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Selected Topics Life Insurance Mathematics

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1 A general life insurance model

1.1 Introduction

The life insurance market offers a wide range of different policies. It is, without expert knowledge, hardly possible to differentiate between all these policies. This is in particular due to the fact that the content of a life insurance is an abstract good.

A life insurance can always be understood as a bet: either one gets a benefit or one pays the premium without getting anything in return. From this point of view life insurance mathematics is a part of probability theory.

Since a life insurance deals with monetary benefits and premiums it is also part of the financial market and the economy. In this context one should note that insurances whose benefits are unit-linked, e.g. the payout depends on the performance of a fond, actually rely on modern theory of financial markets.

Form a legal point of view a life insurance is a contract between the policy holder and the insurer.

As we have note above, life insurances is characterized by its abstract matter and its diversity. Since its content is abstract its value is not intuitively obvious. This is particular due to the fact that a life insurance is usually only bought once or twice during life time. In contrast, for example one buys a loaf of bread on a regular basis, and thus one has acquired a feeling for its correct value.

A life insurance - in particular an individual policy - is a long term contract. Take for example a thirty year old man who buys a permanent life insurance. Now suppose he dies when he reaches ninety, then the contract period was sixty years.

Due to this long duration of the contract and the risks taken - think for example of changing fundamentals - it is necessary to calculate the price of an insurance with care and foresight.

In this chapter we are going to explain classical types of insurance policies. Furthermore we introduce a general model for life insurance, which can be used to price many of the available policies.

1.2 Examples

We start with a description of the most common types of life insurance and the different methods of financing a policy.

1.2.1 Types of life insurance

It is characteristic for every life insurance that the insured event is strongly related to the health of the insured. Thus one can classify life insurance as follows:

- insurance on life or death,
- insurance on permanent disability,
- health insurance.

For an insurance on life or death the essential event is the survival of the insured person up to a certain date or the death before a certain date, respectively. Furthermore these insurance types can be classified by the causes of death which yield a payout (e.g. a life insurance which pays only in the event of an accidental death). Especially, various kinds of survivor's pensions and pure endowments are insurances on life or death.

For a permanent disability insurance the essential criterion is the (dis)ability of the insured at a given date. These insurances have the special feature, that already a certain degree of disability might be sufficient for a claim.

For insurances on health the payout depends on the health of the insured. This class of insurances contains also modern types of policies, like a long term care insurance. The latter only provides benefits if the insured is unable to meet his basic needs (e.g. he is unable to dress himself).

Besides a classification based on the insured event one can also classify the insurance based on the benefit. This can either be paid in annuities or in a lump sum.

In the following we give some typical examples of life insurances.

Pension: A pension policy constitutes that the insurer has to pay annuities to the insured when he reaches a certain age (age of maturity of the policy). Then the pension is paid until the death of the insured. The payment of the annuities is usually done at regular intervals: monthly, quarterly or yearly. Moreover the payment can be done in advance (at the beginning of each interval) or arrears (at the and of each interval). Since the pension is only paid until death, one can additionally agree upon a minimum payment period. In this case the pension is paid at least for the minimum period. (This type of pension contract supplies the desire of the insured to get at least something back for the premiums paid in.)

Pure endowment: A pure endowment insurance provides a payment from the insurer to the insured, if he reaches the age of maturity of the policy. Otherwise there is no payment.

Term/permanent life insurance: A (term/permanent) life insurance is the counterpart to a pure endowment insurance. In contrast to the latter a life insurance does not yield a payout to the insured if the age of maturity of the policy is reached. In the popular case of a term life insurance there is no payout at all if the insured reaches the age of maturity of the policy. But if the insured dies before that age his heirs get a payment. A special case is the permanent life insurance, which yields a payout to the heirs no matter how old the insured is at the time of his death. This insurance is in some countries very popular, since it in a sense an investment into ones offsprings.

Endowment: The endowment insurance is the classic example of a life insurance. It is the sum of a pure endowment insurance and a term or permanent life insurance. This means that it yields a payout in the case of an early death and also in the case of reaching the fixed age of maturity.

Widow's pension: A widow's pension is connected to the life of two persons. This is in contrast to the previous examples, where only the life of one person was considered. For a widow's pension there is the insured (the person whose life is insured, e.g. husband) and the beneficiary person (e.g. spouse). As long as both persons are alive no payment is due. If the insured dies and the beneficiary is still alive, then the beneficiary gets a pension until death. Also for this kind of insurance it is possible to fix a minimum period of payments in the policy.

Orphan's pension: After the death of the father or the mother their child gets a pension until it is of age or until death.

Insurance on two lives: For an insurance on two lives, as in the case of a widow's or orphans's pension, one has to consider the lives of two persons. Here the policy fixes a payment depending on the state of the two persons (insured, beneficiary)$\in \{(**), (*\dagger), (\dagger *), (\dagger\dagger)\}$. Obviously, a widow's or an orphan's pension is just a special case of the insurance on two lives. As before, also in this case one could agree upon a minimum period of payments.

Refund guarantee: The refund guarantee is an additional insurance which is often sold together with a pension or a pure endowment. It is a life insurance whose payment equals the paid-in premiums, possibly reduced by the already received payments. The refund guarantee supplies the same want as the minimum periods of payment for the pensions.

Now we have discussed the main insurance types on life and death. Next we want to give a short description of insurances on permanent disability. For these the ability to work is the main criterion. In this context one should note that the probability of becoming disabled strongly depends on the economic environment. This is due to the fact that in a good economic environment everyone finds a job. But a person with restricted health has a hard time finding a job during an economic downturn. In connection with disability the following types of insurances are most common:

Disability pension: In the case of disability, after an initial waiting period, the insured gets a pension until he reaches a fixed age (or until his death) or until he is able to work again. A disability pension without fixed age for a final payment is called permanent disability pension. Often an initial waiting period is introduced, since in most cases disability occurs after an accident or illness and the person actually recovers quickly thereafter. Thus the initial waiting period reduces the price of these policies. Typical waiting periods are three or six month and one or two years.

Disability capital: The disability capital insurance pays a lump sum to the insured in case of a permanent disability.

Premium waiver: The premium waiver is an additional insurance. It waives the obligation to pay further premiums for the insured in the case of disability. Waiting periods are also common for this type of insurance.

Disability children's pension: This pension is similar to the orphan's pension. The only difference is that the cause for a payment is the disability of the mother or father instead of their death.

1.2.2 Methods of financing

Previously we looked at the different types of insurances. Now we are going to discuss the different financing methods and the ideas they are based on. The main principle of life insurances states that the value of the benefits provided by the insurer is equivalent to the value of the policy for the insured. Obviously one has to discuss this equivalence relation in more detail. We will do this in the next chapters and provide a precise definition of the equivalence principle. Then it will be used to calculate the premiums, but for now we get back to the methods of financing. The two common types are:

- Financing by premiums,
- Financing by a single payment (single premium).

Financing by premiums requires the insured to pay premiums to the insurer at regular intervals. This obligation usually ends either when the age of maturity of the policy is reached or if the insured dies.

The other option to finance a life insurance is a single payment. Often a policy incorporates a mixture of both financing methods.

1.3 The insurance model

In this section we want to introduce the insurance model which we will use thereafter. We attempt to describe the real world by a model. Thus it is important to use a model class which is flexible enough to accommodate this. Figure 1.1 shows the general setup of an insurance model. Here we think of an insured person who, at every time t, is in a state $1, 2, \ldots, n$. State 1 could for example indicate that the person is alive. The state of the person is then given by the stochastic process X with $X_t(\omega) \in S = \{1, 2, \ldots, n\}$. When the insured remains in one state or switches its state a payment, as defined in the insurance policy, is due. For this there are functions $a_i(t)$ and $a_{ij}(t)$ given which correspond to the lines in the figure. They define the amount which the insured gets if he remains in state i (payment $a_i(t)$) or if he switches from state i to j at time t (payment $a_{ij}(t)$). In the following we are going to introduce the necessary concepts for this setup. One distinguishes between the continuous time model, where $(X_t)_{t \in T}$ is defined on an interval in $\mathbb{R}$, and the discrete time model, where $(X_t)_{t \in T}$ is defined on a subset of $\mathbb{N}$. The continuous time model yields the more interesting statements whereas the discrete model is very important in applications, therefore we will discuss both models.

