difference equation) allows the recursive calculation of the necessary reserves and hence of the necessary single premiums:

$$V_i^+(t) = a_i^{Pre}(t) + \sum_{j \in S} v_i(t)\, p_{ij}(t) \big\{ a_{ij}^{Post}(t) + V_j^+(t+1) \big\}. \tag{B.1}$$

The interpretation of the formula is almost the same as in the trivial example at the beginning. In principle the present reserve consists of payments due to the different possible transitions and the discounted values of the future necessary reserves. It can be seen that the above recursion uses only the different benefits, the probabilities and the discount factor. In order to calculate the reserve for a certain age one has to do a backwards recursion starting at the expiration date of the policy. For annuities this is usually the age $\omega$ when everybody has died. Starting the recursion it is necessary to have boundary conditions, which depend on the payment stream at the expiration date. Usually the boundary conditions are taken to be zero for all reserves. It should be pointed out that one has to do this recursion for the reserves of all states simultaneously.

After the calculation of the different reserves one can naturally determine the corresponding necessary single premiums by the principle of equivalence.

We want to end this section with a short proof of the above mentioned Thiele recursion:

We know that $A(t) = \sum_{k \le t} \Delta A(k)$ and also that

$$\Delta V(t, A) = v(t) \left[ \sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t) \right].$$

Hence we have

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \mathbb{E} \left[ \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \,|\, X_t = i \right] \\ &= \frac{1}{v(t)} \mathbb{E} \left[ \sum_{j \in S} I_j(t+1) \times \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \,|\, X_t = i \right], \end{aligned}$$

remarking that $\sum_{j \in S} I_j(t+1) = 1$. If we now consider all the terms in $\Delta A(t)$ for a given $I_j(t+1)$ for $j \in S$, it becomes obvious that the Markov chain changes from $i \to j$ and in consequence only $N_{ik}(t)$ increases by one for $k = j$. If we furthermore use the projection property and the linearity of the conditional expected value and the fact that $\mathbb{E}\left[I_j(t+1) \,|\, X_t = i\right] = p_{ij}(t, t+1)$, together with the Markov property, we get the formula if we split $V_i^+(t)$ as follows:

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \mathbb{E} \left[ \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \,|\, X_t = i \right] \\ &= \frac{1}{v(t)} \mathbb{E} \left[ \left\{ \sum_{\tau=t}^{t} + \sum_{\tau=t+1}^{\infty} \right\} v(\tau) \times \Delta A(\tau) \,|\, X_t = i \right]. \end{aligned}$$

Doing this decomposition we get for the first part:

$$\text{Part}_1 = a_i^{Pre}(t) + \sum_{j \in S} v_i(t)\, p_{ij}(t) a_{ij}^{Post}(t),$$

and for the second:

$$\text{Part}_2 = \sum_{j \in S} v_i(t)\, p_{ij}(t) V_j^+(t+1).$$

Adding the two parts together we get the desired result:

$$V_i^+(t) = a_i^{Pre}(t) + \sum_{j \in S} v_i(t)\, p_{ij}(t) \big\{ a_{ij}^{Post}(t) + V_j^+(t+1) \big\}.$$

More concretely we have

$$\begin{aligned}
V_i^+(t) &= \frac{1}{v(t)} \mathbb{E}\left[ \sum_{j \in S} I_j(t+1) \times \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \,|\, X_t = i \right] \\
&= a_i^{Pre}(t) + \sum_{j \in S} \mathbb{E}\left[ I_j(t+1) \times \sum_{\tau=t}^{\infty} \frac{v(\tau)}{v(t)} \times \Delta A(\tau) \,|\, X_t = i \right] \\
&= a_i^{Pre}(t) + \sum_{j \in S} \mathbb{E}\Bigg[ I_j(t+1) v_i(t) \Bigg\{ a_{ij}^{Post} + \\
&\qquad + \mathbb{E}\Bigg[ \sum_{\tau=t+1}^{\infty} \frac{v(\tau)}{v(t+1)} \times \Delta A(\tau) \,\Bigg|\, X_t = i, X_{t+1} = j \Bigg] \Bigg\} \Bigg| \, X_t = i \Bigg] \\
&= a_i^{Pre}(t) + \sum_{j \in S} v_i(t)\, p_{ij}(t) \big\{ a_{ij}^{Post}(t) + V_j^+(t+1) \big\}.
\end{aligned}$$

We remark that this section can only be a short introduction to this topic and we refer to [Kol10] for a more extensive discussion.

# Appendix C
# Abstract Valuation

This appendix follows closely [Kol10] and creates a link between the Markov chain model for life insurance on the one hand and abstract valuation and the concepts used in this book. Furthermore it aims to explain the concept of replicating portfolios and the cost of capital approach in a more general and abstract manner. We assume that the reader is familiar with elementary functional analysis such as Hilbert spaces and we refer to [DS57], [Con91] or [Ped89] for the corresponding mathematical proofs. Finally it is worth mentioning that [Duf92] covers the theoretical approach in some greater detail and we would encourage everybody to deepen its know-how with respect to this topic.

## C.1 Framework

**Definition 92 (Stochastic Cash Flows)** *A* stochastic cash flow *is a sequence $x = (x_k)_{k\in\mathbb{N}} \in L^2(\Omega, \mathcal{A}, P)^{\mathbb{N}}$, which is $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ adapted.*

**Definition 93 (Regular Stochastic Cash Flows)** *A* regular stochastic cash flow $x$ with respect to $(\alpha_k)_{k\in\mathbb{N}}$, *with $\alpha_k > 0 \forall k$ is a stochastic cash flow such that*

$$Y := \sum_{k\in\mathbb{N}} \alpha_k X_k \in L^2(\Omega, \mathcal{A}, P).$$

*We denote the vector space of all regular cash flows by $\mathcal{X}$.*

**Remark 94** *1. We note that for all $n \in \mathbb{N}$ the image of $\psi : L^2(\Omega, \mathcal{A}, P)^n \to \mathcal{X}, (x_k)_{k=0,\ldots n} \mapsto (x_0, x_1, \ldots, x_n, 0, 0 \ldots)$ is a sub-space of $\mathcal{X}$.*

*2. $\mathcal{X}$ has been defined this way in order to capture cash flow streams where the sum of the cash flows is infinite with a finite present value. In this set up $\alpha_k$ can be interpreted as a majorant of the price of the payment $1$ at time $k$.*

**Proposition 95** *1. For $x, y \in \mathcal{X}$, we define the scalar product as follows:*

$$\begin{aligned} < x, y > &= \sum_{k\in\mathbb{N}} < \alpha_k\, x_k, \alpha_k\, y_k > \\ &= \mathbb{E}[\sum_{k\in\mathbb{N}} \alpha_k^2\, x_k\, y_k], \end{aligned}$$

*and remark that the scalar product exists as a consequence of the Cauchy-Schwarz inequality.*

*2. $\mathcal{X}$ equipped with the above defined scalar product is a Hilbert space with norm $||x|| = \sqrt{< x, x >}$.*

*Proof.* We leave the proof of this proposition to the reader.

In a next step we introduce the concept of a positive valuation functional and we closely follow [Büh95].

**Definition 96 (Positivity)** *1. $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ is called* positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$. In this case we write $x \geq 0$.*

*2. $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ is called* strictly positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$ and there exists a $k \in \mathbb{N}$, such that $x_k > 0$ with a positive probability. In this case we write $x > 0$.*

**Definition 97 (Positive Functionals)** $Q : \mathcal{X} \to \mathbb{R}$ *is called a* positive, continuous and linear functional *if the following hold true:*

1. *If $x > 0$, we have $Q[x] > 0$.*
2. *If $x = \lim_{n\to\infty} x_n$, for $x_n \in \mathcal{X}$ we have $Q[x] = \lim_{n\to\infty} Q[x_n]$.*
3. *For $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{R}$ we have $Q[\alpha x + \beta y] = \alpha Q[x] + \beta Q[y]$.*

**Remark 98** *1. We note that $Q \in \mathcal{X}'$ the dual space of $\mathcal{X}$ equipped with its canonical norm.*

*2. Instead of $L^2$ we can also use the Hilbert space $L^p$, remarking that the dual of $L^p$ can be identified with $L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 99 (Riesz representation theorem)** *For $Q$ a positive, linear functional as defined before, there exists $\phi \in \mathcal{X}$, such that*

$$Q[y] = < \phi, y > \forall y \in \mathcal{X}.$$

*Proof.* This is a direct consequence of Riesz representation theorem of continuous linear functionals of Hilbert spaces.

**Definition 100 (Deflator)** *The $\phi \in \mathcal{X}$ generating $Q[\bullet]$ is called* deflator.

**Proposition 101** *For a positive functional $Q : \mathcal{X} \to \mathbb{R}$, with deflator $\psi \in \mathcal{X}$ we have the following:*

1. *$\phi_k > 0$ for all $k \in \mathbb{N}$.*
2. *$\phi$ is unique.*

*Proof.* 1. Assume $\phi_k = 0$ for some $k \in \mathbb{N}$. In this case we have $Q[(\delta_{kn})_{n\in\mathbb{N}}] = 0$ which is a contradiction.

2. Assume $Q[y] =< \phi, y >=< \phi^*, y >$ for all $y \in \mathcal{X}$. In this case we have $< \phi - \phi^*, y >= 0$, in particular for $y = \phi - \phi^*$. Hence we have $||\phi - \phi^*|| = 0$.

**Definition 102 (Projections)** *For $k \in \mathbb{N}$ we define the following projections:*

1. *$p_k : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\delta_{kn}\, x_n)_{n\in\mathbb{N}}$, the projection on the $k$-th coordinate.*
2. *$p_k^+ : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\chi_{k\le n}\, x_n)_{n\in\mathbb{N}}$, the projection starting on the $k$-th coordinate.*

**Remark 103** • *We remark that the both above defined projections are linear operators with norm $\leq 1$.*

- *As a consequence of that we have for $x \geq 0$ with $x \in \mathcal{X}$ the following two relations:*

$$Q[p_k(x)] \leq Q[x],$$
$$Q[p_k^+(x)] \leq Q[x].$$

**Definition 104 (Valuation at time t)** *For $t \in \mathbb{N}$ we define the valuation of $x \in \mathcal{X}$ at time $t$ by*

$$Q_t[x] = Q[x|\mathcal{F}_t] = \frac{1}{\phi_t}\,\mathbb{E}[\sum_{k=0}^{\infty} \phi_k\, x_k|\mathcal{F}_t].$$

*In the same sense as for mathematical reserves we define the value of the future cash flows at time $t$ by*

$$Q_t^+[x] = Q[p_t^+(x)].$$

**Definition 105 (Zero Coupon Bonds)** *The zero coupon bond $\mathcal{Z}_{(k)} = (\delta_{kn})_{n\in\mathbb{N}}$ is an element of $\mathcal{X}$. We remark that*

$$\pi_0\left(\mathcal{Z}_{(t)}\right) = Q[\mathcal{Z}_{(t)}] = \mathbb{E}[\phi_t].$$

**Definition 106** *The cash flow $x = (x_k)_{k\in\mathbb{N}}$ in the discrete Markov model (cf. appendix B) is given by:*

$$x_k = \sum_{(i,j)\in S^2} \Delta N_{ij}(k-1) a_{ij}^{Post}(k-1) + \sum_{i\in S} I_i(k) a_i^{Pre}(k),$$

*where we assume that $\Delta N_{ij}(-1) = 0$.*

**Proposition 107** *For $x \in \mathcal{X}$, as defined above we have the following:*

1. $\mathbb{E}[\Delta N_{ij}(s)|X_t = k] = p_{ki}(t,s) p_{ij}(s, s+1)$,
2. $\mathbb{E}[I_i(s)|X_t = k] = p_{ki}(t,s)$,
3. $\mathbb{E}[x_s|X_t = k] =$

$$\sum_{(i,j)\in S^2} p_{ki}(t, s-1) p_{ij}(s-1, s) a_{ij}^{Post}(s-1) + \sum_{i\in S} p_{ki}(t,s) a_i^{Pre}(s),$$

*where we assume that $p_{ki}(t, s-1) = 0$ if $t \geq s$.*

*Proof.* We leave the proof of this proposition to the reader as an exercise.

**Definition 108** *The abstract vector space of financial instruments we denote by $\mathcal{Y}$. Elements of this vector space are for example all zero coupon bonds, shares, options on shares etc.*

**Remark 109**
- *We remark we can canonically embed $y \in \mathcal{Y}$ in $\mathcal{X}$, by means of its corresponding cash flows $(\xi(y)_k)_{k\in\mathbb{N}}$. Hence applying $Q[\bullet]$ to $y \in \mathcal{Y}$, is a shortcut for $Q[\xi(y)]$.*
- *Link to the arbitrage free pricing theory: If we assume that $Q$ does not allow arbitrage, see appendix D. In proposition 151 we can see that $\pi(X) = \mathbb{E}^Q[\beta_T X]$, where $\beta_T$ denotes the risk free discount rate. In the context of the above, we would have $\pi_0(x) = Q[x] = \mathbb{E}^P[\phi_T x]$. Hence we can identify $\phi_T = \frac{dQ}{dP} \beta_T$. In consequence we can interpret a deflator as a discounted Radon-Nikodym derivative with respect to the two measures $P$ and $Q$.*

**Proposition 110** *Let $Q$ be a positive, continuous functional $Q : \mathcal{X} \to \mathbb{R}$, and assume $Q[\bullet] =< \phi, \bullet >$, with $\phi = (\phi_t)_{t\in\mathbb{N}}$ $\mathbb{F}$ – adapted. In this case $(\phi_t\, Q_t[x])_{t\in\mathbb{N}}$ is an $\mathbb{F}$-martingale over $P$.*

*Proof.* Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ and the projection property of the conditional expectation we have

$$\mathbb{E}^P[\phi_{t+1}\, Q_{t+1}[x]|\mathcal{F}_t] = \mathbb{E}^P[\mathbb{E}^P[\sum_{k\in\mathbb{N}} \phi_k x_k\, Q_{t+1}[x]|\mathcal{F}_{t+1}]|\mathcal{F}_t]$$

$$= \mathbb{E}^P[\sum_{k \in \mathbb{N}} \phi_k x_k \, Q_{t+1}[x] | \mathcal{F}_t]$$
$$= \phi_t \, Q_t[x].$$

