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Stochastic Models in Life Insurance

June 26, 2018

Springer-Verlag

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To Luisa, Giulia and Anna

Introduction

This book was originally written in German and goes back to a lecture which I have given in 1995 at ETH in Zurich. At this time, just after having done my PhD in abstract mathematics, I was of the opinion that life insurance mathematics would benefit greatly from a mathematically rigid approach, and I decided to write this book in German. Since I like very much to put theory in practise, I also programmed the corresponding algorithms, which are now used in one or the other company for the calculation of mathematical reserves. This English version of the book bases on the second edition of the original German version and has been enlarged: Over the last 15 years since writing the book for the first time, the world has changed considerably, mainly in respect to risk management in insurance companies and also with respect to Solvency II and the computer power now available for models and simulations. In order to reflect these two topics, I have written two additional chapters covering ALM and abstract valuation concepts.

The aim of this book is twofold: On one hand it aims to provide a sound mathematical base for life insurance mathematics and on the other it wants to apply the underlying concepts concretely. This book is mainly written for the advanced mathematical student or actuary and I would expect that the reader understands basic analysis including measure and integration theory. Moreover the reader should understand basic probability theory and functional analysis. The book is written in a typical mathematical approach, where each theorem is followed by its proof. From a very high level point of view the aim of the book is to provide a robust framework for modelling life insurance policies by means of Markov chains. We will see how to calculate the expected present values and higher moments of future cash flows with recursion formulae. In the same sense the underlying probability density functions are calculated. In the following chapters unit linked policies with guarantees and insurance portfolios are covered. In the last two chapters of the book we look at abstract valuation concepts and asset liability management.

At this point I would like to thank some people, mainly the ones who helped me to make this book happen and have helped me to make it better. My particular thanks go to Hans Bühlmann and Josef Kupper, who motivated me in the first instance to write this book in 1995 and also helped me to improve its quality considerably. For the English translation I would want to express my thanks to Bjoern Böttcher who translated the book into English. Finally I would like also to tank my colleagues from the different insurance companies (Swiss Life, Swiss Re, Partner Re, Aviva and Prudential) who helped me to better understand insurance and to improve the quality of the book. Finally I would like to thank my wife Luisa and my two daughters Giulia and Anna for their constant support.

Herrliberg, 1. September 2011 *Michael Koller*

Contents

1. A general life insurance model

1.1 Introduction

The life insurance market offers a wide range of different policies. It is, without expert knowledge, hardly possible to differentiate between all these policies. This is in particular due to the fact that the content of a life insurance is an abstract good.

A life insurance can always be understood as a bet: either one gets a benefit or one pays the premium without getting anything in return. From this point of view life insurance mathematics is a part of probability theory.

Since a life insurance deals with monetary benefits and premiums it is also part of the financial market and the economy. In this context one should note that insurances whose benefits are unit-linked, e.g. the payout depends on the performance of a fond, actually rely on modern theory of financial markets.

Form a legal point of view a life insurance is a contract between the policy holder and the insurer.

As we have note above, life insurances is characterized by its abstract matter and its diversity. Since its content is abstract its value is not intuitively obvious. This is particular due to the fact that a life insurance is usually only bought once or twice during life time. In contrast, for example one buys a loaf of bread on a regular basis, and thus one has acquired a feeling for its correct value.

A life insurance - in particular an individual policy - is a long term contract. Take for example a thirty year old man who buys a permanent life insurance. Now suppose he dies when he reaches ninety, then the contract period was sixty years.

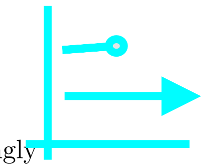
Due to this long duration of the contract and the risks taken - think for example of changing fundamentals - it is necessary to calculate the price of an insurance with care and foresight.

In this chapter we are going to explain classical types of insurance policies. Furthermore we introduce a general model for life insurance, which can be used to price many of the available policies.

1.2 Examples

We start with a description of the most common types of life insurance and the different methods of financing a policy.

1.2.1 Types of life insurance



It is characteristic for every life insurance that the insured event is strongly related to the health of the insured. Thus one can classify life insurance as follows:

– insurance on life or death,

– insurance on permanent disability,

– health insurance.

For an insurance on life or death the essential event is the survival of the insured person up to a certain date or the death before a certain date, respectively. Furthermore these insurance types can be classified by the causes of death which yield a payout (e.g. a life insurance which pays only in the event of an accidental death). Especially, various kinds of survivor's pensions and pure endowments are insurances on life or death.

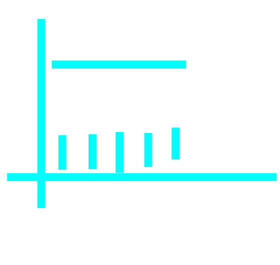
For a permanent disability insurance the essential criterion is the (dis)ability of the insured at a given date. These insurances have the special feature, that already a certain degree of disability might be sufficient for a claim.

For insurances on health the payout depends on the health of the insured. This class of insurances contains also modern types of policies, like a long term care insurance. The latter only provides benefits if the insured is unable to meet his basic needs (e.g. he is unable to dress himself).

Besides a classification based on the insured event one can also classify the insurance based on the benefit. This can either be paid in annuities or in a lump sum.

In the following we give some typical examples of life insurances.

Pension: A pension policy constitutes that the insurer has to pay annuities to the insured when he reaches a certain age (age of maturity of the policy). Then the pension is paid until the death of the insured. The payment of the annuities is usually done at regular intervals: monthly, quarterly or yearly. Moreover the payment can be done in advance (at the beginning of each interval) or arrears (at the and of each interval). Since the pension is only paid until death, one can additionally agree upon a minimum payment period. In this case the pension is paid at least for



the minimum period. (This type of pension contract supplies the desire of the insured to get at least something back for the premiums paid in.)

Pure endowment: A pure endowment insurance provides a payment from the insurer to the insured, if he reaches the age of maturity of the policy. Otherwise there is no payment.

Term/permanent life insurance: A (term/permanent) life insurance is the counterpart to a pure endowment insurance. In contrast to the latter a life insurance does not yield a payout to the insured if the age of maturity of the policy is reached. In the popular case of a term life insurance there is no payout at all if the insured reaches the age of maturity of the policy. But if the insured dies before that age his heirs get a payment. A special case is the permanent life insurance, which yields a payout to the heirs no matter how old the insured is at the time of his death. This insurance is in some countries very popular, since it in a sense an investment into ones offsprings.

Endowment: The endowment insurance is the classic example of a life insurance. It is the sum of a pure endowment insurance and a term or permanent life insurance. This means that it yields a payout in the case of an early death and also in the case of reaching the fixed age of maturity.

Widow's pension: A widow's pension is connected to the life of two persons. This is in contrast to the previous examples, where only the life of one person was considered. For a widow's pension there is the insured (the person whose life is insured, e.g. husband) and the beneficiary person (e.g. spouse). As long as both persons are alive no payment is due. If the insured dies and the beneficiary is still alive, then the beneficiary gets a pension until death. Also for this kind of insurance it is possible to fix a minimum period of payments in the policy.

Orphan's pension: After the death of the father or the mother their child gets a pension until it is of age or until death.

Insurance on two lives: For an insurance on two lives, as in the case of a widow's or orphans's pension, one has to consider the lives of two persons. Here the policy fixes a payment depending on the state of the two persons (insured, beneficiary)$\in \{(**), (*\dagger), (\dagger *), (\dagger\dagger)\}$. Obviously, a widow's or an orphan's pension is just a special case of the insurance on two lives. As before, also in this case one could agree upon a minimum period of payments.

Refund guarantee: The refund guarantee is an additional insurance which is often sold together with a pension or a pure endowment. It is a life insurance whose payment equals the paid-in premiums, possibly reduced by the already received payments. The refund guarantee supplies the same want as the minimum periods of payment for the pensions.

Now we have discussed the main insurance types on life and death. Next we want to give a short description of insurances on permanent disability. For these the ability to work is the main criterion. In this context one should note that the probability of becoming disabled strongly depends on the economic environment. This is due to the fact that in a good economic environment everyone finds a job. But a person with restricted health has a hard time finding a job during an economic downturn. In connection with disability the following types of insurances are most common:

Disability pension: In the case of disability, after an initial waiting period, the insured gets a pension until he reaches a fixed age (or until his death) or until he is able to work again. A disability pension without fixed age for a final payment is called permanent disability pension. Often an initial waiting period is introduced, since in most cases disability occurs after an accident or illness and the person actually recovers quickly thereafter. Thus the initial waiting period reduces the price of these policies. Typical waiting periods are three or six month and one or two years.

Disability capital: The disability capital insurance pays a lump sum to the insured in case of a permanent disability.

Premium waiver: The premium waiver is an additional insurance. It waives the obligation to pay further premiums for the insured in the case of disability. Waiting periods are also common for this type of insurance.

Disability children's pension: This pension is similar to the orphan's pension. The only difference is that the cause for a payment is the disability of the mother or father instead of their death.

1.2.2 Methods of financing

Previously we looked at the different types of insurances. Now we are going to discuss the different financing methods and the ideas they are based on. The main principle of life insurances states that the value of the benefits provided by the insurer is equivalent to the value of the policy for the insured. Obviously one has to discuss this equivalence relation in more detail. We will do this in the next chapters and provide a precise definition of the equivalence principle. Then it will be used to calculate the premiums, but for now we get back to the methods of financing. The two common types are:

- Financing by premiums,
- Financing by a single payment (single premium).

Financing by premiums requires the insured to pay premiums to the insurer at regular intervals. This obligation usually ends either when the age of maturity of the policy is reached or if the insured dies.

The other option to finance a life insurance is a single payment. Often a policy incorporates a mixture of both financing methods.

1.3 The insurance model

In this section we want to introduce the insurance model which we will use thereafter. We attempt to describe the real world by a model. Thus it is important to use a model class which is flexible enough to accommodate this. Figure 1.1 shows the general setup of an insurance model. Here we think of an insured person who, at every time t, is in a state $1, 2, \ldots, n$. State 1 could for example indicate that the person is alive. The state of the person is then given by the stochastic process X with $X_t(\omega) \in S = \{1, 2, \ldots, n\}$. When the insured remains in one state or switches its state a payment, as defined in the insurance policy, is due. For this there are functions $a_i(t)$ and $a_{ij}(t)$ given which correspond to the lines in the figure. They define the amount which the insured gets if he remains in state i (payment $a_i(t)$) or if he switches from state i to j at time t (payment $a_{ij}(t)$). In the following we are going to introduce the necessary concepts for this setup. One distinguishes between the continuous time model, where $(X_t)_{t \in T}$ is defined on an interval in $\mathbb{R}$, and the discrete time model, where $(X_t)_{t \in T}$ is defined on a subset of $\mathbb{N}$. The continuous time model yields the more interesting statements whereas the discrete model is very important in applications, therefore we will discuss both models.

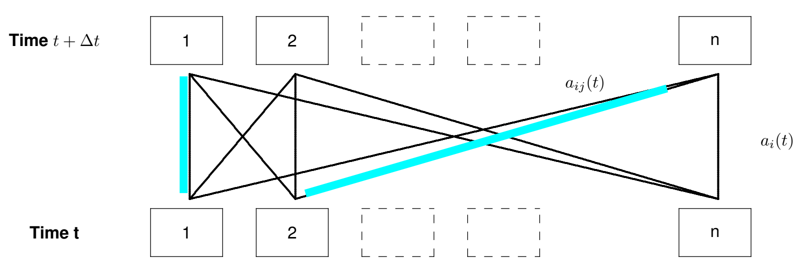


Figure 1.1. Policy setup from t to $t + \Delta t$

Definition 1.3.1 (State space). *We denote by S the state space which is used for the insurance policy. S is finite set.*

Example 1.3.2. For a life insurance or an endowment one often uses the statespace $S = \{*, \dagger\}$.

Example 1.3.3. For a disability insurance one has to consider at least the states: alive (active), dead and disabled. Often one uses more states to get a better model. For example in Switzerland a model is used which uses the states $\{*, \dagger\}$ and the family of states {person became disabled at the age of $x : x \in \mathbb{N}$}.

With the states defined, it is now possible to derive a mathematical model for the payments/benefits. To define the model of the benefits, the so called policy functions, it is necessary to define the time set more precisely. We will define two different time sets since a discrete time set is often used in applications, but the continuous time set yields the neater results.

Definition 1.3.4. – *$a_i(t)$ denotes the sum of the payments to the insured up to time t, given that we know that he has always been in state i. The $a_i(t)$ are called the generalized pension payments. If this pension function is of bounded variation (see Def. 2.1.5) we can also write $a_i(t) = \int_0^t da_i(s)$.*

- *$a_{ij}(t)$ denotes the payments which are due when the state switches from i to j at time t. These benefits are called generalized capital benefits.*
- *In the case of a discrete time set $a_i^{Pre}(t)$ denotes the pension payment which is due at time t, given that the insured is at time t in i.*
- *In the case of a discrete time set $a_{ij}^{Post}(t)$ denotes the capital benefits which are due when switching from i at time t to j at time $t+1$. We are going to assume that the payment is transfered at the end of the time interval.*

The functions $a_i(t)$ are different in the continuous time model and the discrete time model. In the former $a_i(t)$ denotes the sum of the pension payments which are payed up to time t, similar to a mileage meter in a car. In the latter $a_i^{\mathrm{Pre}}(t)$ denotes the single pension payment at time t.

The following example illustrates the interplay between the state space and the functions which define the policy.

Example 1.3.5. Consider an endowment policy with 200,000 USD death benefit and 100,000 USD survival benefit. This insurance shall be financed by a yearly premium of 2,000 USD.

For a age at maturity of 65 the non trivial policy functions are:

$$a_*(x) = \begin{cases} 0, & \text{if } x < x_0, \\ -\int_{x_0}^{x} 2000\, dt, & \text{if } x \in [x_0, 65], \\ -(65 - x_0) \times 2000 + 100000, & \text{if } x > 65, \end{cases}$$

$$a_{*\dagger}(x) = \begin{cases} 0, & \text{if } x < x_0 \text{ or } x > 65, \\ 200000, & \text{if } x \in [x_0, 65], \end{cases}$$

where x_0 is the age of entry into the contract, $*$ and $\dagger$ denote the states alive and dead, respectively.

2. Stochastic processes

2.1 Definitions

In this section we will recall basic definitions from probability theory. These will be used throughout the book.

To understand this chapter a basic knowledge in probability theory, measure theory and analysis is a prerequisite.

Definition 2.1.1 (Sets). *We are going to use the notations:*

$$\begin{aligned} \mathbb{N} &= \text{the set of the natural numbers including } 0, \\ \mathbb{N}_+ &= \{x \in \mathbb{N} : x > 0\}, \\ \mathbb{R} &= \text{the set of the real numbers}, \\ \mathbb{R}_+ &= \{x \in \mathbb{R} : x \geq 0\}. \end{aligned}$$

Furthermore we use the following notations for intervals. For $a, b \in \mathbb{R}, a < b$ we write

$$\begin{aligned} [a,b] &:= \{t \in \mathbb{R} \,:\, a \leq t \leq b\}, \\ ]a,b] &:= \{t \in \mathbb{R} \,:\, a < t \leq b\}, \\ ]a,b[&:= \{t \in \mathbb{R} \,:\, a < t < b\}, \\ [a,b[&:= \{t \in \mathbb{R} \,:\, a \leq t < b\}. \end{aligned}$$

Definition 2.1.2 (Indicator function). *For $A \subset \Omega$ we define the indicator function $\chi_A : \Omega \to \mathbb{R}, \omega \mapsto \chi_A(\omega)$ by*

$$\chi_A(\omega) := \begin{cases} 1, & \text{if} \quad \omega \in A, \\ 0, & \text{if} \quad \omega \notin A. \end{cases}$$

Furthermore δ_{ij} is Kronecker's delta, i.e., it is equal to 1 for $i = j$ and 0 otherwise.

Definition 2.1.3. *Let*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x).$$

We define, if they exist, the left limit and the right limit of f at x by:

$$\begin{aligned} f(x^-) &:= \lim_{\xi \uparrow x} f(\xi), \\ f(x^+) &:= \lim_{\xi \downarrow x} f(\xi). \end{aligned}$$

Definition 2.1.4. *A real valued function $f : \mathbb{R} \to \mathbb{R}$ is said to be of order $o(t)$, if*

$$\lim_{t \to 0} \frac{f(t)}{t} = 0.$$

This is denoted by $f(t) = o(t)$.

Definition 2.1.5 (Function of bounded variation). *Let $I \subset \mathbb{R}$ be a bounded interval. For a function*

$$f : I \to \mathbb{R}, t \mapsto f(t)$$

the total variation of the function f on the interval I is defined by

$$V(f, I) = \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)|,$$

where the supremum is taken with respect to all partitions of the interval I satisfying

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n.$$

The function f is of bounded variation *on I, if $V(f, I)$ is finite. Functions corresponding to a life insurance are usually defined on the interval $[0, \omega]$, where $\omega < \infty$ denotes the last age at which some individuals are alive.*

Properties of functions of bounded variation can be found for example in [DS57].

It is important to note, that functions of bounded variation form an algebra and a lattice. Thus, if f, g are functions of bounded variation and $\alpha \in \mathbb{R}$, then the following functions are also of bounded variation: $\alpha f + g$, $f \times g$, $\min(0, f)$ and $\max(0, f)$.

Definition 2.1.6 (Probability space, stochastic process). *We denote by $(\Omega, \mathcal{A}, P)$ a probability space which satisfies Kolmogorov's axioms.*

Let $(S, \mathcal{S})$ be a measurable space (i.e. S is a set and $\mathcal{S}$ is a σ-algebra on S) and T be a set.
The Borel σ-algebra on the real numbers will be denoted by $\mathcal{R} = \sigma(\mathbb{R})$.

A family $\{X_t : t \in T\}$ *of random variables*

$$X_t : (\Omega, \mathcal{A}, P) \to (S, \mathcal{S}), \omega \mapsto X_t(\omega)$$

is called stochastic process *on* $(\Omega, \mathcal{A}, P)$ *with* state space S.

For each $\omega \in \Omega$ *a sample path of the process is given by the function*

$$X.(\omega) : T \to S, t \mapsto X_t(\omega).$$

We assume that each sample path is right continuous and has left limits.

Definition 2.1.7 (Expectations). *Let* X *be a random variable on* $(\Omega, \mathcal{A}, P)$ *and* $\mathcal{B} \subset \mathcal{A}$ *be a* σ*-algebra. Then we denote by*

- $E[X]$ *the* expectation *of the random variable* X,
- $V[X]$ *the* variance *of the random variable* X,
- $E[X|\mathcal{B}]$ *the* conditional expectation *of* X *with respect to* $\mathcal{B}$.

Definition 2.1.8. *Let* $(X_t)_{t \in T}$ *be a stochastic process on* $(\Omega, \mathcal{A}, P)$ *taking values in a countable set* S*. We define for* $j \in S$ *the indicator function with respect to the process* $(X_t)_{t \in T}$ *at time* t *by*

$$I_j(t)(\omega) = \begin{cases} 1, & \text{if} \quad X_t(\omega) = j, \\ 0, & \text{if} \quad X_t(\omega) \neq j. \end{cases}$$

Analogous, we define for $j, k \in S$ *the number of jumps from* j *to* k *in the time interval* $]0, t[$ *by*

$$N_{jk}(t)(\omega) = \#\left\{\tau \in]0, t[: X_{\tau^-} = j \text{ und } X_\tau = k\right\}.$$

Remark 2.1.9. In the following the function $I_j(t)$ is used to check if the insured person is at time t in state j. Thus one can check if the pension $a_j(t)$ has to be paid. Similarly, a switch from i to j is indicated by an increase of $N_{ij}(t)$ by 1.

Definition 2.1.10 (Normal distribution). *A random variable* X *on* $(\mathbb{R}, \sigma(\mathbb{R}))$ *with density*

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}$$

is called normal distributed *with expectation* μ *and variance* σ^2*. Such a random variable is denoted by* $X \sim \mathcal{N}(\mu, \sigma^2)$.

Examples of stochastic processes are:

Definition 2.1.11 (Brownian motion). *An example of a non trivial stochastic process is Brownian motion. Brownian motion $W = (W_t)_{t\geq 0}$ with continuous time set ($T = \mathbb{R}_+$) and state space $S = \mathbb{R}$ is used to model many real world phenomena.*

The process is defined by the following properties:

1. *$W_0 = 0$ almost surely.*
2. *W has independent increments: the random variables $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\ldots$, $W_{t_n} - W_{t_{n-1}}$ are independent for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$.*
3. *$W_{t+h} - W_t \sim \mathcal{N}(0, h)$ for all $t \geq 0$ and $h \geq 0$.*
4. *Almost all sample paths of $(W_t)_{t\in\mathbb{R}^+}$ are continuous.*

One can show that $(W_t)_{t\in\mathbb{R}^+}$ is nowhere differentiable.

Until now we do not know whether there exists a stochastic process $(W_t)_{t\in\mathbb{R}^+}$, which fulfills the requirements for a Brownian Motion. In the following we will show the existence of such a process. The way we will construct it, will also be helpful to simulate it. In order to do that we need some lemmas:

Lemma 2.1.12. *For $X \sim \mathcal{N}(0, 1)$ and $x > 0$ the following inequalities hold:*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq P[X > x] \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Proof. For the second inequality we have:

$$\begin{aligned} P[X > x] &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\xi^2/2}d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{\xi}{x}e^{-\xi^2/2}d\xi \\ &= \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \end{aligned}$$

For the second inequality we define

$$f(x) = x\,e^{x^2/2} - (x^2+1)\int_x^\infty e^{\xi^2/2}\,d\xi$$

We remark that $f(0) < 0$ and $\lim\limits_{x\to\infty} f(x) = 0$. Furthermore we have the following:

$$f'(x) \quad = \quad -2x\left(\int_x^\infty e^{-\xi^2/2}d\xi - \frac{e^{-x^2/2}}{x}\right),$$

which is positive for $x > 0$ by the first part. Therefore $f(x) \leq 0$.

Lemma 2.1.13. *For X_1 and X_2 two independent normally distributed random variable with $X_i \sim \mathcal{N}(0, \sigma^2)$ we have the following:*

1. *$Y_1 := X_1 + X_2$ and $Y_2 := X_1 - X_2$ are independent and normally distributed.*
2. *$Y_i \sim \mathcal{N}(0, 2\sigma^2)$.*

Proof. We note that

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

is an isometric orthogonal coordinate transformation in $\mathbb{R}^2$. Furthermore we note that $(X_1/\sigma, X_2/\sigma)^T$ is standard Gaussian by definition. Hence the application of A to this vector yields to the result.

Exercise 2.1.14. Complete the proof of lemma 2.1.13.

Theorem 2.1.15 (Existence of Brownian Motion, Wiener 1923). *The Standard Brownian Motion exists.*

Proof. We first construct Brownian motion on the interval $[0, 1]$ as a random element on the space $C[0, 1]$ of continuous functions on $[0, 1]$. The idea is to construct the right joint distribution of Brownian motion step by step on the finite sets

$$\mathcal{D}_n = \{\frac{k}{2^n} : k \in \mathbb{N},\, 0 \leq k \leq 2^n\}$$

of dyadic points for $n \in \mathbb{N}$. We then interpolate the values on $\mathcal{D}_n$ linearly and check that the uniform limit of these continuous functions exists and is a Brownian motion.

To do this let $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ and let $(\Omega, \mathcal{A}, P)$ be a probability space on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent, standard normally distributed random variables can be defined. Let $W(0) := 0$ and $W(1) := Z_1$. For each $n \in \mathbb{N}$ we define the random variables $W(d), d \in \mathcal{D}_n$ such that

1. for all $r < s < t$ in $\mathcal{D}_n$ the random variable $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, and is independent of $W(s) - W(r)$,
2. the vectors $\{W(d) : d \in \mathcal{D}_n\}$ and $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ are independent.

Note that we have already done this for $\mathcal{D}_0 = \{0, 1\}$. Proceeding inductively we may assume that we have succeeded in doing it for some $n - 1$. We then define $W(d)$ for $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ by

$$W(d) \quad = \quad \frac{W(d-2^n)+W(d+2^n)}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

Note that the first term is the linear interpolation of the values of W at the neighbouring points of d in $\mathcal{D}_{n-1}$. Therefore $W(d)$ is independent of $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ and the second property is fulfilled. Moreover, as $\frac{1}{2}\left[W(d+2^{-n})+W(d+2^{-n})\right]$ depends only on $\{Z_t : t \in \mathcal{D}_{n-1}\}$, it is independent of $Z_d/2^{(n+1)/2}$. Both terms are normally distributed with mean zero and variance $2^{-(n+1)}$. Hence their sum $W(d)-W(d-2^{-n})$ and their difference $W(d+2^{-n})-W(d)$ are independent and normally distributed with mean zero and variance 2^{-n} by Lemma 2.1.13.

Hence all increments $W(d)-W(d-2^{-n})$, for $d \in \mathcal{D}_n \setminus 0$, are independent. To see this it suffices to show that they are pairwise independent, as the vector of these increments is Gaussian. We have seen that pairs $W(d) - W(d-2^{-n})$, $W(d+2^{-n})-W(d)$ with $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ are independent. The other possibility is that the increments are over intervals separated by some $d \in \mathcal{D}_{n-1}$. Choose $d \in \mathcal{D}_j$ with this property and minimal j, so that the two intervals are contained in $[d-2^{-j}, d]$, respectively $[d, d+2^{-j}]$. By induction the increments over these two intervals of length 2^{-j} are independent, and the increments over the intervals of length 2^{-n} are constructed from the independent increments $W(d)-W(d-2^{-j})$, respectively $W(d+2^{-j})-W(d)$, using a disjoint set of variables $\{Z_t : t \in \mathcal{D}_n\}$. Hence they are independent and this implies the first property, and completes the induction step.

Having thus chosen the values of the process on all dyadic points, we interpolate between them. Formally, define

$$F_0(t) = \begin{cases} Z_1 & \text{for } t=1, \\ 0 & \text{for } t=0, \text{ and} \\ \text{linear} & \text{in between.} \end{cases}$$

For each $n \geq 0$ we define

$$F_n(t) = \begin{cases} 2^{-(n+1)/2}\, Z_t & \text{for } t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, \\ 0 & \text{for } t \in \mathcal{D}_{n-1}, \text{ and} \\ \text{linear} & \text{between consecutive points in } \mathcal{D}_n. \end{cases}$$

These functions are continuous on $[0,1]$ and for all n and $d \in \mathcal{D}_n$.

$$W(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d) \tag{2.1}$$

This can be seen by induction. It holds for $n=0$. Suppose that it holds for $n-1$. Let $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Since for $0 \leq i \leq n-1$ the function F_i is linear on $[d-2^{-n}, d+2^{-n}]$, we get

$$\sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{n} \frac{F_i(d-2^{-n}) + F_i(d+2^{-n})}{2}$$
$$= \frac{W(d-2^{-n}) + W(d+2^{-n})}{2}$$

Since $F_n(d) = 2^{-(n+1)/2}\, Z_d$, this gives 2.1. On the other hand, we have, by definition of Z_d and by 2.1.12, for $c > 0$ and large n,

$$P[|Z_d| > c\,\sqrt{n}] \leq \exp\left(\frac{-c^2 n}{2}\right),$$

so that the series

$$P[\exists d \in \mathcal{D}_n \text{with} |Z_d| \geq c\sqrt{n}] \leq \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} P[|Z_d| \geq c\sqrt{n}]$$
$$\leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2 n}{2}\right),$$

converges as soon as $c > \sqrt{2\log 2}$. Fix such a $c \in \mathbb{R}$. By the Borel-Cantelli lemma there exists a random (but almost surely finite) $N \in \mathbb{N}$ such that for all $n \geq N$ and $d \in \mathcal{D}_n$ we have $|Z_d| < c\sqrt{n}$. Hence, for all $n \geq N$ we have

$$||F_n||_\infty \leq c\,\sqrt{n} 2^{-n/2}.$$

This upper bound implies that, almost surely, the series $W(t) = \sum n = 0^\infty F_n(t)$ is uniformly convergent on $[0,1]$. We denote the continuous limit by $\{W(t) : t \in [0,1]\}$.

It remains to check that the increments of this process have the right marginal distributions. This follows directly from the properties of W on the dense set $D \subset [0,1]$ and the continuity of the paths. Indeed, suppose that $t_1 < t_2 < \ldots < t_n$ are in $[0,1]$. We find $t_{1,k} \leq t_{2,k} \leq \ldots \leq t_{n,k}$ in $\mathcal{D}$ with $\lim\limits_{k\to\infty} t_{i,k} = t_i$ and conclude from the continuity of W that, for $1 \leq i \leq n-1$,

$$W(t_{i+1}) - W(t_i) = \lim_{k\to\infty} W(t_{i+1,k}) - W(t_{i,k}).$$

As $\lim\limits_{k\to\infty} E[W(t_{i+1,k}) - W(t_{i,k})] = 0$ and

$$\lim_{k\to\infty} Cov\,(W(t_{i+1,k}) - W(t_{i,k}), W(t_{j+1,k}) - W(t_{j,k}))$$
$$= \lim_{k\to\infty} \delta_{ij}\,(t_{i+1,k} - t_{i,k}) = \delta_{ij}\,(t_{i+1} - t_i),$$

the increments $W(t_{i+1}) - W(t_i)$ are independent Gaussian random variables with mean 0 and variance $t_{i+1} - t_i$, as required.

We have thus constructed a continuous process $W : [0,1] \to \mathbb{R}$ with the same marginal distributions as Brownian motion. Take a sequence $W_1, W_2, \ldots$ of independent $C[0,1]$-valued random variables with the distribution of this process, and define $\{W(t) : t \geq 0\}$ by gluing together the parts, more precisely by

$$W(t) \quad = \quad W_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} W_i(1),$$

for all $t \geq 0$. It is easy to show that $(W(t))_{t \in \mathbb{R}_0^+}$ fulfills the requirements for a standard Brownian motion.

Remark 2.1.16. Lévy's construction of the Brownian motion helps us also understanding how to simulate it. The typical approach to simulate it is to choose a $\Delta t = 2^{-n}$ and to define $(W(t))_{t \in \mathcal{D}_n}$ inductively as follows:

– $W(0) = 1$, and
– $W((k+1)\, 2^{-n}) = W(k\, 2^{-n}) + \sqrt{\Delta t}\, Z_k$, for all $k \geq 0$,

where $(Z_k)_{k \in \mathbb{N}}$ denotes a series of independent $\mathcal{N}(0,1)$ Gaussian variables. We note that for this purpose we do not need the dyadic recursion. Similarly one simulates a Brownian bridge. In this case $W(1)$ is also given and the simulation is performed by first defining $W(\frac{1}{2})$, then for $\mathcal{D}_2 \setminus \mathcal{D}_1$, etc.

Exercise 2.1.17. 1. Simulate a Brownian motion for $t \in \mathcal{D}_4$.

2. Modify your simulation to a Brownian bridge, assuming that $W(0) = W(1) = 0$.

Example 2.1.18 (Poisson process). The Poisson process $N = (N_t)_{t \geq 0}$ is a *counting process* with state space $\mathbb{N}$. For example it is used in insurance mathematics to model the number of incurred claims. This process also uses a continuous time set. The homogeneous Poisson process is characterized by the following properties:

1. $N_0 = 0$ almost surely.
2. N has independent and
3. N stationary increments.
4. For all $t > 0$ and all $k \in \mathbb{N}$ gilt: $P[N_t = k] = \exp(-\lambda\, t)\, \frac{(\lambda\, t)^k}{k!}$.

2.2 Markov chains on a countable state space

In the following S is a countable set.

Definition 2.2.1. *Let $(X_t)_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$ with state space S and $T \subset \mathbb{R}$. The process X is called Markov chain, if for all*

$$n \geq 1,\ t_1 < t_2 < \ldots < t_{n+1} \in T,\ i_1, i_2, \ldots, i_{n+1} \in S$$

with

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] > 0$$

the following statement holds:

$$P[X_{t_{n+1}} = i_{n+1} | X_{t_k} = i_k \forall k \leq n] \quad = \quad P[X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n]. \tag{2.2}$$

Remark 2.2.2. 1. Equation (2.2) states that the conditional probabilities only depend on the *last* state. They do not depend on the path which led the chain into that state.

2. Markov chains are very versatile in their applications. This is due to the fact, that on the one hand they are very easy to handle and on the other hand they can model a wide range of phenomena. In the following we are going to model life insurances by Markov chains.

Example 2.2.3. 1. Let $(X_t)_{t\in T}$ be a stochastic process with $S \subset \mathbb{R}$ and $T = \mathbb{N}_+$, for which the random variables $\{X_t : t \in T\}$ are independent. This process is a Markov chain since

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] = \prod_{k=1}^{n} P[X_{t_k} = i_k]$$

for $n \geq 1,\ t_1 < t_2 < \ldots < t_{n+1} \in T,\ i_1, i_2, \ldots, i_{n+1} \in S$.

2. Based on the previous example we define $S_m = \sum_{k=1}^{m} X_k$, where $m \in \mathbb{N}$. This is also an example of a Markov chain.

Proof.

$$\begin{aligned} &P[S_{t_{n+1}} = i_{n+1} \,|\, S_{t_1} = i_1, S_{t_2} = i_2, \ldots, S_{t_n} = i_n] \\ &\quad = P[S_{t_{n+1}} - S_{t_n} = i_{n+1} - i_n] \\ &\quad = P[S_{t_{n+1}} = i_{n+1} \,|\, S_{t_n} = i_n]. \end{aligned}$$

Definition 2.2.4. *Let $(X_t)_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$. Then*

$$p_{ij}(s,t) \quad := \quad P[X_t = j \,|\, X_s = i], \qquad \text{where } s \leq t \text{ and } i, j \in S,$$

is called the conditional probability to switch from state i at time s to state j at time t, or also transition probability for short.

The following theorem of Chapman and Kolmogorov is fundamental for the theory which we will present in the next chapters. The theorem states the relation of $P(s,t)$, $P(t,u)$ and $P(s,u)$ for $s \le t \le u$.

Theorem 2.2.5 (Chapman-Kolmogorov equation). *Let $(X_t)_{t\in T}$ be a Markov chain. For $s \le t \le u \in T$ and $i,k \in S$ such that $P[X_s = i] > 0$ the following equations hold:*

$$p_{ik}(s,u) = \sum_{j\in S} p_{ij}(s,t)\, p_{jk}(t,u), \tag{2.3}$$

$$P(s,u) = P(s,t) \times P(t,u). \tag{2.4}$$

This shows, that one can get $P(s,u)$ by matrix multiplication of $P(s,t)$ and $P(t,u)$ for $s \le t \le u \in T$.

Proof. Obviously, the equation holds for $t = s$ or $t = u$. Thus we can assume $s < t < u$ without loss of generality. We will use the following notation:

$$\begin{aligned} S^* &= \{j \in S : P[X_t = j \mid X_s = i] \neq 0\} \\ &= \{j \in S : P[X_t = j\,,\, X_s = i] \neq 0\}\,. \end{aligned}$$

(The last equality holds since $P[X_s = i] > 0$.) Now the Chapman-Kolmogorov equation can be deduced from the following equation:

$$\begin{aligned} p_{ik}(s,u) &= P[X_u = k \mid X_s = i] \\ &= \sum_{j\in S^*} P[X_u = k,\, X_t = j \mid X_s = i] \\ &= \sum_{j\in S^*} P[X_t = j \mid X_s = i] \times P[X_u = k \mid X_s = i,\, X_t = j] \\ &= \sum_{j\in S^*} p_{ij}(s,t) \times p_{jk}(t,u) \\ &= \sum_{j\in S} p_{ij}(s,t) \times p_{jk}(t,u), \end{aligned}$$

where we applied the Markov property to get equality in the forth line.

After proving the Chapman-Kolmogorov equation we are now able to introduce the abstract concept of transition matrices.

Definition 2.2.6 (Transition matrix). *A family $(p_{ij}(s,t))_{(i,j)\in S\times S}$ is called transition matrix, if the following four properties hold:*

1. $p_{ij}(s,t) \ge 0$.

2. $\sum_{j \in S} p_{ij}(s,t) = 1$.
3. $p_{ij}(s,s) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \end{cases} \qquad$ *if* $P[X_s = i] > 0$.
4. $p_{ik}(s,u) = \sum_{j \in S} p_{ij}(s,t)\, p_{jk}(t,u)$ *for* $s \leq t \leq u$ *and* $P\,[X_s = i] > 0$.

Theorem 2.2.7. *Let* $(X_t)_{t \in T}$ *be a Markov chain. Then* $(p_{ij}(s,t))_{(i,j) \in S \times S}$ *is a transition matrix.*

Proof. This theorem is a direct consequence of the theorem by Chapman and Kolmogorov (T. 2.2.5).

Theorem 2.2.8. *A stochastic process* $(X_t)_{t \in T}$ *is a Markov chain, if and only if*

$$P[X_{t_1} = i_1, \ldots, X_{t_n} = i_n] \quad = \quad P[X_{t_1} = i_1] \prod_{k=1}^{n-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}), \tag{2.5}$$

for all

$$n \geq 1,\ t_1 < t_2 < \ldots < t_{n+1} \in T,\ i_1, i_2, \ldots, i_{n+1} \in S.$$

Proof. Let $(X_t)_{t \in T}$ be a Markov chain satisfying

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] > 0.$$

Then the Markov property implies

$$P[X_{t_1} = i_1, \ldots, X_{t_n} = i_n] = P[X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}] \cdot p_{i_{n-1}, i_n}(t_{n-1}, t_n).$$

This yields (2.5) by induction. The converse statement is trivial.

Theorem 2.2.9 (Markov property). *Let* $(X_t)_{t \in T}$ *be a Markov chain and* n, m *be elements of* $\mathbb{N}$. *Fix* $t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_{n+m}$, $i \in S$ *and sets* $A \subset S^{n-1}$ *(where* S^{n-1} *denotes the* $n-1$ *times Cartesian product of the set* S*) and* $B \subset S^m$ *such that*

$$P\left[(X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] > 0.$$

Then the following equation (Markov property) *holds:*

$$\begin{aligned} P\left[(X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B \,|\, (X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] \\ = P\left[(X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B \,|\, X_{t_n} = i\right]. \end{aligned}$$

Proof. We use the notation $\boldsymbol{i}^n = (i_1, i_2, \ldots, i_n)$. An application of equation (2.5) yields:

$$\begin{aligned}
&P\left[(X_{t_1}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] \\
&\qquad = \sum_{\boldsymbol{i}^{n-1} \in A, i_n = i} P\left[X_{t_1} = i_1\right] \times \prod_{k=1}^{n-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}), \\
&P\left[(X_{t_1}, \ldots, X_{t_{n+m}}) \in A \times \{i\} \times B\right] \\
&\qquad = \sum_{\boldsymbol{i}^{n+m} \in A \times \{i\} \times B} P\left[X_{t_1} = i_1\right] \times \prod_{k=1}^{n+m-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}).
\end{aligned}$$

Finally these two equations imply

$$\begin{aligned}
&P\left[(X_{t_{n+1}}, \ldots, X_{t_{n+m}}) \in B \,|\, (X_{t_1}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] \\
&\quad = \sum_{(i_n, i_{n+1}, \ldots, i_{n+m}) \in \{i\} \times B} \prod_{k=n}^{n+m-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}) \\
&\qquad\qquad \times \frac{\sum_{\boldsymbol{i}^{n-1} \in A} P\left[X_{t_1} = i_1\right] \times \prod_{l=1}^{n-1} p_{i_l, i_{l+1}}(t_l, t_{l+1})}{\sum_{\boldsymbol{i}^{n-1} \in A} P\left[X_{t_1} = i_1\right] \times \prod_{l=1}^{n-1} p_{i_l, i_{l+1}}(t_l, t_{l+1})} \\
&\quad = \sum_{(i_{n+1}, \ldots, i_{n+m}) \in B} \prod_{k=n}^{n+m-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}) \frac{P\left[X_{t_n} = i\right]}{P\left[X_{t_n} = i\right]} \\
&\quad = P\left[(X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B | X_{t_n} = i\right].
\end{aligned}$$

Definition 2.2.10. *A Markov chain $(X_t)_{t \in T}$ is called* homogeneous*, if it is time homogeneous, i.e., the following equation holds for all $s, t \in \mathbb{R}, h > 0$ and $i, j \in S$ such that $P[X_s = i] > 0$ and $P[X_t = i] > 0$:*

$$P[X_{s+h} = j \,|\, X_s = i] \quad = \quad P[X_{t+h} = j \,|\, X_t = i].$$

For a homogeneous Markov chain we use the notation:

$$\begin{aligned}
p_{ij}(h) &:= p_{ij}(s, s+h), \\
P(h) &:= P(s, s+h).
\end{aligned}$$

Remark 2.2.11. 1. A homogeneous Markov chain is characterized by the fact, that the transition probabilities, and therefore also the transition matrices, only depend on the size of the time increment.

2. For a homogeneous Markov chain one can simplify the Chapman-Kolmogorov equations to the semi group property:
$$P(s+t) = P(s) \times P(t).$$
The semi group property is popular in many different areas e.g. in quantum mechanics.
3. The mapping
$$P : T \to M_n(\mathbb{R}), t \mapsto P(t)$$
defines a one parameter *semi group*.

2.3 Markov chains in continuous time and Kolmogorov's differential equations

In the following we will only consider Markov chains on a finite state space. Thus point wise convergence and uniform convergence will coincide on S. This enables us to give some of the proofs in a simpler form.

Definition 2.3.1. *Let $(X_t)_{t \in T}$ be a Markov chain with finite state space S and $T \subset \mathbb{R}$. For $N \subset S$ we define*
$$p_{jN}(s,t) := \sum_{k \in N} p_{jk}(s,t).$$

Definition 2.3.2 (Transition rates). *Let $(X_t)_{t \in T}$ be a Markov chain in continuous time with finite state space S. $(X_t)_{t \in T}$ is called* regular*, if*
$$\mu_i(t) = \lim_{\Delta t \searrow 0} \frac{1 - p_{ii}(t, t+\Delta t)}{\Delta t} \text{ for all } i \in S, \quad (2.6)$$
$$\mu_{ij}(t) = \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \text{ for all } i \neq j \in S \quad (2.7)$$
are well defined and continuous with respect to t.

The functions $\mu_i(t)$ and $\mu_{ij}(t)$ are called transition rates *of the Markov chain. Furthermore we define μ_{ii} by*
$$\mu_{ii}(t) = -\mu_i(t) \text{ for all } i \in S. \quad (2.8)$$

Remark 2.3.3. 1. In the insurance model the regularity of the Markov chain is used to derive the differential equations which are satisfied by the mathematical reserve corresponding to the policy (Thiele's differential equation, e.g. Theorem 5.2.1).

2. One can understand the transition rates as derivatives of the transition probabilities. For example we get for $i \neq j$:

$$\begin{aligned} \mu_{ij}(t) &= \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \\ &= \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t) - p_{ij}(t,t)}{\Delta t} \\ &= \frac{d}{ds} p_{ij}(t,s)\Big|_{s=t}. \end{aligned}$$

3. $\mu_{ij}(t)\,dt$ can be understood as the infinitesimal transition rate from i to j ($i \rightsquigarrow j$) in the time interval $[t, t+dt]$. Similarly, $\mu_i(t)\,dt$ can be understood as the infinitesimal probability of leaving state i in the corresponding time interval. Let us define

$$\Lambda(t) = \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) & \mu_{13}(t) & \cdots & \mu_{1n}(t) \\ \mu_{21}(t) & \mu_{22}(t) & \mu_{23}(t) & \cdots & \mu_{2n}(t) \\ \mu_{31}(t) & \mu_{32}(t) & \mu_{33}(t) & \cdots & \mu_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n1}(t) & \mu_{n2}(t) & \mu_{n3}(t) & \cdots & \mu_{nn}(t) \end{pmatrix}.$$

In a sense, Λ generates the behavior of the Markov chain. That is, for a homogeneous Markov chain the following equation holds:

$$\Lambda(0) = \lim_{\Delta t \to 0} \frac{P(\Delta t) - 1}{\Delta t}.$$

$\Lambda := \Lambda(0)$ is called the *generator of the one parameter semi group*. We can reconstruct $P(t)$ by

$$P(t) = \exp(t\,\Lambda) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda^n.$$

4. In the remainder of the book we will only consider finite state spaces. This enables us to avoid certain technical difficulties with respect to convergence.

Based on the transition rates we can prove Kolmogorov's differential equations. These connect the partial derivatives of p_{ij} with μ:

Theorem 2.3.4 (Kolmogorov). *Let $(X_t)_{t \in T}$ be a regular Markov chain on a finite state space S. Then the following statements hold:*

1. *(Backward differential equations)*

$$\frac{d}{ds}p_{ij}(s,t) = \mu_i(s)p_{ij}(s,t) - \sum_{k\neq i}\mu_{ik}(s)p_{kj}(s,t), \tag{2.9}$$

$$\frac{d}{ds}P(s,t) = -\Lambda(s)P(s,t). \tag{2.10}$$

2. *(Forward differential equations)*

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\mu_j(t) + \sum_{k\neq j}p_{ik}(s,t)\mu_{kj}(t), \tag{2.11}$$

$$\frac{d}{dt}P(s,t) = P(s,t)\Lambda(t). \tag{2.12}$$

Proof. The major part of the proof is based on the equations of Chapman and Kolmogorov.

1. We will prove the matrix version of the statement. This will help to highlight the key properties. Let $\Delta s > 0$ and set $\xi := s + \Delta s$.

$$\begin{aligned}\frac{P(\xi,t) - P(s,t)}{\Delta s} &= \frac{1}{\Delta s}\Big(P(\xi,t) - P(s,\xi)\,P(\xi,t)\Big)\\ &= \left(\frac{1}{\Delta s}(1 - P(s,\xi))\right) \times P(\xi,t)\\ &\longrightarrow -\Lambda(s)P(s,t) \text{ for } \Delta s \searrow 0,\end{aligned}$$

   where we used the Chapman-Kolmogorov equation and the continuity of the matrix multiplication.

2. Analogous one can prove the forward differential equation. Let $\Delta t > 0$.

$$\begin{aligned}\frac{P(s,t+\Delta t) - P(s,t)}{\Delta t} &= \frac{1}{\Delta t}\Big(P(s,t)P(t,t+\Delta t) - P(s,t)\Big)\\ &= P(s,t) \times \frac{1}{\Delta t}\Big(P(t,t+\Delta t) - 1\Big)\\ &\longrightarrow P(s,t)\Lambda(t) \text{ for } \Delta t \searrow 0.\end{aligned}$$

Remark 2.3.5. The primary application of Kolmogorov's differential equations is to calculate the transition probabilities p_{ij} based on the rates μ.

Definition 2.3.6. *Let $(X_t)_{t\in T}$ be a regular Markov chain on a finite state space S. Then we denote the conditional probability to stay during the time interval $[s,t]$ in j by*

$$\bar{p}_{jj}(s,t) := P\left[\bigcap_{\xi\in[s,t]} \{X_\xi = j\} \,|\, X_s = j\right]$$

where $s,t \in \mathbb{R}, s \leq t$ *and* $j \in S$.

In the setting of a life insurance this probability can for example be used to calculate the probability that the insured survives 5 years. The following theorem illustrates how this probability can be calculated based on the transition rates.

Theorem 2.3.7. *Let* $(X_t)_{t\in T}$ *be a regular Markov chain. Then*

$$\bar{p}_{jj}(s,t) = \exp\left(-\sum_{k\neq j}\int_s^t \mu_{jk}(\tau)d\tau\right) \tag{2.13}$$

holds for $s \leq t$, *if* $P[X_s = j] > 0$.

Proof. We define $K_j(s,t)$ by $K_j(s,t) := \bigcap_{\xi\in[s,t]}\{X_\xi = j\}$. Let $\Delta t > 0$. We have $P[A\cap B \,|\, C] = P[B\,|\,C]\,P[A\,|\,B\cap C]$ and thus

$$\begin{aligned}
\bar{p}_{jj}(s,t+\Delta t) &= P[K_j(s,t)\cap K_j(t,t+\Delta t)\,|\,X_s = j] \\
&= P[K_j(s,t)\,|\,X_s = j]\,P[K_j(t,t+\Delta t)\,|\,X_s = j\cap K_j(s,t)] \\
&= P[K_j(s,t)\,|\,X_s = j]\,P[K_j(t,t+\Delta t)\,|\,X_t = j] \\
&= \bar{p}_{jj}(s,t)\,P[K_j(t,t+\Delta t)\,|\,X_t = j],
\end{aligned}$$

where we used the Markov property and the relation $\{X_s = j\}\cap K_j(s,t) = \{X_t = j\}\cap K_j(s,t)$. The previous equation yields

$$\begin{aligned}
\bar{p}_{jj}(s,t+\Delta t) - \bar{p}_{jj}(s,t) &= -\bar{p}_{jj}(s,t)\times\Big(1 - P[K_j(t,t+\Delta t)\,|\,X_t = j]\Big) \\
&= -\bar{p}_{jj}(s,t)\times\left(\sum_{k\neq j} p_{jk}(t,t+\Delta t) + o(\Delta t)\right),
\end{aligned}$$

where we used that the rates $\mu_{..}$ are well defined.

Now taking the limit we get the differential equation

$$\frac{d}{dt}\bar{p}_{jj}(s,t) = -\bar{p}_{jj}(s,t)\times\sum_{k\neq j}\mu_{jk}(t).$$

Solving this equation with the boundary condition $\bar{p}_{jj}(s,s) = 1$ yields the statement of the theorem, (2.13).

2.4 Examples

In this section we want to illustrate the theory of the previous sections by some examples.

Example 2.4.1 (Life insurance). We start with a life insurance, which provides a sum of money to the heirs in case of the death of the insured. Usually one uses for this a model with either two states (alive $*$, dead $\dagger$) or three states (alive, dead (accident), dead (disease)). We will use the model with two states, and the death rate will be exemplary modeled by the function

$$\mu_{*\dagger}(x) \quad = \quad \exp(-9.13275 + 8.09438 \cdot 10^{-2} x - 1.10180 \cdot 10^{-5} x^2). \tag{2.14}$$

The death rate is the transition rate of the state transition $* \rightsquigarrow \dagger$. See section 4.3 for a derivation of the death rate. Based on the death rate and formula (2.13) we are now able to calculate the survival probability of a 35 year old man:

$$\bar{p}_{**}(35, x) = \exp\left(- \int_{35}^{x} \mu_{*\dagger}(\tau) d\tau \right), \qquad \text{for } x > 35.$$

Figure 2.1 shows the transition rate (dotted line) and the survival probability (continuous line) based on $x = 35$.

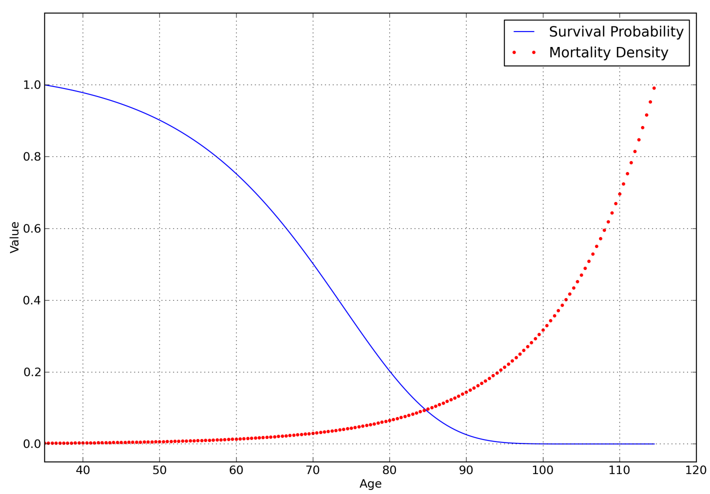


Figure 2.1. Mortality density $\mu_{*\dagger}(x)$ and survival probability $\bar{p}_{**}(35, x)$

Example 2.4.2 (Disability pension). We consider a model of a disability pension with the following three states:

state	**symbol**
active	$*$
disabled	$\diamond$
dead	$\dagger$

The transition rates are defined by

$$\begin{aligned}
\sigma(x) &:= 0.0004 + 10^{(0.060\,x - 5.46)},\\
\mu(x) &:= 0.0005 + 10^{(0.038\,x - 4.12)},\\
\mu_{*\diamond}(x) &:= \sigma(x),\\
\mu_{*\dagger}(x) &:= \mu(x),\\
\mu_{\diamond\dagger}(x) &:= \mu(x).
\end{aligned}$$

The transition rate σ is the infinitesimal probability of becoming disabled and μ is the corresponding probability of dying. We set the other transition rates equal to 0. Thus in particular this model does not incorporate the possibility of becoming active again ($\mu_{\diamond *} = 0$). Moreover one should note that in this model the mortality of disabled persons is equal to the mortality of active persons. This is a simplification, since in reality disabled persons have a higher mortality (they die earlier with a higher probability) than active persons. Therefore, this model yields an overpriced premium for the disability pension.

The explicit knowledge of the transition probabilities p_{ij} is useful for many formulas in insurance mathematics. For the current model they can be calculated by Kolmogorov's differential equations. We get

$$\begin{aligned}
p_{**}(x,y) &= \exp\left(-\int_x^y [\mu(\tau)+\sigma(\tau)]\,d\tau\right),\\
p_{*\diamond}(x,y) &= \exp\left(-\int_x^y \mu(\tau)d\tau\right) \times \left(1 - \exp\left(-\int_x^y \sigma(\tau)d\tau\right)\right),\\
p_{\diamond\diamond}(x,y) &= \exp\left(-\int_x^y \mu(\tau)d\tau\right),
\end{aligned}$$

which solve Kolmogorov's differential equations for this model:

$$
\begin{aligned}
\frac{d}{dt} p_{**}(s,t) &= -p_{**}(s,t) \times (\mu(t) + \sigma(t)), \\
\frac{d}{dt} p_{*\diamond}(s,t) &= -p_{*\diamond}(s,t)\, \mu(t) + p_{**}(s,t)\, \sigma(t), \\
\frac{d}{dt} p_{*\dagger}(s,t) &= (p_{**}(s,t) + p_{*\diamond}(s,t)) \times \mu(t), \\
\frac{d}{dt} p_{\diamond *}(s,t) &= 0, \\
\frac{d}{dt} p_{\diamond\diamond}(s,t) &= -p_{\diamond\diamond}(s,t)\, \mu(t), \\
\frac{d}{dt} p_{\diamond\dagger}(s,t) &= p_{\diamond\diamond}(s,t)\, \mu(t), \\
\frac{d}{dt} p_{\dagger\dagger}(s,t) &= 0,
\end{aligned}
$$

with the boundary conditions $p_{ij}(s,s) = \delta_{ij}$. Note that, if one uses a model with a positive probability of becoming active again, one has to modify the first, second, forth and fifth equation. Obviously one can solve these equations with numerical methods, the solutions for the given example are listed in Table 2.1.

Table 2.1. Transition probabilities for the disability insurance

Initial age	$x_0 = 30$
Algorithm	Runge-Kutta of order 4
Step width	0.001

age x	$p_{**}(x_0,x)$	$p_{*\diamond}(x_0,x)$	$p_{*\dagger}(x_0,x)$	$p_{\diamond\diamond}(x_0,x)$	$p_{\diamond\dagger}(x_0,x)$
30.00	1.00000	0.00000	0.00000	1.00000	0.00000
35.00	0.98743	0.00354	0.00903	0.99097	0.00903
40.00	0.96998	0.00850	0.02152	0.97849	0.02152
45.00	0.94457	0.01620	0.03923	0.96077	0.03923
50.00	0.90624	0.02903	0.06474	0.93526	0.06474
55.00	0.84725	0.05106	0.10169	0.89831	0.10169
60.00	0.75677	0.08832	0.15491	0.84509	0.15491
65.00	0.62287	0.14700	0.23013	0.76987	0.23013

Exercise 2.4.3. Consider the above system of differential equations.

1. Find an exact solution.
2. Find a numerical approximation to the solution.

3. Interest rate

3.1 Introduction

An important part of every insurance contract is the underlying interest rate. The so called technical interest rate describes the interest which the insurer guarantees to the insured. It is a significant factor for the size of the premiums. If the technical interest rate is too low it yields inflated premiums, if it is too high it might yield to insolvency of the insurance company.

For the technical interest rate one can use a deterministic or a stochastic model. In the latter case the interest rate will be coupled to the bond market. In the following we are going to define the main parameters connected to the interest rate and present their relations.

3.2 Definitions

Example 3.2.1. Suppose we put 10,000 USD on a bank account on the first of January. If at the end of the year there are 10,500 USD on the account, then the underlying interest rate was 5 %.

Definition 3.2.2 (Interest rate). *We denote by i the* yearly interest rate. *Furthermore we assume, that it depends on time and write $i_t, t \geq 0$. If we use a stochastic model for the interest rate, then i is a stochastic process $(i_t(\omega))_{t\geq 0}$.*

One should note that this definition is useful in particular for the discrete time model. Since in this case one would use the time intervals which are given by the discretization. The calculation of the future capital based on an interest rate is done by

$$B_{t+1} = (1 + i_t) \times B_t.$$

Here B_t denotes the value of the account at time t. Often also the inverse of this relation is relevant. Thus one defines the discount rate.

Definition 3.2.3 (Discount rate). *Let i_t be the interest rate in year t. Then*

$$v_t = \frac{1}{1+i_t}$$

is the discount rate *in year t.*

The discount rate can be used to calculate the net present value (the present value of the future benefits). If the interest rate is a stochastic process the previous considerations lead to the following problem: suppose we are going to receive 1 USD in one year, what is its present value? Generally there are two possible ways to find an answer.

Valuation principle A: If the interest rate i is known, the present value X is

$$X = \frac{1}{1+i}$$

and thus its mean is

$$X_A = E\left[\frac{1}{1+i}\right].$$

Valuation principle B: If the interest rate is known, the value of the account at the end of the year is $X(1+i)$. Thus

$$1 = E\left[X(1+i)\right] = X \times E\left[1+i\right]$$

holds and the mean is

$$X_B = \frac{1}{E\left[1+i\right]}.$$

But in general X_A is not equal to X_B. To overcome this problem one needs to fix by an assumption the valuation principle of the interest rate: we are always going to determine the net present value by valuation principle A (cf. [Büh92]).

After solving this paradox situation one understands the importance of the discount rate. Obviously, the same problem also arises in continuous time. For models in continuous time we assume that the interest is also payed continuously such that

$$B_{t+s} = \exp\left(\int_t^{t+s} \delta(\xi) d\xi\right) \times B_t.$$

Definition 3.2.4 (Interest intensity). *The* interest intensity *at time t is denoted by $\delta(t)$.*

A yearly interest rate i yields

$$e^{\delta} = 1 + i$$

and thus

$$\delta = \ln(1 + i).$$

In continuous time the discount rate (from t to 0) is

$$v(t) = \exp\left(- \int_0^t \delta(\xi)\, d\xi \right).$$

Here the discount rate is modeled from t to 0 which is contrary to the discrete setting. The following relation holds:

$$v_t = \exp\left(- \int_t^{t+1} \delta(\xi)\, d\xi \right).$$

For a stochastic interest rate the interest intensity δ is also a stochastic process $(\delta_t(\omega))_{t \geq 0}$. Also in the continuous time setting we are going to use valuation principle A.

Finally it is worth to note the difference between $v(t)$ which denotes the discounting back to time 0 and v_t which denotes the discount from time $t+1$ to t. This later quantity is commonly used in actuarial sciences for recursion formulae.

3.3 Models for the interest rate process

Now we want to describe the stochastic behavior of the interest rate. We start with an analysis of the average interest rate of Swiss government bonds in Swiss franc. Figure 3.1 shows the rate from 1948 up to 2009.

The figure shows that during the given period the interest rate of bonds was subject to huge fluctuations. The minimal rate of 1.93 % was reached in 2005. The maximum of 7.41 % was recorded during the "oil crisis" in the autumn of 1974. It is interesting to note, that in the first printing of this book (1999) the minium was actually reached in 1954 at about 2.5 %. At this time nobody expected that the interest rates in Switzerland would drop that far again. Based on this observation the inherent risk of the technical interest rate in insurance policies becomes obvious.

After seeing these values one starts to wonder how the technical interest rate should be determined. First of all, this depends on the purpose of the model in use. One has to differentiate between short term and long term relations. Moreover, the interest rate might only be used for marketing or forecasting

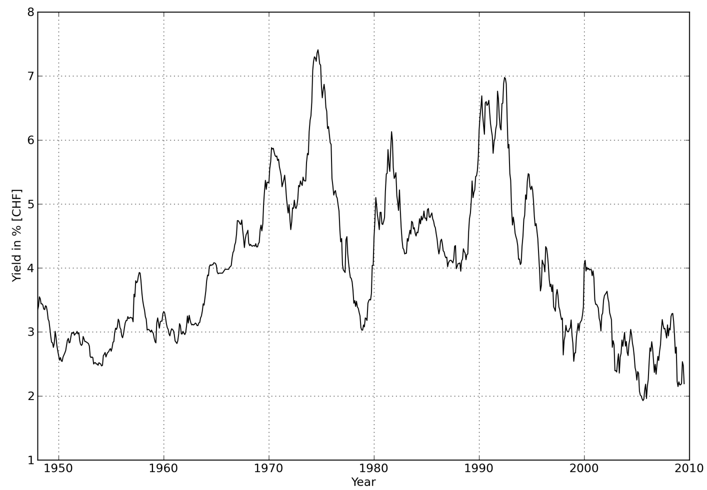


Figure 3.1. Average yield of government bonds in %

purposes. In any case it should be emphasized that it might be risky to fix a constant interest rate above the level of the observed minimum. Since in this case there might be periods during which the interest yield of the assets does not cover the liabilities. Thus one has to take care when fixing the technical interest rate.

Another method to determine the technical interest rate is based on the analysis of the yield-curve or the forward-curve. This curves allow to measure the interest rate structure. The yield-curve can be used to determine the interest rate which one would get for a bond with a fixed investment period. Figure 3.2 shows the yield-curve for various currencies, it indicates that the interest rate is smaller for a bond with a short investment period than for a bond with a long investment period. This is called a 'normal' interest rate structure. Conversely, one speaks of an 'inverse' interest rate structure if the bonds with a short investment period provide a better yield than those with a long investment period.

This point of view provides a realistic evaluation of the interest rate. To utilize this we are going to define the so called zero coupon bond.

Definition 3.3.1 (Zero coupon bond). *Let* $t \in \mathbb{R}$. *Then the* zero coupon bond *with contract period* t *is defined by*

$$\mathcal{Z}_{(t)} = (\delta_{t,\tau})_{\tau \in \mathbb{R}^+}.$$

Thus the zero coupon bound is a security, which has the value 1 at time t.

Definition 3.3.2 (Price of a zero coupon bond). *Let $t \in \mathbb{R}$. Then the* price *of a zero coupon bond $\mathcal{Z}_{(s)}$ at time t with contract period s is denoted by*

$$\pi_t \left(\mathcal{Z}_{(s)} \right).$$

Based on these curves one can calculate the forward rates, i.e. the interest rate for year $n \rightsquigarrow n+1$, of the corresponding investment:

$$(1 + i_k) = \frac{\pi_t \left(\mathcal{Z}_{(k)} \right)}{\pi_t \left(\mathcal{Z}_{(k+1)} \right)}.$$

Here i_k is called the forward rate at time t for the contract period $[k, k+1[$, it is the *expected* rate for this time interval. Therefore the discount rate is given by

$$v_k = \frac{\pi_t \left(\mathcal{Z}_{(k+1)} \right)}{\pi_t \left(\mathcal{Z}_{(k)} \right)}.$$

Thus one can use a time dependent technical interest rate and adapt the necessary elements in the expectation based on the liabilities. This is especially

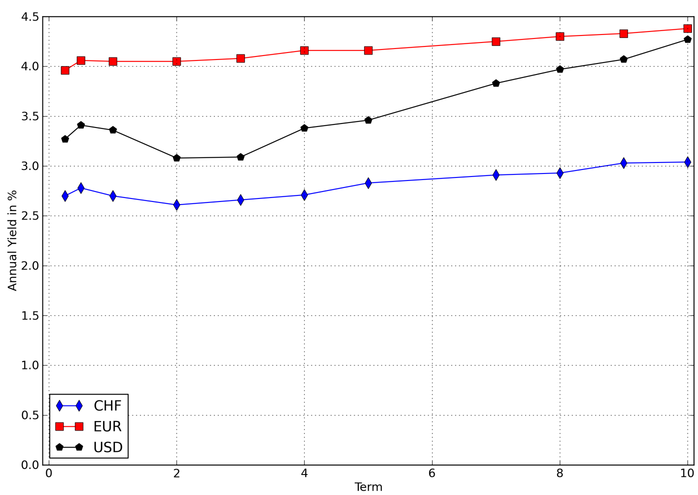


Figure 3.2. Yield curves as at 1.1.2008

useful for short term contracts with cash flows which are assessable. Consider for example the acquisition of a pension portfolio. In this case one takes over the obligation to pay the pensions of a given pension fund. The described method reduces the risks taken by fixing the technical interest rate.

The third method to determine the technical interest rate uses a stochastic interest rate model. This is interesting for practical and theoretical purposes. On the one hand this method is useful for policies which are tied to the performance of fonds and which provide guaranties. On the other hand it enables us to derive models which provide a tool to measure the risk of an insurance portfolio with respect to changes of the interest rate (see Chapter 9). It turns out that for models with stochastic interest rate the corresponding risk does not vanish when the number of policies increases. This is a major difference to the deterministic model. Furthermore it means that the risk induced by the interest rate has a systemic, and thus dangerous, component. The construction of these models requires an analysis of the returns of several investment categories. Figure 3.3 shows the performance of two indices. These are measurements of the mean value of the return on investment in a given category. The indices in Figure 3.3 are the following

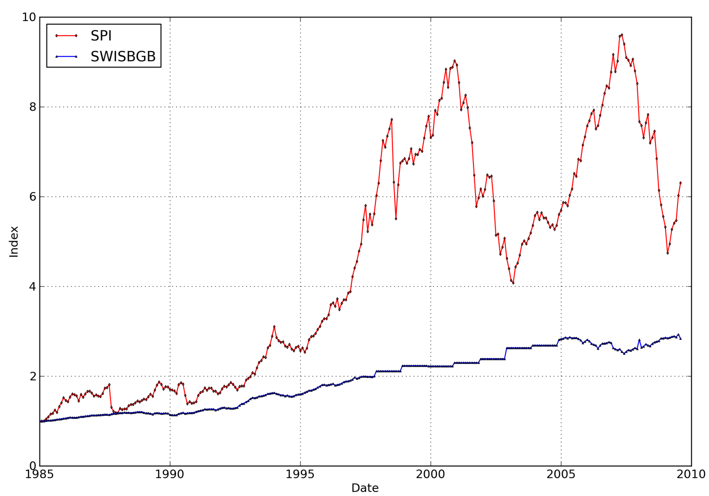


Figure 3.3. Performance of different indices

SPI	Swiss Performance Index: Swiss shares
SWISBGB	Swiss government bonds

Observing the performances of these indices (Figure 3.3) we find a significant difference between shares and bonds. The expected return is larger for shares, but they also have a larger volatility (greater variance).

For a model with stochastic interest rate one has to model processes as depicted in the Figures 3.1, 3.2 and 3.3. The main difficulty in this context is the fact that there is no common standard model. Thus the actuary is responsible for selecting an appropriate model for the given problem.

In the following section we will describe some popular models. The reader interested in further details on the financial market, in particular interest rate models, is referred for example to [Hul12].

3.4 Stochastic interest rate

In the previous section we have seen several possibilities to determine the value of a cash flow based on the interest rate. Furthermore, the basics of a stochastic interest rate model were introduced. In this section we will present explicitly several stochastic interest rate models.

First of all one has to understand the difference between the stochastic behavior of the interest rate and the stochastic component of the mortality. Both create a risk for an insurance company. On the one hand there is the risk induced by the fluctuation of the interest rate and on the other hand there is risk based on the individual mortality. Changes of the interest rate affect all policies to the same degree. But the variation of the risk based on the individual mortality decreases when the number of policies increases. This is due to the law of large numbers and the independence of individual lifetimes.

Now we give a brief survey of stochastic interest rate models. We will concentrate on a description of these models without rating them. Nevertheless one should note that some models (e.g. the random walk model) are unsuitable to realistically describe an interest rate process.

3.4.1 Discrete time interest rate models

Random walk: Let $\mu \in \mathbb{R}_+$ and $t \geq 0$. The interest rate i_t is defined by

$$\begin{aligned} i_t &= \mu + X_t, \mu \in \mathbb{R}, \\ X_t &= X_{t-1} + Y_t, \\ Y_t &\sim \mathcal{N}(0, \sigma^2) \text{ i.i.d.} \end{aligned}$$

Note that this model is too simple to capture the real behavior.

AR(1)-model: In this model the interest rate is an autoregressive process of order 1:

$$\begin{aligned} i_t &= \mu + X_t, \\ X_t &= \phi\, X_{t-1} + Y_t, \text{ with } |\phi| < 1, \\ Y_t &\sim \mathcal{N}(0, \sigma^2). \end{aligned}$$

The model is mostly used by actuaries in England. The main idea is to start with an AR(1)-process as a model for the inflation. Then, in a second step, the models of the other economic values are based on this inflation. The constructed models have many parameters and are difficult to fit. References: [BP80] [Wil86] [Wil95].

3.4.2 Continuous time interest rate models

Brownian motion: $\delta_t = \delta + \sigma\, W_t$, where W_t is a standard Brownian motion.

Vasiček-model: The interest intensity is defined by the following stochastic differential equation

$$d\delta_t = -\alpha\,(\delta_t - \delta)\,dt + \sigma\,dW_t.$$

References: [Vas77].

Cox-Ingersoll-Ross: The interest intensity is defined by the following stochastic differential equation

$$d\delta_t = -\alpha\,(\delta_t - \delta)\,dt + \sigma\,\sqrt{\delta_t}\,dW_t.$$

References: [CIR85].

Markovian interest intensities: This model ([Nor95b]) uses a Markov chain $(X_t)_{t\geq 0}$ on a finite state space and a deterministic function $\delta_j(t)$ for each $j \in S$. Then the interest intensity is defined by

$$\delta_t = \sum_{j\in S} \chi_{\{X_t = j\}} \delta_j(t).$$

This means, that the interest intensity in a given state j at time t is determined by the corresponding deterministic function $\delta_j(.)$ evaluated at t. The model has the advantage, that it can be integrated into the Markov model. Furthermore it is very flexible due to its general state space. Therefore we will focus on this model in the following.

One should note that the Vasiček-model and the CIR-model feature mean reversion. Thus the interest intensity without stochastic noise (dW) converges in the long run toward the the mean intensity δ, since the differential equation without stochastic noise

$$d\delta_t = -\alpha\,(\delta_t - \delta)\,dt$$

has the solution

$$\delta_t = \gamma \times \exp(-\alpha\, t) + \delta.$$

The Vasiček-model and the Cox-Ingersoll-Ross-model are often used to model interest rate processes in applications. We will use these models in Chapter 9.

Brownian motion and the Vasiček-model are problematic, since they allow negative interest rates with positive probability. In the Cox-Ingersoll-Ross model this can be prevented by an appropriate choice of the parameters.

In the following we assume that the presented stochastic differential equations have a solution.

We have seen above various models which are based on fundamentally different ideas. But in addition to the risk in the choice of the model there are further relevant systemic risks which affect the interest rate. These are:

The interest rate paid on an investment is not purely random. It also depends on political decisions. For example a monetary union causes the interest rates to converge, since in this case only one currency with one (random) interest rate exists (e.g. the European monetary union).

4. Cash flows and the mathematical reserve

4.1 Introduction

In the previous two chapters we introduced several types of insurances and their setup. Based on this we will now answer several fundamental questions.

First of all we will decide which general model we are going to use. Afterward we will explain how to value and price an insurance policy.

The present value of an insurance policy, the so called mathematical reserve, has to be determined by an insurance company on a yearly basis for the annual statement. This is necessary since the company has to reserve this value. The mathematical reserve is also important for the insured when he wants to cancel his policy before maturity.

In the remainder of the chapter the insurance model from Chapter 1 will be combined with the stochastic models of Chapter 2. Obviously Markov chains with a countable state space are not the only possible stochastic model, but we will focus on these. On the one hand they are general enough to model many phenomena. On the other hand the corresponding formulas are simple enough to perform explicit calculations.

4.2 Examples

In this section we present some examples which motivate the use of the Markov chain model for insurance policies.

Example 4.2.1 (Life insurance). Usually the state space of a permanent life insurance consists of the states "dead" and "alive". Thus we use for the policy setup and for the stochastic process the state space $S = \{*, \dagger\}$, where $*$ denotes "alive" and $\dagger$ denotes "dead". Based on the benefits of such a policy, as described in Chapter 1, one has to model the corresponding stochastic process. We will use the exemplary life insurance from Chapter 2. A typical sample path of the stochastic process corresponding to this policy is shown in Figure 4.1. It indicates that at the time of death (here at $x = 45$) the

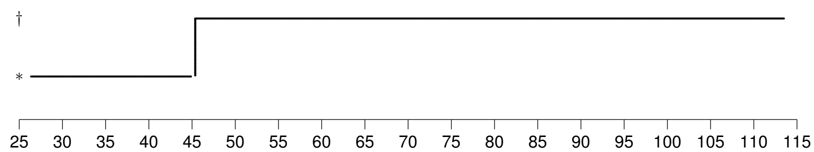


Figure 4.1. Trajectory of a mortality cover

corresponding payment (e.g. 200,000 USD) is due. The mortality at that time is:

$$\begin{aligned}\mu_{*\dagger}(x)\big|_{x=45} &= \exp(-9.13275 + 0.08094x - 0.000011x^2)\big|_{x=45} \\ &= 0.00404.\end{aligned}$$

This means that on average 4 out of 1000 forty-five year old men die per year.

Up to now we are not able to calculate the premiums for the policy in the example above. But we already notice the interplay between the payments and the stochastic processes.

Example 4.2.2 (Temporary disability pension). In this example we consider a policy of a disability pension which corresponds to the sample path in Figure 4.2. We want to record the various cash flows which it induces. For this we take the transition intensities from Example 2.4.2 with the additional assumption $\mu_{\diamond *}(x) = 0.05$. Then the sample path presented in Figure 4.2 causes the cash flows listed in Table 4.1.

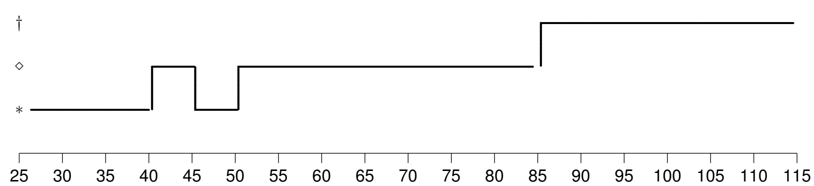


Figure 4.2. Trajectory of a disability cover

Table 4.1. Example of cash flows for a disability pension

time	**state**	**cash flow**	μ
$x \in [0, 40[$	active ($*$)	premiums	
$x = 40$	becomes disabled	disability capital	$\mu_{*\diamond} = 0.00214$
$x \in]40, 45[$	disabled ($\diamond$)	disability pension	
$x = 45$	becomes active	—	$\mu_{\diamond *} = 0.05000$
$x \in [45, 50[$	active ($*$)	premiums	
$x = 50$	becomes disabled	(maybe) disability capital	$\mu_{*\diamond} = 0.00387$
$x \in]50, 85[$	disabled ($\diamond$)	disability pension	
from 65		pension	
$x = 85$	dead	sum payable at death	$\mu_{\diamond \dagger} = 0.12932$

4.3 Fundamentals

In order to derive realistic models we have to know the fundamentals (underlying probabilities and biometric quantities). They are especially needed for the calculation of the premiums and mathematical reserves. As actuary one can look up the fundamentals in published tables. These list the probability of given events, for example the probability to die at a given age.