Definition 1.3.1 (State space). *We denote by S the state space which is used for the insurance policy. S is finite set.*

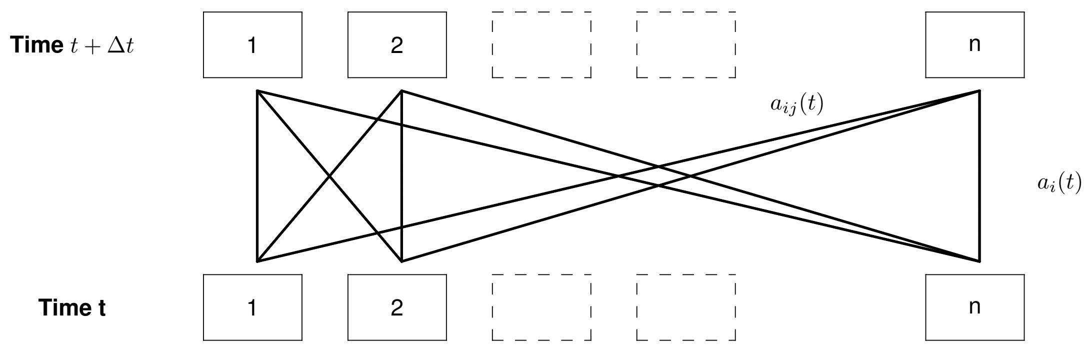

Figure 1.1: Policy setup from t to $t + \Delta t$

Example 1.3.2. For a life insurance or an endowment one often uses the statespace $S = \{*, \dagger\}$.

Example 1.3.3. For a disability insurance one has to consider at least the states: alive (active), dead and disabled. Often one uses more states to get a better model. For example in Switzerland a model is used which uses the states $\{*, \dagger\}$ and the family of states {person became disabled at the age of $x : x \in \mathbb{N}$}.

With the states defined, it is now possible to derive a mathematical model for the payments/benefits. To define the model of the benefits, the so called policy functions, it is necessary to define the time set more precisely. We will define two different time sets since a discrete time set is often used in applications, but the continuous time set yields the neater results.

Definition 1.3.4.
- *$a_i(t)$ denotes the sum of the payments to the insured up to time t, given that we know that he has always been in state i. The $a_i(t)$ are called the generalized pension payments. If this pension function is of bounded variation (see Def. 2.1.5) we can also write $a_i(t) = \int_0^t da_i(s)$.*
- *$a_{ij}(t)$ denotes the payments which are due when the state switches from i to j at time t. These benefits are called generalized capital benefits.*
- *In the case of a discrete time set $a_i^{Pre}(t)$ denotes the pension payment which is due at time t, given that the insured is at time t in i.*
- *In the case of a discrete time set $a_{ij}^{Post}(t)$ denotes the capital benefits which are due when switching from i at time t to j at time $t+1$. We are going to assume that the payment is transfered at the end of the time interval.*

The functions $a_i(t)$ are different in the continuous time model and the discrete time model. In the former $a_i(t)$ denotes the sum of the pension payments which are payed up to time t, similar to a mileage meter in a car. In the latter $a_i^{\mathrm{Pre}}(t)$ denotes the single pension payment at time t.

The following example illustrates the interplay between the state space and the functions which define the policy.

Example 1.3.5. Consider an endowment policy with 200,000 USD death benefit and 100,000 USD survival benefit. This insurance shall be financed by a yearly premium of 2,000 USD.

For a age at maturity of 65 the non trivial policy functions are:

$$
\begin{aligned}
a_*(x) &= \begin{cases} 0, & \text{if } x < x_0, \\ -\int_{x_0}^{x} 2000\,dt, & \text{if } x \in [x_0, 65], \\ -(65 - x_0) \times 2000 + 100000, & \text{if } x > 65, \end{cases} \\
a_{*\dagger}(x) &= \begin{cases} 0, & \text{if } x < x_0 \text{ or } x > 65, \\ 200000, & \text{if } x \in [x_0, 65], \end{cases}
\end{aligned}
$$

where x_0 is the age of entry into the contract, $*$ and $\dagger$ denote the states alive and dead, respectively.

2 Stochastic processes

2.1 Definitions

In this section we will recall basic definitions from probability theory. These will be used throughout the book.

To understand this chapter a basic knowledge in probability theory, measure theory and analysis is a prerequisite.

Definition 2.1.1 (Sets). *We are going to use the notations:*

$$\begin{aligned}
\mathbb{N} \quad &= \quad \textit{the set of the natural numbers including } 0, \\
\mathbb{N}_+ \quad &= \quad \{x \in \mathbb{N} : x > 0\}, \\
\mathbb{R} \quad &= \quad \textit{the set of the real numbers}, \\
\mathbb{R}_+ \quad &= \quad \{x \in \mathbb{R} : x \geq 0\}.
\end{aligned}$$

Furthermore we use the following notations for intervals. For $a, b \in \mathbb{R}, a < b$ we write

$$\begin{aligned}
[a, b] \quad &:= \quad \{t \in \mathbb{R} \,:\, a \leq t \leq b\}, \\
]a, b] \quad &:= \quad \{t \in \mathbb{R} \,:\, a < t \leq b\}, \\
]a, b[\quad &:= \quad \{t \in \mathbb{R} \,:\, a < t < b\}, \\
[a, b[\quad &:= \quad \{t \in \mathbb{R} \,:\, a \leq t < b\}.
\end{aligned}$$

Definition 2.1.2 (Indicator function). *For $A \subset \Omega$ we define the indicator function $\chi_A : \Omega \rightarrow \mathbb{R}, \omega \mapsto \chi_A(\omega)$ by*

$$\chi_A(\omega) \quad := \quad \begin{cases} 1, & \textit{if} \quad \omega \in A, \\ 0, & \textit{if} \quad \omega \notin A. \end{cases}$$

Furthermore δ_{ij} is Kronecker's delta, i.e., it is equal to 1 for $i = j$ and 0 otherwise.

Definition 2.1.3. *Let*

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x).$$

We define, if they exist, the left limit and the right limit of f at x by:

$$f(x^-) := \lim_{\xi \uparrow x} f(\xi),$$
$$f(x^+) := \lim_{\xi \downarrow x} f(\xi).$$

Definition 2.1.4. *A real valued function $f : \mathbb{R} \to \mathbb{R}$ is said to be of order $o(t)$, if*

$$\lim_{t \to 0} \frac{f(t)}{t} = 0.$$

This is denoted by $f(t) = o(t)$.

Definition 2.1.5 (Function of bounded variation). *Let $I \subset \mathbb{R}$ be a bounded interval. For a function*

$$f : I \to \mathbb{R}, t \mapsto f(t)$$

the total variation of the function f on the interval I is defined by

$$V(f, I) = \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)|,$$

where the supremum is taken with respect to all partitions of the interval I satisfying

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n.$$

The function f is of bounded variation *on I, if $V(f, I)$ is finite. Functions corresponding to a life insurance are usually defined on the interval $[0, \omega]$, where $\omega < \infty$ denotes the last age at which some individuals are alive.*

Properties of functions of bounded variation can be found for example in [DS57].

It is important to note, that functions of bounded variation form an algebra and a lattice. Thus, if f, g are functions of bounded variation and $\alpha \in \mathbb{R}$, then the following functions are also of bounded variation: $\alpha f + g$, $f \times g$, $\min(0, f)$ and $\max(0, f)$.

Definition 2.1.6 (Probability space, stochastic process). *We denote by $(\Omega, \mathcal{A}, P)$ a probability space which satisfies Kolmogorov's axioms.*

Let $(S, \mathcal{S})$ be a measurable space (i.e. S is a set and $\mathcal{S}$ is a σ-algebra on S) and T be a set. The Borel σ-algebra on the real numbers will be denoted by $\mathcal{R} = \sigma(\mathbb{R})$.