**Example 111 (Replicating Portfolio Mortality)** *In this first example we consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. For sake of simplicity we assume that the yearly mortality is* $(1 + \frac{x-50}{10} \times 0.1)\%$. *We assume that the death benefit amounts to 100000 EUR and we assume that the premium has been determined with an interest rate* $i = 2\%$. *In this case the premium amounts to* $P = 1394.29$ *EUR. The replicating portfolio in the sense of expected cash flows at inception is therefore given as follows (cf proposition 107). We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively, and that the use of arbitrary yield curves does not imply additional complexity.*

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | $\mathcal{Z}_{(0)}$ | – | *-1394.28* | *-1394.28* | *-1394.28* | *-1394.28* |
| *51* | $\mathcal{Z}_{(1)}$ | *1000.00* | *-1380.34* | *-380.34* | *-372.88* | *-365.71* |
| *52* | $\mathcal{Z}_{(2)}$ | *1089.00* | *-1365.16* | *-276.16* | *-265.43* | *-255.32* |
| *53* | $\mathcal{Z}_{(3)}$ | *1174.93* | *-1348.77* | *-173.84* | *-163.81* | *-154.54* |
| *54* | $\mathcal{Z}_{(4)}$ | *1257.56* | *-1331.24* | *-73.67* | *-68.06* | *-62.97* |
| *55* | $\mathcal{Z}_{(5)}$ | *1336.69* | *-1312.60* | *24.09* | *21.82* | *19.80* |
| *56* | $\mathcal{Z}_{(6)}$ | *1412.12* | *-1292.91* | *119.20* | *105.85* | *94.21* |
| *57* | $\mathcal{Z}_{(7)}$ | *1483.67* | *-1272.23* | *211.44* | *184.07* | *160.67* |
| *58* | $\mathcal{Z}_{(8)}$ | *1551.18* | *-1250.60* | *300.57* | *256.54* | *219.62* |
| *59* | $\mathcal{Z}_{(9)}$ | *1614.50* | *-1228.09* | *386.41* | *323.33* | *271.48* |
| *60* | $\mathcal{Z}_{(10)}$ | *1673.52* | – | *1673.52* | *1372.87* | *1130.57* |
| **Total** | | | | | **0.00** | **-336.47** |

**Exercise 112 (Replicating Portfolio Disability)** *Consider a disability cover and calculate the replicating portfolios for a deferred disability annuity and a disability in payment.*

## C.2 Cost of Capital

In section C.1 we have seen how to abstractly valuate $x \in \mathcal{X}$ by means of a pricing functional $Q$. For some financial instruments $y \in \mathcal{Y}^\star$ we can directly observe $Q[y]$ such as for a lot of zero coupons bonds $\mathcal{Z}_{(\bullet)}$. On the other hand this is not always possible.

**Definition 113** *We denote by $\mathcal{Y}^\star$ the set of all financial instruments in $x \in \mathcal{Y}$ such that $Q[x]$ is observable. With $\tilde{\mathcal{Y}} = span < \mathcal{Y}^\star >$ we denote the vector space generated by $\mathcal{Y}^\star$ and we define:*

1. *$x \in \mathcal{Y}^\star$ is called of level 1.*
2. *$x \in \tilde{\mathcal{Y}}$ is called of level 2.*
3. *$x \in \mathcal{Y} \setminus \tilde{\mathcal{Y}}$ is called of level 3.*

**Remark 114** *It is clear that the model uncertainty and the difficulties to value assets or liabilities increases from level 1 to level 3. Since we are interested in market values only the valuation of level 1 assets and liabilities is really reliable. For level 2 assets and liabilities on has to find a sequence of $x_n = \sum_{k=1}^n \alpha_k\, e_k$ with $e_k \in \mathcal{Y}^\star$ such that $x = \lim_{n\to\infty} x_n$. Since we assume that $Q$ is linear and continuous we can calculate*

$$\begin{aligned} Q[x] &= \lim_{n\to\infty} Q[x_n] \\ &= \lim_{n\to\infty} \sum_{k=1}^n Q[\alpha_k\, e_k] \\ &= \lim_{n\to\infty} \sum_{k=1}^n \alpha_k\, Q[e_k]. \end{aligned}$$

*For level 3 assets and liabilities the situation is even more difficult, since there is no obvious way to do it. The best, which we can be done is to define $\tilde{Q}[x]$ such that $\tilde{Q}[x] = Q[x] \forall x \in \mathcal{Y}^\star$ and hope that $\tilde{Q}[x] \approx Q[x]$ for the $x \in \mathcal{Y}$ we want to valuate. In most cases such $\tilde{Q}[\bullet]$ are based on first economic principles. In the following we want to see how the* Cost of Capital *concept works for insurance liabilities and how we can concretely implement it.*

**Definition 115 (Utility Assumption)** *If we have $x, y \in L^2(\Omega, \mathcal{A}, P)^+$, with $x = \mathbb{E}[y]$. A rational investor would normally prefer $x$, since there is less uncertainty. The way to understand this, is by using utility functions. For $x \in L^2(\Omega, \mathcal{A}, P)^+$ and $u$ a concave function, the utility of $x$ is defined as $\mathbb{E}[u(x)]$. The idea behind utilities is that the first 10000 EUR are higher valued than the one 10000 EUR from 100000 EUR to 110000 EUR. Hence the increase of utility per fixed amount decreases if amounts increase. As a consequence of the Jensen-inequality, we see that the utility of a constant amount is higher than the utility of a random payout with the same expected value.*

**Definition 116** *Let $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ be an insurance cash flow, for example generated by a Markov model.*

1. *In this case we define the expected cash flows by*

$$CF(x) = (\mathbb{E}[x_k])_{k\in\mathbb{N}}.$$

2. *The corresponding portfolio of financial instruments in the vector space $\mathcal{Y}$ we define by*

$$VaPo^{CF}(x) = \sum_{k\in\mathbb{N}} CF(x)_k \mathcal{Z}_{(k)} \in \mathcal{Y}$$

3. *By $R(x)$ we denote the residual risk portfolio given by*

$$\begin{aligned} R(x) &= x - VaPo^{CF}(x) \\ &= \sum_{k\in\mathbb{N}} (x_k - CF(x)_k\, \mathcal{Z}_{(k)}) \in \mathcal{Y} \end{aligned}$$

4. *For a given $x \in \mathcal{X}$ we denote by $VaPo^*(x)$ an approximation $y \in \tilde{\mathcal{Y}}$ of $x$, such that $||x - VaPo^*(x)|| \leq ||x - VaPo^{CF}(x)||$. In analogy to $R(x)$, we can define $R^\star(x)$ with respect to $VaPo^*(x)$.*

*Since we are sometimes interested in conditional expectations, we will also use the following notations for $A \in \mathcal{A}$:*

$$\begin{aligned} CF(x \mid A) &= (\mathbb{E}[x_k \mid A])_{k\in\mathbb{N}}, \\ VaPo^{CF}(x \mid A) &= \sum_{k\in\mathbb{N}} CF(x \mid A)_k \mathcal{Z}_{(k)} \in \mathcal{Y}, \end{aligned}$$

**Proposition 117** *The value of $x \in \mathcal{Y}$ can be decomposed in*

$$Q[x] = Q[VaPo^{CF}(x)] + Q[R(x)],$$

*and we have*

$$Q[VaPo^{CF}(x)] \geq Q[x]$$

*if we use the utility assumption.*

**Remark 118** *1. We will denote $x \in \mathcal{X}$ with $x \leq 0$ as a liability. Proposition 117 hence tells us that we need to reserve more than $Q[VaPo^{CF}(x)]$ for this liability as a consequence of the corresponding uncertainty.*

2. *A risk measure is a functional (not necessarily linear) $\psi : \mathcal{X} \to \mathbb{R}$ which aims to measure the capital needs in an adverse scenario. There are two risk measures, which are commonly used the* Value at Risk *and the* Expected Shortfall *to a given quantile $\alpha \in \mathbb{R}$. The value at risk (VaR) is defined as the corresponding quantile minus the expected value. The expected shortfall is the conditional expectation of the random variable given a loss bigger than the corresponding loss, again minus the expected value. We can hence speak about a 99.5% VaR or a 99% expected shortfall. It is worthwhile to remark that these two concepts are normally applied to losses. Hence in the context introduced above one would strictly speaking*

*calculating the VaR$(-x)$, when considering $x \in \mathcal{X}$. Furthermore in a lot of applications, such as Solvency II, we assume that there is a Dirac measure (aka stress scenario), which just represents the corresponding VaR-level for example. So concretely the stress scenarios, which are used under Solvency II should in principle represent the corresponding point (Dirac) measures at to the confidence level 99.5 %. In the concrete set up, one would for example assume that $q_x(\omega) \in L^2(\Omega, \mathcal{A}, P)$ is a stochastic mortality and one would define the $A, B \in \mathcal{A}$, as the corresponding probabilities in the average and in the tail. In consequence for a policy $x \in \mathcal{X}$, we would have two replicating portfolios, namely $VaPo^{CF}(x \mid A)$ for the average and $VaPo^{CF}(x \mid B)$ for the stressed event according to the risk measure chosen. The corresponding required risk capital is then given (in present value terms) by $Q[VaPo^{CF}(x \mid B) - VaPo^{CF}(x \mid A)]$.*

3. *It is important to remark that the concept of cash flow representation of insurance policies $x \in \mathcal{X}$ makes particularly sense when the corresponding insurance cash flows are independent from the capital market induced stochastic variables. This is the case for non-profit products and also for annuities in payment without discretionary benefits. For other insurance products, such as classical with profits products or also GMDB types of covers, the cash flow representation $VaPo^{CF}(x)$ is not suited to represent $x$. In this case a replicating portfolio needs to take also into consideration the corresponding effects, as we will see in the following. In this set up one has to determine a suitable $VaPo^*(x)$–representation and in consequence use $R^\star(x)$. Also the cost of capital approach has in this case to be performed with respect to $VaPo^*(x)$. Having remarked this we will always the notation $VaPo^{CF}(x)$ even though that in certain of the above mentioned cased we actually mean a suitable $VaPo^*(x)$–representation also taking into consideration dependencies on the capital markets, mutatis mutandis.*

**Definition 119 (Required Risk Capital)** *For a risk measure $\psi_\alpha$ such as VaR or expected shortfall to a security level $\alpha$ we define the required risk capital at time $t \in \mathbb{N}$ by*

$$RC_t(x) = \psi_\alpha(p_k(x - VaPo^{CF}(x))).$$

**Remark 120** *1. If we use $VaR_{99.5\%}$ the required risk capital at time $t$ corresponds to the capital needed to withstand a 1 in 200 year event.*

2. *The definition above could apply to individual insurance policies, but is normally applied to insurance portfolios $\tilde{x} = \sum_{k=1}^{n} x_k$, where $(x_k)_{k=1,\ldots n}$ are the individual insurance policies. As we can see in section 10.3 of [Kol10] the pure diversifiable risk disappears for $n \to \infty$.*
3. *What is more material than the diversifiable risk is the risk, which affects all of the individual insurance policies at the same time, such as a pandemic event, where the overall mortality could increase by 1 % in a certain year such as 1918 (see for example figure 4.3).*

**Definition 121 (Cost of Capital)** *For a unit cost of capital $\beta \in \mathbb{R}^+$ and an insurance portfolio $\tilde{x} \in \mathcal{X}$, we define:*

1. *The present value of the required risk capital by*

$$PVC(\tilde{x}) = Q[\sum_{k \in \mathbb{N}} RC_t(\tilde{x}) \mathcal{Z}_{(k)}].$$

2. *The cost of capital $CoC(\tilde{x})$ is given by:*

$$CoC(\tilde{x}) = \beta \times PVC(\tilde{x}),$$

*and $\tilde{Q}$ is defined by $\tilde{Q}[x] = Q[VaPo^{CF}(\tilde{x})] + \beta PVC(\tilde{x})$.*

**Remark 122** *1. The concept as defined before is somewhat simplified, since one normally assumes that the required capital $C$ from the shareholder is $\alpha \times C$ after tax and investment income on capital. Assume a tax-rate $\kappa$ and a risk-free yield of $i$. In this case we have*

$$\alpha \times C = i \times (1 - \kappa) \times C + \beta \times C,$$

*and hence $\beta = \alpha - i \times (1 - \kappa)$. In reality the calculation can still become more complex since we discount future capital requirements risk-free and because of the fact that the interest rate $i$ is not constant. In order to avoid these technicalities, we will assume for this book that $i$ is constant.*

2. *We remark $\tilde{Q}[\tilde{x}]$ is not uniquely determined, but depends on a lot of assumptions such as $\psi_\alpha$, $\alpha$, $\beta$, ...*
3. *For the moment we do not yet see how to actually model $\tilde{x}$ and we remark that one is normally focusing on the non-diversifiable part of the risks within $\tilde{x}$.*

**Example 123** *We continue with example 111 and we assume that the risk capital is given by a pandemic event where $\Delta q_x = 1\%$ for all ages. This roughly corresponds to the increase in mortality of 1918 as a consequence of the Spanish flu pandemic. The aim of this example is to calculate the required risk capital and the market value of this policy based on the cost of capital method using $\beta = 6\%$. The required risk capital in this context can be calculated as $\Delta q_x \times 100000$ and we get the following results:*

| *Age* | *Unit* | *Units for Risk Capital* | *Units for Benefits* | *Total Units* | $-\tilde{Q}[x]$ $i = 2\%$ | $-\tilde{Q}[x]$ $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | $\mathcal{Z}_{(0)}$ | *1000.00* | *-1394.28* | *-1334.28* | *-1334.28* | *-1334.28* |
| *51* | $\mathcal{Z}_{(1)}$ | *990.00* | *-380.34* | *-320.94* | *-314.65* | *-308.60* |
| *52* | $\mathcal{Z}_{(2)}$ | *979.11* | *-276.16* | *-217.41* | *-208.97* | *-201.01* |
| *53* | $\mathcal{Z}_{(3)}$ | *967.36* | *-173.84* | *-115.80* | *-109.12* | *-102.95* |
| *54* | $\mathcal{Z}_{(4)}$ | *954.78* | *-73.67* | *-16.38* | *-15.14* | *-14.00* |
| *55* | $\mathcal{Z}_{(5)}$ | *941.41* | *24.09* | *80.57* | *72.98* | *66.22* |
| *56* | $\mathcal{Z}_{(6)}$ | *927.29* | *119.20* | *174.84* | *155.25* | *138.18* |
| *57* | $\mathcal{Z}_{(7)}$ | *912.45* | *211.44* | *266.19* | *231.73* | *202.28* |
| *58* | $\mathcal{Z}_{(8)}$ | *896.94* | *300.57* | *354.39* | *302.47* | *258.95* |
| *59* | $\mathcal{Z}_{(9)}$ | *880.80* | *386.41* | *439.26* | *367.55* | *308.61* |
| *60* | $\mathcal{Z}_{(10)}$ | – | *1673.52* | *1673.52* | *1372.87* | *1130.57* |
| **Total** | | | | | **520.69** | **143.98** |