The tables used by insurance companies often incorporate a certain spread. For example one increases the probability of dying at a certain age if one calculates a life insurance. Conversely, one increases the survival rate if one calculates a pension. This spread is used to decrease the risk of default and to cover possible demographic trends. That this is necessary illustrate for example the Tables 4.2 and 4.3. They list the average life expectancy for several generations, as given in the Swiss mortality tables. The average life expectancy is the number of years which a person of given age has on average still ahead. These tables clearly show that the average life expectancy increased during the last one hundred years. Therefore a spread is clearly necessary to cover this trend. Other western countries experience a similar increase of the average life expectancy.

Table 4.2. Average life expectancy based on Swiss mortality tables (male)

Alter	1881-88	1921-30	1939-44	1958-63	1978-83	1998-03	2008-13
1	51.8	61.3	64.8	69.4	72.1	76.6	79.5
20	39.6	45.2	47.9	51.5	53.8	58.0	60.7
40	25.1	28.3	30.4	32.8	35.1	39.0	41.3
60	12.4	13.8	14.8	16.2	17.9	21.1	23.0
80	4.2	4.3	4.8	5.5	6.3	7.5	8.3

Table 4.3. Average life expectancy based on Swiss mortality tables (female)

Alter	1881-88	1921-30	1939-44	1958-63	1978-83	1998-03	2008-13
1	52.8	63.8	68.5	74.5	78.6	82.2	83.8
20	41.0	47.6	51.3	56.2	60.1	63.4	64.9
40	26.7	30.9	33.4	37.0	40.7	43.8	45.3
60	12.7	15.1	16.7	19.2	22.4	25.2	26.4
80	4.2	4.9	5.3	6.1	7.8	9.3	10.0

We note that demographic quantities are constantly changing, like the average life expectancy. But where does this data actually come from and how where these tables calculated?

For the mortality tables one uses either the samples which are owned by the given insurance company or a collection of samples which is obtained jointly by several insurers. Then, to calculate the mortality rate, one counts the number of persons at risk and the number of died subjects for a given period of time (e.g. five years). The following example is based on data obtained by a large Swiss insurance company [PT93]. Figure 4.3 shows the number of alive and dead people at a given age. Figure 4.4 shows the raw mortality and the smoothed mortality.

The smoothed mortality is obtained by a smoothing algorithm. We are not going to discuss these algorithms. But we want to note, that there are various algorithms which greatly differ in their complexity.

On the raw curve one notices for example the accident-bump (i.e. the increased mortality) between 15 and 25 years. This is not visible in the smooth curve. Thus one has to adjust in this region the smoothed mortality.

In this example a polynomial of degree two was fitted to $\log(\mu_{*\dagger})$:

$$\mu_{*\dagger}(x) = \exp(-7.85785 + 0.01538 \cdot x + 5.77355 \cdot 10^{-4} \cdot x^2).$$

Analogous other demographic quantities which are relevant for the calculation of the premiums are obtained. Also for these a smoothed curve is obtained by an application a smoothing algorithm to the raw data.

The relevant probabilities and biometric quantities are collected in a catalog. Then, with the help of such a catalog, one can calculate the premiums and values of various products.

Table 4.4 lists the typically relevant quantities.

Further details about the calculation of mortality tables and disability tables for the German and European market can be found in [DAV09].

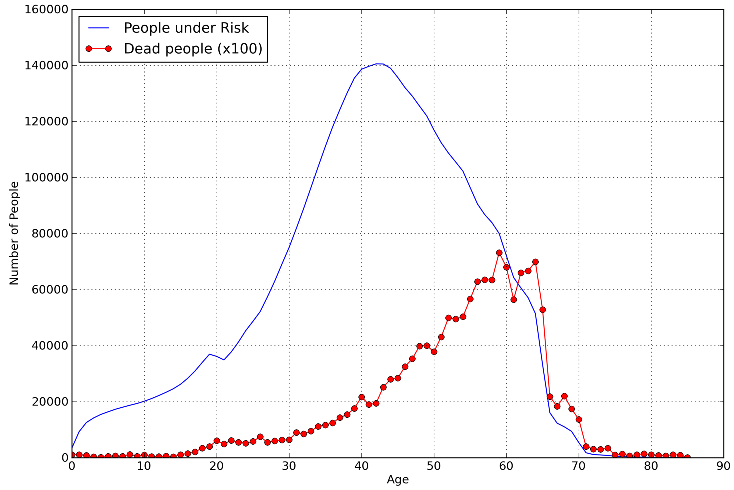


Figure 4.3. Inforce and number of dead people

Table 4.4. Typical quantities for the calculation of premiums

variable	meaning
q_x	mortality, possibly separate for accidents and illness,
i_x	probability of becoming disabled, possibly separate for accidents and illness,
r_x	probability of becoming active again, possibly partitioned by the lengths of the disability period,
g_x	average degree of disability,
h_x	probability of being married at the time of death,
y_x	average age of the surviving marriage partner at the death of the insured.

Exercise 4.3.1. In this exercise we try to calculate the mortality in the Middle Ages. Below are the recorded birth and death dates for the adult royal family of Wales and the associated Marcher relations, beginning with Joanna (the daughter of King John of England) and Llywelyn Fawr (Llywelyn the Great, the Prince of Wales):

– Joanna: 1190-1237 (Daughter of King John of England; wife of Llywelyn Fawr) (47)

– Llywelyn Fawr: 1173-1240 (Prince of Wales) (67)

– Tangwystl: 1168-1206 (Mistress of Llywelyn Fawr) (38)

– Gwladys: 1206-1251 (Princess of Wales) (45)

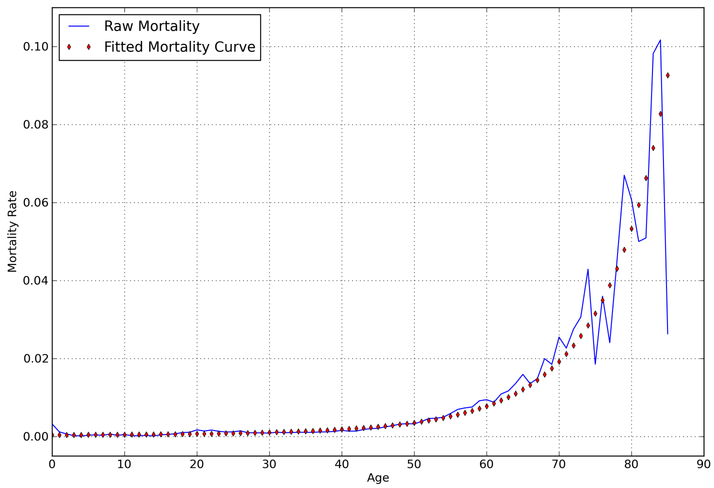


Figure 4.4. Mortality man

- Ralph Mortimer 1198-1246 (Husband of Gladwys) (48)
- Gruffydd: 1196-1244 (Prince of Wales) (Fell from a rope while escaping the Tower of London) (48)
- Roger Mortimer: 1231-1282 (51)
- Maud de Braose: 1224-1300 (76)
- William de Braose: 1198-1230 (Hung by Llywelyn Fawr for sleeping with his wife, Joanna) (32)
- Eve Marshall: 1203-1246 (43)
- Dafydd ap Llywelyn: 1208-1246 (Prince of Wales) (42)
- Isabella de Braose: 1222-1248 (Wife of Dafydd) (26)
- Eleanor de Braose: 1226-1251 (25) (Childbirth)
- Humphrey de Bohun: 1225-1265 (40) (War)
- Edmund Mortimer: 1251-1304 (53)
- Margaret de Fiennes: 1269-1333 (64)
- Humphrey de Bohun: 1249-1298 (49)

- Maud de Fiennes: 1254-1296 (42)
- Llywelyn ap Gruffydd: 1225-1282 (57) (War)
- Elinor de Montfort: 1252-1282 (30) (Childbirth)

The task is to determine a suitable mortality density of the form

$$\mu_{*\dagger}(x) = \exp(a + b \cdot x + c \cdot x^2)$$

by means of a maximum likelihood estimator for the attained age at time of death. In order to do this we remark that

$${}_tp_0 = (1 - q_0) \times (1 - q_1) \times \ldots (1 - q_{t-1}).$$

Moreover it is worth noting that deaths of non-adults have been excluded from the list above. In order to allow for children which have died before adulthood we suggest to amend the list of attained ages by some virtual children. For our example we assume this list to be $(1, 1, 6, 9, 12, 14, 18)$. In order to check the reasonableness of the own results we have provided below the fitted parameters and the respective life expectancies which we have calculated for this exercise.

```
(a,b,c) = (-4.36, 1.01e-02, 4.08e-04)

Age  1 --> ex: 35.2   (x + ex = 36.2)
Age 20 --> ex: 24.7   (x + ex = 44.7)
Age 40 --> ex: 14.5   (x + ex = 54.5)
Age 60 --> ex:  6.5   (x + ex = 66.5)
Age 80 --> ex:  1.8   (x + ex = 81.8)
```

In case further help is needed:

$$\log(MLE(x_1, x_2, \ldots x_n)) = \sum_{k=1}^{n} \log({}_{X_k}p_0).$$

Exercise 4.3.2. Compare the mortality densities as per example 4.2.1 and exercise 4.3.1 to determine the relative reduction of mortality since the Middle ages and interpret the implications on the respective life spans.

4.4 Deterministic cash flows

Definition 4.4.1 (Payout function). *A deterministic payout function A is a function*

$$A : T \to \mathbb{R}, t \mapsto A(t),$$

on $T \subset \mathbb{R}$ with the following properties:

1. *A is right continuous,*
2. *A is of bounded variation.*

The value $A(t)$ represents the total payments from the insurer to the insured up to time t. The payout functions are those functions in the policy setup which represent the benefits for the insured.

Example 4.4.2 (Disability insurance). We continue Example 4.2.2 by calculating the corresponding payout function. For this we assume that the policy does not contain a waiting period and that the disability pension is fixed to 20,000 USD per year (until the age of 65) with premiums of 2,500 USD per year until 65. Furthermore we suppose that the insurance was contracted at the age $x_0 = 25$. The payout function of this example is shown in Figure 4.5.

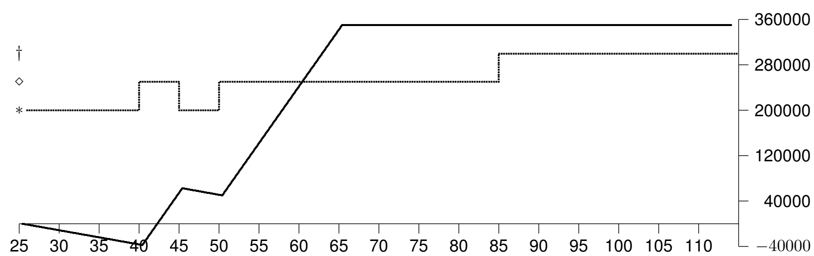


Figure 4.5. Cumulative payout of a disability annuity

Exercise 4.4.3. Derive the payout function for Example 4.4.2.

Remark 4.4.4 (Functions of bounded variation). Functions of bounded variation have the following properties [DS57]:

1. A function A of bounded variation corresponds to a measure on $\sigma(\mathbb{R})$. We will denote the measure again by A. This measure is called *Stieltjes measure*. In our setting it is also called payout measure.
2. Let A be a function of bounded variation on $\mathbb{R}$. Then there exist two positive, increasing and bounded functions B and C such that $A = B - C$. In the insurance model we can interpret B as inflow and C as outflow of cash. This representation is unique if one assumes that the measures corresponding to B and C have disjoint support. (Exercise: Calculate B and C for Example 4.4.2.)
3. Let A be the measure corresponding to a function of bounded variation on $\mathbb{R}$. Then A can be decomposed uniquely into a discrete measure μ and a continuous measure ψ. Furthermore one can decompose ψ into a part which is absolute continuous with respect to the Lebesgue measure and a remainder part. The support of μ is a countable set since A is finite on bounded sets.
4. Let A be a function of bounded variation and $T \in \sigma(\mathbb{R})$. Then $A \times \chi_T$ is also a function of bounded variation. (Here the function χ_T is the indicator function introduced in Definition 2.1.2.)

The above properties also hold for payout functions, since these are just functions of bounded variation. The decomposition of the Stieltjes measures will be useful later on. Thus we introduce the following notation:

Definition 4.4.5 (Decomposition of measures). *Let f be a function of bounded variation with corresponding Stieltjes measure A. Then we define*

$$\mu_f := A.$$

We know that we can decompose this measure uniquely into $A = B - C$, where B and C are positive measures with disjoint support. Therefore we define:

$$\begin{aligned} A^+ &:= B, \\ A^- &:= C. \end{aligned}$$

Furthermore the Stieltjes measure A can be decomposed uniquely into $A = D + E$, where D is discrete and E is continuous. Therefore we define:

$$\begin{aligned} A^{atom} &:= D, \\ A^{cont} &:= E. \end{aligned}$$

Furthermore, let μ be a measure which is absolute continuous with respect to the Lebesgue measure λ. Then we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodym density *of μ with respect to λ.*

Above we have seen the most important properties of deterministic cash flows. This enables us to define their values with the help of the discount rate. Recall, the discount rate is given by

$$v(t) \quad = \quad \exp(-\int_0^t \delta(\tau)\, d\tau).$$

Then the present value of a cash flow is defined as follows.

Definition 4.4.6 (Value of a cash flow). *Let A be a deterministic cash flow and $t \in \mathbb{R}$. We define:*

1. *The value of a cash flow A at time t is*
$$V(t, A) := \frac{1}{v(t)} \int_0^\infty v(\tau)\, dA(\tau).$$
2. *The value of the future cash flow is*
$$V^+(t, A) := V(t, A \times \chi_{]t,\infty]}).$$
It is also called prospective value of the cash flow *or* prospective reserve.

Concerning these definitions one should note:

Remark 4.4.7. 1. The idea of the prospective reserve is to calculate the present value of the future cash flows. Thus a payment of ζ which is due in two years contributes $v(2) \times \zeta$ to the present value. Initially the reserves are defined for deterministic cash flows. To define them also for random cash flows one uses the corresponding conditional expectations.

2. The definition implicitly requires that $v(t)$ is integrable with respect to the measure A, i.e. $v \in L^1(A)$.
3. The equation $A = A^{\text{atom}} + A^{\text{cont}}$ also implies $V(t, A) = V(t, A^{\text{atom}}) + V(t, A^{\text{cont}})$. This decomposition allows us to use different methods of proof for the discrete and the continuous part of the measure.

Example 4.4.8. We want to calculate $V^+(t, A)$ for the cash flow defined in Example 4.4.2 with $\delta(\tau) = \log(1.04)$. The first step is to calculate A^+ and A^-:

$$\begin{aligned} dA^+ &= 20000\,(\chi_{[40,45[} + \chi_{[50,65[})d\tau, \\ dA^- &= 2500\,(\chi_{[25,40[} + \chi_{[45,50[})d\tau. \end{aligned}$$

Then we get for $t \in [25, 65[$

$$\begin{aligned} V^+(t, A) \quad = \quad & 20000 \int_t^{65} (1.04)^{-(\tau - t)}\,(\chi_{[40,45[} + \chi_{[50,65[})d\tau \\ & -2500 \int_t^{65} (1.04)^{-(\tau - t)}\,(\chi_{[25,40[} + \chi_{[45,50[})d\tau. \end{aligned}$$

4.5 Stochastic cash flows

Definition 4.5.1 (Stochastic cash flow). *A* stochastic cash flow *or a* stochastic process of bounded variation *is a stochastic process $(X_t)_{t \in T}$ for which almost all sample paths are functions of bounded variation.*

Let A be a stochastic process of bounded variation such that $t \mapsto A_t(\omega)$ is right continuous and increasing for each $\omega \in \Omega$. Then it is possible to calculate the integral $\int f(\tau) d\mu_{A.(\omega)}(\tau)$ for a bounded Borel function f. Similarly, one can define P-almost everywhere the integral $\int f(\tau, \omega) d\mu_{A.(\omega)}(\tau)$ if $F_t = f(t, \omega)$ is a bounded function which is measurable with respect to the product sigma algebra. The construction of these integrals can be extended to general processes of bounded variation by decomposing the sample path, a function of bounded variation, into its positive (increasing) and negative (decreasing) part.

Definition 4.5.2. *Let $(A_t)_{t \in T}$ be a process of bounded variation on $(\Omega, \mathcal{A}, P)$ and $F : \mathbb{R} \times \Omega \to \mathbb{R}$ be a bounded and product measurable function. Then the description above yields the following definition*

$$(F \cdot A)_t(\omega) = \int_0^t F(\tau, \omega) dA\tau(\omega).$$

We also write this relation in the symbolic notation of stochastic differential equations:

$$d(F \cdot A) = F\, dA.$$

This definition allows us to give a precise definition of the stochastic cash flows in our insurance model.

Definition 4.5.3 (Policy cash flows). *We consider an insurance policy with state space S and payout functions $a_{ij}(t)$ and $a_i(t)$.*[1] *Based on Definition 2.1.8 we can define the stochastic cash flows corresponding to an insurance policy by*

$$\begin{aligned} dA_{ij}(t, \omega) &= a_{ij}(t)\, dN_{ij}(t, \omega), \\ dA_i(t, \omega) &= I_i(t, \omega)\, da_i(t), \\ dA &= \sum_{i \in S} dA_i + \sum_{(i,j) \in S \times S, i \neq j} dA_{ij}. \end{aligned}$$

The quantity $A_{ij}(t, \omega)$ is the sum of the random cash flows which are induced by transitions from state i to state j up to time t. Similarly, $A_i(t, \omega)$ represents the sum of the random cash flows up to time t which are pension payments for being in state i.

[1] Thus the functions are of bounded variation and, in particular, bounded.

Remark 4.5.4. 1. The quantity $dA_{ij}(t,\omega)$ corresponds to the increase of the liabilities by a transition $i \rightsquigarrow j$. Therefore $A_{ij}(t,\omega)$ increases at time t by the capital benefit $a_{ij}(t)$ if at time t a transition $i \rightsquigarrow j$ takes place, i.e. if $N_{ij}(t)$ increases by 1. Similarly, $dA_i(t)$ corresponds to the increase of the liabilities caused by the insured being in state i.

2. The integrals appearing above are well defined since the corresponding processes are, by definition, of bounded variation. Moreover also the pay-out functions have the required regularity.

3. The quantities $(F \cdot A)_t$ are measurable for each t since F was assumed to be product measurable. Therefore also the expectation $E[(F \cdot A)_t]$ is well defined. Similarly, the conditional expectations $E[(F \cdot A)_t \,|\, \mathcal{F}_s]$ are well defined.

4. Thus one can apply Definition 4.4.6 (value of a cash flow) point wise (i.e. for each sample) to a stochastic cash flow. This yields the equation

$$\begin{aligned} dV(t,A) &= v(t)\,dA(t) \\ &= v(t)\left[\sum_{i\in S} I_i(t)da_i(t) + \sum_{(i,j)\in S\times S, i\neq j} a_{ij}(t)dN_{ij}(t).\right] \end{aligned}$$

5. In the discrete Markov model at most two cash flows occur during a time interval $[t, t+1[$. Firstly, if the policy is in state i, $a_i^{\text{Pre}}(t)$ is paid at the beginning of the interval. Secondly, if there is a transition $i \rightsquigarrow j$, $a_{ij}^{\text{Post}}(t)$ is due at the end of the interval. Hence the following equations can be used to calculate the total cash flows:

$$\Delta A_{ij}(t,\omega) = \Delta N_{ij}(t,\omega) a_{ij}^{\text{Post}}(t), \tag{4.1}$$

$$\Delta A_i(t,\omega) = I_i(t,\omega) a_i^{\text{Pre}}(t), \tag{4.2}$$

$$\Delta A(t,\omega) = \sum_{i\in S} \Delta A_i(t,\omega) + \sum_{i,j\in S} \Delta A_{ij}(t,\omega), \tag{4.3}$$

where $\Delta A(t)$ (and similarely ΔA_{ij}, and ΔA_i, respectively) stands for the change of $A(t)$ from t to $t+1$, eg $\Delta A(t) := A(t+1) - A(t)$.

Definition 4.5.5. *Let A and (optionally) also v be stochastic processes on $(\Omega, \mathcal{A}, P)$ which are adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. In this case the* prospective reserve *is defined by:*

$$V_{\mathbb{F}}^{+}(t,A) = E\left[V^{+}(t,A) \,|\, \mathcal{F}_t\right].$$

One should note that also these reserves, like the usual expectations, might not exist, i.e., they might be infinite. In the following we will always assume that

$V_{\mathbb{F}}^{+}(t, A)$ and the other quantities exist. This assumption is always satisfied in applications.

For a Markov chain the conditional expectation with respect to $\mathcal{F}_t$ depends only on the state at time t. Thus we additionally define

$$V_j^{+}(t, A) = E\left[V^{+}(t, A) \,|\, X_t = j\right].$$

The following definition fixes our assumptions on the regularity of an insurance model.

Definition 4.5.6 (Regular insurance model). *A regular insurance model consists of:*

1. *a regular Markov chain $(X_t)_{t \in T}$ with a state space S,*
2. *payout functions $a_{ij}(t)$ and $a_i(t)$,*
3. *right continuous interest intensities $\delta_i(t)$ of bounded variation.*

4.6 Mathematical reserve

The mathematical reserve is the amount of money an insurance company has to reserve for the expected liabilities in order to remain solvent. We assume that the interest intensity δ has the following structure: $\delta_t = \sum_{j \in S} I_j(t)\, \delta_j(t)$. Then the required reserves for the cash flows are defined by:

Definition 4.6.1 (Mathematical reserve). *The mathematical reserve for being in state $g \in S$ within a time interval $T \in \sigma(\mathbb{R})$ under the condition $X_t = j$ is defined by*

$$V_j(t, A_{gT}) = E\left[\frac{1}{v(t)} \times \int_T v(\tau)\, dA_g(\tau) \,|\, X_t = j\right].$$

Similarly, for transitions from g to $h \in S$, we define

$$V_j(t, A_{ghT}) = E\left[\frac{1}{v(t)} \times \int_T v(\tau)\, dA_{gh}(\tau) \,|\, X_t = j\right].$$

We use the notation $V_j(t, A_g)$ and $V_j(t, A_{gh})$ for $V_j(t, A_{g\mathbb{R}})$ and $V_j(t, A_{gh\mathbb{R}})$, respectively.

Remark 4.6.2. The definitions of the mathematical reserve can be translated to the discrete model. One just has to replace the integrals by the corresponding sums:

$$V_j(t, A_{gT}) = E\left[\frac{1}{v(t)} \times \sum_{\tau \in T} v(\tau)\, \Delta A_g(\tau) \,|\, X_t = j\right].$$

Analogous, for transitions from g to $h \in S$ we set:

$$V_j(t, A_{ghT}) = E\left[\frac{1}{v(t)} \times \sum_{\tau \in T} v(\tau + 1)\, \Delta A_{gh}(\tau) \,|\, X_t = j\right],$$

where we assumed that the payments always take place at time $\tau + 1$.

Therefore the total reserve (or mathematical reserve) for a given state j is

$$V_j(t, A) = \sum_{g \in S} V_j(t, A_g) + \sum_{g,h \in S, g \neq h} V_j(t, A_{gh})$$

for the continuous time model and

$$V_j(t, A) = \sum_{g \in S} V_j(t, A_g^{\text{Pre}}) + \sum_{g,h \in S} V_j(t, A_{gh}^{\text{Post}})$$

for the discrete time model. Thus we have defined the mathematical reserves. The next step is to calculate their values. Let us consider the relevant cash flows. On the one hand there are flows of the form $dA_1(t) = a(t) dN_{jk}(t)$ and on the other hand are flows of the form $dA_2(t) = I_j(t) dA(t)$.

The first step is to calculate the integrals $\int dA$ for the partial cash flows. Afterwards we will derive explicit formulas for the mathematical reserves.

Theorem 4.6.3. *Let $(X_t)_{t \in T}$ be a regular Markov chain on $(\Omega, \mathcal{A}, P)$ (cf. Def. 2.3.2). Furthermore let $i, j, k \in S$, $s < t$ and $T \in \sigma(\mathbb{R})$ where $T \subset [s, \infty]$. Then the following statements hold:*

1. $$E\left[\int_T a(\tau)\, dN_{jk}(\tau) \,|\, X_s = i\right] = \int_T a(\tau)\, p_{ij}(s, \tau)\, \mu_{jk}(\tau) d\tau$$

   for $a \in L^1(\mathbb{R})$.

2. *Let A be a function of bounded variation, then*

   $$E\left[\int_T I_j(\tau)\, dA(\tau) \,|\, X_s = i\right] = \int_T p_{ij}(s, \tau)\, dA(\tau).$$

Proof. 1. The step functions are dense in L^1. Therefore it is enough to show the equality for functions of the form $\chi_{[a,b]}$. Moreover, the Borel σ-algebra is generated by the intervals in $\mathbb{R}_+$. Thus we can take $T = [c, d]$. Further we can set $c = a$ and $d = b$ without loss of generality, since the indicator function is equal to zero outside the interval $[a, b]$.

Define the function

$$h(t) := E\left[N_{jk}(t) \mid X_s = i\right].$$

Based on this definition we get

$$\begin{aligned} h(t+\Delta t) - h(t) &= E\left[N_{jk}(t+\Delta t) - N_{jk}(t) \mid X_s = i\right] \\ &= \sum_{l \in S} E\left[\chi_{\{X_t = l\}}(N_{jk}(t+\Delta t) - N_{jk}(t)) \mid X_s = i\right] \\ &= \sum_{l \in S} E\left[N_{jk}(t+\Delta t) - N_{jk}(t) \mid X_t = l\right] \times p_{il}(s,t). \end{aligned}$$

Now we observe that all the terms where $j \neq l$ are of order $o(\Delta t)$. Thus we get

$$= p_{ij}(s,t) \times \mu_{jk}(t) \times \Delta t + o(\Delta t).$$

Therefore $h'(t) = p_{ij}(s,t)\, \mu_{jk}(t)$, and an integration of this equation with initial condition $h(0) = 0$ yields the first statement of the theorem.

2. For the second statement one has to interchange the order of integration, which is allowed by Fubini's theorem.

Remark 4.6.4. Also these statements can be translated easily to the discrete model. One gets the equations

$$E\left[\sum_{\tau \in T} a(\tau)\, \Delta N_{jk}(\tau) \mid X_s = i\right] = \sum_{\tau \in T} a(\tau)\, p_{ij}(s,\tau) p_{jk}(\tau, \tau+1)$$

and

$$E\left[\sum_{\tau \in T} I_j(\tau)\, \Delta A(\tau) \mid X_s = i\right] = \sum_{\tau \in T} p_{ij}(s,\tau)\, \Delta A(\tau).$$

Exercise 4.6.5. Complete the proof of Theorem 4.6.3.

An important consequence of Theorem 4.6.3 is the next theorem.

Theorem 4.6.6. *Let the assumptions of Theorem 4.6.3 be satisfied. Then*

$$dM_{ij}(t) := dN_{ij}(t) - I_i(t)\, \mu_{ij}(t) dt$$

is a martingale.

Proof. We have

$$N_{ij}(t) \in L^1(\Omega, \mathcal{A}, P)$$

and

$$\int_0^t I_j(\tau)\, \mu_{ij}(\tau) d\tau \in L^1(\Omega, \mathcal{A}, P),$$

which implies

$$M_{ij}(t) \in L^1(\Omega, \mathcal{A}, P).$$

Next, we have to prove the equality $E[M_{ij}(t) \,|\, \mathcal{F}_s] = M_{ij}(s)$ for $s < t$. But, since the processes M, N and I are all derived from $(X_t)_{t \in T}$, it is enough to prove $E[M_{ij}(t) \,|\, X_s = k] = M_{ij}(s)$. But this is true, since

$$\begin{aligned} E[M_{ij}(t) \,|\, X_s = k] - M_{ij}(s) &= E\left[\int_s^t dM_{ij}(\tau) \,|\, X_s = k\right] \\ &= E\left[\int_s^t dN_{ij}(t) - I_i(t)\, \mu_{ij}(t) dt \,|\, X_s = k\right] \\ &= 0, \end{aligned}$$

where we used Theorem 4.6.3 in the last step.

Another application of Theorem 4.6.3 yields the following equations for the mathematical reserves in our insurance model:

Theorem 4.6.7. *Let a_{ij} and a_i be payout functions and $(X_t)_{t \in T}$ be a regular Markov chain on $(\Omega, \mathcal{A}, P)$. Then the following equations hold for fixed interest intensities (i.e., $\delta_i = \delta$):*

$$\begin{aligned} & E[V(t, A_{jT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)} \int_T v(\tau)\, p_{ij}(s,\tau) da_j(\tau), \\ & E[V(t, A_{jkT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)} \int_T v(\tau)\, a_{jk}(\tau)\, p_{ij}(s,\tau)\, \mu_{jk}(\tau) d\tau, \\ & E[V(t, A_{jS}) V(t, A_{lT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)^2} \int_{T \times S} v(\theta) v(\tau) \Big\{ \chi_{\{\theta \le \tau\}} p_{ij}(s,\theta) p_{jl}(\theta,\tau) \\ & \qquad\qquad + \chi_{\{\theta > \tau\}} p_{il}(s,\tau) p_{lj}(\tau,\theta) \Big\} da_j(\theta) da_l(\tau), \\ & E[V(t, A_{jkS}) V(t, A_{lmT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)^2} \Bigg[\int_{T \times S} v(\theta) v(\tau) \Big\{ \chi_{\{\theta \le \tau\}} p_{ij}(s,\theta) p_{kl}(\theta,\tau) \\ & \quad + \chi_{\{\theta > \tau\}} p_{il}(s,\theta) p_{mj}(\theta,\tau) \Big\} \mu_{jk}(\theta) \mu_{lm}(\tau) a_{jk}(\theta) a_{lm}(\tau) d\theta d\tau \end{aligned}$$

$$+\delta_{jk,lm} \int_{T\cap S} v(\tau)^2 p_{ij}(s,\tau)\mu_{jk}(\tau) a_{jk}^2 d\tau \Bigg],$$

$$E[V(t, A_{jS})V(t, A_{lmT})|\, X_s = i]$$
$$= \frac{1}{v(t)^2} \int_{T\times S} v(\theta)v(\tau)\Big\{\chi_{\{\theta\le\tau\}} p_{ij}(s,\theta)p_{jl}(\theta,\tau)$$
$$+\chi_{\{\theta>\tau\}} p_{il}(s,\tau)p_{mj}(\tau,\theta)\Big\} da_j(\theta)\mu_{lm}(\tau)a_{lm}(\tau)d\tau.$$

Proof. The first two equations are a direct consequence of Theorem 4.6.3. For a proof of the remaining equations we refer to [Nor91].

Remark 4.6.8. Also this theorem can easily be translated to the discrete setting. The following equalities hold:

$$E[V(t, A_{jT})|\, X_s = i] = \frac{1}{v(t)} \sum_{\tau\in T} v(\tau)\, p_{ij}(s,\tau) a_j^{\text{Pre}}(\tau),$$

$$E[V(t, A_{jkT})|\, X_s = i] = \frac{1}{v(t)} \sum_{\tau\in T} v(\tau+1)\, p_{ij}(s,\tau)\, p_{jk}(\tau,\tau+1)\, a_{jk}^{\text{Post}}(\tau),$$

where we used that, for a transition $j \rightsquigarrow k$, the payments $a_{jk}^{\text{Post}}(\tau)$ are made the end of the period.

Exercise 4.6.9. Complete the proof of Theorem 4.6.7.

Given the transition probabilities one can use Theorem 4.6.7 to calculate the expectations and variances of the prospective reserves for each cash flow. Then, based on these partial reserves, one can calculate the total prospective reserves by the following result.

Theorem 4.6.10. *Let a regular insurance model (Definition 4.5.6) with deterministic interest intensities be given. Then the prospective reserves are given by*

$$V_j^+(t) = \frac{1}{v(t)} \int_{]t,\infty[} v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \times \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau)d\tau \right\}.$$

Remark 4.6.11. The formula of the previous theorem is not very useful, since one has to calculate integrals based on the transition probabilities p_{ij}. This becomes even more complicated by the fact that in applications often only the μ_{ij} are given. In the next section we will find a more elegant way to calculate this quantity.

4.7 Recursion formulas for the mathematical reserves

In this section we will derive a recursion formula for the reserves based on their integral representation. This recursion can be used in two ways. On the one hand it can be used to prove Thiele's differential equation. On the other hand it can be applied to the discrete model. Thereby it provides a way to calculate the values for various types of insurances. We will see in the remaining sections that these recursion equations, difference equations and differential equations are a extremely useful tools for explicit calculations. We adapt our definition of mathematical reserves, in order to simplify the proofs:

Definition 4.7.1. *We define for a regluar insurance model (Definiton 4.5.6):*

$$W_j^+(t) \quad := \quad v(t)\, V_j^+(t).$$

The difference between W and the usual mathematical reserve V is only the discount factor. V resembles the value of the cash flow at time t, whereas W is the value of the cash flow at time 0. Thus W is only a new notation which will help to keep the proofs simple. Based on this we are now able to derive a recursion formula for the prospective reserve.

Lemma 4.7.2. *Let $j \in S$, $s < t < u$ and $(X_t)_{t \in T}$ be a regular insurance model in continuous time with deterministic interest intensities. Then the following equation holds:*

$$\begin{aligned} W_j^+(t) &= \sum_{g \in S} p_{jg}(t,u)\, W_g^+(u) \\ &\quad + \int_{]t,u]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau) \mu_{gh}(\tau) d\tau \right\}. \end{aligned}$$

Proof. The proof is based on the Chapman-Kolmogorov equation. We get

$$\begin{aligned} W_j^+(t) &= \int_{]t,\infty]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau) \mu_{gh}(\tau) d\tau \right\} \\ &= \left(\int_{]t,u]} + \int_{]u,\infty]} \right) v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \\ &\qquad \times \left\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau) \mu_{gh}(\tau) d\tau \right\} \end{aligned}$$

$$
\begin{aligned}
&= \int_{]t,u]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \Big\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \Big\} \\
&\quad + \int_{]u,\infty]} v(\tau) \sum_{g \in S} \Big(\sum_{k \in S} p_{jk}(t,u) p_{kg}(u,\tau) \Big) \\
&\qquad\quad \times \Big\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \Big\} \\
&= \int_{]t,u]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \Big\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \Big\} \\
&\quad + \sum_{k \in S} p_{jk}(t,u) \Big(\int_{]u,\infty]} v(\tau) \sum_{g \in S} p_{kg}(u,\tau) \\
&\qquad\quad \times \Big\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \Big\} \Big) \\
&= \sum_{g \in S} p_{jg}(t,u)\, W_g^+(u) \\
&\quad + \int_{]t,u]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \Big\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \Big\}.
\end{aligned}
$$

The recursion formula can also be translated to the discrete model. For this one assumes that the payments are done at discrete times rather than in continuous time. For example pensions are paid at the beginning of the interval and death capital is paid at the end of the interval. We denote the payments at the beginning of the year by $a_i^{Pre}(t)$ and those at the end of the year by $a_{ij}^{Post}(t)$. Thus, in particular we assume that a transition between states can only occur at the end of the year.

Setting $\Delta t = 1$ in the previous lemma yields the following recursion for the reserves in the discrete setting.

Theorem 4.7.3 (Thiele's difference equation). *For a discrete time Markov model the prospective reserve satisfies the following recursion:*

$$
V_i^+(t) \quad = \quad a_i^{Pre}(t) + \sum_{j \in S} v_t\, p_{ij}(t) \big\{ a_{ij}^{Post}(t) + V_j^+(t+1) \big\}.
$$

Remark 4.7.4. – The formula shows that errors which are introduced by the discretization of time are due to payments between the discretization times.

– The recursion formula of the mathematical reserve is very important for applications, since it provides a way to calculate a single premium and yearly premiums. In fact it is the most important formula for explicit calculations.

– To solve a differential equation or a difference equation one needs a boundary condition. For example, if one calculates a pension, the boundary condition is given by the fact that the reserve has to be equal to zero at the final age ω.

4.8 Calculation of the premiums

In this section we are going to calculate single premiums and yearly premiums for several types of insurance policies. The calculations in the examples are based on the discrete recursion (Theorem 4.7.3). We start with an endowment policy.

Example 4.8.1 (Endowment policy in discrete time). We consider the insurance defined in Example 4.2.1. Thus there is a death benefit of 200,000 USD. Moreover we assume an endowment of 100,000 USD and a starting age of 30 with 65 as fixed age at maturity.

– How much is a single premium for this insurance, given a technical interest rate of 3.5%?

– How much are the corresponding yearly premium?

We use the mortality rates given by (2.14). First we calculate the single premium. The following payout functions are given:

$$
\begin{aligned}
a_{*\dagger}^{\text{Post}}(x) &= \begin{cases} 200000, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases} \\
a_{**}^{\text{Post}}(x) &= \begin{cases} 0, & \text{if} \quad x < 64, \\ 100000, & \text{if} \quad x = 64, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

An application of Theorem 4.7.3 yields the results presented in Table 4.5. Here one has to note that the mathematical reserves for the case of survival and the case of death have to be calculated separately. The reserve in the case of survival is $V_*(t, A_{**\mathbb{R}})$ and the reserve in the case of death is $V_*(t, A_{*\dagger\mathbb{R}})$ (cf. Definition 4.6.1).

The table of the mathematical reserves indicates that the recursion formula was used with the boundary condition $x = 65$. Figure 4.6 shows the necessary reserves for several values of the technical interest rate.

After the calculation of the single premium we will now consider the case of yearly premiums. The yearly payment of the premiums is modeled by the following payout function:

Table 4.5. Reserves for an endowment

age	q_x	**res. for endowment**	**res. for death benefit**	**sums reserve**
65	0.01988	**100000**	**0**	**100000**
64	0.01836	94844	3548	98392
63	0.01696	90083	6647	96730
62	0.01566	85674	9348	95022
61	0.01446	81579	11696	93275
60	0.01336	77768	13730	91498
55	0.00897	62086	20275	82360
50	0.00602	50444	22766	73210
45	0.00404	41470	22956	64426
40	0.00271	34362	21874	56236
35	0.00181	28624	20135	48759
30	0.00121	23928	18116	42044

$$a_*^{\mathrm{Pre}}(x) = \begin{cases} -P, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

P has to be calculated such that the value of the insurance is equal to zero at the beginning of the policy. (Equivalence principle: The expected value of the benefits provided by the insurer and the value of the expected premium payments by the insured coincide.) The simplest method to determine the value of P is to consider V_x^{payout} (given by the first two payout functions of the policy) and V_x^{premiums} (given by the third function of the policy, i.e. by the premiums paid in). The total mathematical reserve is then given by $V_x = V_x^{\text{payout}} + V_x^{\text{premiums}}$. But we know that $V_x^{\text{premiums}} = P \times V_x^{\text{premiums},P=1}$ holds. Thus we can calculate P by the formula

$$P = -V_x^{\text{payout}} / V_x^{\text{premiums},P=1},$$

since V_x at inception of the policy is 0, as a consequence of the equivalence principle. For our example we get

$$P = 2.129,15 \text{ USD per year.}$$

Table 4.6 lists the reserves for this insurance with yearly premiums. Figure 4.7 illustrates the same data in a graph.

Exercise 4.8.2. Do the calculation for the previous example.

In the next example we will consider the simple disability insurances model which we looked at earlier. We will show how to model the disability pension with and without an exemption from payment of premiums.

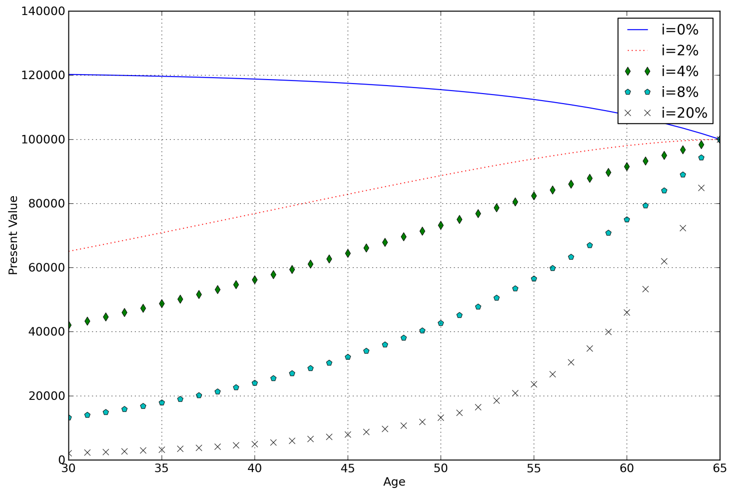


Figure 4.6. Mathematical reserves as a function of interest rates

Table 4.6. Reserves for an endowment with yearly premiums

age	present value premiums	present value payout	reserve
65	0	100000	100000
64	-2129	98392	96263
63	-4149	96730	92581
62	-6069	95022	88952
61	-7901	93275	85374
60	-9653	91498	81845
50	-23928	73210	49282
40	-34292	56236	21943
30	-42044	42044	0

Example 4.8.3 (Disability insurance). We use the model for the disability insurance introduced in Example 2.4.2. Thus in particular we do not incorporate the possibility that insured becomes active again. Moreover we also do not model a waiting period.

– Calculate for a 30-year old man the present value of a (new) disability pension based on 65 as age at maturity and a technical interest rate of 4%.

– Compare for the same person the present value of the premiums for a policy with exemption from payment of premiums and for a policy without this option.

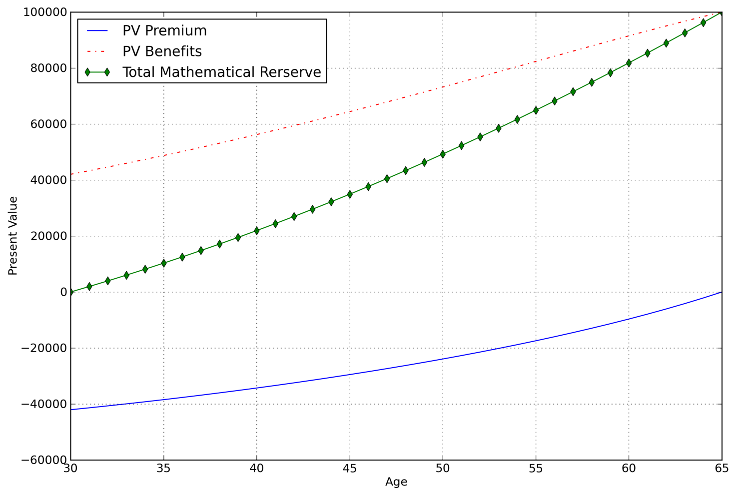


Figure 4.7. Endowment policy against regular premium payment

First we calculate the present value of the payouts of the future disability pension. In this case the non-trivial functions which model the policy are: (where we assumed that the disability pension is payable in advance and has the value 1.)

$$a_{\diamond}^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

The boundary conditions are zero in this case, i.e., there is no payment when the age of maturity is reached. Table 4.7 lists the calculated values for this example. Thus one has to pay 4,396.8 USD as single premium for a future disability pension of 10,000 USD. Furthermore, the reserve is 170,790 USD for a disabled man of 35 years with the same disability pension as above.

Next we consider the present values of the premiums. We have to treat the following two cases separately. On the one hand there is the present value of a policy without exemption from payment of premiums

$$a_{*}^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_{\diamond}^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand there is the present value of a policy with exemption from payment of premiums ("premium raider")

$$a_{*}^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases}$$

Table 4.7. Reserves of a disability pension

age	$p_{*\dagger}$	$p_{*\diamond}$	$V_*(x)$	$V_\diamond(x)$
65	0.02289	0.02794	**0.00000**	**0.00000**
64	0.02101	0.02439	0.00000	1.00000
63	0.01929	0.02129	0.02047	1.94299
62	0.01772	0.01860	0.05372	2.83515
61	0.01628	0.01625	0.09427	3.68174
60	0.01495	0.01420	0.13828	4.48719
55	0.00983	0.00732	0.34176	8.01299
50	0.00653	0.00387	0.46175	10.90260
45	0.00439	0.00214	0.50531	13.31967
40	0.00301	0.00127	0.50178	15.35782
35	0.00212	0.00084	0.47493	17.07904
30	0.00155	0.00062	0.43968	18.53012

$$a_\diamond^{\text{Pre}} = \begin{cases} 0, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

The difference between these two policies is, that in the first case also with status “disabled” the insured has to pay the premiums. In both cases the boundary condition at the age of 65 is 0. An application of Thiele’s difference equations yields the values listed in Table 4.8. We note that the value of the paid-in premiums is smaller in the case of an exemption of premiums. This can also be seen in the plot of these values in Figure 4.8.

Table 4.8. Present value of the premiums

age	$V(x)$ without premium raider	$V(x)$ with premium raider
65	**0.00000**	**0.00000**
64	1.00000	1.00000
63	1.94299	1.92251
62	2.83515	2.78144
61	3.68174	3.58747
60	4.48719	4.34891
55	8.01299	7.67123
50	10.90260	10.44085
45	13.31967	12.81436
40	15.35782	14.85604
35	17.07904	16.60411
30	18.53012	18.09044

Now we are able to calculate the yearly premium for a disability insurance of 10,000 USD with exemption from premiums

$$P = 4396.8/18.09044 = 243.05 \text{ USD per year.}$$

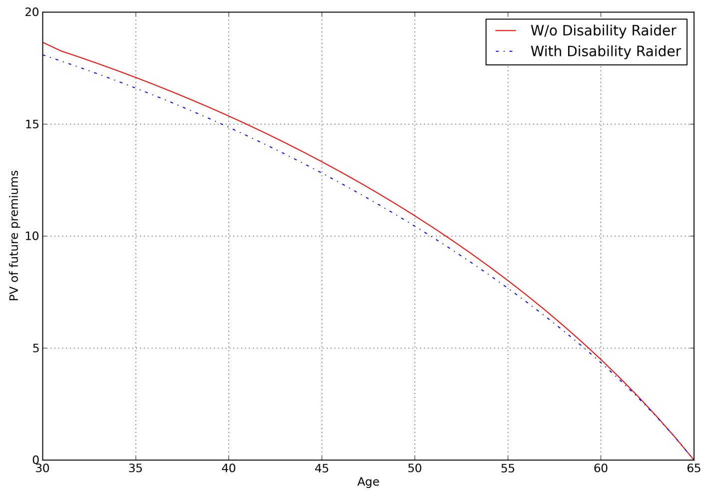


Figure 4.8. Present value of premiums

Exercise 4.8.4. 1. Do the calculations of the above example also for a model which includes the possibility of reactivation (cf. Example 4.2.2).

2. Extend the model by incorporating a waiting period of one year.

Next we consider an insurance on two lives. There are several possible states for which the policy could guarantee a pension.

Example 4.8.5 (Pension on two lives). We start with the calculation of a single premium for several types of an insurance on two lives. For this we assume that the two persons have the same mortality as given in Example 4.2.1 and that $x_1 = 30$ and $x_2 = 35$ are fixed. We set the technical interest rate to 3.5 %, and $\omega = 114$ be the maximal possible age of a living person.

There are three possible pensions: (we denote by x_1 the age of the first person)

state	type	formula
$**$	both persons are alive	$a_{**}^{\text{Pre}}(x) = \alpha_{**} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise,} \end{cases}$
$*\dagger$	second person is dead	$a_{*\dagger}^{\text{Pre}}(x) = \alpha_{*\dagger} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise,} \end{cases}$
$\dagger*$	first person is dead	$a_{\dagger*}^{\text{Pre}}(x) = \alpha_{\dagger*} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise.} \end{cases}$

The definitions of the pensions above are particular, since the pension for the second life (i.e. for $\dagger*$) is paid at the age of 65. Usually this pension would be paid immediately after the death of the first person.

We set $x = (x_1, x_2)$ and suppose that the two insured die independently. In this case the recursion takes the form

$$\begin{aligned} V_{**}(x) &= a_{**}^{\text{Pre}}(x) + p_{x_1}\, p_{x_2}\, v\, V_{**}(x+1) + p_{x_1}\,(1-p_{x_2})\, v\, V_{*\dagger}(x+1) \\ &\quad +(1-p_{x_1})\, p_{x_2}\, v\, V_{\dagger*}(x+1), \\ V_{*\dagger}(x) &= a_{*\dagger}^{\text{Pre}}(x) + p_{x_1}\, v\, V_{*\dagger}(x+1), \\ V_{\dagger*}(x) &= a_{\dagger*}^{\text{Pre}}(x) + p_{x_2}\, v\, V_{\dagger*}(x+1). \end{aligned}$$

This recursion yields the values listed in Table 4.9.

Exercise 4.8.6. 1. Calculate the present values of the premiums for the insurance on two lives. Note that also in this calculation one has to consider three different cases.

2. Create a model for an orphan's pension. For this one has to consider three persons: farther, mother and child. Define the payout functions for a policy which pays 5,000 USD to the child if one of parent dies and 10,000 USD if both die. Assume that the policy matures if the child is 25 years old.

Table 4.9. Mathematical reserves (res.) for an insurance on two lives

Notations

$V_{*\cdot}$(R1)	res. for the pension of 1th life, independent of 2nd life
$V_{\cdot *}$(R2)	res. for the pension of 2nd life, independent of 1th life
V_{**}(R1)	res. for the pension of 1th life, if $X_t = (**)$,
V_{**}(R2)	res. for the pension of 2th life, if $X_t = (**)$,
V_{**}(R $(**)$)	res. for the pension of joint lives, if $X_t = (**)$,.

Alter 1	Alter 2	$V_{*\cdot}$(R1)	$V_{\cdot *}$(R2)	V_{**}(R1)	V_{**}(R2)	V_{**}(R $(**)$)
115	120	**0.00000**		**0.00000**	**0.00000**	**0.00000**
114	119	1.00000		0.00000	0.00000	1.00000
113	118	1.11505		0.11505	0.00000	1.00000
112	117	1.19991		0.19991	0.00000	1.00000
111	116	1.28640		0.28640	0.00000	1.00000
110	**115**	1.37771	**0.00000**	0.37771	0.00000	1.00000
109	114	1.47450	1.00000	0.45824	0.00000	1.01626
108	113	1.57710	1.11505	0.52973	0.06845	1.04736
90	95	4.64366	3.53796	2.04747	0.94178	2.59618
75	80	9.05696	7.43141	3.39065	1.76510	5.66631
65	70	12.54173	10.77780	3.88770	2.12377	8.65403
55	60	7.78663	6.26733	3.37939	1.86010	4.40724
40	45	4.30964	3.34208	2.13044	1.16288	2.17920
30	35	3.00101	2.30680	1.52353	0.82932	1.47748

5. Difference equations and differential equations

5.1 Introduction

In this chapter we focus on the Markov model in continuous time. The differential equations are the continuous counter part to the difference equations of the discret model.

These differential equations where first proved for simple insurance models by Thiele at the end of the 19th century. We are going to derive these equations for the Markov model. They are useful in two ways. On the one hand they help to deepen our understanding of the model. On the other hand they can be used to calculate the premiums for a policy.

5.2 Thiele's differential equations

In this section we are going to derive Thiele's differential equations for the mathematical reserve. For simplicity we consider in this chapter only reserves without jumps. Later on we will also allow jumps, but then the proofs become more involved.

Theorem 5.2.1 (Thiele's differential equation). *Let $(X_t)_{t \in T}$, a_{ij}, a_i and δ_t be a regular insurance model (Definition 4.5.6). Moreover, $da_g(t)$ be absolute continuous with respect to the Lebesgue measure λ, i.e. $da_g(t) = a_g(t)\, d\lambda$. (Thus the payout function $A_g(t)$ is continuous.) Then, assuming a deterministic interest intensity, the following statements hold:*

1. *$W_g^+(t)$ is continuous for all $g \in S$.*
2. $\frac{\partial}{\partial t} W_j^+(t) = -v(t) \left\{ a_j(t) + \sum_{j \neq g \in S} \mu_{jg}(t) a_{jg}(t) \right\}$
   $+\mu_j(t) W_j^+(t) - \sum_{j \neq g \in S} \mu_{jg}(t) W_g^+(t)$. *(Thiele's differential equation)*
3. $V_j^+(t) = \frac{1}{v(t)} \left[\begin{array}{c} \int_t^u v(\tau) \overline{p}_{jj}(t,\tau) \{ a_j(\tau) + \sum\limits_{j \neq g \in S} \mu_{jg}(\tau)(a_{jg}(\tau) \\ +V_g^+(\tau))\} d\tau + v(u) \overline{p}_{jj}(t,u) V_j^+(u^-) \end{array} \right].$

Proof. The proof of the first statement is left as an exercise to the reader. To prove the second statement we fix $j \in S, t \in \mathbb{R}$ and $\Delta t > 0$. Then Lemma 4.7.2 implies

$$\begin{aligned} W_j^+(t) &= v(t)\left\{a_j(t) + \sum_{j \neq g \in S} \mu_{jg}(t) a_{jg}(t)\right\} \Delta t \\ &\quad +(1-\mu_j(t)\Delta t) W_j^+(t+\Delta t) \\ &\quad + \sum_{j \neq g \in S} \mu_{jg}(t) W_g^+(t+\Delta t)\, \Delta t + o(\Delta t), \end{aligned}$$

where we used the following facts

$$\begin{aligned} p_{jj}(t, t+\Delta t) &= 1 - \Delta t\, \mu_j(t) + o(\Delta t), \\ p_{jk}(t, t+\Delta t) &= \Delta t\, \mu_{jk}(t) + o(\Delta t). \end{aligned}$$

The above equation yields

$$\begin{aligned} \frac{W_j^+(t+\Delta t) - W_j^+(t)}{\Delta t} &= -v(t)\left\{a_j(t) + \sum_{j \neq g \in S} \mu_{jg}(t) a_{jg}(t)\right\} \\ &+ \mu_j(t)\, W_j^+(t+\Delta t) \\ &- \sum_{g \neq j} \mu_{jg}(t) W_g^+(t+\Delta t) + \frac{o(\Delta t)}{\Delta t}. \end{aligned}$$

Letting $\Delta t \longrightarrow 0$ we get

$$\begin{aligned} \frac{\partial}{\partial t} W_j^+(t) &= -v(t)\left\{a_j(t) + \sum_{j \neq g \in S} \mu_{jg}(t) a_{jg}(t)\right\} + \mu_j(t) W_j^+(t) \\ &- \sum_{j \neq g \in S} \mu_{jg}(t) W_g^+(t). \end{aligned}$$

For the proof of the third statement we use Thiele's differential equation:

$$\begin{aligned} & \exp\left(-\int_o^t \mu_j(\tau)d\tau\right) \left(-v(t)\{a_j(t) + \sum_{j \neq g \in S} \mu_{jg}(t) a_{jg}(t)\}\right. \\ & \qquad \left. - \sum_{j \neq g \in S} \mu_{jg}(t) W_g^+(t)\right) \\ = \ & \exp\left(-\int_o^t \mu_j(\tau)d\tau\right) \left(\frac{\partial}{\partial t} W_j^+(t) - \mu_j(t) W_j^+(t)\right) \\ = \ & \frac{\partial}{\partial t}\left(\exp\left(-\int_o^t \mu_j(\tau)d\tau\right) W_j^+(t)\right). \end{aligned}$$

An integration $\int_t^u$ of both sides yields

$$\begin{aligned}
&\exp(-\int_o^t \mu_j(\tau)d\tau)\left(\exp(-\int_t^u \mu_j(\tau)d\tau)W_j^+(u) - W_j^+(t)\right)\\
&= \int_t^u \exp(-\int_o^t \mu_j(\xi)d\xi)\exp(-\int_t^\tau \mu_j(\xi)d\xi)\\
&\quad \times \Bigg[- v(\tau)\Big\{ a_j(\tau) + \sum_{j\neq g\in S} \mu_{jg}(\tau)a_{jg}(\tau)\Big\}\\
&\quad - \sum_{j\neq g\in S} \mu_{jg}(\tau)W_g^+(\tau)\Bigg]d\tau.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
V_j^+(t) &= \frac{1}{v(t)}\Bigg[\int_t^u v(\tau)\overline{p}_{jj}(t,\tau)\{a_j(\tau) + \sum_{j\neq g\in S}\mu_{jg}(\tau)\\
&\quad \times (a_{jg}(\tau) + V_g^+(\tau))\}d\tau + v(u)\overline{p}_{jj}(t,u)V_j^+(u^-)\Bigg],
\end{aligned}$$

where we used that $\overline{p}_{jj}(t,\tau) = \exp(-\int_t^\tau \mu_j(\xi)d\xi)$.

Remark 5.2.2. We derived the following integral equation from Thiele's differential equation:

$$V_j^+(t) = \frac{1}{v(t)}\left[\begin{array}{c} \int_t^u v(\tau)\overline{p}_{jj}(t,\tau)\{\underbrace{a_j(\tau)}_{I} + \sum_{j\neq g\in S} \overbrace{\underbrace{\mu_{jg}(\tau)(a_{jg}(\tau)}_{IIa} + V_g^+(\tau))}^{IIb}\}d\tau \\ + \underbrace{v(u)\overline{p}_{jj}(t,u)V_j^+(u^-)}_{III} \end{array}\right].$$

This formula shows the structure of the reserve. The components of the reserve are:

I)		reserve for payments in state j (pensions and premiums),
II)		reserves for state transitions composed of
	IIa)	transition cost (e.g. death benefit) and
	IIb)	necessary reserves in the new state,
III)		reserve, for the case that the insured is still in j after $[t,u]$.

5.3 Examples - Thiele's differential equation

In this section we look at examples related to those in the discrete setting. In the first example the differential equations have an explicit solution.

Example 5.3.1 (Term life insurance). We consider a term life insurance with death benefit b, which is financed by a premium of size c. In this situation the differential equations take the following form:

$$\begin{aligned}
\frac{\partial}{\partial t} W_*(t) &= v^t (c - \mu_{x+t}\, b) + \mu_{x+t} W_*(t) - \mu_{x+t} W_\dagger(t), \\
\frac{\partial}{\partial t} W_\dagger(t) &= 0
\end{aligned}$$

with the boundary condition $W_*(s-x) = W_\dagger(s-x) = 0$, where s denotes the age of maturity of the policy.

Next, we are going to calculate the mathematical reserve. The above equations obviously imply $W_\dagger(t) \equiv 0$. Thus we only have to calculate $W_*(t)$. The homogeneous part of the equation satisfies

$$\frac{dW_*(t)}{W_*(t)} = \mu_{x+t} dt$$

and therefore

$$L_h(t) = A \times \exp\left(\int_0^t \mu_{x+\tau}\, d\tau\right).$$

By variation of constants we get

$$\begin{aligned}
L_p(t) &= A(t) \times L_h(t), \\
\frac{d}{dt} L_p &= A' \times L + A \times L' \\
&= A' \times L + A \times L \\
&= A' \times L + L_p, \\
A' \times L &= v^t (c - \mu_{x+t}\, b), \\
A' &= v^t (c - \mu_{x+t}\, b) \exp\left(- \int \mu_{x+\tau}\, d\tau\right) \\
&= v^t (c - \mu_{x+t} b)\, {}_t p_x, \\
A(t) &= \int_0^t v^\tau (c - \mu_{x+\tau} b)\, {}_\tau p_x d\tau.
\end{aligned}$$

Finally, the boundary condition $W_*(s-x) = 0$ yields

$$\begin{aligned}
W_*(s-x) &= A(s-x) \times L(s-x) \\
&= \left[\int_0^{s-x} v^\tau (c - \mu_{x+\tau} b)\, {}_\tau p_x d\tau\right] \times \left[\exp\left(\int_0^{s-x} \mu_{x+\tau} d\tau\right)\right], \\
c &= b\, \frac{\int_0^{s-x} v^\tau\, {}_\tau p_x\, \mu_{x+\tau} d\tau}{\int_0^{s-x} v^\tau\, {}_\tau p_x d\tau}.
\end{aligned}$$

Example 5.3.2 (Endowment policy). We consider the endowment policy defined in Example 4.8.1. Thus it contains a death benefit of 200,000 USD and an endowment of 100,000 USD. We consider a 30 year old man and 65 as the age of maturity of the policy.

– How much is a single premium for this insurance if the technical interest rate is 3.5%?
– How do these results compare to the values in the corresponding example in discrete time?

We use the mortality rates given by (2.14). For the single premium the following payout function defines the policy:

$$a_{*\dagger}(x) = \begin{cases} 200000, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

Now Thiele's differential equations are

$$\begin{aligned} \frac{\partial}{\partial t} W_*(t) &= v^t (c - \mu_{x+t}\, a_{*\dagger}(x+t)) + \mu_{x+t}\, W_*(t) - \mu_{x+t}\, W_\dagger(t), \\ \frac{\partial}{\partial t} W_\dagger(t) &= 0, \end{aligned}$$

with the boundary conditions $W_*(s-x) = 100000 \times v(s)$ and $W_\dagger(s-x) = 0$. Then Theorem 5.2.1 yields the results listed in Table 5.1.

Table 5.1. Discretization error for an endowment policy

age	$\mu_{*\dagger}(x)$	**reserve discrete model**	**reserve cont. model**	**diff. in %**
65	0.01988	**100000**	**100000**	
64	0.01836	98392	98512	0.12
63	0.01696	96730	96955	0.23
62	0.01566	95022	95341	0.34
61	0.01446	93275	93678	0.43
60	0.01336	91498	91975	0.52
55	0.00897	82360	83092	0.89
50	0.00602	73210	74059	1.16
45	0.00404	64426	65308	1.37
40	0.00271	56236	57096	1.53
35	0.00181	48759	49566	1.65
30	0.00121	42044	42782	1.75

Note that the difference of the reserves in the discrete and continuous model have always the same sign. This is caused by the fact, that people only die at the end of the year in the discrete model. Therefore the required single premium is smaller than in the continuous model.

Exercise 5.3.3. – Calculate yearly premiums for the previous example.

- What happens to the discretization error, if we suppose that people die in the discrete model only at the middle of the year?
- What happens, if we suppose that the interest rates drops linearly from 6% at the age of 30 to 3% at the age of 65?

Exercise 5.3.4. Calculate with the continuous model the premiums for the policy defined in Example 4.8.3.

Example 5.3.5 (Pensions on two lives). 1. Derive Thiele's differential equation for pensions on two lives.

2. Calculate the premiums for the pensions based on the assumption that the husband and his wife die independently.
3. What happens if the the mortality rate for the state $(**)$ (or $(*\dagger) \cup (\dagger *)$) decreases (or increases) by 15 % for each life? (Empirical studies show that the mortality rate of widows and widowers is increased in comparison to the rest of the population.)

For this example we only derive Thiele's differential equations and present the results in a figure. The calculations are done in the setting of Example 4.8.5. (Δt denotes the difference in age of the husband and his wife.)

$$\begin{aligned}
\frac{\partial W_{**}^{+}(t)}{\partial t} &= -v^t\, a_{**}(t) + (\mu_{x+t}^{husband} + \mu_{x+t+\Delta t}^{wife}) W_{**}^{+}(t) \\
&\qquad - \mu_{x+t}^{husband} W_{\dagger *}^{+}(t) - \mu_{x+t+\Delta t}^{wife} W_{*\dagger}^{+}(t), \\
\frac{\partial W_{*\dagger}^{+}(t)}{\partial t} &= -v^t\, a_{*\dagger}(t) + \mu_{x+t}^{wife} \left(W_{*\dagger}^{+}(t) - W_{\dagger\dagger}^{+}(t) \right), \\
\frac{\partial W_{\dagger *}^{+}(t)}{\partial t} &= -v^t\, a_{\dagger *}(t) + \mu_{x+t+\Delta t}^{wife} \left(W_{\dagger *}^{+}(t) - W_{\dagger\dagger}^{+}(t) \right), \\
\frac{\partial W_{\dagger\dagger}^{+}(t)}{\partial t} &= 0.
\end{aligned}$$

Figure 5.1 shows the relation of the present values of the benefits for a change in mortality by $\pm 15\%$. The results are what we expected: a pension on the joint lives becomes more expensive, whereas a pension on the 2nd life becomes cheaper.

Exercise 5.3.6. Complete the previous example.

Exercise 5.3.7. 1. Calculate the present value of the premiums for the insurance on two lives. (Also in the continuous setting one has to treat three cases separately.)

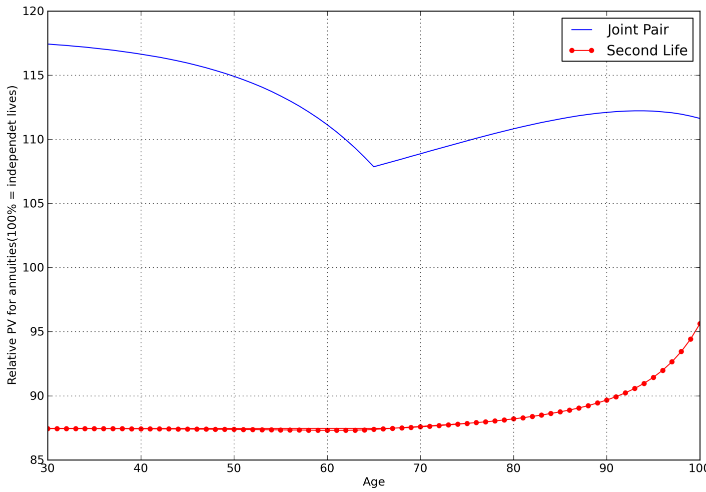


Figure 5.1. Ratio between the present values of benefits for annuities on two lives (100% = independent mortality probabilities).

2. Create a model for an orphan's pension. In the model one has to consider three persons: farther, mother and child. Define the payout functions for a policy which pays 5,000 USD to the child if one of parent dies and 10,000 USD if both die. Assume that the policy matures if the child is 25 years old.

5.4 Differential equations for moments of higher order

Thiele's differential equations characterize the mathematical reserve of an insurance policy. In this section we look at the moments of the mathematical reserve. This will enable us for example to calculate the variance of the reserve, which is a measure for its variation. Furthermore, it can be used to analyze the risk structure of an insurance policy.

We start with the difference equations corresponding to the discrete model. The payout functions in the discrete Markov model have the following form:

$$\Delta B_t = \sum_{j \in J} I_j(t) a_j^{Pre}(t) + \sum_{j,k \in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t),$$

where v_t is the yearly discount from $t+1$ to t, with $v_t = \sum_{j \in J} I_j(t) v_j(t)$.

The prospective reserves are

$$\begin{aligned} V_t^+ &= \sum_{\xi=t}^{\infty} \Big(\prod_{k=t}^{k<\xi} v_k\Big)\, \Delta B_\xi \\ &= \sum_{\xi=t}^{\infty} \Big(\prod_{k=t}^{k<\xi} v_k\Big) \left\{ \sum_{j\in J} I_j(\xi) a_j^{Pre}(\xi) + \sum_{j,l\in J} \Delta N_{jl}(\xi)\, v_\xi\, a_{jl}^{Post}(\xi) \right\}. \end{aligned}$$

Now our aim is to calculate the expectation of the p-th power of the mathematical reserve ($(V_t^+)^p$) conditioned on $\mathcal{F}_t$. The linearity of the integral yields the difference equation

$$V_t^+ = v_t \sum_{j\in J} I_j(t+1) V_{t+1}^+ + \sum_{j\in J} I_j(t) a_j^{Pre}(t) + \sum_{j,k\in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t).$$

This formula indicates that the future reserve is composed of the payments in the period $]t, t+1]$, the payments at time $\{t\}$ and the payments in the period $]t+1, \infty[$. To keep the calculations simple we assume that there is no payment at time $\{t\}$. Furthermore we will use the notation

$$L_t = \sum_{j,k\in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t).$$

Therefore we can simplify the recursion to

$$V_t^+ = v_t \sum_{j\in J} I_j(t+1) V_{t+1}^+ + L_t = \sum_{j\in J} I_j(t+1) \left(v_t\, V_{t+1}^+ + L_t \right).$$

Now the p-th moment is given by

$$\begin{aligned} (V_t^+)^p &= \left(\sum_{j\in J} I_j(t+1)(v_t\, V_{t+1}^+ + L_t) \right)^p \\ &= \sum_{k=0}^{p} \binom{p}{k} \left(\sum_{j\in J} I_j(t+1) v_t\, V_{t+1}^+ \right)^k L_t^{p-k} \\ &= \sum_{k=0}^{p} \binom{p}{k} \sum_{j\in J} I_j(t+1)(v_t\, V_{t+1}^+)^k L_t^{p-k}, \end{aligned}$$

where we used the fact that $I_\alpha(t+1) I_\beta(t+1) = \delta_{\alpha\beta} I_\alpha(t+1)$. Next, we can use $P[A\cap B|C] = P[A|B\cap C] \times P[B|C]$ to simplify the recursion for the expectation. We get

$$
\begin{aligned}
&E\left[(V_t^+)^p \mid X_t = i\right] \\
&\quad = E\left[\sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j\in J} I_j(t+1)(V_{t+1}^+)^k L_t^{p-k} \mid X_t = i\right] \\
&\quad = \sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j\in J} E\left[I_j(t+1)(V_{t+1}^+)^k L_t^{p-k} \mid X_t = i\right] \\
&\quad = \sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j\in J} E\left[I_j(t+1)(V_{t+1}^+)^k \left(v_t\, a_{ij}^{Post}(t)\right)^{p-k} \mid X_t = i\right] \\
&\quad = (v_t^i)^p \sum_{j\in J} p_{ij}(t,t+1) \sum_{k=0}^{p} \binom{p}{k} \left(a_{ij}^{Post}(t)\right)^{p-k} E\left[(V_{t+1}^+)^k \mid X_{t+1} = j\right].
\end{aligned}
$$

This is the difference equation for the higher order moments of the reserve, if there are no payments in advance. Note that the integration of the payments in advance $a_j^{Pre}(t)$ is not complicated. Nevertheless, to keep the presentation clear we continue without them. We summarize our findings in the following theorem.

Theorem 5.4.1 (Differential equation for moments of higher order). *Under the above assumptions the higher order moments of the reserve satisfy the recursion*

$$
\begin{aligned}
&E\left[(V_t^+)^p \mid X_t = i\right] \\
&\quad = (v_t^i)^p \sum_{j\in J} p_{ij}(t,t+1) \sum_{k=0}^{p} \binom{p}{k} \left(a_{ij}^{Post}(t)\right)^{p-k} E\left[(V_{t+1}^+)^k \mid X_{t+1} = j\right].
\end{aligned}
$$

Exercise 5.4.2. Derive the above formula for a model which includes payments in advance.

After treating the discrete case we are now going to derive the analog statements for the continuous setting. The proofs will become more involved, since the model is more general. Before stating the theorem we recall some definitions:

$$
\begin{aligned}
dB &= \sum_{j\in S} I_j(t) dB_j + \sum_{j\neq k} dB_{jk}, \\
dv_t &= -v_t \times \delta_t\, dt, \\
\delta_t &= \sum_{j\in S} I_j(t)\delta_j(t).
\end{aligned}
$$

The p-th moment of the prospective reserve is defined by

$$
\begin{aligned}
V_j^{(p)}(t) &:= E\left[(V_t^+)^p \mid X_t = j\right] \\
&= E\left[\left(\frac{1}{v_t}\int_t^\infty v\, dB\right)^p \mid X_t = j\right],
\end{aligned}
$$

where we implicitly assumed that $V_t^+ \in L^p(\Omega, \mathcal{A}, P)$ and that the functions δ, a_i and a_{jk}, μ_{jk} are piecewise continuous. Then the following theorem holds.

Theorem 5.4.3 (Differential equations for moments of higher order). *Under the above assumptions the functions $V_j^{(p)}(t)$ satisfy the differential equation*

$$
\begin{aligned}
\frac{\partial}{\partial t} V_j^{(p)}(t) &= \left(p\, \delta_j(t) + \sum_{S \ni k \neq j} \mu_{jk}(t)\right) V_j^{(p)}(t) - p\, a_j(t) V_j^{(p-1)}(t) \\
&- \sum_{j \neq k \in S} \mu_{jk}(t) \sum_{k=0}^{p} \binom{p}{k} (a_{jk}(t))^{p-k}\, V_j^{(k)}(t)
\end{aligned}
$$

for all $t \in]0, n[\setminus \mathcal{D}$ with the boundary condition

$$
V_j^{(p)}(t^-) = \sum_{k=0}^{p} \binom{p}{k} (\Delta a_j(t))^{p-k}\, V_j^{(k)}(t)
$$

for all $t \in \mathcal{D}$. Here $\mathcal{D}$ is the set of discontinuities of the payout function B.

Remark 5.4.4. – The differential equation given above also holds at points where the functions $V_j^{(p)}(t)$ are not differentiable. In these points one gets a valid interpretation by considering the differentials which are given by a formal multiplication with the factor dt.

– The idea of the proof is to represent a suitable martingale in two different ways. These representations will then be used to find a stochastic differential equation for the martingale. Then, since the drift term of the differential equation is zero for a martingale, one obtains an ordinary differential equation.

Proof. It turns out to be more convenient to show the differential equation for $W_j^{(p)}(t) = v_t^p\, V_j^{(p)}(t)$. We have

$$
\begin{aligned}
dW_j^{(p)}(t) &= d(v_t^p)\, V_j^{(p)}(t) + v_t^p\, dV_j^{(p)}(t) && (5.1)\\
&= -p v_t^p \sum_{j \in S} I_j(t) \delta_j(t) dt\, V_j^{(p)}(t) + v_t^p\, dV_j^{(p)}(t). && (5.2)
\end{aligned}
$$

Now we define the martingale

$$\begin{aligned} M^{(p)}(t) &:= E\left[\left(\int_0^\infty v\,dB\right)^p \mid \mathcal{F}_t\right] \\ &= E\left[\left\{\left(\int_0^t + \int_t^\infty\right) v\,dB\right\}^p \mid \mathcal{F}_t\right] \end{aligned}$$

and the function

$$U_t = \int_0^t v\,dB.$$

The Markov property implies

$$E\left[\left\{\int_t^\infty v\,dB\right\}^{p-k} \mid \mathcal{F}_t\right] = \sum_{j\in S} I_j(t) W_j^{(p-k)}(t),$$

and using the Binomial Theorem we get

$$M^{(p)}(t) = \sum_{k=0}^{p} \binom{p}{k} \sum_{j\in S} U_t^k I_j(t) W_j^{(p-k)}(t).$$

By choosing a right continuous modification of $M^{(p)}$ we can ensure that U and $I_j(t)$ are right continuous.