A family $\{X_t : t \in T\}$ of random variables

$$X_t : (\Omega, \mathcal{A}, P) \to (S, \mathcal{S}), \omega \mapsto X_t(\omega)$$

is called stochastic process *on $(\Omega, \mathcal{A}, P)$ with* state space *S.*

For each $\omega \in \Omega$ a sample path of the process is given by the function

$$X_\cdot(\omega) : T \to S, t \mapsto X_t(\omega).$$

We assume that each sample path is right continuous and has left limits.

Definition 2.1.7 (Expectations). *Let X be a random variable on $(\Omega, \mathcal{A}, P)$ and $\mathcal{B} \subset \mathcal{A}$ be a σ-algebra. Then we denote by*

- $E[X]$ *the* expectation *of the random variable* X*,*
- $V[X]$ *the* variance *of the random variable* X*,*
- $E[X|\mathcal{B}]$ *the* conditional expectation *of* X *with respect to* $\mathcal{B}$*.*

Definition 2.1.8. *Let $(X_t)_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$ taking values in a countable set S. We define for $j \in S$ the indicator function with respect to the process $(X_t)_{t\in T}$ at time t by*

$$I_j(t)(\omega) = \begin{cases} 1, & \text{if} \quad X_t(\omega) = j, \\ 0, & \text{if} \quad X_t(\omega) \neq j. \end{cases}$$

Analogous, we define for $j, k \in S$ the number of jumps from j to k in the time interval $]0, t[$ by

$$N_{jk}(t)(\omega) = \# \left\{ \tau \in]0, t[: \; X_{\tau^-} = j \text{ und } X_\tau = k \right\}.$$

Remark 2.1.9. In the following the function $I_j(t)$ is used to check if the insured person is at time t in state j. Thus one can check if the pension $a_j(t)$ has to be paid. Similarly, a switch from i to j is indicated by an increase of $N_{ij}(t)$ by 1.

Definition 2.1.10 (Normal distribution). *A random variable X on $(\mathbb{R}, \sigma(\mathbb{R}))$ with density*

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}$$

is called normal distributed *with expectation μ and variance σ^2. Such a random variable is denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$.*

Examples of stochastic processes are:

Definition 2.1.11 (Brownian motion). *An example of a non trivial stochastic process is Brownian motion. Brownian motion $W = (W_t)_{t\geq 0}$ with continuous time set ($T = \mathbb{R}_+$) and state space $S = \mathbb{R}$ is used to model many real world phenomena.*

The process is defined by the following properties:

1. $W_0 = 0$ *almost surely.*
2. W *has independent increments: the random variables $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$.*
3. $W_{t+h} - W_t \sim \mathcal{N}(0, h)$ *for all $t \geq 0$ and $h \geq 0$.*
4. *Almost all sample paths of $(W_t)_{t\in\mathbb{R}^+}$ are continuous.*

One can show that $(W_t)_{t\in\mathbb{R}^+}$ is nowhere differentiable.

Until now we do not know whether there exists a stochastic process $(W_t)_{t\in\mathbb{R}^+}$, which fulfills the requirements for a Brownian Motion. In the following we will show the existence of such a process. The way we will construct it, will also be helpful to simulate it. In order to do that we need some lemmas:

Lemma 2.1.12. *For $X \sim \mathcal{N}(0,1)$ and $x > 0$ the following inequalities hold:*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq P[X > x] \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Proof. For the second inequality we have:

$$\begin{aligned} P[X > x] &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\xi^2/2}d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{\xi}{x}e^{-\xi^2/2}d\xi \\ &= \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \end{aligned}$$

For the second inequality we define

$$f(x) = x\,e^{x^2/2} - (x^2+1)\int_x^\infty e^{\xi^2/2}\,d\xi$$

We remark that $f(0) < 0$ and $\lim_{x\to\infty} f(x) = 0$. Furthermore we have the following:

$$f'(x) \quad = \quad -2x\left(\int_x^\infty e^{-\xi^2/2}d\xi - \frac{e^{-x^2/2}}{x}\right),$$

which is positive for $x > 0$ by the first part. Therefore $f(x) \leq 0$.

Lemma 2.1.13. *For X_1 and X_2 two independent normally distributed random variable with $X_i \sim \mathcal{N}(0,\sigma^2)$ we have the following:*

1. *$Y_1 := X_1 + X_2$ and $Y_2 := X_1 - X_2$ are independent and normally distributed.*
2. *$Y_i \sim \mathcal{N}(0, 2\sigma^2)$.*

Proof. We note that

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

is an isometric orthogonal coordinate transformation in $\mathbb{R}^2$. Furthermore we note that $(X_1/\sigma, X_2/\sigma)^T$ is standard Gaussian by definition. Hence the application of A to this vector yields to the result.

Exercise 2.1.14. Complete the proof of lemma 2.1.13.

Theorem 2.1.15 (Existence of Brownian Motion, Wiener 1923). *The Standard Brownian Motion exists.*

Proof. We first construct Brownian motion on the interval $[0,1]$ as a random element on the space $C[0,1]$ of continuous functions on $[0,1]$. The idea is to construct the right joint distribution of Brownian motion step by step on the finite sets

$$\mathcal{D}_n \quad = \quad \{\frac{k}{2^n} : k \in \mathbb{N}, 0 \le k \le 2^n\}$$

of dyadic points for $n \in \mathbb{N}$. We then interpolate the values on $\mathcal{D}_n$ linearly and check that the uniform limit of these continuous functions exists and is a Brownian motion.

To do this let $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ and let $(\Omega, \mathcal{A}, P)$ be a probability space on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent, standard normally distributed random variables can be defined. Let $W(0) := 0$ and $W(1) := Z_1$. For each $n \in \mathbb{N}$ we define the random variables $W(d), d \in \mathcal{D}_n$ such that

1. for all $r < s < t$ in $\mathcal{D}_n$ the random variable $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, and is independent of $W(s) - W(r)$,
2. the vectors $\{W(d) : d \in \mathcal{D}_n\}$ and $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ are independent.

Note that we have already done this for $\mathcal{D}_0 = \{0, 1\}$. Proceeding inductively we may assume that we have succeeded in doing it for some $n - 1$. We then define $W(d)$ for $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ by

$$W(d) \quad = \quad \frac{W(d - 2^n) + W(d + 2^n)}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

Note that the first term is the linear interpolation of the values of W at the neighbouring points of d in $\mathcal{D}_{n-1}$. Therefore $W(d)$ is independent of $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ and the second property is fulfilled. Moreover, as $\frac{1}{2}[W(d + 2^{-n}) + W(d + 2^{-n})]$ depends only on $\{Z_t : t \in \mathcal{D}_{n-1}\}$, it is independent of $Z_d/2^{(n+1)/2}$. Both terms are normally distributed with mean zero and variance $2^{-(n+1)}$. Hence their sum $W(d) - W(d - 2^{-n})$ and their difference $W(d + 2^{-n}) - W(d)$ are independent and normally distributed with mean zero and variance 2^{-n} by Lemma 2.1.13.

Hence all increments $W(d) - W(d - 2^{-n})$, for $d \in \mathcal{D}_n \setminus 0$, are independent. To see this it suffices to show that they are pairwise independent, as the vector of these increments is Gaussian. We have seen that pairs $W(d) - W(d - 2^{-n})$, $W(d + 2^{-n}) - W(d)$ with $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ are independent. The other possibility is that the increments are over intervals separated by some $d \in \mathcal{D}_{n-1}$. Choose $d \in \mathcal{D}_j$ with this property and minimal j, so that the two intervals are contained in $[d - 2^{-j}, d]$, respectively $[d, d + 2^{-j}]$. By induction the increments over these two intervals of length 2^{-j} are independent, and the increments over the intervals of length 2^{-n} are constructed from the independent increments $W(d) - W(d - 2^{-j})$, respectively $W(d + 2^{-j}) - W(d)$, using a disjoint set of variables $\{Z_t : t \in \mathcal{D}_n\}$. Hence they are independent and this implies the first property, and completes the induction step.