*We remark that the value of the policy at inception becomes positive, which means nothing else, that the insurance company does need equity capital to cover the economic loss. It is obvious that this is the case for* $i = 2\%$*, since the premium principle did not allow for a compensation of the risk capital. More interestingly even at the higher interest rate the compensating effect is not big enough to turn this policy into profitability.*

**Exercise 124** *In the same sense as for the mortality example calculate the respective risk capitals and* $\tilde{Q}$ *for a disability cover.*

**Example 125 (GMDB mortality cover)** *In this example we want to have a look at a unit linked insurance policy (*$x \in \mathcal{X}$*) where the death benefit amounts to the maximum of the fund value and of a fixed (minimal) death benefit (see also section D.2.2).*

*We assume that the policyholder aged 40 invests 100000 EUR single premium in a fund* $\mathcal{U}$ *with a volatility* $\sigma = 20\%$*. The term of the policy is 10 years, the guaranteed mortality benefit 150000 EUR and we assume a flat yield curve of 2 %. The mortality follows example 111. We want to calculate the following things:*

- *Calculate* $VaPo^*(x)$ *for this GMDB death benefit, where we assume that the policyholders die according to the given mortality law.*
- *Calculate the single premium of this GMDB cover (w/o CoC).*
- *Calculate some sensitivities for* $Q[VaPo^*(x)]$*, namely for a volatility of 30 % and for an interest rate of 1 %.*

*In a first step we need to calculate the expected number of death people by* ${}_tp_x\, q_{x+t}$*. For this people we need to be able to sell the funds value* $S_t$ *at a value of 150000 EUR. This represents a put option* $\mathcal{P}_t \in \mathcal{Y}$ *with a current funds value* $S_0 = 100000$*, maturity* $t$ *and strike 150000. The price for a* put*-option with payout* $C(T, P) = \max(K - S; 0)$ *at time* $t$ *and strike price* $K$ *and equity price* $S$ *is given by:*

$$\begin{aligned}
P &= K \times e^{-r \times T} \times \Phi(-d_2) - S_0 \times \Phi(-d_1), \\
d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2) \times T}{\sigma \times \sqrt{T}}, \\
d_2 &= d_1 - \sigma \times \sqrt{T}, \\
\Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(\frac{-\zeta^2}{2}) d\zeta,
\end{aligned}$$

*and we get the following for $VaPo^*(x)$:*

| Age | Unit | Expected Deaths | Units Death | Value Units | Value $\sigma = 30\%$ | Value $i = 1\%$ |
|---|---|---|---|---|---|---|
| 50 | $\mathcal{P}_0$ | 0.01000 | 0.01000 | – | – | – |
| 51 | $\mathcal{P}_1$ | 0.01089 | 0.01089 | 472.78 | 487.26 | 487.26 |
| 52 | $\mathcal{P}_2$ | 0.01174 | 0.01174 | 497.45 | 539.39 | 526.90 |
| 53 | $\mathcal{P}_3$ | 0.01257 | 0.01257 | 524.96 | 592.78 | 569.82 |
| 54 | $\mathcal{P}_4$ | 0.01336 | 0.01336 | 552.39 | 644.22 | 613.24 |
| 55 | $\mathcal{P}_5$ | 0.01412 | 0.01412 | 578.51 | 692.85 | 655.91 |
| 56 | $\mathcal{P}_6$ | 0.01483 | 0.01483 | 602.75 | 738.31 | 697.20 |
| 57 | $\mathcal{P}_7$ | 0.01551 | 0.01551 | 624.83 | 780.41 | 736.70 |
| 58 | $\mathcal{P}_8$ | 0.01614 | 0.01614 | 644.58 | 819.02 | 774.14 |
| 59 | $\mathcal{P}_9$ | 0.01673 | 0.01673 | 661.93 | 854.08 | 809.32 |
| 60 | $\mathcal{P}_{10}$ | – | – | 676.85 | 885.55 | 842.09 |
| **Total** | | | | **5837.06** | **7033.92** | **6712.62** |

*We see here very clearly the dependency of the value of the replicating portfolio on the different parameters.*

## C.3 Inclusion in the Markov Model

In this section we want to have a look how we could concretely use the recursion technique for the calculation of the cost of capital in a Markov chain similar environment. In order to do that we look at an insurance policy with a term of one year.

We assume that we have a mortality of $q_x$ in case of a "normal" year with a probability of $(1-\alpha)$ and an excess mortality of $\Delta q_x$ in an extreme year with probability $\alpha$. We denote with $\Gamma = \frac{q_x + \Delta q_x}{q_x}$. Furthermore we assume a mortality benefit of 100000. In this case we get the following by some simple calculations:

$$VaPo^{CF}(x) = (\delta_{1k}(q_x + \alpha(\Gamma - 1)q_x \times 100000))_{k \in \mathbb{N}},$$

$$\begin{aligned}
RC_1(x) &= (\delta_{1k}(1-\alpha)(\Gamma-1)q_x \times 100000)_{k\in\mathbb{N}}, \\
\tilde{Q}[x] &= Q[(\delta_{1k}(q_x + \alpha(\Gamma-1)q_x \times 100000 + \\
&\quad + \beta(1-\alpha)(\Gamma-1) \times 100000))_{k\in\mathbb{N}}].
\end{aligned}$$

We see that the price of this insurance policy with only payments at time 1 can be decomposed into a part representing best estimate mortality:

$$\delta_{1k}\{q_x\,(1+\alpha\,(\Gamma-1))\},$$

where we can arguably say that this $\tilde{q}_x = q_x\,(1+\alpha\,(\Gamma-1))$ is our actual best-estimate mortality. On top of that we get a charge for the excess mortality $\Delta q_x$ with an additional cost of $\beta$. Hence we get the following:

1. There is a contribution to the reserve from the people surviving the year with a probability $p_x$.
2. There is a contribution to the reserve from the people dying in normal years with probability $q_x$ and the defined benefit $a_{*\dagger}^{\text{post}}$, and
3. There is finally a contribution of the people dying in extreme years with probability $\Delta q_x$ and the additional cost of defined benefit of $\beta \times a_{*\dagger}^{\text{post}}$.

The interesting fact is that we can actually use the same recursion of the reserves for the Markov chain model as in formula 107 with 3 states and the exception that now the "transition probabilities" do not fulfil anymore the requirement that their sum equals 1. However this method provides a pragmatic way to implement the cost of capital in legacy admin systems.

The main problem for the determining of the corresponding Markov chain model is the underlying stochastic mortality model. For the QIS 5 longevity model a similar calculation can be used. In this model it is assumed that the mortality drops by 25 % in an extreme scenario. Hence the calculation goes along the following process:

1. Determine $x_1 = VaPo^{CF}(\tilde{x}|A)$ for standard mortality $A$.
2. Determine $x_2 = VaPo^{CF}(\tilde{x}|B)$ for stressed mortality $B$.
3. $\tilde{Q}[x] = Q[x_1] + \beta\, Q[x_2 - x_1]$

**Example 126** *In this example we want to revisit the exercise 111 and we want again to calculate the market value of the insurance liability using the cost of capital approach, but this time with the recursion. We get the following results:*

| *Age* | *Benefit Normal* | *Benefit Premium* | *Excess Risk* | *Math Res.* $i = 2\%$ | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | *100000* | *-1394.28* | *6000* | **0.00** | **520.69** | **143.98** |
| *51* | *100000* | *-1394.28* | *6000* | *426.43* | *901.09* | *542.82* |
| *52* | *100000* | *-1394.28* | *6000* | *765.56* | *1193.21* | *861.67* |
| *53* | *100000* | *-1394.28* | *6000* | *1015.22* | *1394.79* | *1096.96* |
| *54* | *100000* | *-1394.28* | *6000* | *1172.95* | *1503.20* | *1244.68* |
| *55* | *100000* | *-1394.28* | *6000* | *1235.88* | *1515.45* | *1300.33* |
| *56* | *100000* | *-1394.28* | *6000* | *1200.79* | *1428.16* | *1258.89* |
| *57* | *100000* | *-1394.28* | *6000* | *1064.00* | *1237.49* | *1114.74* |
| *58* | *100000* | *-1394.28* | *6000* | *821.42* | *939.18* | *861.64* |
| *59* | *100000* | *-1394.28* | *6000* | *468.45* | *528.45* | *492.63* |
| *60* | | | | *0* | *0* | *0* |

*We remark that this calculation is much faster to calculate since it is based on Thiele's difference equation for the mathematical reserves, and we get at the same time the corresponding results for the classical case and also for the case using the cost of capital approach.*

*As seen in the calculation above there is a small second order effect, which we can detect, when looking more closely. The results below correspond to the* $2\%$ *valuation:*

```
Direct Method  520.698380872792
Recursion      520.698380872793
```

**Exercise 127** *Perform the corresponding calculation for the disability example.*

## C.4 Asset Liability Management

Until now we have looked only at insurance liabilities as an $x \in \mathcal{Y}$. An insurance company needs to cover its insurance liabilities $l = \sum x_\iota \in \mathcal{X}$ with corresponding assets, which are also elements in $\mathcal{Y} \subset \mathcal{X}$.

**Definition 128 (Assets and Liabilities)** *An* $x \in \mathcal{X}$ *with a valuation functional* $Q$ *is called*

1. *an asset if* $Q[x] \geq 0$ *and*
2. *a liability if* $Q[x] \leq 0$.

**Remark 129** *At this point is important to have a closer look at the convention what is an asset and what is a liability and the corresponding signs of the cash flows. In* *a actuarial context payments from the insurance company to the policyholder have a "+" sign and premium payments a "−". Hence the whole appendix up to here is based on this convention. Now we are looking at an entire balance sheet and we will in consequence apply the respective accounting conventions, where assets have a "+" sign and liabilities a "−". Hence the reserves and stochastic cash flows as defined and calculated until now represent actually the* negative *liability in term of a balance sheet.*

**Definition 130 (Insurance balance sheet)** *An insurance balance sheet consists of a set of assets $(a_i)_{i\in I}$ and a set of liabilities $(l_j)_{j\in J}$. The equity of an insurance balance sheet is defined as*

$$e = \sum_{i\in I} a_i + \sum_{j\in J} l_j.$$

*The insurance entity is called bankrupt if $Q[e] < 0$.*

**Definition 131** *In a regulated insurance market, each insurance company is required to hold an adequate amount of risk capital in order to absorb shocks. In order to do that, the regulator defines a risk measure $\psi_\alpha$ to a security level $\alpha$. In this context an insurance company is called solvent if:*

$$Q[e] \geq \psi_\alpha(e).$$

**Remark 132** *Note that an insurance regulator may not want to use a market consistent approach. Never the less the above definition can be used, be suitably adjust $\psi$.*

**Definition 133 (Asset Liability Management)** *Under asset liability management we understand the process of analysing $(l_j)_{j\in J}$ and the (dynamic) management of $(a_i)_{i\in I}$ in order to achieve certain target , such as remaining solvent.*

**Definition 134** *For an insurance liability $l \in \mathcal{X}$ an asset portfolio $(a_i)_{i\in I}$ is called:*

1. Matching *if $\sum_{i\in I} a_i + l = 0$, and*
2. Cash flow matching *if $\sum_{i\in I} a_i + VaPo^{CF}(l) = 0$.*

**Remark 135** *We remark that is normally not feasible to do a perfect matching, and hence one normally uses a cash flow matching to a achieve a proxy for a perfect match. We also remark that in this case the shareholder equity needs still be able to absorb the basis risk $l - VaPo^{CF}(l)$.*

**Definition 136 (Macaulay Duration)** *The duration for an $x \in \mathcal{Y}$ with*

$$x = \sum_{k\in\mathbb{N}} \alpha_k\, \mathcal{Z}_{(k)} \text{ and } \alpha_k \geq 0$$

*is defined by*

$$d(x) = \frac{Q[\sum_{k\in\mathbb{N}} \alpha_k \times k \times \mathcal{Z}_{(k)}]}{Q[\sum_{k\in\mathbb{N}} \alpha_k \times \mathcal{Z}_{(k)}]}$$

*We say that an asset portfolio $(a_i)_{i\in I}$ is duration matching a liability $l$ if the following two conditions are fulfilled:*

1. $Q[\sum_{i\in I} a_i + l] = 0$, *and*
2. $d(\sum_{i\in I} a_i) = d(-VaPo^{CF}(l))$.