Now we want to simplify the differential form

$$dM^{(p)}(t) = \sum_{k=0}^{p} \binom{p}{k} \sum_{j\in S} d\left(U_t^k I_j(t) W_j^{(p-k)}(t)\right).$$

Recall that for a function of bounded variation A we denote by A^{cont} the continuous part and by A^{atom} the discontinuous part. An application of Itô's formula yields

$$\begin{aligned} &d\left(U_t^{(k)} I_j(t) W_j^{(p-k)}(t)\right) \\ &\quad = kU_t^{(k-1)} dU_t^{cont} I_j(t) W_j^{(p-k)}(t) \\ &\quad +U_t^k dI_j^{cont}(t) W_j^{(p-k)}(t) + U_t^k I_j(t) dW_j^{(p-k),cont}(t) \\ &\quad + \left\{U_t^k I_j(t) W_j^{(p-k)}(t) - U_{t-}^k I_j(t^-) W_j^{(p-k)}(t^-)\right\}. \end{aligned} \tag{5.3}$$

To simplify this formula we use for the first line the identities

$$
\begin{aligned}
dU_t^{cont} &= v_t \sum_{l\in S} I_l(t) a_l(t),\\
I_\alpha(t) I_\beta(t) &= \delta_{\alpha\beta} I_\alpha(t).
\end{aligned}
$$

For the second line of (5.3) note that the continuous part of $I_\alpha(t)$ vanishes. Finally we have to deal with the jump part in the third line. The jumps have two possible origins. On the one hand they might be caused by a transition:

$$
\sum_{j\neq l\in S} \left(\{U_{t^-} + v_t a_{jl}(t)\}^k W_l^{(p-k)}(t) - U_{t^-}^k W_l^{(p-k)}(t^-) \right) dN_{jl}(t).
$$

On the other hand they might be due to a jump in a pension:

$$
\sum_{l\in S} I_l(t) \left(\{U_{t^-} + v_t \Delta a_l(t)\}^k W_l^{(p-k)}(t) - U_{t^-}^k W_l^{(p-k)}(t^-) \right).
$$

Moreover, we know that jumps can only occur on the set $\mathcal{D}$ and on this set one can replace $I_l(t)$ by $I_l(t^-)$, since these coincide with probability 1. We also know that a simultaneous jump of both components occurs with probability 0. Finally, $W_l^{(p-k)}(t)$ is continuous and does not induce any jumps, since $\int_t^n v\, dB$ is almost surely continuous on $t \notin \mathcal{D}$. Therefore we get

$$
\begin{aligned}
d&\left(U_t^{(k)} I_j(t) W_j^{(p-k)}(t)\right)\\
&= \sum_{l\in S} I_l(t) \left(k U_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t) + U_t^k dW_j^{(p-k),cont}(t)\right)\\
&\quad + \sum_{j\neq l\in S} \left(\{U_{t^-} + v_t a_{jl}(t)\}^k W_l^{(p-k)}(t) - U_{t^-}^k W_l^{(p-k)}(t^-) \right) dN_{jl}(t)\\
&\quad + \sum_{l\in S} I_l(t^-) \left(\{U_{t^-} + v_t \Delta a_l(t)\}^k W_l^{(p-k)}(t) - U_{t^-}^k W_l^{(p-k)}(t^-) \right).
\end{aligned}
$$

Applying the fact $X_{t^-}dt = X_t\, dt$ and the previous formula to (5.3) we derive

$$
\begin{aligned}
dM^{(p)}(t) &- \sum_{j\neq l\in S} \sum_{k=0}^{p} \binom{p}{k} \Big(\{U_{t^-} + v_t a_{jl}(t)\}^k W_l^{(p-k)}(t)\\
&\qquad\qquad - U_{t^-}^k W_l^{(p-k)}(t^-) \Big) dM_{jl}(t)\\
&= \sum_{j\in S} I_j(t) \sum_{k=0}^{p} \binom{p}{k} \Bigg[k U_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t) dt + U_t^k dW_j^{(p-k),cont}(t)\\
&+ \sum_{j\neq l\in S} \left(\{U_t + v_t a_{jl}(t)\}^k W_l^{(p-k)}(t) - U_t^k W_j^{(p-k)}(t) \right) \mu_{jl}(t) dt \Bigg]\\
&+ \sum_{j\in S} I_l(t^-) \sum_{k=0}^{p} \binom{p}{k} \left(\{U_{t^-} + v_t \Delta a_j(t)\}^k W_j^{(p-k)}(t) - U_{t^-}^k W_j^{(p-k)}(t^-) \right),
\end{aligned}
\tag{5.4}
$$

where we used the identity

$$dM_{ij}(t) \quad = \quad dN_{ij}(t) - I_i(t)\,\mu_{ij}(t).$$

Note that

$$dW_j^{(p-k),cont}(t) \quad = \quad -(p-k)v_t^{(p-k)}\delta_j(t)\,dt\,V_j^{(p-k)}(t) + v_t^{(p-k)}dV_j^{(p-k),cont}(t). \tag{5.5}$$

The left hand side of (5.4) is the differential of a sum of martingales. Thus also the right hand side is the differential of a martingale. Now this has to be constant, since it is previsible and of bounded variation. Therefore the increments of the continuous and the discrete part have to be equal to zero. But this is only possible if

$$\begin{aligned} 0 \quad = \quad & \sum_{k=1}^{p} \binom{p}{k} k U_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t) dt \\ & + \sum_{k=0}^{p} \binom{p}{k} U_t^k W_l^{(p-k)}(t) \\ & + \sum_{k=0}^{p} \binom{p}{k} \sum_{j \neq l \in S} \sum_{r=0}^{k} \binom{k}{r} U_t^{(r)} \left\{ v_t a_{jl}(t) \right\}^{k-r} W_l^{(p-k)}(t) \mu_{jl}(t) dt \\ & - \sum_{k=0}^{p} \binom{p}{k} U_t^k W_j^{(p-k)}(t) \left(\sum_{j \neq l \in S} \mu_{jl}(t) \right) dt \end{aligned} \tag{5.6}$$

holds for all $j \in S$ and all $t \in]0,n[\setminus \mathcal{D}$. For $x \in \mathcal{D}$ we get

$$0 = \sum_{k=0}^{p} \binom{p}{k} \left(\sum_{r=0}^{k} \binom{k}{r} U_{t^-}^{(r)} \left\{ v_t \Delta a_l(t) \right\}^{k-r} W_j^{(p-k)}(t) - U_{t^-}^{(k)} W_j^{(p-k)}(t^-) \right).$$

Using the identity

$$\binom{p}{k} k = \binom{p}{k-1} (p-(k-1))$$

we can transform the first line of Equation (5.6) into

$$\begin{aligned} & \sum_{k=1}^{p} \binom{p}{k-1} (p-(k-1)) U_t^{(k-1)} v_t a_j(t) W_j^{(p-1-(k-1))}(t) dt \\ & \quad = \sum_{k=0}^{p} \binom{p}{k} (p-k) U_t^{(k)} v_t a_j(t) W_j^{(p-1-k)}(t) dt, \end{aligned}$$

where we set $W_j^{(-1)} \equiv 0$. Hence the third line of (5.6) becomes

$$\sum_{k=0}^{p} U_t^{(r)} \sum_{r=0}^{k} \binom{p}{k} \binom{k}{r} \sum_{j \neq l \in S} \{v_t a_{jl}(t)\}^{k-r} W_l^{(p-k)}(t) \mu_{jl}(t) dt.$$

This can be transformed by the identity

$$\binom{p}{k}\binom{k}{r} = \binom{p}{r}\binom{p-r}{k-r}$$

into the representation

$$\sum_{r=0}^{p} \binom{p}{r} U_t^{(r)} \sum_{k=r}^{p} \binom{p-r}{k-r} \sum_{j \neq l \in S} \{v_t a_{jl}(t)\}^{r} W_l^{(p-k-r)}(t) \mu_{jl}(t) dt,$$

i.e.,

$$\sum_{k=0}^{p} \binom{p}{k} U_t^{(k)} \sum_{r=0}^{p-k} \binom{p-k}{r} \sum_{j \neq l \in S} \{v_t a_{jl}(t)\}^{r} W_l^{(p-k-r)}(t) \mu_{jl}(t) dt.$$

If we now gather the powers of U_t we get

$$0 = \sum_{k=0}^{p} \binom{p}{k} U_t^{(k)} dQ_j^{(p-k)}(t),$$

where

$$\begin{aligned} dQ_j^{(q)}(t) &= q v_t a_j(t) W_j^{(q-1)}(t) dt + dW_j^{(q),cont}(t) \\ &\quad + \sum_{k=0}^{q} \binom{q}{k} \sum_{j \neq l \in S} \{v_t a_{jl}(t)\}^{k} W_l^{(q-k)}(t) \mu_{jl}(t) dt \\ &\quad - W_j^{(q)}(t) \sum_{j \neq l \in S} \mu_{jl}(t)\, dt. \end{aligned}$$

This equation implies $dQ_j^{(0)}(t) \equiv 0$ and thus $dQ_j^{(q)}(t) \equiv 0$ by induction. Finally, the formula above and Equation (5.1) and (5.5) imply the result.

5.5 The distribution of the mathematical reserve

The distribution function can be used to answer questions which only depend on the tail of the distribution. Thus it is important for the estimation of extremal risks.

This section has the same structure as the previous section. In the beginning, we solve the problem for the discrete time set. Afterward we treat the continuous time model.

Recall that the cash flows in the discrete Markov model are given by

$$\Delta B_t = \sum_{j\in J} I_j(t) a_j^{Pre}(t) + \sum_{j,k\in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t).$$

We want to calculate the distribution function of the discounted future cash flows

$$P_i(t,u) = P\left[\sum_{j=t}^{\infty}(\prod_{k=t}^{k<j} v_k)\, \Delta B_j < u | X_t = i\right].$$

The following identities hold:

$$\begin{aligned}
P_i(t,u) &= P\left[\sum_{j=t}^{\infty} D_{t,j}\, \Delta B_j < u | X_t = i\right] \\
&= \sum_{l\in J} p_{il}(t)\, P\left[\sum_{j=t}^{\infty} D_{t,j}\, \Delta B_j < u | X_t = i, X_{t+1} = l\right] \\
&= \sum_{l\in J} p_{il}(t) P\Bigg[v_{i,t} \sum_{j=t+1}^{\infty} D_{t,j}\, \Delta B_j < u - a_i^{Pre}(t) \\
&\qquad\qquad - v_{i,t}\, a_{il}^{Post}(t) | X_{t+1} = l\Bigg] \\
&= \sum_{l\in J} p_{il}(t)\, P_l(t+1, v_{i,t}^{-1}(u - a_i^{Pre}(t)) - a_{il}^{Post}(t)), \text{ where} \\
D_{t,j} &= \prod_{k=t}^{k<j} v_k.
\end{aligned}$$

These relations are summarized in the following theorem.

Theorem 5.5.1 (Distribution of the reserves). *The distribution function of the reserves satisfy the recursion*

$$P_i(t,u) = \sum_{j\in J} p_{ij}(t)\, P_j(t+1, v_{i,t}^{-1}(u - a_i^{Pre}(t)) - a_{ij}^{Post}(t)).$$

Besides the recursion formula also boundary conditions are required. These are, in contrast to the previous problems, now given in form of distributions rather than fixed values. For an insurance whose mathematical reserve is equal to zero at maturity the boundary condition is for example given by: (Here ω denotes the maximal age at which insured persons are alive.)

$$P_i(\omega+1,u) = \begin{cases} 0, & \text{if } u \leq 0, \\ 1, & \text{if } u > 0. \end{cases}$$

The distribution function of a pension in the discrete model is shown in Figure 5.2. The jumps, which are caused by the discrete model, are clearly visible.

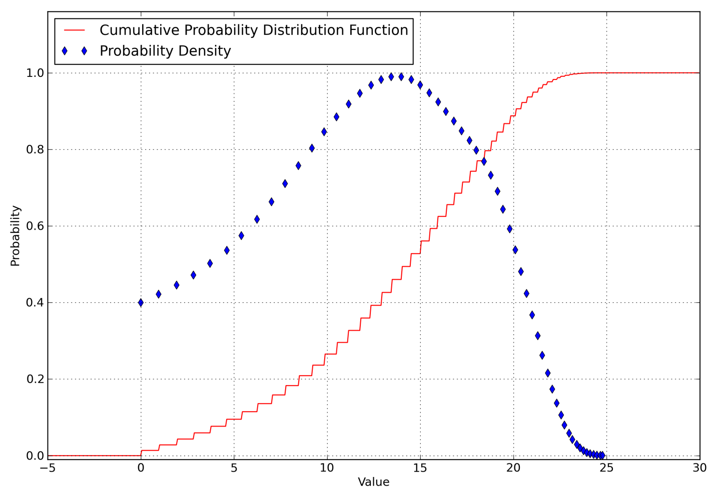


Figure 5.2. Probability distribution function for the present value of an immediate payout annuity (x=65)

In the previous sections we have seen that one can use differential equations to find the moments of the reserves. For cash flows of sufficient regularity one could also prove that the moments are differentiable.

Analogous one could try to find differential equations for the distribution functions. But the following example shows, that for distribution functions this is not an easy task.

Example 5.5.2. This example will illustrate that the distribution function can be discontinuous even for relatively simple insurance policies. We consider

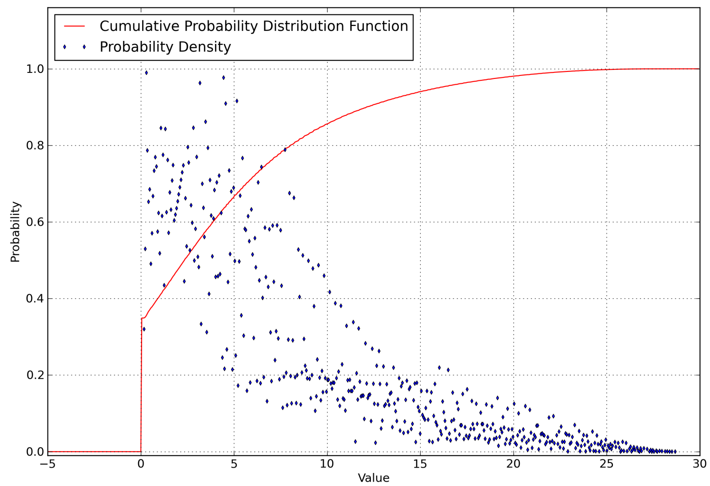


Figure 5.3. Probability distribution function of the present value of a deferred widows pension (x=65)

an endowment policy, with a death benefit of 100,000 USD and an endowment of 200,000 USD. Now we want to calculate the reserve for an insured of 30 years of age. Here T_x will denote the future life span. The following equations hold:

$$V_{30} = \begin{cases} 100000 \times v^{T_x}, & \text{if} \quad T_x < 35, \\ 200000 \times v^{65-30}, & \text{if} \quad T_x \geq 35. \end{cases}$$

Now assume a technical interest rate of 1.5 %. Then the following equation holds for $0 < \alpha \leq 100000$:

$$\begin{aligned} P[V_{30} < \alpha] &= P[100000 v^{T_x} < \alpha, T_x < 35] \\ &= P[35 > T_X > \log(\alpha/100000)/\log(v)] \\ &= {}_{\gamma}p_x - {}_{35}p_x, \text{where} \\ \gamma &= \log(\alpha/100000)/\log(v). \end{aligned}$$

This calculation shows that the distribution function has a jump of size ${}_{35}p_x$ at $200000 v^{35}$, and thus it is discontinuous.

The next theorem shows that the distribution functions satisfy an integral equation. Note that the recursion in discrete time provides an approximation to this integral equation.

Theorem 5.5.3. *The conditional distribution functions of the reserves*

$$P_j(t,u) = P\left[\int_t^\infty \exp\left(-\int_t^\xi \delta_\tau d\tau\right) dB(\xi) \le u | X_t = j\right]$$

satisfy the integral equation

$$\begin{aligned}P_j(t,u) &= \sum_{k\neq j}\int_t^\infty \exp\left(-\int_t^\xi \sum_{l\neq j}\mu_{jl}(\tau)d\tau\right)\mu_{jk}(\xi)\\ &\quad\times P_k\left(\xi, \exp(\delta_j(\xi-t))u - \int_t^\xi \exp(\delta_j(\xi-\tau))dB_j(\tau) - a_{jk}(s)\right)d\xi\\ &\quad+\exp\left(-\int_t^\infty \sum_{l\neq j}\mu_{jl}(\tau)d\tau\right)\chi_{\left[\int_t^n \exp(-\delta_j(\tau-t))dB_j(\tau)\le u\right]}.\end{aligned} \tag{5.7}$$

Proof. The proof is analogous to the discrete setting. One considers

$$A = \left\{\int_t^\infty \exp\left(-\int_t^\xi \delta\right) dB(\xi) \le u\right\}$$

and treats, as in the discrete setting, the various cases separately. We leave the proof to the reader and refer to [HN96].

Exercise 5.5.4. Complete the proof of the previous theorem. [HN96]

After deriving the integral equations for the distribution function, we will modify these slightly. The equations are still hard to handle, since the right hand side depends on t. To overcome this problem we define

$$Q_j(t,u) := P_j\left(t, \exp(\delta_j t)\left(u - \int_0^t \exp(-\delta_j\,\tau)dB_j(\tau)\right)\right).$$

The mapping from P to Q can be inverted by

$$P_j(t,u) = Q_j\left(t, \exp(-\delta_j t)u + \int_0^t \exp(-\delta_j\,\tau)dB_j(\tau)\right).$$

Using Q_j one can easily derive the equation

$$\begin{aligned}\exp\left(-\int_0^t \mu_j\right) Q_j\,(t,u) &= \int_t^n \exp\left(-\int_0^s \mu_j\right)\sum_{k\neq j}\mu_{jk}(s)\\ &\quad\times Q_k\Bigg(s, \exp((\delta_j-\delta_k)s)u + \int_0^s \exp(-\delta_k\tau)dB_k(\tau)\end{aligned}$$

$$- \exp((\delta_j - \delta_k)s) \int_0^s \exp(-\delta_j \tau) dB_j(\tau) - \exp(-\delta_k s) a_{jk}(s) \Bigg) ds$$
$$+ \exp\left(- \int_0^n \mu_j\right) \chi_{\left[\int_0^n \exp(-\delta_j(\tau)) dB_j(\tau) \leq u\right]}.$$

Theorem 5.5.5. *The functions Q satisfy (in the sence of Stieltje's differentials) the following differential equations*

$$\begin{aligned} d_t Q_j(t,u) &= \mu_j \, dt \, Q_j(t,u) - \sum_{k \neq j} \mu_{jk} \, dt \\ &\times Q_k(t, \exp((\delta_j - \delta_k)t) \, u + \int_0^t \exp(-\delta_k \, \tau) dB_k(\tau) \\ &- \exp((\delta_j - \delta_k)t) \int_0^t \exp(-\delta_j \, \tau) dB_j(\tau) - \exp(-\delta_k \, t) \, a_{jk}(t)), \end{aligned}$$

with the boundary conditions

$$Q_j(n,u) = \chi_{\left[\int_0^n \exp(-\delta_j \, \tau) dB_j(\tau) \leq u\right]}.$$

Remark 5.5.6. Theorem 5.5.5 proves useful, since the equations given therein are easier to solve by numerical methods. The main idea is to derive Q in a first step and then calculate P based on Q.

6. Examples and problems from applications

6.1 Introduction

In this chapter we take a closer look at problems which appear in applications. Moreover, the examples will show the scope of the Markov model and illustrate some special modeling tricks. The examples are based on the discrete model, since this is most popular in applications.

Besides the structural conditions given by the model one has to consider the actual conditions fixed in the policies. This is necessary since on the one hand the model should also be able to represent existing policies, on the other hand formulas are often based on explicit contract terms. Therefore, in the following we will try to understand the formulas based on the policy setup.

6.2 Monthly and quarterly payments

We start with the problem of monthly and quarterly payments. For this note that in applications one often considers payment periods of one year, but the actual payments are done monthly or quarterly. As an example we look at a 4/4 pension with payments in advance. That is a pension which pays $\frac{1}{4}$ quarterly. Let us suppose for the moment that the mortality rate for a given year is q_x and that the quarterly mortality $q_x^{[4]}$ is given by

$$(1 - q_x^{[4]})^4 = 1 - q_x.$$

Thus the mortality is constant during the whole year. The current, paid in advance, pension for the discret time model with time intervals of 3 month is given by

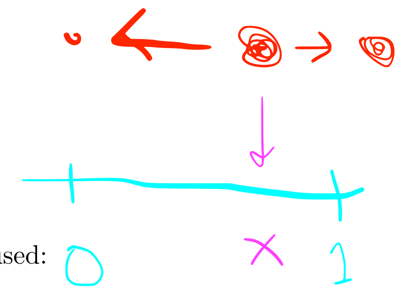
$$a_*(t) = \frac{1}{4},$$

and we have the recursion

$$V_*(t) = a_*(t) + (1 - q_x^{[4]}) v^{\frac{1}{4}} \, V_*(t + \frac{1}{4}).$$

Now we can derive the recursion for a time interval of one year. We get

$$\begin{aligned}
V_*(t) &= a_*(t) + (1 - q_x^{[4]}) v^{\frac{1}{4}}\, V_*(t + \frac{1}{4}) \\
&= a_*(t) \times \sum_{k=0}^{3} \left((1 - q_x^{[4]}) v^{\frac{1}{4}} \right)^k + (1 - q_x) v\, V_*(t+1) \\
&= a_*(t) \times \frac{1 - (1 - q_x) v}{1 - \left((1 - q_x^{[4]}) v^{\frac{1}{4}} \right)} + (1 - q_x) v\, V_*(t+1) \\
&\approx \frac{5}{8} + (1 - q_x) v \left(\frac{3}{8} + V_*(t+1) \right),
\end{aligned}$$



where, in the last step, the following Taylor expansion of f at $z = 1$ was used:

$$\begin{aligned}
f(z) &= \frac{1}{4} \left(1 + z^{0.25} + z^{0.50} + z^{0.75} \right), \\
f(1) &= 1, \\
\frac{d}{dz} f(z)|_{z=1} &= \frac{3}{8}, \\
f(z) &\approx 1 + \frac{3}{8}(z - 1) \\
&= \frac{5}{8} + \frac{3}{8} z.
\end{aligned}$$

The above example shows that one models quarterly payments also by a model with a time interval of one year. Note that the approximation which we derived is in fact the same as the one which is used for a model with commutation functions. Moreover one should note, that these arguments can also be applied to other policies, e.g. insurances on two lives.

Exercise 6.2.1. 1. How do you adapt the method from above, which uses a time interval of one year, to a disability pension with a 3 month waiting period?

2. Calculate the corresponding approximation for a quarterly pension on two lives (for the state $**$) with payments in advance.

3. One can also solve the previous example by decomposing the exact solution into two terms of the form $(1 - q_x)$ and v, respectively. What is the solution in this case?

Note, the formula above does not require that all payments within the year are of the same amount.

Example 6.2.2. Now we calculate the size of the error in the mathematical reserve, which is caused by the approximation of a quarterly pension with payments in advance. We use the mortalities defined by (2.14). The results are listed in Table 6.1. This shows that in the usual range, up to 85 years, the error is very small.

Table 6.1. Approximation error for a quarterly pension

x	part per year exact	part per year approx.	total exact	total approx.	error total
114	0.4436	0.6421	0.4436	0.6421	44.7533%
113	0.5298	0.6681	0.5808	0.7420	27.7533%
112	0.5874	0.6922	0.6915	0.8253	19.3363%
111	0.6323	0.7145	0.7973	0.9115	14.3198%
110	0.6692	0.7351	0.9033	1.0027	11.0058%
105	0.7917	0.8170	1.4893	1.5472	3.8884%
100	0.8614	0.8724	2.2214	2.2597	1.7228%
95	0.9047	0.9098	3.1358	3.1630	0.8654%
90	0.9325	0.9351	4.2484	4.2687	0.4773%
85	0.9508	0.9521	5.5575	5.5733	0.2846%
80	0.9629	0.9636	7.0436	7.0564	0.1817%
75	0.9709	0.9714	8.6713	8.6820	0.1231%
70	0.9763	0.9766	10.3937	10.4028	0.0879%
65	0.9799	0.9801	12.1588	12.1667	0.0656%
60	0.9823	0.9825	13.9156	13.9227	0.0508%
50	0.9850	0.9851	17.2341	17.2399	0.0335%

6.3 Pensions with guaranteed payment periods

Now we want to consider the problem of pensions with a guaranteed minimal number of payment periods. This type of insurance is offered, since an insured does not want that all his paid in premiums are lost in the case of an early death. This means that, when reaching the age of maturity of the policy, the insured is guaranteed a fixed number of pension payments. Technically this corresponds to an adaptation of the mortality rate for the insured during the guarantee period.

Another approach to this problem is the following. We consider a pension with guaranteed payment periods (10 years guaranteed after the age of 65). Hence 65 is the age of maturity of the policy. For the standard pension the non trivial functions which define the policy are given by

$$a_*(t) = \begin{cases} 0, & \text{if} \quad t < 65, \\ 1, & \text{if} \quad t \geq 65. \end{cases}$$

For the pension with guarantee one has to split the state $\dagger$ into dying before 65 (denoted by: $\dagger_<$) and dying after 65 ($\dagger_\geq$). In this case the relevant transition probabilities are

$$\begin{aligned}
p_{**}(x) &= 1 - q_x, \\
p_{*\dagger_<}(x) &= \begin{cases} q_x, & \text{if} \quad t < 65, \\ 0, & \text{if} \quad t \geq 65, \end{cases} \\
p_{*\dagger_\geq}(x) &= \begin{cases} 0, & \text{if} \quad t < 65, \\ q_x, & \text{if} \quad t \geq 65, \end{cases} \\
p_{\dagger_<\dagger_<}(x) &= 1, \\
p_{\dagger_\geq\dagger_\geq}(x) &= 1.
\end{aligned}$$

The non trivial functions which define the policy take now the following form (For a 1/1 pension with payments in advance.)

$$\begin{aligned}
a_*(t) &= \begin{cases} 0, & \text{if} \quad t < 65, \\ 1, & \text{if} \quad t \geq 65, \end{cases} \\
a_{\dagger_\geq}(t) &= \begin{cases} 1, & \text{if} \quad t \in [65; 75[, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

We will give an example which illustrate these pensions with a guaranteed minimal number of payments.

Example 6.3.1. This example shows the development of the mathematical reserve for a 65 year old man. We assume a guarantee period of 15 years.

age	**premium with guarantee**	**premium without guarantee**	**ratio in %**
121	10000	10000	100 %
120	14694	14694	100 %
110	27143	27143	100 %
100	42031	42031	100 %
90	65298	65298	100 %
80	91840	91840	100 %
75	124057	107387	116 %
70	151184	124817	121 %
65	174024	142454	122 %

Figure 6.1 illustrates the mathematical reserve of a term pension (20 years) with 10 years guarantee period, which starts to payout immediately.

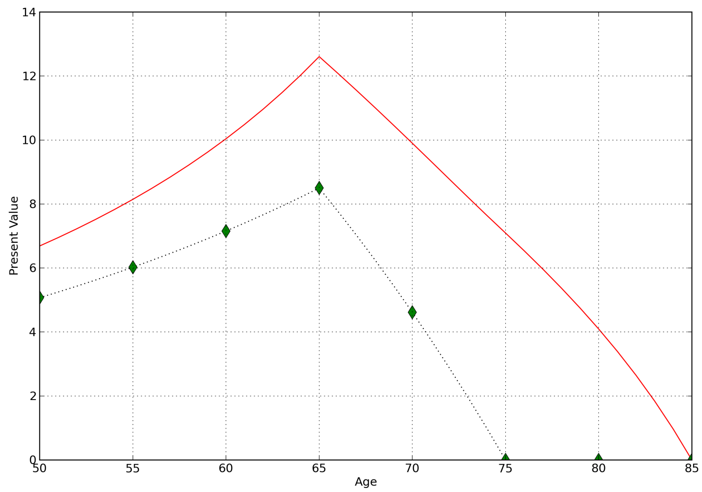


Figure 6.1. Guaranteed annuity($\diamondsuit$ = guaranteed part)

6.4 Refund guarantee

A pension with refund guarantee is a special policy, which guaranties a refund of a certain part of the premiums to the insured in case of an early death. There are several possibilities for the guaranteed refund:

1. refund of the paid in premiums only before maturity of the policy,
2. refund of the difference of paid in premiums and paid out benefits, (total refund)
3. refund of the current mathematical reserve before or after the beginning of the pension payments.

One can understand the first two types of refund guarantees as an additional death benefit. These types of refund guarantees have been part of policies for a long time, and they are well studied. Therefore we will concentrate here on the third type of refund guarantee. Also this type of refund can be modeled in several ways.

Example 6.4.1 (Mathematical reserve refund guarantee). We want to calculate the premiums for the policy with a mathematical reserve refund guarantee. But first we consider an insurance whose death benefit coincides with the mathematical reserve. In this case the following recursion holds:

$$V_*(x) = 1 + p_{**}(x)\ v\ V_*(x+1) + p_{*\dagger}(x)\ V_*(x).$$

(We assumed that the mathematical reserve for the refund guarantee is paid at the beginning of the period.) Now a simple transformation yields the modified recursion equation

$$V_*(x) = \frac{1 + p_{**}(x)\, v\ V_*(x+1)}{(1 - p_{*\dagger}(x))}.$$

This solves the problem in the case of a single premium. To calculate yearly premiums one has to solve a more complicated equation, which can be done easily by numerical methods.

Table 6.2 shows a comparison of single premiums for pensions with various types of refund guarantee. In the case of refund of the mathematical reserve the guarantee ends when the pension payments start. Figure 6.2 illustrates the same comparison for yearly premiums.

Table 6.2. Comparison of single premiums for refund guarantees (KT 1995, $s = 65$, male)

age	refund mathematical reserve	refund premium	total refund
40	5.86921	5.50680	5.58673
45	6.97078	6.64053	6.78605
50	8.27910	8.01191	8.27932
55	9.83297	9.65914	10.15667
60	11.67848	11.61117	12.55151
65	13.87038	13.87038	15.69276

Example 6.4.2. This example will be a bit more involved than the previous. We consider a widow's pension with a single premium. Let additionally the policy guarantee a refund of the mathematical reserve until the husband is 85 years, which is paid if the wife dies or in case of a simultaneous (in the same year) death of the couple. The corresponding recursion is

$$V_{(**)}(x) = \frac{v\,(p_{(**)(**)}\, V_{(**)}(x+1) + p_{(**)(\dagger *)}\, V_{(\dagger *)}(x+1))}{(1 - p_{(**)(*\dagger)}(x) - p_{(**)(\dagger\dagger)}(x))}.$$

Figure 6.3 depicts the solution to this equation. It clearly indicates that the refund guarantee only contributes until 85. Afterward the shown solution is just based on the standard recursion equation.

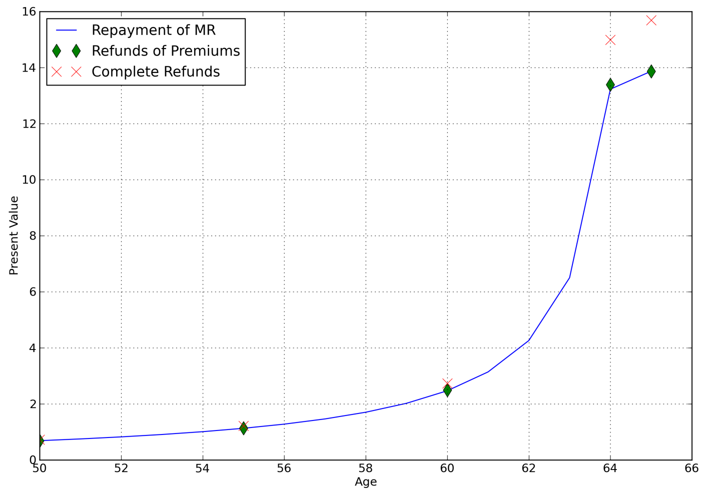


Figure 6.2. Mathematical reserve for permanent pensions with refund guarantee based on yearly premiums

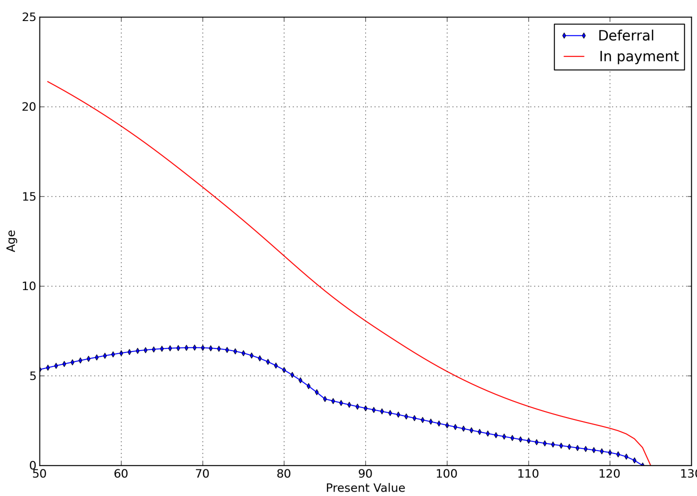


Figure 6.3. Mathematical reserve of a widow's pension with refund guarantee

6.5 Insurances with stochastic interest rate

In this section we consider insurances with stochastic interest rate. We want to focus on the application of methods which were introduced above. Only exemplary interest rate models are used. They are not meant to resemble reality. We consider the endowment policy defined in Example 4.2.1. Thus we have a policy for a 30 year old man, with a death benefit of 200,000 USD and an endowment of 100,000 USD. The aim is to calculate both: the single premium and yearly premiums.

The following interest rate models are used:

1. a technical interest rate fixed at 5%,
2. an interest rate which models a periodic economy,
3. an interest rate given by a random walk model.

Example 6.5.1 (Interest rate models). In order to calculate the premiums we have to set up the three interest rate models. This is trivial for the constant interest rate, and for the remaining two note:

Periodic economy: We assume that the economy behaves periodic and a period consists of 8 states. The interest rate and the transition probabilities for the states are given by

state	**comment**	**interest rate**	p_{ii}	p_{ii+1}	p_{ii+2}
0	start	5.0 %	0.1	0.7	0.2
1	increasing interest	5.5 %	0.1	0.7	0.2
2	max. interest	6.0 %	0.1	0.7	0.2
3	decreasing interest	5.5 %	0.1	0.7	0.2
4	average	5.0 %	0.1	0.7	0.2
5	degreasing interest	4.5 %	0.1	0.7	0.2
6	min. interest	4.0 %	0.1	0.7	0.2
7	increasing interest	4.5 %	0.1	0.7	0.2

This model yields a periodic interest rate, where the duration of a period is random.

Random walk: We consider a random walk on a finite set, whose transition probabilities are modified at the boundary:

state	**comment**	**interest rate**	p_{ii-1}	p_{ii}	p_{ii+1}
0	min. interest	4.0 %	0.0	0.5	0.5
1		4.3 %	0.4	0.2	0.4
2		4.7 %	0.4	0.2	0.4
3	starting position	5.0 %	0.4	0.2	0.4
4		5.3 %	0.4	0.2	0.4
5		5.7 %	0.4	0.2	0.4
6	max. interest	6.0 %	0.5	0.5	0.0

In both models we assume that the current interest rate is 5%.

We are going to simplify the model by using the stochastic interest rate only for the transition $* \rightsquigarrow *$. Thus we always use the technical interest rate of 5% for the transition $* \rightsquigarrow \dagger$.

Example 6.5.2 (Single and yearly premiums). To calculate the premiums we apply the recursions to the interest rate models. This yields:

Fixed interest rate: In this case the premium is $P = 24755/16.77946 = 1475.30$ USD per year, and we get the following results (pv = present value):

age	**benefits pv**	**premiums pv**	**mathematical reserve single premium**	**mathematical reserve yearly premiums**
65	100000	0.00000	100000	100000
64	96510	1.00000	96510	95035
60	83599	4.45585	83599	77026
55	69535	7.84428	69535	57963
50	57483	10.49434	57483	42000
45	47219	12.59982	47219	28631
40	38554	14.28589	38554	17478
35	31305	15.64032	31305	8231
31	25974	16.57022	25974	1528
30	24755	16.77946	24755	0

Periodic interest rate: In this case the premium is $P = 24630/16.65234 = 1479.07$ USD per year, and we get:

age	benefits pv $i_t = 4\%$	benefits pv $i_t = 5\%$	benefits pv $i_t = 6\%$	mathematical reserve yearly premiums $i_t = 5\%$
65	100000	100000	100000	100000
64	97414	95624	96510	95035
60	83854	83369	82536	76004
55	70204	68904	68923	57431
50	57834	57170	57128	41766
45	47482	46996	46812	28368
40	38832	38316	38233	17323
35	31517	31130	31072	8181
31	26145	25838	25761	1509
30	24914	24630	24558	0

Random walk: In this case the premium is $P = 24936/16.81204 = 1483.20$ USD per year, and we get:

age	benefits pv $i_t = 4\%$	benefits pv $i_t = 5\%$	benefits pv $i_t = 6\%$	mathematical reserve yearly premiums $i_t = 5\%$
65	100000	100000	100000	100000
64	97414	96510	95624	95027
60	86558	83611	80775	77002
55	73429	69588	65924	57951
50	61470	57581	53886	42008
45	50934	47356	43963	28652
40	41854	38717	35746	17503
35	34154	31482	28952	8247
31	28476	26155	23959	1531
30	27171	24936	22822	0

6.6 Disability insurance

We will look at a term disability insurance based on a Markov model. The model for a disability insurance consists at least of three states $\{*, \diamond, \dagger\}$ and the corresponding transition probabilities.

Since the reactivation probability (i.e. the probability of becoming active after a disability incurred) depends on the duration of the disability, we going to decompose the state $\diamond$ into states $\diamond_1, \diamond_2, \dots, \diamond_n$. Here the state $\diamond_k$ represents persons whose duration of disability is in $[k-1, k[$. The state $\diamond_n$ has a special meaning: we suppose that persons in this state have a permanent disability.

The temporal behavior of the reactivation probabilities is shown in Figure 6.4. Note that the initial reactivation probability is high for a young person, but it decreases with the age of the insured.

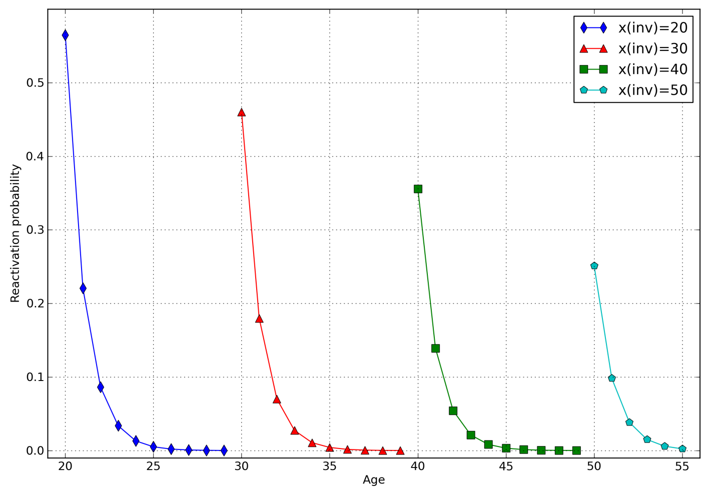


Figure 6.4. Reactivation probability

Moreover for any fixed age at which the disability occurs, the reactivation probability decreases more or less exponentially. Because of this rapid drop in the reactivation probability, one often agrees on some initial waiting period. These induce a minor modification to the model. The state space with the possible transitions for the disability insurance model is show in Figure 6.5.

Since we want to decompose $\diamond$ into the states $\diamond_1, \ldots, \diamond_n$ we are faced with an obvious question: how large must be n such that the error of the model is less than a given bound? We will answer this question for a model with the following transition probabilities:

$$\begin{aligned}
p_{*\dagger}(x) &= \exp(-7.85785 + 0.01538x + 0.000577355x^2), \\
p_{*\diamond_1}(x) &= 3 \times 10^{-4} \times (8.4764 - 1.0985x + 0.055x^2), \\
p_{\diamond_k *}(x) &= \begin{cases} \exp(-0.94(k-1)) \times \alpha(x,k), & \text{if} \quad k < n, \\ 0, & \text{otherwise}, \end{cases} \\
\alpha(x,k) &= 0.773763 - 0.01045(x - k + 1), \\
p_{\diamond_k \dagger}(x) &= 0.008 + p_{*\dagger}(x),
\end{aligned}$$

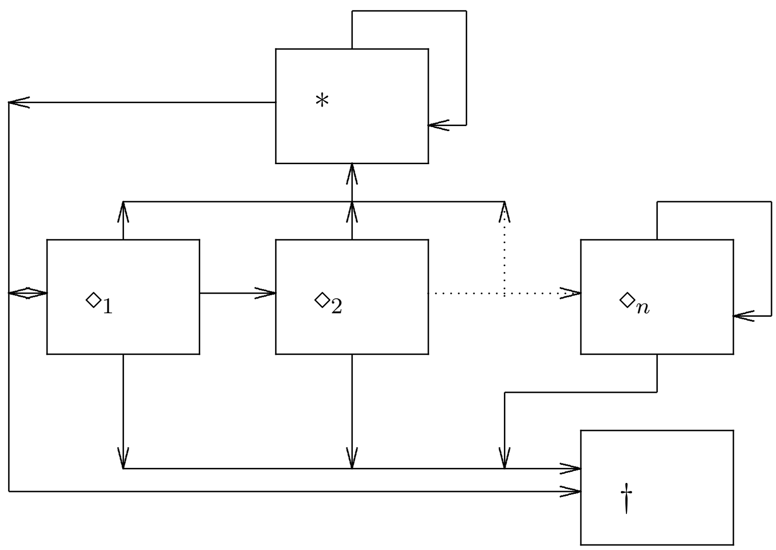


Figure 6.5. State space diagram for a disability insurance

$$\begin{aligned} p_{**}(x) &= 1 - p_{*\diamond_1}(x) - p_{*\dagger}(x), \\ p_{\diamond_k \diamond_{k+1}}(x) &= 1 - p_{\diamond_k *}(x) - p_{\diamond_k \dagger}(x). \end{aligned}$$

For the calculations we further assume that we consider a 1/1 disability pension with payments in advance and with policy function

$$a_{\diamond_k}^{\mathrm{Pre}}(x) = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

The mathematical reserve for an active person for different values of n is show in Figure 6.6.

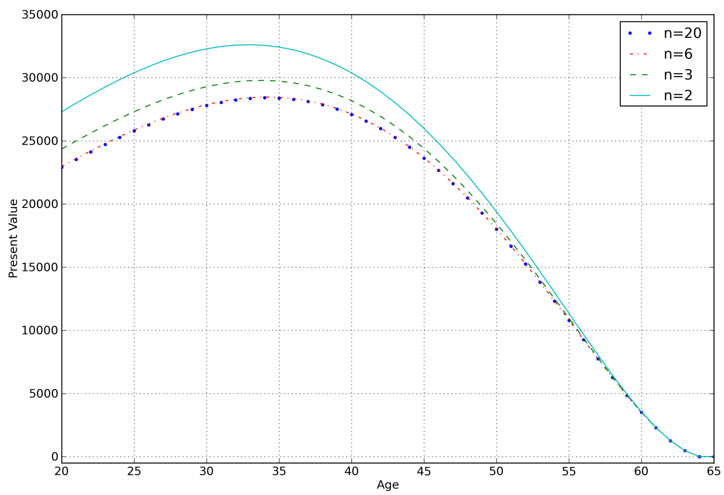


Figure 6.6. Mathematical reserve for active persons for various n

It turns out that for $n = 6$ the error for people of 25 to 65 years of age is less than 5%. Figure 6.7 shows the mathematical reserve for different ages for the disability insurance model with $n = 6$.

6.7 Long Term Care

In this section we want now to focus on long term care business. In order to do this, we need to understand the corresponding cover and how to value it. Afterwards we want to have a look at the risks of this cover.

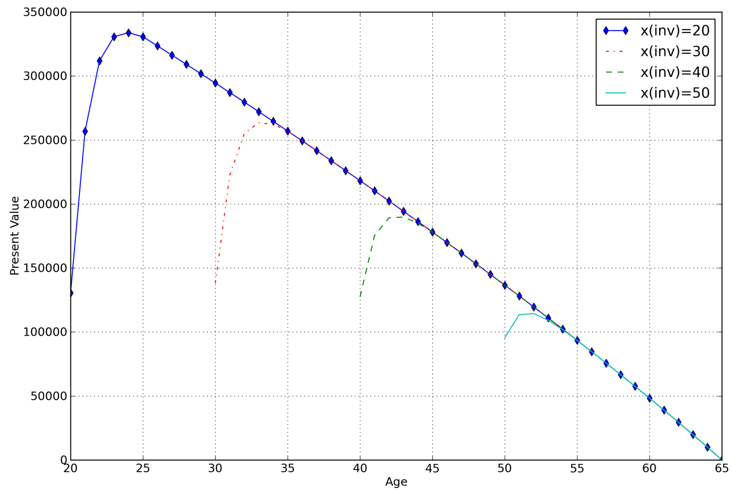


Figure 6.7. Mathematical reserve for disabled persons for $n = 6$

Assume you are a healthy person living at home, able to feed yourself, to wash yourself et cetera. Hence you are able to perform the essential daily living activities (DLA) without help. Once you get older this may not be possible anymore and you are threatened to go into care, which you may not want. You would rather have home help. The long term care (LTC) cover aims to protect you from this, by paying for long term care support. How does this work in practise?

First the insurer defines the main daily living activities which you should be able to perform yourself and an amount which is paid if the person is not able to perform these anymore. Technically speaking we have, for example, 8 DLA which are monitored and you can perform between 0 and 8 of them. One could have a cover where you do not receive anything if your ability is 6 and above and gradually increase for the fewer DLAs you can perform yourself. In the concrete example the respective states are numbered from 1 to 6, where 1 indicates that everything can be autonomously and 6 represents the fact that we need help for all daily living activities. Formally the states are called $S = \{\dagger, 1, 2, 2a, 3, 3a, 4, 5, 6\}$. Assume that the benefits are given by the following table:

Number of DLA	Amount payable pa.
$DLA \geq 7$ (S1, S2)	No benefit, only premium payment
$DLA = 6$ (S2a, S3)	6000
$DLA \leq 5$	12000

– The Markov model consists of the following states $S = \{\dagger, 1, 2, \ldots 8\}$, where $\dagger$ stands for the state of being dead.

– In a next step one needs to define the corresponding transition probabilities $p_{ij}(t, t+1)$, for $(i, j) \in S \times S$.

– For a market consistent valuation the discount rates follow risk free curves as seen before.

– Finally the table above has to be translated in payment functions $a_{ij}^{\text{Post}}(t)$ and $a_i^{\text{Pre}}(t)$. In order not to be overly complicated we assume that the benefits defined above are paid at the beginning of the year. Furthermore we denote with P the premium and we assume that this is only paid in states S1 and S2.

Based on the above we have the following:

$$\begin{aligned} a_{ij}^{\text{Post}}(t) &= 0, \\ a_i^{\text{Pre}}(t) &= \begin{cases} -P & \text{if} \quad i \in \{S1, S2\}, \\ 6000 & \text{if} \quad i \in \{S2a, S3\}, \\ 12000 & \text{else.} \end{cases} \end{aligned}$$

In order to determine the premium and the mathematical reserves we use the Thiele recursion and assume that the person has currently an age $x = 65$. We want to have a closer look at the following questions:

1. What is the price and the mathematical reserves?
2. What happens if we consider an increase in life span and we assume that the remaining probabilities are reduced proportionally?
3. What does it mean if people become older and the time they are healthy remains constant?

Using the elements defined above and Thiele's difference equation, we can calculate the premium P for the two states S1 and S2, where we see an obvious difference in the present value of a premium 1. Figure 6.8 shows this effect. A person buying this cover at age 75 would have to pay about 1000 if he is in state S1 and about 4300 if he is in state S2. This difference shows clearly the risk the company is assuming, as a person who is not able to perform 1 DLA has a materially higher risk. This also explains why the underwriting of this type of policy is of utmost importance. You can imagine what would happen if a person is assumed in state S1 soon becomes unable to perform his daily living activities. Figure 6.8 also shows the relative size of the premiums between states S1 and S2, and we see that there is at age 65 a factor of about 4.5 between the two.

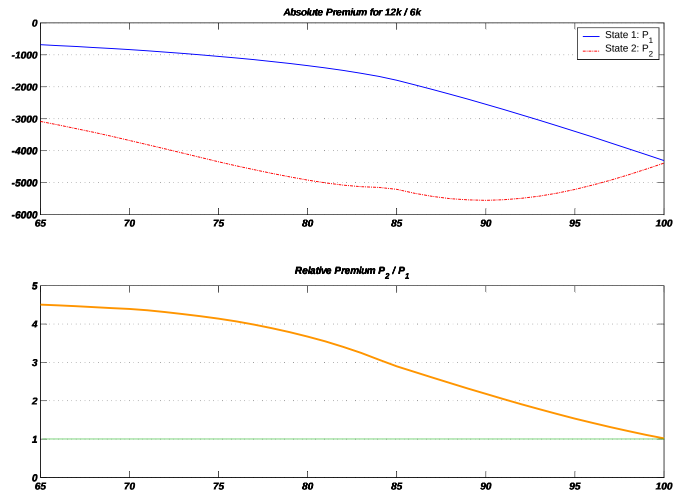


Figure 6.8. Level Premiums for LTC cover

With the same calculation, we can also determine the present value of future cash flows, as shown in Figure 6.9. In order to have a comparison, the mathematical reserves have been scaled relative to state S1. Note that state S4 is the one which is the most expensive and that both states S5 and S6 are cheaper. This is because people in these two states have a higher probability of death and therefore, the time the insurer has to pay is shorter. We will see, that the future development of mortality could impact this. Finally figure 6.10 shows the distribution of the losses. It also shows the dependency of the corresponding state. Such a calculation can either be performed by recursion, or as in the concrete case by a simulation. From this figure we can for example see that the probability of never receiving a benefit for a person starting in state S1 is about 34%. In the same sense we see that the death probability is higher in state S4 than in state S3.

Next we want to look what happens, if we assume that the mortality reduces faster as shown in Figure 6.11. We see that this improvement has a considerable impact. More concretely two versions have been calculated, one (variant 1) where the people remain healthy and stay longer in state S1. In the other, the reduction in mortality goes in parallel with an increased time where the people are not anymore able to perform the different DLAs. Obviously this has a material impact, which needs to be considered when constructing and pricing this type of product. We finally see in figure 6.12 the way the reduction in mortality leads to higher claims. In respect to variant 1 we see that

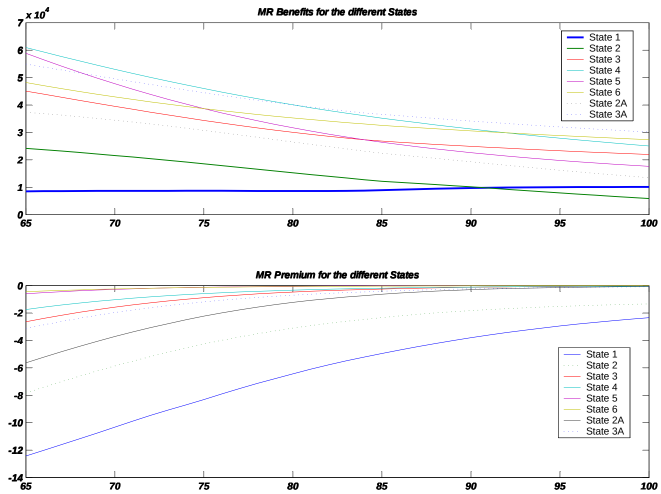


Figure 6.9. Relative Mathematical Reserves for LTC cover

the main additional cash flows are a consequence of living longer, starting at about age 80. We also see that for variant 2 the higher losses start soon after age 70.

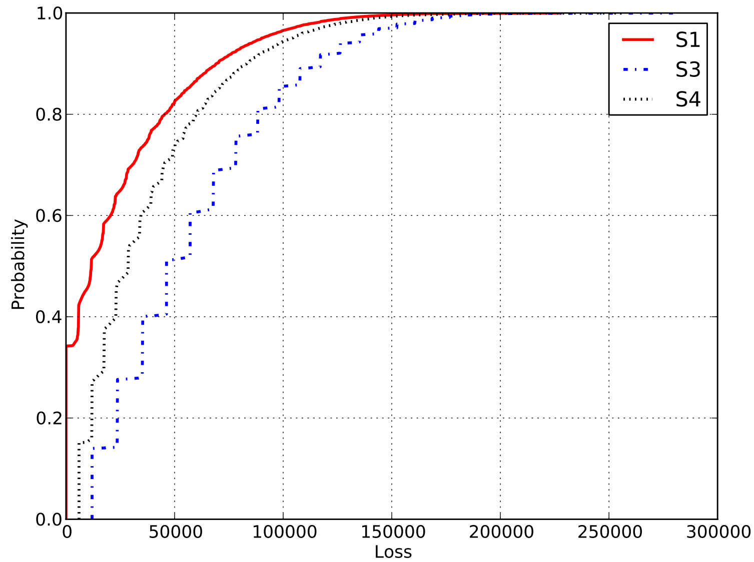


Figure 6.10. Distribution of Mathematical Reserves for LTC cover

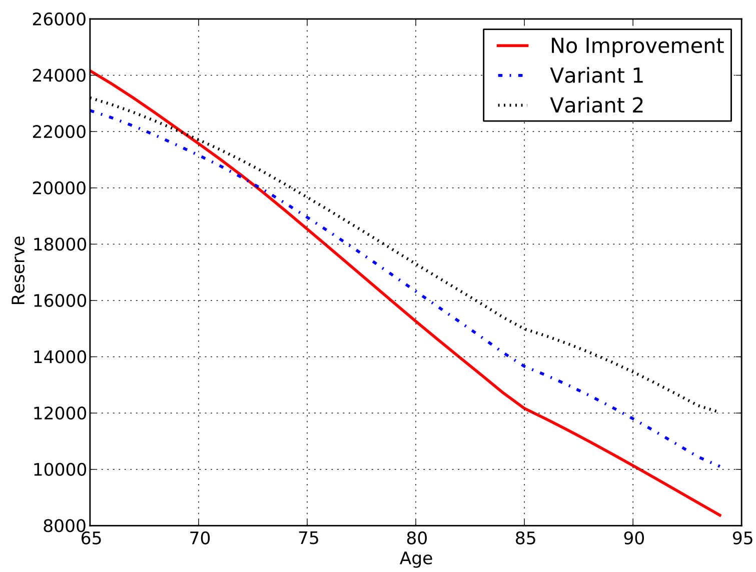


Figure 6.11. LTC Mathematical reserves when reducing mortality

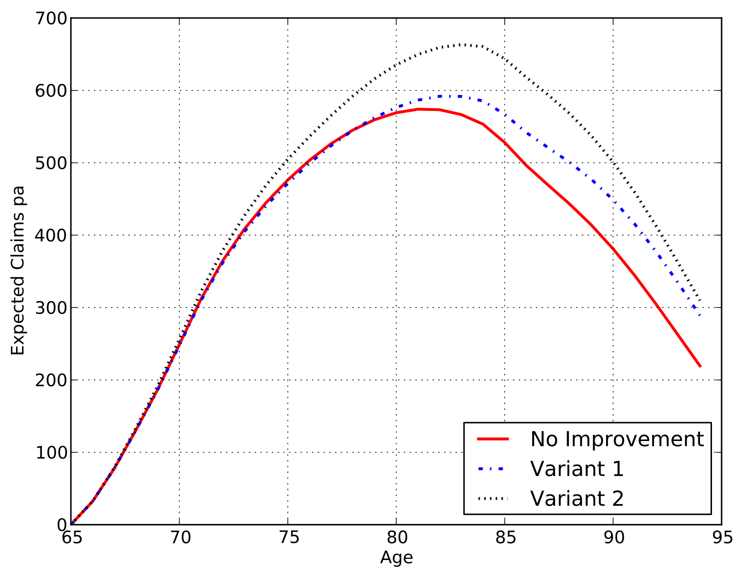


Figure 6.12. LTC expected losses when reducing mortality

7. Hattendorff's Theorem

7.1 Introduction

Hattendorff's Theorem (1868) states that the losses which incur for an insurance policy in different years are uncorrelated and have mean zero. When this theorem was first discovered it caused many discussions. Nowadays it is part of every introduction to stochastic modeling in insurance. In the following we present the theorem in its general form and its version for the Markov model.

7.2 Hattendorff's Theorem - general setting

We start with a cash flow B which is discounted by v. Thus the present value is given by $V = \int v_t \, dB$. Further we assume that v and B are adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Then the prospective mathematical reserve $V_{\mathbb{F}}^{+}(t)$ is given by:

$$V_{\mathbb{F}}^{+}(t) = E\left[\frac{1}{v(t)} \int_{]t,\infty]} v \, dB \middle| \mathcal{F}_t \right].$$

Definition 7.2.1 (Loss for the insurer). *The* loss for the insurer *in the time interval* $]s,t]$ *(with* $s < t$*) discounted to the time* 0 *of a cash flow* B *is defined by*

$$L_{]s,t]} := \int_{]s,t]} v \, dB + v(t) \, V_{\mathbb{F}}^{+}(t) - v(s) \, V_{\mathbb{F}}^{+}(s).$$

The loss is composed of the following parts:

$\int_{]s,t]} v \, dB$	payments in the interval $]s,t]$,
$v(t) \, V_{\mathbb{F}}^{+}(t)$	value of the policy at the end of the period,
$-v(s) \, V_{\mathbb{F}}^{+}(s)$	value of the policy at the beginning of the period.

Remark 7.2.2. The main tool in the proof of Hattendorff's Theorem is the fact that

$$\begin{aligned} M(t) &= E\left[V \middle| \mathcal{F}_t\right] \\ &= \int_{[0,t]} v\, dB + v(t)\, V_{\mathbb{F}}^{+}(t) \end{aligned}$$

is an $\mathbb{F}$–adapted martingale, which we can assume (without loss of generality) to be right continuous. Hence, L can be represented by M in the following way:

$$L_{]s,t]} = M(t) - M(s).$$

Definition 7.2.3. *Let $(T_i)_{i \in \mathbb{N}}$ be a non decreasing sequence of stopping times ($0 = T_0 < T_1 < \ldots$). Then we set*

$$L_i^T = L_{]T_{i-1},T_i]}.$$

Remark 7.2.4. The most important sequences of stopping times $(T_i)_{i \in \mathbb{N}}$ are:

1. $T_i = \alpha_i$ is an increasing sequence of numbers (e.g. the ends of the policy periods),
2. the T_i are the times of a state transition.

To unify the notation, we also set

$$L_0 = B(0) + V_{\mathbb{F}}(0) - E\left[V\right].$$

Theorem 7.2.5 (Hattendorff). *Let $V \in L^2(dP)$, i.e. V has finite moments up to order 2. Then the following statements are equivalent:*

1. $E\left[L_i^T\right] = 0$ *and* $(L_i^T)_{i \in \mathbb{N}}$ *are uncorrelated,*
2. $E\left[L_j^T \middle| \mathcal{F}_i\right] = 0$ *and* $Cov\left(L_j^T, L_k^T \middle| \mathcal{F}_i\right) = 0$ *for all* $i < j, k$*,*
3. $Var\left(\sum_{k=j}^{\infty} L_k^T \middle| \mathcal{F}_{T_i}\right) = \sum_{k=j}^{\infty} Var\left(L_k^T \middle| \mathcal{F}_{T_i}\right)$,
4. $$\begin{aligned} Var\left(L_j^T \middle| \mathcal{F}_{T_i}\right) &= E\left[[M,M](T_j) - [M,M](T_{j-1}) \middle| \mathcal{F}_{T_i}\right] \\ &= E\left[\int_{]T_{j-1},T_j]} d[M,M] \middle| \mathcal{F}_{T_i}\right], \end{aligned}$$

   where $d[M,M](t) = E\left[(dM(t))^2 \middle| \mathcal{F}_{t-}\right]$.

The first two statements are due to Hattendorff.

Proof. $M(t)$ is a martingale, since $V \in L^1(dP)$. Thus $L_{]s,t]} = M(t) - M(s)$ holds, since B and v are $\mathbb{F}$–adapted. Now we can apply Doob's optional sampling theorem to the martingale M, i.e. for stopping times $S \le T$ we get

$$M_S = E\left[M_T \middle| \mathcal{F}_S\right].$$

1. Let the stopping times $S \le T$ be given. Then
$$\begin{aligned} E[L_{]S,T]}] &= E[M_T - M_S] \\ &= E\left[E\left[M_T - M_S \middle| \mathcal{F}_S\right]\right] \\ &= E\left[M_S - M_S\right] = 0. \end{aligned}$$
   For the second part of the statement we consider the corresponding stopping times $S \le T \le U \le V$ and find
$$\begin{aligned} &E\left[L_{]S,T]}L_{]U,V]}\right] \\ &\quad = E\left[(M_T - M_S)(M_V - M_U)\right] \\ &\quad = E\left[M_T\, E\left[M_V - M_U \middle| \mathcal{F}_T\right]\right] - E\left[M_S\, E\left[M_V - M_U \middle| \mathcal{F}_S\right]\right] \\ &\quad = E\left[M_T\left(M_T - M_T\right)\right] - E\left[M_S\left(M_S - M_S\right)\right] = 0. \end{aligned}$$
2. This is essentially proved in the same way as the the first statement.
3. For a finite sum the statement is a consequence of the previous two. Furthermore, $X \mapsto E[X|\mathcal{F}]$ is continuous. Thus the statement is a consequence of the monotone convergence theorem.
4. Let $U \le S \le T$ be stopping times such that $L_i = M_T - M_S$. We define
$$\begin{aligned} H_t &= \begin{cases} 1, & \text{if} \quad t \in]S,T], \\ 0, & \text{otherwise} \end{cases} \\ &= 1_{[0,T]} - 1_{[0,S]}. \end{aligned}$$
   Hence $L_i = (H \cdot M)_\infty$ holds and we get
$$\begin{aligned} \mathrm{Var}\left(L_j \middle| \mathcal{F}_{T_i}\right) &= E\left[(H\cdot M)^2_\infty \middle| \mathcal{F}_{T_i}\right] \\ &= E\left[[M,M](T_j) - [M,M](T_{j-1}) \middle| \mathcal{F}_{T_i}\right] \\ &= E\left[\int_{]T_{j-1},T_j]} d[M,M] \middle| \mathcal{F}_{T_i}\right]. \end{aligned}$$

Exercise 7.2.6. Complete the proof of the theorem above.

7.3 Hattendorff's Theorem - Markov model

Now we want to state Hattendorff's Theorem for the Markov model which we developed in the previous chapters. We concentrate on a regular insurance model (Definition 4.5.6) for which the payout functions a_i are absolute continuous with respect to the Lebesgue measure and the interest intensities ($\delta_i = \delta$) are deterministic.

Definition 7.3.1. *Let $j \in S$ and $n \in \mathbb{N}$. Then $S_n^{(j)}$ denotes the n-th arrival time in state j. Similarly, $T_n^{(j)}$ denotes the n-th departure time from state j.*

S_n and T_n are stopping times, since S is discrete. Based on these stopping times we can define the total loss in state j :

Definition 7.3.2. *For each $j \in S$ we set*

$$\begin{aligned} L^j &= \sum_{n=1}^{\infty} \left[M(T_n^{(j)}) - M(S_n^{(j)}) \right] \\ &= \sum_{n=1}^{\infty} \int_{]S_n^{(j)}, T_n^{(j)}]} dM. \end{aligned}$$

The following lemma is needed in order to determine the variance of the loss.

Lemma 7.3.3. *1. $\{X(t^-) = j\} = \cup_{n=1}^{\infty} \{S_n^{(j)} < t \le T_n^{(j)}\}$.*

2. $L^j = \int \chi_{\{X(t^-)=j\}} dM$.

3.

$$\begin{aligned} Var\left(L^j\right) &= E\left[\int_0^{\infty} \chi_{\{X(t^-)=j\}} (dM(t))^2\right] \\ &= \int_0^{\infty} E\left[\chi_{\{M(t^-)=j\}} E\left[dM[M,M] \middle| \mathcal{F}_{t^-}\right]\right]. \end{aligned}$$

Proof. 1. The proof is left as an exercise to the reader.

2. This is a consequence of the relation $\int_A dM = \int \chi_A \, dM$.

3. $L^j = \int_0^{\infty} \chi_{\{X(t^-)=j\}} dM$ holds and therefore

$$\begin{aligned} Var(L^j) &= E\left[\left[L^j, L^j\right]_{\infty}\right] \text{ by [Pro90] Cor. 2.6.4} \\ &= E\left[\int_0^{\infty} \chi^2_{\{X(t^-)=j\}} d[M,M]\right] \text{ by [Pro90] Thm. 2.6.28} \\ &= E\left[\int_0^{\infty} \chi_{\{X(t^-)=j\}} d[M,M]\right] \\ &= \int_0^{\infty} E\left[\chi_{\{X(t^-)=j\}} \, E\left[(dM(t))^2 \mid \mathcal{F}_{t^-}\right]\right], \end{aligned}$$

where the last equality holds in general, under certain assumptions. See also [Nor96a] and [Nor92].

We are going to apply the previous lemma to the Markov model, but first recall the following statements:

Remark 7.3.4. – $[A+B, A+B] = [A,A] + 2[A,B] + [B,B]$,
– $|[A,B]| \le \sqrt{[A,A]} \times \sqrt{[B,B]}$,
– $[A,B] = AB - \int A_- dB - \int B_- dA$.
– A is of bounded variation $\Longleftrightarrow [A,A] = 0$.
– For a process of quadratic variation $\Delta[A,A] = (\Delta A)^2$ holds.

Next, we are going to calculate $d[M,M]$ and $d[M,M]|\mathcal{F}_{t^-}$. We know that

$$\begin{aligned} M(t) &= \int_0^t v(\tau) \left[\sum_{j \in S} a_j(\tau) I_j(\tau) d\tau + \sum_{S \ni k \neq j} a_{jk}(\tau) dN_{jk}(\tau) \right] \\ &+ E\left[\sum_{j \in S} I_j(t) W_j(t) \big| \mathcal{F}_{t^-} \right], \text{ where} \\ W_j(t) &= v(t)\, V_j(t) \end{aligned}$$

and

$$M(t) = A_0(0) + \int_{]0,t]} \left(\sum_j I_j(\tau) dA_j(\tau) + \sum_{j \neq k} a_{jk}(\tau) dN_{jk}(\tau) \right) + \sum_j I_j(t) W_j(t). \tag{7.1}$$

To proceed we will use partial integration for semimartingales. For this consider the variation process, and note that

$$[I_j, W_j] = 0,$$

since on the one hand $[W_j, W_j] = 0$ (due to $t \mapsto W_j(t)$ beeing of bounded variation) and on the other hand

$$0 \le |[I_j, W_j]| \le \sqrt{[I_j, I_j]} \times \sqrt{[W_j, W_j]} = 0.$$

Thus partial integration yields

$$I_j(t)\,W_j(t) = I_j(0)W_j(0) + \int_{]0,t]} I_j(\tau)dW_j(\tau) + \int_{]0,t]} W_j(\tau^-)dI_j(\tau).$$

To calculate dI_j we can use the identities

$$I_j(t) = \sum_{k \neq j} (N_{kj}(t) - N_{jk}(t))$$

and

$$dI_j(t) = \sum_{k \neq j} (dN_{kj}(t) - dN_{jk}(t)) \,.$$

Finally, we get

$$\begin{aligned}
\sum_j I_j(t)\,W_j(t) &= \sum_j I_j(0)W_j(0) + \sum_j \int_{]0,t]} I_j(\tau)dW_j(\tau) \\
&+ \sum_j \int_{]0,t]} W_j(\tau^-) \sum_{j \neq k} (dN_{kj}(\tau) - dN_{jk}(\tau)) \\
&= W_0(0) + \sum_j \int_{]0,t]} I_j(\tau)dW_j(\tau) \\
&+ \sum_{j \neq k} \int_{]0,t]} \big(W_k(\tau^-) - W_j(\tau^-)\big)\, dN_{jk}(\tau).
\end{aligned}$$

This equation tells us, that the change of the prospective mathematical reserve is equal to

- the change of the mathematical reserve, if there is no state transition
- the difference of the old and the new mathematical reserve corresponding to a transition.

Now, by combining the formula which we derived and Equation (7.1) we get the following theorem.

Theorem 7.3.5. *The martingale M can be calculated by the formula*

$$\begin{aligned}
M(t) &= A_0(0) + W_0(0) + \int_0^t \sum_j I_j(t) d\left(A_j(\tau) + W_j(\tau)\right) \\
&+ \sum_j \int_{]0,t]} \sum_{k \neq j} \big(a_{jk}(\tau) + W_k(\tau^-) - W_j(\tau^-)\big)\, dN_{jk}(\tau).
\end{aligned}$$

Based on this we can calculate the quadratic variation of M. First of all we know that A and W are continuous and we introduce the notations:

$$
\begin{aligned}
A &:= \int_0^t \sum_j I_j(t) d\left(A_j(\tau) + W_j(\tau)\right), \\
B &:= \sum_j \int_{]0,t]} \sum_{k \neq j} \left(a_{jk}(\tau) + W_k(\tau^-) - W_j(\tau^-)\right) dN_{jk}(\tau).
\end{aligned}
$$

Then

$$[M, M] = [A + B, A + B] = [A, A] + 2[A, B] + [B, B],$$

where $|[A, B]| \leq \sqrt{[A, A]} \times \sqrt{[B, B]}$. Next, we define $Z_j = A_j(\tau) + W_j(\tau)$ and note that Z is continuous, since

$$
\begin{aligned}
Z_j(\tau^-) &= A_j(\tau^-) + W_j(\tau^-) \\
&= A_j(\tau) - \left(A_j(\tau) - A_j(\tau^-)\right) + W_j(\tau^-) \\
&= A_j(\tau) + W_j(\tau) \\
&= Z_j(\tau).
\end{aligned}
$$

The continuity of $Z_j(\tau)$ implies $[Z_j, Z_j] = 0$ and therefore also the following statements hold:

$$
\begin{aligned}
[A, B] &= 0, \\
[A, A] &= 0.
\end{aligned}
$$

These equations yield the following theorem.

Theorem 7.3.6 (Hattendorff).

$$d[M, M](t) = \sum_{k \neq j} \left(a_{jk}(t) + V_k(t^-) - V_j(t^-)\right)^2 v(t)^2 dN_{jk}(t).$$

Proof. The statement is simple if we recall that we defined $v(\tau) = 1$. Then we just have to rewrite the previous equations with the substitutions

$$
\begin{aligned}
a_{jk} &\longrightarrow v(\tau)\, a_{jk}(\tau), \\
v(\tau) V_k(\tau) &= W_k(\tau).
\end{aligned}
$$

The value

$$R_{jk}(t) = a_{jk}(t) + V_k(t^-) - V_j(t^-)$$

is called value at risk. It is composed of

1. the payment, which is due when a transition from j to k occurs,
2. the difference of the mathematical reserves corresponding to the transition.

A further consequence of the calculations above is the next theorem, which provides a formula for the variance of the loss.

Theorem 7.3.7. *The following statements hold:*

1. *Let* $X(0) = 1$, *then*

$$Var\ L^j(0) = \int_{]0,\infty[} v(\tau)^2 p_{1j}(0,\tau) \sum_{j \neq k} \mu_{jk}(\tau) R_{jk}^2(\tau) d\tau.$$

2. *The variance of the future loss* $L_j^k(t)$, *conditioned on* $X(t) = j$, *is*

$$Var\ L_j^k(t) = \frac{1}{v(t)^2} \int_{]t,\infty[} v(\tau)^2\, p_{jk}(t,\tau) \sum_{l \neq k} \mu_{kl}(\tau) R_{kl}^2(\tau) d\tau.$$

8. Unit-linked policies

8.1 Introduction

Up to now we have mostly considered models with deterministic interest rates or with an interest rate given by a Markov chain on a finite state space. This helped us to keep the calculations simple. In this chapter we have a look at variable annuities. We consider models for policies whose actual value depends on the performance of an underlying unit (usually a funds). Since we do not need at the beginning the entire complexity of a Markov model for the underlying demographic process, we will start using the simpler traditional approach. In case the reader is not aware of this we will formally introduce this below.

Definition 8.1.1. *A classical life insurance model consists of a Markov model with a state space $S = \{\star, \dagger\}$, where $\star$ and $\dagger$ represent the states of being alive or death, respectively. We assume that the state $\dagger$ is absorbing (ie $p_{\dagger,\star} \equiv 0$. If we assume that the person is alive at age x (ie $X_x = \star$), we can define the future life span T_x of a person aged x as*

$$T_x = min\{\xi > x | X_\xi = \star\} - x,$$

and we use the following (common) notations (for $\Delta_x > 0$, $t > 0$):

$$\begin{aligned} {}_t q_{x+\Delta_x} &= P(T_x \leq t + \Delta_x | T_x > \Delta_x) \\ {}_t p_{x+\Delta_x} &= P(T_x > t + \Delta_x | T_x > \Delta_x) \\ \mu_{x+\Delta_x} &= \lim_{\Delta \to 0} \frac{{}_\Delta q_{x+\Delta_x}}{\Delta} \end{aligned}$$

Remark 8.1.2. In the same sense we can allow for a slightly refined insurance model allowing for lapse (or also called surrender). In this case the state space consists of $S = \{\star, \dagger\, \ddagger\}$, where $\ddagger$ represents the additional state of lapse. It is worth noting that also this state is absorbing. Furthermore, we have in this case (as also in the above case)

$$
\begin{aligned}
{}_t p_x &= p_{\star,\star}(x, x+t) \\
&= \bar{p}_{\star,\star}(x, x+t).
\end{aligned}
$$

We begin with a look at policies whose value is tied to a bond or a funds. The payout of these so called "unit-linked policies" usually consists of a certain number of shares of a funds (the underlying unit) to the insured, in case of an occurrence of the insured event.

These policies have the characteristic feature that the level of the benefits (endowments or death benefits) are not deterministic, but random, depending on the underlying funds. A unit-linked policy is usually financed by a single premium. This type of financing is preferred due to the management of these policies. Note that this is in contrast to traditional policies. Moreover one has to note that the value at risk is constant for a traditional policy, but for a unit-linked policy it depends on the underlying funds.

To analyse unit-linked policies, we introduce the following notation:

$N(t)$	number of shares at time t,
$S(t)$	value of a share at time t.