Having thus chosen the values of the process on all dyadic points, we interpolate between them. Formally, define

$$F_0(t) = \begin{cases} Z_1 & \text{for } t = 1, \\ 0 & \text{for } t = 0, \text{ and} \\ \text{linear} & \text{in between.} \end{cases}$$

For each $n \geq 0$ we define

$$F_n(t) = \begin{cases} 2^{-(n+1)/2}\, Z_t & \text{for } t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, \\ 0 & \text{for } t \in \mathcal{D}_{n-1}, \text{ and} \\ \text{linear} & \text{between consecutive points in } \mathcal{D}_n. \end{cases}$$

These functions are continuous on $[0, 1]$ and for all n and $d \in \mathcal{D}_n$.

$$W(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d) \tag{2.1}$$

This can be seen by induction. It holds for $n = 0$. Suppose that it holds for $n - 1$. Let $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Since for $0 \leq i \leq n - 1$ the function F_i is linear on $[d - 2^{-n}, d + 2^{-n}]$, we get

$$\begin{aligned} \sum_{i=0}^{n} F_i(d) &= \sum_{i=0}^{n} \frac{F_i(d - 2^{-n}) + F_i(d + 2^{-n})}{2} \\ &= \frac{W(d - 2^{-n}) + W(d + 2^{-n})}{2} \end{aligned}$$

Since $F_n(d) = 2^{-(n+1)/2}\, Z_d$, this gives 2.1. On the other hand, we have, by definition of Z_d and by 2.1.12, for $c > 0$ and large n,

$$P[|Z_d| > c\sqrt{n}] \quad \leq \quad \exp\left(\frac{-c^2 n}{2}\right),$$

so that the series

$$\begin{aligned} P[\exists d \in \mathcal{D}_n \text{with} |Z_d| \geq c\sqrt{n}] &\leq \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} P[|Z_d| \geq c\sqrt{n}] \\ &\leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2 n}{2}\right), \end{aligned}$$

converges as soon as $c > \sqrt{2 \log 2}$. Fix such a $c \in \mathbb{R}$. By the Borel-Cantelli lemma there exists a random (but almost surely finite) $N \in \mathbb{N}$ such that for all $n \geq N$ and $d \in \mathcal{D}_n$ we have $|Z_d| < c\sqrt{n}$. Hence, for all $n \geq N$ we have

$$||F_n||_\infty \quad \leq \quad c\sqrt{n} 2^{-n/2}.$$

This upper bound implies that, almost surely, the series $W(t) = \sum n = 0^\infty F_n(t)$ is uniformly convergent on $[0, 1]$. We denote the continuous limit by $\{W(t) : t \in [0, 1]\}$.

It remains to check that the increments of this process have the right marginal distributions. This follows directly from the properties of W on the dense set $D \subset [0,1]$ and the continuity of the paths. Indeed, suppose that $t_1 < t_2 < \ldots < t_n$ are in $[0,1]$. We find $t_{1,k} \leq t_{2,k} \leq \ldots \leq t_{n,k}$ in $\mathcal{D}$ with $\lim_{k\to\infty} t_{i,k} = t_i$ and conclude from the continuity of W that, for $1 \leq i \leq n-1$,

$$W(t_{i+1}) - W(t_i) \quad = \quad \lim_{k\to\infty} W(t_{i+1,k}) - W(t_{i,k}).$$

As $\lim_{k\to\infty} E[W(t_{i+1,k}) - W(t_{i,k})] = 0$ and

$$\lim_{k\to\infty} Cov\left(W(t_{i+1,k}) - W(t_{i,k}), W(t_{j+1,k}) - W(t_{j,k})\right) \\ = \lim_{k\to\infty} \delta_{ij}\,(t_{i+1,k} - t_{i,k}) = \delta_{ij}\,(t_{i+1} - t_i),$$

the increments $W(t_{i+1}) - W(t_i)$ are independent Gaussian random variables with mean 0 and variance $t_{i+1} - t_i$, as required.

We have thus constructed a continuous process $W : [0,1] \to \mathbb{R}$ with the same marginal distributions as Brownian motion. Take a sequence $W_1, W_2, \ldots$ of independent $C[0,1]$-valued random variables with the distribution of this process, and define $\{W(t) : t \geq 0\}$ by gluing together the parts, more precisely by

$$W(t) \quad = \quad W_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} W_i(1),$$

for all $t \geq 0$. It is easy to show that $(W(t))_{t\in\mathbb{R}_0^+}$ fulfills the requirements for a standard Brownian motion.

Remark 2.1.16. Lévy's construction of the Brownian motion helps us also understanding how to simulate it. The typical approach to simulate it is to choose a $\Delta t = 2^{-n}$ and to define $(W(t))_{t\in\mathcal{D}_n}$ inductively as follows:

- $W(0) = 1$, and
- $W((k+1)\,2^{-n}) = W(k\,2^{-n}) + \sqrt{\Delta t}\,Z_k$, for all $k \geq 0$,

where $(Z_k)_{k\in\mathbb{N}}$ denotes a series of independent $\mathcal{N}(0,1)$ Gaussian variables. We note that for this purpose we do not need the dyadic recursion. Similarly one simulates a Brownian bridge. In this case $W(1)$ is also given and the simulation is performed by first defining $W(\frac{1}{2})$, then for $\mathcal{D}_2 \setminus \mathcal{D}_1$, etc.

Exercise 2.1.17. 1. Simulate a Brownian motion for $t \in \mathcal{D}_4$.

2. Modify your simulation to a Brownian bridge, assuming that $W(0) = W(1) = 0$.

Example 2.1.18 (Poisson process). The Poisson process $N = (N_t)_{t\geq 0}$ is a *counting process* with state space $\mathbb{N}$. For example it is used in insurance mathematics to model the number of incurred claims. This process also uses a continuous time set. The homogeneous Poisson process is characterized by the following properties:

1. $N_0 = 0$ almost surely.
2. N has independent and
3. N stationary increments.
4. For all $t > 0$ and all $k \in \mathbb{N}$ gilt: $P[N_t = k] = \exp(-\lambda\, t)\, \frac{(\lambda\, t)^k}{k!}$.

2.2 Markov chains on a countable state space

In the following S is a countable set.

Definition 2.2.1. *Let $(X_t)_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$ with state space S and $T \subset \mathbb{R}$. The process X is called Markov chain, if for all*

$$n \geq 1,\ t_1 < t_2 < \ldots < t_{n+1} \in T,\ i_1, i_2, \ldots, i_{n+1} \in S$$

with

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] > 0$$

the following statement holds:

$$P[X_{t_{n+1}} = i_{n+1} | X_{t_k} = i_k \forall k \leq n] \quad = \quad P[X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n]. \tag{2.2}$$

Remark 2.2.2.
1. Equation (2.2) states that the conditional probabilities only depend on the *last* state. They do not depend on the path which led the chain into that state.
2. Markov chains are very versatile in their applications. This is due to the fact, that on the one hand they are very easy to handle and on the other hand they can model a wide range of phenomena. In the following we are going to model life insurances by Markov chains.

Example 2.2.3.
1. Let $(X_t)_{t\in T}$ be a stochastic process with $S \subset \mathbb{R}$ and $T = \mathbb{N}_+$, for which the random variables $\{X_t : t \in T\}$ are independent. This process is a Markov chain since
$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] = \prod_{k=1}^{n} P[X_{t_k} = i_k]$$
for $n \geq 1,\ t_1 < t_2 < \ldots < t_{n+1} \in T,\ i_1, i_2, \ldots, i_{n+1} \in S$.
2. Based on the previous example we define $S_m = \sum_{k=1}^{m} X_k$, where $m \in \mathbb{N}$. This is also an example of a Markov chain.