**Example 137** *In this example we want to further elaborate on the example 111 and we want to see how the replicating scenario changes in case a pandemic occurs in year three, with an excess mortality of 1 %. We want also to have a look on what risk is implied in this, assuming that the pandemic at the same time leads to a reduction of interest rates down from 2% to 0 %. Finally we want to see an example how we could do a perfect cash flow matching portfolio.*

**Definitions** *We assume that $A \in \mathcal{A}$ represents the information that we have going to have average mortality after year 3 and three and that the person survived until then (year 2). In the same sense we assume that $B \in \mathcal{A}$ represents the same as $A$ but with the exception that we assume a pandemic event in the year 3 with an average excess mortality of 1%. For simplicity reasons (to avoid notation) we use $x, y \in \mathcal{X}$ as abbreviations for the corresponding conditional random variables.*

**Calculation of the Replicating Portfolios** *In a first step we will calculate the replicating portfolios (starting at time 2) with respect to both $A$ and $B$. Doing this we get the following results for case $A$:*

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *52* | $\mathcal{Z}_{(0)}$ | – | *-1394.28* | *-1394.28* | *-1394.28* | *-1394.28* |
| *53* | $\mathcal{Z}_{(1)}$ | *1200.00* | *-1377.55* | *-177.55* | *-174.07* | *-170.72* |
| *54* | $\mathcal{Z}_{(2)}$ | *1284.40* | *-1359.64* | *-75.24* | *-72.32* | *-69.57* |
| *55* | $\mathcal{Z}_{(3)}$ | *1365.21* | *-1340.61* | *24.60* | *23.18* | *21.87* |
| *56* | $\mathcal{Z}_{(4)}$ | *1442.25* | *-1320.50* | *121.75* | *112.48* | *104.07* |
| *57* | $\mathcal{Z}_{(5)}$ | *1515.33* | *-1299.37* | *215.95* | *195.59* | *177.49* |
| *58* | $\mathcal{Z}_{(6)}$ | *1584.27* | *-1277.28* | *306.99* | *272.59* | *242.61* |
| *59* | $\mathcal{Z}_{(7)}$ | *1648.95* | *-1254.29* | *394.65* | *343.57* | *299.90* |
| *60* | $\mathcal{Z}_{(8)}$ | *1709.23* | – | *1709.23* | *1458.81* | *1248.91* |
| **Total** | | | | | **765.56** | **460.30** |

*For case $B$ we get:*

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i=2\%$ | *Value* $i=4\%$ |
|---|---|---|---|---|---|---|
| *52* | $\mathcal{Z}_{(0)}$ | – | *-1394.28* | *-1394.28* | *-1394.28* | *-1394.28* |
| *53* | $\mathcal{Z}_{(1)}$ | *1200.00* | *-1377.55* | *-177.55* | *-174.07* | *-170.72* |
| *54* | $\mathcal{Z}_{(2)}$ | **2272.40** | *-1345.87* | **926.52** | *890.54* | *856.62* |
| *55* | $\mathcal{Z}_{(3)}$ | *1351.38* | *-1327.03* | *24.35* | *22.95* | *21.65* |
| *56* | $\mathcal{Z}_{(4)}$ | *1427.64* | *-1307.12* | *120.51* | *111.34* | *103.01* |
| *57* | $\mathcal{Z}_{(5)}$ | *1499.97* | *-1286.21* | *213.76* | *193.61* | *175.70* |
| *58* | $\mathcal{Z}_{(6)}$ | *1568.22* | *-1264.34* | *303.88* | *269.83* | *240.16* |
| *59* | $\mathcal{Z}_{(7)}$ | *1632.24* | *-1241.58* | *390.65* | *340.09* | *296.86* |
| *60* | $\mathcal{Z}_{(8)}$ | *1691.91* | – | *1691.91* | *1444.03* | *1236.26* |
| **Total** | | | | | **1704.05** | **1365.28** |

*We note two things:*

- *The pandemic happens when the person is aged* $53$ *and we see the impact in* $\mathcal{Z}_{(2)}$ *at age 54. This has to do with the convention that we assume that the deaths occur at the end on the year, hence just before the person gets 54.*
- *We see that the difference in reserves amounts to* $1704.05 - 765.56 = 938.49$ *which represents the economic loss as a consequence of the pandemic. The biggest contributor to this loss is the increased death benefit, e.g.* $926.52 - 1284.40 = 962.87$.

**Matching asset portfolios** *Based on the above it is now easy to calculate the cash flow matching portfolio, by just investing the different amounts of liabilities into the corresponding assets, such as buying* $24.60\mathcal{Z}_{(3)}$. *We remark that consequently we would have to sell* $-177.55\mathcal{Z}_{(1)}$. *In normal circumstances for mature businesses this will not occur, since it is a consequence that we consider a endowment insurance policy and not for example an endowment.*

**Mismatch in case of a pandemic** *The table below finally shows the cash flow mismatch as a consequence of the pandemic and we see that in this case the present values do not have a big impact since the main difference is at time* $1$.

| *Age* | *Unit* | *Units Normal* | *Units Stress* | *Difference Units* | *Value* $i=2\%$ | *Value* $i=0\%$ |
|---|---|---|---|---|---|---|
| *52* | $\mathcal{Z}_{(0)}$ | *-1394.28* | *-1394.28* | *0.00* | *0.00* | *0.00* |
| *53* | $\mathcal{Z}_{(1)}$ | *-177.55* | *-177.55* | *0.00* | *0.00* | *0.00* |
| *54* | $\mathcal{Z}_{(2)}$ | *-75.24* | *926.52* | *1001.77* | *962.87* | *1001.77* |
| *55* | $\mathcal{Z}_{(3)}$ | *24.60* | *24.35* | *-0.24* | *-0.23* | *-0.24* |
| *56* | $\mathcal{Z}_{(4)}$ | *121.75* | *120.51* | *-1.23* | *-1.13* | *-1.23* |
| *57* | $\mathcal{Z}_{(5)}$ | *215.95* | *213.76* | *-2.18* | *-1.98* | *-2.18* |
| *58* | $\mathcal{Z}_{(6)}$ | *306.99* | *303.88* | *-3.11* | *-2.76* | *-3.11* |
| *59* | $\mathcal{Z}_{(7)}$ | *394.65* | *390.65* | *-3.99* | *-3.48* | *-3.99* |
| *60* | $\mathcal{Z}_{(8)}$ | *1709.23* | *1691.91* | *-17.31* | *-14.78* | *-17.31* |
| **Total** | | | | | **938.49** | **973.67** |

**Example 138 (Lapses)** *In this example we want to see how lapses can influence the replicating portfolios. In order to do that we have to change the example 111 a little bit, as follows:*

- *We consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. We assume that the yearly mortality is* $(1 + \frac{x-50}{10} \times 0.1)\%$. *We assume that the benefit amounts to 100000 EUR and we assume that the premium has been determined with an interest rate* $i = 2\%$.
- *In this case the premium amounts to* $P = 9562.20$ *EUR.*
- *In addition the policyholder can surrender the policy at any time and gets back 98 % of the expected future cash flows valued at the pricing interest rate of 2%. We remark here that this is a risk since the surrenders can happen in case the market value of the corresponding units is below the surrender value.*
- *We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively.*

*In order to calculate this example we will perform the following steps:*

1. *Calculation of the cash flow matching portfolio in case of no surrenders.*
2. *Calculation of the cash flow including lapses with an average lapse rate of 7 %*
3. *Calculation of the cash flows at time 2, assuming average lapses, lapses at 25 % at time 2.*

**Calculation of the cash flow matching portfolio in case of no surrenders:**

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | $\mathcal{Z}_{(0)}$ | – | *-9562.20* | *-9562.20* | *-9562.20* | *-9562.20* |
| *51* | $\mathcal{Z}_{(1)}$ | *1000.00* | *-9466.57* | *-8466.57* | *-8300.56* | *-8140.94* |
| *52* | $\mathcal{Z}_{(2)}$ | *1089.00* | *-9362.44* | *-8273.44* | *-7952.17* | *-7649.26* |
| *53* | $\mathcal{Z}_{(3)}$ | *1174.93* | *-9250.09* | *-8075.16* | *-7609.40* | *-7178.79* |
| *54* | $\mathcal{Z}_{(4)}$ | *1257.56* | *-9129.84* | *-7872.27* | *-7272.76* | *-6729.25* |
| *55* | $\mathcal{Z}_{(5)}$ | *1336.69* | *-9002.02* | *-7665.32* | *-6942.72* | *-6300.34* |
| *56* | $\mathcal{Z}_{(6)}$ | *1412.12* | *-8866.99* | *-7454.87* | *-6619.71* | *-5891.69* |
| *57* | $\mathcal{Z}_{(7)}$ | *1483.67* | *-8725.12* | *-7241.45* | *-6304.11* | *-5502.90* |
| *58* | $\mathcal{Z}_{(8)}$ | *1551.18* | *-8576.79* | *-7025.61* | *-5996.29* | *-5133.54* |
| *59* | $\mathcal{Z}_{(9)}$ | *1614.50* | *-8422.41* | *-6807.90* | *-5696.55* | *-4783.14* |
| *60* | $\mathcal{Z}_{(10)}$ | – | *88080.30* | *88080.30* | *72256.53* | *59503.90* |
| **Total** | | | | | **0** | **-7368.19** |

*We remark that the there is considerable value in the policy if we assume no lapses, in case we earn a higher interest rate, such as 4 %.*

**Calculation of the cash flow matching portfolio in case of 7% surrenders:**

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | $\mathcal{Z}_{(0)}$ | – | -9562.20 | -9562.20 | -9562.20 | -9562.20 |
| *51* | $\mathcal{Z}_{(1)}$ | 1019.67 | -8797.22 | -7777.54 | -7625.04 | -7478.40 |
| *52* | $\mathcal{Z}_{(2)}$ | 1594.01 | -8084.65 | -6490.63 | -6238.59 | -6000.95 |
| *53* | $\mathcal{Z}_{(3)}$ | 2066.98 | -7421.70 | -5354.72 | -5045.87 | -4760.33 |
| *54* | $\mathcal{Z}_{(4)}$ | 2449.23 | -6805.70 | -4356.47 | -4024.71 | -3723.93 |
| *55* | $\mathcal{Z}_{(5)}$ | 2750.73 | -6234.02 | -3483.29 | -3154.92 | -2863.01 |
| *56* | $\mathcal{Z}_{(6)}$ | 2980.77 | -5704.13 | -2723.35 | -2418.26 | -2152.31 |
| *57* | $\mathcal{Z}_{(7)}$ | 3147.94 | -5213.57 | -2065.63 | -1798.25 | -1569.71 |
| *58* | $\mathcal{Z}_{(8)}$ | 3260.18 | -4759.99 | -1499.81 | -1280.07 | -1095.89 |
| *59* | $\mathcal{Z}_{(9)}$ | 3324.77 | -4341.11 | -1016.34 | -850.42 | -714.06 |
| *60* | $\mathcal{Z}_{(10)}$ | 5486.26 | 42159.27 | 47645.53 | 39085.93 | 32187.61 |
| **Total** | | | | | **-2912.44** | **-7733.21** |

*We remark that at that time, the company makes still some additional gains as a consequence of the 2% surrender penalty.*

**Calculation of the cash flow matching portfolio in case of high surrenders:** *We assume that there has been observed an exceptional lapse rate at time 2 of 25% of the portfolio.*

| *Age* | *Unit* | *Units for Mortality* | *Units for Premium* | *Total Units* | *Value* $i = 2\%$ | *Value* $i = 4\%$ |
|---|---|---|---|---|---|---|
| *50* | $\mathcal{Z}_{(0)}$ | – | -9562.20 | -9562.20 | -9562.20 | -9562.20 |
| *51* | $\mathcal{Z}_{(1)}$ | 1019.67 | -8797.22 | -7777.54 | -7625.04 | -7478.40 |
| *52* | $\mathcal{Z}_{(2)}$ | 1594.01 | -8084.65 | -6490.63 | -6238.59 | -6000.95 |
| *53* | $\mathcal{Z}_{(3)}$ | 4773.16 | -5966.47 | -1193.30 | -1124.48 | -1060.84 |
| *54* | $\mathcal{Z}_{(4)}$ | 1968.98 | -5471.25 | -3502.26 | -3235.55 | -2993.75 |
| *55* | $\mathcal{Z}_{(5)}$ | 2211.37 | -5011.66 | -2800.29 | -2536.31 | -2301.63 |
| *56* | $\mathcal{Z}_{(6)}$ | 2396.30 | -4585.67 | -2189.36 | -1944.09 | -1730.28 |
| *57* | $\mathcal{Z}_{(7)}$ | 2530.70 | -4191.30 | -1660.60 | -1445.65 | -1261.92 |
| *58* | $\mathcal{Z}_{(8)}$ | 2620.93 | -3826.66 | -1205.73 | -1029.08 | -881.01 |
| *59* | $\mathcal{Z}_{(9)}$ | 2672.85 | -3489.91 | -817.05 | -683.67 | -574.05 |
| *60* | $\mathcal{Z}_{(10)}$ | 4410.52 | 33892.74 | 38303.27 | 31422.02 | 25876.31 |
| **Total** | | | | | **-4002.67** | **-7968.76** |

**ALM Risk of mass lapses** *Finally we want to look what happens when we have mass lapses as indicated before, but if we have invested in the cash flow matching portfolio according to average 7 % lapses. Hence we have to calculate the assets according to 7 % lapses and the liabilities according 25 % lapses.*

| Age | Unit | Units for Assets | Units for Liability | Total Units | Value $i = 2\%$ | Value $i = 4\%$ |
|---|---|---|---|---|---|---|
| *52* | $\mathcal{Z}_{(0)}$ | *-6490.63* | *6490.63* | *0* | *0* | *0* |
| *53* | $\mathcal{Z}_{(1)}$ | *-5354.72* | *1193.30* | *-4161.42* | *-4079.82* | *-4001.36* |
| *54* | $\mathcal{Z}_{(2)}$ | *-4356.47* | *3502.26* | *-854.21* | *-821.04* | *-789.76* |
| *55* | $\mathcal{Z}_{(3)}$ | *-3483.29* | *2800.29* | *-682.99* | *-643.60* | *-607.18* |
| *56* | $\mathcal{Z}_{(4)}$ | *-2723.35* | *2189.36* | *-533.99* | *-493.32* | *-456.45* |
| *57* | $\mathcal{Z}_{(5)}$ | *-2065.63* | *1660.60* | *-405.02* | *-366.84* | *-332.90* |
| *58* | $\mathcal{Z}_{(6)}$ | *-1499.81* | *1205.73* | *-294.08* | *-261.13* | *-232.41* |
| *59* | $\mathcal{Z}_{(7)}$ | *-1016.34* | *817.05* | *-199.28* | *-173.48* | *-151.43* |
| *60* | $\mathcal{Z}_{(8)}$ | *47645.53* | *-38303.27* | *9342.26* | *7973.53* | *6826.29* |
| **Total** | | | | | **1134.26** | **254.76** |

*Now we see that the lapses induce quite a big risk for the company since it loses in case of mass lapses almost 1 % of the face value of the policy, more concretely* $1134.26 - 254.76 = 879.50$.

The above example shows very clearly how the behaviour of the policyholders can change the cash flow matching portfolio and in consequence induces a risk. As a consequence the risk minimising portfolio in the sense of $VaPo^*(x)$ for an insurance portfolio $x \in \mathcal{X}$ does also consist of additional assets offsetting the corresponding risks. In the above example the corresponding asset would be a (complex) put option, which allows to sell the bond portfolio at the predefined (book-) values. So in reality insurance companies aim to model these risks in order to determine the corresponding assets and to reduce the undesired risk.

In the example above we have assumed that at a given year 25% of the policies in force lapse. In practise one models the dynamic lapse behaviours. Eg the lapse rate is a function of the interest differential between market and book yields. Normally the corresponding lapse rates stay below 1, which is interesting. Assuming a market efficient behaviour, one would expect that there is a binary decision of the policyholders to stick to the contract or to lapse as a funtion of the beforementioned interest differential. In consequence the underlying theory how to model such policyholder behaviour is not as crisp and transparent as with the arbitrage free pricing theory, since market efficient behaviours is normally not observed. As a corollary there is a lot of model risk intrinsic to these calculations and it is important to test the results from the models with different scenarios.