We assume that $N(t)$ is deterministic. The relevant quantities for a life insurance in this setting and in the traditional setting are summarised in the following table:

	traditional	(pure) unit-linked
death benefit	$C(t) = 1$	$C(t) = S(t)$
value (time 0)	$\pi_0(t) = \exp(-\delta t)$	$\pi_0(t) = S(0)$ (assuming a "normal" economy)
single premium	$E\left[\int_0^T \pi_0(t) d(\chi_{T_x \le t})\right]$ $= \int_0^T \exp(-\delta t) {}_t p_x \mu_{x+t} dt$	$E\left[\int_0^T \pi_0(t) d(\chi_{T_x \le t})\right]$ $= S(0) \int_0^T {}_t p_x \mu_{x+t} dt$ $= (1 - {}_T p_x) S(0)$

The above terms indicate that the financial risk taken by the insurer is smaller for a unit-linked product than for a traditional product with a fixed technical interest rate[1]. Furthermore, note that in the calculation of the single premium

[1] Actually, this is not true in general. Here we implicitly assumed that the capital market risks of a unit-linked policy are minimised by an appropriate trading strategy. For a classical insurance such a trading strategy replicates the cash flows by zero coupon bonds with the corresponding maturities.

we implicitly assumed that the mean of the discounted (to time 0) value of the funds at time t coincides with the value of the funds at time 0. This means, that we have to start with a discussion of the value or price of a funds.

Also the model did not include any guarantees. But generally one would like to add a guarantee (e.g. a refund guarantee for the paid in premiums) to the policy. In this case we have a unit linked insurance with a guarantee also known as variable annuity. For example, the guarantee could be of the form

$$G(t) = \int_0^t \bar{p}(s) ds,$$

where $\bar{p}(s)$ denotes the density of the premiums at time s. More generally one could be interested in a refund guarantee of the paid in premiums with an additional interest at a fixed rate:

$$G(t) = \int_0^t \exp(r(t-s)) \bar{p}(s) ds.$$

In these examples the payout function would be

$$C(t) = \max(S(t), G(t)).$$

Let us assume that the value of the funds is given by a stochastic process (with distribution P). What is, in this setting, the value of the discounted payment $C(t)$ at time t? A first guess might be

$$\pi_0 \left(C(t)\right) = E^P \left[\max(S(t), G(t))\right].$$

But it is not that simple! If this would be the value of the payment, there would be the possibility to make a profit without risk (arbitrage). In order to prevent this possibility one has to change the measure P. In mathematical finance it is proved that an equivalent martingale measure exists, such that there is no arbitrage. Then in a "fair" market we have

$$\pi_0 \left(C(t)\right) = E^Q \left[\max(S(t), G(t))\right],$$

where Q is a measure equivalent to P such that the discounted value of the underlying funds is a martingale.

Furthermore, in mathematical finance payments like $C(t)$ are called the payouts of an option. To determine the price of an option, one uses the arbitrage free pricing theory. A quick introduction to this theory will be given in the next section.

8.2 Pricing theory

In this section we look at modern financial mathematics. It is not our aim to give a comprehensive exposition with proofs of every detail, which would easily fill a whole book. We only want to give a brief survey which illustrates the theory. The reader interested in more details is referred to [Pli97], [HK79], [HP81] and [Duf92]. In the same sense we suggest [Oks03] for further reading regarding stochastic integrals and stochastic differential equations.

In this context we clearly also have to mention the paper of Black and Scholes [BS73] with their famous formula for the pricing of options.

8.2.1 Definitions

First we start with an example which illustrates the use of the pricing theory. The price of a share, modelled by a geometric Brownian motion $(S_t(\omega))$, might develop as shown in Figure 8.1.

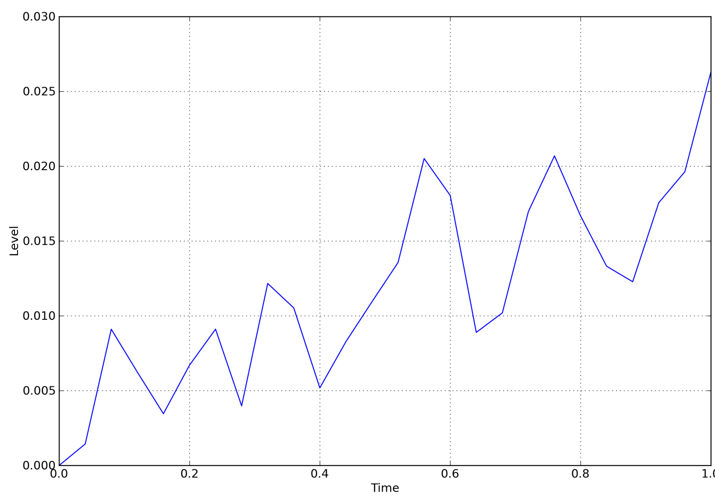


Figure 8.1. Movement of a share price

A European call option for a certain share is the right to buy these shares at a fixed price c (strike price) at a fixed time T. The value of this right at time T is

$$H = \max\left(S_T - c, 0\right).$$

Now a bank would like to know the value (i.e., the fair price) of this option at time 0. As noted in the previous section, taking the expectation would systematically yield the wrong values. In many cases this would provide the possibility to make profits without risk. Thus there would be arbitrage opportunities.

To simplify the exposition we will consider the most simple models of the economy, i.e. finite models. In particular the time set will be discrete. The reader interested in the corresponding theorems for continuous time can for example find these in [HP81]. The ideas and concepts of the pricing theory are presented, as follows:

Let $(\Omega, \mathcal{A}, P)$ be a probability space where Ω is a finite set. Moreover, we assume that $P(\omega) > 0$ holds for all $\omega \in \Omega$.

We also fix a finite time T, as the time at which all trading is finished. The σ-algebra of the observable events at time t is denoted by $\mathcal{F}_t$, and the shares are traded at the times $\{0, 1, 2, \ldots, T\}$. We note that $(\mathcal{F}_t)_{t \in T}$ is a filtration, ie $\mathcal{F}_t \subseteq \mathcal{F}_s$, for $t \leq s$.

We suppose that there are $k < \infty$ stochastic processes, which represent the prices of the shares $1, \ldots, k$, i.e.,

$$S = \{S_t, t = 0, 1, 2, \ldots, T\} \text{ with components } S^0, S^1, \ldots S^k.$$

As usual, we assume that each S^j is adapted to $(\mathcal{F}_t)_t$. Here S_t^j can be understood as the price of the jth share at time t. The fact that the price process has to be adapted reflects the necessity that one has to know at time t the previous price of S. The share S^0 plays a special role. We suppose that $S_t^0 = (1+r)^t$, i.e., we have the possibility to make risk free investments which provide interest rate r. The risk free discount factor is defined by

$$\beta_t = \frac{1}{S_t^0}.$$

Next, we are going to define what is meant by a trading strategy.

Definition 8.2.1. *A* trading strategy *is a previsible ($\phi_t \in \mathcal{F}_{t-1}$) process $\Phi = \{\phi_t, t = 1, 2, \ldots, T\}$ with components ϕ_t^k.*

We understand ϕ_t^k as the number of shares of type k which we own during the time interval $[t-1, t[$. Therefore ϕ_t is called the portfolio at time $t-1$.

Notation 8.2.2. *Let X, Y be vector valued stochastic processes. Then we use the notations:*

$$\begin{aligned} < X_s, Y_t > \quad &= \quad X_s \cdot Y_t = \sum_{k=0}^{n} X_s^k \times Y_t^k, \\ \Delta X_t \quad &= \quad X_t - X_{t-1}. \end{aligned}$$

Next, we want to determine the value of the portfolio at time t:

time	value of the portfolio
$t-1$	$\phi_t \cdot S_{t-1}$
t^-	$\phi_t \cdot S_t$

Thus, the return in the interval $[t-1, t[$ is $\phi_t \cdot \Delta S_t$, and hence the total return is

$$G_t(\phi) = \sum_{\tau=1}^{t} \phi_\tau \cdot \Delta S_\tau.$$

We fix $G_0(\phi) = 0$, and $(G_t)_{t \geq 0}$ is called return process.

Theorem 8.2.3. *G is an adapted and real valued stochastic process.*

Proof. The proof is left as an exercise to the reader.

Definition 8.2.4. *A trading strategy is* self financing*, if*

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \qquad \forall t = 1, 2, \ldots, T-1.$$

A self financing trading strategy is just a trading strategy where at no time further money is added to or deduced from the portfolio.

Definition 8.2.5. *A trading strategy is* admissible*, if it is self financing and*

$$V_t(\phi) := \begin{cases} \phi_t \cdot S_t, & \text{if} \quad t = 1, 2, \ldots, T, \\ \phi_1 \cdot S_0, & \text{if} \quad t = 0 \end{cases}$$

is non negative. (In other words, one is not allowed to become bankrupt.) The set of admissible trading strategies is denoted by Φ.

Remark 8.2.6. The idea of admissible trading strategies is to only consider portfolios which neither lead to bankruptcy nor allow an addition or deduction of money. This also indicates that the value of the trading strategies remains constant when the portfolio is rearranged. Thus a trading strategy, which generates the same cash flow as an option, can be used to determine the value of the option.

Definition 8.2.7. *A contingent claim is a positive random variable X. The set of all contingent claims is denoted by $\mathcal{X}$.*

A random variable X is attainable, if there exists an admissible trading strategy $\phi \in \Phi$ which replicates it, i.e.

$$V_T(\phi) = X.$$

In this case one says "ϕ replicates X".

Definition 8.2.8. *The price of an attainable contingent claim, which is replicated by ϕ, is denoted by*

$$\pi = V_0(\phi)$$

(We will see later, that this price is not necessarily unique. It coincides with the initial value of the portfolio.)

8.2.2 Arbitrage

We say, the model offers arbitrage opportunities, if there exists

$$\phi \in \Phi \text{ with } V_0(\phi) = 0 \text{ and } V_T(\phi) \text{ positive and } P[V_T(\phi) > 0] > 0,$$

i.e., money is generated out of nothing. If such a strategy exists, one can make a profit without taking any risks. One of the axioms of modern economy says that there are no arbitrage opportunities. This is fundamental for some important facts in the option pricing theory.

Now, we are going to define what is meant by a price system.

Definition 8.2.9. *A mapping*

$$\pi : \quad \mathcal{X} \to [0, \infty[, \quad X \mapsto \pi(X)$$

is called price system *if and only if the following conditions hold:*

– $\pi(X) = 0 \iff X = 0$,
– π *is linear.*

A price system is consistent*, if*

$$\pi(V_T(\phi)) = V_0(\phi) \qquad \textit{for all } \phi \in \Phi.$$

The set of all consistent price systems is denoted by Π, and $\mathbb{P}$ denotes the set

$$\mathbb{P} = \{Q \textit{ is a measure equivalent to } P, \textit{ s.th. } \beta \times S \textit{ is a martingale w.r.t. } Q\},$$

where β is the discount factor from time t to 0. The measures $\mu \in \mathbb{P}$ are called equivalent martingale measures.

Theorem 8.2.10. *There is a bijection between the consistent price systems $\pi \in \Pi$ and the measures $Q \in \mathbb{P}$. It is given by*

1. $\pi(X) = E^Q[\beta_T X]$.

2. $Q(A) = \pi(S_T^0\, \chi_A)$ *for all* $A \in \mathcal{A}$.

Proof. Let $Q \in \mathbb{P}$. We define $\pi(X) = E^Q\left[\beta_T\, X\right]$. Then π is a price system, since P is strictly positive on Ω and Q is equivalent to P. Thus it remains to show, that π is consistent. For $\phi \in \Phi$ we get

$$\begin{aligned}
\beta_T\, V_T(\phi) &= \beta_T\, \phi_T\, S_T + \sum_{i=1}^{T-1} (\phi_i - \phi_{i+1})\, \beta_i\, S_i \\
&= \beta_1\, \phi_1\, S_1 + \sum_{i=2}^{T} \phi_i\, \left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right),
\end{aligned}$$

where we used that ϕ is self financing. This yields

$$\begin{aligned}
\pi\left(V_T(\phi)\right) &= E^Q\left[\beta_T\, V_T(\phi)\right] \\
&= E^Q\left[\beta_1\, \phi_1\, S_1\right] + E^Q\left[\sum_{i=2}^{T} \phi_i\, \left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right)\right] \\
&= E^Q\left[\beta_1\, \phi_1\, S_1\right] + \sum_{i=2}^{T} E^Q\left[\phi_i\, E^Q\left[\left(\beta_i\, S_i - \beta_{i-1} S_{i-1}\right) | \mathcal{F}_{i-1}\right]\right] \\
&= \phi_1\, E^Q\left[\beta_1\, S_1\right] \\
&= \phi_1\, \beta_0\, S_0,
\end{aligned}$$

since ϕ is previsible and $\beta\, S$ is a martingale with respect to Q.

Thus, π is a consistent price system.

Now let $\pi \in \Pi$ be a consistent price system and Q be defined as above. Then $Q(\omega) = \pi(S_t^0\, \chi_{\{\omega\}}) > 0$ holds for all $\omega \in \Omega$, since $S_t^0\, \chi_{\{\omega\}} \neq 0$. Moreover, we have $\pi(X) = 0 \iff X = 0$ and therefore Q is absolutely continuous with respect to P.

In the next step, we are going to show that Q is a probability measure. We define

$$\phi^0 = 1 \qquad \text{and} \qquad \phi^k = 0 \quad \forall k \neq 0.$$

Hence, by the consistency of π, we get

$$\begin{aligned}
1 &= V_0(\phi) \\
&= \pi(V_T(\phi)) \\
&= \pi(S_T^0 \cdot 1) \\
&= Q(\Omega).
\end{aligned}$$

The prices of positive contingent claims are positive and Q is additive. Therefore, Kolmogorov's axioms are satisfied, since Ω is finite. We have $Q(\omega) = \pi(S_T^0 \cdot \chi_{\{\omega\}})$ by definition. Hence, also

$$E[f] = \sum_{\omega} \pi(S_T^0 \cdot \chi_{\{\omega\}}) \cdot f(\omega) = \pi(S_T^0 \cdot \sum_{\omega} f(\omega)).$$

Thus, with $f = \beta_t X$, we have

$$E^Q[\beta_T X] = \pi(S_T^0 \cdot \beta_T \cdot X) = \pi(X).$$

Now we still have to show that $\beta_T S_T^k$ is a martingale for all k. Let k be a coordinate and τ be a stopping time, and set

$$\begin{array}{rcl} \phi_t^k & = & \chi_{\{t \leq \tau\}}, \\ \phi_t^0 & = & \left(S_\tau^k / S_\tau^0\right) \chi_{\{t > \tau\}}. \end{array}$$

(We keep the share k up to time τ, then it is sold and the money is used for a risk free investment.) It is easy to show, that the strategy ϕ is previsible and self financing. Finally, for an arbitrary stopping time τ,

$$\begin{array}{rcl} V_0(\phi) & = & S_0^k, \\ V_T(\phi) & = & \left(S_\tau^k / S_\tau^0\right) S_T^0 \\ \text{and} & & \\ S_0^k & = & \pi(S_T^0 \cdot \beta_\tau \cdot S_\tau^k) \\ & = & E^Q\left[\beta_\tau \cdot S_\tau^k\right]. \end{array}$$

Thus $\beta_T S_T^k$ is a martingale with respect to Q.

Above we have proved one of the main theorems in the option pricing theory. Next, we will present further statements without proofs. They all can be found for example in [HP81].

Theorem 8.2.11. *The following statements are equivalent*

1. *There is no arbitrage opportunity,*
2. $\mathbb{P} \neq \emptyset$,
3. $\Pi \neq \emptyset$.

Lemma 8.2.12. *Suppose there exists a self financing strategy $\phi \in \Phi$ such that*

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0,\ E[V_T(\phi)] > 0.$$

Then there exists an arbitrage opportunity.

Example 8.2.13. We are going to calculate the price of an option for a simple example. Consider a market with two shares $Z = (Z_1, Z_2)$ which are traded at the times $t = 0$, $t = 1$ and $t = 2$. Figure 8.2 shows the possible behaviour of these shares in form of a tree. To calculate the price of the option we suppose that all nine possibilities have the same probability.

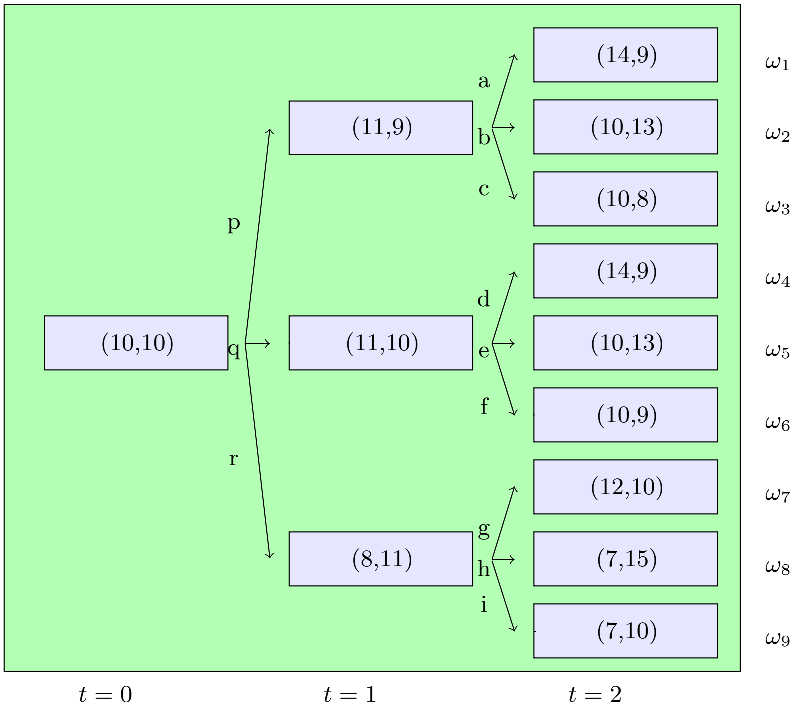


Figure 8.2. Example calculation of an option price

We want to calculate the price of a complex option given by

$$X = \left\{2Z_1(2) + Z_2(2) - [14 + 2\min\left(\min\left\{Z_1(t), Z_2(t)\right\}, 0 \leq t \leq 2\right)]\right\}^+ .$$

First of all, we have to find an equivalent martingale measure. Thus we have to solve for the times $t = 0$ and $t = 1$ the following equations:

$$\begin{aligned}
10 &= 11p + 11q + 8r, && \text{(martingale condition for } Z_1) \\
10 &= 9p + 10q + 11r, && \text{(martingale condition for } Z_2) \\
1 &= p + q + r.
\end{aligned}$$

The solution to these equations is $p = q = r = \frac{1}{3}$.

Here we can see explicitly which circumstances imply the existence and uniqueness of a martingale measure. In this example the martingale measure is, from a geometric point of view, defined as the intersection of three hyper-planes. Depending on their orientation, there is either one or there are many or there is none equivalent martingale measure.

Next, we can derive the xsequations for the times $t = 1$ and $t = 2$. These are

$$\begin{aligned} 11 &= 14a + 10b + 10c, \\ 9 &= 9a + 13b + 8c, \\ 1 &= a + b + c, \end{aligned}$$

$$\begin{aligned} 11 &= 14d + 10e + 10f, \\ 10 &= 9d + 13e + 9f, \\ 1 &= d + e + f, \end{aligned}$$

$$\begin{aligned} 8 &= 12g + 7h + 7i, \\ 11 &= 10g + 15h + 10i, \\ 1 &= g + h + i, \end{aligned}$$

and they are solved by

$$\begin{aligned} (a, b, c) &= (\frac{1}{4}, \frac{3}{20}, \frac{3}{5}) \\ (d, e, f) &= (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \\ (g, h, i) &= (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}) \end{aligned}$$

Now we know the transition probabilities with respect to the martingale measure, which enables us to calculate the martingale measure Q itself. The results of these calculations are summarised in the following table:

state	$X(\omega_i)$	$Q(\omega_i)$
ω_1	5	1/12
ω_2	1	1/20
ω_3	0	1/5
ω_4	5	1/12
ω_5	0	1/12
ω_6	0	1/6
ω_7	4	1/15
ω_8	1	1/15
ω_9	0	1/5

Finally, we can calculate the price of the option as expectation with respect to Q. The result is $\frac{73}{60}$. This calculation can also be done recursively as per figure 8.3. Here one calculates the conditional expectations backwards. This in the end results in the same result – $\frac{73}{60}$. In a next step we want to determine the replicating portfolio from $t = 0 \rightsquigarrow t = 1$.

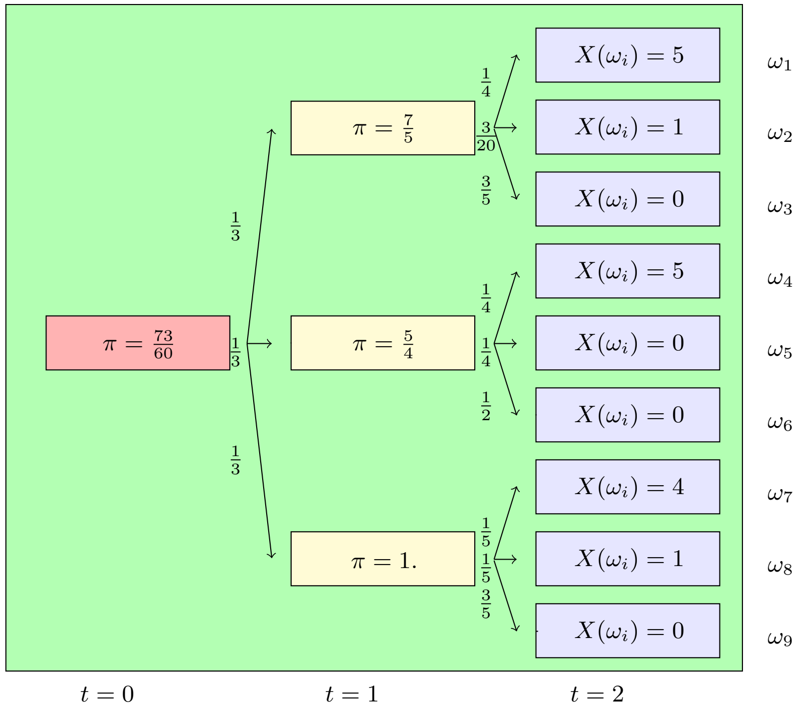


Figure 8.3. Calculation of price

Assume that we hold at time $t = 0$ a portfolio of α cash and β (resp. γ) units of security Z_1 (resp. Z_2). Then the following equation holds:

$$\begin{pmatrix} 1 & 11 & 9 \\ 1 & 11 & 10 \\ 1 & 8 & 11 \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{5}{4} \\ 1 \end{pmatrix}$$

Since the matrix is invertible the solution for the equation is unique:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{60} \begin{pmatrix} 143 \\ 2 \\ -9 \end{pmatrix}$$

Hence we need to hold at time $t = 0$ cash in the amount of 143, 2 shares Z_1 and -9 shares Z_2. We note that the value of the replicating portfolio at time $t = 0$

$$\frac{143 + 2 \times 10 - 9 \times 10}{60} = \frac{73}{60}$$

equals again the value of the option. This calculation can be performed for the entire tree resulting in the respective replicating portfolios (exercise). Figure 8.4 shows the corresponding replicating portfolios per state.

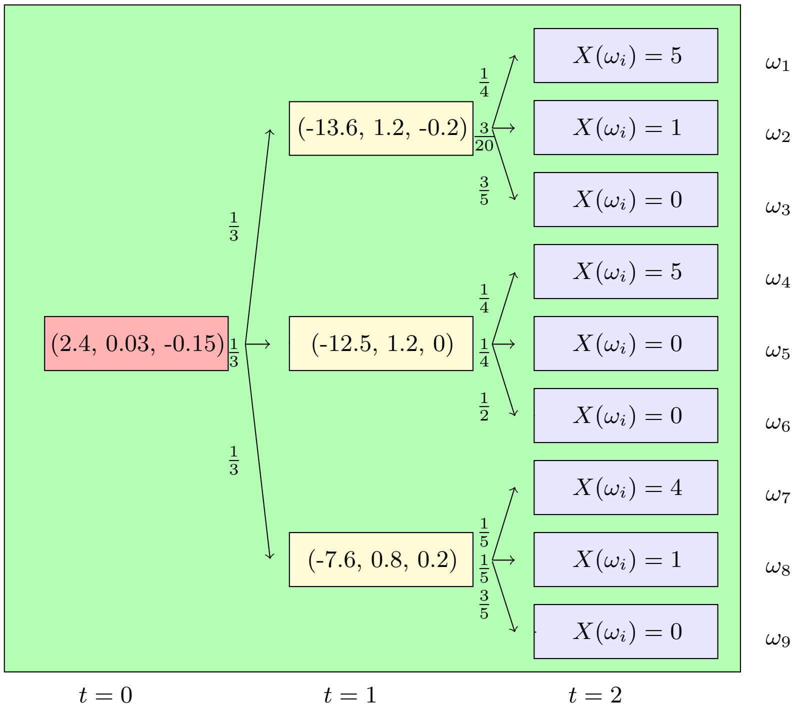


Figure 8.4. Calculation of replicating portfolios

8.2.3 Continuous time models

For models in continuous time we restrict our exposition to the statements, the proofs can be found in the references previously mentioned. A major difference between the discrete and the continuous setting is that we are going to *assume* that $\mathbb{P} \neq \emptyset$ holds for the continuous time model.

We start with some basic definitions.

Definition 8.2.14. – *A trading strategy ϕ is a locally bounded, previsible process.*

– *The value process corresponding to a trading strategy ϕ is defined by*

$$V : \mathit{\Pi} \to \mathbb{R}, \phi \mapsto V(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{k} \phi_t^k \cdot S_t^k.$$

– *The return process G is defined by*

$$G : \mathit{\Pi} \to \mathbb{R}, \phi \mapsto G(\phi) = \int_0^\tau \phi \, dS = \int_0^\tau \sum_{i=0}^{k} \phi^k dS^k.$$

– *ϕ is self financing, if $V_t(\phi) = V_0(\phi) + G_t(\phi)$.*

– *To define admissible trading strategies we use the notation:*

$$\begin{aligned} Z_t^i &= \beta_t \cdot S_t^i, & &\textit{discounted value of share } i \\ G^*(\phi) &= \int \sum_{i=1}^{k} \phi^i \, dZ^i, & &\textit{discounted return} \\ V^*(\phi) &= \beta \, V(\phi) = \phi^0 + \sum_{i=1}^{k} \phi^i \, Z^i. \end{aligned}$$

A trading strategy is called admissible, if it has the following three properties:

1. *$V^*(\phi) \geq 0$,*
2. *$V^*(\phi) = V^*(\phi)_0 + G^*(\phi)$,*
3. *$V^*(\phi)$ is a martingale with respect to Q.*

Theorem 8.2.15. *1. The price of a contingent claim X is given by $\pi(X) = E^Q[\beta_T \, X]$.*

2. A contingent claim is attainable $\iff$ $V^* = V_0^* + \int H dZ$ for all H.

Definition 8.2.16. *The market is called complete, if every integrable contingent claim is attainable.*

Although this theory is very important, we only gave a brief sketch of the main ideas. It is therefore recommended that the reader extends his knowledge of financial mathematics by consulting the references.

8.3 The Black-Scholes Model and the Itô-Formula

As we have seen in the previous sections, we need an underlying economic model to calculate the price of an option. In principle one can use various different economic models. Exemplary, we are going to consider the most common model: geometric Brownian motion.

The following references are a good source for various aspects of the economic model: [Dot90], [Duf88], [Duf92], [CHB89], [Per94], [Pli97].

Convention 8.3.1 (General conventions). *For the remainder of this chapter we will use the following notations and conventions:*

- T_x *denotes the future lifespan of an* x *year old person.*
- *The* σ*-algebras generated by* T_x *are denoted by* $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right)$.
- *We assume, that the values of the shares in the portfolio are given by standard Brownian motions* W. *(Compare with Figure 8.5.).*
- $\mathcal{G}_t$ *denotes the* σ*-algebra generated by* W *augmented by the* P*-null sets.*

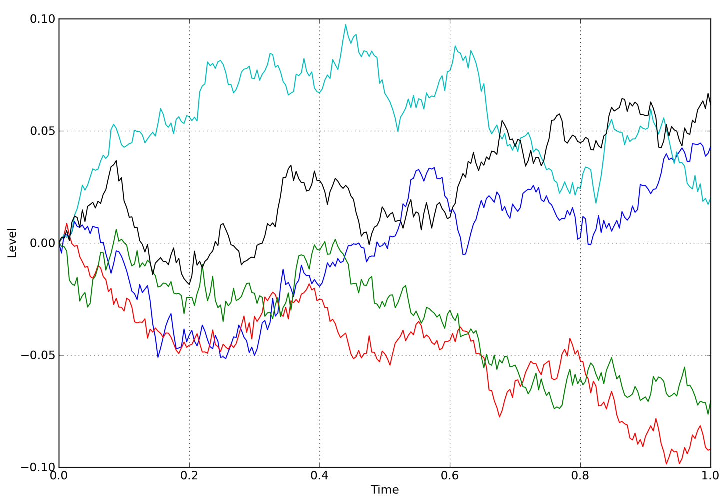


Figure 8.5. 5 Simulations of a Brownian motion

Convention 8.3.2 (Independence of the financial variables). – *We assume that* $\mathcal{G}_t$ *and* $\mathcal{H}_t$ *are stochastically independent. This means, that the financial variables are independent of the future lifespan.*

- $\mathcal{F}_t = \sigma\left(\mathcal{G}_t, \mathcal{H}_t\right)$ *denotes the* σ*-algebra generated by* $\mathcal{G}_t$ *and* $\mathcal{H}_t$.

Next step we look at specific semimartingales which are of particular interest for stochastic finance and variable annuities. For these the Itô-calculus is simpler. First we define the corresponding class of stochastic processes:

Definition 8.3.3 (Money Market Account; Risk Free Bond). *We assume that an investor can invest in a money market account B_t which earns interest rate δ, ie*

$$\begin{aligned} B_{\tau+\Delta\tau} &= \exp(\Delta\tau\,\delta)\;B_\tau, \;\; \textit{or equivalently} \\ dB_\tau &= \delta\,B_\tau\,d\tau. \end{aligned}$$

We remark that this investment is risk free and that we do not assume any defaults.

Definition 8.3.4 (Forward Price of a Zero Coupon Bond). *For $t < T$, we denote by $B_0(t,T)$ the expected price of a $\mathcal{Z}_{(T)}$ as at time 0 , ie*

$$B_0(t,T) = E\left[\pi_t(\mathcal{Z}_{(T)})\middle|\,\mathcal{F}_0\right].$$

Theorem 8.3.5. *For $t < T$ the following formula holds:*

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

In the context of the model of Definition 8.3.3 this leads to

$$\begin{aligned} B(t,T) &= \exp(-\delta\,(T-t)) \;\; \textit{and} \;, \\ dB(\tau,T) &= \delta\,B(\tau,T)\,d\tau. \end{aligned}$$

Proof. In an arbitrage free environment we can compare the following two strategies:

– Invest at $\tau = 0$ in one unit of $\mathcal{Z}_{(t)}$ and invest at $\tau = t$ the proceeds into $\mathcal{Z}_{(T)}$. At time $\tau = T$ we get the following proceeds:

$$\frac{1}{\pi_0(\mathcal{Z}_{(t)})}\,\frac{1}{\pi_t(\mathcal{Z}_{(T)})}.$$

– Alternatively we can invest the same value (ie $\pi_0(\mathcal{Z}_{(t)})$ into a T-year zero coupon bon ($\mathcal{Z}_{(T)}$), which will yield 1.

Since $\mathcal{Z}_{(t)}$ is risk free and because prices are linear, under the assumption of absence of arbitrage, this yields to

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

Definition 8.3.6 (Itô process (1-dim)). *Let $(W_t)_{t \in \mathbb{R}^+}$ be a 1-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$. A (1-dimensional) Itô process or stochastic integral is a stochastic process $(X_t)_{t \in \mathbb{R}^+}$ on $(\Omega, \mathcal{A}, P)$ of the form:*

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dW_s,$$

where $v \in \mathcal{W}_{\mathcal{H}}$ (see [Oks03] chapter 4 for the precise meaning), such that

$$P\left[\int_0^t v(s, \omega)^2 ds < \infty \forall t \geq 0\right] = 1.$$

We assume that u is $\mathcal{H}_t$-adapted and

$$P\left[\int_0^t |u(s, \omega)| ds < \infty \forall t \geq 0\right] = 1.$$

Remark 8.3.7. Note that writing

$$dX = u(s)ds + v(s)dW_s$$

is an abbreviation for:

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_o^t v(s, \omega) dW_s.$$

Definition 8.3.8 (Share Price Process). *We assume that the share price S_t follows the following stochastic differential equation:*

$$dS_t \quad = \quad \eta \, S_t \, dt + \sigma \, S_t \, dW,$$

where $\eta \in \mathbb{R}$ is the real world equity return drift term. $\sigma \geq 0$ represents the equity volatility and $(W_t)_{t \in \mathbb{R}_0^+}$ is a one-dimensional standard Brownian motion on $(\Omega, \mathcal{A}, P)$.

Remark 8.3.9. – When considering

$$\frac{dS}{S} \quad = \quad \eta \, dt + \sigma \, dW,$$

we see that the Black-Scholes-Merton Model is based on a geometric Brownian motion, where we denote with W the Wiener measure or Brownian motion.

– The equity drift term η means that the value of the equity price would follow:

$$S_{t+\Delta t} \quad = \quad S_t \, \exp\left(\eta \, \Delta t\right)$$

in absence of volatility (ie for $\sigma = 0$). At the same time the risk free account value would yield

$$B_{t+\Delta t} \quad = \quad B_t \, \exp\left(\delta \, \Delta t\right).$$

In consequence the difference $\eta - \delta$ is called equity risk premium. It is typically positive and represents the additional return one can expect when investing in shares by assuming the additional risk induced by the volatility.

– The stochastic differential equation above is to be understood in sense of a stochastic integral (see appendix A) and the Itô-calculus.

– We note that $(W_t)_{t \in \mathbb{R}_0^+}$ has P-a.e. continuous sample paths and is nowhere differentiable. As a consequence of this the stochastic differential equation is to be understood as an integral equation, ie

$$dS_t \quad = \quad \eta \, S_t \, dt + \sigma \, S_t \, dW$$

is a short form for

$$S_t \quad = \quad S_0 + \int_0^t \eta \, S_\tau \, d\tau + \int_0^t \sigma \, S_\tau \, dW_\tau,$$

where the integral is akin to a Riemann integral, eg one can define

$$\int_0^t X_\tau dW_\tau \quad = \quad \lim_{n \to \infty} \sum_{j=0}^n X_{t_j^n} \left(W_{t_{j+1}^n} - W_{t_j^n} \right),$$

with limit taken over arbitrary partitions of the interval [O,T], into n pieces

$$0 = t_0^n < t_1^n < t_2^n \ldots < t_n^n = T \qquad \text{with}$$
$$\lim_{n \to \infty} \max\{|t_j^n - t_{j-1}^n| : j = 1, \ldots n\} = 0.$$

Definition 8.3.10 (Black-Scholes-Merton model). *This economic model consists of two investment options:*

$B(t)$	$=$	$\exp(\delta \, t)$	*risk free investment.*
$S(t)$	$=$	$S(0) \, \exp\left[\left(\eta - \frac{1}{2}\sigma^2\right) \, t \, + \, \sigma \, W(t)\right]$	*shares, modeled by a geometric Brownian motion (cf. Figure 8.6).*

S is the solution to the following stochastic differential equation:

$$dS = \eta \, S \, dt \, + \, \sigma \, S \, dW.$$

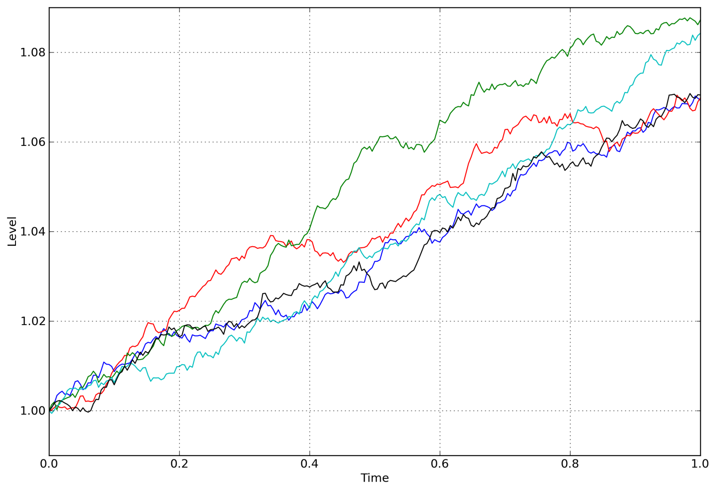


Figure 8.6. 5 Simulations of a geometric Brownian motion

Exercise 8.3.11. Prove that S solves the stochastic differential equation given above.

Next we will calculate the discounted values of B and S:

$$\begin{aligned} B^*(t) = \quad & \tfrac{B(t)}{B(t)} = \quad 1, \\ S^*(t) = \quad & \tfrac{S(t)}{B(t)} = \quad S(0)\, \exp\left[\left(\eta - \delta - \tfrac{1}{2}\sigma^2\right)\, t \,+\, \sigma\, W(t)\right]. \end{aligned}$$

Thus we have defined the investment options. To calculate the option prices we need to find an equivalent martingale measure.

Note that we interpret dX in the sense of a stochastic integral, to which we can apply Itô's formula:

Theorem 8.3.12 (Itô). *Assume X being an Itô process with*

$$dX = a\, dt + b\, dW$$

and let $g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, (x,t) \mapsto g(x,t)$ be a function, where the following partial derivatives are continuous: $\frac{\partial}{\partial x} g$, $\frac{\partial^2}{\partial x^2} g$, and $\frac{\partial}{\partial t} g$. In this case $Y_t = g(X_t, t)$ is also an Itô process with the following stochastic differential equation:

$$dY \quad = \quad \left(\frac{\partial g}{\partial x}\, a + \frac{\partial g}{\partial t} + \frac{1}{2}\,\frac{\partial^2 g}{\partial x^2}\, b^2\right) dt + \frac{\partial g}{\partial x}\, b\, dW$$

Proof. For the proof we refer to [Oks03], or also [Pro90] or [IW81].

We can now apply Itô's lemma to the geometric Brownian motion, as follows:

Theorem 8.3.13. *Let a stock price S be modelled by a geometric Brownian motion*

$$dS = \eta\, S dt + \sigma\, S\, dW$$

and let

$$V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}, (s,t) \mapsto V(x,t)$$

be a function fulfilling the regularity criteria of the function g of the Itô lemma. In this case $V := V(S,t)$ is also an Itô process with the following stochastic differential equation:

$$dV = (\frac{\partial V}{\partial s}\, \eta\, S + \frac{\partial V}{\partial t} + \frac{1}{2}\, \frac{\partial^2 V}{\partial S^2}\, \sigma^2\, S^2)\, dt + \frac{\partial V}{\partial S}\, \sigma\, S\, dW.$$

Proof. This follows immediately from the Itô formula keeping in mind that

$$\begin{aligned} a &= \eta\, S, \text{and} \\ b &= \sigma\, S. \end{aligned}$$

In order to simulate a geometric Brownian motion, the following corollary is helpful:

Theorem 8.3.14. *For a geometric Brownian motion*

$$dS = \eta\, S\, dt + \sigma\, S\, dW$$

we define $Y = \log(S)$. Then we have the following:

1. *$dY = (\eta - \frac{1}{2}\sigma^2)\, dt + \sigma\, dW$ is the unique solution to the above stochastic differential equation.*
2. *We can calculate S_t by:*

$$S_t = S_0 \exp\left(\left(\eta - \frac{1}{2}\sigma^2\right) \times t + \sigma\, W_t\right)$$

3. *For a series of times $t_0 = 0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots$, and $(X_k)_{k\in\mathbb{N}}$ standard normally distributed independent variables we can simulate $(S(t_k))_{k\in\mathbb{N}}$ with $S(0) = 1$ by the following recursion $\forall n \in \mathbb{N}$:*

$$S(t_n) = S(t_{n-1}) \times \exp\left[(\eta - \frac{\sigma^2}{2}) \times (t_n - t_{n-1}) + \sqrt{t_n - t_{n-1}} \times \sigma \times X_n\right].$$

Proof. Define

$$dX_t = (\eta - \frac{1}{2}\sigma^2)\, dt + \sigma\, dW,$$

with $X_0 = 0$. We know that $\tilde{S}_t := g(X_t)$, with

$$g : \mathbb{R} \to \mathbb{R}, x \mapsto g(x) = \exp(x),$$

and that $g = \frac{d}{dx} g = \frac{d^2}{dx^2} g$. As a consequence of the Itô-formula we get

$$\begin{aligned} d\tilde{S}_t &= dg(X_t) \\ &= g'(X_t)\,(\eta - \frac{1}{2}\sigma^2)\, dt + g'(X_t)\, \sigma\, dW_t \\ &\quad + \frac{1}{2} g''(X_t)\sigma^2\, dt \\ &= g(X_t)\ (\eta\, dt + \sigma\, dW_t) \\ &= \tilde{S}_t\ (\eta\, dt + \sigma\, dW_t)\,. \end{aligned}$$

Hence both S and $\tilde{S}$ fulfill the same stochastic differential equation with identical boundary condition. The proposition follows as a consequence of a general result for stochastic differential equations an the Itô-calculus, stating the uniqueness if a stochastic differential equation if the coefficients of the stochastic differential equations are Lipschitz continuous.

Theorem 8.3.15. *Under the natural filtration $\mathcal{F}$ induced by $(W_t)_{t\in\mathbb{R}_0^+}$ the following hold:*

1. *The Brownian motion is a martingale, and*
2. $M_t = \exp\left(\sigma\, W_t - \frac{1}{2}\sigma^2\, t\right)$ *is a martingale.*

Proof. Let $Z \sim \mathcal{N}(0,1)$ and let $u \leq t$. For the first equation we have

$$\begin{aligned} E^P\left[W_t|\mathcal{F}_u\right] &= E^P\left[W_t - W_u|\mathcal{F}_u\right] + E\left[W_u|\mathcal{F}_u\right] \\ &= E\left[(t-u)\, Z\right] + W_u \\ &= W_u. \end{aligned}$$

For the second equation we have:

$$\begin{aligned} E^P\left[M_t|\mathcal{F}_u\right] &= E^P\left[\exp\left(\sigma\, W_t - \frac{1}{2}\sigma^2\, t\right)|\mathcal{F}_u\right] \\ &= E^P\left[\exp\left(\sigma\left((W_t - W_u) + W_u\right) - \frac{1}{2}\sigma^2\left((t-u)+u\right)\right)|\mathcal{F}_u\right] \\ &= \exp\left(\sigma\, W_u - \frac{1}{2}\sigma^2\, u\right) \times E^P\left[\exp\left(\sigma\,(W_t - W_u) - \frac{1}{2}\sigma^2\,(t-u)\right)|\mathcal{F}_u\right] \end{aligned}$$

$$
\begin{aligned}
&= M_u \times E^P \left[\exp\left(\sigma \left(W_t - W_u \right) - \frac{1}{2} \sigma^2 \left(t - u \right) \right) | \mathcal{F}_u \right] \\
&= M_u \times E^P \left[\exp\left(\sigma \sqrt{t-u}\, Z - \frac{1}{2} \sigma^2 \left(t - u \right) \right) | \mathcal{F}_u \right] \\
&= M_u \times E^P \left[\exp\left(\sigma \sqrt{t-u}\, Z - \frac{1}{2} \sigma^2 \left(t - u \right) \right) \right] \\
&= M_u
\end{aligned}
$$

where we used the defining charcateristics of the Brownian motion and where $E^P \left[\exp\left\{ \sigma \sqrt{t-u}\, Z \right\} \right] = \exp\left(\frac{1}{2} \sigma^2 \left(t - u \right) \right)$ can be shown by elementary calculus (exercise).

Corollary 8.3.16. $(S_t)_{t \in \mathbb{R}_0^+}$ *follows a martingale under* P *if and only if* $\eta = 0$.

Remark 8.3.17. We note that S_t is a time continuous Markov-process under P with respect to $\mathcal{F}$ and we have

$$
\begin{aligned}
p_S(u,x,t,y) &:= P\left[S_t = y | S_u = s \right] \\
&= \frac{1}{\sqrt{2\pi (t-u)}\, \sigma\, y} \\
&\quad \times \exp\left\{ - \frac{\left(\ln\left(\frac{y}{x} \right) - \eta \left(t-u \right) + \frac{1}{2} \sigma^2 \left(t-u \right) \right)^2}{2\sigma^2 \left(t-u \right)} \right\}
\end{aligned}
$$

Definition 8.3.18 (Trading Strategy). *A trading strategy in the Black-Scholes-Merton framework is a pair* $\Phi = (\Phi^1, \Phi^2)$ *of* $\mathcal{F}$*- previsible, progressively measurable stochastic processes. on* $(\Omega, \mathcal{A}, P)$.

Definition 8.3.19. *A trading strategy* $\Phi = (\Phi^1, \Phi^2)$ *over* $[0,T]$ *is self-financing if the corresponding value process (also wealth process)* $(V_t(\Phi))_{t\in[0,t]}$,

$$
V_t(\Phi) = \Phi_t^1\, S_t + \Phi_t^2\, B_t, (\forall t \in [0,T])
$$

satisfies the following condition:

$$
V_t(\Phi) = V_0(\Phi) + \int_0^t \Phi_\tau^1\, dS_\tau + \int_0^t \Phi_\tau^2 dB_\tau .
$$

Remark 8.3.20. – As usual, we implicitly assume the existence of an integral if we write $\int X dW$. The following two conditions are sufficient that the integral in Definition 8.3.18 exists:

$$
\begin{aligned}
P\left[\int_0^T (\Phi_\tau^1)^2 d\tau < \infty \right] &= 1, \text{ and} \\
P\left[\int_0^T |\Phi_\tau^2| d\tau < \infty \right] &= 1.
\end{aligned}
$$

– Note that we will denote by * discounted quantities, with respect to the money market account, eg
$$S_t^* = \frac{S_t}{B_t}.$$

– Since $B_t = \exp(t\,\delta)\,B_0$, we get
$$S_t^* \quad = \quad S_0^* \exp\left(\left(\eta - \delta - \frac{1}{2}\sigma^2\right) t + \sigma\, W_t\right),$$
or equivalently
$$dS_t^* \quad = \quad (\eta - \delta) S_t^*\, dt + \sigma S_t^* dW_t,$$
with $S_0^* = S_0$.

Corollary 8.3.21. *The discounted stock price process* $(S_t^*)_{t\in[0,T]}$ *is a martingale under* P *if and only if* $\eta = \delta$.

Definition 8.3.22 (Martingale Measure). *A probability measure* Q *on* $(\Omega, \mathcal{A})$ *equivalent to* P *is called martingale measure for* S^* *if* $(S_t^*)_{t\in[0,T]}$ *is a local martingale under* Q.

Definition 8.3.23 (Spot Martingale Measure). *A probability measure* P^* *on* $(\Omega, \mathcal{A}, P)$ *on* $(\Omega, \mathcal{A}, P)$ *equivalent to* P *is called spot martingale measure, if the discounted value process* $V_t^*(\Phi) = \frac{V_t(\Phi)}{B_t}$ *of any self-financing trading strategy is a local martingale under* P^*.

Lemma 8.3.24. *A probability measure is a spot martingale measure if and only if it is a martingale measure for the discounted stock price* S^*.

Proof. Let Φ be a self-financing strategy and denote $V^* := V^*(\Phi)$. Using Itô's product rule and the fact that $dB_t^{-1} = -\delta\, B_t^{-1}\, dt$ we the following:
$$\begin{aligned} dV_t^* &= d(V_t\, B_t^{-1}) \\ &= V_t\, dB_t^{-1} + B_t^{-1}\, dV_t \\ &= \left(\Phi_t^1\, S_t + \Phi_t^2 B_t\right) dB_t^{-1} + B_t^{-1} \left(\Phi_t^1\, dS_t + \Phi_t^2 dB_t\right) \\ &= \Phi_t^1 \left(B_t^{-1}\, dS_t + S_t\, dB_t^{-1}\right) \\ &= \Phi_t^1\, dS_t^*. \end{aligned}$$

Hence we get
$$V_t^*(\Phi) \quad = \quad V_0^*(\Phi) + \int_0^t \Phi_\tau^1\, dS_\tau^*$$

Therefore the lemma follows

Lemma 8.3.25. *The unique martingale measure Q for the discounted stock price process S^* is given by the Radon-Nikodym density*

$$\begin{aligned} \xi_t &= \frac{dQ}{dP} \\ &= \exp\left(-\frac{1}{2}\left(\frac{\eta-\delta}{\sigma}\right)^2 t - \frac{\eta-\delta}{\sigma} W(t)\right) \qquad \textit{for all } t \in [0,T]. \end{aligned}$$

The discounted stock price S^* satisfies under Q the following:

$$dS_t^* = \sigma\, S_t^*\, dW_t^*,$$

and the continuous $\mathcal{F}$-adapted process W^* is given by

$$W_t^* = W_t - \frac{\delta-\eta}{\sigma}\, t \;\forall t \in [0,T].$$

– **Exercise 8.3.26.** Prove the following statements:

1. $E[\xi_t] = 1$,
2. $Var[\xi_t] = \exp\left(\left(\frac{\eta-\delta}{\sigma}\right)^2 t\right) - 1$,
3. $\xi_t > 0$.

(Hint: $W(t) \sim \mathcal{N}(0,t)$.)

Proof. An application of Girsanov's theorem – a theorem in the theory of stochastic integration (e.g. [Pro90] Theorem 3.6.21) – shows that

$$\hat{W}_t = W(t) + \frac{\eta-\delta}{\sigma}\, t$$

is a Brownian motion with respect to $Q = \xi \cdot P$.

Naturally, after this transformation we want to prove that

$$S^*(t) = S(0)\, \exp\left(-\frac{1}{2}\sigma^2\, t + \sigma\, \hat{W}(t)\right)$$

is a martingale with respect to Q. (Then the price of the option is given by its expectation with respect to Q.)

We have to show the equality

$$E^Q\left[S^*(u)|\mathcal{F}_t\right] = S^*(t)$$

for $t, u \in \mathbb{R}, u > t$. With the notation $u = t + \Delta t$, $W_u = W_t + \Delta W$ and $Z \sim \mathcal{N}(0,1)$ we have

$$
\begin{aligned}
&E^Q\left[S^*(u)|\mathcal{F}_t\right] \\
&\quad = E^Q\left[S(0)\,\exp\left(-\frac{1}{2}\sigma^2\,t + \sigma\,\hat{W}(t) + (-\frac{1}{2}\sigma^2\,\Delta t + \sigma\,\Delta\hat{W})\right)|\mathcal{F}_t\right] \\
&\quad = S(0)\,\exp\left(-\frac{1}{2}\sigma^2\,t + \sigma\,W(t)\right)\,E^Q\left[\exp\left(-\frac{1}{2}\,\sigma^2\Delta t + \sigma\sqrt{\Delta t}Z\right)|\mathcal{F}_t\right] \\
&\quad = S^*(t).
\end{aligned}
$$

Therefore the measure Q is equivalent to P, and S^* is a martingale with respect to Q. An economist would say, "it exists (at least) one consistent price system". For the uniqueness of Q we refer to [MR07] lemma 3.1.3.

As a next step we want to understand the Black-Scholes-Merton partial differential equation from a theoretical aspect. The following theorem sheds light into this question. We note that this partial differential equation will be useful, when we will look at Δ-hedging.

Theorem 8.3.27 (Black-Scholes-Merton Differential Equation). *Let δ be the risk-free interest rate, η the equity drift rate and σ the volatility of a asset S_t in the Black-Scholes-Merton framework (eg the share price satisfying the following stochastic differential equation $dS = \eta\,S\,dt + \sigma\,S\,dW$). Suppose we have a contingent claim with expiry date T and underlying asset S_t, with value function:*

$$v : \mathbb{R}^+ \times]0,T[, \to \mathbb{R}, (x,t) \mapsto v(x,t),$$

where $v \in C^{2,1}(\mathbb{R}^+ \times [0,T])$. Then the following partial differential equation holds:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} = \delta v.$$

Moreover the portfolio $\mathcal{P} = (V - \Delta \times S, \Delta)$ consisting $V - \Delta \times S$ units cash (valued at 1) and $\Delta = \frac{\partial v}{\partial S}$ shares replicates the contingency claim.

Proof. Since this theorem is very important we offer in a first step an *outline the proof.* The proof is an application of Itô's lemma. We have the following:

$$
\begin{aligned}
dS &= \eta\,S\,dt + \sigma\,S\,dW, \text{ and} \\
dv &= (\frac{\partial v}{\partial S}\,\eta\,S + \frac{\partial v}{\partial t} + \frac{1}{2}\,\frac{\partial^2 v}{\partial S^2}\,\sigma^2\,S^2)\,dt + \frac{\partial v}{\partial S}\,\sigma\,S\,dW
\end{aligned}
$$

We define $\Pi := -v + \Delta\,S$ and we see that for $\Delta = \frac{\partial v}{\partial S}$ the stochastic part of the SDE cancels out and we get:

$$d\Pi = \left\{-\frac{\partial v}{\partial t} - \frac{1}{2}\,\frac{\partial^2 v}{\partial S^2}\,\sigma^2\,S^2\right\}\,dt$$

On the other hand Π can be interpreted as the value of an investment portfolio, shorting $-v$ of cash and buying Δ shares. One could also invest the respective value in cash, by means of absence of arbitrage. This needs to result in the same value and in the same increments, hence:

$$d\Pi = \delta \, \Pi \, dt$$

When now using the definition of Π for the right hand side and using the above formula for the right hand side, this results in:

$$\left\{ -\frac{\partial v}{\partial t} - \frac{1}{2} \frac{\partial^2 v}{\partial S^2} \sigma^2 S^2 \right\} dt = \delta \left\{ -v + \frac{\partial v}{\partial S} S \right\} dt.$$

In consequence we get the Black-Scholes-Merton differential equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} = \delta v$$

Proof. In a next step we offer a more formal proof. It follows essentially [MR07] theorem 3.1.1 where further details can be found. We will proof this theorem by directly determining the replicating strategy and note that the price v_t of the contingency claim is a function $v_t = v(S_t, t)$. We may assume that the replicating strategy Φ has the form:

$$\Phi_t = (\Phi_t^{(1)}, \Phi_t^{(2)}) = (h(S_t, t), g(S_t, t)),$$

for $t \in [0, T]$ and $g, h : \mathbb{R}^+ \times [0, T] \to \mathbb{R}$ unknown functions. Note that $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ represent the amount of cash and the number for shares which we hold at time t. Since Φ is assumed self-financing, its wealth process $V(\Phi)$ given by

$$V_t(\Phi) = g(S_t, t) S_t + h(S_t, t) B_t = v(S_t, t), \tag{8.1}$$

needs to satisfy the following:

$$dV_t(\phi) = g(S_t, t) dS_t + h(S_t, t) dB_t. \tag{8.2}$$

Given that we operate in the Black-Scholes-Merton framework, we can conclude from (8.2) that

$$\begin{aligned} dV_t(\Phi) &= g(S_t, t)(\eta S_t dt + \sigma S_t dW_t) + h(S_t, t) \delta B_t dt \\ &= (\eta - \delta) S_t g(S_t, t) dt + \sigma S_t g(S_t, t) dW_t + \delta S_t g(S_t, t) dt + \delta h(S_t, t) B_t dt \\ &= (\eta - \delta) S_t g(S_t, t) dt + \sigma S_t g(S_t, t) dW_t + \delta S_t v(S_t, t) dt. \end{aligned} \tag{8.3}$$

The application of the Itô lemma to v results in

$$dv(S_t, t) = \left(\frac{\partial v}{\partial t}(S_t, t) + \eta S_t \frac{\partial v}{\partial S}(S_t, t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t, t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t, t) dW_t.$$

Combining the above expression with (8.2), results in the following expression for the Itô differential of the process $Y_t = v(S_t, t) - V_t(\Phi)$:

$$\begin{aligned} dY_t &= \left(\frac{\partial v}{\partial t}(S_t,t) + \eta S_t \frac{\partial v}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t,t) dW_t \\ &\quad +(\delta - \eta) S_t g(S_t,t)dt - \sigma S_t g(S_t,t) dW_t - \delta v(S_t,t)dt \end{aligned}$$

On the other hand, in view (8.1), the process Y vanishes identically (i.e. $Y \equiv 0$), and thus $dY_t = 0$. By virtue of the uniqueness of the canonical decomposition of continuous semimartingales, the diffusion term in the above decomposition of Y vanishes. In our case, this means that we have, for every $t \in [0, T]$,

$$\int_0^t \sigma S_\tau \left(g(S_\tau, \tau) - \frac{\partial v}{\partial S}(S_\tau, \tau) \right) dW\tau = 0 \qquad P - a.s.$$

In view of the properties of the Itô integral (isometry used for the construction of the Itô integral), this is equivalent to:

$$\int_0^T S_\tau^2 \left(g(S_\tau, \tau) - \frac{\partial v}{\partial S}(S_\tau, \tau) \right)^2 d\tau = 0. \tag{8.4}$$

For (8.4) to hold, it is sufficient and necessary that g satisfies

$$g(s,t) = \frac{\partial v}{\partial S}(s,t) \qquad \forall (s,t) \in \mathbb{R}^+ \times [0,T] \tag{8.5}$$

Strictly speaking, the equality above should hold $P \otimes \lambda-$almost surely, where λ is the Lebesgue measure. Using (8.5), results in still another equation of Y:

$$Y_t = \int_0^t \left\{ \frac{\partial v}{\partial \tau}(S_\tau, \tau) + \frac{1}{2}\sigma^2 S_\tau^2 \frac{\partial^2 v}{\partial S^2}(S_\tau, \tau) + \delta S_\tau \frac{\partial v}{\partial S}(S_\tau, \tau) - \delta v(S_\tau, \tau) \right\} d\tau$$

It is thus apparent that $Y \equiv 0$ whenever v satisfies the following partial differential equation, referred to as Black-Scholes PDE:

$$\frac{\partial v}{\partial t}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) + \delta S_t \frac{\partial v}{\partial S}(S_t,t) - \delta v(S_t,t) = 0 \tag{8.6}$$

This terminates the first part of the proof. It thus remains to check that Φ is an admissible trading strategy.

If the contingent claim is attainable, it thus remains to check that the replicating strategy Φ, given by the formula

$$\begin{aligned} \Phi^1 &= g(S,t) = \frac{\partial}{\partial S} v(S_t, t), \\ \Phi^2 &= h(S_, t) = B_t^{-1} \left(v(S_t,t) - g(S_t,t) S_t \right), \end{aligned}$$

is admissible. Let us first check that Φ is self-financing. We need to check that

$$dV_t(\Phi) = \Phi_t^1 \, dS_t + \Phi_t^2 \, dB_t.$$

Since $V_t(\Phi) = \Phi_t^1 \, S_t + \Phi_t^2 \, B_t = v(S_t, t)$, by applying Itô's formula, we get

$$dV_t(\Phi) \quad = \quad \frac{\partial}{\partial S} v(S_t, t) \, dS_t + \frac{1}{2} \sigma^2 S_t^2 \, \frac{\partial^2}{\partial S^2} v(S_t, t) \, dt + \frac{\partial}{\partial t} v(S_t, t) \, dt.$$

In view of (8.6), the last equality can also be given the following form

$$dV_t(\Phi) \quad = \quad \frac{\partial}{\partial S} v(S_t, t) \, dS_t + \delta \, v(S_t, t) \, dt - \delta \, S_t \frac{\partial}{\partial S} v(S_t, t) \, dt,$$

and thus

$$\begin{aligned} dV_t(\Phi) \quad &= \quad \frac{\partial}{\partial S} v(S_t, t) dS_t + \delta v(S_t, t) dt - \delta S_t \frac{\partial}{\partial S} v(S_t, t) dt \\ &= \quad \Phi_t^1 \, dS_t + \delta \, B_t \frac{\frac{\partial}{\partial t} v(S_t, t) - \Phi_t^1 \, S_t}{B_t} \\ &= \quad \Phi_t^1 \, dS_t + \Phi_t^2 \, dB_t. \end{aligned}$$

This ends the verification of the self-financing property.

Note that the Δ-portfolio is not permanently risk free, but only *instantaneously*. For the interested reader we suggest as further reading: [Ped89] for the analytical basics such as integration theory, Banach and Hilbert spaces etc, and [Per94] and [Oks03] for stochastic integration and stochastic differential equations. Finally we would also suggest [Hul12] and [MR07] as general valuable reference.

Remark 8.3.28. The proof of the above theorem is very helpful for understanding the concept of dynamical hedging. Assume a given contingency claim in the Black-Scholes framework which has been sold by an insurance company or a bank. Theorem 8.3.27 shows a way to calculate the value V via solving the corresponding partial differential equation. More importantly it helps us also to understand the intrinsic risk to this contract in case of changes in equity prices (S). In many instances the bank or insurance company is not willing to take this risk on its balance sheet (eg letting fluctuate the shareholder equity) and tries to mitigate this risk by a suitable hedging strategy.

One classical hedging strategy is to buy at each point of time assets with the same partial derivative as V with respect to S. Such a strategy is called δ-hedging. Figure 8.7 shows this. Note that the δ-hedging strategy has this name, because the first partial derivative of V with respect of S is called Δ,

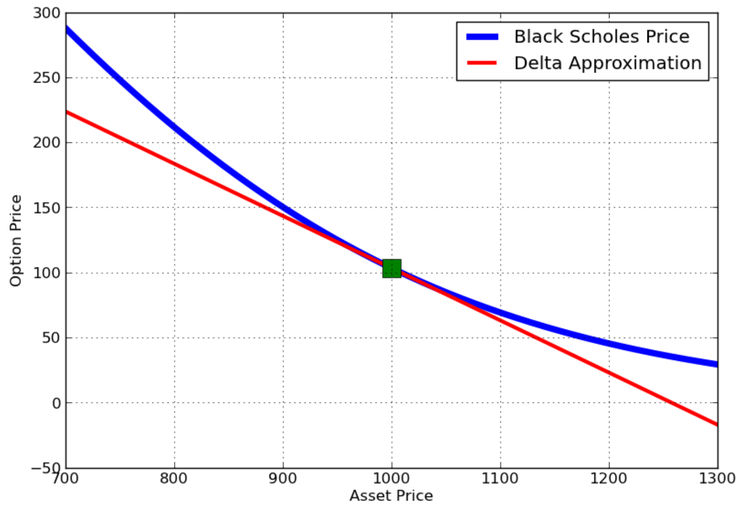


Figure 8.7. Delta Hedge

similarly Γ is the second partial derivative of V with respect to S, ie. we have the following

$$\begin{aligned} \Delta &= \frac{\partial}{\partial S} V \\ \Gamma &= \frac{\partial^2}{\partial S^2} V \end{aligned}$$

Turning back to the proof of theorem 8.3.27, we see that for a pure δ strategy the value process of the hedge portfolio becomes deterministic, following

$$d\Pi = \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right\} dt.$$

This however means that a pure δ strategy leads to a deterministic difference in price as a consequence of the Γ term. This effect is known under the name γ-bleed. Figure 8.8 shows the effect of using a pure δ-strategy vs using a $\delta - \gamma$ strategy for a simulated trajectory. The red line shows the true value of V, the blue one using a *delta*-hedge and the green one a $\delta - \gamma$-hedge.

Theorem 8.3.29. *Let the economic model defined above be given, i.e. it is defined by* $(\Omega, \mathcal{A}, P)$*,* S *and* B*. Then at time* t *the price of a policy with death benefit* $C(T)$ *is*

$$\pi_t(T) = E^Q \left[\exp\left(-\delta(T-t)\right) C(T) | \mathcal{F}_t\right].$$

Remark 8.3.30. The main difference of this model in comparison to the classical model is that one has to calculate the expectation with respect to Q and not with respect to P. Moreover one should note, that we have not proved the uniqueness of the price system.

The following two formulas are an important consequence of the previous considerations.

Theorem 8.3.31. *A single premium for a policy based on the economic model defined above is given by the following formulas.*

Endowment policy:

$$V(0) = E^Q \left[\exp(-\delta T)\, C(T)\right] \cdot {}_T p_x.$$

Term life insurance:

$$V(0) = \int_0^T E^Q \left[\exp(-\delta t)\, C(t)\right] {}_t p_x \, \mu(x+t) dt.$$

8.4 Calculation of single premiums

Up to now the calculations have been relatively simple, since we did not include any guarantees in our policy model. Next, we will consider a unit-linked policy with an additional guarantee. We recall some of the notations from the previous sections:

$N(\tau)$	Number of shares at time τ,
$S(\tau)$	value of a share at time τ,
$G(\tau)$	guaranteed benefits at time τ,
$C(\tau) = \max\{N(\tau)S(\tau), G(\tau)\}$	value of the insurance at time τ.

8.4.1 Pure endowment policy

Theorem 8.4.1. *Let the Black-Scholes model be given. Then the single premium for a pure endowment policy with payout*

$$C(T) = \max\{N(T)S(T), G(T)\}$$

is given by

$$_TG_x = {}_Tp_x \left[G(T)\, \exp(-\delta T)\Phi(-d_2^0(T)) + S(0)N(T)\Phi(d_1^0(T))\right],$$

where

$$\begin{aligned} \Phi(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \exp(-\frac{x^2}{2})dx, \\ d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s>t), \\ d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s>t). \end{aligned}$$

Proof. In the following we denote by J^* the discounted value of a random variable J. The value of the pure endowment policy at time zero is $E^Q[C^*(T)]$. We set $Z = S^*(T)$. Then the following equations hold

$$_TG_x = {}_Tp_x\, E^Q\left[\max\{N(T)Z, G^*(T)\}\right]$$

and

$$Z = S(0)\, \exp\left(-\frac{1}{2}\sigma^2 T + \sigma \hat{W}(T)\right) \qquad \text{where} \qquad \hat{W}(T) \sim \mathcal{N}(0,T).$$

Thus we get

$$\begin{aligned} _TG_x &= {}_Tp_x \int_{-\infty}^{\infty} \max\left[N(T)S(0)\exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi), G^*(T)\right] f(\xi)d\xi, \\ f(\xi) &= \frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{1}{2T}\xi^2\right). \end{aligned}$$

Next we define $\bar{\xi} = \frac{1}{\sigma}\left[\ln\left(\frac{G^*(T)}{N(T)S(0)}\right) + \frac{1}{2}\sigma^2 T\right]$ and note that $\xi > \bar{\xi}$ implies $N(T)Z > G^*(T)$. Therefore, the single premium is given by

$$\begin{aligned} _TG_x &= {}_Tp_x\Bigg(G^*(T)\int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\ &\qquad + N(T)S(0)\int_{\bar{\xi}}^{\infty} \exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi)f(\xi)d\xi\Bigg) \\ &= {}_Tp_x\Bigg(G^*(T)\int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\ &\qquad + N(T)S(0)\int_{\bar{\xi}}^{\infty} \frac{1}{\sqrt{2\pi T}}\exp(-\frac{1}{2T}(\xi - \sigma T)^2 d\xi\Bigg). \end{aligned}$$

This equation, with adapted notation, yields the statement of the theorem.

8.4.2 Term life insurance

Theorem 8.4.2. *Let the Black-Scholes model be given. Then the single premium for a term life insurance with death benefit*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

is given by

$$G^1_{x:T} = \int_0^T \left(G(t)\exp(-\delta t)\Phi(-d_2^0(t)) + S(0)N(t)\Phi(d_1^0(t))\right) {}_tp_x\mu_{x+t}\, dt,$$

where

$$\begin{aligned} \Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx, \\ d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\ d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \end{aligned}$$

for $s > t$.

Exercise 8.4.3. Prove the previous theorem by the same methods which we used for the pure endowment policy.

Remark 8.4.4. For the calculations above we have defined the guarantee for both temporary death benefit ("GMDB") and for the pure endowment ("GMAB") by

$$C(T) = \max\{N(T)S(T), G(T)\}.$$

This means that we have value before the total product, eg the value of the underlying funds plus the corresponding variable annuity guarantee. In practise the total value is often split into the value of the underlying funds ($N(T)S(T)$) and the value of the variable annuity guarantee or also called variable annuity rider. Most of the examples in the following will actually calculate the value of the variable annuity rider. Note that the split of the entire value of the variable annuity (including the value of the underlying funds) is simple as a consequence of the linearity of the expected value under the measure Q. Moreover, one can define the guaranteed part of the insurance benefit as follows:

$$\begin{aligned} C^G(T) &= \max\{N(T)S(T), G(T)\} - N(T)S(T) \\ &= \max\{0, G(T) - N(T)S(T)\}. \end{aligned}$$

In order to compare the value of the variable annuity rider on often compares it with the value of the underlying funds. Depending on lapse assumptions the value of the guarantee can often exceed 10% of the underlying funds value.

8.5 Thiele's differential equation

Now we want to derive Thiele's differential equation. For this we need to determine premiums for the policies. We introduce the notation $\bar{p}(t)$ for the density of the premiums at time t. Then the equivalence principle yields the following two equations:

$$ {}_T G_x \quad = \quad \int_0^T \bar{p}(t) \exp(-\delta\, t)\, {}_t p_x dt $$

and

$$ G^1_{x:T} \quad = \quad \int_0^T \bar{p}(t) \exp(-\delta\, t)\, {}_t p_x dt. $$

Also in this section the pure endowment policy and the term life insurance will be considered separately. The mathematical reserve for these policies is given by:

$$ \begin{aligned} \text{Pure endowment:} \quad V(t) = \quad & {}_{T-t}p_{x+t}\pi_t(T) \\ & - \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t}\, d\xi. \\ \text{Life insurance:} \quad V(t) = \quad & \int_t^T (\pi_t(\xi)\mu_{x+\xi} - \bar{p}(\xi) \exp(-\delta(\xi - t))) \\ & \times_{\xi-t}p_{x+t}\, d\xi, \end{aligned} $$

where

$$ \begin{aligned} \pi_t(s) = \quad & G(s) \exp(-\delta(s - t))\Phi(-d_2^t(s)) \\ & + N(s)S(t)\Phi(d_1^t(s)), \\ d_1^t(s) \quad = \quad & \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s - t)}{\sigma\sqrt{s - t}}, \\ d_2^t(s) \quad = \quad & \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s - t)}{\sigma\sqrt{s - t}}, \end{aligned} $$

for $s > t$.

Remark 8.5.1. – In the classical setting the reserves were deterministic, but here they depend on the underlying share S.

– Note that we are beyond the deterministic theory of differential equations. In particular we have to use Itô's formula, which takes the following form for the purely continuous case of a standard Brownian motion W:

$$ df(W) = f'\, dW + \frac{1}{2} f''\, ds. $$

For the policies defined above we have the following theorem.

Theorem 8.5.2. *1. The differential equation for the price of a pure endowment policy is:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\, V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.$$

2. The differential equation for the price of a term life insurance is:

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\, V(t) - C(t)\mu_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.$$

Before we prove this theorem, we want to make some comments on the formulas.

Remark 8.5.3. 1. One obtains Black-Scholes formula by setting $\mu_{x+t} = \bar{p}(t) = 0\, \forall t$.

2. The first terms in the differential equations in the theorem above coincide with the classical case (see section 5.3), i.e. the dependence of the values on the premiums, on the mortality and on the interest rate. Due to the shares in the model a further term appears: $-\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}$. It represents the fluctuations of the underlying shares.

Proof. We have

$$\pi_t^*(T) = \exp(-\delta t)\pi_t(T).$$

Hence, by the definition of V, we get

$$V(t) = {}_{T-t}p_{x+t}\pi_t^*(T)\exp(\delta t) - \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t} d\xi$$

and

$$\pi_t^*(T) = \Psi(t)\left[V(t) + \int_t^T \bar{p}(\xi)\exp(-\delta(\xi - t))_{\xi-t}p_{x+t} d\xi\right],$$

where

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

Now we can apply Itô's formula to the function $\pi_t^*(t, S)$, since π_t^* is a function of S and t. We get

$$\begin{aligned} dY_t &= U(t + dt, X_t + dX_t) - U(t, X_t) \\ &= \left(U_t dt + \frac{1}{2}U_{xx} b^2 dt\right) + U_x dX_t \\ &= \left(U_t + \frac{1}{2}U_{xx} b^2\right) dt + U_x\, b\, dB_t \end{aligned}$$

and

$$d\pi^* = \left(\frac{\partial \pi^*}{\partial t} + \frac{\partial \pi^*}{\partial S}\, a + \frac{1}{2}\,\frac{\partial^2 \pi^*}{\partial S^2}\, b^2 \right) dt + \frac{\partial \pi^*}{\partial S} b\, d\hat{W}.$$

Furthermore we know that

$$dS = \delta S(t)dt + \sigma S(t) d\hat{W},$$

and thus we have $a = \delta S(t)$ and $b = \sigma S(t)$. In the next step we want to determine the two terms:

$$\begin{aligned} \frac{\partial \pi_t^*}{\partial S} &= \Psi(t)\, \frac{\partial V}{\partial S}, \\ \frac{\partial^2 \pi_t^*}{\partial S^2} &= \Psi(t)\, \frac{\partial^2 V}{\partial S^2}. \end{aligned}$$

To get $\frac{\partial \pi^*}{\partial t}$, we start with

$$\begin{aligned} \frac{\partial}{\partial t} {}_{\xi-t}p_{x+t} &= \mu_{x+t}\; {}_{\xi-t}p_{x+t}, \\ \frac{\partial}{\partial t} \Psi(t) &= \left(\frac{A}{B} \right)' = \frac{A'}{B} - \frac{A}{B^2}\, B' \\ &= -\left(\mu_{x+t} + \delta\right) \Psi(t). \end{aligned}$$

Now, with the formula from above we get

$$\begin{aligned} \frac{\partial \pi^*}{\partial t} &= \frac{\partial \Psi}{\partial t} \left(V(t) + \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt \right) \\ &+ \Psi(t) \left(\frac{\partial V}{\partial t} + \frac{\partial}{\partial t} \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt \right) \\ &= \Psi(t) \left(\frac{\partial V}{\partial t} - \left(\mu_{x+t} + \delta\right) V(t) - \bar{p}(t) \right), \end{aligned}$$

where we applied the chain rule to

$$\frac{\partial}{\partial t} \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt.$$

Thus we get

$$\begin{aligned} \pi_s^*(T) &= \pi_t^*(T) + \int_t^s \Psi(\xi) \frac{\partial V}{\partial S} \sigma S d\hat{W}(\xi) \\ &+ \int_t^s \Psi(\xi) \left[\frac{\partial V}{\partial S} \delta S + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (\mu_{x+\xi} + \delta) V(\xi) \right. \\ &\qquad \left. + \frac{\partial V}{\partial t}(\xi) - \bar{p}(\xi) \right] d\xi. \end{aligned}$$

Now the drift term is equal to zero, since $\pi_\cdot^*(T)$ is a martingale. Therefore we finally get

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\, V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.$$

Exercise 8.5.4. Prove the second part of the theorem above.

Remark 8.5.5. Thiele's differential equation for pure endowments and term insurance (Theorem 8.5.2), has been stated assuming no lapses. In reality a lot of the corresponding GMDB and GMAB policies are lapse supported. Depending on the context the surrender value of the policy varies. If we assume that the surrender value in case of lapse is given by $f(x)$ for a function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$$

it is easy to show that the mathematical reserves for the model including lapsed with a lapse density $\mu^l(x)$ fulfill the following partial differential equations:

1. The differential equation for the price of a pure endowment policy is:

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - f(x)\mu^l_{x+t} \\ &\quad - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

2. The differential equation for the price of a term life insurance is:

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} - f(x)\mu^l_{x+t} \\ &\quad - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

It is worth mentioning that in many cases we have $f \equiv 0$, meaning that the option value of the variable annuity guarantee is lost in case of a surrender.