   Proof.
$$\begin{aligned} &P[S_{t_{n+1}} = i_{n+1} \,|\, S_{t_1} = i_1, S_{t_2} = i_2, \ldots, S_{t_n} = i_n] \\ &\quad = P[S_{t_{n+1}} - S_{t_n} = i_{n+1} - i_n] \\ &\quad = P[S_{t_{n+1}} = i_{n+1} \,|\, S_{t_n} = i_n]. \end{aligned}$$

Definition 2.2.4. *Let $(X_t)_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$. Then*

$$p_{ij}(s,t) \quad := \quad P[X_t = j \,|\, X_s = i], \qquad \text{where } s \leq t \text{ and } i, j \in S,$$

is called the conditional probability to switch from state i at time s to state j at time t, or also transition probability for short.

The following theorem of Chapman and Kolmogorov is fundamental for the theory which we will present in the next chapters. The theorem states the relation of $P(s,t)$, $P(t,u)$ and $P(s,u)$ for $s \le t \le u$.

Theorem 2.2.5 (Chapman-Kolmogorov equation). *Let $(X_t)_{t\in T}$ be a Markov chain. For $s \le t \le u \in T$ and $i,k \in S$ such that $P\left[X_s = i\right] > 0$ the following equations hold:*

$$p_{ik}(s,u) = \sum_{j\in S} p_{ij}(s,t)\, p_{jk}(t,u), \tag{2.3}$$

$$P(s,u) = P(s,t) \times P(t,u). \tag{2.4}$$

This shows, that one can get $P(s,u)$ by matrix multiplication of $P(s,t)$ and $P(t,u)$ for $s \le t \le u \in T$.

Proof. Obviously, the equation holds for $t = s$ or $t = u$. Thus we can assume $s < t < u$ without loss of generality. We will use the following notation:

$$\begin{aligned} S^* &= \{j \in S : P\left[X_t = j \mid X_s = i\right] \neq 0\} \\ &= \{j \in S : P\left[X_t = j\,,\ X_s = i\right] \neq 0\}\,. \end{aligned}$$

(The last equality holds since $P\left[X_s = i\right] > 0$.) Now the Chapman-Kolmogorov equation can be deduced from the following equation:

$$\begin{aligned} p_{ik}(s,u) &= P\left[X_u = k \mid X_s = i\right] \\ &= \sum_{j\in S^*} P\left[X_u = k,\ X_t = j \mid X_s = i\right] \\ &= \sum_{j\in S^*} P\left[X_t = j \mid X_s = i\right] \times P\left[X_u = k \mid X_s = i,\ X_t = j\right] \\ &= \sum_{j\in S^*} p_{ij}(s,t) \times p_{jk}(t,u) \\ &= \sum_{j\in S} p_{ij}(s,t) \times p_{jk}(t,u), \end{aligned}$$

where we applied the Markov property to get equality in the forth line.

After proving the Chapman-Kolmogorov equation we are now able to introduce the abstract concept of transition matrices.

Definition 2.2.6 (Transition matrix). *A family $(p_{ij}(s,t))_{(i,j)\in S\times S}$ is called transition matrix, if the following four properties hold:*

1. $p_{ij}(s,t) \ge 0$.
2. $\sum_{j\in S} p_{ij}(s,t) = 1$.

3. $p_{ij}(s,s) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$ *if* $P[X_s = i] > 0$.
4. $p_{ik}(s,u) = \sum_{j \in S} p_{ij}(s,t)\, p_{jk}(t,u)$ *for* $s \leq t \leq u$ *and* $P\,[X_s = i] > 0$.

Theorem 2.2.7. *Let* $(X_t)_{t \in T}$ *be a Markov chain. Then* $(p_{ij}(s,t))_{(i,j) \in S \times S}$ *is a transition matrix.*

Proof. This theorem is a direct consequence of the theorem by Chapman and Kolmogorov (T. 2.2.5).

Theorem 2.2.8. *A stochastic process* $(X_t)_{t \in T}$ *is a Markov chain, if and only if*

$$P[X_{t_1} = i_1, \ldots, X_{t_n} = i_n] \;=\; P[X_{t_1} = i_1] \prod_{k=1}^{n-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}), \tag{2.5}$$

for all

$$n \geq 1,\; t_1 < t_2 < \ldots < t_{n+1} \in T,\; i_1, i_2, \ldots, i_{n+1} \in S.$$

Proof. Let $(X_t)_{t \in T}$ be a Markov chain satisfying

$$P[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n] > 0.$$

Then the Markov property implies

$$P[X_{t_1} = i_1, \ldots, X_{t_n} = i_n] = P[X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}] \cdot p_{i_{n-1}, i_n}(t_{n-1}, t_n).$$

This yields (2.5) by induction. The converse statement is trivial.

Theorem 2.2.9 (Markov property). *Let* $(X_t)_{t \in T}$ *be a Markov chain and* n, m *be elements of* $\mathbb{N}$. *Fix* $t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_{n+m}$, $i \in S$ *and sets* $A \subset S^{n-1}$ *(where* S^{n-1} *denotes the* $n-1$ *times Cartesian product of the set* S*) and* $B \subset S^m$ *such that*

$$P\left[(X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] > 0.$$

Then the following equation (Markov property) *holds:*

$$\begin{aligned} P\left[(X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B \,|\, (X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] \\ = P\left[(X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B \,|\, X_{t_n} = i\right]. \end{aligned}$$

Proof. We use the notation $\boldsymbol{i}^n = (i_1, i_2, \ldots, i_n)$. An application of equation (2.5) yields:

$$\begin{aligned} & P\left[(X_{t_1}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right] \\ & \quad = \sum_{\boldsymbol{i}^{n-1} \in A, i_n = i} P\left[X_{t_1} = i_1\right] \times \prod_{k=1}^{n-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}), \\ & P\left[(X_{t_1}, \ldots, X_{t_{n+m}}) \in A \times \{i\} \times B\right] \\ & \quad = \sum_{\boldsymbol{i}^{n+m} \in A \times \{i\} \times B} P\left[X_{t_1} = i_1\right] \times \prod_{k=1}^{n+m-1} p_{i_k, i_{k+1}}(t_k, t_{k+1}). \end{aligned}$$

Finally these two equations imply

$$
\begin{aligned}
&P\left[(X_{t_{n+1}},\ldots,X_{t_{n+m}})\in B\,|\,(X_{t_1},\ldots,X_{t_{n-1}})\in A, X_{t_n}=i\right]\\
&= \sum_{(i_n,i_{n+1},\ldots,i_{n+m})\in\{i\}\times B}\ \prod_{k=n}^{n+m-1} p_{i_k,i_{k+1}}(t_k,t_{k+1})\\
&\qquad \times \frac{\sum_{\boldsymbol{i}^{n-1}\in A} P[X_{t_1}=i_1]\times\prod_{l=1}^{n-1} p_{i_l,i_{l+1}}(t_l,t_{l+1})}{\sum_{\boldsymbol{i}^{n-1}\in A} P[X_{t_1}=i_1]\times\prod_{l=1}^{n-1} p_{i_l,i_{l+1}}(t_l,t_{l+1})}\\
&= \sum_{(i_{n+1},\ldots,i_{n+m})\in B}\ \prod_{k=n}^{n+m-1} p_{i_k,i_{k+1}}(t_k,t_{k+1})\frac{P[X_{t_n}=i]}{P[X_{t_n}=i]}\\
&= P\left[(X_{t_{n+1}},X_{t_{n+2}},\ldots,X_{t_{n+m}})\in B|X_{t_n}=i\right].
\end{aligned}
$$

Definition 2.2.10. *A Markov chain $(X_t)_{t\in T}$ is called* homogeneous, *if it is time homogeneous, i.e., the following equation holds for all $s,t\in\mathbb{R}, h>0$ and $i,j\in S$ such that $P[X_s=i]>0$ and $P[X_t=i]>0$:*

$$P[X_{s+h}=j\,|\,X_s=i] \quad=\quad P[X_{t+h}=j\,|\,X_t=i].$$

For a homogeneous Markov chain we use the notation:

$$
\begin{aligned}
p_{ij}(h) &:= p_{ij}(s,s+h),\\
P(h) &:= P(s,s+h).
\end{aligned}
$$

Remark 2.2.11. 1. A homogeneous Markov chain is characterized by the fact, that the transition probabilities, and therefore also the transition matrices, only depend on the size of the time increment.