**Remark 139** *At the end of this section a remark on how to determine a* $VaPo^*(x)$ *for an* $x \in \mathcal{X}$ *concretely: One normally models an* $l \in \mathcal{X}$ *and simulates* $l(\omega)$ *together with some test assets* $D \subset \mathcal{Y}$ *observable prices and cash flows. We denote* $D = \{d_1, \dots d_n\}$. *Hence at the end of this process we have a vector*

$$\mathcal{W} := (l(\omega_i), d_1(\omega_i), \dots d_n(\omega_i))_{i \in I}.$$

*Now the process is quite canonical:*

1. *We define a distance between two $x, y \in \mathcal{X}$, for example by means of $||x||$ as defined.*
2. *We solve the numerical optimisation problem, for minimising the distance between $l$ and the target $y \in$ span $< \mathcal{D} >$.*

*We note two things:*

- *The numerical procedures to determine $y$ can sometimes prove to be difficult since the corresponding design matrix can be near to a singular matrix, and hence additional care is needed.*
- *In case of the $|| \bullet ||$ defined before, we remark that it has been deducted from the Hilbert space $\mathcal{X}$. Hence what we actually doing is to use the projection $\tilde{p} : \mathcal{X} \to$ span $< \mathcal{D} >$, which can be expressed by means of $< \bullet, \bullet >$. We remark that $y = \tilde{p}(x)$.*

## Appendix D
# An Introduction to Arbitrage Free Pricing

## D.1 Price Systems

In this section we want to provide an introduction to modern financial market theory. The purpose of the section is not to cover each possible detail, but rather to give an overview. For an in depth study we refer to [Pli97], [HK79], [HP81] and [Duf92].

This section would be incomplete without mentioning the work of Black and Scholes [BS73] with its famous formula for pricing equity options.

### *D.1.1 Definitions*

Firstly we need to explain the rationale for this theory. If we consider the market value of an equity price, which is modelled with a geometric Brownian motion $(S_t(\omega))$.

A European call-option is a security, which allows to purchase the underlying asset at a predefined price $c$ (strike price at a fixed time $T$. At time $T$ the value of this paper is known:

$$H = \max\left(S_T - c, 0\right).$$

An bank now wants to know the value or the price of this option at time $0$. If the price is chosen in a wrong way, such as by taking the expected present value with respect to the original measure, it is possible to make a risk free gain, and we speak about arbitrage.

We consider the simplest possible economy, considering finite models. This implies that we consider discrete time. The interested reader can find analogous results in continuous time for example in [HP81]. This section should therefore illustrate the ideas and concepts of the arbitrage free pricing theory.

We consider a probability space $(\Omega, \mathcal{A}, P)$ with $\Omega$ finite. Furthermore we assume that $P(\omega) > 0 \, \forall \omega \in \Omega$. We fix a finite time horizon $T$, at which all trading activities stop. With $\mathcal{F}_t$ we denote the $\sigma$-algebra of the – at time $t$ – observable outcomes. The securities are traded at times $\{0, 1, 2, \ldots, T\}$. We assume that there exist $k < \infty$ stochastic processes, which represent the development of the value of the securities $1, \ldots, k$.

$$S = \{S_t, t = 0, 1, 2, \ldots, T\} \text{ with components } S^0, S^1, \ldots S^k.$$

As usual we assume that each $S^j$ is adapted with respect to $(\mathcal{F}_t)_t$. We interpret $S_t^j$ as price of security $j$ at time $t$. The condition of adaptedness represents the necessity to know the trajectory of $S$ before time $t$, being at time $t$. The $0$. security has a particular role. We assume that $S_t^0 = (1 + r)^t$. This means that we can invest risk free at an interest rate of $r$. The risk free discount factor is hence given by:

$$\beta_t = \frac{1}{S_t^0}.$$

In a next step we want to understand the meaning of a trading strategy:

**Definition 140** *A trading strategy is a previsible ($\phi_t \in \mathcal{F}_{t-1}$) process $\Phi = \{\phi_t, t = 1, 2, \ldots, T\}$ with components $\phi_t^k$.*

*We interpret $\phi_t^k$ as number of security k, which we hold between $[t-1, t[$. This is the reason why $\phi_t$ is called portfolio at time t.*

**Definition 1.** Let $X, Y$ be two multi–dimensional stochastic processes. In this case we denote:

$$< X_s, Y_t > = X_s \cdot Y_t = \sum_{k=0}^{n} X_s^k \times Y_t^k,$$
$$\Delta X_t = X_t - X_{t-1}.$$

In a next step we want to determine the value of a portfolio at time $t$:

| Time | Value of the portfolio |
|---|---|
| $t-1$ | $\phi_t \cdot S_{t-1}$ |
| $t^-$ | $\phi_t \cdot S_t$ |

This shows that the profit in the interval $[t-1, t[$ amounts to $\phi_t \cdot \Delta S_t$. Hence we can calculate the total profit in the interval $[0, t]$ by:

$$G_t(\phi) = \sum_{\tau=1}^{t} \phi_\tau \cdot \Delta S_\tau.$$

We set $G_0(\phi) = 0$ can call $(G_t)_{t \geq 0}$ profit process.

**Proposition 141** *$G$ is an adapted, real valued stochastic process.*

*Proof.* We leave the proof as an exercise.

**Definition 142** *A trading strategy is called self–financing, if we have*

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \qquad \forall t = 1, 2, \ldots, T-1.$$

A self-financing strategy indicates that no money is injected or withdrawn from the portfolio at any time.

**Definition 143** *A trading strategy is admissible, if it is self–financing and is*

$$V_t(\phi) := \begin{cases} \phi_t \cdot S_t, \text{ falls } t = 1, 2, \ldots, T, \\ \phi_1 \cdot S_0, \text{falls } t = 0 \end{cases}$$

*positive. (With other words: we must not go bankrupt.) With $\Phi$ we denote the set of all admissible trading strategies.*

**Remark 144** *The idea of admissible trading strategies consists in the fact that we do not want to inject or withdraw money from the portfolio and we can only re-base the portfolio. If we could find trading strategies which have at the end always (eg $\forall \omega \in \Omega$) the same payout as an option, we would say that the value of the option is the value of the corresponding portfolio at inception.*

**Definition 145** *Under a contingency claim we understand a positive random variable $X$, The set of all contingency claims we denote by $\mathcal{X}$.*

*A random variable $X$ is attainable, if there exists an admissible trading strategy $\phi \in \Phi$ such that*

$$V_T(\phi) = X.$$

*In this case we say "$\phi$ generates $X$".*

**Definition 146** *For an attainable contingency claim $X$, which is generated by $\phi$ we denote by*

$$\pi = V_0(\phi)$$

*its price. (This price does not need to be unique, as we will see later, and it corresponds to the value of the corresponding portfolio at inception.)*

### *D.1.2 Arbitrage*

Under an arbitrage opportunity we understand

$$\phi \in \Phi \text{ with } V_0(\phi) = 0 \text{ and } V_T(\phi) \text{ positive and } P[V_T(\phi) > 0] > 0$$

(Money is created from nil). If such a strategy exists, we can create economic profit without any risk. One of the axioms of modern economy postulates the absence of such arbitrage opportunities. From this axiom we can deduct some important learning for determining the price

In a next step we want to understand the concept of a pricing system.

**Definition 147** *A map*

$$\pi : \qquad \mathcal{X} \to [0, \infty[, \qquad X \mapsto \pi(X)$$

*is called pricing system, if the following two conditions are fulfilled:*

- $\pi(X) = 0 \iff X = 0$,
- $\pi$ *is linear.*

*A pricing system is called consistent if the following holds:*

$$\pi(V_T(\phi)) = V_0(\phi) \qquad \text{for all } \phi \in \Phi.$$

*With $\Pi$ we denote the set of all consistent pricing systems. With $\mathbb{P}$ we denote the set*

$$\mathbb{P} = \{Q \text{ measure equivalent to } P, \text{ under which } \beta \times S \text{ is a martingale}\},$$

*where we denote by $\beta$ the discount factor from time $t$ to $0$. These measures are called equivalent martingale measures.*

**Proposition 148** *Between the sets of consistent pricing systems $\pi \in \Pi$ and the measures $Q \in \mathbb{P}$ there exists a bijection, given by*

1. $\pi(X) = \mathbb{E}^Q[\beta_T X]$.
2. $Q(A) = \pi(S_T^0 \chi_A)$ *for all* $A \in \mathcal{A}$.

*Proof.* For $Q \in \mathbb{P}$ we define $\pi(X) = \mathbb{E}^Q[\beta_T X]$. $\pi$ is a pricing system, since $P$ is strictly positive on $\Omega$ and $Q$ is equivalent to $P$. It remains to show that $\pi$ is consistent. To this end let $\phi \in \Phi$. In this case we have the following:

$$\begin{aligned} \beta_T V_T(\phi) &= \beta_T \phi_T S_T + \sum_{i=1}^{T-1} (\phi_i - \phi_{i+1}) \beta_i S_i \\ &= \beta_1 \phi_1 S_1 + \sum_{i=2}^{T} \phi_i \left(\beta_i S_i - \beta_{i-1} S_{i-1}\right), \end{aligned}$$

where we have used that $\phi$ is self–financing. Hence

$$
\begin{aligned}
\pi\left(V_T(\phi)\right) &= \mathbb{E}^Q\left[\beta_T\, V_T(\phi)\right] \\
&= \mathbb{E}^Q\left[\beta_1\, \phi_1\, S_1\right] + \mathbb{E}^Q\left[\sum_{i=2}^{T} \phi_i\, \left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right)\right] \\
&= \mathbb{E}^Q\left[\beta_1\, \phi_1\, S_1\right] + \sum_{i=2}^{T} \mathbb{E}^Q\left[\phi_i\, \mathbb{E}^Q\left[\left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right) | \mathcal{F}_{i-1}\right]\right] \\
&= \phi_1\, \mathbb{E}^Q\left[\beta_1\, S_1\right] \\
&= \phi_1\, \beta_0\, S_0,
\end{aligned}
$$

where we have used that $\phi$ is previsible and that $\beta\, S$ is a martingale under $Q$. Hence we have proofed, that $\pi$ is a consistent pricing system.

Let $\pi \in \Pi$ be a consistent pricing system and define $Q$ as above. In this case we have $Q(\omega) = \pi(S_t^0\, \chi_{\{\omega\}}) > 0$, for all $\omega \in \Omega$, since $S_t^0\, \chi_{\{\omega\}} \neq 0$. Moreover we have $\pi(X) = 0 \iff X = 0$ and hence $Q$ is absolutely continuous with respect to $P$.

In a next step we need to show that $Q$ is a probability measure. To this end we define

$$
\phi^0 = 1 \qquad \text{and} \qquad \phi^k = 0 \quad \forall k \neq 0.
$$

Since $\pi$ is consistent we have

$$
\begin{aligned}
1 &= V_0(\phi) \\
&= \pi(V_T(\phi)) \\
&= \pi(S_T^0 \cdot 1) \\
&= Q(\Omega).
\end{aligned}
$$

Since the prices of positive contingency claims are positive and since $Q$ is additive, we can show Kolmogorov's axioms, since $\Omega$ is finite. Per definition we have $Q(\omega) = \pi(S_T^0 \cdot \chi_{\{\omega\}})$ and hence also

$$
\mathbb{E}[f] = \sum_{\omega} \pi(S_T^0 \cdot \chi_{\{\omega\}}) \cdot f(\omega) = \pi(S_T^0 \cdot \sum_{\omega} f(\omega)).
$$

For $f = \beta_T\, X$ we hence have

$$
\mathbb{E}^Q[\beta_T\, X] = \pi(S_T^0 \cdot \beta_T \cdot X) = \pi(X).
$$

It remains to show that $\beta_T\, S_T^k$ is a martingale for all $k$. Let $k$ be a coordinate and $\tau$ a stopping time. We define

$$
\begin{aligned}
\phi_t^k &= \chi_{\{t \leq \tau\}}, \\
\phi_t^0 &= \left(S_\tau^k / S_\tau^0\right)\, \chi_{\{t > \tau\}}.
\end{aligned}
$$

(We hold security $k$ until time $\tau$ and invest the proceeds into the risk free asset.) it is easy to show that this strategy $\phi$ is both self–financing and previsible. The following equations hold:

$$V_0(\phi) = S_0^k,$$
$$V_T(\phi) = \left(S_\tau^k / S_\tau^0\right) S_T^0$$

and moreover

$$\begin{aligned} S_0^k &= \pi(S_T^0 \cdot \beta_\tau \cdot S_\tau^k) \\ &= \mathbb{E}^Q\left[\beta_\tau \cdot S_\tau^k\right]. \end{aligned}$$

Since the above equation is valid for an arbitrary stopping time $\tau$, we know that $\beta_T\, S_T^k$ is a martingale with respect to $Q$.

After this important theorem we want to list some properties without proof and we refer to [HP81] for details.

**Theorem 149** *The following three statements are equivalent:*

1. *The market model does not allow for arbitrage,*
2. $\mathbb{P} \neq \emptyset$,
3. $\Pi \neq \emptyset$.

**Lemma 4.** *If there exists a self–financing trading strategy $\phi \in \Phi$ with*

$$V_0(\phi) = 0,\, V_T(\phi) \geq 0,\, \mathbb{E}[V_T(\phi)] > 0$$

*the market model allows arbitrage.*

### *D.1.3 Continuous case*

We assume that $\mathbb{P} \neq \emptyset$.

In a next step we need to define the different concepts:

**Definition 150** • *A trading strategy $\phi$ is a locally bounded, previsible stochastic process.*

- *The value process with respect to a trading strategy $\phi$ is given by*

$$V : \Pi \to \mathbb{R}, \phi \mapsto V(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{k} \phi_t^k \cdot S_t^k.$$

- *The gain process $G$ is defined by*

$$G : \Pi \to \mathbb{R}, \phi \mapsto G(\phi) = \int_0^\tau \phi \, dS = \int_0^\tau \sum_{i=0}^{k} \phi^k dS^k.$$

- *$\phi$ is self–financing, if $V_t(\phi) = V_0(\phi) + G_t(\phi)$.*
- *In order to define admissible trading strategies we need the following notation:*

$$\begin{aligned} Z_t^i &= \beta_t \cdot S_t^i, && \textit{discounted value of security } i \\ G^*(\phi) &= \int \sum_{i=1}^{k} \phi^i \, dZ^i, && \textit{discounted profit} \\ V^*(\phi) &= \beta \, V(\phi) = \phi^0 + \sum_{i=1}^{k} \phi^i \, Z^i. \end{aligned}$$

*A trading strategy is called admissible if it has the following three properties:*

1. *$V^*(\phi) \geq 0$,*
2. *$V^*(\phi) = V^*(\phi)_0 + G^*(\phi)$,*
3. *$V^*(\phi)$ is a martingale under $Q$.*

**Proposition 151** *1. The price of a contingency claim $X$ is given by $\pi(X) = \mathbb{E}^Q[\beta_T \, X]$.*

*2. A contingency claim is attainable $\iff$ $V^* = V_0^* + \int H dZ$ for all $H$.*

**Definition 152** *The market is complete if all integrable contingency claims are attainable.*

Since the arbitrage free pricing theory is very important we would suggest that the reader familiarised with it.