Exercise 8.5.6. Prove the modified partial differential equations as per Remark 8.5.5. Note that in this context one has to replace

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

by

$$\Psi(t) = \frac{\exp(-\delta t)}{\bar{p}_{**}(x+t, x+T)}$$

(see definition 2.3.6). By theorem 2.3.7 we can use the following equation:

$$\bar{p}_{**}(s,t) = \exp\Big(- \sum_{k \neq *} \int_s^t \mu_{*k}(\tau) d\tau \Big). \tag{8.7}$$

Remark 8.5.7. Until now we have mainly considered insurance covers which pay a lump sum in case a person survives a certain number of years (endowment) or in case of death (term insurance). The same methodology shown above can be used to model unit-linked annuities (GMWB variable annuities). Since an annuity can be considered as a negative insurance premium, we have actually already included this case in Theorem 8.5.2 and in Remark 8.5.5. We will use the same notation as in Remark 8.5.5. Furthermore we denote by $\bar{r}(t)$ the annuity density being paid to the policyholder. Keeping in mind that $\bar{p}(t)$ in Theorem 8.5.2 needs to be replaced by $\bar{p}(t) - \bar{r}(t)$ we get the following Thiele partial differential equation for a GMWB / GMDB combined insurance product:

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) - \bar{r}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} \\ &\quad - f(x)\mu^l_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

By using a Taylor expansion we get the following movement of $V(t)$ during the interval $[t, t + \Delta t]$.

$$\begin{aligned}V(t) + \bar{p}(t)\Delta t &= \bar{r}(t)\,\Delta t + \sum_i \left(\mu^i_{x+t}\, C_i(t)\right)\,\Delta t \\ &\quad - \left(V(t) - S(t)\,\frac{\partial}{\partial S} V(t)\right)\,\delta\Delta t \\ &\quad - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2}\Delta t \\ &\quad + \left(1 - \left(\sum_i \mu^i_{x+t}\right)\,\Delta t\right)\, V(t + \Delta t)\end{aligned}$$

We will see in the following that the above roll forward of $V(t)$ is useful when introducing the concept of a hedge P&L.

8.6 Example

In this section we consider a concrete example for which the corresponding calculations can be explicitly performed. We consider the following steps:

1. Description of the product,
2. Valuation of the product, and

The value of a variable annuity from a policyholder point of view consists of two parts:

1. The value of the underlying fund investment, and
2. the value of the variable annuity rider (eg GMDB, GMAB, GMIB and GM(L)WB).

From an insurer's point of view the fund investment is normally classified as a separate account and rather easy to value. The focus on valuation of variable annuities regards the actual variable annuity riders, on which we will focus. Figure 8.9 shows the different parts of a balance sheet of an insurance company which sells unit linked products with guarantees (aka "variable annuities").

In order to value such variable annuities there are different ways to determine the value of the underlying guarantee:

– Explicit formula or recursion (only for insurance valuation and very simple variable annuities),
– Solution of Black-Scholes-Merton differential equation / Thiele's differential equation (different methods including tree method),
– Monte Carlo Simulation (this is the approach most often used for variable annuities).

Monte Carlo is most commonly used for variable annuities since it is very versatile and can also cope with very complex option structures, such as ratchets. For this example, we will however limit ourselves to a product which can be valued explicitly. In order to keep things simple we look in a first step at a discrete time version of Theorems 8.4.1 and 8.4.2.

Theorem 8.6.1. *Assume that the economy follows the Black-Scholes-Merton model and assume that deaths occur at time $K = \lfloor T \rfloor$. For*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

we can calculate the single premiums as follows:

$$\begin{aligned} {}_T G_x &= {}_T p_x \, \pi_0^*(T), \\ G_{x:T}^1 &= \sum_{k=0}^{T-1} {}_k p_x \, q_{x+k} \, \pi_0^*(k), \end{aligned}$$

where

$$\begin{aligned}
\Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}$$

for $s > t$.

Exercise 8.6.2. Prove the previous theorem.

8.6.1 Definition of the product

In the following we consider a very simple variable annuity product consisting of a term assurance ("Guaranteed Minimal Death Benefit (GMDB)") and of a pure endowment policy ("Guaranteed Minimal Accumulation Benefit (GMAB)"). This is exactly the set up which we have considered in section 8.4.

- Variable annuity for a 30 year old man consisting of GMDB and GMAB for a term of 25 years.
- Variable annuity guarantee level according to a "doubler", eg
  - At maturity guarantees twice the initial fund value,
  - In case of death, at least double the value after 10 years.
- Single premium for both funds value and guarantee premium.

In order to illustrate things clearer we will base our calculations on the following decrement table ($l_{35} = 100000$, $l_{x+1} = l_x\,(1 - q_x)$, and $d_x = l_x\,q_x$):

x	l_x	d_x
35	100000.0	147.7
36	99852.3	157.5
37	99694.8	169.5
38	99525.4	183.8
39	99341.5	200.8
40	99140.8	220.3
...		
60	86622.9	1394.0

We note that one can calculate

$$
\begin{aligned}
{}_t p_x &= \frac{l_{x+t}}{l_x}, \text{ and,} \\
{}_t p_x \, q_{x+t} &= \frac{d_{x+t}}{l_x}.
\end{aligned}
$$

In a next step we want to do this example also allowing for lapses. For sake of simplicity, we have assumed a level lapse rate of 4% with an exception for year 10 where the assumed lapse rate is 12%. These uneven lapse rates are typical – after 10 years (or so, product dependent), the policyholder has the ability to lapse without surrender penalty, leading to higher lapses in this year.

The following table shows the decrement table including lapses:

x	l_x	d_x	l_x incl. lapse	d_x incl. lapse
35	100000.0	147.7	100000.0	147.7
36	99852.3	157.5	95852.3	151.2
37	99694.8	169.5	91860.7	156.2
38	99525.4	183.8	88016.8	162.6
...				
44	98114.7	326.8	67604.5	225.2
45	97787.9	360.9	60517.2	223.4
46	97426.9	398.4	57735.6	236.1
...				
60	86622.9	1394.0	23716.1	381.7

Note that also in this context we can calculate ${}_t p_x$ and ${}_t p_x \, q_{x+t}$ by means of l_x and d_x. Based on the above table we see that the valuation of a variable annuities heavily depends on lapses and also that the guarantee gets cheaper when considering lapses, since less people profit from it. In consequence most variable annuity products are lapse supported. "Lapse supported" means that the pricing of the product relies on the fact that some of the policyholders will lapse early.

8.6.2 Valuation of the Product / Replicating of a Variable Annuity

Based on the example introduced before we now determine the value or price for this variable annuity guarantee. In order to do this, three steps are involved:

Determine the number of people which benefit: Since only a tiny percentage of the whole inforce dies within a given year, one only needs to provide the respective GMDB cover only to them. Similarly the GMAB cover is paid only to the people surviving the entire term of the policy. Hence we need to determine the respective percentages. This is done by means of life decrement tables, as introduced before.

Calculate what these people receive: We need to know what the respective policyholders are entitled to. Assume for example the people dying at age 40. They are entitled to get a GMDB at a certain level. Hence we need to determine the number of the corresponding units of guarantees. For our 40 year old dying person, this would be put options at a strike price.

Calculate the value: We know the valuation portfolio of guarantees representing the variable annuity guarantee (eg number of instruments and their characteristics). We now need to value them. For our example this is done via the Black-Scholes formula.

Note: only in this simple example we can easily distinguish between steps 2 and 3. Normally one performs 2 and 3 together using a Monte Carlo simulation. For our concrete example the table below provides the portfolio of guarantees at inception.

Instrument	Strike	Amount
Fund		100000
Put Fund value t=0 at	100000	0.1 %
Put Fund value t=1 at	107177	0.2 %
Put Fund value t=2 at	114870	0.2 %
Put Fund value t=3 at	123114	0.2 %
Put Fund value t=4 at	131951	0.2 %
Put Fund value t=5 at	141421	0.2 %
...		
Put Fund value t=9 at	186607	0.3 %
Put Fund value t=10 at	200000	0.4 %
...		
Put Fund value t=25 at	200000	86.6 %

Over time, as the policy matures more and more of these instruments are used to pay the guarantees of the corresponding period. Deviations from this modeled guarantee portfolio result in a profit or loss.

Based on the valuation portfolio we can now calculate the various metrics for the policy. In particular we can value the variable annuity guarantee of the policy by valuing each individual instrument:

	Instrument	Strike	Amount %age	**Value**
0	Put Fund	100000	0.1 %	8.7
1	Put Fund	107177	0.2 %	17.7
2	Put Fund	114870	0.2 %	27.7
3	Put Fund	123114	0.2 %	39.5
4	Put Fund	131951	0.2 %	53.6
5	Put Fund	141421	0.2 %	70.6
...				
9	Put Fund	186607	0.3 %	146.0
10	Put Fund	200000	0.4 %	181.8
...				
25	Put Fund	200000	86.6 %	34789.4
	Total			**40311.7**

We need to look at the consequence of different market shocks at inception, such as lower equity prices, lower interest rates and higher volatility. Note that the valuation portfolio does not change and remains (in terms of respective types of guarantees and amounts) the same! Concretely we look at the following three shocks:

– Equity Drop by 10%,

– Interest Lower by 1%,

– Volatility up by 1%.

The following table summarises the corresponding results. We can observe the high dependency of the value on the market variables.

	Instrument	Value Normal	Value Equity -10%	Value Interest -1%	Value Volatility $+1\%$
0	Put Fund	8.7	16.5	9.5	14.5
1	Put Fund	17.7	26.5	19.6	26.5
2	Put Fund	27.7	37.6	31.3	39.4
3	Put Fund	39.5	50.6	45.3	54.1
4	Put Fund	53.6	66.0	62.4	71.4
5	Put Fund	70.6	84.6	83.4	92.0
...					
9	Put Fund	146.0	165.8	178.2	180.8
10	Put Fund	181.8	204.0	224.2	222.3
...					
25	Put Fund	34789.4	38092.7	54782.4	50112.1
	Total	**40311.7**	**44239.3**	**62445.2**	**57454.8**

Finally we look how the value of the valuation portfolio changes over time. There are two effects which affect its value, namely that, over time, parts of it are used to finance the claims which have occurred in the past and also as a consequence of the market movement of the underlying fund. Figure 8.10 shows how the guarantee of the hedge liability moves over time:

– Upper figure shows movement in fund value. The lower figure shows the changing value of the underlying guarantee.
– Note that at inception the value of the guarantee equals the value determined above (40311).
– Over time the valuation portfolio becomes smaller, since part of its instruments are used to finance the claims which have occurred in the past.
– Moreover the valuation portfolio changes its value as a consequence of changing fund levels, interest rates and volatilities.
– For this example interest rates and volatilities have been keep constant.
– It becomes obvious that the value of the guarantee increases each time fund value decreases and vice versa.

It is important to understand that none, except the most simple variable annuity guarantee structures can be calculated explicitly by the Black-Scholes formula. The examples in this text have been designed in such a manner that they still allow to use the Black-Scholes formula.

8.6.3 Value of a variable annuity as a function of equity level

Finally, the following table shows the dependency of the value (π) of the variable annuity guarantee as a function of the equity level and is called a trading grid. The lower the equity level, the more valuable the variable annuity guarantee.

Equity Level	π $V(S)$	δ $\frac{\partial}{\partial S}V$	γ $\frac{\partial^2}{\partial S^2}V$	ρ $\frac{\partial}{\partial r}V$	ν $\frac{\partial}{\partial \sigma}V$
-50 %	65777	-33818	39673	-2341692	86714
-40 %	59397	-36051	42961	-2241519	111455
-30 %	53734	-37319	44743	-2136639	133559
-20 %	48710	-37826	45431	-2030460	152596
-10 %	44254	-37744	45168	-1925345	168470
-5 %	42219	-37531	44705	-1873714	175247
0 %	40301	-37230	44052	-1822905	181284
+5 %	38494	-36854	43243	-1773034	186616
+10 %	36789	-36419	42315	-1724192	191283
+20 %	33662	-35417	40235	-1629845	198789
+30 %	30871	-34295	38026	-1540199	204141
+40 %	28372	-33102	35808	-1455379	207655
+50 %	26130	-31872	33640	-1375369	209618

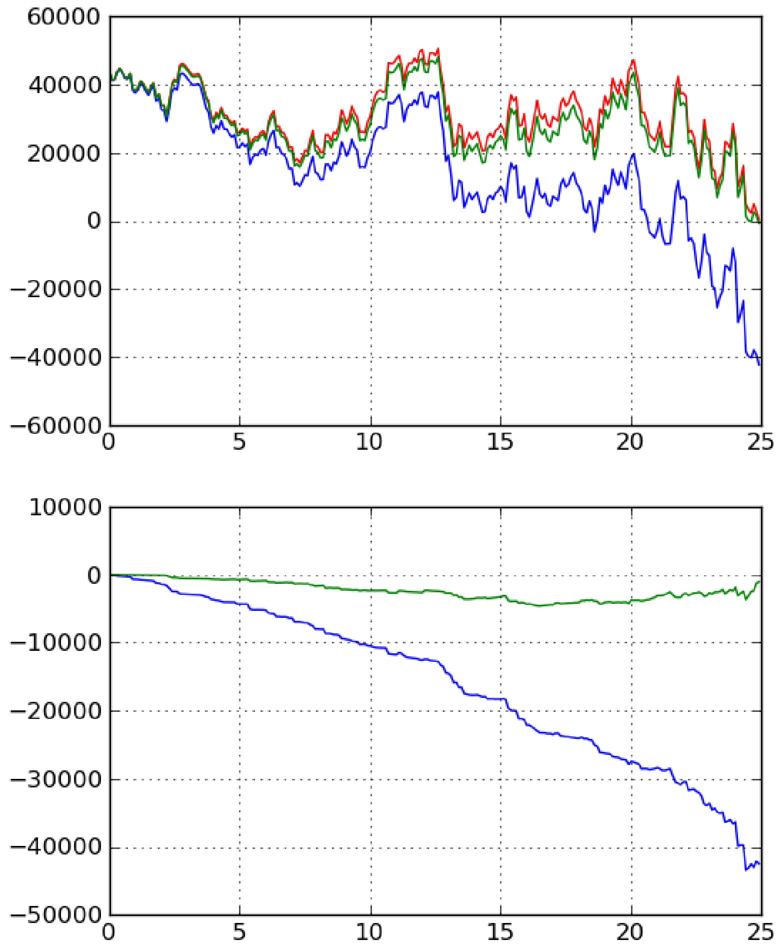


Figure 8.8. Delta Hedge

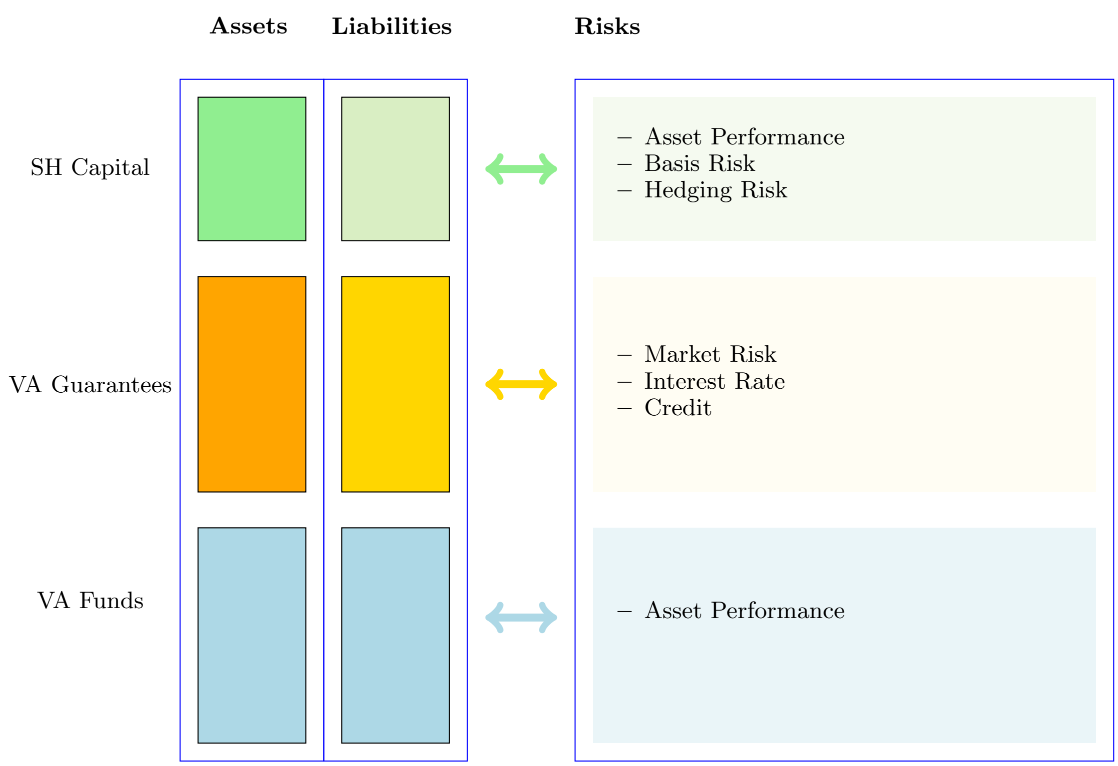


Figure 8.9. Balance sheet of an insurance company

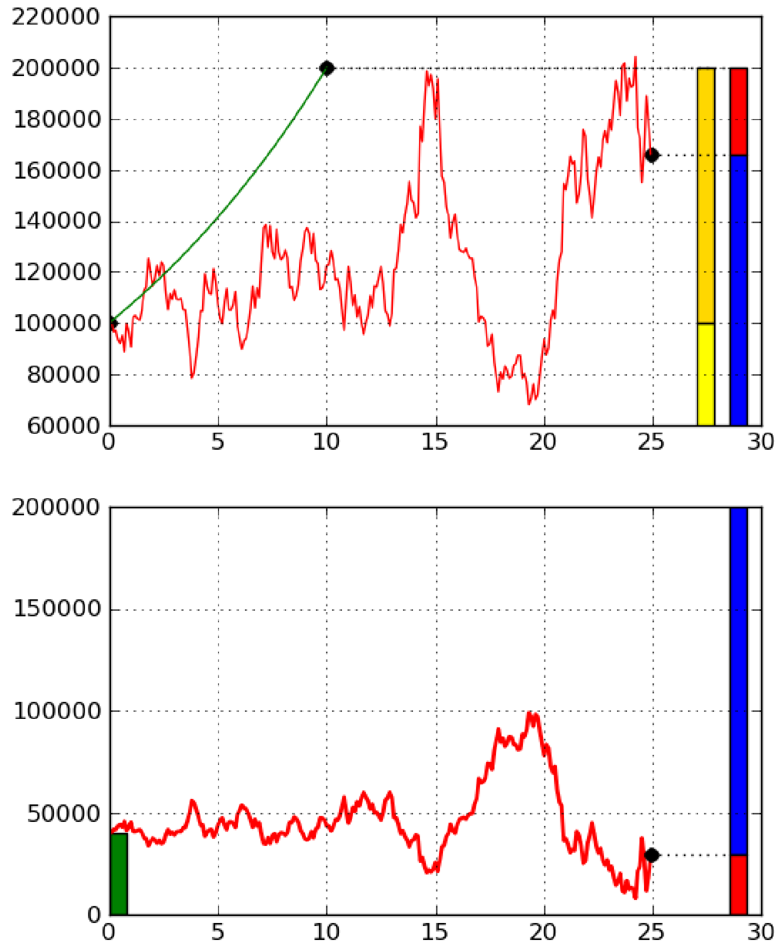


Figure 8.10. Value of Hedge Liability over time

9. Policies with stochastic interest rate

9.1 Introduction

In the previous chapter we looked at unit-linked policies. Based on these we will now consider policies with a technical interest rate modeled by a stochastic process, which is for example given by a stochastic differential equation.

As before we will assume that the random variables of the economic model are independent of the random variables which describe the state of the insured. We also assume that the market does not provide arbitrage opportunities. Thus one has to find an equivalent martingale measure to price the option which models the policy. The price of the option is then given as expectation with respect to this new measure.

Definition 9.1.1 (Spotrate). *In the following we will denote the spot interest rate by r_t, this is the interest rate which is currently paid. The cumulated interest for an interval $[0, t]$ is given by*

$$\gamma_t := \beta_t^{-1} = \exp\left(\int_0^t r_s ds\right),$$

which is the solution to the stochastic differential equation

$$d\gamma_t = r_t \, \gamma_t \, dt,$$

with $\gamma_0 = 1$.

The value of a zero coupon bond at time t, paying 1 at time s, is denoted by $B_t(s)$.

9.2 The Vasicek model

In the following we look at a particular interest rate model, the Vasiček model. It is defined by the stochastic differential equation

$$dr_t = \alpha\,(\rho - r_t)\,dt + \sigma\,dW_t, \tag{9.1}$$

where $\alpha > 0$ and $\rho, \sigma \in \mathbb{R}$. Thus, the interest rate process is induced by a Brownian motion W. The model has the feature, that in the absence of stochastic noise the interest rate would converge towards ρ (mean reversion). The stochastic process given by (9.1) is also known as Ornstein-Uhlenbeck process.

We chose the Vasiček model as primary example, since for this model it is possible to calculate the quantities relevant to us (e.g. the discount rate) explicitly. Thus one can use the values given by this model directly. Moreover, it is easily possible to extend the model by replacing the Brownian motion W_t by another stochastic process [Nor98]. A shortcoming of this model is the fact, that it allows with positive probability negative interest rates.

Theorem 9.2.1. *The following statements hold for the Vasiček model:*

1. $r_t \sim \mathcal{N}\left(\rho + \exp(-\alpha\,t)\,(r_0 - \rho)\,, \frac{\sigma^2}{2\alpha}\,(1 - \exp(-2\alpha t))\right)$,
2. $Cov(r_s, r_t) = \exp\left(-\alpha(s+t)\right)\frac{\sigma^2}{2\alpha}\left(\exp(2\alpha t) - 1\right)$, *for* $s \le t$.

Proof. [Nor98], [Pro90], [KS88].

Exercise 9.2.2. Calculate a 95% confidence interval for r_t with $\alpha = 0.1$, $\rho = 0.05$, $r_0 = 0.03$ and $\sigma = 0.01$ for a period of 20 years.

Next, we have a look at the cumulated interest $y(t) = \exp(\int_0^t r_s\,ds)$ on the interval $[0, t]$.

Theorem 9.2.3. *The following equations hold for the Vasiček model:*

1. $E\left[y(t)\right] = \rho\,t + (r_0 - \rho)\frac{1-\exp(-\alpha t)}{\alpha}$,
2. $Var\left[y(t)\right] = \frac{\sigma^2}{\alpha^2}t + \frac{\sigma^2}{2\alpha^3}\left[-3 + 4\exp(-\alpha t) - \exp(-2\alpha t)\right]$,
3. $Cov(y(s), y(t)) = \frac{\sigma^2}{\alpha^2}\min(s,t) + \frac{\sigma^2}{2\alpha^3}\Big[-2 + 2\exp(-\alpha s)$
$+2\exp(-\alpha t) - \exp(-\alpha|t-s|) - \exp(-\alpha(t+s))\Big]$.

Proof. This theorem is a consequence of Theorem 9.2.1. One just has to change the order of integration, which is allowed by Fubini's theorem.

Exercise 9.2.4. Complete the proof of the theorem above.

9.3 Value of the portfolio

In this section we will analyze the influence of the stochastic interest rate on the value of the portfolio. For insurance policies with deterministic interest rate we know that the present values of the cash flows of the policies are independent and their joint variance $V_{Tot} = \sum_{k=1}^{n} V_k$ converges with rate $1/n$ to zero (central limit theorem). This does not hold, if the interest rate is a stochastic process. In this case the present values are dependent, since the same interest rate process is used for all policies. In the following we want to concentrate on the most important aspects of this setup. Therefore we make the following assumptions:

– We work with a discrete time model. Thus, we do not have to consider double integrals.

– We assume that the state in which benefits are payed to the insured is absorbing. Furthermore we also assume that the insured event can only occur at the end of each period. These assumptions are very common for pensions or life insurances or their combination in an endowment policy.

In order to be able to derive some results, we start with setting some notations.

Definition 9.3.1. *In the following Z denotes the present value of the future benefits discounted to time 0. The time of the occurrence of the insured event is denoted by K. Moreover we assume, that a payment b_k is due at time $t(k)$, if the insured event occurs at time $K = k$.*

As usual, we denote the probability to die within t years (to survive t years, respectively) for an x year old person by

$$\begin{aligned} {}_tq_x &= P[K \leq t], \\ {}_tp_x &= 1 - {}_tq_x, \end{aligned}$$

respectively. In the following we will always consider "death" as the insured event, even if in the concrete example another event is insured.

Theorem 9.3.2. *With the notation introduce above the following equation holds:*

$$A := E[Z] = \sum_{k=0}^{\infty} b_k E[\exp(-y(t(k)))]_k p_x q_{x+k},$$

where $\exp(-y(t)) = \exp(-\int_0^t r_s ds)$ denotes the (stochastic) discount from time t to time 0.

Proof. This identity is a direct consequence of the previous definitions and the tower property of the expectation (i.e., $E[Z] = E[E[Z \mid K]]$).

In the same manner one can calculate moments of higher order of Z:

Theorem 9.3.3. *Let $m \in \mathbb{N}$. Then the m-th moment of the present value of Z is*

$$\begin{aligned} E[Z^m] &= \sum_{k=0}^{\infty} b_k^m E[\exp(-\tilde{y}(t(k)))]_k p_x q_{x+k}, \\ \text{where } \tilde{y}(t) &= \int_0^t m\, r_\tau d\tau \end{aligned}$$

denotes the cumulated interest for the m-th power of the interest intensity. In the case of an insurance whose payouts are either 0 *or* 1 *the previous equation simplifies to*

$$E[Z^m] = \sum_{k=0}^{\infty} b_k E[\exp(-\tilde{y}(t(k)))]_k p_x q_{x+k}.$$

Proof. These equations can be obtained by calculating the conditional expectation $E[Z^m \mid K]$:

$$\begin{aligned} E[Z^m] &= E[E[Z^m \mid K]] \\ &= \sum_{k=0}^{\infty} E[Z^m \mid K = k]_k p_x q_{x+k} \\ &= \sum_{k=0}^{\infty} E[\{b_k \exp(-y(t(k)))\}^m]_k p_x q_{x+k} \\ &= \sum_{k=0}^{\infty} b_k^m E[\exp(-m\, y(t(k)))]_k p_x q_{x+k} \\ &= \sum_{k=0}^{\infty} b_k^m E[\exp(-\tilde{y}(t(k)))]_k p_x q_{x+k}. \end{aligned}$$

The second equation is implied by the facts, that $1^m = 1$ und $0^m = 0$.

Definition 9.3.4 (Portfolio assumptions). *We consider a portfolio with c identical policies, and we use the notations:*

K_i	*time of the occurrence of the insured event for policy i,*
Z_i	*present value of the future benefits of policy i.*

The sum of the present values of all c policies is

$$Z(c) := \sum_{k=1}^{c} Z_i.$$

The aim of the next part is to analyze Z. For this we have to make certain assumptions about our portfolio:

A1 *The times of the occurrences of the insured events are independent and identical distributed.*

A2 *The random variables of the economic model are independent of the times of the occurrences of the insured events. In other words: the family $\{K_i : i = 1, ...c\}$ is independent of the family $\{\delta_\tau : \tau > 0\}$.*

A3 *All policies have an identical structure, i.e. the conditional present values $\{Z_i \mid (y_n)_{n\in\mathbb{N}}\}_{i=1,...c}$ are independent and identical distributed.*

Remark 9.3.5. Note that the present values are not independent, even though the conditional present values are independent. This is due to the fact, that all present values are subject to the same discount $\exp(-y(t))$.

We begin with a calculation of the present values of the whole portfolio.

Theorem 9.3.6. *Let the assumptions A1, A2 and A3 hold. Then the present value of the portfolio is given by*

$$E[Z(c)] = c\,E[Z_1].$$

Proof.

$$E[Z(c)] = E[\sum_{k=1}^{c} Z_i] = \sum_{k=1}^{c} E[Z_i] = c\,E[Z_1].$$

Analogous, one can also calculate the second moment.

Theorem 9.3.7. *Let the assumptions A1, A2 and A3 hold. Then the second moment of the present value of the portfolio is*

$$E[Z(c)^2] = c(c-1)E[Z_1\,Z_2] + c\,E[Z_1^2],$$

where

$$E[Z_1\,Z_2] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_i\,b_j E[\exp(-y(t(i)) - y(t(j)))]_i p_x q_{x+i\;j} p_x q_{x+j}.$$

Proof. The theorem is proved by a direct verification of the formula

$$\begin{aligned} E[Z(c)^2] &= E[(\sum_{i=1}^{c} Z_i)^2] \\ &= \sum_{i,j=1}^{c} E[Z_i\, Z_j] \\ &= \sum_{i=1}^{c} E[Z_i^2] + 2\sum_{i<j} E[Z_i\, Z_j]. \end{aligned}$$

Now each part can be calculated. We get

$$\begin{aligned} E[Z_i^2] &= E[E[Z_i^2 \mid (y_n)_{n\in\mathbb{N}}]] \\ &= E[E[Z_1^2 \mid (y_n)_{n\in\mathbb{N}}]] \\ &= E[Z_1^2] \end{aligned}$$

and similarly for $i \neq j$

$$\begin{aligned} E[Z_i\, Z_j] &= E[E[Z_i\, Z_j \mid (y_n)_{n\in\mathbb{N}}]] \\ &= E[E[Z_i \mid (y_n)_{n\in\mathbb{N}}] \times E[Z_j \mid (y_n)_{n\in\mathbb{N}}]] \\ &= E[E[Z_1 \mid (y_n)_{n\in\mathbb{N}}] \times E[Z_2 \mid (y_n)_{n\in\mathbb{N}}]] \\ &= E[Z_1\, Z_2]. \end{aligned}$$

Replacing these terms in the above equation yields the statement of the theorem.

The second moment of the portfolio can be used to calculate its asymptotic variance.

Theorem 9.3.8. *The asymptotic variance satisfies the equation:*

$$\lim_{c\to\infty} \mathit{Var}(\frac{Z(c)}{c}) = E[Z_1\, Z_2] - E[Z_1]^2.$$

Proof. The following identities hold:

$$\begin{aligned} \mathrm{Var}\left(\frac{Z(c)}{c}\right) &= E[(\frac{Z(c)}{c})^2] - E[\frac{Z(c)}{c}]^2 \\ &= \frac{c(c-1)}{c^2} E[Z_1\, Z_2] + \frac{c}{c^2} E[Z_1^2] - E[Z_1]^2 \\ &\to E[Z_1\, Z_2] - E[Z_1]^2 \text{ for } c \to \infty. \end{aligned}$$

Remark 9.3.9. Contrary to the classical model with deterministic interest rate, here the asymptotic variance (for $c \to \infty$) is not equal to zero. This shows, that a stochastic oscillation of the bonds induces a positive asymptotic variance. Furthermore, this risk can not be reduced by raising the number of policies in the portfolio.

Theorem 9.3.10. *Let assumptions A1, A2 and A3 hold. Then the third moment of the present value of the portfolio is*

$$\begin{aligned} E[Z(c)^3] &= c(c-1)(c-2)\, E[Z_1\, Z_2\, Z_3] \\ &+ c(c-1)\, E[Z_1^2\, Z_2] \\ &+ c\, E[Z_1^3], \end{aligned}$$

where

$$E[Z_1\, Z_2\, Z_3] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_i\, b_j\, b_k\; {}_ip_x q_{x+i}\; {}_jp_x q_{x+j}\; {}_kp_x q_{x+k} \\ \times E[\exp(-y(t(i)) - y(t(j)) - y(t(k)))],$$

$$E[Z_1^2\, Z_2] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_i^2\, b_j\; {}_ip_x q_{x+i}\; {}_jp_x q_{x+j} E[\exp(-2y(t(i)) - y(t(j)))],$$

$$E[Z_1^3] = \sum_{i=0}^{\infty} b_i^3\, {}_ip_x q_{x+i} E[\exp(-3y(t(i)))].$$

Proof. The proof is analogous to the proof of the formula for the second moment. One starts by decomposing $E[Z(c)^3]$ into

$$E[Z(c)^3] = \sum_{i,j,k=1}^{c} E[Z_i\, Z_j\, Z_k].$$

Then the terms are sorted based on their structure, and one proves the following identities:

$$\begin{aligned} E[Z_i\, Z_j\, Z_k] &= E[Z_1\, Z_2\, Z_3], \text{ if } i \neq j \neq k \neq i, \\ E[Z_i^2\, Z_j] &= E[Z_1^2\, Z_2], \text{ if } i \neq j. \end{aligned}$$

Using these identities in the decomposition yields the statement of the theorem.

Theorem 9.3.11. *The formula of Theorem 9.3.10 can also be used to calculate the asymptotic skewness $Sk(Z) := \frac{E[(Z-E[Z])^3]}{Var(Z)^{3/2}}$. We have*

$$\lim_{c\to\infty} Sk(Z(c)) = \frac{E[Z_1\, Z_2\, Z_3] - 3E[Z_1\, Z_2] \times E[Z_1] + 2E[Z_1]^3}{(E[Z_1\, Z_2] - E[Z_1]^2)^{3/2}}.$$

Proof. To prove this theorem we set $W = \frac{Z(c)}{c}$. Then the following identities hold:

$$\begin{aligned}
Sk(Z) &= \frac{E[(Z-E[Z])^3]}{Var(Z)^{3/2}} \\
&= \frac{E[W^3-3W^2E[W]+3WE[W]^2-E[W]^3]}{(Var(W))^{3/2}} \\
&= \frac{1}{(Var(W))^{3/2}} \times \Bigg[\frac{c(c-1)(c-2)}{c^3}E[Z_1Z_2Z_3] + \frac{c(c-1)}{c^3}E[Z_1^2Z_2] \\
&\qquad + \frac{c}{c^3}E[Z_1^3] - 3\frac{cE[Z_1]}{c}\left\{\frac{c(c-1)}{c^2}E[Z_1Z_2] + \frac{c}{c^2}E[Z_1^2]\right\} \\
&\qquad + 3\frac{c^2E[Z_1]^2}{c^2}\,\frac{cE[Z_1]}{c} - \left\{\frac{cE[Z_1]}{c}\right\}^3\Bigg] \\
&\to \frac{E[Z_1\,Z_2\,Z_3]-3E[Z_1\,Z_2]\times E[Z_1]+2E[Z_1]^3}{(E[Z_1\,Z_2]-E[Z_1]^2)^{3/2}}, \text{ for } c\to\infty.
\end{aligned}$$

The previous formulas are all based on direct calculations of the values in question. This is in contrast to the recursion techniques for the Markov model, where we based statements for time n on statements for time $n-1$.

In order to derive also recursion equations in the current setting we have to make further assumptions.

Definition 9.3.12 (Portfolio assumptions). *We consider a portfolio with c identical policies with state space S, and we use the following notations:*

${}_iX$	*the Markov chain on the state space S, which represents the state of the i-th policy,*
W	*the Markov chain on the state space $\tilde{S}$, which determines the bond process. Here, $v_t = \sum_{\iota\in\tilde{S}} I_\iota^W(t)v_\iota(t)$ holds.*

These notations and the following definitions will enable us to derive recursion formulas for policies modeled by Markov chains.

We set the following notations:

$$\begin{aligned}
V_{il}^{\alpha}(t) &= E[{}_1V^{+,\alpha}|{}_1X=i,\, W=l], \\
V_{ijl}^{\alpha\beta}(t) &= E[{}_1V^{+,\alpha}\,{}_2V^{+,\beta}|{}_1X=i,\, {}_2X=j,\, W=l], \\
V_{ijkl}^{\alpha\beta\gamma}(t) &= E[{}_1V^{+,\alpha}\,{}_2V^{+,\beta}\,{}_3V^{+,\gamma}|{}_1X=i,\, {}_2X=j,\, {}_3X=k,\, W=l],
\end{aligned}$$

where ${}_1V^{+,\alpha}$ denotes α-th power of the prospective reserve of the first policy. The sum of all present values of all c policies is

$$Z(c) := \sum_{k=1}^{c} {}_kV^+.$$

The aim of the next part is to analyze Z. For this we have to make certain assumptions about our portfolio:

A1 *The processes $\{{}_iX | i = 1, \dots, c\}$ are independent and identical distributed.*

A2 *The random variables of the economic model are independent of the times of the occurrences of the insured events.*

A3 *All policies have an identical structure, i.e. the conditional present values $\{{}_iV^+ \mid (y_n)_{n\in\mathbb{N}}, {}_iX = \iota\}_{i=1,\dots c}$ are independent and identical distributed.*

Under these assumptions on the portfolio we are able to derive several recursion formulas.

Theorem 9.3.13. *In the above setting the following formulas hold:*

$$E[Z(c)|\{{}_\kappa X(t) = i : \kappa = 1 \dots c, W(t) = w\}] \quad = \quad c\, E[V_1|{}_1X(t) = i, W(t) = w],$$

$$\begin{aligned} &E[Z(c)^2|\{{}_\kappa X(t) = i : \kappa = 1, \dots, c, W(t) = w\}] \\ &\quad = cE[{}_1V^2|{}_1X(t) = i, W(t) = w] \\ &\qquad\quad + c(c-1)E[{}_1V\, {}_2V|{}_1X(t) = i, {}_2X(t) = i, W(t) = w], \end{aligned}$$

$$\begin{aligned} &E[Z(c)^3|\{{}_\kappa X(t) = i : \kappa = 1, \dots, c, W(t) = w\}] \\ &\quad = c(c-1)(c-2)E[{}_1V\, {}_2V\, {}_3V|{}_1X(t) = i, {}_2X(t) = i, {}_3X(t) = i, W(t) = w] \\ &\qquad\quad + c(c-1)E[{}_1V^2\, {}_2V|{}_1X(t) = i, {}_2X(t) = i, W(t) = w] \\ &\qquad\quad + c\, E[{}_1V^3|{}_1X(t) = i, W(t) = w]. \end{aligned}$$

Proof. The proof is analogous to the proofs of Theorem 9.3.7 and Theorem 9.3.10.

Now we have seen that recursion formulas also hold in this setting. Next, we want to calculate explicitly the terms appearing therein. By algebraic transformation, values at time t will be calculated based on the values at time $t+1$. We start with the calculation of $V_{ijl}^{1\,1}(t)$.

Theorem 9.3.14. *In the discrete time Markov model the prospective reserves $V_{ijl}^{1\,1}(t)$ with $i, j \in S$ and $l \in \tilde{S}$ satisfy the following recursion:*

$$\begin{aligned}
V_{ijl}^{1\,1}(t) &= \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\,p_{j\tilde{j}}(t,t+1)\,p_{l\tilde{l}}^{W}(t,t+1)\,v_l^2\,V_{\tilde{i},\tilde{j},\tilde{l}}^{1\,1}(t+1)\\
&+ \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\,p_{j\tilde{j}}(t,t+1)\,p_{l\tilde{l}}^{W}(t,t+1)\,v_l\,V_{\tilde{i},\tilde{l}}^{1}(t+1)\,\Theta_{j\tilde{j}l}(t)\\
&+ \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\,p_{j\tilde{j}}(t,t+1)\,p_{l\tilde{l}}^{W}(t,t+1)\,v_l\,V_{\tilde{j},\tilde{l}}^{1}(t+1)\,\Theta_{i\tilde{i}l}(t)\\
&+ \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\,p_{j\tilde{j}}(t,t+1)\,p_{l\tilde{l}}^{W}(t,t+1)\,\Theta_{i\tilde{i}l}(t)\,\Theta_{j\tilde{j}l}(t),
\end{aligned}$$

where we used the notation

$$\Theta_{ijk}(t) = a_i^{Pre}(t) + v_k(t)\,a_{ij}^{Post}(t).$$

Proof. The proof is based on a general principle: it considers on the one hand the current period and on the other hand the future periods. Thereby the sum is decomposed into two parts, which leads to the result.

We want to calculate the quantity $E[{}_1V^1\,{}_2V^1|{}_1X_t = i,\,{}_2X_t = j,\,W_t = l]$. First we get the formula

$${}_1V^1 = \sum_{k=0}^{\infty}\left(\prod_{l=0}^{l<k} v_l\right)\left(\sum_{\iota\in I} I_\iota(k)a_\iota^{\mathrm{Pre}}(k) + v_k \sum_{\iota,\kappa\in I} \Delta N_{\iota\kappa}(k)a_{\iota\kappa}^{\mathrm{Post}}(k)\right).$$

Similarly, one can also calculate ${}_2V^1$. In a next step this formula is transformed to

$${}_1V^1 = \sum_{k=0}^{\infty}\left(\prod_{l=0}^{l<k} v_l\right) \times \left(\sum_{\iota\kappa\in I} \Delta N_{\iota\kappa}(k)\Theta_{\iota\kappa\bullet}(k)\right).$$

Now ${}_1V^1 \times {}_2V^1$ can be calculated. For this we denote the term in the first sum by $\Upsilon^\iota(k)$. Thus the formula above simplifies to

$${}_1V^1 = \sum_{k=0}^{\infty} \Upsilon^1(k), \tag{9.2}$$

and we can do the following calculation:

$$\begin{aligned}
{}_1V^1(x) \times {}_2V^1(x) &= \left(\sum_{k_1=x}^{\infty} \Upsilon^1(k_1)\right) \times \left(\sum_{k_2=x}^{\infty} \Upsilon^2(k_2)\right)\\
&= \left(\Upsilon^1(x) + \sum_{k_1=x+1}^{\infty} \Upsilon^1(k_1)\right) \times \left(\Upsilon^2(x) + \sum_{k_2=x+1}^{\infty} \Upsilon^2(k_2)\right)\\
&= (A_1 + A_2)(B_1 + B_2)\\
&= A_1B_1 + A_2B_1 + A_1B_2 + A_2B_2.
\end{aligned}$$

Finally, taking expectations of these terms yields the statement of the theorem.

The above calculation shows that one could apply the same method to moments of arbitrary order, just the formulas would become more involved. For example one can prove the following.

Theorem 9.3.15. *The following recursion formulas hold:*

$$
\begin{aligned}
V_{ijl}^{12}(t) = & \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, \Theta_{i\tilde{i}l}(t)\Theta_{j\tilde{j}l}(t)^2 \\
+ & \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, v_l\, V^1_{\tilde{i},\tilde{l}}(t+1)\Theta_{j\tilde{j}l}(t)^2 \\
+ & \; 2\sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, v_l\, V^1_{\tilde{j},\tilde{l}}(t+1)\Theta_{i\tilde{i}l}(t)\Theta_{j\tilde{j}l}(t) \\
+ & \; 2\sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, v_l^2\, V^1_{\tilde{i},\tilde{l}}(t+1)\, V^1_{\tilde{j},\tilde{l}}(t+1)\Theta_{j\tilde{j}l}(t) \\
+ & \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, v_l^2\, V^2_{\tilde{j},\tilde{l}}(t+1)\Theta_{j\tilde{j}l}(t) \\
+ & \sum_{\tilde{i},\tilde{j},\tilde{l}} p_{i\tilde{i}}(t,t+1)\, p_{j\tilde{j}}(t,t+1)\, p^W_{l\tilde{l}}(t,t+1)\, v_l^3\, V^{12}_{\tilde{i}\tilde{j},\tilde{l}}(t+1),
\end{aligned}
$$

where we used the notation

$$\Theta_{ijk}(t) = a_i^{Pre}(t) + v_k(t)\, a_{ij}^{Post}(t).$$

Exercise 9.3.16. Prove the previous theorem and try to find a general formula for the derivation of the equation given above. For this it is in each case necessary to decompose the sum (9.2) in a part corresponding to the benefits which are due in the current period and in a part for the future periods. For example, if we want to calculate $E\left[XY^2Z^3\right]$, we have to consider

$$(X_1 + X_2)(Y_1 + Y_2)^2(Z_1 + Z_2)^3,$$

where $\bullet_1$ denotes the payments in the current period and $\bullet_2$ denotes the future payments of the corresponding process. How does the recursion formula for $V^{111}_{ijkl}(t)$ look like?

9.4 A model for the interest rate structure

In the previous section we have seen, how a stochastic interest rate influences a portfolio. In this section we will focus on a concrete interest rate model. The interest rate, which we are going to use, is given by the following stochastic differential equation:

$$r_t = r_0 + \int_0^t \eta_t(r_s, s)ds + \int_0^t \sigma(r_s, s)dW,$$

where r_0 denotes the spot rate at time 0, and we assumed that η and σ are of sufficient regularity. Furthermore we assume, that $\hat{Z}$ is an $n-1$ dimensional Itô process. We set $Z = \begin{pmatrix} r \\ \hat{Z} \end{pmatrix}$ for

$$Z_t = Z_0 + \int_0^t \eta_Z(Z_s, s)ds + \int_0^t \sigma_Z(Z_s, s)ds,$$

where $\eta_Z : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_Z : \mathbb{R}^n \to \mathbb{R}^{n\times d}$. Finally we assume that $B_t(s) = B_s(Z_t, t)$ for a B of sufficient regularity.

In this setting the following theorem holds.

Theorem 9.4.1. *B satisfies the stochastic differential equation*

$$B_t(s) = B_0(s) + \int_0^t \eta_B(u, s)B_u(s)du + \int_0^t B_u(s)\sigma_B dW_u,$$

where

$$\begin{aligned} \eta_B(t,s) &= \frac{1}{B_t(s)}\left\{\frac{\partial B^T}{\partial Z}\eta_Z + \frac{\partial B}{\partial t} + \frac{1}{2}trace(\sigma_Z\,\sigma_Z^T\frac{\partial^2 B}{\partial Z^2})\right\}, \\ \sigma_B(t,s) &= \frac{1}{B_t(s)}\sigma_Z^T\frac{\partial B}{\partial Z}. \end{aligned}$$

Proof. This theorem follows by a direct application of Itô's formula (Thm. A.3.13) to the given setting.

Lemma 9.4.2. *Suppose the economic model does not allow arbitrage opportunities. Then there exist functions $(\lambda_i(Z_t, t))_{i=1,\ldots,d}$ which satisfy*

$$\lambda^T\sigma_B(t,s) = \eta_B(t,s) - r_t, \text{ with } 0 \le s < t.$$

Proof. [Vas77] and [CIR85].

Remark 9.4.3. – In the following we assume that the functions λ are independent of the occurrence of the insured event.

– We also assume that they satisfy Novikov's condition:

$$E^{P_1}\left[\exp(\frac{1}{2}\int_0^T \lambda^T\lambda du)\right] < \infty. \tag{9.3}$$

Theorem 9.4.4. *Suppose the economic model for the interest rates does not allow arbitrage and Novikov's condition holds. Then the present value of a zero coupon bond with maturity s at time t is*

$$B_t(s) = E\left[\exp(-\int_t^s r_u du) \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du) \mid \mathcal{G}_t\right].$$

Remark 9.4.5. After proving the above theorem, we can define the equivalent martingale measure by

$$\xi_t = \frac{dQ}{dP} = \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du)$$

and it satisfies:

1. $\xi_t \geq 0$,
2. $E^{P_1}[\xi_t] = 1$,
3. $Q_1(D) = E[\chi_D\, \xi_T]$ for all $D \in \mathcal{G}$.

Proof. We define

$$\begin{aligned} A(u) &= -\int_t^u \left(r_v + \frac{1}{2}\lambda^T \lambda\right) dv - \int_t^u \lambda^T dW, \\ Y(u) &= B_u(s)\, \exp(A(u)). \end{aligned}$$

Next we consider $Y(u) := h(B_u(s), \exp(A(u)))$ and calculate the drift dY, using the fact that

$$dA = -(r + \frac{1}{2}\lambda^T \lambda)du + \lambda^T dW$$

and

$$dB = \eta_B\, B du + B\, \sigma_B^T dW$$

hold. It turns out that this term is equal to zero. Therefore, $Y(u)$ is a martingale and we get:

$$E[Y(s) \mid \mathcal{F}_t] = Y(t).$$

Now, $Y(t) = B_t(s)$ and

$$Y(s) = E\left[\exp(-\int_t^s r_u)\, du \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du) \mid \mathcal{G}_t\right],$$

which implies the result.

Using the previous theorem we can calculate the price of a zero coupon bond.

Lemma 9.4.6.

$$B_t(s) = E^{Q_1}\left[\exp(-\int_t^s r_u du) \mid \mathcal{G}_t\right].$$

Proof. We set

$$\xi_{t,s} = \exp(-\int_t^s \lambda\, dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du).$$

One can calculate the expectation of this, and it turns out to be $E^{P_1}\left[\xi_{t,T}\right] = 1$ for $t \leq T$. Furthermore, we know that

$$\begin{aligned} &E^{Q_1}\left[\exp(-\int_t^s r_u du) \mid \mathcal{G}_t\right] \\ &\quad = E\left[\exp(-\int_t^s r_u du)\xi_{0,T} \mid \mathcal{G}_t\right] / E\left[\xi_{0,T} \mid \mathcal{G}_t\right]. \end{aligned}$$

The individual terms of this can be calculated by

$$\begin{aligned} E^{P_1}\left[\xi_{0,T} \mid \mathcal{G}_t\right] &= \xi_{0,t} \times E^{P_1}\left[\xi_{t,T} \mid \mathcal{G}_t\right] \\ &= \xi_{0,t} \end{aligned}$$

and

$$\begin{aligned} &E^{P_1}\left[\exp(-\int_t^s r_u du)\xi_{0,T} \mid \mathcal{G}_t\right] \\ &\quad = E^{P_1}\left[\exp(-\int_t^s r_u du)\xi_{0,s} E^{P_1}[\xi_{s,T} | \mathcal{G}_s] \mid \mathcal{G}_t\right] \\ &\quad = E^{P_1}\left[\exp(-\int_t^s r_u du)\xi_{0,s} \mid \mathcal{G}_t\right] \\ &\quad = \xi_{0,t} \times E^{P_1}\left[\exp(-\int_t^s r_u du)\xi_{t,s} \mid \mathcal{G}_t\right]. \end{aligned}$$

Finally, we get

$$\begin{aligned} B_t(s) &= \frac{\xi_{0,t} \times E^{P_1}\left[\exp(-\int_t^s r_u du)\xi_{t,s} \mid \mathcal{G}_t\right]}{\xi_{0,t}} \\ &= E^{Q_1}\left[\exp(-\int_t^s r_u du) \mid \mathcal{G}_t\right]. \end{aligned}$$

Thus, we have seen that it is possible to choose $\frac{dP}{dQ}$ as shown above. Now, this equation and an application of Girsanov's theorem yield the following representations with respect to Q:

$$
\begin{aligned}
Z_t &= Z_0 + \int_0^t [\eta_Z \sigma_Z \, \lambda] \, du + \int_0^t \sigma_Z^T d\widehat{W}, \\
B_t(s) &= B_0(s) + \int_0^t r_u \, B_u(s) du + \int_0^t B_u(s) \sigma_B^T d\widehat{W},
\end{aligned}
$$

where $\widehat{W}_s$ is a standard Brownian motion with respect to Q.

9.5 Thiele's differential equation

We use the interest rate model from the previous section. Furthermore we assume that the insurance model is regular (Definition 4.5.6). The mathematical reserve is defined by

$$
V_t^g = \int_t^T \Pi_t^g(u) du,
$$

where

$$
\begin{aligned}
\Pi_t^g(u) &= B_t(u) \times P_t^g(u), \\
P_t^g(u) &= \sum_j p_{gj}(t,u) \left\{ a_j(u) + \sum_{k \neq j} \mu_{jk}(u) a_{jk}(u) \right\}.
\end{aligned}
$$

To derive Thiele's differential equation, we calculate by the chain rule the partial derivative of the mathematical reserve with respect to time:

$$
\begin{aligned}
\frac{\partial}{\partial t} V_t^g &= \frac{\partial}{\partial t} \int_t^T \Pi_t^g(u) du, \\
\int_t^T \frac{\partial}{\partial t} \Pi_t^g(u) du &= \frac{\partial V_t^g}{\partial t} + a_g(t) + \sum_{k \neq g} \mu_{gk}(t) a_{gk}(t).
\end{aligned}
$$

Moreover

$$
B_t(u) = \frac{\Pi_t^g(u)}{P_t^g}.
$$

Now Kolmogorov's theorem implies

$$
\frac{\partial}{\partial t} p_{gj}(t,u) = \sum_{k \neq g} \mu_{gk}(t) \left\{ p_{gj}(t,u) - p_{kj}(t,u) \right\}
$$

and we get

$$\begin{aligned}\frac{\partial}{\partial t} B_t(u) &= \frac{\partial}{\partial t} \frac{\Pi_t^g(u)}{P_t^g} \\ &= \frac{1}{P_t^g} \left(\sum_{k \neq g} \mu_{gk}(t) \left\{ \Pi^k(u) - \Pi^g(u) \right\} + \frac{\partial}{\partial t} \Pi_t^g \right).\end{aligned}$$

But we also know that Π_t^g is a function of Z. Thus the drift term of B_t can, by Itô's formula, be calculated as follows:

$$\begin{aligned}\frac{1}{P_t^g} \Bigg(&\frac{\partial \Pi^g(u)^T}{\partial Z} (\eta_Z - \sigma_Z \, \lambda) + \frac{1}{2} \mathrm{trace}(\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi_t^g(u)}{\partial Z^2}) \\ &+ \sum_{k \neq g} \mu_{gk}(t) \left\{ \Pi^k(u) - \Pi^g(u) \right\} + \frac{\partial}{\partial Z} \Pi_t^g \Bigg).\end{aligned}$$

Using the stochastic differential equation which defines B, we know that this term has also the representation:

$$r_t B_t(u) = \frac{1}{P_t^g} \Pi_t^g(u) \, r_t.$$

Finally, the uniqueness of the drift term implies the following partial differential equation for Π^g:

$$\begin{aligned}&\frac{\partial \Pi^g(u)^T}{\partial Z} (\eta_Z - \sigma_Z \, \lambda) + \frac{1}{2} \mathrm{trace}(\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi_t^g(u)}{\partial Z^2}) \\ &\quad + \sum_{k \neq g} \mu_{gk}(t) \left\{ \Pi^k(u) - \Pi^g(u) \right\} + \frac{\partial}{\partial Z} \Pi_t^g - \Pi^g(u) \, r_u = 0.\end{aligned}$$

This equation yields Thiele's differential equation for the given setting.

Theorem 9.5.1 (Thiele's differential equation). *Let a regular insurance model and the interest rate process defined in Section 9.4 be given. Then the following partial differential equation holds:*

$$\begin{aligned}\frac{\partial V_t^g}{\partial t} &= r_t \, V_t - a_g(t) - \sum_{k \neq g} \mu_{gk}(t) \left\{ a_{gk}(t) + V_t^k - V_t^g \right\} \\ &\quad - \left[\frac{\partial V_t^g}{\partial Z} (\eta_Z - \sigma_Z \, \lambda) + \frac{1}{2} \mathit{trace}(\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi_t^g(u)}{\partial Z^2}) \right].\end{aligned}$$

Remark 9.5.2. The differential equation in the theorem above consists of three parts:

1. classical part: $r_t \, V_t^g - a_g(t) - \sum_{k \neq g} \mu_{gk}(t) \left\{ a_{gk}(t) + V_t^k - V_t^g \right\}$
2. component corresponding to the stochastic interest rate:

$$- \left[\frac{\partial V_t^g}{\partial Z} (\eta_Z - \sigma_Z \, \lambda) + \frac{1}{2} \mathrm{trace}(\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi_t^g(u)}{\partial Z^2}) \right]$$

3. The additional term $-\sigma_Z\lambda$ appears in the formula, since we had to use the equivalent martingale measure instead of the original measure. If one would calculate everything with respect to the original measure, then this term would disappear.

Example 9.5.3. We look again at the interest rate model of Vasiček [Vas77] (Section 9.2). Therein η and σ are given by the functions

$$\begin{aligned}\eta(r_t,t) &= \alpha\,(\rho - r_t),\\ \sigma(r_t,t) &= \sigma.\end{aligned}$$

Thus we have the stochastic differential equation

$$dr_t = \alpha(\rho - r_t)dt + \sigma\,dW_t.$$

If we assume that $\lambda = \lambda(r_t,t)$, then we get

$$B_0(t) = E^Q[\exp(-\int_0^t r_s ds)] = G_t\,\exp(-H_t\,r_t),$$

where

$$\begin{aligned}H_t &= \frac{1-\exp(-\alpha t)}{\alpha},\\ G_t &= \exp((\rho - \frac{\lambda\,\sigma}{\alpha} - \frac{1}{2}(\frac{\sigma}{\alpha})^2)(H_t - t) - \frac{1}{\alpha}(\frac{\sigma H_t}{2})^2)\end{aligned}$$

(compare with [Vas77]).

Now these formulas together with Thiele's differential equation yield for an endowment policy the equation

$$\frac{\partial \Pi}{\partial t} = (\mu_{x+t} + r_t)\Pi_t + \bar{p}_t - c_t\,\mu_{x+t} - \left[\frac{\partial \Pi}{\partial r}(\alpha(\rho - r) - \sigma\lambda) + \frac{1}{2}\sigma^2\frac{\partial^2 \Pi}{\partial r^2}\right].$$

10. Technical analysis

In this chapter we discuss the technical analysis of life insurance policies. This is the actual technical analysis which has to be done at the end of each fiscal year. It is concerned with the question, if the underlying loaded assumptions resemble the reality or if these have to be adapted. This analysis is used for yearly inspections of the current portfolio.

At the end of this chapter we will look at profit testing and the calculation of the present value of the future profits. These techniques are required for the calculation of the profit potential of the insurance products.

10.1 Classical technical analysis

In the technical analysis of an insurance policy one compares, for a given period, the incurred losses with the the funds available to cover these.

First, one needs methods to calculate the effective incurred losses and the premiums for the policies. The latter will be split into a risk premium and a savings premium.

In order to begin the technical analysis one has to define the normal subsequent state for each state $j \in S$. The definition of the subsequent state depends on the model. This state describes the transitions without incurred losses. In the setting of a classical life insurance with state space $S = \{*, \dagger\}$ the subsequent state of $*$ is usually defined to be $*$.

Definition 10.1.1 (Normal subsequent state). *The normal subsequent state is defined by a function*

$$\phi : S \to S, i \mapsto \phi(i),$$

which assigns to each state i, a state which is subsequent to it and for which, according to the policy setup, no payout to the insured is due.

One can derive the following quantities based on the normal subsequent state.

Definition 10.1.2 (Savings premium). *The savings premium for state i during the time interval $]t, t+1]$ is denoted by $\Pi_i^{(s)}$. It can be calculated by the following formula:*

$$\Pi_i^{(s)}(t) = v_t^i \, V_{\phi(i)}(t+1) - V_i(t).$$

The savings premium is the amount of money, which one has to add to the mathematical reserve at time t in order to have the necessary mathematical reserve in the subsequent state $\phi(i)$ at time $t+1$.

Definition 10.1.3 (Technical pension dept). *The regular cash flow or the technical pension dept is*

$$\Pi_i^{(tr)}(t) = a_i^{Pre}(t) + v_t^i \, a_{i\phi(i)}^{Post}(t).$$

For a policy with premiums the technical pension dept $\Pi_i^{(tr)}(t)$ and the premiums coincide.

Definition 10.1.4 (Value at risk and risk premium). *Let $i \in S$ and $j \neq \phi(i)$. Then the value at risk $R_{ij}(t)$ is given by*

$$R_{ij}(t) = V_j(t+1) + a_{ij}^{Post}(t) - (V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t)).$$

It corresponds to the loss, which incurs for a transition $i \rightsquigarrow j$. The risk premium corresponding to the transition $i \rightsquigarrow j$ is given by

$$\Pi_{ij}^{(r)}(t) = p_{ij}(t) \, v_t^i \, R_{ij}(t).$$

Furthermore, $\Pi_i^{(r)}(t) = \sum_{j \neq \phi(i)} \Pi_{ij}^{(r)}(t)$ denotes the total risk premium.

Based on these quantities one can perform a technical analysis by comparing the loss, which effectively incurred in the year, and the risk premium for the corresponding transition. Usually one also considers the loss quotient $\frac{\text{loss}(i \rightsquigarrow j)}{\Pi_{ij}^{(r)}}$.

Example 10.1.5. If for a life insurance the mortality rate is 15% above the effectively observed mortality, then the loss qoutient is roughly 85 % ($= 1/1.15$).

We have seen that one can use the risk premium and the losses to check if the underlying best estimate assumptions fit to the current situation.

The following theorem relates the two kinds of premiums.

Theorem 10.1.6. *The regular cash flow is related to the savings premium and the risk premium by the following equation:*

$$-\Pi_i^{(tr)}(t) = \Pi_i^{(r)}(t) + \Pi_i^{(s)}(t).$$

This decomposition of the premiums is called technical decomposition.

Proof. The statement is proved by a direct calculation:

$$\begin{aligned}
&\Pi_i^{(r)}(t) + \Pi_i^{(s)}(t) \\
&\quad = \sum_{j \in S} v_t \, p_{ij}(t) \left(V_j(t+1) + a_{ij}^{Post}(t) - V_{\phi(i)}(t+1) - a_{i\phi(i)}^{Post}(t) \right) \\
&\qquad\quad + \left(v_t V_{\phi(i)}(t+1) - V_i(t) \right) \\
&\quad = -v_t \left(V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t) \right) \times p_{i\phi(i)}(t) - v_t \left(V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t) \right) \\
&\qquad\quad \times (1 - p_{i\phi(i)}(t)) + v_t V_{\phi(i)}(t+1) - a_i^{Pre}(t) \\
&\quad = -\Pi_i^{(tr)}(t),
\end{aligned}$$

where we used Thiele's difference equations.

Remark 10.1.7. Theorem 10.1.6 states that the premium can be decomposed into a savings premium and a risk premium. This holds for net premiums without any administration charge. The gross premium is composed of three parts: savings premium, risk premium, cost premiums.

10.2 Profit testing

The analysis of an insurance product with respect to a given period is called profit testing. For this one tries to calculate for a period $[t, t+1[$ the return of a policy depending on various parameters.

The return, which is generated by selling a policy, mainly depends on the following effects:

– The major part of the return, which is induced by the insurance contract, is usually due to the return on investment. For example a mathematical reserve of 100,000 USD and a return on investment of 5.1% (technical interest rate: 3.5 %) yields 100 USD per year, with an payed out surplus of 1.5 %.

– Besides the interest surplus also the possible risk surplus is relevant. Also in this case it is possible, that the insured does not receive the total risk surplus.

– Finally one also has to consider the effect of a redemption of the insurance. This yields a return, since in the case of redemption the insured does not get back the total mathematical reserve. He has to accept a reduction. For example suppose that 5 % of all insurances are redeemed each year and that the redemption value is 97% of the mathematical reserve. Then the expected return of a redemption is 0.15% of the mathematical reserve.
– Furthermore, return might be generated by the difference of the charged administration costs and the effective administration costs.

One has to note, that not all of the above returns have to be positive, i.e. some might actually reduce the total return. Consider for example an insurance company which guarantees an interest surplus which is much higher then the effectively realized return. Then clearly the return on investment is negative.

The aim of profit testing is to derive a profile of the returns with respect to time. Furthermore the effects of various parameters on the return profile are determined. Thus a model is used to calculate the expected yearly returns. Then one looks for example at the effect of an 10% increase of the mortality.

Certainly we can not provide in this book a general model for the return of an insurance company, since for this specific details of the each company have to be taken into account. Nevertheless, note the following list of effects which are usually considered:

Profit and loss based on

– return on investment,
– risk,
– redemption,
– acquisition costs,
– administration costs,
– taxes.

10.3 Embedded value

In this section the concept of the embedded value will be introduced. This quantity is used to measure the value of an insurance company, of a division of the company or of a product. Therefore the embedded value is of major interest to investors and to the management of the company. Besides measuring the value of the company this quantity can also be used to determine the salaries of the management or of the sales staff. In fact there are more and more companies which link the salaries of the management to the embedded value.

The concept of the embedded value is very popular in Anglo-Saxon countries. Also in continental Europe this method is becoming increasingly important.

The embedded value is composed of the present value of the future profits of all insurance policies and of the net asset value of the company. The net asset value is the value of the insurance company, which are not bound to outstanding debts, especially they are not part of the mathematical reserves. They originate form parts of the profits which were not distributed to the shareholders and the paid in share caital. Usually one can determine the net asset value with the help of the accounts of the company. Besides the net asset value the second component of the embedded value is the so called pvfp (present value of future profits). This quantity has to be calculated with the help of models. Here the Markov model will be used to derive a model for the return process.

At this point one should note, that the model for the calculation of the pvfp varies for the various types of policies. Thus its calculation strongly depends on the policy. Moreover the complexity of the chosen model might vary strongly with the aims of the calculation, it also depends on regional characteristics and the specific setting of the policy and the insurance company. Therefore the difficulty in calculating the pvfp is not caused directly by the model, but it is due to the diversity of possible models and the wide range of insurance products.

We start with a collection of factors which have an influence on the return of an insurance company. One might have to find a model which incorporates one or more of the following:

– return on investment for bonds and shares,
– return based on risks,
– return based on costs,
– reinsurance,
– redemption,
– surplus.

To incorporate these effects into a Markov model the following terms have to be defined accordingly:

– state space,
– discount functions,
– policy functions.

Given these functions one can easily apply the theory which we developed before. For example one can calculate the distribution or the moments of the pvfp. These quantities capture the temporal variability of the pvfp. Moreover, an analysis could also consider a whole portfolio, and thus it could determine the pvfp and its characteristics for all policies hold by an insurance company.

Now we will give precise definitions of the components of the model. Afterward, we have a look at some examples.

10.3.1 State space

To calculate the pvfp one has to consider a state space which is suitable for the problem at hand. Thus, first of all the state space depends on the underlying insurance policy. It also depends on the modeling techniques which we want to apply, i.e. do we use a deterministic or a stochastic model for the return on investment. In the latter case the state space would be larger. For example, consider a life insurance with state space $\{*, \dagger\}$. Now suppose we want to include the possibility of redemption and a stochastic interest rate model with Markovian interest intensities (state space S_W). Then, for the calculation of the pvfp, we have to use the state space $\{*, \dagger, \ddagger\} \times S_W$, where $\ddagger$ denotes the state of a lapsed policy.

To ensure that the model (state space S_1) for the calculation of the premiums and the model (state space S_2) for the calculation of the embedded value are consistent, we have to assume that a mapping

$$\Phi : S_2 \to S_1, s_2 \mapsto s_1$$

exists. In order to satisfy this assumption it might be necessary to enlarge the state space S_1 with states which are actually not required for the calculation of the premiums.

In the above example the state $\ddagger$ has to be added to the state space S_1, since redemption was not explicitly considered in the calculation of the premiums. Then one can define the mapping $\Phi : S_2 = S_1 \times S_W \to S_1$ by $(i, j) \in S_1 \times S_W \mapsto i$. Thus S_2 can be visualized as a decomposition of the states of S_1, where Φ determines the relation of the states within the models.

10.3.2 Discount function

In this section we want to define the discount function. First of all one should note that there are several approaches to this problem. The classical approach tries to determine the value of the insurance company directly. For this one imagines an investor who wants to get a return on his investment, and the return should be reasonably related to the risks involved. Thus one defines the risk discount rate, which is basically the return of an investment into something (e.g. shares) with a similar risk profile. In fact, the return of shares of the corresponding market is usually a good reference point. Note that this incorporates the fact that the risk discount rate depends on the development of the economy.

The risk discount rate is just one of several possibilities to calculate the discount. For example, if one wants to calculate the expected value of an insurance company and, at the same time, derive its risk profile, then one would have to use a stochastic interest rate model. Here the models which were introduced in the previous chapters are applicable.

In any case, one should note that the pvfp strongly depends on the choice of the discount function.

10.3.3 Definition of the policy functions

Next, given the state space and the discount function, the policy functions will be defined. For this one calculates the profit or loss during a given time interval for every transition between states. Depending on the required precision and on the purpose of the whole model one decides which effects should be incorporated into the model. Usually one focuses on those effects, which have a high impact on the return. Hence, for many policies the focus is on the modeling of the return of investment in shares and bonds and the integration of the risk of redemption.

Now, the question is how to calculate the profit and loss for a given state transition of the policy. For this one has to analyze the cash flows, which occur within a year. It is helpful to assume that the insurance company has the mathematical reserve $V_{\Phi(i)}(t)$ at the beginning of the year and also receives at that time the corresponding premiums. At the end of the year the mathematical reserve $V_{\Phi(j)}(t+1)$ has to be stored and various expenses like administration costs and insurance benefits have to be financed. The following table illustrates this situation:

time	t	$t+1$	comment
t	$V_{\Phi(i)}(t) + P$		income: reserve and premium
t	$-a^{Pre}_{\Phi(i)}(t)$		expenses: pension
t	costs		expenses: costs
t	$\pm \ldots$		other
$t+1$		$-V_{\Phi(j)}(t+1)$	expenses: reserve
$t+1$		$-a^{Post}_{\Phi(i)\Phi(j)}(t)$	expenses: capital
$t+1$		$-$ surplus	expenses: surplus
$t+1$		$\pm \ldots$	other

10.3.4 Examples

Previously we have discussed the general setup for the calculation of the pvfp. Now we will consider concrete examples.

Example 10.3.1. The first example is an endowment policy, which was bought for a single premium of 100,000 USD. We suppose the insured is forty years old and the policy matures in 10 years. Since the policy was bought for a single premium, we suppose there is no reduction in the case of redemption, i.e., the insurance company does not make any additional profit if the policy is redeemed. Furthermore we assume:

– The yearly administration costs are fixed within the policy to 0.6 % of the mathematical reserve plus an additional 100 USD.
– The costs for the contract are 5% of the single premium plus an addtional 200 USD.
– The effective yearly administration costs are 210 USD, and by inflation they increase by 3.25 % per year.
– The effective costs for the contract (commission) are 7 % of the single premium.
– We assume a technical interest rate of 4 % and a return on investment for shares of 5 %. The insured gets 95 % of the interest surplus, the remaining 5 % are profit for the insurer.
– The risk spread is 10 %, and we assume that 90 % of this are credited to the insured.
– We assume a risk discount rate of 7 % and denote the yearly redemption probability by s_x.

Based on these assumptions 130,064 USD is the death benefit and endowment, respectively, for the given policy. Further values of this calculation are shown in Table 10.1, the resulting profits and losses are listed in Table 10.2. An analysis of these values shows a correlation between the return and the age of the policy. Especially the loss in the first year stands out, it is caused by the commission. Figure 10.1 illustrates the same values in form of a graph.

Table 10.1. Technical values of an endowment policy

age	q_x	${}_1p_x$	death benefit	mathematical reserve	cost premium	risk premium
40	0.0011	0.9989	130064	100000	5,950.00	35.04
41	0.0012	0.9988	130064	97,12	736.87	34.46
42	0.0013	0.9987	130064	100927	755.56	33.39
43	0.0015	0.9985	130064	104149	774.89	31.92
44	0.0016	0.9984	130064	107480	794.88	29.81
45	0.0018	0.9982	130064	110927	815.56	27.10
46	0.0020	0.9980	130064	114492	836.95	23.08
47	0.0022	0.9978	130064	118183	859.10	17.44
48	0.0025	0.9975	130064	122004	882.02	9.82
49	0.0027	0.9973	130064	125962	905.77	-

Table 10.2. Profit/loss profile of an endowment policy with a single premium

age	return	s_x	${}_tp_x$	p/l costs	p/l risk	p/l investment	p/l total
40	5%	0.2000	1.0000	-1260.00	3.50	46.11	-1,210.39
41	5%	0.1400	0.7989	371.84	2.75	38.76	413.36
42	5%	0.0800	0.6860	294.48	2.29	34.35	331.12
43	5%	0.0800	0.6302	257.22	2.01	32.57	291.80
44	5%	0.0800	0.5789	221.49	1.73	30.87	254.09
45	5%	0.0800	0.5316	187.17	1.44	29.26	217.88
46	5%	0.0800	0.4882	154.14	1.13	27.74	183.00
47	5%	0.0800	0.4481	122.28	0.78	26.28	149.35
48	5%	0.0800	0.4113	91.51	0.40	24.90	116.82
49	5%	0.0800	0.3773	61.73	-	23.59	85.33
total				122.82	13.54	245.51	**381.87**

For the previous example the present value of the future profits is 381.87 USD, of theses 122.82 USD are due to profits with respect to administration, 13.54 USD are due to the risks and 245.51 US are due to the return on

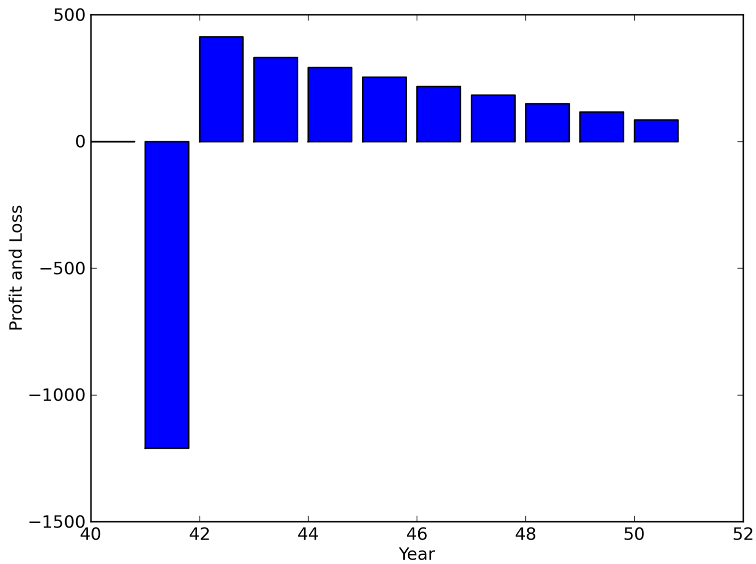


Figure 10.1. Profit/loss profile of a endowment policy

investment in shares. The irr (internal rate of return) for this policy is about 8.7 %. In a next step one can try to analyze the sensitivity of the pvfp with respect to various parameters. Most of the profit in this example is due to the adminstration of the policy. If we are interested in the relation between the present value of the future profits and s_x we just multiply s_x by a factor γ. Then we note, that the profit would increases to 773.44 USD, if one could reduce the redemptions by 20 % ($\gamma = 0.8$). On the other hand a 20 % increase of the redemption probability would yield a profit of only 28.05 USD. This values illustrate the usefulness of the calculations, since they provide a tool for the management of the company to assess the consequencess of certain decisions.

After the calculation of the pvfp for the endowment policy we will now look at a more complicated example. We consider a disability pension as introduced in Section 6.6. This type of insurance is particular difficult to handle, since the probability of becoming disabled depends on the state of the economy and it varies over time. We will develop a model for this type of policy in the following example.