2. For a homogeneous Markov chain one can simplify the Chapman-Kolmogorov equations to the semi group property:
$$P(s+t)=P(s)\times P(t).$$
The semi group property is popular in many different areas e.g. in quantum mechanics.

3. The mapping
$$P: T\to M_n(\mathbb{R}), t\mapsto P(t)$$
defines a one parameter *semi group*.

2.3 Markov chains in continuous time and Kolmogorov's differential equations

In the following we will only consider Markov chains on a finite state space. Thus point wise convergence and uniform convergence will coincide on S. This enables us to give some of the proofs in a simpler form.

Definition 2.3.1. *Let $(X_t)_{t\in T}$ be a Markov chain with finite state space S and $T \subset \mathbb{R}$. For $N \subset S$ we define*

$$p_{jN}(s,t) := \sum_{k\in N} p_{jk}(s,t).$$

Definition 2.3.2 (Transition rates). *Let $(X_t)_{t\in T}$ be a Markov chain in continuous time with finite state space S. $(X_t)_{t\in T}$ is called* regular, *if*

$$\mu_i(t) = \lim_{\Delta t \searrow 0} \frac{1 - p_{ii}(t, t+\Delta t)}{\Delta t} \text{ for all } i \in S, \tag{2.6}$$

$$\mu_{ij}(t) = \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \text{ for all } i \neq j \in S \tag{2.7}$$

are well defined and continuous with respect to t.

The functions $\mu_i(t)$ and $\mu_{ij}(t)$ are called transition rates *of the Markov chain. Furthermore we define μ_{ii} by*

$$\mu_{ii}(t) = -\mu_i(t) \text{ for all } i \in S. \tag{2.8}$$

Remark 2.3.3. 1. In the insurance model the regularity of the Markov chain is used to derive the differential equations which are satisfied by the mathematical reserve corresponding to the policy (Thiele's differential equation, e.g. Theorem 5.2.1).

2. One can understand the transition rates as derivatives of the transition probabilities. For example we get for $i \neq j$:

$$\begin{aligned} \mu_{ij}(t) &= \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \\ &= \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t) - p_{ij}(t,t)}{\Delta t} \\ &= \frac{d}{ds} p_{ij}(t,s)\Big|_{s=t}. \end{aligned}$$

3. $\mu_{ij}(t)\, dt$ can be understood as the infinitesimal transition rate from i to j ($i \rightsquigarrow j$) in the time interval $[t, t+dt]$. Similarly, $\mu_i(t)\, dt$ can be understood as the infinitesimal probability of leaving state i in the corresponding time interval. Let us define

$$\Lambda(t) = \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) & \mu_{13}(t) & \cdots & \mu_{1n}(t) \\ \mu_{21}(t) & \mu_{22}(t) & \mu_{23}(t) & \cdots & \mu_{2n}(t) \\ \mu_{31}(t) & \mu_{32}(t) & \mu_{33}(t) & \cdots & \mu_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n1}(t) & \mu_{n2}(t) & \mu_{n3}(t) & \cdots & \mu_{nn}(t) \end{pmatrix}.$$

In a sense, Λ generates the behavior of the Markov chain. That is, for a homogeneous Markov chain the following equation holds:

$$\Lambda(0) \quad = \quad \lim_{\Delta t \to 0} \frac{P(\Delta t) - 1}{\Delta t}.$$

$\Lambda := \Lambda(0)$ is called the *generator of the one parameter semi group*. We can reconstruct $P(t)$ by

$$P(t) = \exp(t\,\Lambda) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda^n.$$

4. In the remainder of the book we will only consider finite state spaces. This enables us to avoid certain technical difficulties with respect to convergence.

Based on the transition rates we can prove Kolmogorov's differential equations. These connect the partial derivatives of p_{ij} with μ:

Theorem 2.3.4 (Kolmogorov). *Let $(X_t)_{t \in T}$ be a regular Markov chain on a finite state space S. Then the following statements hold:*

1. (Backward differential equations)

$$\frac{d}{ds} p_{ij}(s,t) \quad = \quad \mu_i(s) p_{ij}(s,t) - \sum_{k \neq i} \mu_{ik}(s) p_{kj}(s,t), \tag{2.9}$$

$$\frac{d}{ds} P(s,t) \quad = \quad -\Lambda(s) P(s,t). \tag{2.10}$$

2. (Forward differential equations)

$$\frac{d}{dt} p_{ij}(s,t) \quad = \quad -p_{ij}(s,t)\mu_j(t) + \sum_{k \neq j} p_{ik}(s,t)\mu_{kj}(t), \tag{2.11}$$

$$\frac{d}{dt} P(s,t) \quad = \quad P(s,t)\Lambda(t). \tag{2.12}$$

Proof. The major part of the proof is based on the equations of Chapman and Kolmogorov.

1. We will prove the matrix version of the statement. This will help to highlight the key properties. Let $\Delta s > 0$ and set $\xi := s + \Delta s$.

$$\begin{aligned} \frac{P(\xi,t) - P(s,t)}{\Delta s} \quad &= \quad \frac{1}{\Delta s}\Big(P(\xi,t) - P(s,\xi)\,P(\xi,t)\Big) \\ &= \quad \left(\frac{1}{\Delta s}(1 - P(s,\xi))\right) \times P(\xi,t) \\ &\longrightarrow \quad -\Lambda(s)P(s,t) \text{ for } \Delta s \searrow 0, \end{aligned}$$

where we used the Chapman-Kolmogorov equation and the continuity of the matrix multiplication.

2. Analogous one can prove the forward differential equation. Let $\Delta t > 0$.

$$\begin{aligned}\frac{P(s,t+\Delta t)-P(s,t)}{\Delta t} &= \frac{1}{\Delta t}\Big(P(s,t)P(t,t+\Delta t)-P(s,t)\Big)\\ &= P(s,t)\times\frac{1}{\Delta t}\Big(P(t,t+\Delta t)-1\Big)\\ &\longrightarrow P(s,t)\Lambda(t) \text{ for } \Delta t\searrow 0.\end{aligned}$$

Remark 2.3.5. The primary application of Kolmogorov's differential equations is to calculate the transition probabilities p_{ij} based on the rates μ.

Definition 2.3.6. *Let $(X_t)_{t\in T}$ be a regular Markov chain on a finite state space S. Then we denote the conditional probability to stay during the time interval $[s,t]$ in j by*

$$\bar{p}_{jj}(s,t) := P\left[\bigcap_{\xi\in[s,t]}\{X_\xi = j\}\,|\,X_s = j\right]$$

where $s,t\in\mathbb{R}, s\leq t$ and $j\in S$.

In the setting of a life insurance this probability can for example be used to calculate the probability that the insured survives 5 years. The following theorem illustrates how this probability can be calculated based on the transition rates.

Theorem 2.3.7. *Let $(X_t)_{t\in T}$ be a regular Markov chain. Then*

$$\bar{p}_{jj}(s,t) = \exp\left(-\sum_{k\neq j}\int_s^t \mu_{jk}(\tau)d\tau\right) \tag{2.13}$$

holds for $s\leq t$, if $P[X_s = j] > 0$.

Proof. We define $K_j(s,t)$ by $K_j(s,t) := \bigcap_{\xi\in[s,t]}\{X_\xi = j\}$. Let $\Delta t > 0$. We have $P[A\cap B\,|\,C] = P[B\,|\,C]\,P[A\,|\,B\cap C]$ and thus

$$\begin{aligned}\bar{p}_{jj}(s,t+\Delta t) &= P[K_j(s,t)\cap K_j(t,t+\Delta t)\,|\,X_s = j]\\ &= P[K_j(s,t)\,|\,X_s=j]\,P[K_j(t,t+\Delta t)\,|\,X_s = j\cap K_j(s,t)]\\ &= P[K_j(s,t)\,|\,X_s=j]\,P[K_j(t,t+\Delta t)\,|\,X_t = j]\\ &= \bar{p}_{jj}(s,t)\,P[K_j(t,t+\Delta t)\,|\,X_t=j],\end{aligned}$$

where we used the Markov property and the relation $\{X_s = j\}\cap K_j(s,t) = \{X_t = j\}\cap K_j(s,t)$. The previous equation yields

$$\begin{aligned}\bar{p}_{jj}(s,t+\Delta t)-\bar{p}_{jj}(s,t) &= -\bar{p}_{jj}(s,t)\times\Big(1-P[K_j(t,t+\Delta t)\,|\,X_t=j]\Big)\\ &= -\bar{p}_{jj}(s,t)\times\Big(\sum_{k\neq j}p_{jk}(t,t+\Delta t)+o(\Delta t)\Big),\end{aligned}$$

where we used that the rates $\mu_{..}$ are well defined.