## D.2 The Black-Scholes set up

As we have seen in the previous section, we need to pick an economic model for the calculation of option prices and we remark that there are in principle different possible choices. In this section we want to look at the Black-Scholes set up and we would like to refer to the following sources for additional information: [Dot90], [Duf88], [Duf92], [CHB89], [Per94], [Pli97].

**Definition 153 (General conventions for Black-Scholes set up)** *For the remainder of this chapter we use the following notation and conventions:*

- *$T_x$ denotes the future life span of a person of age $x$.*
- *With $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right)$ we denote the $\sigma$-algebras induced by $T_x$.*
- *For the assets we assume that their value develops according to a standardised Brownian motion $W$.*
- *With $\mathcal{G}_t$ we denote the $\sigma$-algebras induced by $W$, enlarged by all $P$-null sets.*

**Definition 154 (Independence of financial variables)** • *We assume that$\mathcal{G}_t$ and $\mathcal{H}_t$ stochastically independent. Hence we assume that the financial variables are independent of the future life span of the considered persons.*

- *With $\mathcal{F}_t = \sigma\left(\mathcal{G}_t, \mathcal{H}_t\right)$ we denote the $\sigma$-algebra generated by $\mathcal{G}_t$ and $\mathcal{H}_t$.*

**Definition 155 (Black-Scholes-Model)** *In this model the market consists of two investment vehicles:*

| | |
|---|---|
| $B(t) = \exp(\delta\, t)$ | *Risk free asset.* |
| $S(t) \; = S(0)\, \exp\left[\left(\eta - \frac{1}{2}\sigma^2\right)\, t \, + \, \sigma\, W(t)\right]$ | *Units, modelled by a geometric Brownian motion .* |

$S$ solves the following stochastic differential equation (SDE):

$$dS = \eta\, S\, dt \, + \, \sigma\, S\, dW.$$

**Exercise 156** *Proof the above SDE.*

In a next step we need to calculate the discounted values of $B$ and $S$:

$$B^*(t) = \tfrac{B(t)}{B(t)} = 1,$$
$$S^*(t) = \tfrac{S(t)}{B(t)} = S(0)\, \exp\left[\left(\eta - \delta - \tfrac{1}{2}\sigma^2\right)\, t \, + \, \sigma\, W(t)\right].$$

After the definition of the investment possibilities, we need to calculate the equivalent martingale measures in order to be able to calculate the corresponding option prices' Hence we need to determine a equivalent measure $Q$ such that $S^*$ is a martingale with respect to $Q$. In order to do this we define the following Radon-Nikodym-density:

$$\xi_t = \exp\left(-\frac{1}{2}\left(\frac{\eta - \delta}{\sigma}\right)^2 t - \frac{\eta - \delta}{\sigma}\, W(t)\right) \qquad \text{for all } t \in [0, T].$$

**Exercise 157** *Proof the following properties*

1. $\mathbb{E}[\xi_t] = 1$,
2. $Var[\xi_t] = \exp\left(\left(\frac{\eta-\delta}{\sigma}\right)^2 t\right) - 1$,
3. $\xi_t > 0$.

*(Remark: $W(t) \sim \mathcal{N}(0,t)$.)*

As a consequence of a corollary of the Girsanov-Theorem (see for example [Pro90] Theorem 3.6.21) it follows that

$$\hat{W}_t = W(t) + \frac{\eta - \delta}{\sigma}\, t$$

is a standardised Brownian motion under $Q = \xi \cdot P$ .

After the introduction of this transformation we want to show that

$$S^*(t) = S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, \hat{W}(t)\right)$$

is a $Q$–martingale. (In this case we can calculate the prices of options as expected values with respect to $Q$.)

*Proof.* For $t, u \in \mathbb{R}, u > t$ we need to show the following equality:

$$\mathbb{E}^Q\left[S^*(u)|\mathcal{F}_t\right] = S^*(t).$$

We use the following notation: $u = t + \Delta t$, $W_u = W_t + \Delta W$ and $Z \sim \mathcal{N}(0,1)$.

$$\begin{aligned}
&\mathbb{E}^Q\left[S^*(u)|\mathcal{F}_t\right] \\
&= \mathbb{E}^Q\left[S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, \hat{W}(t) + (-\frac{1}{2}\sigma^2\, \Delta t + \sigma\, \Delta\hat{W})\right)|\mathcal{F}_t\right] \\
&= S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, W(t)\right)\, \mathbb{E}^Q\left[\exp\left(-\frac{1}{2}\,\sigma^2 \Delta t + \sigma\sqrt{\Delta t} Z\right)|\mathcal{F}_t\right] \\
&= S^*(t).
\end{aligned}$$

We have hence shown that $Q$ is an equivalent measure to $P$, with respect to which $S^*$ is a martingale. With the words of the economist: there exists at least one equivalent pricing system.

**Theorem 158** *In the above defined economy given by $(\Omega, \mathcal{A}, P)$, $S$ and $B$, we can calculate the price of a mortality benefit $C(T)$ at time t by*

$$\pi_t(T) = \mathbb{E}^Q\left[\exp\left(-\delta(T-t)\right) C(T)|\mathcal{F}_t\right].$$

**Remark 159** *The important difference in respect to the classical model is that we take expected values with respect to $Q$ and not with respect to $P$.*

In the following we want to have a look at unit–linked insurance policies with guarantees. We use the following notation:

| | |
|---|---|
| $C(\tau)$ | Insured sum at time $\tau$, |
| $N(\tau)$ | Number of units at time $\tau$, |
| $S(\tau)$ | price of the units at time $\tau$, |
| $G(\tau)$ | Guaranteed amount at time $\tau$, |
| $C(\tau) = \max\{N(\tau)S(\tau), G(\tau)\}$ | Amount insured. |

### *D.2.1 Endowment Policies*

**Proposition 160** *Given the Black-Scholes-model. In this case we can calculate the single premium for a pure endowment policy with*

$$C(T) = \max\{N(T)S(T), G(T)\}$$

*by*

$$_TG_x = {}_Tp_x \left[G(T)\, \exp(-\delta T)\Phi(-d_2^0(T)) + S(0)N(T)\Phi(d_1^0(T))\right],$$

*where*

$$\begin{aligned}
\Phi(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{x^2}{2})dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s > t), \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s > t).
\end{aligned}$$

*Proof.* In the following we always denote by $J^*$ the discounted value of the random variable $J$. The value of the endowment policy at time 0 amounts to $\mathbb{E}^Q[C^*(T)]$. If we denote by $Z = S^*(T)$, we get the following:

$$_TG_x = {}_Tp_x\, \mathbb{E}^Q\left[\max\{N(T)Z, G^*(T)\}\right]$$

and

$$Z = S(0)\, \exp\left(-\frac{1}{2}\sigma^2 T + \sigma \hat{W}(T)\right) \qquad \text{with} \qquad \hat{W}(T) \sim \mathcal{N}(0,T).$$

And hence we get

$$
\begin{aligned}
{}_TG_x &= {}_Tp_x \int_{-\infty}^{\infty} \max\left[N(T)S(0)\exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi), G^*(T)\right] f(\xi)d\xi,\\
f(\xi) &= \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}\xi^2\right).
\end{aligned}
$$

In a next step we set $\bar{\xi} = \frac{1}{\sigma}\left[\ln\left(\frac{G^*(T)}{N(T)S(0)}\right) + \frac{1}{2}\sigma^2 T\right]$ and remark that for all $\xi > \bar{\xi}$ we have $N(T)Z > G^*(T)$. Hence we can calculate the corresponding single premium as follows:

$$
\begin{aligned}
{}_TG_x = {}_Tp_x \Bigg( & G^*(T) \int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\
&+ N(T)S(0) \int_{\bar{\xi}}^{\infty} \exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi) f(\xi)d\xi \Bigg)\\
= {}_Tp_x \Bigg( & G^*(T) \int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\
&+ N(T)S(0) \int_{\bar{\xi}}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp(-\frac{1}{2T}(\xi - \sigma T)^2 d\xi \Bigg).
\end{aligned}
$$

From the above equation the desired result follows.

### *D.2.2 Term Insurance*

**Proposition 161** *Given the Black-Scholes-Model, we can calculate the net single premium for a lump sum*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

*by*

$$G^1_{x:T} = \int_0^T \left(G(t)\exp(-\delta t)\varPhi(-d_2^0(t)) + S(0)N(t)\varPhi(d_1^0(t))\,{}_tp_x\mu_{x+t}\right) dt,$$

*where*

$$
\begin{aligned}
\varPhi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx,\\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}
$$

$$d_2^t(s) = \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},$$

*for* $s > t$.

**Exercise 162** *Proof the above proposition using the methods learnt for the endowment cover.*

## D.3 Thiele's Differential Equation

In order to deduct Thiele's differential equation we need to introduce premium payments in a first step. By $\bar{p}(t)$ we denote the premium density at time $t$. As a consequence of the equivalence principle we have the following two equations:

$$_TG_x = \int_0^T \bar{p}(t)\exp(-\delta\, t)_t p_x dt,$$

respectively

$$G_{x:T}^1 = \int_0^T \bar{p}(t)\exp(-\delta\, t)_t p_x dt.$$

In this section we want to treat the pure endowment cover and the term insurance separately. For these two types of insurance cover the mathematical reserves are given as follows:

$$\begin{aligned}
\text{Endowment: } V(t) &= {}_{T-t}p_{x+t}\pi_t(T) \\
&\quad - \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}\, d\xi. \\
\text{Term insurance: } V(t) &= \int_t^T (\pi_t(\xi)\mu_{x+\xi} - \bar{p}(\xi)\exp(-\delta(\xi-t))) \\
&\quad \times_{\xi-t}p_{x+t}\, d\xi,
\end{aligned}$$

where

$$\begin{aligned}
\pi_t(s) &= G(s)\exp(-\delta(s-t))\Phi(-d_2^t(s)) \\
&\quad + N(s)S(t)\Phi(d_1^t(s)), \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}$$

$$d_2^t(s) \quad = \quad \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},$$

for $s > t$.

**Remark 163** • *In contrast to the classical case the reserves are not anymore deterministic, but crucially depend on the value of the underlying asset $S$.*

- *We remark that we need now to apply Itô-calculus, where we have for the pure continuous for a Brownian motion the following:*

$$df(W) = f'\,dW + \frac{1}{2}f''\,ds.$$

For the two insurance types we have the following theorem:

**Theorem 164** *1. The differential equation for the market value of a pure endowment policy is given by:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\,V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}.$$

*2. The differential equation for the market value of a term insurance policy is given by:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\,V(t) - C(t)\mu_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}.$$

Before proofing the theorem we would like to add some comments:

**Remark 165** *1. For $\mu_{x+t} = \bar{p}(t) = 0\,\forall t$ we get the Black-Scholes-formula.*

*2. The first term of the above differential equation correspond to the classical set–up. This relates to the dependence on premiums, mortality and interest. As a consequence of the fluctuation of the value of the units $S$ we get an additional term: $-\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}$.*

*Proof.* Since

$$\pi_t^*(T) = \exp(-\delta t)\pi_t(T),$$

we get the following equation as a consequence of the definition of $V$:

$$V(t) = {}_{T-t}p_{x+t}\pi_t^*(T)\exp(\delta t) - \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}d\xi$$

and hence

$$\pi_t^*(T) = \Psi(t)\left[V(t) + \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t}d\xi\right],$$

where

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

Since $\pi_t^*$ is a function of $S$ and $t$, we can apply the Itô-formula on the function $\pi_t^*(t,S)$:

$$\begin{aligned}
dY_t &= U(t+dt, X_t + dX_t) - U(t, X_t) \\
&= \left(U_t dt + \frac{1}{2} U_{xx} b^2 dt\right) + U_x dX_t \\
&= \left(U_t + \frac{1}{2} U_{xx} b^2\right) dt + U_x \, b \, dB_t
\end{aligned}$$

and we get:

$$d\pi^* = \left(\frac{\partial \pi^*}{\partial t} + \frac{\partial \pi^*}{\partial S}\, a + \frac{1}{2}\, \frac{\partial^2 \pi^*}{\partial S^2}\, b^2\right) dt + \frac{\partial \pi^*}{\partial S}\, b \, d\hat{W},$$

knowing that

$$dS = \delta S(t) dt + \sigma S(t) d\hat{W}.$$

Hence we have $a = \delta S(t)$ and $b = \sigma S(t)$. In a next step we want to determine the different terms for the above formula:

$$\begin{aligned}
\frac{\partial \pi_t^*}{\partial S} &= \Psi(t)\, \frac{\partial V}{\partial S}, \\
\frac{\partial^2 \pi_t^*}{\partial S^2} &= \Psi(t)\, \frac{\partial^2 V}{\partial S^2}.
\end{aligned}$$

In order to calculate $\frac{\partial \pi^*}{\partial t}$ , we first calculate:

$$\begin{aligned}
\frac{\partial}{\partial t} {}_{\xi-t}p_{x+t} &= \mu_{x+t}\, {}_{\xi-t}p_{x+t}, \\
\frac{\partial}{\partial t} \Psi(t) &= \left(\frac{A}{B}\right)' = \frac{A'}{B} - \frac{A}{B^2}\, B' \\
&= -\left(\mu_{x+t} + \delta\right) \Psi(t).
\end{aligned}$$

If we now compile the different parts we get:

$$\begin{aligned}
\frac{\partial \pi^*}{\partial t} &= \frac{\partial \Psi}{\partial t} \left(V(t) + \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt\right) \\
&\quad + \Psi(t) \left(\frac{\partial V}{\partial t} + \frac{\partial}{\partial t} \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt\right) \\
&= \Psi(t) \left(\frac{\partial V}{\partial t} - \left(\mu_{x+t} + \delta\right) V(t) - \bar{p}(t)\right),
\end{aligned}$$

by using the chain rule

$$\frac{\partial}{\partial t}\int_t^T \bar{p}(\xi)\exp(-\delta(\xi-t))_{\xi-t}p_{x+t}dt.$$

Finally we get:

$$\begin{aligned}\pi_s^*(T) = \pi_t^*(T) &+ \int_t^s \Psi(\xi)\frac{\partial V}{\partial S}\sigma S d\hat{W}(\xi)\\ &+ \int_t^s \Psi(\xi)\left[\frac{\partial V}{\partial S}\delta S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - (\mu_{x+\xi}+\delta)V(\xi)\right.\\ &\qquad\qquad\left. + \frac{\partial V}{\partial t}(\xi)\bar{p}(\xi)\right]d\xi.\end{aligned}$$

Sine $\pi_\cdot^*(T)$ is a martingale, the drift term is zero, and get the required result:

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t}+\delta)\,V(t) - \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S^2} - \delta\,S(t)\frac{\partial V}{\partial S}.$$

**Exercise 166** *Proof the second part of the theorem.*

# Appendix E

# An Introduction to Stochastic Integration

The aim of this appendix is to provide all necessary definition and results with respect to Martingales and stochastic integration. Since some of the underlying properties and theorems require a lot of advanced mathematics, we do not aim to proof the different theorems. For valuable literature we refer to [Pro90] and [IW81].