Example 10.3.2. To analyze the influence of various effects one has to generalize the disability pension model. The state $*$ will be decomposed into a

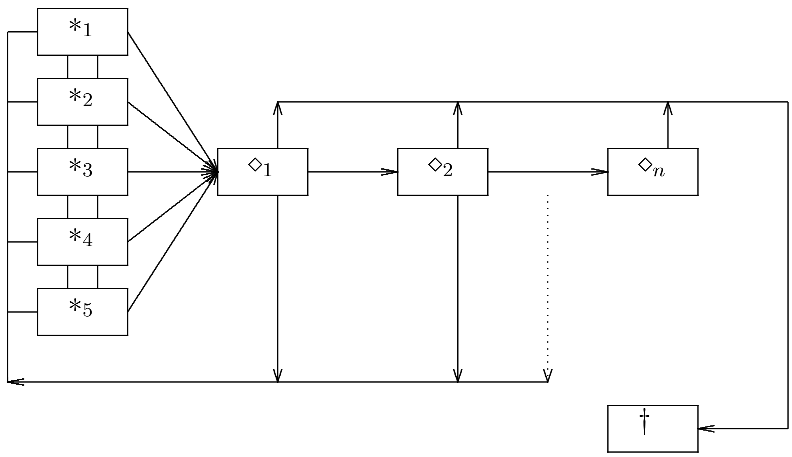


Figure 10.2. Disability model

set of states $*_1$ to $*_m$. For a state $*_k$ the probability of becoming disabled $p_{*_k \diamond_1}(x)$, denoted by i_x, is much larger than i_x^{Ref}, which is the assumed probability of becoming disabled (used for the calculation of the premiums). Additionally we assume that the transitions between the states $*_k$ are given by a homogeneous Markov chain. The disability pension model, which we will use for the calculation of the premium, is the same as in the example in Section 6.6. We work with $m = 6$ states of disability. For the calculation of the pvfp we have to define the corresponding model. We only look at those profits and losses which are due to a change of the probability of becoming disabled. The corresponding state space is depicted in Figure 10.2. Further we make the following assumptions:

– The disability pension is fixed to 10,000 USD and it has to be payed yearly in advance.

– Technical interest rate 4 %. Risk discount rate 8 %.

– We consider the case $m = 6$ and assume that $i_x^k = \gamma_k \times i_x^{\text{Ref}}$, where $\gamma = \{1.5, 1.3, 1.1, 0.9, 0.7, 0.5\}$. This means for example that the effective probability of becoming disabled in state $*_4$ is 90 % of the corresponding probability used for the calculation of the premiums.

– The transition probabilities $p_{*_i,*_k}$ are given by

$$P = \begin{pmatrix} 0.70 & 0.20 & 0.10 & -- & -- & -- \\ 0.10 & 0.70 & 0.10 & 0.10 & -- & -- \\ 0.10 & 0.10 & 0.60 & 0.10 & 0.10 & -- \\ -- & 0.10 & 0.10 & 0.60 & 0.10 & 0.10 \\ -- & -- & 0.10 & 0.10 & 0.70 & 0.10 \\ -- & -- & -- & 0.10 & 0.20 & 0.70 \end{pmatrix}.$$

– In case of a reactivation the new state is one of the six active states. The state is chosen with probability 1/6.

– We consider a 30 year old person and suppose that 65 is the age of maturity of the policy.

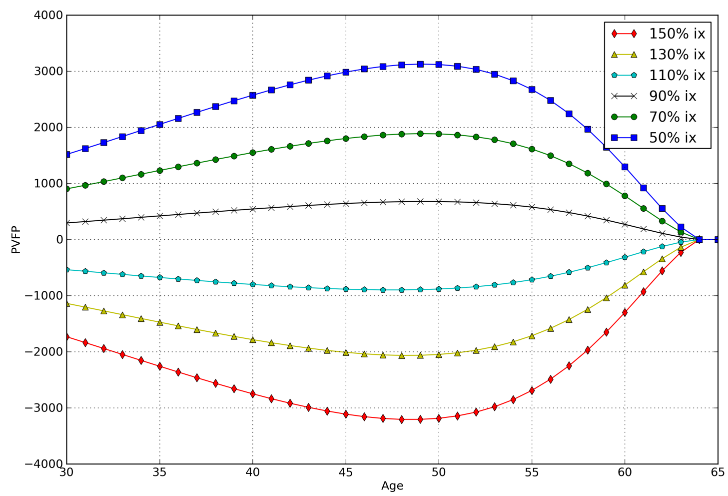


Figure 10.3. Profit/loss profile of a disability insurance

It is useful to look at an example to understand the calculation of the pvfp. The mathematical reserve for a 55 year old active person is 9,607 USD. Then for a 56 year old active person it is 8,352 USD, but in case of disability it is 57,353 USD. Thus, if the man is 55 years old he might make the following loss or profit (at the end of the year):

from	to	profit/loss
$*_k$	$*_l$	1,702 USD
$*_k$	$\dagger$	10,376 USD
$*_k$	$\diamond_1$	-49,183 USD

At the end of the year, the loss which incurred by becoming disabled can be calculated as follows:

	on January 1st	on December 31st
reserve begin of year	9,607 USD	
reserve for loss	-55,147 USD	-57,353 USD
sum begin of year	-45,540 USD	
sum end of year		**- 49,183 USD**

Using this scheme one can calculate the profit or loss for every transition, and Thiele's difference equations yield for $x = 30$ the following values:

state	level	pvfp
$*_1$	$150\% \times i_x$	-1732.50 USD
$*_2$	$130\% \times i_x$	-1138.26 USD
$*_3$	$110\% \times i_x$	-538.06 USD
$*_4$	$90\% \times i_x$	295.12 USD
$*_5$	$70\% \times i_x$	902.39 USD
$*_6$	$50\% \times i_x$	1515.75 USD

This calculations show that the present value of future profits strongly depends on the probability of becoming disabled. Suppose a loss ratio of 90 %, then the pvfp is about 300 USD, which is about 3 % of one pension payment. If the loss ratio increases by 20 %, the present value of the loss is already about 5 % of one pension payment.

This analysis can be used to determine the necessary spread in the premium rate. The dependence of the pvfp on the level of the probability of becoming disabled is shown in Figure 10.3.

For further informations related to the embedded value we refer to [CFO08].

11. Abstract Valuation

In this chapter we look at abstract valuation methods. These methods will be useful, when looking at variable annuities in detail. We also explain the concept of a deflator. During this chapter, we assume that the reader is familiar with the basic concepts of functional analysis such as Banach and Hilbert spaces. For the interested reader we suggest as further reading: [Ped89] for the analytical basics such as integration theory, Banach- and Hilberspaces.

11.1 Framework: Valuation Portfolios

Definition 11.1.1 (Stochastic Cash Flows). *A* Stochastic Cash Flow *is a sequence* $x = (x_k)_{k\in\mathbb{N}} \in L^2(\Omega, \mathcal{A}, P)^{\mathbb{N}}$*, which is* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *adapted.*

Definition 11.1.2 (Regular Stochastic Cash Flows). *A* Regular Stochastic Cash Flow x with respect to $(\alpha_k)_{k\in\mathbb{N}}$*, with* $\alpha_k > 0 \forall k$ *is a stochastic cash flow such that*

$$Y := \sum_{k\in\mathbb{N}} \alpha_k \, X_k \in L^2(\Omega, \mathcal{A}, P).$$

We denote the vector space of all regular cash flows by $\mathcal{X}$*.*

Remark 11.1.3.
1. We note that for all $n \in \mathbb{N}_0$ the image of

$$\psi : L^2(\Omega, \mathcal{A}, P)^n \to \mathcal{X}, (x_k)_{k=0,\dots n} \mapsto (x_0, x_1, \dots, x_n, 0, 0 \dots)$$

   is a sub-space of $\mathcal{X}$.
2. $\mathcal{X}$ has been defined this way in order to capture cash flow streams where the sum of the cash flows is infinite with a finite present value. In this set up α_k can be interpreted as a majorant of the price of the payment 1 at time k.

Theorem 11.1.4. *1. For* $x, y \in \mathcal{X}$*, we define the scalar product as follows:*

$$\begin{aligned} < x, y > &= \sum_{k \in \mathbb{N}} < \alpha_k\, x_k, \alpha_k\, y_k > \\ &= E[\sum_{k \in \mathbb{N}} \alpha_k^2\, x_k\, y_k], \end{aligned}$$

and remark that the scalar product exists as a consequence of the Cauchy-Schwartz inequality .

2. *$\mathcal{X}$ equipped with the above defined scalar product is a Hilbert space with norm $||x|| = \sqrt{< x, x >}$.*

Proof. We leave the proof of this proposition to the reader.

In a next step we introduce the concept of a positive valuation functional and we closely follow [Büh95].

Definition 11.1.5 (Positivity). 1. *$x = (x_k)_{k \in \mathbb{N}} \in \mathcal{X}$ is called* positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$. In this case we write $x \geq 0$.*

2. *$x = (x_k)_{k \in \mathbb{N}} \in \mathcal{X}$ is called* strictly positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$ and there exists a $k \in \mathbb{N}$, such that $x_k > 0$ with a positive probability. In this case we write $x > 0$.*

Definition 11.1.6 (Positive functionals). *$Q : \mathcal{X} \to \mathbb{R}$ is called a* positive, continuous and linear functional *if the following hold true:*

1. *If $x > 0$, we have $Q[x] > 0$.*
2. *If $x = \lim_{n \to \infty} x_n$, for $x_n \in \mathcal{X}$ we have $Q[x] = \lim_{n \to \infty} Q[x_n]$.*
3. *For $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{R}$ we have $Q[\alpha x + \beta y] = \alpha Q[x] + \beta Q[y]$.*

Theorem 11.1.7 (Riesz representation theorem). *For Q a positive, linear functional as defined before, there exists $\phi \in \mathcal{X}$, such that*

$$Q[y] \quad = \quad < \phi, y > \forall y \in \mathcal{X}.$$

Proof. This is a direct consequence of Riesz representation theorem of continuous linear functionals of Hilbert spaces.

Definition 11.1.8 (Deflator). *The $\phi \in \mathcal{X}$ generating $Q[\bullet]$ is called* deflator.

Theorem 11.1.9. *For a positive functional $Q : \mathcal{X} \to \mathbb{R}$, with deflator $\psi \in \mathcal{X}$ we have the following:*

1. *$\phi_k > 0$ for all $k \in \mathbb{N}$.*

2. ϕ is unique.

Proof. 1. Assume $\phi_k = 0$. In this case we have $Q[(\delta_{kn})_{n\in\mathbb{N}}] = 0$ which is a contradiction.

2. Assume $Q[y] =< \phi, y >=< \phi^*, y >$ for all $y \in \mathcal{X}$. In this case we have $< \phi - \phi^*, y >= 0$, in particular for $y = \phi - \phi^*$. Hence we have $||\phi - \phi^*|| = 0$.

Definition 11.1.10 (Projections). *For $k \in \mathbb{N}$ we define the following projections:*

1. *$p_k : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\delta_{kn}\, x_n)_{n\in\mathbb{N}}$, the projection on the k-th coordinate.*
2. *$p_k^+ : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\chi_{k\leq n}\, x_n)_{n\in\mathbb{N}}$, the projection starting on the k-th coordinate.*

Definition 11.1.11 (Valuation at time t, pricing functionals). *For $t \in \mathbb{N}$ we define the valuation of $x \in \mathcal{X}$ at time t by*

$$Q_t[x] \quad = \quad Q[x|\mathcal{F}_t] = \frac{1}{\phi_t}\, E[\sum_{k=0}^{\infty} \phi_k\, x_k | \mathcal{F}_t]$$

In the same sense as for mathematical reserves we define the value of the future cash flows at time t by

$$Q_t^+[x] \quad = \quad Q[p_t^+(x)].$$

The operators Q_t and Q_t^+ are called pricing functionals.

Definition 11.1.12 (Zero Coupon Bonds). *The Zero Coupon Bond $\mathcal{Z}_{(k)} = (\delta_{kn})_{n\in\mathbb{N}}$ is an element of $\mathcal{X}$. We remark that*

$$\pi_0\left(\mathcal{Z}_{(t)}\right) \quad = \quad Q[\mathcal{Z}_{(t)}] = E[\phi_t].$$

Theorem 11.1.13. *The cash flow $x = (x_k)_{k\in\mathbb{N}}$ in the discrete Markov model (cf. proposition 4.7.3) on a finite time interval $T \subset \mathbb{N}$ is given by:*

$$x_k \quad = \quad \sum_{(i,j)\in S^2} \Delta N_{ij}(k-1) a_{ij}^{Post}(k-1) + \sum_{i\in S} I_i(k) a_i^{Pre}(k),$$

where we assume that $\Delta N_{ij}(-1) = 0$.

Proof. The form of the cash flow follows from the calculations of the earlier chapters. It remains to show that $(x_k)_{k\in\mathbb{N}}$ is in L^2. This is however easy, since the benefit functions and the state space are finite. Given the fact that also the time considered for a life insurance is finite, the required property follows.

Theorem 11.1.14. *For $x \in \mathcal{X}$, as defined above we have the following:*

1. $E[\Delta N_{ij}(s)|X_t = k] = p_{ki}(t,s)p_{ij}(s,s+1)$,
2. $E[I_i(s)|X_t = k] = p_{ki}(t,s)$,
3. $E[x_s|X_t = k] =$

$$\sum_{(i,j)\in S^2} p_{ki}(t,s-1)p_{ij}(s-1,s)a_{ij}^{Post}(s-1) + \sum_{i\in S} p_{ki}(t,s)a_i^{Pre}(s),$$

and we assume that $p_{ki}(t,s-1) = 0$ if $t = s$.

Proof. We leave the proof of this proposition to the reader as an exercise.

Definition 11.1.15. *The abstract vector space of financial instruments we denote by $\mathcal{Y}$. Elements of this vector space are for example all zero coupon bonds, shares, options on shares etc.*

Remark 11.1.16. – Link to the arbitrage free pricing theory: If we assume that Q does not allow arbitrage we are in the set up of chapter 8. In proposition 8.2.15 we have seen that $\pi(X) = E^Q[\beta_T X]$, where β_T denotes the risk free discount rate. In the context of the above, we would have $\pi_0(x) = Q[x] = E^P[\phi_T x]$. Hence we can identify $\phi_T = \frac{dQ}{dP}\beta_T$. In consequence we can interpret a deflator as a discounted Radon-Nikodym density with respect to the two measures P and Q.

– In the same sense the concepts of definition 11.1.11 have a lot in common with the definition of the present values of a cash flow stream as defined in chapter 4.6.

– For the interest rate model in section 9.4 we know that

$$\begin{aligned} B_t(s) &= \pi_t\left(\mathcal{Z}_{(s)}\right) \\ &= E\left[\exp(-\int_t^s r_u du) \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du) \mid \mathcal{G}_t\right], \end{aligned}$$

and hence we have actually calculated the corresponding deflators as follows:

$$\phi_t(s) = \exp(-\int_t^s r_u du) \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du).$$

Theorem 11.1.17. *Let Q be a positive, continuous functional $Q : \mathcal{X} \to \mathbb{R}$, and assume $Q[\bullet] =< \phi, \bullet >$, with $\phi = (\phi_t)_{t\in\mathbb{N}}$ $\mathbb{F}$ – adapted. In this case $(\phi_t\, Q_t[x])_{t\in\mathbb{N}}$ is an $\mathbb{F}$-martingale over P.*

Proof. Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ and the projection property of the conditional expectation we have

$$\begin{aligned} E^P[\phi_{t+1} \, Q_{t+1}[x] | \mathcal{F}_t] &= E^P[E^P[\sum_{k \in \mathbb{N}} \phi_k x_k \, Q_{t+1}[x] | \mathcal{F}_{t+1}] | \mathcal{F}_t] \\ &= E^P[\sum_{k \in \mathbb{N}} \phi_k x_k \, Q_{t+1}[x] | \mathcal{F}_t] \\ &= \phi_t \, Q_t[x]. \end{aligned}$$

Example 11.1.18 (Replicating Portfolio Mortality). In this first example we consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. For sake of simplicity we assume that the yearly mortality is $(1 + \frac{x-50}{10} \times 0.1)\%$. We assume that the death benefit amounts to 100,000 USD and we assume that the premium has been determined with an interest rate $i = 2\%$. In this case the premium amounts to $P = 1394.29$. The replicating portfolio in the sense of expected cash flows at inception is therefore given as follows (cf proposition 11.1.14). We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively, and remark the the use of arbitrary yield curves does not imply additional complexity.

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
51	$\mathcal{Z}_{(1)}$	1000.00	-1380.34	-380.34	-372.88	-365.71
52	$\mathcal{Z}_{(2)}$	1089.00	-1365.16	-276.16	-265.43	-255.32
53	$\mathcal{Z}_{(3)}$	1174.93	-1348.77	-173.84	-163.81	-154.54
54	$\mathcal{Z}_{(4)}$	1257.56	-1331.24	-73.67	-68.06	-62.97
55	$\mathcal{Z}_{(5)}$	1336.69	-1312.60	24.09	21.82	19.80
56	$\mathcal{Z}_{(6)}$	1412.12	-1292.91	119.20	105.85	94.21
57	$\mathcal{Z}_{(7)}$	1483.67	-1272.23	211.44	184.07	160.67
58	$\mathcal{Z}_{(8)}$	1551.18	-1250.60	300.57	256.54	219.62
59	$\mathcal{Z}_{(9)}$	1614.50	-1228.09	386.41	323.33	271.48
60	$\mathcal{Z}_{(10)}$	1673.52	–	1673.52	1372.87	1130.57
Total					**0.00**	**-336.47**

Exercise 11.1.19 (Replicating Portfolio Disability). Consider a disability cover and calculate the replicating portfolios for a deferred disability annuity and a disability in payment.

11.2 Cost of Capital

In section 11.1 we have seen how to abstractly valuate $x \in \mathcal{X}$ by means of a pricing functional Q . For some financial instruments $y \in \mathcal{Y}^\star$ we can directly observe $Q[y]$ such as for a lot of zero coupons bonds $\mathcal{Z}_{(\bullet)}$. On the other hand this is not always possible.

Definition 11.2.1. *We denote by $\mathcal{Y}^\star$ the set of all stochastic cash flows in $x \in \mathcal{Y}$ such that $Q[x]$ is observable. With $\tilde{\mathcal{Y}} = span < \mathcal{Y}^\star >$ we denote the vector space generated by $\mathcal{Y}^\star$ and we define:*

1. *$x \in \mathcal{Y}^\star$ is called of level 1.*
2. *$x \in \tilde{\mathcal{Y}}$ is called of level 2.*
3. *$x \in \mathcal{Y} \setminus \tilde{\mathcal{Y}}$ is called of level 3.*

Remark 11.2.2. It is clear that the model uncertainty and the difficulties to value assets or liabilities increases from level 1 to level 3. Since we are interested in market values only the valuation of level 1 assets and liabilities are really reliable. For level 2 assets and liabilities on has to find a sequence of $x_n = \sum_{k=1}^n \alpha_k \, e_k$ with $e_k \in \mathcal{Y}^\star$ such that $x = \lim_{n\to\infty} x_n$. Since we <u>assume</u> that Q is linear and continuous we can calculate

$$\begin{aligned} Q[x] &= \lim_{n\to\infty} Q[x_n] \\ &= \lim_{n\to\infty} \sum_{k=1}^n Q[\alpha_k \, e_k] \\ &= \lim_{n\to\infty} \sum_{k=1}^n \alpha_k \, Q[e_k]. \end{aligned}$$

For level 3 assets and liabilities the situation is even more difficult, since there is no obvious way to do it. The best, which we can be done is to <u>define</u> $\tilde{Q}[x]$ such that $\tilde{Q}[x] = Q[x] \forall x \in \mathcal{Y}$ and <u>hope</u> that $\tilde{Q}[x] \approx Q[x]$ for the $x \in \mathcal{Y}$ we want to valuate. In most cases such $\tilde{Q}[\bullet]$ are based on first economic principles. In the following we want to see how the *Cost of Capital* concept works for insurance liabilities and how we can concretely implement it.

Definition 11.2.3 (Utility Assumption). *If we have $x, y \in L^2(\Omega, \mathcal{A}, P)^+$, with $x = E[y]$. A rational investor would normally prefer x, since there is less uncertainty. The way to understand this, is by using utility functions. For $x \in L^2(\Omega, \mathcal{A}, P)^+$ and u a concave function, the utility of x is defined as $E[u(x)]$. The idea behind utilities is that the first 10,000 USD are higher valued than the one 10,000 USD from 100,000 USD to 110,000 USD. Hence the increase of utility per fixed amount decreases if amounts increase. As a* *consequence of the Jensen's inequality, we see that the utility of a constant amount is higher than the utility of a random payout with the same expected value.*

Definition 11.2.4. *Let $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ be an insurance cash flow, for example generated by a Markov model.*

1. *In this case we define the expected cash flows by*
$$CF(x) = (E[x_k])_{k\in\mathbb{N}}.$$
2. *The corresponding portfolio of financial instruments in the vector space $\mathcal{Y}$ we define by*
$$VaPo^{CF}(x) = \sum_{k\in\mathbb{N}} CF(x)_k \mathcal{Z}_{(k)} \in \mathcal{Y}$$
3. *By $R(x)$ we denote the residual risk portfolio given by*
$$\begin{aligned} R(x) &= x - VaPo^{CF}(x) \\ &= \sum_{k\in\mathbb{N}} (x_k - CF(x)_k)\, \mathcal{Z}_{(k)} \in \mathcal{Y} \end{aligned}$$
4. *For a given $x \in \mathcal{X}$ we denote by $VaPo^*(x)$ an approximation $y \in \tilde{\mathcal{Y}}$ of x, such that $||x - VaPo^*(x)|| \leq ||x - VaPo^{CF}(x)||$.*

Since we are sometimes interested in conditional expectations, we will also use the following notations for $A \in \mathcal{A}$:
$$\begin{aligned} CF(x \,|\, A) &= (E[x_k \,|\, A])_{k\in\mathbb{N}}, \\ VaPo^{CF}(x \,|\, A) &= \sum_{k\in\mathbb{N}} CF(x \,|\, A)_k \mathcal{Z}_{(k)} \in \mathcal{Y}, \end{aligned}$$

Theorem 11.2.5. *The value of $x \in \mathcal{X}$ can be decomposed in*
$$Q[x] = Q[VaPo^{CF}(x)] + Q[R(x)],$$
and we have
$$Q[VaPo^{CF}(x)] \geq Q[x]$$
if we use the utility assumption.

Remark 11.2.6. 1. We will denote $x \in \mathcal{X}$ with $x \leq 0$ as a liability. Proposition 11.2.5 hence tells us that we need to reserve more than $Q[VaPo^{CF}(x)]$ for this liability as a consequence of the corresponding uncertainty.

2. A risk measure is a functional (not necessarily linear) $\psi : \mathcal{X} \to \mathbb{R}$ which aims to measure the capital needs in an adverse scenario. There are two risk measures, which are commonly used the *Value at Risk* and the *Expected Shortfall* to a given quantile $\alpha \in \mathbb{R}$. The value at risk (VaR) is defined as the corresponding quantile minus the expected value. The expected shortfall is the conditional expectation of the random variable given a loss bigger than the corresponding loss, again minus the expected value. We can hence speak about a 99.5% VaR or a 99% expected shortfall. It is worthwhile to remark that these two concepts are normally applied to losses. Hence in the context introduced above one would strictly speaking calculating the VaR$(-x)$, when considering $x \in \mathcal{X}$. Furthermore in a lot of applications, such as Solvency II, we assume that there is a Dirac measure (aka stress scenario), which just represents the corresponding VaR-level for example. So concretely the stress scenarios, which are used under Solvency II should in principle represent the corresponding point (Dirac) measures at to the confidence level 99.5 %. In the concrete set up, one would for example assume that $q_x(\omega) \in L^2(\Omega, \mathcal{A}, P)$ is a stochastic mortality and one would define the $A, B \in \mathcal{A}$, as the corresponding probabilities in the average and in the tail. In consequence for a policy $x \in \mathcal{X}$, we would have two replicating portfolios, namely $VaPo^{CF}(x \mid A)$ for the average and $VaPo^{CF}(x \mid B)$ for the stressed event according to the risk measure chosen. The corresponding required risk capital is then given (in present value terms) by $Q[VaPo^{CF}(x \mid B) - VaPo^{CF}(x \mid A)]$.

Definition 11.2.7 (Required Risk Capital). *For a risk measure ψ_α such as VaR or expected shortfall to a security level α we define the required risk capital at time $t \in \mathbb{N}$ by*

$$RC_t(x) \quad = \quad \psi_\alpha(p_k(x - VaPo^{CF}(x))).$$

Remark 11.2.8. 1. If we use $VaR_{99.5\%}$ the required risk capital at time t corresponds to the capital needed to withstand a 1 in 200 year event.

2. The definition above could apply to individual insurance policies, but is normally applied to insurance portfolios $\tilde{x} = \sum_{k=1}^{n} x_k$, where x_k are the individual insurance policies. As we have seen in section 9.3 the pure diversifiable risk disappears for $n \to \infty$.

3. What is more material than the diversifiable risk is the risk, which affects all of the individual insurance policies at the same time, such as a pandemic event, where the overall mortality could increase by 1 % in a certain year such as 1918.

Definition 11.2.9 (Cost of Capital). *For a unit cost of capital $\beta \in \mathbb{R}^+$ and an insurance portfolio $\tilde{x} \in \mathcal{X}$, we define:*

1. *The present value of the required risk capital by*

$$PVC(\tilde{x}) \quad = \quad Q[\sum_{k \in \mathbb{N}} RC_t(\tilde{x}) \mathcal{Z}_{(k)}].$$

2. *The cost of capital $CoC(\tilde{x})$ is given by:*

$$CoC(\tilde{x}) \quad = \quad \beta \times PVC(\tilde{x}),$$

and $\tilde{Q}$ is defined by $\tilde{Q}[\tilde{x}] = Q[VaPo^{CF}(\tilde{x})] + \beta PVC(\tilde{x})$.

Remark 11.2.10. 1. The concept as defined before is somewhat simplified, since one normally assumes that the required capital C from the shareholder is $\alpha \times C$ after tax and investment income on capital, Assume a tax-rate κ and a risk-free yield of i. In this case we have

$$\alpha \times C = i \times (1 - \kappa) \times C + \beta \times C,$$

and hence $\beta = \alpha - i \times (1 - \kappa)$. In reality the calculation can still become more complex since we discount future capital requirements risk-free and because of the fact that the interest rate i is not constant. In order to avoid these technicalities, we will assume for this book that i is constant.

2. We remark $\tilde{Q}[\tilde{x}]$ is not uniquely determined, but depends on a lot of assumptions such as $\psi\alpha$, α, β, ...
3. For the moment we did not yet see how to actually model $\tilde{x}$ and we remark that one is normally focusing on the non-diversifiable part of the risks within $\tilde{x}$.

Example 11.2.11. We continue with example 11.1.18 and we assume that the risk capital is given by a pandemic event where $\Delta q_x = 1\%$ for all ages. This roughly corresponds to the increase in mortality of 1918 as a consequence of the Spanish flu pandemic. The aim of this example is to calculate the required risk capital and the market value of this policy based on the cost of capital method using $\beta = 6\%$. The required risk capital in this context can be calculated as $\Delta q_x \times 100,000$ and we get the following results:

Age	Unit	Units for Risk Capital	Units for Benefits	Total Units	$-\tilde{Q}[x]$ $i=2\%$	$-\tilde{Q}[x]$ $i=4\%$
50	$\mathcal{Z}_{(0)}$	1000.00	-1394.28	-1334.28	-1334.28	-1334.28
51	$\mathcal{Z}_{(1)}$	990.00	-380.34	-320.94	-314.65	-308.60
52	$\mathcal{Z}_{(2)}$	979.11	-276.16	-217.41	-208.97	-201.01
53	$\mathcal{Z}_{(3)}$	967.36	-173.84	-115.80	-109.12	-102.95
54	$\mathcal{Z}_{(4)}$	954.78	-73.67	-16.38	-15.14	-14.00
55	$\mathcal{Z}_{(5)}$	941.41	24.09	80.57	72.98	66.22
56	$\mathcal{Z}_{(6)}$	927.29	119.20	174.84	155.25	138.18
57	$\mathcal{Z}_{(7)}$	912.45	211.44	266.19	231.73	202.28
58	$\mathcal{Z}_{(8)}$	896.94	300.57	354.39	302.47	258.95
59	$\mathcal{Z}_{(9)}$	880.80	386.41	439.26	367.55	308.61
60	$\mathcal{Z}_{(10)}$	–	1673.52	1673.52	1372.87	1130.57
Total					**520.69**	**143.98**

We remark that the value of the policy at inception becomes positive, which means nothing else, that the insurance company does need equity capital to cover the economic loss. It is obvious that this is the case for $i = 2\%$, since the premium principle did not allow for a compensation of the risk capital. More interestingly even at the higher interest rate the compensating effect is not big enough to turn this policy into profitability.

Exercise 11.2.12. In the same sense as for the mortality example calculate the respective risk capitals and the $\tilde{Q}$ for a disability cover.

11.3 Inclusion in the Markov Model

In this section we want to have a look how we could concretely use the recursion technique for the calculation of the cost of capital in a Markov chain similar environment. In order to do that we look at an insurance policy with a term of one year.

We assume that we have a mortality of q_x in case of a "normal" year with a probability of $(1-\alpha)$ and an excess mortality of Δq_x in an extreme year with probability α. We denote with $\Gamma = \frac{q_x + \Delta q_x}{q_x}$. Furthermore we assume a mortality benefit of 100,000. In this case we get the following by some simple calculations:

$$\begin{aligned}
VaPo^{CF}(x) &= (\delta_{1k}(q_x + \alpha(\Gamma-1)q_x \times 100000))_{k\in\mathbb{N}} \\
RC_1(x) &= (\delta_{1k}(1-\alpha)(\Gamma-1)q_x \times 100000)_{k\in\mathbb{N}} \\
\tilde{Q}[x] &= Q[(\delta_{1k}(q_x + \alpha(\Gamma-1)q_x \times 100000 + \\
&+ \beta(1-\alpha)(\Gamma-1) \times 100000))_{k\in\mathbb{N}}]
\end{aligned}$$

We see that the price of this insurance policy with only payments at time 1 can be decomposed into a part representing best estimate mortality:

$$\delta_{1k}\{q_x\,(1+\alpha\,(\Gamma-1))\},$$

where we can arguably say that this $\tilde{q}_x = q_x\,(1+\alpha\,(\Gamma-1))$ is our actual best-estimate mortality. On top of that we get a charge for the excess mortality Δq_x with an additional cost of β. Hence we get the following:

1. There is a contribution to the reserve from the people surviving the year with a probability p_x.
2. There is a contribution to the reserve from the people dying in normal years with probability q_x and the defined benefit $a_{*\dagger}^{\text{post}}$, and
3. There is finally a contribution of the people dying in extreme years with probability Δq_x and the additional cost of defined benefit of $\beta \times a_{*\dagger}^{\text{post}}$.

The interesting fact is that we can actually use the same recursion of the reserves for the Markov chain model as in proposition 4.7.3 with the exception that now the “transition probabilities” do not fulfil anymore the requirement that their sum equals 1. However this method provides a pragmatic way to implement the cost of capital in legacy admin systems.

The main problem for the determining of the corresponding Markov chain model is the underlying stochastic mortality model. For the QIS 5 longevity model a similar calculation can be used. In this model it is assumed that the mortality drops by 25 % in an extreme scenario. Hence the calculation goes along the following process:

1. Determine $x_1 = VaPo^{CF}(\tilde{x})$.
2. Determine $x_2 = VaPo^{CF}(\tilde{x})$ for stressed mortality.
3. $\tilde{Q}[x] = Q[x_1] + \beta\,Q[x_2 - x_1]$

Example 11.3.1. In this example we want to revisit the exercise 11.1.18 and we want again to calculate the market value of the insurance liability, but this time with the recursion. We get the following results:

Age	Benefit Normal	Benefit Premium	Excess Risk	Math Res. $i = 2\%$	Value $i = 2\%$	Value $i = 4\%$
50	100000	-1394.28	6000	**0.00**	**520.69**	**143.98**
51	100000	-1394.28	6000	426.43	901.09	542.82
52	100000	-1394.28	6000	765.56	1193.21	861.67
53	100000	-1394.28	6000	1015.22	1394.79	1096.96
54	100000	-1394.28	6000	1172.95	1503.20	1244.68
55	100000	-1394.28	6000	1235.88	1515.45	1300.33
56	100000	-1394.28	6000	1200.79	1428.16	1258.89
57	100000	-1394.28	6000	1064.00	1237.49	1114.74
58	100000	-1394.28	6000	821.42	939.18	861.64
59	100000	-1394.28	6000	468.45	528.45	492.63
60				0	0	0

We remark that this calculation was much faster to calculate since it is based on Thiele's difference equation for the mathematical reserves, and we get at the same time the corresponding results for the classical case and also for the case using the cost of capital approach.

As seen in the calculation above there is a small second order effect, which we can detect, when looking more closely. The results below correspond to the 2% valuation:

```
Direct Method  520.698380872792
Recursion      520.698380872793
```

Exercise 11.3.2. Perform the corresponding calculation for the disability example.

11.4 Asset Liability Management

Until now we have looked only at insurance liabilities as an $x \in \mathcal{X}$. An insurance company needs to cover its insurance liabilities $l = \sum x\iota \in \mathcal{X}$ with corresponding assets, which are also elements in $\mathcal{X}$.

Definition 11.4.1 (Assets and Liabilities). *An $x \in \mathcal{X}$ with a valuation functional Q is called*

1. *an asset if $Q[x] \geq 0$ and*
2. *a liability if $Q[x] \leq 0$.*

Definition 11.4.2 (Insurance balance sheet). *An insurance balance sheet consists of a set of assets $(a_i)_{i \in I}$ and a set of liabilities $(l_j)_{j \in J}$. The equity of an insurance balance sheet is defined as*

$$e = \sum_{i \in I} a_i + \sum_{j \in J} l_j.$$

The insurance entity is called bankrupt if $Q[e] < 0$.

Definition 11.4.3. *In an insurance market, each insurance company is required to hold an adequate amount of risk capital in order to absorb shocks. In order to do that, the regulator defines a risk measure $\psi\alpha$ to a security level α. In this context an insurance company is called solvent if:*

$$Q[e] \geq \psi\alpha(e).$$

Remark 11.4.4. Note that an insurance regulator may not want to use a market consistent approach. Never the less the above definition can be used, be suitably adjust ψ.

Definition 11.4.5 (Asset Liability Management). *Under Asset Liability Management we understand the process of analysing $(l_j)_{j \in J}$ and the (dynamic) management of $(a_i)_{i \in I}$ in order to achieve certain targets , such as remaining solvent.*

Definition 11.4.6. *For an insurance liability $l \in \mathcal{X}$ an asset portfolio $(a_i)_{i \in I}$ is called:*

1. matching *if $\sum_{i \in I} a_i + l = 0$, and*
2. cash flow matching *if $\sum_{i \in I} a_i + VaPo^{CF}(l) = 0$.*

Remark 11.4.7. We remark that is normally not feasible to do a perfect matching, and hence one normally uses a cash flow matching to a achieve a proxy for a perfect match. We also remark that in this case the shareholder equity needs still be able to absorb the basis risk $l - VaPo^{CF}(l)$.

Definition 11.4.8 (Duration). *The duration for an $x \in \mathcal{X}$ with $x = \sum_{i \in \mathbb{N}} \alpha_i \, \mathcal{Z}_{(i)}$ and $\alpha_i \geq 0$ is defined by*

$$d(x) = \frac{Q[\sum_{i \in \mathbb{N}} \alpha_i \times i \times \mathcal{Z}_{(i)}]}{Q[\sum_{i \in \mathbb{N}} \alpha_i \times \mathcal{Z}_{(i)}]}$$

We say that an asset portfolio $(a_i)_{i \in I}$ is duration matching a liability l if the following two conditions are fulfilled:

1. *$Q[\sum_{i \in I} a_i + l] = 0$, and*
2. *$d(\sum_{i \in I} a_i) = d(-l)$.*

Example 11.4.9. In this example we want to further elaborate on the example 11.1.18 and we want to see how the replicating scenario changes in case a pandemic occurs in year three, with an excess mortality of 1 %. We want also to have a look on what risk is implied in this, assuming that the pandemic at the same time leads to a reduction of interest rates down from 2% to 0.5 %. Finally we want to see an example how we could do a perfect cash flow matching portfolio and duration matched portfolio.

Definitions We assume that $A \in \mathcal{A}$ represents the information that we have going to have average mortality after year 3 and three and that the person survived until then (year 2). In the same sense we assume that $B \in \mathcal{A}$ represents the same as A but with the exception that we assume a pandemic event in the year 3 with an average excess mortality of 1%. For simplicity reasons (to avoid notation) we use $x, y \in \mathcal{X}$ as abbreviations for the corresponding conditional random variables.

Calculation of the Replicating Portfolios In a first step we will calculate the replicating portfolios (starting at time 2) with respect to both A and B. Doing this we get the following results for case A:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
52	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
53	$\mathcal{Z}_{(1)}$	1200.00	-1377.55	-177.55	-174.07	-170.72
54	$\mathcal{Z}_{(2)}$	1284.40	-1359.64	-75.24	-72.32	-69.57
55	$\mathcal{Z}_{(3)}$	1365.21	-1340.61	24.60	23.18	21.87
56	$\mathcal{Z}_{(4)}$	1442.25	-1320.50	121.75	112.48	104.07
57	$\mathcal{Z}_{(5)}$	1515.33	-1299.37	215.95	195.59	177.49
58	$\mathcal{Z}_{(6)}$	1584.27	-1277.28	306.99	272.59	242.61
59	$\mathcal{Z}_{(7)}$	1648.95	-1254.29	394.65	343.57	299.90
60	$\mathcal{Z}_{(8)}$	1709.23	–	1709.23	1458.81	1248.91
Total					**765.56**	**460.30**

For case B we get:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
52	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
53	$\mathcal{Z}_{(1)}$	1200.00	-1377.55	-177.55	-174.07	-170.72
54	$\mathcal{Z}_{(2)}$	**2272.40**	-1345.87	**926.52**	890.54	856.62
55	$\mathcal{Z}_{(3)}$	1351.38	-1327.03	24.35	22.95	21.65
56	$\mathcal{Z}_{(4)}$	1427.64	-1307.12	120.51	111.34	103.01
57	$\mathcal{Z}_{(5)}$	1499.97	-1286.21	213.76	193.61	175.70
58	$\mathcal{Z}_{(6)}$	1568.22	-1264.34	303.88	269.83	240.16
59	$\mathcal{Z}_{(7)}$	1632.24	-1241.58	390.65	340.09	296.86
60	$\mathcal{Z}_{(8)}$	1691.91	–	1691.91	1444.03	1236.26
Total					**1704.05**	**1365.28**

We note two things:

– The pandemic happens when the person is aged 53 and we see the impact in $\mathcal{Z}_{(2)}$ at age 54. This has to do with the convention that we assume that the deaths occur at the end on the year, hence just before the person gets 54.

– We see that the difference in reserves amounts to $1704.05 - 765.56 = 938.49$ which represents the economic loss as a consequence of the pandemic. The biggest contributor to this loss is the increased death benefit, e.g. $926.52 - 1284.40 = 962.87$.

Matching asset portfolios Based on the above it is now easy to calculate the cash flow matching portfolio, by just investing the different amounts of liabilities into the corresponding assets, such as buying $24.60\mathcal{Z}_{(3)}$. We remark that consequently we would have to sell $-177.55\mathcal{Z}_{(1)}$. In normal circumstances for mature businesses this will not occur, since it is a consequence that we consider a term insurance policy and not for example an endowment.

Mismatch in case of a pandemic The table below finally shows the cash flow mismatch as a consequence of the pandemic and we see that in this case the present values do not have a big impact since the main difference is at time 1.

Age	Unit	Units Normal	Units Stress	Difference Units	Value $i = 2\%$	Value $i = 0\%$
52	$\mathcal{Z}_{(0)}$	-1394.28	-1394.28	0.00	0.00	0.00
53	$\mathcal{Z}_{(1)}$	-177.55	-177.55	0.00	0.00	0.00
54	$\mathcal{Z}_{(2)}$	-75.24	926.52	1001.77	962.87	1001.77
55	$\mathcal{Z}_{(3)}$	24.60	24.35	-0.24	-0.23	-0.24
56	$\mathcal{Z}_{(4)}$	121.75	120.51	-1.23	-1.13	-1.23
57	$\mathcal{Z}_{(5)}$	215.95	213.76	-2.18	-1.98	-2.18
58	$\mathcal{Z}_{(6)}$	306.99	303.88	-3.11	-2.76	-3.11
59	$\mathcal{Z}_{(7)}$	394.65	390.65	-3.99	-3.48	-3.99
60	$\mathcal{Z}_{(8)}$	1709.23	1691.91	-17.31	-14.78	-17.31
Total					**938.49**	**973.67**

Example 11.4.10 (Lapses). In this example we want to see how lapses can influence the replicating portfolios. In order to do that we have to change the example 11.1.18 a little bit, as follows:

- We consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. For sake of simplicity we assume that the yearly mortality is $(1 + \frac{x-50}{10} \times 0.1)\%$. We assume that the benefit amounts to 100.000 USD and we assume that the premium has been determined with an interest rate $i = 2\%$.
- In this case the premium amounts to $P = 9562.20$.
- In addition the policyholder can surrender the policy at any time and gets back 98 % of the expected future cash flows valued at the pricing interest rate of 2%. We remark here that this is a risk since the surrenders can happen in case the market value of the corresponding units is below the surrender value.
- We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively.

In order to calculate this example we will perform the following steps:

1. Calculation of the cash flow matching portfolio in case of no surrenders.
2. Calculation of the cash flow including lapses with an average lapse rate of 7 %
3. Calculation of the cash flows at time 2, assuming average lapses, lapses at 25 % at time 2.

Calculation of the cash flow matching portfolio in case of no surrenders:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1000.00	-9466.57	-8466.57	-8300.56	-8140.94
52	$\mathcal{Z}_{(2)}$	1089.00	-9362.44	-8273.44	-7952.17	-7649.26
53	$\mathcal{Z}_{(3)}$	1174.93	-9250.09	-8075.16	-7609.40	-7178.79
54	$\mathcal{Z}_{(4)}$	1257.56	-9129.84	-7872.27	-7272.76	-6729.25
55	$\mathcal{Z}_{(5)}$	1336.69	-9002.02	-7665.32	-6942.72	-6300.34
56	$\mathcal{Z}_{(6)}$	1412.12	-8866.99	-7454.87	-6619.71	-5891.69
57	$\mathcal{Z}_{(7)}$	1483.67	-8725.12	-7241.45	-6304.11	-5502.90
58	$\mathcal{Z}_{(8)}$	1551.18	-8576.79	-7025.61	-5996.29	-5133.54
59	$\mathcal{Z}_{(9)}$	1614.50	-8422.41	-6807.90	-5696.55	-4783.14
60	$\mathcal{Z}_{(10)}$	–	88080.30	88080.30	72256.53	59503.90
Total					**0**	**-7368.19**

We remark that the there is considerable value in the policy if we assume no lapses, in case we earn a higher interest rate, such as 4 %.

Calculation of the cash flow matching portfolio in case of 7% surrenders:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1019.67	-8797.22	-7777.54	-7625.04	-7478.40
52	$\mathcal{Z}_{(2)}$	1594.01	-8084.65	-6490.63	-6238.59	-6000.95
53	$\mathcal{Z}_{(3)}$	2066.98	-7421.70	-5354.72	-5045.87	-4760.33
54	$\mathcal{Z}_{(4)}$	2449.23	-6805.70	-4356.47	-4024.71	-3723.93
55	$\mathcal{Z}_{(5)}$	2750.73	-6234.02	-3483.29	-3154.92	-2863.01
56	$\mathcal{Z}_{(6)}$	2980.77	-5704.13	-2723.35	-2418.26	-2152.31
57	$\mathcal{Z}_{(7)}$	3147.94	-5213.57	-2065.63	-1798.25	-1569.71
58	$\mathcal{Z}_{(8)}$	3260.18	-4759.99	-1499.81	-1280.07	-1095.89
59	$\mathcal{Z}_{(9)}$	3324.77	-4341.11	-1016.34	-850.42	-714.06
60	$\mathcal{Z}_{(10)}$	5486.26	42159.27	47645.53	39085.93	32187.61
Total					**-2912.44**	**-7733.21**

We remark that at that time, the company makes still some additional gains as a consequence of the 2% surrender penalty.

Calculation of the cash flow matching portfolio in case of high surrenders:

We assume that there has been observed an exceptional lapse rate at time 2 of 25% of the portfolio.

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1019.67	-8797.22	-7777.54	-7625.04	-7478.40
52	$\mathcal{Z}_{(2)}$	1594.01	-8084.65	-6490.63	-6238.59	-6000.95
53	$\mathcal{Z}_{(3)}$	4773.16	-5966.47	-1193.30	-1124.48	-1060.84
54	$\mathcal{Z}_{(4)}$	1968.98	-5471.25	-3502.26	-3235.55	-2993.75
55	$\mathcal{Z}_{(5)}$	2211.37	-5011.66	-2800.29	-2536.31	-2301.63
56	$\mathcal{Z}_{(6)}$	2396.30	-4585.67	-2189.36	-1944.09	-1730.28
57	$\mathcal{Z}_{(7)}$	2530.70	-4191.30	-1660.60	-1445.65	-1261.92
58	$\mathcal{Z}_{(8)}$	2620.93	-3826.66	-1205.73	-1029.08	-881.01
59	$\mathcal{Z}_{(9)}$	2672.85	-3489.91	-817.05	-683.67	-574.05
60	$\mathcal{Z}_{(10)}$	4410.52	33892.74	38303.27	31422.02	25876.31
Total					**-4002.67**	**-7968.76**

ALM Risk of mass lapses Finally we want to look what happens when we have mass lapses as indicated before, but if we have invested in the cash flow matching portfolio according to average 7 % lapses. Hence we have to calculate the assets according to 7 % lapses and the liabilities according 25 % lapses.

Age	Unit	Units for Assets	Units for Liability	Total Units	Value $i = 2\%$	Value $i = 4\%$
52	$\mathcal{Z}_{(0)}$	-6490.63	6490.63	0	0	0
53	$\mathcal{Z}_{(1)}$	-5354.72	1193.30	-4161.42	-4079.82	-4001.36
54	$\mathcal{Z}_{(2)}$	-4356.47	3502.26	-854.21	-821.04	-789.76
55	$\mathcal{Z}_{(3)}$	-3483.29	2800.29	-682.99	-643.60	-607.18
56	$\mathcal{Z}_{(4)}$	-2723.35	2189.36	-533.99	-493.32	-456.45
57	$\mathcal{Z}_{(5)}$	-2065.63	1660.60	-405.02	-366.84	-332.90
58	$\mathcal{Z}_{(6)}$	-1499.81	1205.73	-294.08	-261.13	-232.41
59	$\mathcal{Z}_{(7)}$	-1016.34	817.05	-199.28	-173.48	-151.43
60	$\mathcal{Z}_{(8)}$	47645.53	-38303.27	9342.26	7973.53	6826.29
Total					**1134.26**	**254.76**

Now we see that the lapses induce quite a big risk for the company since they lose in case of mass lapses almost 1 % of the face value of the policy, more concretely $1134.26 - 254.76 = 879.50$.

The above example shows very clearly how the behaviour of the policyholders can change the cash flow matching portfolio and in consequence induces a risk. As a consequence the risk minimising portfolio in the sense of $VaPo^*(x)$ for an insurance portfolio $x \in \mathcal{X}$ does also consist of additional assets offsetting the corresponding risks. In the above example the corresponding asset would be a (complex) put option, which allows to sell the bond portfolio at the predefined (book-) values. So in reality insurance companies aim to model these risk in order to determine the corresponding assets to manage and reduce the undesired risk.

In the example above we have assumed that at a given year 25% of the policies in force lapse. In practise one models the dynamic lapse behaviours. Eg the lapse rate is a function of the interest differential between market and book yields. Normally the corresponding lapse rates stay below 1, which is interesting. Assuming a market efficient behaviour, one would expect that there is a binary decision of the policyholders to stick to the contract or to lapse as a function of the before mentioned interest differential. In consequence the underlying theory how to model such policyholder behaviour is not as crisp and transparent as with the arbitrage free pricing theory, since market efficient behaviours is normally not observed. As a corollary there is a lot of model risk intrinsic to these calculations and it is important to test the results from the models with different scenarios.

Remark 11.4.11. At the end of this section a remark on how to determine a $VaPo^*(x)$ for an $x \in \mathcal{X}$: One normally models an $l \in \mathcal{X}$ and simulates $l(\omega)$ together with some test assets $D \subset \mathcal{Y}$ observable prices and cash flows. We denote $D = \{d_1, \dots d_n\}$. Hence at the end of this process we have a vector

$$\mathcal{W} := (l(\omega_i), d_1(\omega_i), \dots d_n(\omega_i))_{i \in I}.$$

Now the process is quite canonical:

1. We define a distance between two $x, y \in \mathcal{X}$, for example by means of $||x||$ as defined.
2. We solve the numerical optimisation problem, for minimising the distance between l and the target $y \in \text{span} < \mathcal{D} >$.

We note two things:

– The numerical procedures to determine y can sometimes prove to be difficult since the corresponding design matrix can be near to a singular matrix, and hence additional care is needed.

– In case of the $||\bullet||$ defined before, we remark that it has been deducted from the Hilbert space $\mathcal{X}$. Hence what we actually doing is to use the projection $\tilde{p} : \mathcal{X} \to \text{span} < \mathcal{D} >$, which can be expressed by means of $< \bullet, \bullet >$. We remark that $y = \tilde{p}(x)$.

12. Policyholder Bonus Mechanism

The aim of this chapter is to introduce the concept of policyholder bonus and the corresponding effects.

12.1 Concept of Surplus: Traditional Approach and Legal Quote

Lets reconsider example 11.4.10. We have used in this context a interest rate for pricing of 2%. According to the EU laws the technical interest rate must normally not exceed 60% of the average government bond yields. This would mean that at that time the average government bond yield was above 3.3 % – assume 4 %. If we redo the corresponding calculations, we get the following:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1000.00	-9466.57	-8466.57	-8300.56	-8140.94
52	$\mathcal{Z}_{(2)}$	1089.00	-9362.44	-8273.44	-7952.17	-7649.26
53	$\mathcal{Z}_{(3)}$	1174.93	-9250.09	-8075.16	-7609.40	-7178.79
54	$\mathcal{Z}_{(4)}$	1257.56	-9129.84	-7872.27	-7272.76	-6729.25
55	$\mathcal{Z}_{(5)}$	1336.69	-9002.02	-7665.32	-6942.72	-6300.34
56	$\mathcal{Z}_{(6)}$	1412.12	-8866.99	-7454.87	-6619.71	-5891.69
57	$\mathcal{Z}_{(7)}$	1483.67	-8725.12	-7241.45	-6304.11	-5502.90
58	$\mathcal{Z}_{(8)}$	1551.18	-8576.79	-7025.61	-5996.29	-5133.54
59	$\mathcal{Z}_{(9)}$	1614.50	-8422.41	-6807.90	-5696.55	-4783.14
60	$\mathcal{Z}_{(10)}$	–	88080.30	88080.30	72256.53	59503.90
Total					**0**	**-7368.19**

Looking at these results we see that the difference in value using the technical pricing rate of 2 % and the market yield of 4 % amounts to 7368.19, which represents 7.3 % of the maturity benefit. Since the corresponding differences can also become bigger than that consumer protection regulation was introduced in the form of legal quotes. The idea there was that for example 90 %

of the profit has to allocated to the policyholder and the shareholder receives maximally the remaining 10 %.

Assume that in a concrete year, we have:

– Mathematical reserve: 45’000 USD,
– Investment return: 4.2 %,
– Required technical interest rate: 2.0 %.

In this case we get the following income statement:

Item	Amount
Investment Income	1890 USD
Technical Interest	- 900 USD
Gross Profit	990 USD
of which PH	891 USD
of which SH	99 USD

and we remark the following:

1. The biggest part of the gross profit is allocated to the policyholder and the shareholder receives only the smaller part.
2. Since the return on the assets is random, this is also true for the bonus payment, which however can not become negative.
3. As a consequence the shareholder assumes an additional downside risk if he operates an investment strategy which also allows investment returns below the technical interest rate with a positive probability.
4. It is worth mentioning that the insurance company can also give a higher policyholder bonus than the legal minimum defined by the legal quote. Obviously the shareholder return is then reduced accordingly.

After the split of the gross investment return into policyholder bonus and shareholder return, one needs to look how the bonus is used. Here there are different possibilities. The bonus can be used to

– Reduce the premium. Hence assuming a regular premium of 4900 USD, the policyholder would only need to pay 4001 USD.
– Accumulate it on a bank account. The bonus is invested in a bank account type of investment where a yearly interest rate is credited. At the end of the policy the insured receives the value of this account.
– Increase the benefits. In this case the policyholder bonus is used as a single premium to increase the insurance benefits.

We finally want to revisit the above example with different investment returns:

Return	-5%	0%	2%	5%
Investment Income	-2250	0	900	2250
Technical Interest	-900	-900	-900	-900
Gross Profit	-3150	-900	0	1350
of which PH	0	0	0	1215
of which SH	-3150	-900	0	135

Finally we want to remark that there are two slightly different approaches to legal quotes. In the UK context the insurance company balance sheet is split into policyholder and shareholder funds. The legal quote applies to the policyholder funds. In case of an under-coverage of the policyholder funds, the shareholder is required to compensate for this just as in the example above for an investment return of -5%. In continental Europe, such as in Germany and France the legal quote is applied to the whole balance sheet. The difference between the two concepts is that in the UK context the shareholder receives the full return on his assets. On the other hand the shareholder has to share the return on “his” assets with the policyholder according to the legal quote requirements.

In a next step we will look at insurance policies from a different point of view, namely from the policyholders' point of view. Also here we want to apply a market consistent approach and hence the value of the premium before and after paying them to the insurance company is equal, but it is allocated to different stakeholders. We want to illustrate this concept based on an example. We assume that

1. the policy is defined as above as an endowment policy with the parameters set of example 11.4.10, and that
2. the pricing interest rate amounts to $i = 2\%$ with a market interest rate of $i = 4\%$.
3. Furthermore we assume that there is a legal quote of 90 % of the difference in present values as shown above.
4. Finally we assume that the present value of the administration charges amounts to 0.2 % of the present value of the benefits and that the tax rate of the insurance company amounts to 35 %.

In this case we know that:

1. The premium amounts to $P = 1.02 \times 9562.20 = 9753.44$.
2. The present value of the premiums amounts (at 4%) to -1.02×58033.16.

3. The present value of death benefits amounts (at 4%) to 7285.59.
4. The present value of maturity benefits amounts (at 4%) to 28481.29.
5. The present value of surrender payments amounts (at 4%) to 14533.06.
6. The present value of gross profit amounts (at 4%) to 7368.19.

Based on this information we can now decompose the present value of the payments of the policyholder (-59k USD) into its parts:

Type	Stakeholder	Amount	%-age
Mortality	PH	7285.59	12.3 %
Maturity	PH	28481.29	48.1 %
Surrender	PH	14897.09	25.1 %
PH Bonus	PH	6631.37	11.2 %
Subtotal PH	PH	57295.34	**96.7 %**
Admin	Employees of Insurer	1160.66	1.9 %
Tax	Tax authorities	257.88	0.4 %
Profit	SH	478.93	**0.8 %**
Total	(equals PV Prem.)	59193.82	100.0 %

After having seen why policyholder bonus is generated, we need to better understand what is done with the bonus allocated to a policy, since this materially impacts the underlying investment strategies and the corresponding risks.

There are from a high level point of view two different ways how gross bonus can be used, namely for the benefit of an individual policyholder or for the entire collective of policyholders together. In the first case the individual policyholder benefits from the excess funds allocated to his policy, be in the form of a reduction of premiums or be it that his benefits are increased.

Besides the direct allocation of the gross bonus to individual policyholders, there is also the possibility that a part of the money is used for the entire insurance portfolio together, in the sense of mutualisation of insurance risk. This is known as reserve strengthening and is a consequence that the mathematical reserves for an insurance portfolio are estimated based on statistical methods, which involve some uncertainty. In case a new estimation of the expected reserves needed turns out to be higher than the reserves within the balance sheet, it is necessary to strengthen them in order to be able to honour the corresponding commitments. Now there are two possibilities how such a reserve strengthening is financed, by the shareholder, or as mentioned before by the collective of policyholders by using a part of the gross profits stemming from the in-force portfolio.

12.2 Portfolio Calculations

When doing ALM it is normally important to use efficient calculation processes since a lot of simulations are needed. Let's look at the moment at an insurance portfolio with 1 million polices. During the year end calculation, one needs normally to calculate the mathematical reserves for the past, the current and the next year, in order to save these values in the data base to be able to interpolate them for a possible policy surrender.

Assume for the moment that your tariff engine is performing such a calculation in 0.01 seconds. Hence the whole year end calculation takes you 8h 20min. When doing ALM this is obviously too long when requiring 10000 simulations, since this would result in about 3500 years run-time on the same infrastructure. Some acceleration can be gained by using a grid, but even then it seems to make sense to look for faster methods to do the above task.

In this section we will look at such approaches, which can concretely be implemented. If we consider a simple set up we have a set of policies $\mathcal{P}$, and each policy can be characterised by its Markov representation, eg the state space S_i, the discount factors (which is normally the same for all polices), the transition probabilities (which are normally structurally similar per different state space) and finally the benefits vectors $a_{ij}(t)$ and $a_i(t)$. In a lot of cases one can also restrict the state space to a common one, which we call S.

In order to be concrete we want to have a look at the set of all insurance policies on one life, be it a lump sum or an annuity. In this case we choose $S = \{*, \dagger, \ddagger\}$ as corresponding state space for a person with age ξ, where $\ddagger$ represents the state of a surrendered policy. In this set up we introduce the linear vector space of all insurance policies for this person ξ by

$$\mathcal{F}_\xi = \{x_\xi = (a_{ij}(t), a_i(t)) : i, j \in S \text{ and } t \in \mathbb{N}\}.$$

We now remark that both the mathematical reserve and the expected cash flow operators

$$\begin{aligned}
\Phi_{t,j}(x_\xi) &: \mathcal{F}_\xi \to \mathbb{R}, x_\xi \mapsto V_j(t)[x_\xi] \\
\Psi_{t,j}(x_\xi) &: \mathcal{F}_\xi \to \mathbb{R}, x_\xi \mapsto \mathbb{E}[CF(s)[x_\xi]|X_{t_0} = j]
\end{aligned}$$

are linear (continuous) functionals from $\mathcal{F}_\xi \to \mathbb{R}$, where ξ denotes the policy considered, t and s the respective times and $j \in S$ a state. When recognising this fact we can now construct the space of all insurance policies for a given portfolio $\mathcal{P} = \{\xi_1, \xi_2, \ldots \xi_n\}$ by defining the respective Cartesian product such as:

$$\begin{aligned} S &= \prod_{i \in \mathcal{P}} S_i, \text{ and} \\ \mathcal{F} &= \prod_{i \in \mathcal{P}} \mathcal{F}_i. \end{aligned}$$

In the same sense we can now define the mathematical reserve and expected cash flow operator of the whole insurance portfolio as the corresponding sum:

$$\Phi_t(x) : \mathcal{F} \to \mathbb{R}, x((\xi_i)_{i \in \mathcal{P}}) \mapsto \sum_{i \in \mathcal{P}} V_{j(\xi_i)}(t)[x_{\xi_i}] \text{ and}$$

$$\Psi_t(x) : \mathcal{F} \to \mathbb{R}, x((\xi_i)_{i \in \mathcal{P}}) \mapsto \sum_{i \in \mathcal{P}} \mathbb{E}[CF(s)[x_{\xi_i}] | X_{t_0} = j(\xi_i)].$$

Until now we have gained nothing except for a more complex representation of what we already know. The way we can now make the whole thing much more efficient is to use the given structure and the fact that the two operators defined above are linear. In the concrete set up where each policy is characterised by the three states above we can define a new "pseudo" state space

$$\tilde{S} = \{x0, x1, \dots, x120, y0, y1, \dots, \dagger, \ddagger\},$$

where the states $x0, \dots, x120$ stand for males which are alive and have the respective age at time t_0. Similarly $y0, \dots, y120$ stand for the respective females. There is only a need to map the respective $x_\xi \in \mathcal{F}_\xi$ into $\tilde{\mathcal{F}}$.

Assume that $\mathcal{R}50 \subset \mathcal{P}$ is a homogeneous set of policies representing the males ages 50. In this case we have the following benefit functions:

$$\begin{aligned} a_{x50}^{Pre} &= \sum_{\xi \in \mathcal{R}50} a_{*}^{Pre}(\xi) \\ a_{x50,x50}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,*}^{Post}(\xi) \\ a_{x50,\dagger}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,\dagger}^{Post}(\xi) \\ a_{x50,\ddagger}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,\ddagger}^{Post}(\xi) \end{aligned}$$

Please note that in order to implement the approach described above, one needs to be careful with respect to the definition of time. In the usual Markov model the time t is calculated with respect to the age of each policyholder in a certain year. For the above purpose, it is useful to enumerate by the number of years into the future, starting at $t = 0$. Hence one needs to correspondingly adjust the time of the benefit functions. We remark that for all the other states in $\tilde{S} \setminus \{\dagger, \ddagger\}$ the same approach is used. It is also worth noting that this "pseudo" Markov chains can be interpreted as "normal" Markov chains, where the initial state of the portfolio is given by a probability distribution, over which one integrates.

After having done so, we can now perform a lot of the calculations much faster. After having applied the above mapping into $\tilde{\mathcal{F}}$, the calculations do not depend anymore on the actual size of the portfolio. By this we gain considerable amounts of times when doing the actual calculations. Looking at the sample portfolio above we would possibly use 1h calculation time to perform the mapping on $\tilde{\mathcal{F}}$, including the data base queries. Once this is done the calculation of the mathematical value and expected cash flow operator take some 1 to 2 seconds on a common laptop. Hence doing ALM in a lot of cases result in run-times of several minutes. In the same sense stress scenarios can be calculated much faster.

Finally I would like to mention that this approach has been applied concretely for the examples in section 12.3 using 402 states for $\tilde{\mathcal{F}}$ by also splitting annuities from capital insurance. Using this approach one can also map deferred widows pension using a collective approach and hence one can cover the vast majority of traditional life insurance covers sold by a life insurance company. Since structurally the method is a slight variation of the Markov recursion, this method can actually be implemented using the same core Markov calculation objects.

12.3 Portfolio Dynamics and ALM

The aim of this section is to look at the portfolio dynamics induced by the relationship between assets and liabilities and the corresponding asset liability management (ALM). To this end we fix $(\Omega, \mathcal{A}, P)$, together with a filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{N}}$. We assume that we have represented the benefits and premiums in a suitable ("pseudo") Markov model with state space $\tilde{S}$ and benefits vector space $\tilde{F}$.

In this case we can represent the benefits for the entire portfolio by $x = ((a_{i,j}^{Post}(t))_{i,j \in \tilde{S}}, (a_i^{Pre}(t))_{i \in \tilde{S}})_{t \in \mathbb{N}} \in \tilde{F}$, and we know that we can decompose this into

$$x = x^{Benefits} + x^{Premium},$$

where $x^{Benefits}$ and $x^{Premium}$ represent the corresponding benefit vectors in $\tilde{\mathcal{F}}$ for benefits and premiums, respectively. At this point the bonus concept enters. If for example the benefits are increased, the corresponding $x^{Benefits}$ is increased accordingly. Formally one introduces hence a random vector α_t which represents the relative benefit level. This means that $\alpha_0 = 1$ and also that α is previsible. For traditional bonus promises, the bonus allocated to the individual policy become a guarantee, means that α is increasing in t for each trajectory and hence we define the new benefit representation of the policy at time t as follows:

$$\hat{x}^{Benefits} = ((\alpha_t \times a_{i,j}^{Post}(t))_{i,j \in \tilde{S}}, (\alpha_t \times a_i^{Pre}(t))_{i \in \tilde{S}})_{t \in \mathbb{N}},$$

remarking that this is now a random quantity, since α is a random vector. In consequence the entire portfolio after bonus allocation is given by

$$\hat{x} = \hat{x}^{Benefits} + x^{Premium}.$$

The mechanism to increase α is performed by using the allocated bonus as a single premium. In a next step we want have a look at a concrete example. The first step is to define a stochastic model, which generates the corresponding states of the world. In the concrete set up we are assuming a world with a constant interest rate with one risky asset (say a share) with has a non constant, stochastic volatility. We are using the Heston model, which is given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t^1 \\ dS_t &= \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^2. \end{aligned}$$

This model is given by two stochastic differential equations, where the first equation describes the volatility of the shares. This volatility process has a structural similarity with the interest rate models we have seen before, in the sense that the volatility process is also mean reverting.

In order to solve this stochastic differential equation system we can use a numerical method such as the Milstein scheme (see for example [KP92]). After having done this, it is important to understand how the simulation works. In principle one does first a loop over the different simulation and calculates the quantities, which need to be analysed. The following code performs the corresponding task:

```
# 4. Stepper
# ---------------------------
for i in 1... n:
     sim.vNewTrajectory()
    (pl, cf, ... ) = CalcPV(sim, lvp, lvl)

 ... keep and analyse results for run i
```

Hence first one generates a new trajectory of the world (line 4, in this case based on the Heston model), and secondly on calculates the development of the insurance portfolio, as outlined above in the subroutine `CalcPV` (line 5). Hence it makes sense to look at this part of the code in order to understand what happens:

```
... code left away
 1  for i in 1... MaxTime:
 2      TargetSurplus  = ... target surplus level required
 3      perf = ((1+iRF) * (1-EquityProportion) +
           EquityProportion* sim.dGetValue(1,float(i)) / sim.dGetValue(1,float(i-1.)))
 4      cf[i] = psi[i-1] * lvl.dGetCF(i) + lvp.dGetCF(i)
 5      l[i] = psi[i-1] * lvl.dGetDKTilde(i) + lvp.dGetDKTilde(i) - cf[i]
 6      a[i] = a[i-1] * perf - cf[i]
 7      ExAssets = a[i] - l[i]
 8     # Calculate Excess Assets and Pl, afterwards adjust assets by pl for SH
 9      if ExAssets > 0.:
10          ExAssets = max(0., ExAssets - TargetSurplus)
11          Bonus = ExAssets * RelBonusAllocation * LegalQuote
12          pl[i-1] = ExAssets * RelBonusAllocation * (1-LegalQuote)
13      else:
14          Bonus = 0.
15          pl[i-1] = ExAssets
16      a[i] -= pl[i-1]
17      e[i] += pl[i-1]
18      psi[i] = psi[i-1] + Bonus / (psi[i-1] * (lvl.dGetDKTilde(i) - lvl.dGetCF(i)))
```

What this code does, is the following:

1. In a first step, the minimal required surplus of the policyholder funds (line 1) and the performance of the assets (line 2) is calculated. We see that in the concrete set up we have a mixture of assets. A part of them yielding risk free and the reminder, the equity portion having an equity yield.
2. In lines 4,5 and 6 the cash-flows and the assets and liabilities are calculated at the end of the period. We see that

$$\begin{aligned} \Psi_i(\hat{x}^{Benefits}) &= \texttt{psi[i-1] * lvl.dGetCF(i)}, \text{ and} \\ \Psi_i(x^{Premium}) &= \texttt{lvp.dGetCF(i)}. \end{aligned}$$

3. In a next (lines 7 to 15) step the excess assets are calculated in order to determine, whether there is a bonus in the corresponding year. If there are excess assets (line 9), the bonus is calculated.
4. Finally the benefit level for the subsequent year is calculated (line 18)

We remark that the initialisation of the code plus the analysis have been left in order to focus on the essential parts of the calculation.

After having done this, we want to have a look at some sample output of the program. Figure 12.1 and 12.2 show the mathematical reserves and the expected cash flows for the benefits and the premiums respectively. Moreover figure 12.3 show the quantiles of the profit and loss over time for the 5%, 10%, 33%, 50 %, 67%, 90% and 95% quantiles for both the profit and loss account and the corresponding dividends. We remark that this figure shows that the underlying portfolio is insufficiently financed and suffers considerable losses between the time 5 and 15, and also that the losses start earlier in cases where we observe an adverse equity performance.

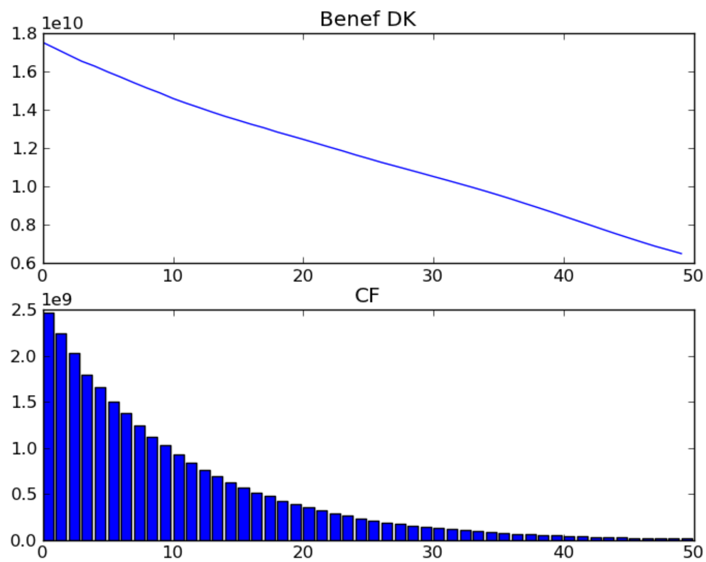


Figure 12.1. Mathematical reserve and expected cash flows of portfolio (benefits only)

The output of the program looks as follows:

```
Input Parameter / Main Results
------------------------------

Simulations                                          5000
>>   PV Benefits ..... .....DK 1   =  22,786,998,031
>>   PV Premium. ..... .....DK 0   =  -4,091,249,171
>>   Mathematical Reserves..DK     =  18,695,748,860
>>   Underlying Assets .....A0     =  22,635,000,000
>>   Shareholder Equity.....E0     =   1,584,450,000
```

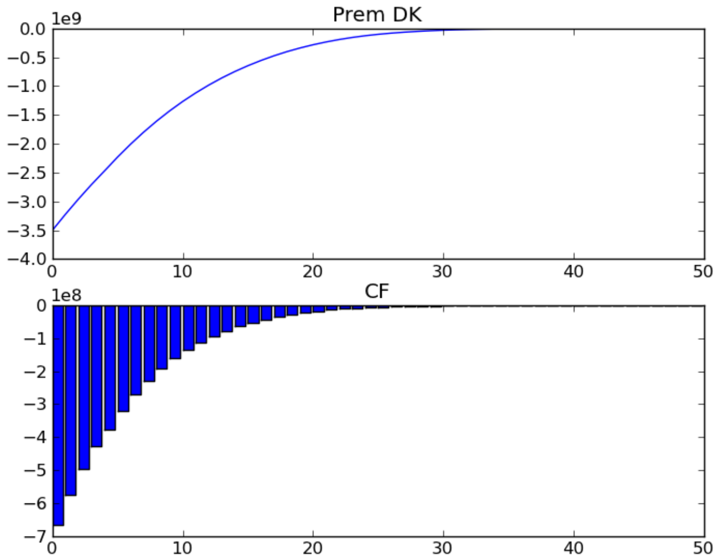


Figure 12.2. Mathematical reserve and expected cash flows of portfolio (premiums only)

```
Distribution of Economic Profit
-------------------------------

Min Return                                          0.0962
Max Return                                          0.2473
Average                                             0.1792
 0.5%  Quantile                                     0.1288
 1.0%  Quantile                                     0.1324
 5.0%  Quantile                                     0.1452
10.0%  Quantile                                     0.1529
25.0%  Quantile                                     0.1654
99.5%  Quantile                                     0.2289


Time used
---------

Step             Calc EP Time used     2.3515 s
Step             Calc PF Time used     3.6497 s
```

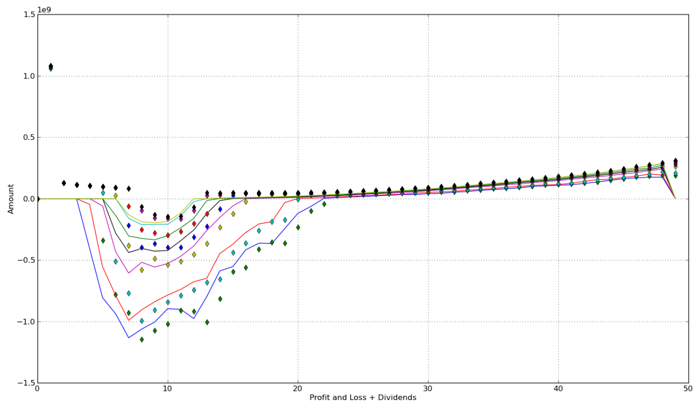


Figure 12.3. Quantiles for profits and losses

```
Step              Calc PF Time used     3.2491 s
Step            A0 and E0 Time used     0.0190 s
Step   Simulation Stepper Time used    81.8312 s

Time used for preparation               9.2 s
Time used for simulations              81.8 s

Time per simulation                     0.0163 s
```

We see in particular that the implementation via the “pseudo” Markov model results in an extremely fast calculation of the portfolio in about 9 seconds. Also the simulation is performed very fast, despite the fact that a relatively slow laptop was used. The output of the above run is summarised also in figure 12.4.

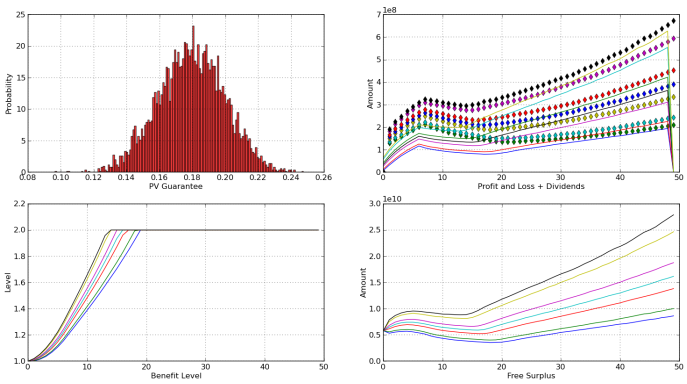


Figure 12.4. Summary of ALM analysis

13. Variable Annuities - Application of Markov Chains

We will see that allowing for lapses reduces the people receiving the variable annuity benefits. The price for the variable annuity guarantee for our sample policy (see section 8.6) reduces, when including lapses, from 40311.70 down to 12222.80. This big difference raises the question, what happens if policyholders become more efficient and lapse less. To understand this it is important to know that the insurer will price this guarantee possibly at 15000, to have some safety and profit margin. However the 15000 could not cover the guarantees in case no lapses occur. Such effects have led to losses in some variable annuity portfolios. To illustrate this, assume that the best estimate lapses ("BE") indicated above (eg 4% for all years, except for year 10 where lapses are 12%) were inaccurate and need to be revalued to 8% at year 10 and 2% thereafter ("New BE"). The following table shows the value of the valuation portfolio as at time 2. We note that maturity is now in 23 years.