Now taking the limit we get the differential equation

$$\frac{d}{dt}\bar{p}_{jj}(s,t) \quad = \quad -\bar{p}_{jj}(s,t) \times \sum_{k \neq j} \mu_{jk}(t).$$

Solving this equation with the boundary condition $\bar{p}_{jj}(s,s) = 1$ yields the statement of the theorem, (2.13).

2.4 Examples

In this section we want to illustrate the theory of the previous sections by some examples.

Example 2.4.1 (Life insurance). We start with a life insurance, which provides a sum of money to the heirs in case of the death of the insured. Usually one uses for this a model with either two states (alive $*$, dead $\dagger$) or three states (alive, dead (accident), dead (disease)). We will use the model with two states, and the death rate will be exemplary modeled by the function

$$\mu_{*\dagger}(x) \quad = \quad \exp(-9.13275 + 8.09438 \cdot 10^{-2}x - 1.10180 \cdot 10^{-5}x^2). \tag{2.14}$$

The death rate is the transition rate of the state transition $* \rightsquigarrow \dagger$. See section 4.3 for a derivation of the death rate. Based on the death rate and formula (2.13) we are now able to calculate the survival probability of a 35 year old man:

$$\bar{p}_{**}(35,x) = \exp\left(-\int_{35}^{x} \mu_{*\dagger}(\tau)d\tau\right), \qquad \text{for } x > 35.$$

Figure 2.1 shows the transition rate (dotted line) and the survival probability (continuous line) based on $x = 35$.

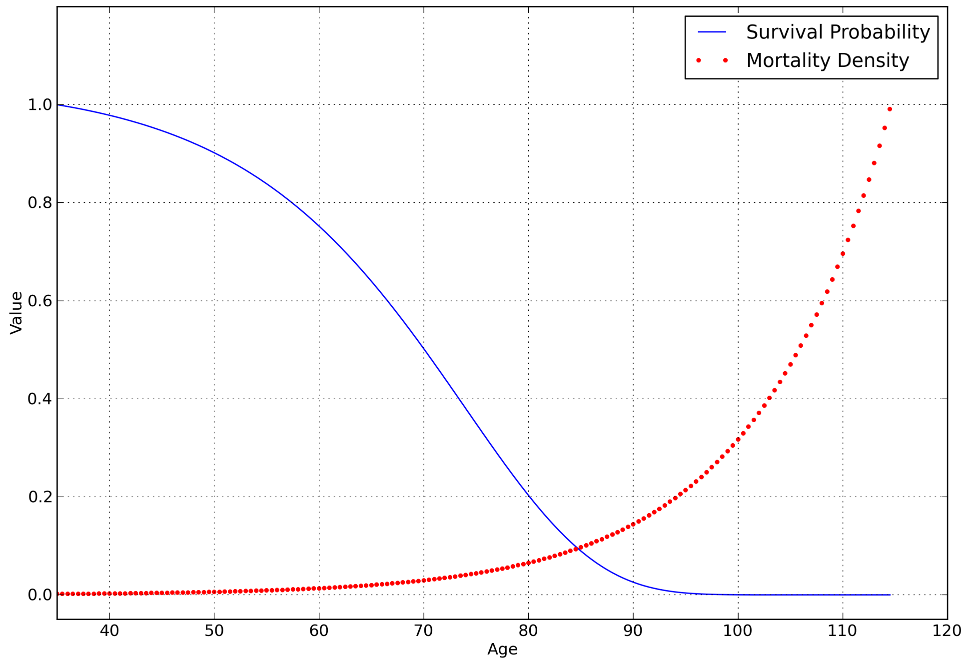

Figure 2.1: Mortality density $\mu_{*\dagger}(x)$ and survival probability $\bar{p}_{**}(35, x)$

Example 2.4.2 (Disability pension). We consider a model of a disability pension with the following three states:

state	**symbol**
active	$*$
disabled	$\diamond$
dead	$\dagger$

The transition rates are defined by

$$\begin{aligned}
\sigma(x) &:= 0.0004 + 10^{(0.060\,x - 5.46)}, \\
\mu(x) &:= 0.0005 + 10^{(0.038\,x - 4.12)}, \\
\mu_{*\diamond}(x) &:= \sigma(x), \\
\mu_{*\dagger}(x) &:= \mu(x), \\
\mu_{\diamond\dagger}(x) &:= \mu(x).
\end{aligned}$$

The transition rate σ is the infinitesimal probability of becoming disabled and μ is the corresponding probability of dying. We set the other transition rates equal to 0. Thus in particular this model does not incorporate the possibility of becoming active again ($\mu_{\diamond *} = 0$). Moreover one should note that in this model the mortality of disabled persons is equal to the mortality of active persons. This is a simplification, since in reality disabled persons have a higher mortality (they die earlier with a higher probability) than active persons. Therefore, this model yields an overpriced premium for the disability pension.

The explicit knowledge of the transition probabilities p_{ij} is useful for many formulas in insurance mathematics. For the current model they can be calculated by Kolmogorov's differential equations. We get

$$\begin{aligned}
p_{**}(x,y) &= \exp\left(-\int_x^y [\mu(\tau)+\sigma(\tau)]\, d\tau\right), \\
p_{*\diamond}(x,y) &= \exp\left(-\int_x^y \mu(\tau) d\tau\right) \times \left(1-\exp\left(-\int_x^y \sigma(\tau) d\tau\right)\right), \\
p_{\diamond\diamond}(x,y) &= \exp\left(-\int_x^y \mu(\tau) d\tau\right),
\end{aligned}$$

which solve Kolmogorov's differential equations for this model:

$$\begin{aligned}
\frac{d}{dt} p_{**}(s,t) &= -p_{**}(s,t) \times (\mu(t)+\sigma(t)), \\
\frac{d}{dt} p_{*\diamond}(s,t) &= -p_{*\diamond}(s,t)\, \mu(t) + p_{**}(s,t)\, \sigma(t), \\
\frac{d}{dt} p_{*\dagger}(s,t) &= (p_{**}(s,t) + p_{*\diamond}(s,t)) \times \mu(t), \\
\frac{d}{dt} p_{\diamond *}(s,t) &= 0, \\
\frac{d}{dt} p_{\diamond\diamond}(s,t) &= -p_{\diamond\diamond}(s,t)\, \mu(t), \\
\frac{d}{dt} p_{\diamond\dagger}(s,t) &= p_{\diamond\diamond}(s,t)\, \mu(t), \\
\frac{d}{dt} p_{\dagger\dagger}(s,t) &= 0,
\end{aligned}$$

with the boundary conditions $p_{ij}(s,s) = \delta_{ij}$. Note that, if one uses a model with a positive probability of becoming active again, one has to modify the first, second, forth and fifth equation. Obviously one can solve these equations with numerical methods, the solutions for the given example are listed in Table 2.1.

Exercise 2.4.3. Consider the above system of differential equations.