## E.1 Stochastic Processes and Martingales

**Definition 167** *A probability space* $(\Omega, \mathcal{A}, P)$ *is called filtered, if* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *there exists a family of* $\sigma-$*algebras with*

1. $\mathcal{F}_0 \supset \{A \in \mathcal{A} | P(A) = 0\}$,
2. $\mathcal{F}_s \subset \mathcal{F}_t$ *for* $s \leq t$.

*The filtration is called continuous from the right side, if* $\mathcal{F}_t = \bigcap_{t'>t} \mathcal{F}_{t'}, \forall t \geq 0$.

**Definition 168** *A random variable* $T : \Omega \rightarrow [0, \infty]$ *is called* stopping time, *if* $\{T \leq t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$.

**Proposition 169** $T$ *is a stopping time if and only if* $\{T < t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$. *([Pro90] Thm. 1.1.1.)*

**Definition 170** *Let* $X, Y$ *be two stochastic processes.* $X$ *and* $Y$ *are called* modifications *if*

$$X_t = Y_t \qquad P\text{–almost everywhere}\forall t.$$

$X$ *and* $Y$ *are called* identical,

$$X_t = Y_t, \forall t \qquad P\text{–almost everywhere}.$$

**Definition 171** *1. A stochastic process is called càdlàg (continue à droite, limites à gauche), if its trajectories are right-continuous, with limits from the left.*

*2. A stochastic process is called càglàd its trajectories are left-continuous, with limits from the right.*

*3. A stochastic process is called* adapted*, if $X_t \in \mathcal{F}_t$ ($X_t$ is $\mathcal{F}_t$-measurable).*

**Proposition 172** *1. Let $\Lambda$ be an open set and $X$ an adapted càdlàg-process. In this case is $T := \inf\{t \in \mathbb{R}_+ : X_t \in \Lambda\}$ a stopping time.*

*2. Let $S, T$ be two stopping times and $\alpha > 1$. In this case the following random variables are also stopping times $\min(S, T)$, $\max(S, T)$, $S + T$, $\alpha \cdot T$.*

*Proof.* [Pro90] Thm. 1.1.3 and Thm. 1.1.5.

**Definition 173** *Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtrated Probability space. A stochastic process is called Martingale, if*

- *$X_t \in L^1(\Omega, \mathcal{A}, P)$, d.h. $\mathbb{E}[|X_t|] < \infty$,*
- *For $s < t$ follows $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$.*

**Remark 174** *I we replace "$=$" in the above equation by "$\leq$" (resp. "$\geq$"), $X$ is called Super-martingale (resp. Sub-martingale).*

**Theorem 175** *Let $X$ be a super-martingale. In this case the following conditions are equivalent*

*1. The map $T \to \mathbb{R}, t \mapsto \mathbb{E}[X_t]$ is right continuous.*

*2. There exists a unique modification $Y$ of $X$, which is càdlàg.*

*Proof.* [Pro90] Thm. 1.2.9.

**Proposition 176** *Let $X$ be a martingale. In this case there exists a unique modification $Y$ of $X$, which is càdlàg.*

**Theorem 177 (Doob's stopping theorem)** *Let $X$ be a right-continuous martingale, which is closed by $X_\infty$, i.e. $X_t = \mathbb{E}[X_\infty | \mathcal{F}_t]$. Moreover let $S$ and $T$ be two stopping times with $S \leq T$ P–a.e. Then we have the following:*

*1. $X_S,\ X_T \in L^1(\Omega, \mathcal{A}, P)$,*

*2. $X_S = \mathbb{E}[X_T | \mathcal{F}_S]$.*

*Proof.* [Pro90] Thm. 1.2.16.

**Definition 178** *Let $X$ be a stochastic process and $T$ a stopping time. With $(X_t^T)_{t\geq 0}$ we denote the stopped stochastic process $X_t^T$, defined by $X_t^T = X_{\min(t,T)}$ for $t \geq 0$.*

**Theorem 179 (Jensen-inequality)** *Let $\phi : \mathbb{R} \to \mathbb{R}$ a convex function and $X \in L^1(\Omega, \mathcal{A}, P)$ with $\phi(X) \in L^1(\Omega, \mathcal{A}, P)$. Moreover let $\mathcal{G}$ be a $\sigma$-algebra. In this case we have the following inequality*

$$\phi \circ \mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

*Proof.* [Pro90] Thm. 1.2.19.

## E.2 Stochastic Integral

The aim of this section is to provide a short introduction into the theory of stochastic integrals. We closely follow [Pro90].

In principle we can consider the stochastic integration of semi-martingales as trajectory wise Stieltjes-integration, as we know them from typical lectures in analysis. The idea is to form the integral as a limit of sums of the form

$$\sum f(T_k)\ (T_{k+1} - T_k)$$

for finer re-participations. In the following let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtrated probability space satisfying the usual regularity conditions.

**Definition 180** *1. A stochastic process $H$ is called simple predictable, if it is of the form*

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i,T_{i+1}]}(t)$$

*with*

$$0 = T_1 \leq \ldots \leq T_{n+1} < \infty$$

*a finite family of stopping times and $H_i \in \mathcal{F}_t$, $(H_i)_{i=0,\ldots,n}$ P-a.e finite.*

*With $\mathbb{S}$ we denote the set of all simple predictable stochastic processes and with $\mathbb{S}_u$ the set $\mathbb{S}$, equipped with the topology of uniform convergence in $(t, \omega)$ on $\mathbb{R} \times L^\infty(\Omega, \mathcal{A}, P)$.*

2. *With $\mathbb{L}^0$ we denote the vector space of all finite, real-valued random variables, equipped with the topology induced by the convergence in probability.*

In a next step we define a sense for the expression $\int H\,dX$ for certain processes $(X_t)_{t\in\mathbb{R}}$ and $(H_t)_{t\in\mathbb{R}}$. In order that such an operator $I_X$ devotes the name integral, we would expect that it is linear and fulfils a sort of the Lebesgue theorem.

We require for the convergence theorem the following continuity: If $H^n$ converges uniformly to $H$, we require that $I_X(H^n)$ converges in probability to $I_X(H)$.

For a stochastic process $X$ we then define $I_X : \mathbb{S} \to \mathbb{L}^0$ as follows:

$$I_X(H) = H_0\,X_0 + \sum_{i=1}^{n} H_i \left(X^{T_i} - X^{T_{i+1}}\right),$$

where

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i,T_{i+1}]}(t).$$

The above definition of $I_X(H)$ is independent from its representation of $H$.

**Definition 181 (Total Semimartingale)** *A stochastic process $(X_t)_{t\geq 0}$ is called total Semi-martingale, if we have the following:*

1. *$X$ càdlàg and*
2. *$I_X$ is a continuous map from $\mathbb{S}^u$ to $\mathbb{L}^0$.*

**Definition 182 (Semimartingale)** *A stochastic process $(X_t)_{t\geq 0}$ is called Semi-martingale, if $X^t$ (cf. definition 178) is a total semi-martingale for all $t \in [0,\infty[$.*

**Remark 183** *Semi-martingales are hence defined as "good" integrators.*

The following proposition summarises the most important properties of the operator $I_X$:

**Proposition 184** *1. The set of all semi-martingales is a vector space.*

2. *Let $Q$ be a measure which is absolutely continuous with respect to $P$. In this case each $P$-semi-martingale is also a $Q$-semi-martingale.*
3. *For a sequence $(P_n)_{n\in\mathbb{N}}$ a probability measures, for which $(X_t)_{t\geq 0}$ is a $P_n$-semi-martingale, we define $R = \sum_{n\in\mathbb{N}} \lambda_n\,P_n$, with $\sum_{n\in\mathbb{N}} \lambda_n = 1$. In this case $(X_t)_{t\geq 0}$ is also $R$-semi-martingale.*
4. *(Stricker's Theorem) Let $X$ be semi-martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $(\mathcal{G}_t)_{t\geq 0}$ a sub-filtration of $(\mathcal{F}_t)_{t\geq 0}$ such that $X$ is adapted with respect to $(\mathcal{G}_t)_{t\geq 0}$. In this case $X$ is a $\mathcal{G}$-semi-martingale.*

*Proof.* The above properties follow from the definition of semi-martingales. The proofs can be found in [Pro90] chapter II.2.

In a next step we want to characterise the class of semi-martingales.

**Theorem 185** *Each adapted process with càdlàg-paths and finite variation on compact sets is a semi-martingale.*

*Proof.* This proposition follows from the fact that

$$|I_X(H)| \leq ||H||_u \int_0^\infty |dX_s|,$$

where we denote with $\int_0^\infty |dX_s|$ the total variation.

**Theorem 186** *Each quadratic integrable martingale with càdlàg-paths is a semi-martingale.*

*Proof.* Let $X$ be a quadratic integrable martingale with $X_0 = 0$, $H \in \mathbb{S}$. In order to show the continuity of the operator $I_X$, it is sufficient to proof the following inequality:

$$\begin{aligned}
E\left[(I_X(H))^2\right] &= E\left[\left(\sum_{i=0}^n H_i\left(X^{T_i} - X^{T_{i+1}}\right)\right)^2\right] \\
&= E\left[\sum_{i=0}^n H_i^2\left(X^{T_i} - X^{T_{i+1}}\right)^2\right] \\
&\leq ||H||_u^2\, E\left[\sum_{i=0}^n \left(X^{T_i} - X^{T_{i+1}}\right)^2\right] \\
&= ||H||_u^2\, E\left[\sum_{i=0}^n \left({X^{T_i}}^2 - {X^{T_{i+1}}}^2\right)\right] \\
&= ||H||_u^2\, E\left[{X_{T^{n+1}}}^2\right] \\
&\leq ||H||_u^2\, E\left[{X_{T^\infty}}^2\right].
\end{aligned}$$

**Example 187** *The Brownian motion is a semi-martingale.*

After characterising semi-martingales we want to enlarge in a next step the class of integrands. A very suitable class are càglàd-processes. We choose them in order that the proofs remain relatively simple.

**Definition 188** *With $\mathbb{D}$ (resp. $\mathbb{L}$) we denote the set of all adapted càdlàg (resp. càglàd)-processes. With $b\mathbb{L}$ we denote all $X \in \mathbb{L}$, with bounded paths.*

Until now we have seen the topology of uniform convergence (on $\mathbb{S}_u$) and the topology of convergence in probability on $\mathbb{L}^0$. we introduce another topology:

**Definition 189** *For $t \geq 0$ and a stochastic process $H$ we define*

$$H_t^* = \sup_{0 \leq s \leq t} |H_s|.$$

*A sequence $(H^n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets in probability (we refer this topology as ucp-topology) to $H$, if*

$$(H^n - H)_t^* \to 0$$

*in probability for $n \to \infty$ and all $t \geq 0$.*

*With $\mathbb{D}_{ucp}$, $\mathbb{L}_{ucp}$ and $\mathbb{S}_{ucp}$ we denote the respective sets, equipped with the above defined topology.*

**Remark 190** *1. The ucp-topology can be defined by a metric. An equivalent metric is for example:*

$$d(X,Y) = \sum_{i=1}^{\infty} \frac{1}{2^n} E\left[\min(1, (X-Y)_n^*\right].$$

*2. $\mathbb{D}_{ucp}$ is a complete metric space.*

in order to extend $I_X$, we need the following theorem:

**Theorem 191** *The vector space $\mathbb{S}$ is dense with respect to the ucp-topology in $\mathbb{L}$.*

*Proof.* [Pro90] Thm. 2.4.10.

If we now can show that $I_X$ is continuous, we can extend $I_X$. In order to do this, we define:

**Definition 192** *For $H \in \mathbb{S}$ and $X$ a semi-martingale we define $J_X : \mathbb{S} \to \mathbb{D}$ by*

$$J_X(H) = H_0\, X_0 + \sum_{i=0}^{n} H_i \left(X^{T_i} - X^{T_{i+1}}\right),$$

*where*

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i, T_{i+1}]}(t).$$

*With $H_i \in \mathcal{F}_{T_i}$, $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$ stopping times.*

**Definition 193 (Stochastic Integral)** *For $H \in \mathbb{S}$ and $X$ a càdlàg-process we call $J_X(H)$ stochastic integral of $H$ with respect to $X$ and denote*

$$H \cdot X := \int H_s \, dX_s := J_X(H).$$

After the definition of the stochastic integral $\mathbb{S}$, we want to extend it to $\mathbb{L}$. In order to do that we need the following theorem:

**Theorem 194** *For a semi-martingale $X$ the map $J_X : \mathbb{S}_{ucp} \to \mathbb{D}_{ucp}$ is continuous. The extension of $J_X$ on $\mathbb{S}_{ucp}$ is also called a stochastic integral and we use the notation introduced in definition 193.*

*Proof.* [Pro90] Thm. 2.4.11.

**Remark 195** *In order to extend $J_X$ to $\mathbb{D}$, we use the fact that the space $\mathbb{D}_{ucp}$ is a complete metric space.*

With

$$H \cdot X_{\,t} := \int_0^t H_s \, dX_s := \int_{[0,t]} H_s \, dX_s$$

denote the stochastic process $J_X(H) = \int H_s \, dX_s$, at $t \geq 0$.

## E.3 Properties of the Stochastic Integral

After having defined the stochastic integral we want to have a look at its properties.

**Proposition 196** *1. Let $T$ be a stopping time. In this case we have $(H \cdot X)^T = H \cdot \chi_{[0,T]} \cdot X = H \cdot X^T$.*

*2. Let $G, H \in \mathbb{L}$ and $X$ a semi-martingale. in this case $Y := H \cdot X$ is also a semi-martingale. Moreover we have:*

$$G \cdot Y = G \cdot (H \cdot X) = (G \cdot H) \cdot X.$$

*Proof.* [Pro90] Thm.2.5.12 and 2.5.19.