	Instrument	Value Normal	Value BE	Value New BE	Value P&L
1	Put $t = 1$	25.6	23.5	23.5	–
2	Put $t = 2$	39.7	35.1	35.1	–
3	Put $t = 3$	55.7	47.3	47.3	–
...					
7	Put $t = 7$	156.7	112.6	112.6	–
8	Put $t = 8$	195.4	134.6	134.6	–
9	Put $t = 9$	241.2	149.3	149.3	0.0
...					
23	Put $t = 23$	36986.0	10138.3	14802.2	-4663.9
	Total	42844.5	12973.2	18006.3	**-5033.1**

In this example the loss of the updated lapse assumptions exceeds the assumed profit and safety margin. We wish to remark that a lot of companies have more refined lapse assumptions, in the sense that they make them dynamic depending on the value of the underlying guarantee. The higher the fund value the more likely the policyholder is going to lapse the insurance policy. In such instances the best estimate lapses could be, for example 50% of normal lapse levels. In case of low fund values the dynamic lapses try to model the reduced lapse levels and hence the policyholder behaviour. There is a high incentive for the policyholder to lapse in times of good fund performance and to stick in times of adverse fund development.

From a theoretical background the inclusion of the policyholder behaviour makes things much more complicated. We recall convention 8.3.2, where we assumed that $\mathcal{G}_t$ and $\mathcal{H}_t$ are stochastically independent. This means, that the financial variables are independent of the future lifespan and from lapses. Whereas we can defend the independence of mortality from the equity market prices, this is not quite the case if we take lapses into consideration. To model policyholder behaviour better, it is in a first step necessary to understand what a policyholder can do with his policy over time. This will be the topic of the following section.

13.1 What can a policyholder do during the lifetime of his policy

In this section we look at the actions a policyholder can take with respect to his policy. From an insurer's point of view this list indicates some of the policyholder behaviour risks:

Change Asset Allocation: The policyholder can change his asset allocation and invest in different assets, which are more or less risky.

Top up investment: The policyholder can invest an additional amount in the underlying fund. This can change the guarantees:

Lapse: policyholder can end the policy.

Start withdrawing: The policyholder can start to withdraw money from the fund.

Change amount of withdrawal: Within a given period, the policyholder can decide to withdraw more or less.

Partial Surrender: Withdrawing more than regularly allowed.

Sell Policy: He can sell the policy to a third party to monetize the value of the policy.

Policyholder behaviour is a risk that needs to be considered, in particular for the product design. Here one needs to avoid product designs, which promote crystallisation of losses for many policyholders at the same time.

There are different ways how policyholder behaviour can be considered. One school of thought is to assume that the policyholder behaves efficient, in the same way the price of an option is determined. In reality one can observe that policyholders do not behave fully efficient and therefore implicitly have

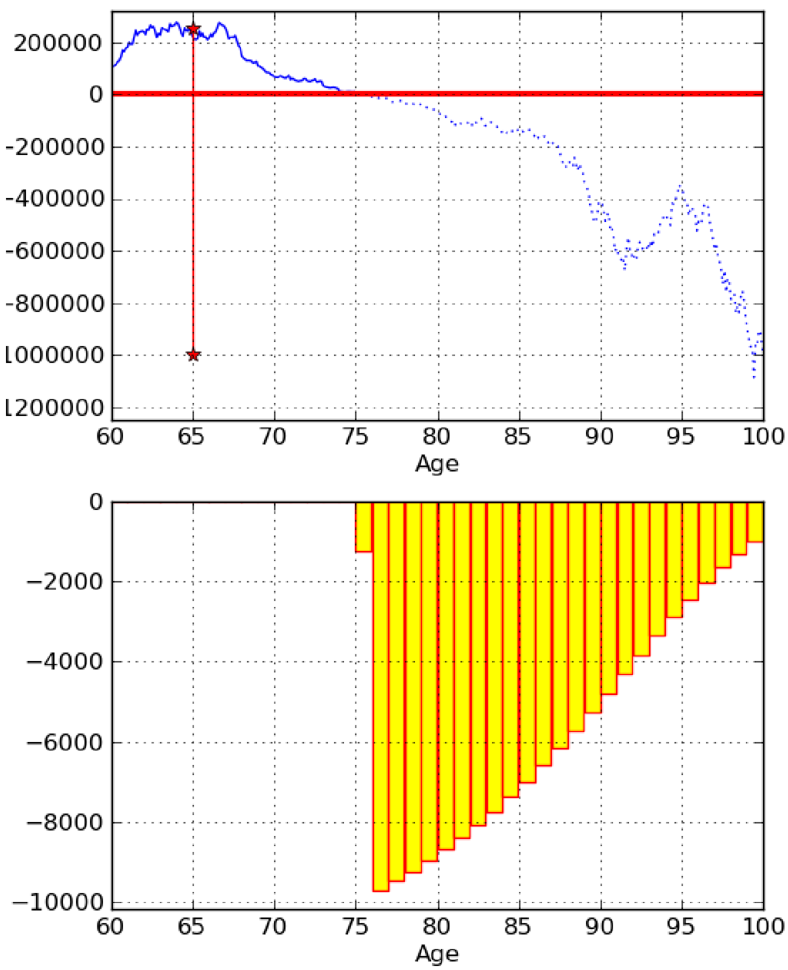


Figure 13.1. Example of GMWB contingency claims one trajectory

another utility function which determines their behaviour (see for example [AFV15]). Ultimately the different ways to consider policyholder behaviour boils down to the way the respective models are defined. In the following we will try to incorporate observed policyholder behaviour in our models.

One term which important in the way policyholder behaviour is modelled is the in-the-moneyness (“itm”). The “itm” is defined as one minus the ratio between the fund value “FV” and the corresponding remaining guarantee, hence $itm = 1 - \frac{FV}{Guarantee}$ (as proxy for the guarantee one could for example use $\alpha\,\ddot{a}_x$). The higher this ratio, the higher the value of the underlying guarantee and the lower the likelihood that the policyholder lapses.

With respect to withdrawing (GMWB policies) the policyholder has usually two additional choices to make. He has to decide at which point in time he will start to consume his funds (aka withdraw) and also the amount. A lot of GMWB policies have step-up (“ratchets”) features - ie the benefit base (B) increases at certain points in time $(\tau_1, \tau_2, \ldots, \tau_N)$ if the at this time the fund value (FV) exceeds the benefit base. Since the maximum withdrawal (R_t) amount is usually a percentage of the benefit base it is normally beneficial for the policyholder to wait withdrawing since this way he can expect an overall higher level of guaranteed withdrawals. On the other hand, this effect can be materially be muted in case of equities having fallen considerably. In this case the policyholder would be better advised to start withdrawing earlier. For the above case we have the following formulae:

$$\begin{aligned} B_0 &= FV_0 \\ dB_t &= \sum_{k=1}^{N} \delta_{\tau_k,t}\,(FV_t - B_t)^+ \\ R_t &= \beta_t \times B_t \end{aligned}$$

Though far from exhaustive, we discuss some of the main considerations and drivers for policyholders of the four options in the table below:

Policyholder Option	Description	Pros	Cons	Drivers
Leave the account untouched for another year	Account remains invested. Guarantee can be utilised on next anniversary. Some benefits may become more onerous.	Opportunity to benefit from potentially better returns to come as remain invested; Potentially higher income can be drawn in future	Increases reliance on other sources of income	Guarantee is not itm; Expectation of strong equity market performance; Income not needed; Deferred retirement; Lack of interesting alternative investments.
Utilise the guarantee and draw an income	Policyholder draws income in the form of a withdrawal (GMWB)	Income level is secured; GMWB: Future income withdrawal can be deferred.	GMIB: Income is fixed based on benefits base; GMIB: Control of principle is lost; GMWB: Level of income is restricted to the contractually agreed maximum.; GMWB: Higher withdrawals can come with a surrender charge.	Guarantee is itm; Lack of income from other sources; Immediate annuity rates are competitive – low interest rates.
Surrender the account value for cash	Policyholders can surrender the account value at the end of the waiting period for cash to be reinvested or consumed as wished	Flexibility; Control of principle.	Variable annuity features not utilised; Lack of income; Exposure to reinvestment and inflation risk.	Guarantee is not itm; Bird in the hand theory; Interesting alternative investment opportunities.
Surrender the account value for cash and purchase an immediate annuity	Policyholders can surrender the account value at the end of the waiting period for cash to be reinvested in an immediate annuity	Income level is secured; May offer more attractive income.	Variable annuity features not used; Income is fixed based on account value and prevailing IA rates; Control of principle is lost.	Guarantee is not itm; Prevailing immediate annuity rates are attractive – high interest rates.

13.2 Dynamic Policyholder Lapses for GMWB

In this example we want to look at a Guaranteed Minimum Withdrawal benefit for life, where we model the lapse behaviour dependent from the underlying equity market returns. We will look at how to model lapse rates which depend on the moneyness of the underlying guarantee. A GMWB is the variable annuity analogon of an immediate or deferred annuity. For this type of insurance we want see how the independence assumption (convention 8.3.2) interacts with the calculations: Hence we look at the following two different cases:

1. Assuming only mortality as decrement (where we assume that convention 8.3.2 holds), and
2. Including as a second effect the dynamic lapse behaviour.

In a first step we want to have a look at the stochastic differential equation which governs the dynamics of the fund value $(FV_t)_{t \in \mathbb{R}^+}$. Hence we assume as before that the equity value follows

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dW_t,$$

with $(W_t)_{t \in \mathbb{R}^+}$ the Brownian motion, r the interest rate density and σ the equity volatility. Assume also that $(A_t)_{t \in \mathbb{R}^+}$ and $(P_t)_{t \in \mathbb{R}^+}$ represent the cumulative annuity paid out and the cumulative guarantee fee respectively. In this case the fund value FV_t is governed by the following equation:

$$dFV_t = r\, FV_t dt + \sigma\, FV_t dW + dA_t + dP_t.$$

We note that the product is the defined via the concrete choices of A and P. We could for example envisage a product where the guarantee fee is constant multiple α of the fund value and where the annuity is a constant β as long as the person is alive. In this case we would have:

$$\begin{aligned} dP_t &= -\alpha\, FV_t\, dt \text{ and} \\ dA_t &= -\beta\, I_\star(t)\, dt, \end{aligned}$$

where $I_\star(t)$ refers to definition 2.1.8.

A GMWB is a unit linked annuity for a person of age x. We use the following notations, assuming a time discrete model (eg $t \in \mathbb{N}_0$):

Variable	Meaning
S_t	Denotes the equity price process introduced in chapter 8.
X_t	Process with values $\star, \dagger, \ddagger$ to reflect the state of the policy, whereby $\star$ stands for alive, policy in force, $\dagger$ stands for death and $\ddagger$ stands for surrender. The set up is much the same as for the Markov chain models which we encountered before, however depending of the use of convention 8.3.2, the stochastic process is assumed to be Markovian conditional S_t.
FV_t	The fund value at time t; we assume that FV_0 represents the fund value at inception of the policy.
R_t	The annuity which is withdrawn at time t in case the policy is in state $\star$. Please note that there are a variety of possibilities for R in practice. The easiest one is to assume that $R_t = \alpha \times FV_0$, which means that a constant proportion of the funds can be withdrawn per period. In general R_t will be more complex (in particular if taking ratcheting options into account), and will depend on S_t.
C_t	The contingent claim in relation to the GMWB variable annuity at time t. The value of the GMWB option is then the expected value of the sum of the present values of $(C_t)_{t \in \mathbb{N}_0}$ under the risk neutral measure Q.

We use the same notation as in chapter 8. This means in particular that:

– The σ-algebras generated by X_t are denoted by $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right).$

– We assume, that the values of the shares in the portfolio are given by standard Brownian motions W. (Compare with Figure 8.5.).

– $\mathcal{G}_t$ denotes the σ-algebra generated by W augmented by the P-null sets.

– $\mathcal{F}_t$ denotes the sigma algebra generated by $\mathcal{G}_t$ and $\mathcal{H}_t$.

This means that we assume, as in chapter 8, that $(S_t)_{t \in \mathbb{N}_0}$ is adapted with respect to $(\mathcal{G}_t)_{t \in \mathbb{N}_0}$. We also assume that the annuity to be paid out (conditionally that $X_t = \star$) is previsible with respect to $(S_t)_{t \in \mathbb{N}_0}$, which can be interpreted in the sense that R_n is determined by the sigma algebra $\mathcal{G}_{n-1}$ hence at the beginning of the respective period.

In order to make things simple we use a flat risk free interest rate of r per time step and define $v = \frac{1}{1+r}$. This will help to make formulae simpler. The inclusion of an interest rate term structure does not pose additional complexities (except the formulae get more convoluted). Please note that we assume that the annuity R_t is deducted and paid out at the beginning of the time interval $[t, t+1[$. Since we are looking at an annuity financed by a single premium, one would normally consider $R_0 = 0$. In the same sense we can model deferred annuities by setting $R_k = 0$ for $k < m$, where m denotes the deferral period. We assume that in case of a lapse the policyholder can get hold of the residual fund value FV_t but has no right to claim back a residual value of the GBWB contingency claim.

Before diving into the two distinct cases, lets look at the mechanics and interaction of the different processes. We first note that the annuity paid out at time t has the form

$$I_\star(t) \times R_t,$$

where $I_\star(t)$ refers to definition 2.1.8. Hence we have the following recursion for the fund:

$$FV_{t+1} = \max\left\{0, \frac{S_{t+1}}{S_t} \times (FV_t - I_\star(t) \times R_t)\right\}.$$

Based on this we can now determine the contingency claim at time t by

$$\begin{aligned} C_t &= \max\{0, (R_t - FV_t) \times I_\star(t)\} \\ &= \max\{0, (R_t - FV_t)\} \times I_\star(t)\}. \end{aligned}$$

As a next step we need to calculate the price of the GMWB guarantee by taking the expected present values under the risk neutral measure Q of the contingency claims $(C_t)_{t \in \mathbb{N}_0}$. Hence we have the following:

$$\begin{aligned} \pi(GMWB) &= E^Q\left[\sum_{k \in \mathbb{N}_0} v^k \max\{0, (R_k - FV_k)\} \times I_\star(k)\right] \\ &= \sum_{k \in \mathbb{N}_0} v^k E^Q\left[\max\{0, (R_k - FV_k)\} \times I_\star(k)\right] \\ &= \sum_{k \in \mathbb{N}_0} v^k E^Q\left[E^Q\left[\max\{0, (R_k - FV_k)\} \times I_\star(k) | \mathcal{G}_k\right]\right] \\ &= \sum_{k \in \mathbb{N}_0} v^k E^Q\left[\max\{0, (R_k - FV_k)\} \times E^Q\left[I_\star(k) | \mathcal{G}_k\right]\right]. \end{aligned}$$

We note that the last equality holds be construction of FV and R and the definition of the sigma algebra $\mathcal{G}$. At this point things become interesting. Until now we did not assume stochastic independence between $\mathcal{G}$ and $\mathcal{H}$, as we did in chapter 8. If we assume stochastic independence we can take $E^Q\left[I_\star(k) | \mathcal{G}_k\right]$ out of the outer expectation and get:

$$\begin{aligned}
\pi(GMWB) &= E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \max\{0,(R_k - FV_k)\} \times I_\star(k)\right] \\
&= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\max\{0,(R_k - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right] \\
&= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\max\{0,(R_k - FV_k)\}\right] \times E^Q\left[I_\star(k)\right] \\
&= \sum_{k\in\mathbb{N}_0} v^k\, {}_kp_x\, E^Q\left[\max\{0,(R_k - FV_k)\}\right].
\end{aligned}$$

This is exactly the same formula we reached in chapter 8. In this case it is possible to first calculate the value of all the contingency claims assuming the person lives for ever and then weigh the respective values with the respective survival probabilities. This makes the calculation simpler. In practice one can not assume this stochastic independence, particularly when taking lapses in consideration. It is known that the lapse rates reduce more when the variable annuity guarantees are in the money. Normally one models this conditional lapse behaviour to determine $E^Q\left[I_\star(k)|\mathcal{G}_k\right]$ via the so called "in-the-moneyness". Here one compares the ratio between the guaranteed value of the variable annuity and the underlying funds. The lower the funds value the more the guarantee is "in-the-money" and consequently a lower lapse rate. Conversely if the fund value is above the nominal guarantee value the policy is called "out-of-the-money" resulting in higher lapse rates.

We end this example with some figures which show the mechanics intrinsic to GMWB's. Figure 13.1 illustrates the interaction between the fund value (FV_k in red) and the corresponding expected contingency claims (in yellow) for one trajectory. One can see that the contingency payments decrease after the payment in proportion to the respective decrement table. In order to value the entire contingency claim one needs to look at the corresponding risk-neutral expected contingency claim cash flows (measure Q). Figure 13.2 shows the respective expected cash flows. For the example they have been simulated.

Please note that example 13.2 focuses on the GMWB part of a typical US variable annuity only. Normally this type of product is sold together with a GMDB part. The questions regarding the dynamic policyholder behaviour remain the same. It makes sense however to look at the following example which considers both parts together. In contrast to example 13.2, we will be slightly more concrete in the definition of the respective benefits and we will show the corresponding formulae.

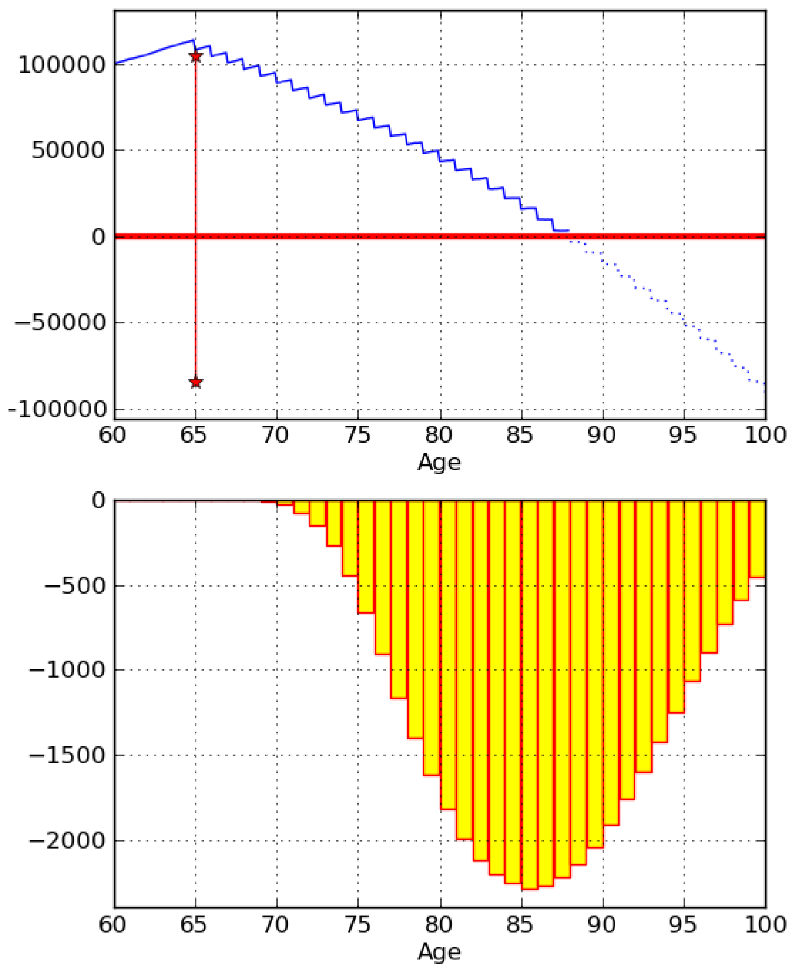


Figure 13.2. Example of GMWB contingency claims - expected value under Q

13.3 Dynamic Policyholder Lapses for GMWB/GMDB Policy

We assume that this product is sold to x year old man, which pays a single premium of V_0. In order to make things a little bit easier, we assume that all payments take place annually, and that the ratchet feature is also annual. This will allow us to use $t \in \mathbb{N}_0$ as time elapsed in years. The product has the following features:

- The fund management fees amount to $\beta = 1.5\%$ part of the fund value at the beginning of each period,
- The guarantee fee (for financing the variable annuity benefits) amounts to $\alpha = 0.9\%$ times the nominal value of the funds $\tilde{FV}_k$ to be charged at the beginning of the period.
- The annuity to be withdrawn equals $\gamma = 5\%$ of the nominal fund value $\tilde{FV}_k$ staring at $t = 1$ until the death of the policyholder.
- The GMDB benefits amount to the maximum of the nominal fund value and FV_0 and is paid in case of death before age 85.
- The nominal fund value is ratcheted up by taking the maximum of the fund value of the previous year and the fund value at the end of the period before deduction of the charges mentioned before.
- The charges are deducted in the order of the list above.

We denote by $\pi(x)$ for $x \in \{$ "GMDB", "GMWB", "GP", "Total" $\}$ the risk neutral prices of the GMDB, GMWB and Guarantee Premium's respectively. "Total" represents the sum of the first three components. By $N_{\star\dagger}(k)$ we denote the event of the policyholder dying during period k, ie

$$N_{\star\dagger}(k) \quad = \quad I_{\star}(k-1)\, I_{\dagger}(k)$$

We will use all the notation of example 13.2. In case we have the following recursions

$$\begin{aligned} FV_{t+1} &= \max\left\{0, \frac{S_{t+1}}{S_t} \times \Big(FV_t - I_{\star}(t) \times (R_t + \tilde{FV}_t \times (\alpha+\beta))\Big)\right\} \text{ and} \\ \tilde{FV}_{t+1} &= \max\left\{\tilde{FV}_t, \frac{S_{t+1}}{S_t} \times FV_t\right\}. \end{aligned}$$

Based in the above definitions, we can now calculate $R_t = \gamma\, \tilde{FV}_t$. For the different contingency claims we define by C_k^i for $i \in \{$ "GMDB", "GMWB", "GP", "Total" } the respective cash flows. We have the following relationships:

$$\begin{aligned} C_t^{GMWB} &= \max\left\{0, \left(R_t - (FV_t - \tilde{FV}_t \times (\alpha+\beta)\right)\right\} \times I_\star(t), \\ C_t^{GMDB} &= \max\left\{0, \tilde{FV}_t - FV_t\right\} \times N_{\star\dagger}(t), \\ C_t^{GP} &= -\min\left\{FV_t, \tilde{FV}_t \times \alpha\right\} \times I_\star(t) \end{aligned}$$

and C_k^{total} being the sum of the three other ones. Now the standard argument as above applies also to this more general situation:

$$\pi(i) = E^Q\left[\sum_{k\in\mathbb{N}_0} v^k\, C_k^i\right],$$

for $i \in \{$ "GMDB", "GMWB", "GP", "Total" $\}$. We refine the formula for each of the three basis types as follows:

$$\begin{aligned} \pi(GMWB) &= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\max\{0,(R_k - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right], \\ \pi(GP) &= -\sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\min\left\{FV_k, \tilde{FV}_k \times \alpha\right\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right], \text{ and} \\ \pi(GMDB) &= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\max\left\{0, \tilde{FV}_k - FV_k\right\} \times E^Q\left[I_\star(k-1)|\mathcal{G}_k\right]\right] \times q_{x+k}, \end{aligned}$$

where we have assumed for the GMDB that mortality is independent from capital markets. We finally remark that the last formula for the GMDB benefit is equivalent to the same formula in chapter 8, if we assume stochastic independence. In this case we get:

$$\pi(GMDB) = \sum_{k\in\mathbb{N}_0} {}_kp_x \times q_{x+k}\, v^k\, E^Q\left[\max\left\{0, \tilde{FV}_k - FV_k\right\}\right].$$

Exercise 13.3.1. Calculate the present value of each of the contingency claims for example 13.3 by simulation.

13.4 Model for dynamic lapses

As we have seen, the lapses - ie the conditional probabilities for X to move from $\star$ to $\ddagger$ are dependent on $\mathcal{G}_t$. Normally this is modeled in practice by the definition of the lapse rates via the "in-the-moneyness" (itm), which is defined as $itm = 1 - \frac{FundValue}{ValueofGuarantee}$. This means that the higher the guarantee relative to the funds value, the higher the "in-the-moneyness". A policy which is deep in the money (eg $itm \approx 100\%$) means that the guarantee is very valuable and in turn less people lapse. Figure 13.3 shows the lapse level in function of duration (d) of the policy for different levels of itm, and we can observe the above mentioned relationship. A higher "in-the-moneyness" results in a higher persistency of the policyholders, eg more policyholder will stay up to older ages and in consequence the variable annuity guarantee becomes more valuable. Figure 13.5 shows this effect. It is important to understand how

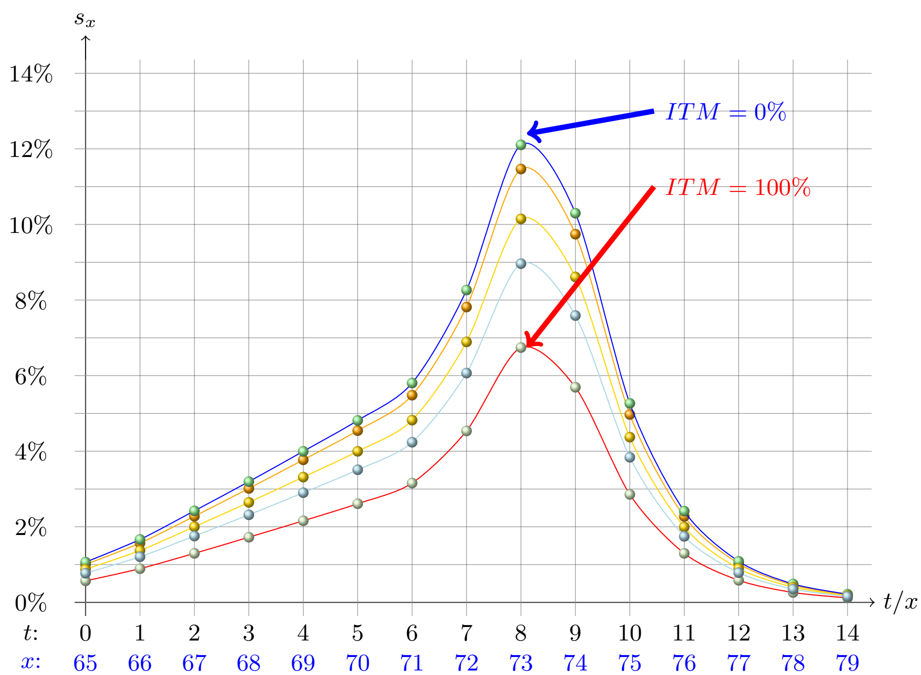


Figure 13.3. Dynamic Lapses

this dynamical lapse assumption is determined. In principle one has to start with observed policy data and one has to estimate the lapse probability for a period based on a set of explanatory variables. The aim is to find a dynamic lapse function which depends on these explanatory variables. To fit these one can choose a standard statistical methodology. The examples presented here are based on a realistic model portfolio and the underlying analysis has been performed using general additive models ("GAM") using a so called "logit" link:

$$\mathrm{logit}(x) \quad = \quad \ln\left\{\frac{x}{1-x}\right\}, \text{ for } x \in]0,1[.$$

This means that the logit transformation maps the parameters for a binomial distribution into a linear space, where we can determine the underlying probabilities. In the context of the dynamic lapse function above we have used a "GAM" model, where each of the dependent variables was used via a cubic spline. Hence we have the following relationship:

$$\mathrm{logit}(s_x) \quad = \quad c + s_1(x) + s_2(d) + s_3(itm),$$

where $c \in \mathbb{R}$ denotes the intercept, and where s_i, $i \in \{1,2,3\}$ are the three cubic splines representing the impact of age ("x"), duration ("d") and "in-the-moneyness" ("itm"). Based on the intercept and the three splines it is then possible to calculate the yearly lapse rate ("sx") by application of the inverse logit function:

$$\mathrm{logit}^{-1}(x) \quad = \quad \frac{e^x}{1+e^x}.$$

Instead of showing the three splines, we separately show in figure 13.4 the impact of changes in age and duration. We have set the 100% level at age 73. This is the age where most lapses occur, as a consequence of tax rules in the country where these variable annuities are offered[1].

Besides the actual model it is also important to discuss possible explanatory variables. The following table lists some possible explanatory variables together with a reason why this could be relevant. The underlying statistical analysis normally shows, that some of the discussed variables are more or less relevant for the concrete context. This has to do with the fact that the dynamic lapse behaviour is dependent on the tax environment and also on the general structure of the underlying products. Hence it is *dangerous* to just take some assumption which is readily available without checking its suitability for the concrete application!

[1] Note that the lapse rates shown do *not* represent a concrete existing portfolio. Hence they must not be taken to value a concrete variable annuity portfolio

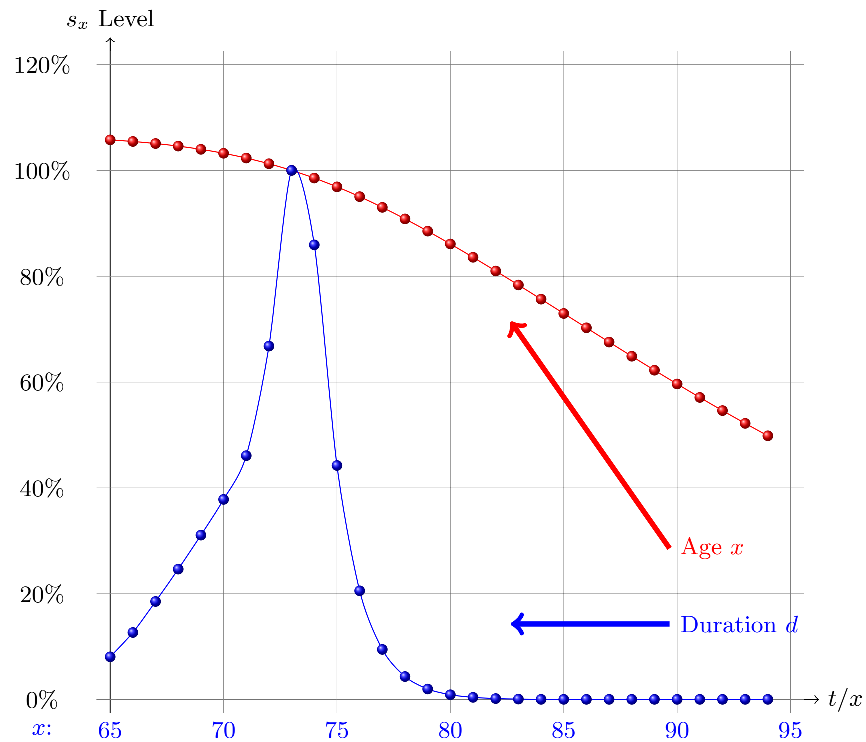


Figure 13.4. Impact of age and duration on lapses

Quantity	Description
itm	The "in-the-moneyness" drives the lapse behaviour considerably, hence is present in most dynamic lapse assumptions for variable annuities.
Duration	Lapse behaviour mainly depends on the duration of a contract, since in a lot of countries tax benefits emerge only if the money is for a certain minimal time in an insurance policy.
Age	Evidence suggests that lapses are normally dependent on age, tending to decrease for old ages.
Gender	Males and females often show different lapse behaviour.
Product	Lapse rates are normally quite dependent on the underlying product, also because different products have different surrender charges and tax treatments.
Interest Rate	In particular for GMWB one would expect that lapse rates decrease in a low interest environment because the underlying variable annuity guarantees get more costly.
Volatility	Implied volatility (such as the VIX index) tracks the implied volatility. Again economics would suggest that policyholders lapse less in a high volatility environment as a consequence of the increased value of the underlying guarantees.

The above list does not aim to be comprehensive and there are other factors which should be considered. What is quite clear is the underlying analysis is quite complex and it is often advisable to use a statistical software packet such as "R" to do the corresponding analyses. We end this example by remarking that based on the dynamic lapse assumptions from period to period, it is possible to construct the corresponding decrements $(E^Q [I_\star(k)|\mathcal{G}_k])$. Since the decrements are actually path dependent, it is rather difficult to plot them. Assuming a constant *itm* level it is however possible to calculate the decrements. Figure 13.5 shows such an example for a policy with GMWB for a $x = 65$ year old man. We remark the underlying GAM model depends on the following explanatory variables: duration, age and "in-the-moneyness".

Before moving to utilisation it is worth noting that the inclusion of dynamic lapses versus flat lapse assumptions means that deeply in-the-money policies become more expensive and that in turn a higher amount of hedging is needed. Figure 13.6 shows this effect. We conclude that the inclusion of dynamic lapse assumption is essential in modeling variable annuities.

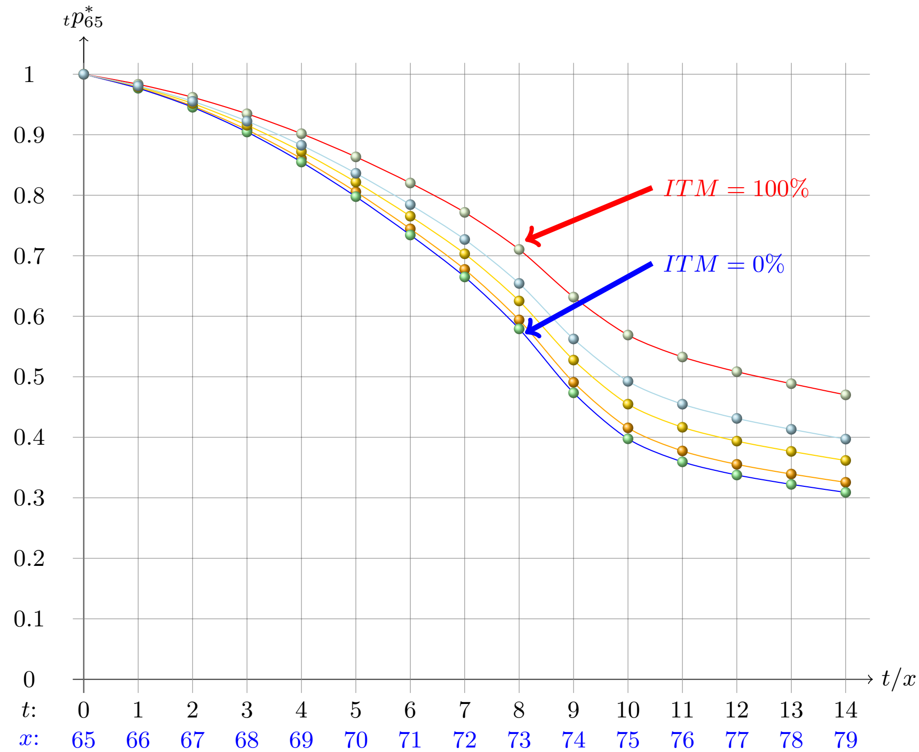


Figure 13.5. Decrement using Dynamic Lapses

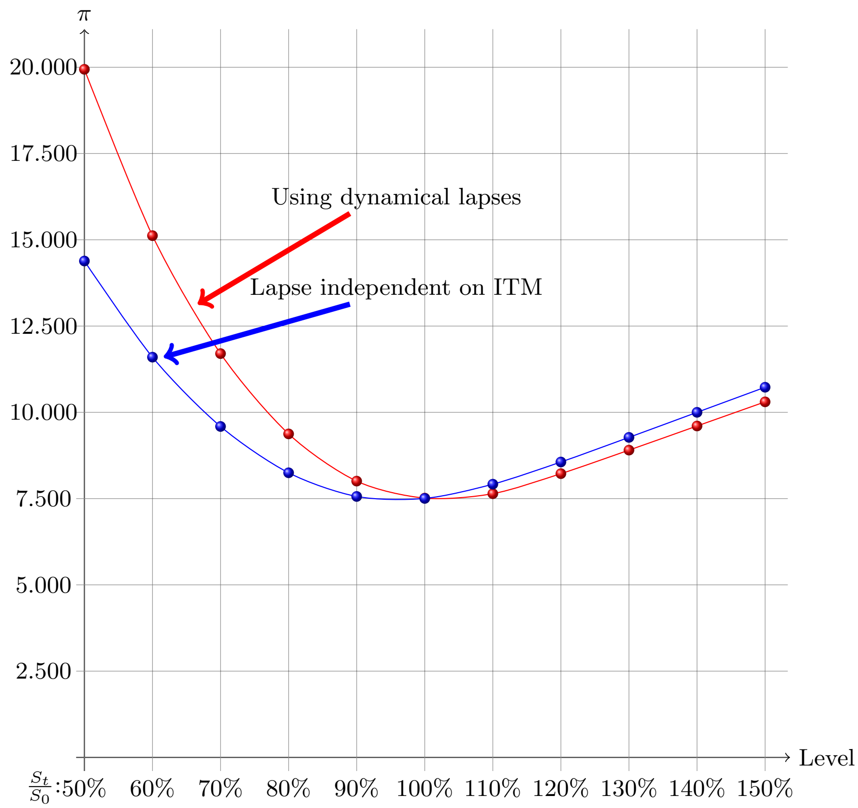


Figure 13.6. Comparison of trading grid with and without dynamic lapses

13.5 Modelling of Utilisation for GMWB

In example 13.2 we have seen the importance of modeling the lapses dynamically. In reality variable annuities are more complex than the example would suggest. In this section we want to focus on a product where utilisation depends on policyholder behaviour. In example 13.2, we used the following set up for R:

$$I_\star(t) \times R_t,$$

with the assumption that R_t depends on the financial market only (ie. $\mathcal{G}$ previsible). This covers all the cases where the annuity is deterministic dependent on the capital market and where the start of the annuity payment is already known at inception. In a lot of American GMWB's the policyholder has the option to start withdrawing at a point in time which suits them and they also have the right to use only a part of the maximal withdrawal benefit. If we denote with $\psi_k \in [0,1]$ the ratio which the Policyholder chooses to with withdraw, the above formula has to be modified to

$$I_\star(t) \times \psi_t \times R_t.$$

The process $(\psi_k)_{k\in\mathbb{N}_0}$ is called the utilisation process. Again, this process is normally modeled in a very similar manner as the dynamic lapse assumption by modeling it with a suitable set of explanatory variables. This time however the process may be more complex. It can for example be seen that once the policyholder starts to withdraw the likelihood that he reduces utilisation is quite low and in turn ψ_{t-1} is a good predictor for ψ_t. One possibility to model the process ψ is to assume stochastic independence of utilisation process from capital market variables (represented by $(\mathcal{G}_t)_{t\in\mathbb{N}_0}$). In this case one could choose a Markov process (see section 2.2) using a discretised version of the observed utilisation. Denote with $S = \{0\%, 25\%, 50\%, 75\%, 100\%\}$ the underlying state space, and denote ψ_t the Markov chain on S. We note that $t \in \mathbb{N}_0$ denotes the actual policy year and identifies it with the respective ages. If we assume that the current policyholder's age is x_0 and that the policyholder withdraws currently $i \in S$, it is then possible to calculate the probability $P_{ij}(0, t0)$ to withdraw $j \in S$ at age x_0+t by the standard approach using the Chapman-Kolmogorov equation, eg

$$P(s,u) \quad = \quad P(s,t)\, P(t,u),$$

for $s \le t \le u$. It is worth noting that it makes sense to use a time-inhomogeneous Markov chain, because it can be expected that older people normally behave in average differently from younger policy holders. Reconsidering example 13.2, we see that we need to calculate the following quantity:

$$\begin{aligned}\pi(GMWB|\psi_o = i) &= E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \max\{0,(R_k \times \psi_k - FV_k)\} \times I_\star(k)|\psi_o = i\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[E^Q\left[\max\{0,(R_k \times \psi_k - FV_k)\} \times I_\star(k)|\mathcal{G}_k\right]|\psi_o = i\right].\end{aligned}$$

Since we have assumed stochastic independence of ψ from the capital market variables, we need to first calculate

$$\begin{aligned}E^Q\left[\psi_t|\mathcal{G}_t\right] &= E^Q\left[\psi_t\right]\\ &= \sum_{j\in S} j \times p_{ij}(0,t).\end{aligned}$$

Moreover if we assume that ψ_t and $I_\star(t)$ are independent, we can calculate $\pi(GMWB)$ as follows:

$$\pi(GMWB|\psi_o = i) = E^Q\left[\sum_{k\in\mathbb{N}_0} v^k \max\{0,(R_k \times \psi_k - FV_k)\} \times I_\star(k)\big|\psi_o = i\right]$$

$$\begin{aligned}&= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[E^Q\left[\max\{0,(R_k \times \psi_k - FV_k)\} \times I_\star(k)|\mathcal{G}_k\right]|\psi_o = i\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k\, E^Q\left[\max\{0,(R_k \times E[\psi_k|\psi_o = i] - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]\right]\\ &= \sum_{k\in\mathbb{N}_0} v^k\, E^Q[\max\{0,(R_k \times \{\sum_{j\in S} j \times p_{ij}(0,t)\} - FV_k)\} \times E^Q\left[I_\star(k)|\mathcal{G}_k\right]].\end{aligned}$$

Finally we look at an abstract example of the above concept. We assume for the sake of simplicity a time-homogeneous Markov chain and we consider a $x = 65$ year old man. To model the transition matrix $P(1)$ we assume the following:

1. After 5 years 80% of the policyholders withdraw.
2. For each year the percentage of people not taking higher withdrawals and withdrawing one step less (eg 50% instead of 75%) is 5%, the reminder (95%) of the policyholders not withdrawing more remains at the same level.
3. After starting to withdraw we assume that after 5 years, 75% withdraw more.

Note that the above set of assumptions do not allow determining $P(1)$. Which other assumptions have been made (excise), in order for the assumptions

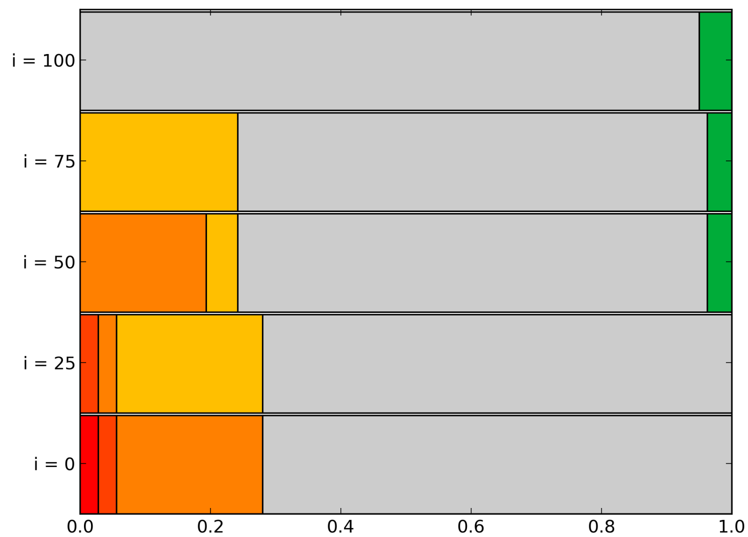


Figure 13.7. Decomposition of $P(1)$ into its parts

result in the following one year transition matrix $P(1)$? Figure 13.7 shows this matrix in graphical form. It shows for each state $i \in S$ the decomposition of the probabilities $\{p_{ij}(1)|j \in S\}$. We know that $\sum_{j \in S} p_{ij}(1) = 1$ and we use reddish colors for transitions with $j < i$ (ie higher utilisation in the next period), gray for the probability $p_{ii}(1)$ and greenish colors for $j > i$ (ie lower utilisation in the next period).

	100%	75 %	50%	25%	0%
100%	0.95	0.05	–	–	–
75%	$1-\alpha$	0.95α	0.05α	–	–
50%	$0.8\,(1-\alpha)$	$0.2\,(1-\alpha)$	0.95α	0.05α	–
25%	$0.1\,(1-\beta)$	$0.1\,(1-\beta)$	$0.8\,(1-\beta)$	β	–
0%	$0.1\,(1-\beta)$	$0.1\,(1-\beta)$	$0.8\,(1-\beta)$	–	β

with

$$
\begin{aligned}
\alpha &= \sqrt[5]{1-75\%} \\
&\approx 0.758 \\
\beta &= \sqrt[5]{1-80\%} \\
&\approx 0.724
\end{aligned}
$$

Figure 13.8 shows the utilisation for the Markov model specified above. As in reality, we can observe in figure 13.8 that the utility increases for a closed book of business. For example we have assumed that the Markov chain is time homogeneous and we would like to remark, that in reality an accelerated increase of the utilisation for older ages would occur.

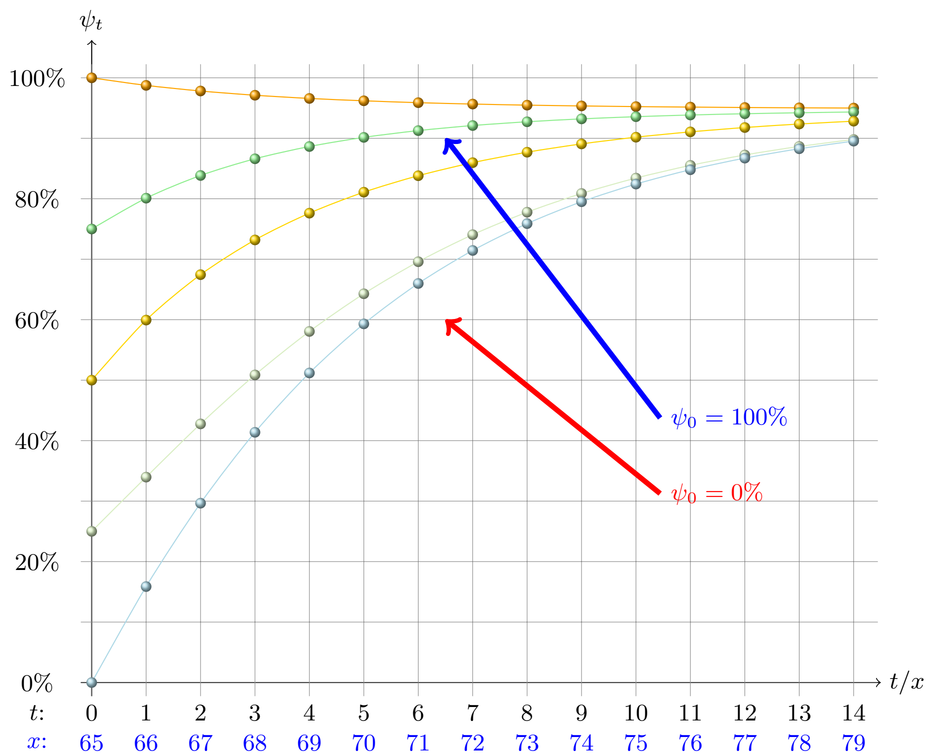


Figure 13.8. GMWB utilisation over time

We end this section with a concrete example of the Markov chain model, which we have introduced before.

Example 13.5.1. We consider the following variable annuity product consisting of a GMDB / GMWB combination:

Product definition: We assume a GMWB product where the benefit base is defined via a ratchet of the funds value, in the sense that at certain periods the benefit base can ratchet up to the funds value. Concretely we consider a product where the ratcheting is either quarterly, half-yearly or yearly. To this end we introduce the following notation:

$FV(t)$	denotes fund value at time t (before withdrawal),
$GV(t)$	denotes benefit base (guaranteed) value at time t,
$R(t)$	annuity paid at time t
$\psi(t, t+\Delta t)$	denotes the fund performance from time t to $t+\Delta t$, and
$\mathfrak{T} \subset \mathbb{R}^+$	denotes the set of times at which a ratchet takes place.

In this set up we have the following (assuming for sake of simplicity that $\mathfrak{T} \subset \{k \times \Delta t | k \in \mathbb{N}_0\}$ and also that annuity payments only take place at direct times $k\,\Delta t$ for some $k \in \mathbb{N}_0$):

$$
\begin{aligned}
FV(0) &= EE > 0 \\
GV(0) &= FV(0) \\
FV((k+1)\Delta t) &= (FV(k\,\Delta t) - R(k\,\Delta t))\,\psi(k\,\Delta t, (k+1)\,\Delta t) \\
GV((k+1)\Delta t) &= \begin{cases} \max\left(GV(k\Delta t), FV((k+1)\Delta t) - R((k+1)\,\Delta t)\right) & \text{if } (k+1)\,\Delta t \in \mathfrak{T}, \\ GV(k\Delta t) & \text{else.} \end{cases}
\end{aligned}
$$

The death benefit is defined as the maximum of the current funds value and the difference between the current $GV(t)$ and the annuities paid out until this point in time (ie $\sum_{k \in \mathbb{N}_0}^{k\,\Delta t \le t} R(k\,\Delta t)$) until age 85. Afterwards there is no death benefit.

Annuity Definition: The annuity can be withdrawn at times $\mathfrak{S} \subset \{k \times \Delta t | k \in \mathbb{N}_0\}$ and it amounts at time $t \in \mathfrak{S}$ to $\rho(\xi_0) \times GV(t)$, where ξ_0 is the first time $\xi_0 \in \mathfrak{S}$ where the person can withdraw. The person is allowed to withdraw less than this amount in line with the model as defined before.

Fund dynamics: The fund performance follows a geometric Brownian Motion with a risk free interest rate of 2.5% and a volatility of 16.5%.

Policy: We assume a 65 year old policyholder which invests 100'000 $.

Based on this product two different values for the guarantee have been calculated, namely the guarantee based on the best estimation parameters for the Markov chain X_t and with respect to X_t but assuming that the policyholder will never reduce the utilisation. The following table shows the respective result for the best estimate utilisation:

GMDB			
Start state	100%	50%	0%
Ratchet			
Quarterly	2855.3	3432.0	4572.3
Half-yearly	2650.9	3193.0	4257.1
Yearly	2450.5	2904.1	3888.6
GMWB			
Start state	100%	50%	0%
Ratchet			
Quarterly	4439.0	4321.6	4392.1
Half-yearly	4219.0	4107.4	4182.7
Yearly	3966.0	3744.4	3820.7
Total			
Start state	100%	50%	0%
Ratchet			
Quarterly	7294.3	7753.6	8964.5
Half-yearly	6869.9	7300.5	8439.9
Yearly	6416.5	6648.5	7709.3

If we assume in a next step a more conservative utilisation assumption, where the policyholder does not reduce its consumption we get the following results:

GMDB			
Start state	100%	50%	0%
Ratchet			
Quarterly	2256.0	2874.7	4196.2
Half-yearly	2101.9	2663.6	3908.4
Yearly	1928.9	2428.5	3563.6
GMWB			
Start state	100%	50%	0%
Ratchet			
Quarterly	8378.6	8043.2	8069.0
Half-yearly	7989.1	7637.0	7623.3
Yearly	7470.0	7065.1	7037.9
Total			
Start state	100%	50%	0%
Ratchet			
Quarterly	10634.6	10918.0	12265.3
Half-yearly	10091.0	10300.7	11531.8
Yearly	9398.9	9493.7	10601.5

We see for both Markov chain models that the value of the variable annuity guarantee depends on the initial state. Figure 13.9 shows an example of the respective cash flows for the best estimate version. Finally figure 13.10 shows the response function with respect to both models. The x-axis shows the relative equity level together with the values of the GMWB (red) and GMDB (blue) cover for the best estimate cash flows. The black line represents the total value of the variable annuity under the maximum utility model. It can be clearly seen that this second assumption increases the price of the variable annuity and also the hedging costs. The latter one can be seen by comparing the relative changes in value for the two models for example for a -40% downwards stress (at eg going down from 1.0 to 0.6).

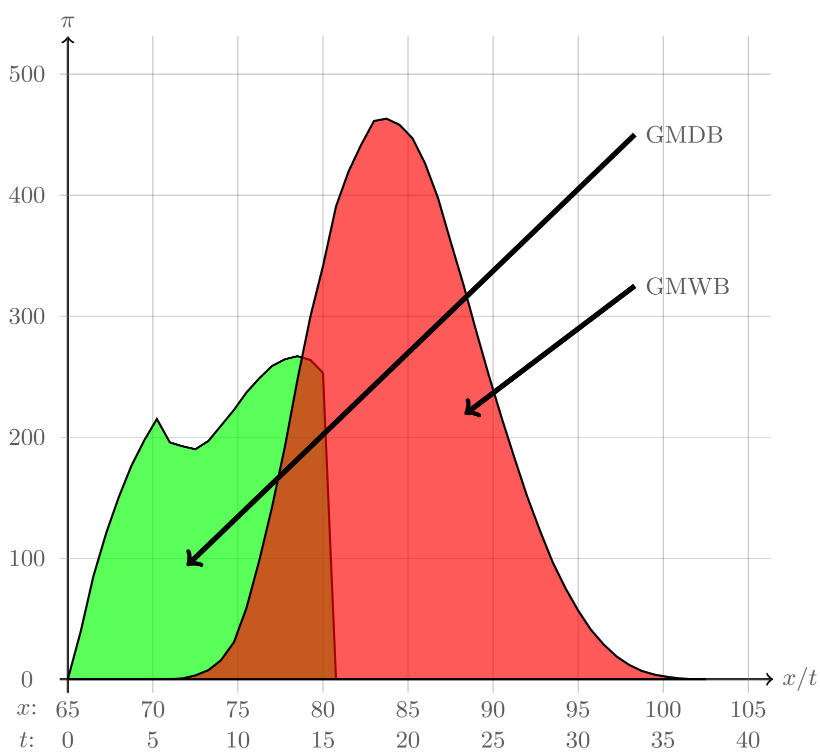


Figure 13.9. Risk neutral cash flows for quarterly ratchet and $\psi_0 = 50\%$ and $x = 65$

Based on this last example we can now look at the different response functions of the various types of ratchets and states. Figure 13.12 compares them for a 60 year old policyholder. We see that for a downward stress (eg equity levels < 1), the different types of ratchets start to convert to the same value.

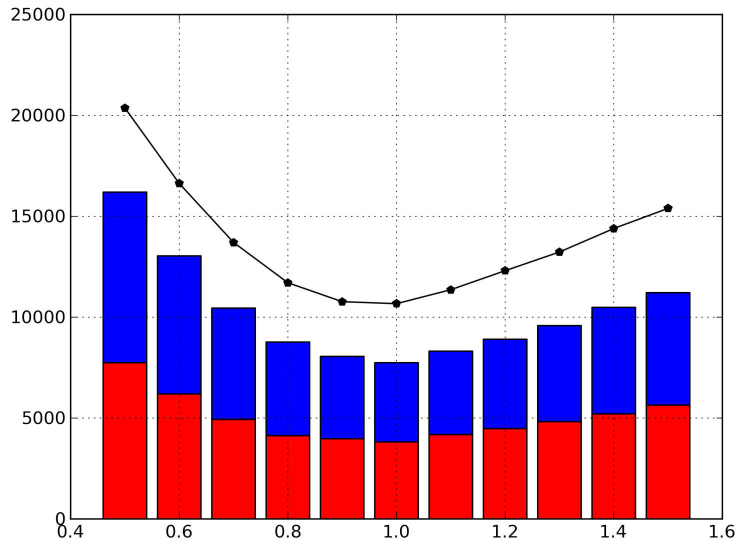


Figure 13.10. Trading Grid for quarterly ratchet and starting state 50%

This is due to the reduced likelihood of a speedy recovery to allow the policy to ratchet up again. For the upward stress we can see that all the nine different combinations yield to different liability values (and also to different hedging requirements). It is also worth pointing out that there is considerable dependency on the starting state of the Markov model – the value of the variable annuity guarantee varies depending on whether the policyholder starts to immediately withdraw his funds. We see that the policyholder who starts withdrawing immediately has the highest hedging requirement for material downwards stresses. At the same time the value of the variable annuity guarantee is the smallest (with respect to the other cases) for a market which has a strong positive performance. These results are in line with the comments of the table in section 13.1.

We note it is quite simple to combine the effects of dynamic lapsing and utilisation. The simpler possibility is to assume that these two effects are indecent and to calculate the hedge liability this way. It is also possible to estimate (and then simulate) a conditional Markov chain for utilisation and lapse, conditional to explanatory variables such as product type, in-the-moneyness etc. If choosing this approach, one needs to consider the state ‡ (lapse) explicitly as part of the Markov chain.

Example 13.5.2. We want to end this section by providing a link between the deterministic function ψ_t and the class of Markov models introduced pre-

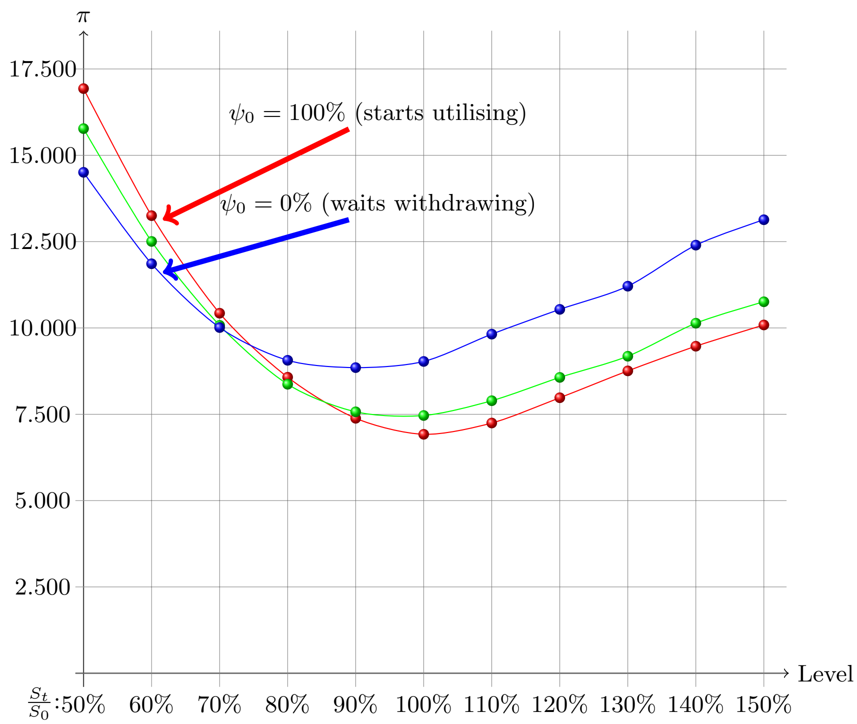


Figure 13.11. Comparing Trading Grids for 65 year old Policyholder (different states)

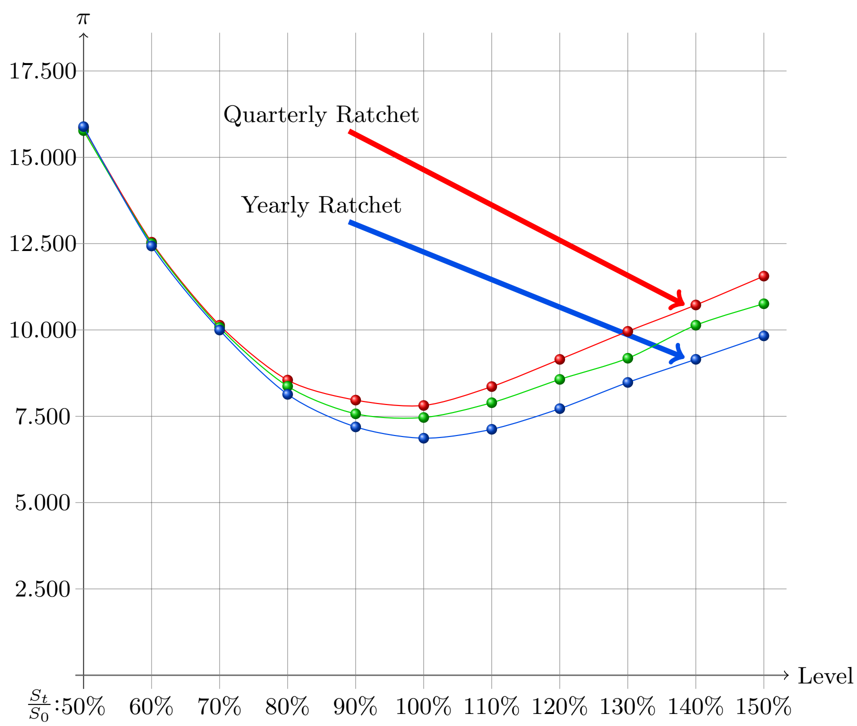


Figure 13.12. Comparing Trading Grids for 65 year old Policyholder (different ratchets)

viously. Such a link may be useful in interpreting deterministic utilisation assumptions $(\psi_t)_{t\in\mathbb{N}_0^+}$. Such assumptions can be based either on expert judgment or observed utilisation pattern. In order to illustrate this link we use a Markov model with two states, namely "N" for the inforce which is not utilising at a certain age and "U" for the inforce utilising. We assume that each policyholder either utilises at 100% or not at all, noting that these 4 model assumptions can easily be generalised. Furthermore we assume that we have given an utilisation pattern $(\psi_t)_{t\in\mathbb{N}_0^+}$ and the task is to determine the Markov probabilities intrinsic to these assumptions. Assume for example:

x	ψ_x	ψ_{x+1}	ψ_{x+2}	ψ_{x+3}	ψ_{x+4}
65	0.2500	0.3354	0.4207	0.5061	0.5914
70	0.7518	0.7663	0.7663	0.7663	0.7663
75	0.7663	0.7663	0.7963	0.7963	0.7963
80	0.7963	0.7963	0.7963	0.7963	0.7963
85	0.7963	0.7963	0.7963	0.7963	0.7963
90	0.7963	0.7963	0.7963	0.7963	0.7963

Hence the task is to determine the initial distribution of the states and the respective transition probabilities. In order to do this we denote with $w_0(x)$ and $w_1(x)$ the probability distribution of the Markov chain at age x. Furthermore we denote with $r(x)$ and $q(x)$ the probabilities at age x to not utilising. Hence we have:

$$
\begin{aligned}
p_{00}(x) &= r(x), \\
p_{01}(x) &= 1 - r(x), \\
p_{10}(x) &= q(x), \text{ and} \\
p_{11}(x) &= 1 - q(x).
\end{aligned}
$$

We also denote by

$$P(x) = (p_{ij}(x))_{(i,j)\in S\times S}$$

the corresponding transition matrix. By definition of the transitions matrix we know that

$$(w_0(x+1), w_1(x+1)) = (w_0(x), w_1(x)) \cdot P(x),$$

or in coordinate form

$$\begin{aligned} w_0(x+1) &= w_0(x) \times p_{00}(x) + w_1(x) \times p_{10}(x), \text{ and} \\ w_1(x+1) &= w_0(x) \times p_{01}(x) + w_1(x) \times p_{11}(x). \end{aligned}$$

We see that there are two unknown variables for each age, namely $r(x)$ and $q(x)$ and one boundary condition ψ_x. Therefore the equation system will be under-determined and we therefore need an additional assumption. Based on the estimation of the transition matrices and the raw inforce data, it is reasonable to assume $q(x) = 5\% \, \forall x$. With this additional assumption we can now solve the problem, by recognising that ψ can be expressed by $\boldsymbol{w}$ via

$$\begin{aligned} \psi_x &= 0 \times w_0(x) + 1 \times w_1(x) \\ &= w_1(x) \end{aligned}$$

Therefore we immediately get the boundary condition (for age 65)

$$\begin{aligned} w_0(65) &= 0.75, \text{ and} \\ w_1(65) &= 0.25. \end{aligned}$$

By means of the recursion we can now solve for $r(65)$ again considering the needed boundary condition for $\boldsymbol{w}(66)$:

$$\begin{aligned} 0.6646 = w_0(66) &= 0.75 \times r(65) + 0.25 \times q(65), \\ 0.3354 = w_1(66) &= 0.75 \times (1 - r(65)) + 0.25 \times (1 - q(65)), \text{ and} \\ q(65) &= 0.05. \end{aligned}$$

This results in $r(65) \approx 87\%$. Figure 13.13 shows the values for $1 - r(x)$ and we can observe two marked spikes in the utilisation uptake probability $(1 - r(x))$ at ages of about 70 and 75, which stem from the tax treatment and the structure of the underlying product. The following table shows the numerical values when solving for r, q and $\boldsymbol{w}$ respectively:

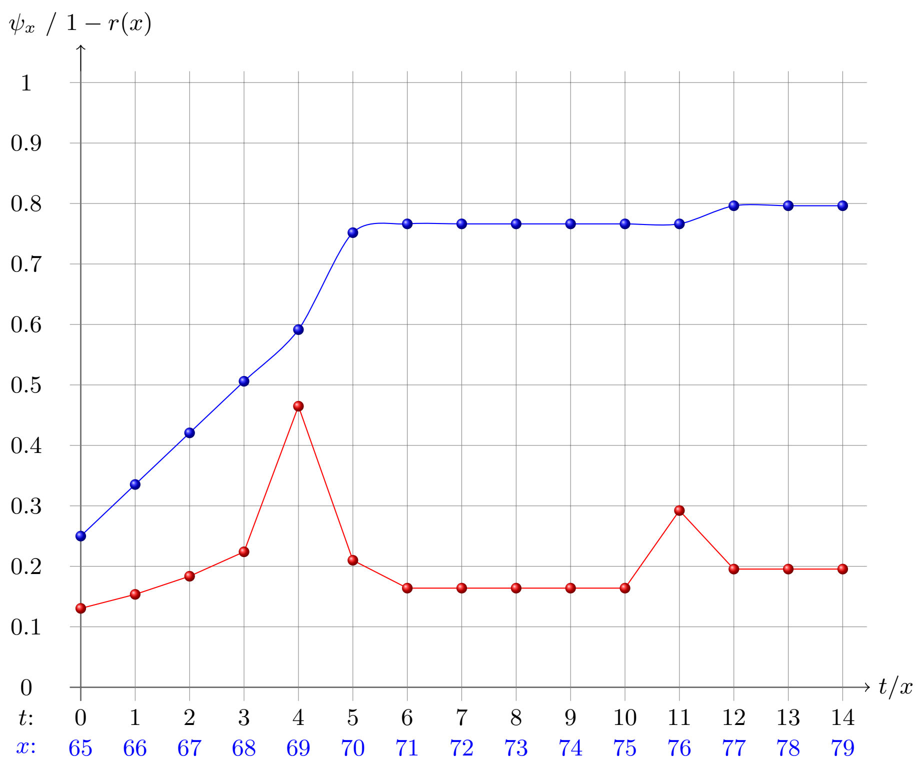


Figure 13.13. Implicit Markov Transition Probability $r(x)$

x	ψ_x	$q(x)$	$r(x)$	$w_0(x)$	$w(1)$
65	**0.2500**	0.0500	0.8695	0.7500	**0.2500**
66	0.3353	0.0500	0.8463	0.6646	0.3353
67	0.4207	0.0500	0.8163	0.5793	0.4207
68	0.5060	0.0500	0.7759	0.4939	0.5060
69	0.5914	0.0500	0.5351	0.4086	0.5914
70	0.7517	0.0500	0.7899	0.2482	0.7517
71	0.7663	0.0500	0.8360	0.2337	0.7663
80	0.7963	0.0500	0.8045	0.2037	0.7963
90	0.7963	0.0500	0.8045	0.2037	0.7963

13.6 Other types of policyholder behaviour

Besides lapse and utilisation there are some more types of policyholder behaviour which we mention mainly for the sake of completeness. It is important to understand policyholder behaviour in the context of the product, since policyholder behaviour impact is intrinsically linked with the way the product works. From this point of view the treatment in this section is necessarily limited. We also note that the techniques that we have seen previously can also be applied in such a context as “mutatis-mutandis”, eg one needs to change the things that need to be changed.

The following lists some types of policyholder behaviour and the corresponding impact:

Demographic anti-selection: Under demographic anti-selection we understand that people buy certain products because they have a better understanding of their health status than an insurance company. Assume you have a GMWB benefit for life, where people can elect to withdraw their money or to leave it with the company. If a person perceives himself particularly healthy he will most likely opt for a GMWB benefit for life, because he is of the opinion that he will get this benefit cheaply. On the other hand a rather ill person will buy a GMDB cover in order to protect their inheritance. This will lead to an anti-selection, which is also present in most traditional products. In order to avoid this arbitrage opportunity it is essential to design the products in such a way that the financial benefit of such an arbitrage remains small. Moreover for death benefits it is normal that a material increase in the level of death benefits trigger a medical examination of the policyholder. A good example of a benefit design which is rather immune for arbitrage is a guaranteed annuity in payment. Assume a 65 year old policyholder has purchased an immediate payout annuity which is guaranteed for 15 years. One year later he decides to repurchase (lapse) the policy. The canonical method to avoid arbitrage is to pay back the guaranteed part of the annuity (eg after a year the 14 year guaranteed part of the annuity). After this partial lapse the policyholder will still receive a 14 year deferred annuity. Hence if he survives the remaining 14 years of the guarantee, he will receive the respective annuity payments. This product design is rather immune against anti-selection when ill people try to lapse the policy.

Change in asset allocation: Some variable annuity writers offer the possibility to switch between investments funds. We have seen that the price of the variable annuity guarantee is a function of the underlying volatility (σ) of the fund. If the guarantee fee for the variable annuity does not depend on the actual asset allocation of the policyholders funds, the insurance company has to bear the corresponding policyholder behaviour risk. Concretely, the values of the variable riders will increase if the policyholder chooses more risky funds. Again this type of risk is not existent for all types of variable annuities and it can be (partially) migrated by limiting the funds' choice or by charging the effective economic price of the rider in function of the concrete asset allocation.

Top up's: In many variable annuity products the policyholder has the ability to pay in additional funds at his digression and he can potentially benefit from the pricing level at inception of the policy, also in case the economic environment has changed. Assume for example that the pricing level has risen since inception of the policy due to a higher level of volatility and lower interest rates. In this case it would be economically beneficial for the policyholder to invest additional money since he would get the guarantee cheaper.

Partial Withdrawals: Partial withdrawals are the converse of top-ups. Here also it might be economical for the policyholder to partially withdraw money at certain times. Again this question is best addressed (as with top-ups) by an adequate product design.

Premium holidays: Typically, variable annuities are single premium products. In case they are financed by a regular premium, the policyholder also has the possibility to take premium holidays.

In order to better understand the effect of the strategic asset allocation of the policyholder we look in a first step at the response function of the hedge liability with respect to example 13.5.1. Figure 13.14 shows the dependency of the value of the hedge liability with respect to varying fund volatility σ. We see that higher volatility means a higher value of the variable annuity and also the shape changes. It becomes obvious that a lower volatility means in general a steeper response function for a equal amount of equity level shift. This has however to be put into context, since a material reduction in

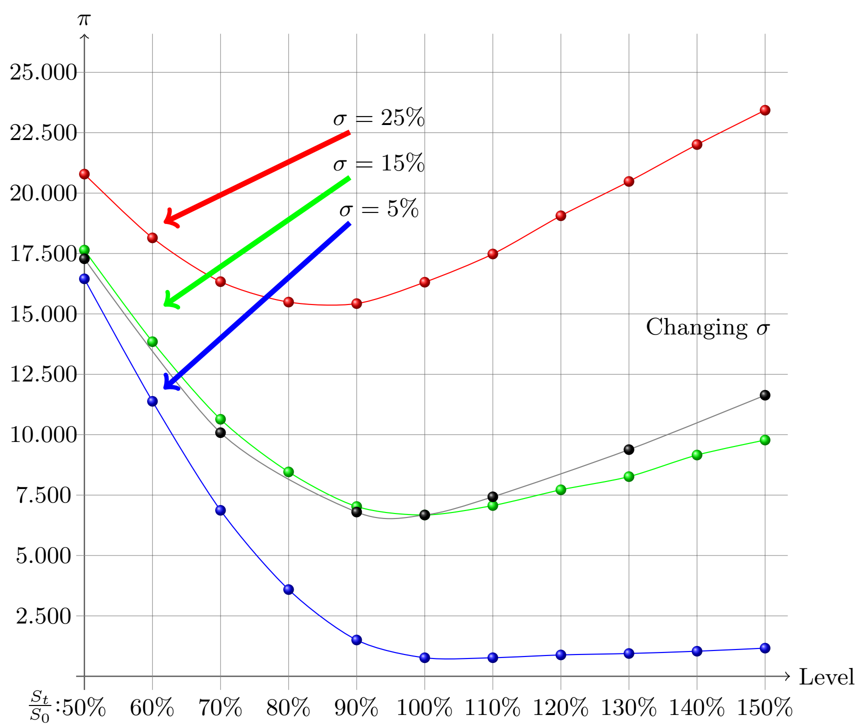


Figure 13.14. Response function for a 65 year old with $\psi_0 = 50\%$

equity values (eg -50%) is much less likely with a lower volatility. In turn the recovery in a low volatility environment would be commensurately more rare.

In respect to policyholder behaviour the asset allocation piece depends crucially on the approach the policyholder is taking. In principle he can take an active or a passive approach. When taking a active approach the policyholder would actively re-balance his investments in order to maintain the desired asset allocation. Adopting a passive approach would mean that he does not re-balance the his assets. This is best explained with an example.

Assume that there are two investment at disposition of the policyholder: cash (with $\sigma = 0\%$) and equities (with $\sigma = 20\%$). Also we assume that at time $T = 0$ the policyholder invests 75% of his assets in equities, the remainder in cash. This leads to a volatility of his investments of $\sigma = \frac{3}{4} \times 20\% = 15\%$.

Adopting an active approach the equity portion will stay at 75% and in consequence $\sigma = 15\%$. The following table shows how the volatility will change for an immediate change in equity values, assuming a passive approach. We see that now the volatility can not be assumed to be constant.

	Bond Value	Equity Value	Volatility
-50 %	0.250	0.375	12.0%
-30 %	0.250	0.525	13.6%
-10 %	0.250	0.675	14.6%
–	0.250	0.750	15.0%
10 %	0.250	0.825	15.4%
30 %	0.250	0.975	15.9%
50 %	0.250	1.125	16.4%

Figure 13.14 shows the consequence of adopting a passive re-balancing approach in terms of the value of the hedge liability. We can observe that a passive re-balancing approach necessitates a higher amount of protection for the ratchets than using an active re-balancing. As a consequence of this there are variable annuity products in the market where the underlying funds is actively re-balanced in order to have lower hedging costs.

13.7 Summary

In this section we want to summarise the formalism we have learned in this section to model policyholder behaviour and to put it in context to the formulation of the respective processes in terms of stochastic integrals. We will consider the following two effects for policyholder behaviour: lapse and utilisation. As before we will assume the Black-Scholes-Merton framework for the share-price process and we denote with $(W_t)_{t \in \mathbb{R}^+}$ the standard Brownian motion. The demographic and policyholder behaviour process is based on a

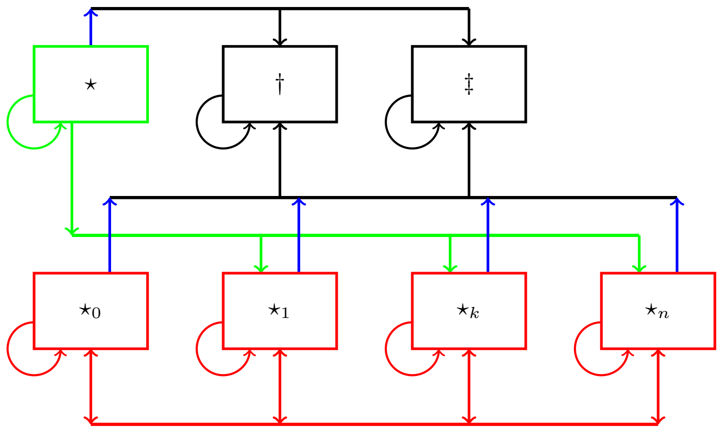


Figure 13.15. Modelling Policyholder Behaviour

Markov Chain (eventually conditional to the moneyness as we have seen in section 13.4. In this context we use the following states:

Alive – Never Utilised: This is the canonical starting state and we denote it by $\star$.