**Definition 197** *For a càdlàg-process $X$ we denote*

$$X_-(t) = \lim_{s \uparrow t} X(s),$$
$$\Delta X(t) = X(t) - X_-(t).$$

**Definition 198** *A random partition $\sigma$ of $\mathbb{R}$ is a finite sequence of stopping times with*

$$0 = T_0 \leq T_1 \leq \ldots \leq T_n < \infty.$$

*A sequence $(\sigma_n)_{n\in\mathbb{N}}$ of random partitions of $\mathbb{R}$ converges to the identity, if the following conditions are fulfilled:*

1. $\lim_{n\to\infty} (\sup_k T_k^n) = \infty$ *$P$-a.e.,*
2. $||\sigma_n|| := \sup_k |T_{k+1}^n - T_k^n|$ *converges $P$-a.e. to* $0$.

*For a process $Y$ and a random partition $\sigma$ we define*

$$Y^\sigma := Y_0 \cdot \chi_{\{0\}} + \sum_k Y_{T_k} \cdot \chi_{]T_k, T_{k+1}]}.$$

**Remark 199** *It is easy to show that*

$$\int Y_s^\sigma \, dX_s = Y_0 \, X_0 + \sum_k Y_{T_k} \left( X^{T_{k+1}} - X^{T_k} \right)$$

*for all semi-martingales $X$ and for all $Y$ in $\mathbb{S}$, $\mathbb{D}$ and $\mathbb{L}$.*

With the help of random partition we can calculate the stochastic integral as follows

**Theorem 200** *Let $X$ be a semi-martingale, $Y \in \mathbb{D}$ and $(\sigma_n)_{n\in\mathbb{N}}$ a sequence of random partitions, which converges to the identity. In this case*

$$\int_{0^+} Y_s^{\sigma_n} \, dX_s = \sum_k Y_{T_k^n} \left( X^{T_{k+1}^n} - X^{T_k^n} \right)$$

*converges with respect to the ucp-topology to the stochastic integral $\int (Y_-) \, dX$.*

*Proof.* [Pro90] Thm. 2.5.21.

**Definition 201** *Let $X$ and $Y$ be two semi-martingales. In this case we denote*

$[X,X] = ([X,X]_t)_{t\geq 0}$ *the quadratic variation process,*

$$[X,X] := X^2 - 2 \int X_- dX,$$

*resp*

$$[X,Y] := XY - \int X_- \, dY - \int Y_- \, dX$$

*the covariance process.*

**Proposition 202** *Let $X$ be a semi-martingale. Then we have the following:*

1. *$[X,X]$ is càdlàg, monotonously increasing and adapted.*
2. *$[X,X]_0 = X_0^2$ and $\Delta[X,X] = (\Delta X)^2$.*
3. *For a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of random partitions converging to $1$, we have the following*
$$X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})^2 \longrightarrow [X,X] \text{ with respect to ucp for } n \to \infty.$$
4. *Let $T$ be a stopping time. In this case we have $[X^T,X] = [X,X^T] = [X^T,X^T] = [X,X]^T$.*

*Proof.* [Pro90] Thm. 2.6.22.

**Remark 203** • *The map $(X,Y) \mapsto [X,Y]$ is bilinear and symmetric.*

• *We have the following polarisation identity:*
$$[X,Y] = \frac{1}{2}\left([X+Y,X+Y] - [X,X] - [Y,Y]\right).$$

**Proposition 204** *The bracket process $[X,Y]$ of two semi-martingales $X$ and $Y$ has paths of bounded variation on compact sets and is a semi-martingale.*

*Proof.* [Pro90] Cor. 2.6.1.

**Proposition 205 (Partial Integration)**
$$d(XY) = X_-\,dY + Y_-\,dX + d[X,Y].$$

*Proof.* [Pro90] Cor. 2.6.2.

**Proposition 206** *Let $M$ be a local martingale. In this case 1 and 2 are equivalent and 3 follows from1 and 2.*

1. *$M$ is a martingale with $\mathbb{E}[M_t^2] \leq \infty\ \forall t \geq 0$,*
2. *$E\left[[M,M]_t\right] < \infty\ \forall t \geq 0$,*
3. *$\mathbb{E}[M_t^2] = E\left[[M,M]_t\right] \forall t \geq 0$.*

*Proof.* [Pro90] Cor. 2.6.4.

**Theorem 207** *Let $X, Y$ be two semi-martingales and $H, K \in \mathbb{L}$. Then we have the following:*

*1.* $[H \cdot X, K \cdot Y]_t = \int_0^t H_s K_s d[X,Y]_s \forall t \geq 0$,
2. $[H \cdot X, H \cdot X]_t = \int_0^t H_s^2 d[X,X]_s \forall t \geq 0$.

*Proof.* [Pro90] Thm. 2.6.29.

**Theorem 208 (Itô-ormula)** *Let $X$ be a semi-martingale, $f \in C^2(\mathbb{R})$. In this case the Itô-formula holds:*

$$f(X_t) - f(X_0) = \int_{0^+}^t f'(X_s^-) dX_s + \frac{1}{2} \int_{0^+}^t f''(X_s^-) d[X,X]_s^{cont} + \sum_{0<s\leq t} \left\{ f(X_s) - f(X_s^-) - f'(X_s^-) \Delta X_s \right\}.$$

*Proof.* [Pro90] Thm. 2.7.32.

**Remark 209** *For a function $f \in C^2(\mathbb{R})$, we have*

$$f(t) - f(0) = \int_0^t f'(s) ds.$$

*For stochastic integration there are two additional terms. The term*

$$\frac{1}{2} \int_{0^+}^t f''(X_s^-) d[X,X]_s^{cont}$$

*is a consequence of the quadratic variation of the process and*

$$\sum_{0<s\leq t} \left\{ f(X_s) - f(X_s^-) - f'(X_s^-) \Delta X_s \right\}$$

*is induced by the jumps.*

**Proposition 210 (Variable transformation)** *Let $V$ be a stochastic process with bounded variation and right continuous paths. For $f \in C^1(\mathbb{R})$ the process $(f(V_t))_{t\geq 0}$ has bounded variation and the we have the following:*

$$f(V_t) - f(V_0) = \int_{0^+}^t f'(V_{s_-}) \, dV_s + \sum_{0<s\leq t} \left( f(V_s) - f(V_{s_-}) - f'(V_{s_-}) \Delta V_s \right).$$

**Proposition 211 (Itô-Formula)** *Let $X$ be a continuous martingale and $f \in C^2(\mathbb{R})$. In this case $f(X)$ is a semi martingale and we have:*

$$f(X_t) - f(X_0) = \int_{0^+}^t f'(X_s) dX_s + \frac{1}{2} \int_{0^+}^t f''(X_s) d[X,X]_s.$$

# Appendix F
# CERA Comparision

This section will provide a comparision between the topics covered in this book and the respective requirements of the Interntional Actuarial Association IAA, in order to meet the requirements of the Global Enterprise Risk Management Designation Recognition Treaty.

## F.1 Enterprise Risk Management Concept and Framework

| | Requirements | Reference |
|---|---|---|
| (a) | Describe the concept of ERM, the drivers behind it and the resulting value to organisations. (2-3) | 1 |
| (b) | Explain the principal terms in ERM. (2-3) | 1 |
| (c) | Analyse an appropriate framework for an organisation's enterprise risk management and an acceptable governance structure. (4-5) | **??** |
| (d) | Evaluate an organisation's risk management culture including: risk consciousness, accountabilities, discipline, collaboration, incentive compensation, and communication. (4-5) | 1 & **??** |
| (e) | Demonstrate an understanding of governance issues including market conduct, audit, and legal risk. (3-4) | **??** |
| (f) | Demonstrate an understanding of risk frameworks in regulatory and other environments (e.g. Basel II, Solvency II, Sarbanes-Oxley, COSO, Aus/NZ 4360, ISO 31000) and their underlying principles. (3-4) | 16 |
| (g) | Demonstrate an understanding of the perspectives of regulators, rating agencies, stock analysts, and company stakeholders and how they evaluate the risks and the risk management of an organisation. (3-4) | 16 |
| (h) | Propose how an ERM process can create value for an organisation through better assessment of the organisation's risk profile, possible reduction in economic capital, improvement in rating, etc. (5) | 1 |
| (i) | Relate the risk and return trade-offs that result from changes in the organisation's risk profile. (3-4) | **??** – 15 |

## F.2 ERM Process (Structure of the ERM Function and Best Practices)

| | Requirements | Reference |
|---|---|---|
| (j) | Demonstrate how to articulate an organisation's risk appetite, quantified risk tolerances, risk philosophy and risk objectives. (3-4) | 5 |
| (k) | Demonstrate how to articulate a desired risk profile and appropriate risk filters. (3-4) | 5 |
| (l) | Assess the overall corporate risk exposure arising from financial and non-financial risks. (6) | **??** – 15 |
| (m) | Compare the relevance of risk measurement and management to various stakeholders including customers, regulators, government, company directors, professional advisors, shareholders and the general public. (4) | **??** – 15 |
| (n) | Demonstrate an understanding of contagion and how it affects different stakeholders. (3-4) | **??** – 15 |
| (o) | Evaluate the elements of a successful risk management function and a structure for an organisation's risk management function. (4-5) | 1 & **??** |
| (p) | Determine how financial and other risks and opportunities influence the selection of strategy and how ERM can be appropriately embedded in an entity's strategic planning. (4-5) | **??** – 15 |
| (q) | Demonstrate the application of a risk control process such as the Risk Management Control Cycle or other similar approach. (3) | 1 |
| (r) | Propose ERM solutions or strategies to address real (case study) and hypothetical situations. (5-6) | 14 |

## F.3 Risk Categories and Identification

| | Requirements | Reference |
|---|---|---|
| (s) | Explain what is meant by risk and uncertainty. (2) | 1 |
| (t) | Describe different definitions and concepts of risk. (2) | 1 |
| (u) | Discuss risk taxonomy. (2-3) | 1 |
| (v) | Investigate and interpret financial and non-financial risks faced by an entity, including but not limited to: currency risk, credit risk, spread risk, liquidity risk, interest rate risk, equity risk, hazard/insurance risk, pricing risk, reserving risk, other product risk, operational risk, project risk and strategic risk. (3-4) | **??** – 15 |

## F.4 Risk Modelling and Aggregation of Risks

| | Requirements | Reference |
|---|---|---|
| (w) | Demonstrate how each of the financial and non-financial risks faced by an entity can be amenable to quantitative analysis. (3-4) | **??** – 15 |
| (x) | Demonstrate enterprise-wide risk aggregation techniques incorporating the use of correlation. (3-4) | **??** – 15 |
| (y) | Evaluate and select appropriate copulas as part of the process of modelling multivariate risks. (4-5) | 12 |
| (z) | Demonstrate the use of scenario analysis and stress testing in the risk measurement process. (3-4) | **??** – 15 |
| (aa) | Examine the use of extreme value theory to help model risks. (4) | |
| (bb) | Demonstrate the importance of the tails of distributions, tail correlations, and low frequency / high severity events. (3-4) | **??** – 15 |
| (cc) | Demonstrate an understanding of model and parameter risk. (3-4) | **??** – 15 |
| (dd) | Evaluate and select appropriate models to handle diverse risks, including the stochastic approach. (4-5) | **??** – 15 |

## F.5 Risk Measures

| | Requirements | Reference |
|---|---|---|
| (ee) | Apply risk metrics to quantify major types of risk exposure and tolerances in the context of an integrated risk management process. (3-4) | **??** – 15 |
| (ff) | Demonstrate the properties of risk measures (e.g. VaR and TVaR) and their limitations. (3-4) | 5 |
| (gg) | Analyse quantitative financial and insurance data using modern statistical methods (including asset prices, credit spreads and defaults, interest rates, incidents, causes and losses). (4-5) | **??** – 15 |
| (hh) | Evaluate best practices in risk measurement, modelling, and management of various financial and non-financial risks faced by an entity. (4-5) | **??** – 15 |
| (ii) | Analyse credit risk as related to fixed income securities. (4-5) | 6 |

## F.6 Risk Management Tools and Techniques

| | Requirements | Reference |
|---|---|---|
| (jj) | Relate the rationale for managing risk and the selection of the appropriate degree of hedging of risk. (3-4) | **??** – 15 |
| (kk) | Demonstrate risk optimisation and the impact on an organisation's value of an ERM strategy. (3-4) | **??** – 15 |
| (ll) | Demonstrate means for transferring risk to a third party, and estimate the costs and benefits of doing so. (3) | **??** – 15 |
| (mm) | Demonstrate means for reducing risk without transferring it. (3-4) | **??** – 15 |
| (nn) | Demonstrate how derivatives, synthetic securities, and financial contracting may be used to reduce risk or to assign it to the party most able to bear it. (3-4) | **??** – 15 |
| (oo) | Determine an appropriate choice of hedging strategy for a given situation (e.g., reinsurance, derivatives, financial contracting), which balances benefits with inherent costs, including exposure to credit risk, basis risk, moral hazard, and other risks. (4-5) | **??** – 15 |
| (pp) | Demonstrate an understanding of the practicalities of market risk hedging, including dynamic hedging. (3-4) | **??** – 15 |
| (qq) | Define credit risk as related to derivatives; define credit risk as related to reinsurance ceded; define counter-party risk and demonstrate the use of comprehensive due diligence and aggregate counter-party exposure limits. (3-4) | **??** – 15 |
| (rr) | Apply funding and portfolio management strategies to control equity and interest rate risk, including key rate risks. Explain the concepts of immunisation including modern refinements and practical limitations. (3-4) | **??** – 15 |
| (ss) | Analyse application of ALM principles to the establishment of investment policy and strategy including asset allocation. (4-5) | **??** – 15 |
| (tt) | Identify and interpret other key risks (e.g. operational, strategic, legal, and insurance risks) and uncertainty and demonstrate possible mitigation strategies. (3-4) | **??** – 15 |

## F.7 Economic Capital

| | Requirements | Reference |
|---|---|---|
| (uu) | Interpret the concept of economic measures of value (e.g., EVA, embedded value, economic capital) and demonstrate their uses in corporate decision-making processes. (3-4) | **??** – 15 |
| (vv) | Apply risk measures and demonstrate how to use them in economic capital assessment. (3-4) | **??** – 15 |
| (ww) | Propose techniques of allocating/appropriating the "cost" of risk/capital/hedge strategy to business units in order to gauge performance (e.g. returns on marginal capital). (5-6) | **??** – 15 |
| (xx) | Develop an economic capital model for a representative financial firm. (5-6) | **??** – 15 |

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