Alive – Utilising: We denote this states with $\star_0$ to $\star_n$ with corresponding utilisation $\psi_t(j)$. We note that we have introduced this model in section 13.5 and that the state $\star_0$ is considered not utilising after having utilised beforehand. For convenience reasons we denote the set of all utilising states with $S = \{\star_k \,|\, k = 0, 1, \ldots, n\}$.

Death: We denote this state $\dagger$.

Lapse: We denote this state $\ddagger$.

State Space: The state space of the entire demographic model is denoted $\tilde{S} = S \cup \{\star, \dagger, \ddagger\}$. Figure 13.15 shows the state-space diagram for this model.

We note that the model above can be chosen more generic that in the prior section by making all transition intensities μ_{ij} conditional to the moneyness. Until now we have used this technique only for lapsing eg for the intensities $\mu_{i\ddagger}$, $i \in \tilde{S} \setminus \{\ddagger\}$. It is worth pointing out that such a generalised model heuristically makes sense, since policyholder may tend to withdraw more if the funds starts to be depleted and in consequence the policy becomes more "in-the-money". We note that the main complexity of such a model is the need for a complex estimation of the respective intensities as a function of moneyness (eg $\mu_{ij}(itm)$).

Fund return: The fund return is given in the Black-Scholes-Merton context via

$$dF = \eta\, F\, dt + \sigma\, F\, dW.$$

Benefit Base: The benefit base B fulfils the following

$$B_0 \quad = \quad f(F_0).$$

This means that the start benefit base depends on the initial fund value. The propagation of the Benefit base depends on the underlying guarantee. There are various possibilities such as:

$$\begin{aligned} dB &= \bar{\eta}\, B\, dt \text{ Geometric Increase, or} \\ dB &= \bar{\eta}\, dt \text{ Linear Increase, or} \\ dB &= \sum_{k=1}^{N} \delta_{\cdot, t_n}((F_{t_n} - F_{t_{n-1}}) - (A_{t_n} - A_{t_{n-1}}) - (P_{t_n} - P_{t_{n-1}}))^+, \end{aligned}$$

where we have a series of ratchets at times $\{t_1, t_2, \dots t_N\}$ in the latest case.

Guarantee Premium: The guarantee premium $P_0 = 0$ and

$$dP \quad = \quad \frac{P}{F}\, dF + \bar{\xi}\, \chi_{\{F-A-P>0\}} B\, dt.$$

Withdrawals (A): There are two withdrawal processes. The first one in order to get the right guarantee premium. Since we do not deduct the annuity paid out from the fund when this happens, but as a sum (integral), we have to allow the total amount of annuities paid out to grow in line with fund return:

$$dA \quad = \quad \frac{A}{F}\, dF + \left(\sum_{j \in S} I_j\, \psi_t(j) \right) B\, dt.$$

Since the death benefit (below) is defined as the reminder of the benefit base we need also keep track of the sum of annuities paid out:

$$dAS \quad = \quad \left(\sum_{j \in S} I_j\, \psi_t(j) \right) B\, dt.$$

Since the death

GMWB Guarantee Process (G):

$$G = -(F - A - P)^+.$$

Death Benefit (DB), GMDB Guarantee: We limit ourselves to a very death benefit, the return of the Benefit base in case of death.

$$dDB = \sum_{i \in S \backslash \{\dagger, \ddagger\}} \left((B - AS)^+ - (F - A - P)^+ \right)^+ \times dN_{i\dagger}.$$

Residual Guarantee Premium / Value:

$$\pi_t(G + DB) = E^Q[e^{\delta\, t} \times \int_t^\infty e^{-\delta\, \tau} (dG_\tau + dDB_\tau) | \mathcal{F}_t].$$

We note that $-\pi_t(G + DB)$ is the economic value of this policy from the company's point of view. In case of using the equivalance pricicple, one would have the boundary condition

$$\pi_0(G + DB) = 0.$$

We would like to finally note that the above quantities are normally calculated by using a simulation approach.

Exercise 13.7.1. 1. Use the formalism above to model a GMDB/GMWB product in discrete time.

2. Use the formulae to calculate the value of this product by means of simulation.

14. Hedging of Variable Annuities – Application of Arbitrage Free Pricing

14.1 Introduction

In this section we look at hedging a variable annuity and how this can be achieved. From an abstract point of view the value of the variable annuity depends on several factors. The aim of hedging is to reduce the risk for the insurance company in case of market movements and market shocks.

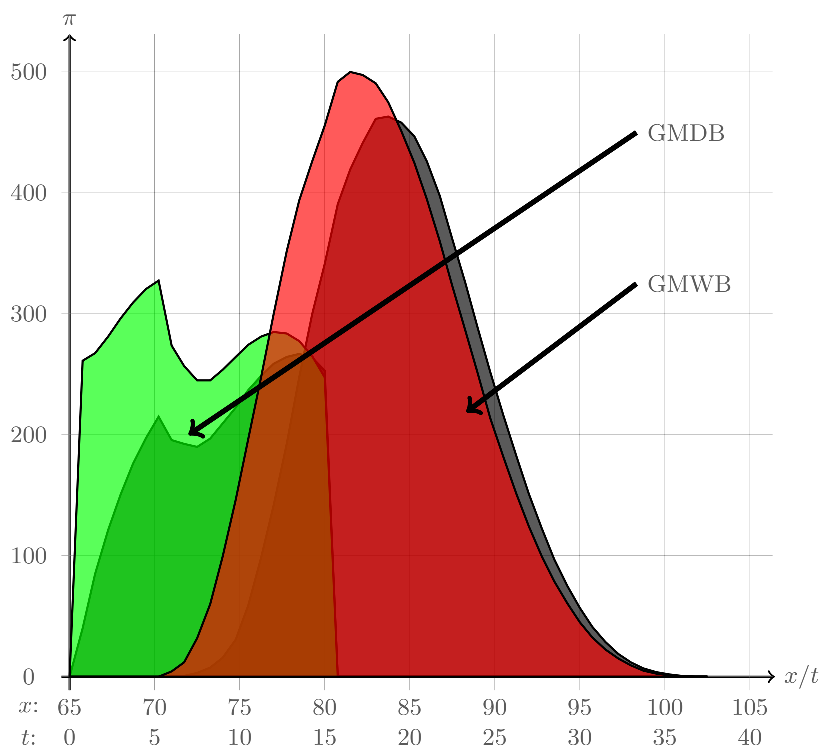


Figure 14.1. Cash-flows under stress (-30%) with $\psi_0 = 0.5$ and $x = 65$

Let's look again at example 13.5.1 and analyze what happens to the expected cash flows under the risk neutral measure Q if we allow for an immediate 30% equity fall. Figure 14.1 illustrates this. One can see the original (ie unstressed cash flows) together with the cash flows under stressed conditions (in green and red) versus the unstressed cash flows as per figure 13.9. The value of the product under the neutral position amounts to 7'166 (3'090 for GMDB and 4'076 for GMWB). Under the stress the corresponding option value increases by c2'200 to 9'422 (4'485 for GMDB and 4'936 for GMWB). This becomes apparent when considering figure 14.1. We see that the GMDB cash-flows in particular spike at the early years. Also the GMWB cash-flows increase in size and the maximum is reached earlier.

This clearly shows there is a need to off-set the corresponding potential loss of c2'200 by using a corresponding hedging strategy. The following table shows how the option prices vary with changing equity stresses:

Sensitivity

	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
DB	6'570	5'463	4'485	3'676	3'189	**3'090**	3'249	3'545	3'845	4'189	4'522
WB	7'343	5'969	4'936	4'352	4'104	**4'076**	4'299	4'823	5'224	5'542	5'871
Total	13'913	11'432	9'422	8'028	7'294	**7'166**	7'549	8'369	9'069	9'732	10'394

We see that the value of the variable annuity increases both for falling equity prices (as a consequence of the guarantee being further in the money) and also for increasing equity levels (as a consequence of the respective ratchets). The idea of hedging is to reduce the impact of changing equity levels by having a mitigation strategy in place. There are various possible objectives for such hedging strategies in terms of:

– the general underlying hedging strategy,

– the instruments to be used, and

– the cost for the respective hedging strategy.

In order to define a hedging strategy, one has to find a suitable set of assets which replicate the liabilities as best as possible in order to then define a concrete hedge. It is key to define the meaning of a hedging strategy and the respective metrics to be considered. A hedging strategy needs to consider and establish objectives in respect of:

Economic Risks: Which economic risks are hedged and to what extent?

Financial Statement Risks: How important are the risks regarding the publicly stated accounts and to what extent is there a need to hedge them?

Regulatory Capital Risks: What are the regulatory capital risks and to which extent need they to be hedged?

Therefore a hedging strategy needs to establish objectives for the different dimensions and define a corresponding risk appetite.

14.2 Hedge Strategies

To better understand the different hedging strategies, it is in a first step important to remember the fundamental principle of replicating a contingency claim. For this we refer back to section 8.3, where we have seen the Black-Scholes-Merton differential equation and Theorem 8.3.27 which also states that a contingency claim can be replicated perfectly by a suitable dynamic hedging strategy $\Phi = (\Phi^1, \Phi^2)$ (see definition 8.3.18). Given that that the amount of shares we hold equals $\Delta = \frac{\partial v}{\partial S}$, the partial derivative of the price of the contingency claim, such a hedging stratgey is called a *replicating* Δ-strategy. Refinements of these dynamic replicating strategies are

Trivial Hedge: Nothing is hedged and the insurance company keeps the entire risk;

Δ-hedge: Only the equity is hedged, no interest rate hedge. A Δ–Γ–hedge is a variant of this, where equities are hedged more accurately than with a pure Δ–hedge;

Δ–ρ–hedge: Interest rates and equities are hedged; and

3 greeks hedge: Δ, ρ and the equity volatility ν is hedged.

It is intrinsic to all such strategies that the contingency claim is replicated by a continuous re-balancing of the underlying hedging portfolios and in the theoretical set up Theorem 8.3.27 ensures perfect replication. There is however a practical problem with these strategies, which becomes very apparent, when looking at the days after 11.9.2001. We remember that on the that day there was a terrorist attack on New York which resulted in the collapse of the twin towers and the New York Stock exchange was closed for several days. The following table shows the S&P 500 index for these days:

Date	S&P 500	Change to next trading day
10.9.2001	1092.54	-11.6 %
11.9.2001	closed	n/a
12.9.2001	closed	n/a
13.9.2001	closed	n/a
14.9.2001	closed	n/a
15.9.2001	closed	n/a
16.9.2001	closed	n/a
17.9.2001	1038.77	-3.40 %
18.9.2001	1032.74	-1.98 %
19.9.2001	1016.10	-0.89 %

It becomes obvious that as a consequence of the closed market, a continuous rebalancing was not possible, resulting in a corresponding non-nil hedge P&L. We will in the following look at the effect of material daily index movements in further detail. We remark again that there were several trading days in the autumn 2008 where we observed daily losses of 2 % or more. Even bigger equity market swings have been observed beginning May 2010, after fears of state bankruptcies in the Euro zone. On Thursday 6.5.2010 the NYSE (Dow Jones Industrial Average) fell temporarily over 9 %, because of such fears and automated trading. The same day Procter and Gamble lost temporarily more than 35 % of its value. On Monday 10.5.2010 the Euro Stoxx index performed 10.35 % within one day, after the announcement of a EUR 750 bn bail-out plan. Assuming a volatility of 20 % and log-normally distributed equity-market returns, this represents a $9.8 \times \sigma$–event. Such an event has a return period of 5.9×10^{17} years. This number is considerably bigger than the age of the universe of 1.375×10^{10} years and hence it is obvious that the log-normally distributed model is not correct in the tails. Figure 14.2 shows how the S&P 500 moved between 1.8.2008 and 1.4.2010.

To see the effects of such market movements, mean for a dynamic replicating strategy we will have a look at the following trading grid, which has been normed to 1M USD as at staring date (1.8.2008).

Figure 14.2. S&P 500 performance during 2008 financial crisis

Change in Index	Date	Date
	09.03.09	10.03.09
Index	-46 %	-43% (+c6%)
-50%	3'418'116	3'433'834
-45%	**3'066'963**	**3'082'383**
-40%	2'743'110	2'758'206
-35%	2'444'987	2'459'706
-30%	2'171'849	2'186'127
-25%	1'923'926	1'937'682
-20%	1'702'022	1'715'171
-15%	1'507'336	1'519'804
-10%	1'341'427	1'353'103
-5%	1'206'280	1'217'046
0%	1'102'989	1'112'803
5%	1'027'233	1'036'330
10%	972'206	980'768
15%	933'233	941'424
20%	906'734	914'666
25%	890'216	897'973
30%	881'652	889'293
35%	879'287	886'878
40%	881'766	889'348
45%	888'049	895'669
50%	897'369	905'055

Figure 14.3 shows how the underlying dynamic Δ-hedging strategy (red) has performed over this period and we see that the risk mitigation is moderate. To better understand the dynamics, it is important to understand, that we have assumed to rebalance the corresponding replicating portfolio once a day. Let us look of what would have happened between the 8.3.2009 and 9.3.2009. The following table summarises the relevant facts:

```
S&P Date                 09.03.09        10.03.09
Trading Grid Date        09.03.09        10.03.09
Index (100%=1.1.08)      -46%            -43%
alpha (weight grid)      0.735930        0.419412
alpha*(weight delta gr)  0.235930        0.919412

Liabilities
-----------
Price current           3'159'692       2'946'419
Price next day          2'931'135
Delta current           6'894'246       6'521'056

Assets
------
Short futures          -6'894'246
Cash                   10'053'939
```

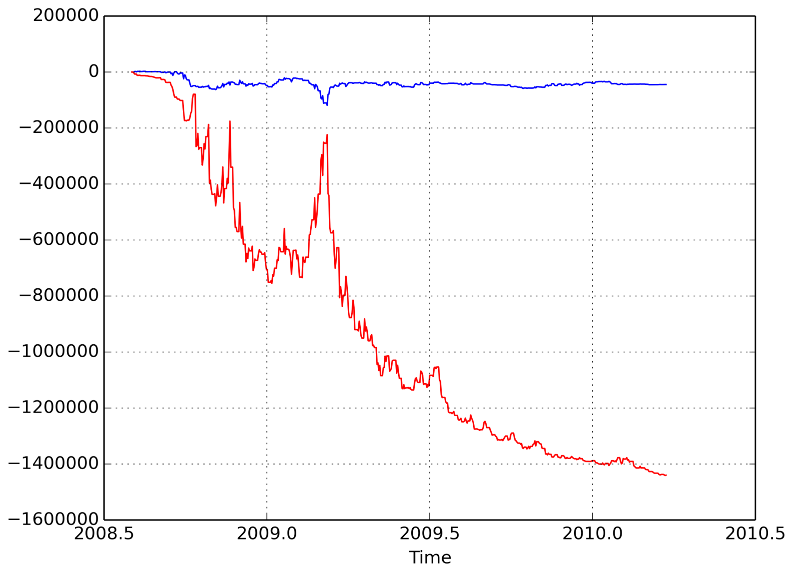


Figure 14.3. Performance of a dynamic Δ-hedging strategy during the 2008 financial crisis

```
Profit and Loss (day)
---------------------

Change in Liabilities                       228'557
Mark on Futures                            -438'909

                                          ---------
P&L w/o interest                           -210'352
                                          =========
```

We can see from this example that the loss of the dynamic Δ–hedging strategy is a consequence of the changing Δ in function of the underlying equity levels. To improve the quality of the hedge there are various possibilities. In the context of a dynamic replicating strategy, one could in a first instance decrease the times between rebalancing the dynamic asset portfolio. The problem is that this does not help in case the equity market being closed. The other canonical approach to mitigate the underlying risk is to use a higher order Taylor approximation of the trading grid:

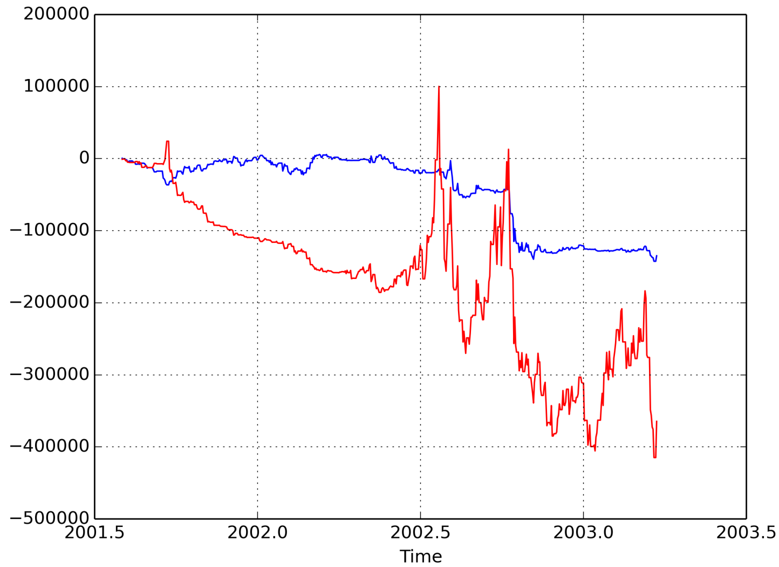


Figure 14.4. Performance of a dynamic Δ-hedging strategy during the 2001 financial crisis

$$\begin{aligned}\Delta V(S,t,r,\sigma) &= \underbrace{\frac{\partial V}{\partial S}}_{Delta}\Delta S + \frac{1}{2}\underbrace{\frac{\partial^2 V}{\partial S^2}}_{Gamma}(\Delta S)^2 + \underbrace{\frac{\partial V}{\partial t}}_{Theta}\Delta t \\ &\quad + \underbrace{\frac{\partial V}{\partial r}}_{Rho}\Delta r + \underbrace{\frac{\partial V}{\partial \sigma}}_{Vega}\Delta\sigma + \ldots\end{aligned}$$

Another solution to the same problem is to complement the dynamic hedging strategy by static hedges. Hence the idea is to buy an asset (hedging) portfolio which approximates the liability trading grid. Such a approximation does not need to be perfect, since the reminder can still be dynamically hedged. Such a hedging strategy is called semi-static replicating hedging strategy or also replicating macro-hedging strategy. Hence the overarching aim is still to replicate the variable annuity liabilities by a replicating $\Phi_L = (\Phi_L^1, \Phi_L^2)$. In case static hedges $(\mathcal{A}_i)_{i=1,\ldots n}$ have been chosen to approximate the insurance liabilities, we know that each $\mathcal{A}_i$ can be generated by Φ_i. In consequence the dynamic part of the hedging strategy can be calculated as

$$\Phi_D = \Phi_L - \sum_{i=1}^{n} \Phi_i$$

since

$$\Phi_L = \Phi_D + \sum_{i=1}^{n} \Phi_i.$$

In principle one can still define a more general macro-hedging strategy by not requiring a perfect replication, but instead requiring that the underlying P&L remains within certain levels for a given time period. The blue line of figure 14.3 shows how the corresponding macro-hedging strategy would have performed and we note that this type of strategy is in this case much more resilient versus tail events. Figure 14.4 compares a pure daily rebalanced Δ–strategy with the above mentioned macro-hedging strategy.

In summary we can distinuish between the following hedging strategies:

Trivial Hedge: Nothing is hedged and the insurance company keeps the entire risk.

Dynamic Replicating Hedging Strategy: Δ-hedge, Δ–Γ–hedge; Δ–ρ–hedge: interest rates and equities are hedged; 3 greeks hedge: Δ, ρ and the equity volatility ν is hedged.

Semi-dynamic hedging strategy: This is a variant of the dynamic hedging strategy, where the dynamic strategy is complemented by static hedging assets.

Macro Hedge: The aim here is to rather hedge the tail (or big movement) risks, potentially however trading-off protection against the accuracy of the hedge for smaller magnitude movements.

14.3 Comparison of different hedging strategies

In this section we will compare and contrast some different hedging strategies. The base for determining the hedge assets is the trading grid. We use the following trading grid (see section 8.6.3) for which we will determine the respective hedge assets for the following hedging strategies:

1. Trivial hedging strategy
2. Δ-hedge
3. $\Delta - \Gamma$-hedge
4. Macro/tail hedge

Equity Level	π	Δ	Γ	ρ	ν
	$V(S)$	$\frac{\partial}{\partial S} V$	$\frac{\partial^2}{\partial S^2} V$	$\frac{\partial}{\partial r} V$	$\frac{\partial}{\partial \sigma} V$
-30 %	8442.9	n/a	n/a	n/a	n/a
-20 %	5931.9	n/a	n/a	n/a	n/a
-10 %	4051.7	n/a	n/a	n/a	n/a
0 %	2784.5	-9925.5	-45.31	325170	47226
+10 %	2004.0	n/a	n/a	n/a	n/a
+20 %	1531.3	n/a	n/a	n/a	n/a
+30 %	1232.2	n/a	n/a	n/a	n/a

As we have seen before a hedging strategy has to fulfill are variety of requirements and has to be seen in the context of risk appetite and regulatory capital. In remark 8.3.28 we have explained that a $\Delta - \Gamma$, which is updated in continuous time, can eliminate all market risk and that in consequence the variable annuity liabilities can be perfectly replicated. There are however some impediments to this:

– There is no continuous trading, hence there remains always some residual risk. More importantly stock markets are normally closed outside business hours and there may be considerable changes in equity prices between the closing of the market and the opening next morning.
– The whole concept is based on the principal assumptions made withing the Black-Scholes-Merton model, namely deep, friction less markets and the equity prices following a geometric Brownian motion. While some of these assumptions are at least approximately fulfilled, there are always transaction costs.

As a consequence of these difficulties a variety of hedging strategies has been developed to overcome some of the problems intrinsic to the dynamic $\Delta - \Gamma$ hedging strategy. When considering a a $\Delta - \Gamma$-hedge it is obvious that the protection of the balance sheet takes place at the body of the risk neutral probability distribution given current prices. Hence such a strategy performs normally best in normal markets with only minor changes in equity prices from day to day. If one want to protect the tail, one rather would concentrate on the entries of the trading grid at the more extreme end (such as for a equity price fall of 30%). Hence from an abstract point of view one wants to solve a minimisation problem. Eg given values

$$x_k = f_k(S_k, r_k, \sigma_k, \ldots), \text{ for k } = 1 \ldots n,$$

and weights $(w_k)_{k=1\ldots n}$, $w_k > 0$ one seeks hedge assets $(\mathcal{A}_j)_{j=1\ldots m}$ with corresponding response values

$$y_k(\mathcal{A}_j) = g_k(\mathcal{A}_j, S_k, r_k, \sigma_k, \ldots) \text{ for } k = 1 \ldots n, \text{and for } j = 1 \ldots m,$$

such that

$$\sum_{k=1}^{n} w_k \left(x_k - \sum_{j=1}^{m} y_k(\mathcal{A}_j) \right)^2 \stackrel{!}{=} min.$$

In a concrete set up f_k could for example represent the Δ at current market levels or also the pro-forma loss in case of an immediate equity fall by 20% (ie. $\pi(0.8 \times S(0)) - \pi(S(0)))$. Hence one picks some of the entries of the trading grid with corresponding weights, and seeks for the correct amount of hedge assets such that the differences between variable annuity liabilities and hedge assets becomes minimal. Such an optimisation can also consider a maximal amount of spend to be used for buying hedge assets. In the following we will use the above trading grid to explain this concept in further detail. The minimisation problem was in the concrete set up solved using a numerical method for minimisation.

In order to chose adequate hedge assets one needs to give the optimiser an adequate set of candidate assets. We consider the following hedge assets:

Asset	Remarks
$\mathcal{B}$	This is an investment in cash (ie."bank"). In consequence all Greeks $(\Delta, \Gamma, \ldots)$ are zero.
$\mathcal{F}$	Futures. At current market level the value of the future $\pi(\mathcal{F}) = 0$ with $\Delta = 1$ and $\Gamma = 0$.
$\mathcal{P}(c, n)$	European put option with strike price c and term n.
$\mathcal{C}(c, n)$	European call option with strike price c and term n.

14.3.1 Trivial hedge

The connotation of a "Trivial Hedge" is a synonym for not hedging at all and assuming full absorption capacity of the market value changes by the shareholder equity. It is obvious from the above example that this is a very risky strategy and during the 2008' financial crises some fallout has been seen for some of the companies utilising a trivial hedging strategy. In the best case the lost a sizable amount of money and had to close their variable annuity portfolio for new business.

14.3.2 Delta - hedge

The Δ hedge is the most simple of all possible non trivial hedging strategies. In the set-up of above there is just one function f which is the $\Delta(S(0))$. Most commonly one uses futures as corresponding hedge assets. The Δ of a future is 1, with $\Gamma = 0$. Hence we have the following set up:

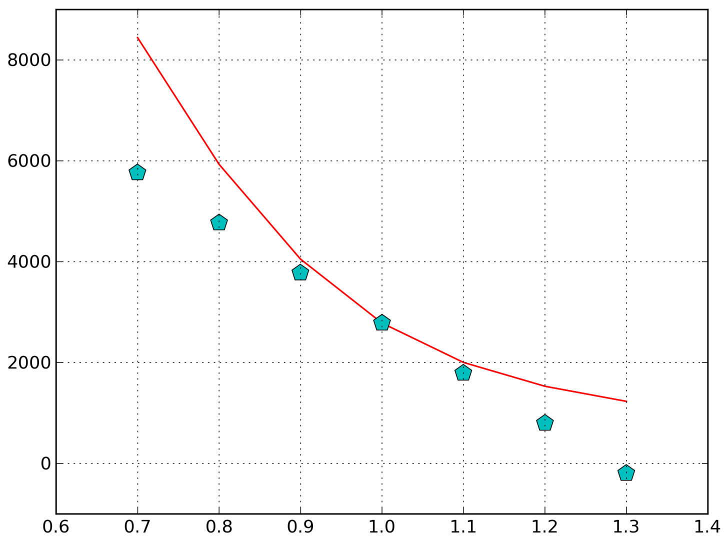


Figure 14.5. Adjusted Comparison

	$\frac{S_t}{S_0}$	Δr	$\Delta\nu$	Type	Value	w_k
1	0.70	0.00	0.00	π	8'442.90	0.00
2	0.80	0.00	0.00	π	5'931.90	0.00
3	0.90	0.00	0.00	π	4'051.70	0.00
4	1.00	0.00	0.00	Δ	-9'925.50	1.00
5	1.00	0.00	0.00	Γ	-45.31	0.00
6	1.00	0.00	0.00	π	2'784.50	10'000.00
7	1.00	0.00	0.00	ρ	325'170.00	0.00
8	1.00	0.00	0.00	ν	47'226.00	0.00
9	1.10	0.00	0.00	π	2'004.00	0.00
10	1.20	0.00	0.00	π	1'531.30	0.00
11	1.30	0.00	0.00	π	1'232.20	0.00

As explained above the candidate assets consist of $\mathcal{T} = \{\mathcal{B}, \mathcal{F}\}$. The optimiser determines the following hedge portfolio:

	Type	n	S_t	Strike	Number
1	$\mathcal{B}$	0.00	–	–	0.00
2	$\mathcal{F}$	–	1'841.10	1'841.10	-5.39

This results in the following response function (trading grid for the hedge assets):

	π	Δ	Γ	ρ	ν
1.30	-2’975.70	-12’903.00	0.00	0.00	0.00
1.20	-1’983.10	-11’911.00	0.00	0.00	0.00
1.10	-990.55	-10’918.00	0.00	0.00	0.00
1.00	2.00	-9’925.50	0.00	0.00	0.00
0.90	994.55	-8’933.00	0.00	0.00	0.00
0.80	1’987.10	-7’940.40	0.00	0.00	0.00
0.70	2’979.70	-6’947.90	0.00	0.00	0.00

The following pictures (14.5 and 14.6) show how variable annuities and hedge assets move for various equity level movements. One can clearly see that the futures do not have any convexity (ie. $\Gamma = 0$) and that in consequence the hedge is more effective around the current market levels, than a more extreme levels.

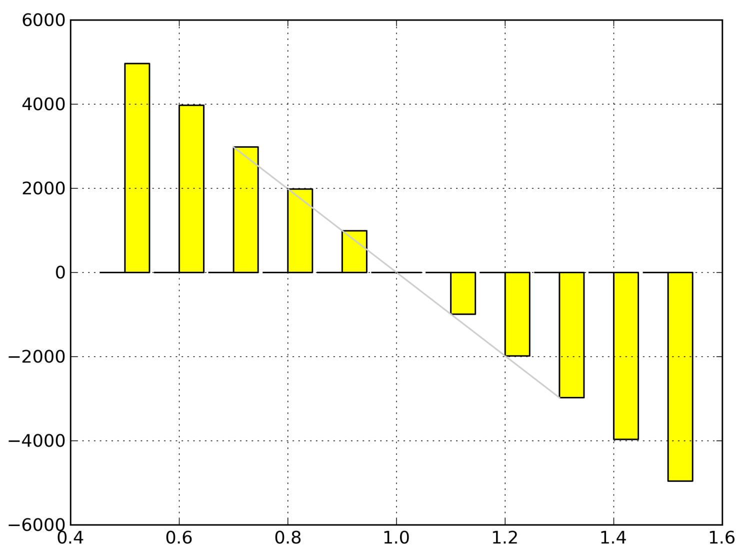


Figure 14.6. Decomposition P&L

14.3.3 Delta - Gamma - hedge

For a $\Delta - \Gamma$-hedge the approach is similar, namely we try to approximate both Δ and Γ at current market levels. Since futures do not have any convexity we still need to add an option to the candidate assets such that the optimiser can optimise for Γ. Hence we have the following set up:

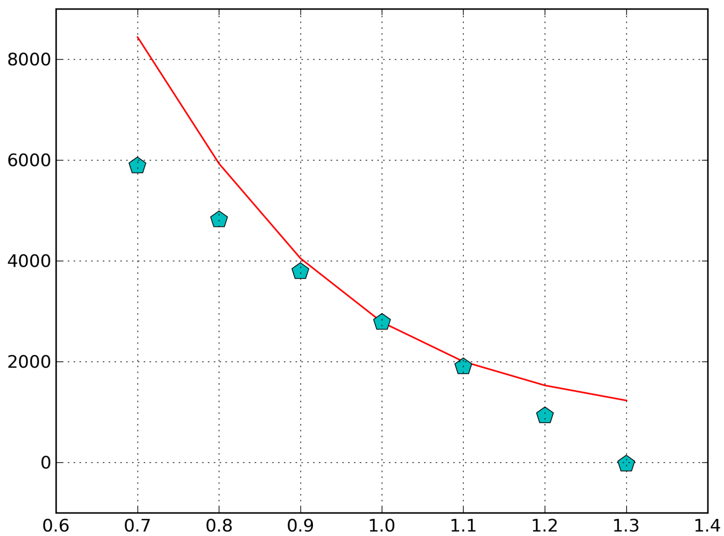


Figure 14.7. Adjusted Comparison

	$\frac{S_t}{S_0}$	Δr	$\Delta \nu$	Type	Value	w_k
1	0.70	0.00	0.00	π	8'442.90	0.00
2	0.80	0.00	0.00	π	5'931.90	0.00
3	0.90	0.00	0.00	π	4'051.70	0.00
4	1.00	0.00	0.00	Δ	-9'925.50	1.00
5	1.00	0.00	0.00	Γ	-45.31	400.00
6	1.00	0.00	0.00	π	2'784.50	10'000.00
7	1.00	0.00	0.00	ρ	325'170.00	0.00
8	1.00	0.00	0.00	ν	47'226.00	0.00
9	1.10	0.00	0.00	π	2'004.00	0.00
10	1.20	0.00	0.00	π	1'531.30	0.00
11	1.30	0.00	0.00	π	1'232.20	0.00

In this case we consider the following test assets consisting of cash, futures and one option:

$$\mathcal{T} = \{\mathcal{B}, \mathcal{F}, \mathcal{P}_1(c \in [0.7, 0.9], t \in [1, 2])\}.$$

The optimiser determines the following hedge portfolio:

	Type	t	S_t	Strike	Number
1	$\mathcal{B}$	0.00	–	–	5.00
2	$\mathcal{F}$	–	1’841.10	1’841.10	-5.13
3	$\mathcal{P}_1$	2.00	1’841.10	1’657.00	1.07

This results in the following response function (trading grid for the hedge assets):

	π	Δ	Γ	ρ	ν
1.30	-2’810.30	-12’421.00	0.40	-315.35	398.60
1.20	-1’851.50	-11’555.00	0.60	-500.57	552.51
1.10	-883.09	-10’722.00	0.87	-771.61	722.66
1.00	99.97	-9’925.50	1.15	-1’145.10	874.49
0.90	1’104.40	-9’158.20	1.40	-1’619.40	952.11
0.80	2’137.40	-8’394.20	1.48	-2’157.40	895.51
0.70	3’204.00	-7’587.90	1.29	-2’677.90	684.21

The following pictures (14.7 and 14.8) show how variable annuities and hedge assets move for various equity level movements. In comparison to the pure Δ-hedge assets and liabilities are moving more in parallel and the some of the tail risk has been removed.

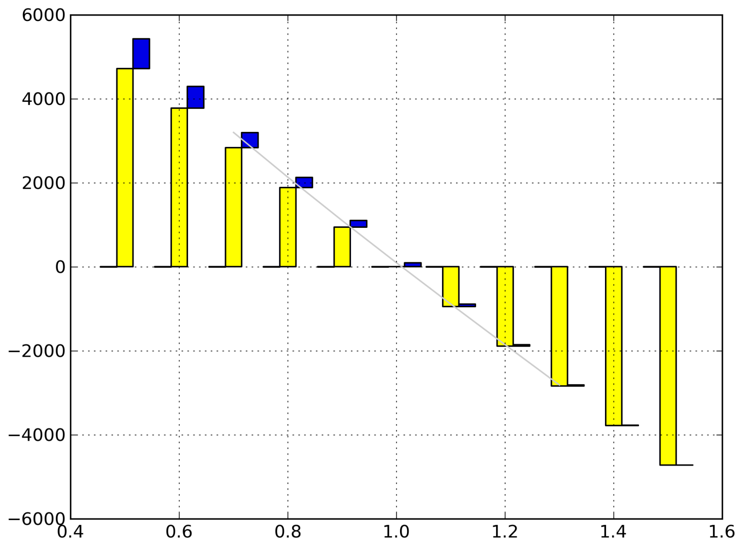


Figure 14.8. Decomposition P&L

14.3.4 Tail-hedge

We have seen that both the Δ and the $\Delta - \Gamma$ hedging strategy focus on the body of the distribution. Another approach is to try to protect the balance sheet for more extreme stresses. Hence one tries to immunise for a wider range of market stresses such as for stresses up to $\pm 30\%$. In consequence more options are needed as candidate assets. The following table shows how we choose the weights in this case:

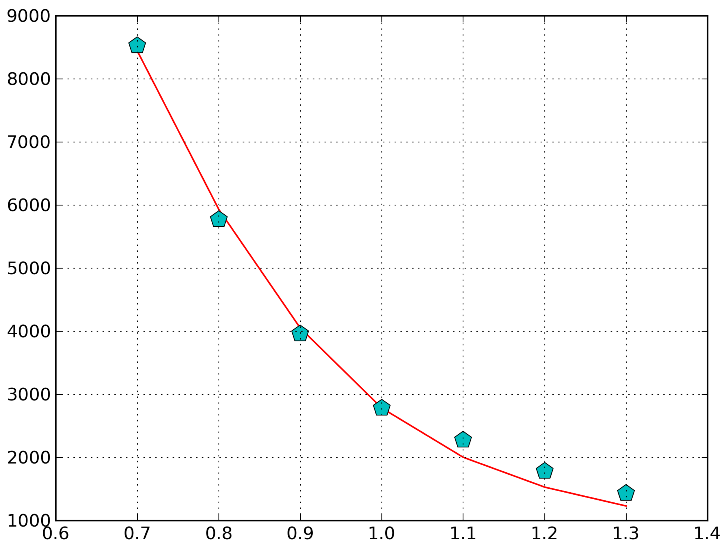


Figure 14.9. Adjusted Comparison

	$\frac{S_t}{S_0}$	Δr	$\Delta \nu$	Type	Value	w_k
1	0.70	0.00	0.00	π	8'442.90	1.00
2	0.80	0.00	0.00	π	5'931.90	1.00
3	0.90	0.00	0.00	π	4'051.70	1.00
4	1.00	0.00	0.00	Δ	9'925.50	0.00
5	1.00	0.00	0.00	Γ	45.31	0.00
6	1.00	0.00	0.00	π	2'784.50	10'000.00
7	1.00	0.00	0.00	ρ	325'170.00	0.00
8	1.00	0.00	0.00	ν	47'226.00	0.00
9	1.10	0.00	0.00	π	2'004.00	1.00
10	1.20	0.00	0.00	π	1'531.30	1.00
11	1.30	0.00	0.00	π	1'232.20	1.00

In this example we also consider a rich portfolio of test assets. The following table shows the test assets which we use ($\mathcal{T}$):

Asset	Freedom
$\mathcal{P}_1$	$t \in [4,5], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_2$	$t \in [9,10], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_3$	$t \in [1,2], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_4$	$t \in [9,10], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_5$	$t \in [1,2], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_6$	$t \in [4,5], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{C}_1$	$t \in [9,9.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{C}_2$	$t \in [4,4.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{C}_3$	$t \in [1,1.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{B}$	
$\mathcal{F}$	
Total Value	$\sum_i \pi(\mathcal{A}_i) \leq 250$

For this minimisation problem the optimiser found the following result:

	Type	t	S_t	Strike	Number
1	$\mathcal{B}$	0.00	–	–	75.55
2	$\mathcal{C}$	1.19	1'841.10	2'419.00	0.66
3	$\mathcal{C}$	4.01	1'841.10	2'211.90	-1.48
4	$\mathcal{C}$	9.12	1'841.10	2'209.40	-1.59
5	$\mathcal{F}$	–	1'841.10	1'841.10	1.05
6	$\mathcal{P}$	1.00	1'841.10	920.57	64.37
7	$\mathcal{P}$	1.38	1'841.10	1'657.00	15.24
8	$\mathcal{P}$	4.29	1'841.10	1'657.00	0.00
9	$\mathcal{P}$	4.36	1'841.10	921.19	47.81
10	$\mathcal{P}$	9.00	1'841.10	1'472.90	0.01
11	$\mathcal{P}$	9.00	1'841.10	1'051.50	0.00

We get the following response function for the hedge assets:

	π	Δ	Γ	ρ	ν
1.30	-1'349.50	-3'927.20	5.27	-24'718.00	857.25
1.20	-1'000.10	-4'894.80	9.31	-25'599.00	3'803.20
1.10	-504.29	-6'625.00	15.17	-28'053.00	7'884.70
1.00	250.00	-9'335.50	22.67	-32'875.00	12'892.00
0.90	1'426.00	-13'087.00	30.89	-40'904.00	18'276.00
0.80	3'239.40	-17'762.00	39.98	-53'010.00	23'848.00
0.70	5'996.00	-23'731.00	57.61	-70'853.00	31'398.00

The following pictures (14.9 and 14.10) show how variable annuities and hedge assets move for various equity level movements. We see that this tail hedging strategy protects the balance sheet best in case of material adverse developments.

Example 14.3.1. In a next step we will apply this technique to example 13.5.1.. In this case we have the following variable annuity response function

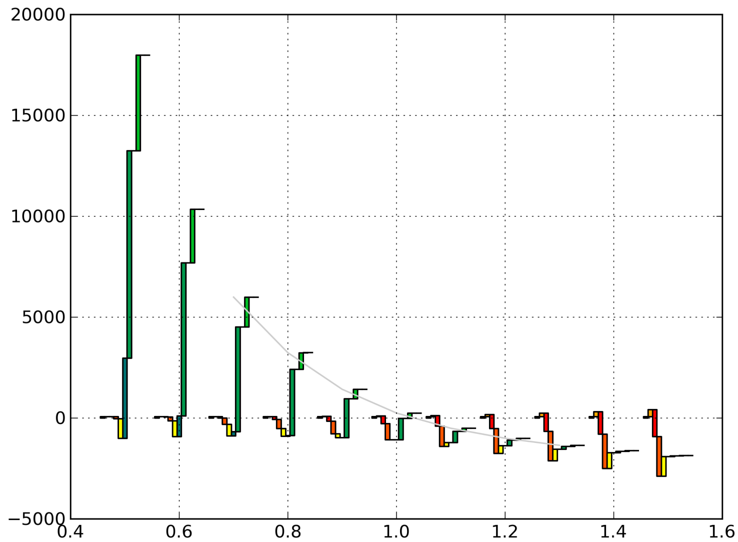


Figure 14.10. Decomposition P&L

	$\frac{S_t}{S_0}$	Δr	$\Delta \nu$	Type	Value	w_k
1	0.50	0.00	0.00	π	15'837.00	1.00
2	0.60	0.00	0.00	π	12'532.00	1.00
3	0.70	0.00	0.00	π	10'129.00	1.00
4	0.80	0.00	0.00	π	8'541.80	1.00
5	0.90	0.00	0.00	π	7'962.10	1.00
6	1.00	0.00	0.00	π	7'806.50	10'000.00
7	1.10	0.00	0.00	π	8'353.20	1.00
8	1.20	0.00	0.00	π	9'140.40	1.00
9	1.30	0.00	0.00	π	9'952.20	1.00
10	1.40	0.00	0.00	π	10'715.00	1.00
11	1.50	0.00	0.00	π	11'551.00	1.00

As before we first define a rich portfolio of test assets. The following table shows the test assets which we use ($\mathcal{T}$):

Asset	Freedom
$\mathcal{P}_1$	$t \in [4,5], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_2$	$t \in [9,10], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_3$	$t \in [1,2], n \in [0.5, 0.6], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_4$	$t \in [9,10], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_5$	$t \in [1,2], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{P}_6$	$t \in [4,5], n \in [0.8, 0.9], \pi(\bullet) \in [0, 200]$
$\mathcal{C}_1$	$t \in [9, 9.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{C}_2$	$t \in [4, 4.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{C}_3$	$t \in [1, 1.2], n \in [1.2, 1.4], \pi(\bullet) \in [-2, 2]$
$\mathcal{B}$	
$\mathcal{F}$	
Total Value	$\sum_i \pi(\mathcal{A}_i) \leq 250$

For this minimisation problem the optimiser found the following result:

	Type	t	S_t	Strike	Number
1	$\mathcal{B}$	0.00	None	None	64.27
2	$\mathcal{C}$	1.20	1'841.10	2'238.70	-2.00
3	$\mathcal{C}$	4.20	1'841.10	2'209.40	-1.99
4	$\mathcal{C}$	9.20	1'841.10	2'221.50	-1.99
5	$\mathcal{F}$	None	1'841.10	1'841.10	9.61
6	$\mathcal{P}$	1.00	1'841.10	1'657.00	6.99
7	$\mathcal{P}$	1.00	1'841.10	920.57	66.28
8	$\mathcal{P}$	4.00	1'841.10	1'634.50	9.92
9	$\mathcal{P}$	5.00	1'841.10	920.57	38.77
10	$\mathcal{P}$	9.00	1'841.10	1'472.90	0.00
11	$\mathcal{P}$	9.00	1'841.10	1'071.80	0.00

In a first step we calculate the (asset) trading grid for the assets which we have found:

	π	Δ	Γ	ρ	ν
1.30	2'148.00	9'541.30	0.92	-43'607.00	1'562.20
1.20	1'426.30	8'427.60	3.64	-44'763.00	4'525.00
1.10	764.27	6'680.90	8.53	-47'550.00	8'755.20
1.00	249.97	3'970.40	15.64	-52'845.00	14'077.00
0.90	28.52	108.40	23.99	-61'586.00	19'850.00
0.80	303.40	-4'860.70	33.40	-74'685.00	25'691.00
0.70	1'370.80	-11'370.00	51.88	-93'739.00	33'203.00

Exercise 14.3.2. Calculate the trading grid for example 13.5.1 and identify a hedging portfolio alongside the example 14.3.1.

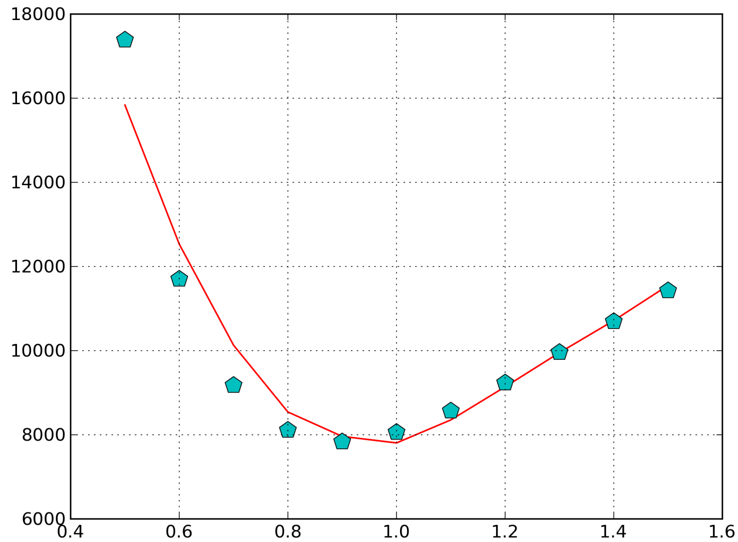


Figure 14.11. Adjusted Comparison

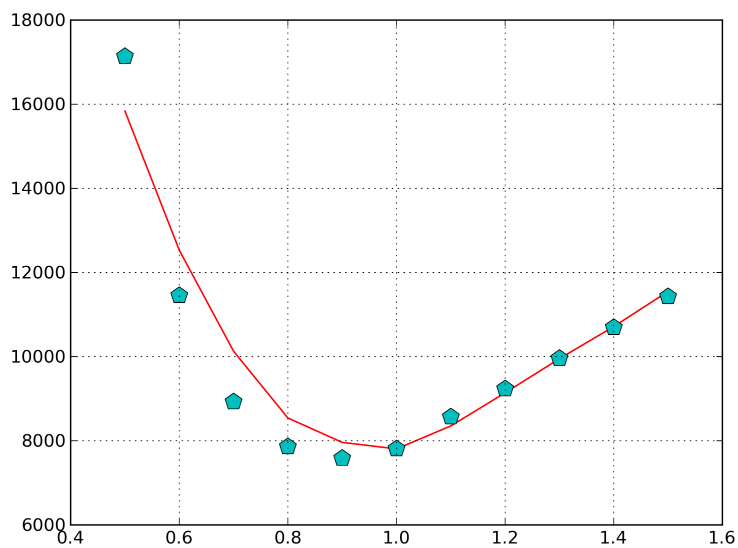


Figure 14.12. Adjusted Comparison

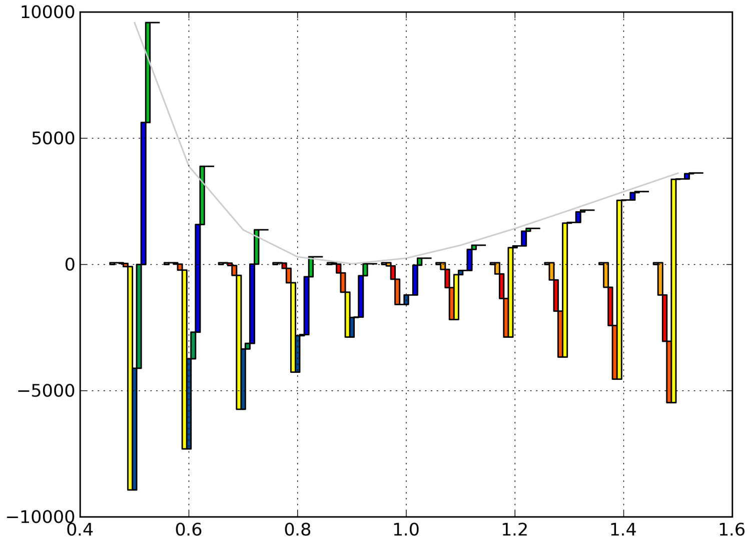


Figure 14.13. Decomposition P&L

A. Notes on stochastic integration

A.1 Stochastic processes and martingales

The following appendices summarize definitions and results in the area of stochastic integration and martingales. For obvious reasons we will not present proofs for all results. In fact we will only give a survey of various results and then refer to the corresponding literature. Foremost we refer to the monographs [Pro90] and [IW81].

Definition A.1.1. *A probability space* $(\Omega, \mathcal{A}, P)$ *is called* filtered*, if there exists a family of* σ*-algebras* $F = (\mathcal{F}_t)_{t\geq 0}$ *such that*

1. $\mathcal{F}_0 \supset \{A \in \mathcal{A} | P(A) = 0\}$,
2. $\mathcal{F}_s \subset \mathcal{F}_t$ *for* $s \leq t$.

The filtration is called right continuous, if $\mathcal{F}_t = \bigcap_{t'>t} \mathcal{F}_{t'}, \forall t \geq 0$.

Definition A.1.2. *A random variable* $T : \Omega \to [0, \infty]$ *is a* stopping time*, if* $\{T \leq t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$.

Theorem A.1.3. T *is a stopping time if and only if* $\{T < t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$. *([Pro90] Thm. 1.1.1.)*

Definition A.1.4. *Let* X, Y *be stochastic processes.* X *is called* modification of Y*, if*

$$X_t = Y_t \qquad\qquad P\text{–almost surely for all } t.$$

X *and* Y *are* indistinguishable*, if*

$$X_t = Y_t, \forall t \qquad\qquad P\text{–almost surely.}$$

Definition A.1.5. *1. A stochastic process is called* càdlàg *(continue à droite, limites à gauche), if its trajectories are right continuous with left limits.*

2. *A stochastic process is called* càglàd *(continue à gauche, limites à droite), if its trajectories are left continuous with right limits.*
3. *A stochastic process is* adapted, *if $X_t \in \mathcal{F}_t$ (X_t is $\mathcal{F}_t$-measurable).*

Theorem A.1.6. *1. Let Λ be an open set and X be an adapted càdlàg process. Then $T := \inf\{t \in \mathbb{R}_+ : X_t \in \Lambda\}$ is a stopping time.*

2. *Let S, T be stopping times and $\alpha > 1$. Then the following random variables are also stopping times: $\min(S,T)$, $\max(S,T)$, $S+T$, $\alpha \cdot T$.*

Proof. [Pro90] Thm. 1.1.3 and Thm. 1.1.5.

Definition A.1.7. *Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A stochastic process X is called* martingale, *if*

– *$X_t \in L^1(\Omega, \mathcal{A}, P)$, i.e. $E[|X_t|] < \infty$,*
– *X is adapted (ie X_t is $\mathcal{F}_t$ measurable for all $t \in \mathbb{R}^+$),*
– *$E[X_t|\mathcal{F}_s] = X_s$ holds for $s < t$.*

Remark A.1.8. If one replaces in the previous equation "=" by "≤" ("≥"), the process X is called supermartingale (submartingale).

Theorem A.1.9. *Let X be a supermartingale. Then the following conditions are equivalent:*

1. *The mapping $T \to \mathbb{R}, t \mapsto E[X_t]$ is right continuous.*
2. *There exists a unique modification Y of X which is càdlàg.*

Proof. [Pro90] Thm. 1.2.9.

Theorem A.1.10. *Let X be a martingale. Then there exists a unique càdlàg modification Y of X.*

Theorem A.1.11 (Doob's stopping theorem). *Let X be a right continuous martingale with closure X_∞, i.e. $X_t = E[X_\infty|\mathcal{F}_t]$. Moreover, let S and T be stopping times such that $S \leq T$ P–almost surely. Then the following statements hold:*

1. *$X_S,\ X_T \in L^1(\Omega, \mathcal{A}, P)$,*
2. *$X_S = E[X_T|\mathcal{F}_S]$.*

Proof. [Pro90] Thm. 1.2.16.

Definition A.1.12. *Let X be a stochastic process and T be a stopping time. The stopped stochastic process $(X_t^T)_{t\geq 0}$ is defined by $X_t^T = X_{\min(t,T)}$ for all $t \geq 0$.*

Theorem A.1.13 (Jensen's inequality). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $X \in L^1(\Omega, \mathcal{A}, P)$ with $\phi(X) \in L^1(\Omega, \mathcal{A}, P)$. Furthermore let $\mathcal{G}$ be a σ-algebra. Then the following inequality holds:*

$$\phi \circ E[X|\mathcal{G}] \leq E[\phi(X)|\mathcal{G}].$$

Proof. [Pro90] Thm. 1.2.19.

A.2 Stochastic integrals

In this section we present a short introduction to the theory of stochastic integration. We follow the approach of [Pro90].

Essentially one can understand a stochastic integral with respect to a semimartingale as a path wise Stieltjes integral, and the latter should be known from lectures in analysis. The main idea for this type of integrals is to consider the limit of sums of the form

$$\sum f(T_k)\ (T_{k+1} - T_k)$$

for partitions with decreasing mesh size. In the following we assume that a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$, which satisfies the common regularity conditions, is given.

Definition A.2.1. 1. *A stochastic process H is called* simple predictable, *if it can be represented as*

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i, T_{i+1}]}(t),$$

where

$$0 = T_1 \leq \ldots \leq T_{n+1} < \infty$$

is a finite family of stopping times and the $H_i \in \mathcal{F}_t$, $(H_i)_{i=0,\ldots,n}$ are finite P-almost everywhere.

The set of simple predictable processes will be denoted by $\mathbb{S}$. Furthermore, $\mathbb{S}_u$ denotes the set $\mathbb{S}$ equipped with the topology of uniformly convergence in (t, ω) on $\mathbb{R} \times L^\infty(\Omega, \mathcal{A}, P)$.

2. *The vector space of finite, real valued random variables equipped with the convergence in probability is denoted by* $\mathbb{L}^0$.

Next, we want to define the expression $\int H\,dX$ for certain processes $(X_t)_{t\in\mathbb{R}}$ and $(H_t)_{t\in\mathbb{R}}$. In order to be able to call such an operator, denoted by I_X, integral, it should at least be linear and a theorem like the Lebesgue convergence theorem should hold.

For the convergence theorem we assume the following continuity: If H^n converges uniformly to H, then $I_X(H^n)$ should converge in probability to $I_X(H)$.

Given a process X. We define $I_X : \mathbb{S} \to \mathbb{L}^0$ by

$$I_X(H) = H_0\,X_0 + \sum_{i=1}^{n} H_i\left(X^{T_i} - X^{T_{i+1}}\right),$$

where

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i,T_{i+1}]}(t).$$

This definition of $I_X(H)$ is independent of the representation of H.

Definition A.2.2 (Total semimartingale). *A stochastic process* $(X_t)_{t\geq 0}$ *is called* total semimartingale, *if*

1. X *is càdlàg and*
2. I_X *is a continuous mapping from* $\mathbb{S}_u$ *to* $\mathbb{L}^0$.

Definition A.2.3 (Semimartingale). *A stochastic process* $(X_t)_{t\geq 0}$ *is a* semimartingale, *if* X^t *(compare with Definition A.1.12) is a total semimartingale for all* $t \in [0,\infty[$.

Remark A.2.4. Thus semimartingales are defined as well behaving integrators.

The following theorem summarizes the most important properties of the operator I_X:

Theorem A.2.5. 1. *The set of all semimartingales is a vector space.*

2. *Let* Q *be a measure which is absolutely continuous with respect to* P. *Then every* P*-semimartingale is also a* Q*-semimartingale.*

3. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures such that $(X_t)_{t\geq 0}$ is a P_n-semimartingale for each n. Then $(X_t)_{t\geq 0}$ is an R-semimartingale, for $R = \sum_{n\in\mathbb{N}} \lambda_n P_n$, where $\sum_{n\in\mathbb{N}} \lambda_n = 1$.*
4. *(Stricker's Theorem) Let X be a semimartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and $(\mathcal{G}_t)_{t\geq 0}$ be a sub-filtration of $(\mathcal{F}_t)_{t\geq 0}$ such that X is adapted to $(\mathcal{G}_t)_{t\geq 0}$. Then X is a $\mathcal{G}$-semimartingale.*

Proof. The statements follow from the definition of a semimartingale. The proofs, which are recommended as an exercise to the reader, can be found in [Pro90] Chapter II.2.

Now we want to characterize the class of semimartingales.

Theorem A.2.6. *Every adapted process with càdlàg paths and finite variation on compact sets is a semimartingale.*

Proof. This theorem is based on the fact that

$$|I_X(H)| \leq ||H||_u \int_0^\infty |dX_s|,$$

where $\int_0^\infty |dX_s|$ denotes the total variation.

Theorem A.2.7. *Every square integrable martingale with càdlàg paths is a semimartingale.*

Proof. Let X be a square integrable martingale with $X_0 = 0$, $H \in \mathbb{S}$. The continuity of the operators I_X is a consequence of the following inequality:

$$\begin{aligned}
E\left[(I_X(H))^2\right] &= E\left[\left(\sum_{i=0}^n H_i\left(X^{T_i} - X^{T_{i+1}}\right)\right)^2\right] \\
&= E\left[\sum_{i=0}^n H_i^2\left(X^{T_i} - X^{T_{i+1}}\right)^2\right] \\
&\leq ||H||_u^2\, E\left[\sum_{i=0}^n \left(X^{T_i} - X^{T_{i+1}}\right)^2\right] \\
&= ||H||_u^2\, E\left[\sum_{i=0}^n \left({X^{T_i}}^2 - {X^{T_{i+1}}}^2\right)\right] \\
&= ||H||_u^2\, E\left[{X_{T^{n+1}}}^2\right] \\
&\leq ||H||_u^2\, E\left[{X_{T^\infty}}^2\right].
\end{aligned}$$

Example A.2.8. Brownian motion is a semimartingale.

Now, after defining semimartingales, we want to enlarge the class of integrands. A class which is very well suited for this purpose is the set of càglàd processes. We will use this class, since for it the proofs remain relatively simple.

Definition A.2.9. *The set of adapted càdlàg (càglàd) processes is denoted by $\mathbb{D}$ ($\mathbb{L}$, respectively). Further, $b\mathbb{L}$ denotes the set of processes $X \in \mathbb{L}$ with bounded paths.*

Up to now we are familiar with the topology of the uniform convergence (on $\mathbb{S}_u$) and the topology of the convergence in probability on $\mathbb{L}^0$. We define a further notion of convergence.

Definition A.2.10. *Let $t \geq 0$ and H be a stochastic process. Then we set*

$$H_t^* = \sup_{0 \leq s \leq t} |H_s|.$$

A sequence $(H^n)_{n \in \mathbb{N}}$ converges uniformly on compact sets in probability (short: convergence in ucp-topology) to H, if

$$(H^n - H)_t^* \to 0$$

in probability for $n \to \infty$ and all $t \geq 0$.

$\mathbb{D}_{ucp}$, $\mathbb{L}_{ucp}$ and $\mathbb{S}_{ucp}$ denote the corresponding sets equipped with the ucp-topology.

Remark A.2.11.

1. The ucp-topology is metrisable. An equivalent metric is for example:

$$d(X, Y) = \sum_{i=1}^{\infty} \frac{1}{2^n} E\left[\min(1, (X - Y)_n^*\right].$$

2. $\mathbb{D}_{ucp}$ is a complete metric space.

The following theorem is essential for the extension of the integral I_X.

Theorem A.2.12. *The vector space $\mathbb{S}$ is dense in $\mathbb{L}$ with respect to the ucp-topology.*

Proof. [Pro90] Thm. 2.4.10.

Note that, I_X can be extended if we can show that I_X is continuous. To show the continuity we start with a definition.

Definition A.2.13. *Let $H \in \mathbb{S}$ and X be a semimartingale. Then we define $J_X : \mathbb{S} \to \mathbb{D}$ by*

$$J_X(H) = H_0\, X_0 + \sum_{i=0}^{n} H_i \left(X^{T_i} - X^{T_{i+1}} \right),$$

where

$$H_t = H_0 \cdot \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \cdot \chi_{]T_i, T_{i+1}]}(t)$$

for $H_i \in \mathcal{F}_{T_i}$ and stopping times $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$.

Definition A.2.14 (Stochastic integral). *Let $H \in \mathbb{S}$ and X be a càdlàg process. Then $J_X(H)$ is called the stochastic integral of H with respect to X and we use the notations:*

$$H \cdot X := \int H_s\, dX_s := J_X(H).$$

Now we have defined the stochastic integral on $\mathbb{S}$, and we want to extend it onto $\mathbb{L}$. For the extension we need the following theorem.

Theorem A.2.15. *Let X be a semimartingale. Then the mapping J_X : $\mathbb{S}_{ucp} \to \mathbb{D}_{ucp}$ is continuous. Also the extension of J_X onto $\mathbb{S}_{ucp}$ will be called stochastic integral, and the notations of Definition A.2.14 will be used also for the extension.*

Proof. [Pro90] Thm. 2.4.11.

Remark A.2.16. To extend J_X onto $\mathbb{D}$ we use the fact that $\mathbb{D}_{ucp}$ is a *complete* metric space.

The process $J_X(H) = \int H_s\, dX_s$ evaluated at the time $t \geq 0$ will be denoted by

$$H \cdot X_{\ t} := \int_0^t H_s\, dX_s := \int_{[0,t]} H_s\, dX_s.$$

A.3 Properties of the stochastic integral

After defining the stochastic integral we will now summarize its properties. We will concentrate on the statements without giving proofs.

Theorem A.3.1. *1. Let T be a stopping time. Then $(H \cdot X)^T = H \cdot \chi_{[0,T]} \cdot X = H \cdot X^T$.*

2. *Let $G, H \in \mathbb{L}$ and X be a semimartingale. Then also $Y := H \cdot X$ is a semimartingale. Furthermore, we have*

$$G \cdot Y = G \cdot (H \cdot X) = (G \cdot H) \cdot X.$$

Proof. [Pro90] Thm. 2.5.12 and 2.5.19.

Definition A.3.2. *Let X be a càdlàg process. Then we define*

$$\begin{aligned} X_-(t) &= \lim_{s \uparrow t} X(s), \\ \Delta X(t) &= X(t) - X_-(t). \end{aligned}$$

Definition A.3.3. *A random partition σ of $\mathbb{R}$ is a finite sequence of stopping times such that*

$$0 = T_0 \leq T_1 \leq \ldots \leq T_n < \infty.$$

A sequence $(\sigma_n)_{n \in \mathbb{N}}$ of random partitions of $\mathbb{R}$ converges to the identity, if the following conditions hold:

1. $\lim_{n \to \infty} (\sup_k T_k^n) = \infty$ *P-almost surely,*
2. $||\sigma_n|| := \sup_k |T_{k+1}^n - T_k^n|$ *converges P-almost surely to* 0.

Let Y be a process and σ be a random partition. Then we define

$$Y^\sigma := Y_0 \cdot \chi_{\{0\}} + \sum_k Y_{T_k} \cdot \chi_{]T_k, T_{k+1}]}.$$

Remark A.3.4. It is easy to show that

$$\int Y_s^\sigma \, dX_s = Y_0 \, X_0 + \sum_k Y_{T_k} \left(X^{T_{k+1}} - X^{T_k} \right)$$

for all semimartingales X and all Y in $\mathbb{S}$, $\mathbb{D}$ and $\mathbb{L}$.

Using random partitions one calculate the stochastic integral by the following theorem.

Theorem A.3.5. *Let X be a semimartingale, $Y \in \mathbb{D}$ and $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence of random partitions which converges to the identity. Then*

$$\int_{0+} Y_s^{\sigma_n} \, dX_s = \sum_k Y_{T_k^n} \left(X^{T_{k+1}^n} - X^{T_k^n} \right)$$

converges in ucp-topology towards the stochastic integral $\int (Y_-) \, dX$.

Proof. [Pro90] Thm. 2.5.21.

Definition A.3.6. *Let X and Y be semimartingales. Then we define*

$$\begin{aligned} [X,X] &= ([X,X]_t)_{t\geq 0} \text{ \textit{the} quadratic variation process by} \\ [X,X] &:= X^2 - 2\int X_- dX, \end{aligned}$$

and accordingly

$$[X,Y] := XY - \int X_-\, dY - \int Y_-\, dX$$

is the covariation process*.*

Theorem A.3.7. *Let X be a semimartingale. Then the following statements hold:*

1. *$[X,X]$ is càdlàg, monotone increasing and adapted.*
2. *$[X,X]_0 = X_0^2$ and $\Delta[X,X] = (\Delta X)^2$.*
3. *If a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of random partitions converges to 1, then*
$$X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})^2 \longrightarrow [X,X] \text{ in ucp-topology for } n\to\infty.$$
4. *Let T be a stopping time. Then $[X^T,X] = [X,X^T] = [X^T,X^T] = [X,X]^T$.*

Proof. [Pro90] Thm. 2.6.22.

Remark A.3.8. – The mapping $(X,Y)\mapsto[X,Y]$ is bilinear and symmetric.
– The polarization identity holds:

$$[X,Y] = \frac{1}{2}\left([X+Y,X+Y] - [X,X] - [Y,Y]\right).$$

Theorem A.3.9. *The* bracket process *$[X,Y]$ of two semimartingales X and Y has paths of bounded variation on compact sets and it is a semimartingale.*

Proof. [Pro90] Cor. 2.6.1.

Theorem A.3.10 (Partial integration).

$$d(XY) = X_-\, dY + Y_-\, dX + d[X,Y].$$

Proof. [Pro90] Cor. 2.6.2.

Theorem A.3.11. *Let M be a local martigale.*

1. *M is martingale with $E[M_t^2] \leq \infty$ $\forall t \geq 0$,*
2. *$E\left[[M,M]_t\right] < \infty$ $\forall t \geq 0$,*
3. *$E[M_t^2] = E\left[[M,M]_t\right]$ $\forall t \geq 0$.*

Proof. [Pro90] Cor. 2.6.4.

Theorem A.3.12. *Let X, Y be semimartingales and $H, K \in \mathbb{L}$. Then the following statements hold:*

1. *$\left[H \cdot X, K \cdot Y\right]_t = \int_0^t H_s K_s d[X,Y]_s$ $\forall t \geq 0$,*
2. *$\left[H \cdot X, H \cdot X\right]_t = \int_0^t H_s^2 d[X,X]_s$ $\forall t \geq 0$.*

Proof. [Pro90] Thm. 2.6.29.

Theorem A.3.13 (Itô-formula). *Let X be a semimartingale and $f \in C^2(\mathbb{R})$. Then*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_s^-) dX_s + \frac{1}{2} \int_{0+}^{t} f''(X_s^-) d[X,X]_s^{cont} \\ &+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_s^-) - f'(X_s^-) \Delta X_s \right\}. \end{aligned}$$

Proof. [Pro90] Thm. 2.7.32.

Remark A.3.14. A function $f \in C^2(\mathbb{R})$ has the deterministic integral representation

$$f(t) - f(0) = \int_0^t f'(s) ds.$$

For a stochastic integral two further terms appear. The term

$$\frac{1}{2} \int_{0+}^{t} f''(X_s^-) d[X,X]_s^{cont}$$

is due to the quadratic variation of the process and the term

$$\sum_{0<s\leq t} \left\{ f(X_s) - f(X_s^-) - f'(X_s^-) \Delta X_s \right\}$$

is due to the jumps of the process.

Theorem A.3.15 (Transformation theorem). *Let V be a stochastic process with right continuous paths of bounded variation. Furthermore, let $f \in C^1(\mathbb{R})$. Then $(f(V_t))_{t\geq 0}$ is a process of bounded variation and*

$$f(V_t) - f(V_0) = \int_{0^+}^t f'(V_{s_-})\, dV_s + \sum_{0<s\leq t} \left(f(V_s) - f(V_{s_-}) - f'(V_{s_-}) \Delta V_s \right).$$

Theorem A.3.16 (Itô-formula). *Let X be a continuous semimartingale and $f \in C^2(\mathbb{R})$. Then also $f(X)$ is a semimartingale and it satisfies*

$$f(X_t) - f(X_0) = \int_{0^+}^t f'(X_s) dX_s + \frac{1}{2} \int_{0^+}^t f''(X_s) d[X, X]_s.$$

B. Java code for the calculation of the Markov model

The following code is an example of a (non optimized) implementation of the Markov recursion in Java. Please note, that we removed some consistency checks from the original code in order to increase the readability of this printed version.

```
/*
 * MarkovClass.java
 */

public class MarkovClass {
    double  dPij[][];     /* Transition probability [Age][Index]    */
    double  dPost[][];    /* Postnumerando benefits [Age][Index]    */
    double  dPre[][];     /* Prenumerando benefits  [Age][State]    */
    double  dDisc[][];    /* Discount Factor        [Age][State]    */
    int     iFrom[];      /* [Index]         */
    int     iTo[];        /* [Index]         */
    int     iMat2Idx[][]; /* [From][To]      */
    double  dV[][];       /* [Age][States]   */
    double  dVTemp[];     /* [States]        */

    int    iMaxTimes;
    int    iMaxIndex;
    int    iMaxStates;
    int    iStates;
    int    iStart;
    int    iStop;
    int    iAdvancedDisc;
    int    iT;
    int    iNrIndex = 0;
    int    iC1, iC2;
    int    iErr;

    boolean bDKCalculated = false;
    boolean bGetData = false;

    /** Creates a new instance of MarkovClass */
    /*****************************************/

    public MarkovClass(int iMaxStateInput, int iMaxTime) {

       /* Define Max allocatable memory */

       iMaxTimes  = iMaxTime;
       iMaxStates = iMaxStateInput;
       iMaxIndex  = iMaxStates * iMaxStates;
       dPij       = new double[iMaxTimes][iMaxIndex];     /* [Age][Index] */
       dPost      = new double[iMaxTimes][iMaxIndex];     /* [Age][Index] */
       dPre       = new double[iMaxTimes][iMaxStates];    /* [Age][State] */
       dDisc      = new double[iMaxTimes][iMaxStates];    /* [Age][State] */
```

```
   iFrom      = new int[iMaxIndex];                  /* [Index] */
   iTo        = new int[iMaxIndex];                  /* [Index] */
   iMat2Idx   = new int[iMaxStates][iMaxStates];;    /* [From][To] */
   dV         = new double[iMaxTimes][iMaxStates];   /* [State] */
   dVTemp     = new double[iStates];                 /* [State] */

   for(iC1=0; iC1 < iMaxStates; ++ iC1)
    {
        for(iC2=0; iC2 < iMaxStates; ++ iC2)
        {
        iMat2Idx[iC1][iC2] = -1;
        }
    }

bDKCalculated = false;
bGetData = false;

}
/** Member functions  */
/** 1. vReset  */
public void  vReset(){
bDKCalculated = false;}

/** 2. vSetStartTime */
public void           vSetStartTime(int iTime){
iStart = iTime;
bDKCalculated = false;}

/** 3. vSetStopTime */
public void           vSetStopTime(int iTime){
iStop = iTime;
bDKCalculated = false;}

/** 4. vSetNrStates */
public void           vSetNrStates(int iNrStatesIpt){
iStates = iNrStatesIpt;
bDKCalculated = false;}

/** 5. vSetGetData */
public void           vSetGetData(boolean bStatus){
bGetData = bStatus;}

/** 6. vSetPre (Sets Prenumerando Cash Flows) */
public double         dSetPre(int iTimeInput, int iFromInput,
                                 int iToInput, double dValue){
  if (bGetData == true)
    {
      return(dPre[iTimeInput][iFromInput]);
    }
    bDKCalculated = false;
    dPre[iTimeInput][iFromInput] = dValue;
    return(dValue);}


/** 7. vSetPost (Sets Postnumerando Cash Flows) */
public double         dSetPost(int iTimeInput, int iFromInput,
                                  int iToInput, double dValue){
  if (bGetData == true)
    { if (iMat2Idx[iFromInput][iToInput] == -1) return(0.);
        return(dPost[iTimeInput][iMat2Idx[iFromInput][iToInput]]);
    }
    bDKCalculated = false;
    if (iMat2Idx[iFromInput][iToInput] == -1)
        {
        iMat2Idx[iFromInput][iToInput] = iNrIndex;
```

```
          iFrom[iNrIndex] = iFromInput;
          iTo[iNrIndex] = iToInput;
          ++iNrIndex;
          }
      dPost[iTimeInput][iMat2Idx[iFromInput][iToInput]] = dValue;
      return(dValue);}

  /** 8. vSetPij (Sets Probabilities ) */
public double         dSetPij(int iTimeInput, int iFromInput,
                                       int iToInput, double dValue){
    if (bGetData == true)
      { if (iMat2Idx[iFromInput][iToInput] == -1) return(0.);
          return(dPij[iTimeInput][iMat2Idx[iFromInput][iToInput]]);
      }
      bDKCalculated = false;
      if (iMat2Idx[iFromInput][iToInput] == -1)
          {
          iMat2Idx[iFromInput][iToInput] = iNrIndex;
          iFrom[iNrIndex] = iFromInput;
          iTo[iNrIndex] = iToInput;
          ++iNrIndex;
          }
      dPij[iTimeInput][iMat2Idx[iFromInput][iToInput]] = dValue;
      return(dValue); }

  /** 9. vSetDisc (Sets Discounts) */
public double         dSetDisc(int iTimeInput, int iFromInput,
                                        int iToInput, double dValue){
    if (bGetData == true)
      {
        return(dDisc[iTimeInput][iFromInput]);
      }
     bDKCalculated = false;
     dDisc[iTimeInput][iFromInput] = dValue;
     return(dValue);}

  /*  $$$$$$$$$$$$$$$$$$$$$$$$$$$ 10. MAIN MARKOV CALCULATOR $$$$$$$$$$$$$$$$$$*/

/* The following routine calculates the mathematics reseve, which is called
   also premium reserve or "Deckungskapital" DK in German */

 public double  dGetDK(int iTimeInput, int iStateInput){

/*===========================================================================*/
     if (bDKCalculated == true)
         return (dV[iTimeInput][iStateInput]);
     bDKCalculated = true;

/*==== 0. Reset =============================================================*/
     for (iT = 0; iT < iMaxTimes; ++ iT)
     {
         for(iC1= 0; iC1 < iMaxStates; ++ iC1)
             dV[iT][iC1] = 0.;
     }
/*==== 1. Recursion =========================================================*/
     for(iT = iStart -1; iT >= iStop; --iT)
     {
       /* 1. fill dVTemp with Pre payments */
       for(iC1= 0; iC1 < iStates; ++iC1){dVTemp[iC1]= 0.;}
       /* 2. Post payments and the mathematical reserve           */
       for(iC1= 0; iC1 < iNrIndex; ++iC1)
         {dVTemp[iFrom[iC1]] +=  dPij[iT][iC1] * (dPost[iT][iC1]
                                                 + dV[iT+1][iTo[iC1]]);}
       /* 2a. Pre payments and discount */
       for(iC1= 0; iC1 < iStates; ++iC1){dV[iT][iC1]= dVTemp[iC1]*dDisc[iT][iC1]
                                                         + dPre[iT][iC1];}
```

```
            }
   return (dV[iTimeInput][iStateInput]);
}}
```

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Notation

Index