1. Find an exact solution.
2. Find a numerical approximation to the solution.

Table 2.1: Transition probabilities for the disability insurance

Initial age	$x_0 = 30$
Algorithm	Runge-Kutta of order 4
Step width	0.001

age x	$p_{**}(x_0, x)$	$p_{*\diamond}(x_0, x)$	$p_{*\dagger}(x_0, x)$	$p_{\diamond\diamond}(x_0, x)$	$p_{\diamond\dagger}(x_0, x)$
30.00	1.00000	0.00000	0.00000	1.00000	0.00000
35.00	0.98743	0.00354	0.00903	0.99097	0.00903
40.00	0.96998	0.00850	0.02152	0.97849	0.02152
45.00	0.94457	0.01620	0.03923	0.96077	0.03923
50.00	0.90624	0.02903	0.06474	0.93526	0.06474
55.00	0.84725	0.05106	0.10169	0.89831	0.10169
60.00	0.75677	0.08832	0.15491	0.84509	0.15491
65.00	0.62287	0.14700	0.23013	0.76987	0.23013

3 Interest rate

3.1 Introduction

An important part of every insurance contract is the underlying interest rate. The so called technical interest rate describes the interest which the insurer guarantees to the insured. It is a significant factor for the size of the premiums. If the technical interest rate is too low it yields inflated premiums, if it is too high it might yield to insolvency of the insurance company.

For the technical interest rate one can use a deterministic or a stochastic model. In the latter case the interest rate will be coupled to the bond market. In the following we are going to define the main parameters connected to the interest rate and present their relations.

3.2 Definitions

Example 3.2.1. Suppose we put 10,000 USD on a bank account on the first of January. If at the end of the year there are 10,500 USD on the account, then the underlying interest rate was 5 %.

Definition 3.2.2 (Interest rate). *We denote by i the* yearly interest rate*. Furthermore we assume, that it depends on time and write $i_t, t \geq 0$. If we use a stochastic model for the interest rate, then i is a stochastic process $(i_t(\omega))_{t\geq 0}$.*

One should note that this definition is useful in particular for the discrete time model. Since in this case one would use the time intervals which are given by the discretization. The calculation of the future capital based on an interest rate is done by

$$B_{t+1} = (1 + i_t) \times B_t.$$

Here B_t denotes the value of the account at time t. Often also the inverse of this relation is relevant. Thus one defines the discount rate.

Definition 3.2.3 (Discount rate). *Let i_t be the interest rate in year t. Then*

$$v_t = \frac{1}{1 + i_t}$$

is the discount rate *in year t.*

The discount rate can be used to calculate the net present value (the present value of the future benefits). If the interest rate is a stochastic process the previous considerations lead to the following problem: suppose we are going to receive 1 USD in one year, what is its present value? Generally there are two possible ways to find an answer.

Valuation principle A: If the interest rate i is known, the present value X is

$$X = \frac{1}{1+i}$$

and thus its mean is

$$X_A = E\left[\frac{1}{1+i}\right].$$

Valuation principle B: If the interest rate is known, the value of the account at the end of the year is $X(1+i)$. Thus

$$1 = E\left[X(1+i)\right] = X \times E\left[1+i\right]$$

holds and the mean is

$$X_B = \frac{1}{E\left[1+i\right]}.$$

But in general X_A is not equal to X_B. To overcome this problem one needs to fix by an assumption the valuation principle of the interest rate: we are always going to determine the net present value by valuation principle A (cf. [Büh92]).

After solving this paradox situation one understands the importance of the discount rate. Obviously, the same problem also arises in continuous time. For models in continuous time we assume that the interest is also payed continuously such that

$$B_{t+s} = \exp\left(\int_t^{t+s} \delta(\xi) d\xi\right) \times B_t.$$

Definition 3.2.4 (Interest intensity). *The* interest intensity *at time t is denoted by $\delta(t)$.*

A yearly interest rate i yields

$$e^{\delta} = 1 + i$$

and thus

$$\delta = \ln(1+i).$$

In continuous time the discount rate (from t to 0) is

$$v(t) = \exp\left(-\int_0^t \delta(\xi)\, d\xi\right).$$

Here the discount rate is modeled from t to 0 which is contrary to the discrete setting. The following relation holds:

$$v_t = \exp\left(-\int_t^{t+1} \delta(\xi)\, d\xi\right).$$

For a stochastic interest rate the interest intensity δ is also a stochastic process $(\delta_t(\omega))_{t\geq 0}$. Also in the continuous time setting we are going to use valuation principle A.

Finally it is worth to note the difference between $v(t)$ which denotes the discounting back to time 0 and v_t which denotes the discount from time $t+1$ to t. This later quantity is commonly used in actuarial sciences for recursion formulae.

3.3 Models for the interest rate process

Now we want to describe the stochastic behavior of the interest rate. We start with an analysis of the average interest rate of Swiss government bonds in Swiss franc. Figure 3.1 shows the rate from 1948 up to 2009.

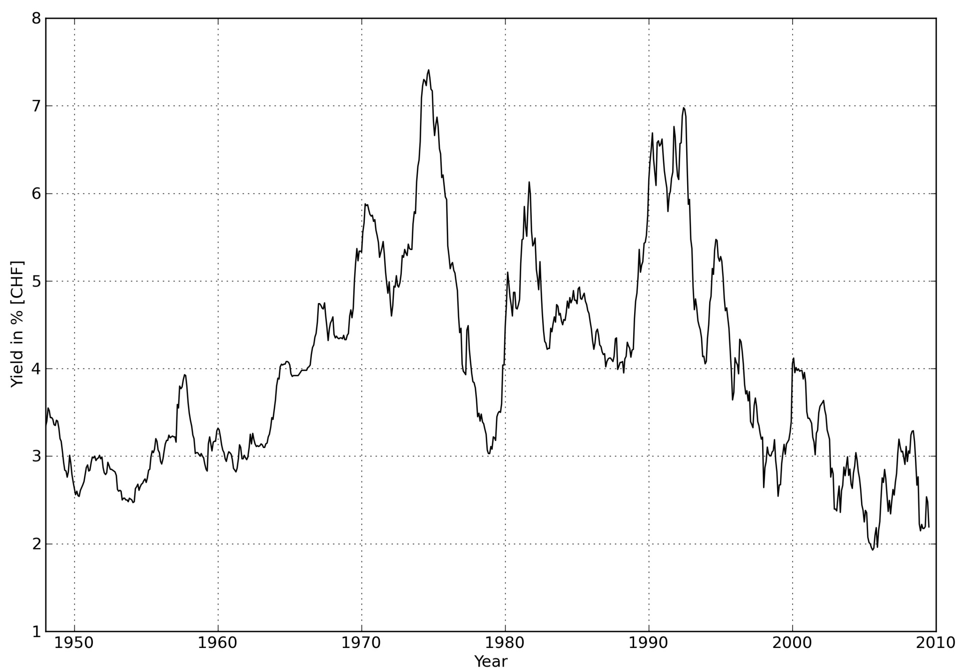

Figure 3.1: Average yield of government bonds in %

The figure shows that during the given period the interest rate of bonds was subject to huge fluctuations. The minimal rate of 1.93 % was reached in 2005. The maximum of 7.41 % was recorded during the "oil crisis" in the autumn of 1974. It is interesting to note, that in the first printing of this book (1999) the minium was actually reached in 1954 at about 2.5 %. At this time nobody expected that the interest rates in Switzerland would drop that far again. Based on this observation the inherent risk of the technical interest rate in insurance policies becomes obvious.

After seeing these values one starts to wonder how the technical interest rate should be determined. First of all, this depends on the purpose of the model in use. One has to differentiate between short term and long term relations. Moreover, the interest rate might only be used for marketing or forecasting purposes. In any case it should be emphasized that it might be risky to fix a constant interest rate above the level of the observed minimum. Since in this case there might be periods during which the interest yield of the assets does not cover the liabilities. Thus one has to take care when fixing the technical interest rate.

Another method to determine the technical interest rate is based on the analysis of the yield-curve or the forward-curve. This curves allow to measure the interest rate structure. The yield-curve can be used to determine the interest rate which one would get for a bond with a fixed investment period. Figure 3.2 shows the yield-curve for various currencies, it indicates that the interest rate is smaller for a bond with a short investment period than for a bond with a long investment period. This is called a 'normal' interest rate structure. Conversely, one speaks of an 'inverse' interest rate structure if the bonds with a short investment period provide a better yield than those with a long investment period.

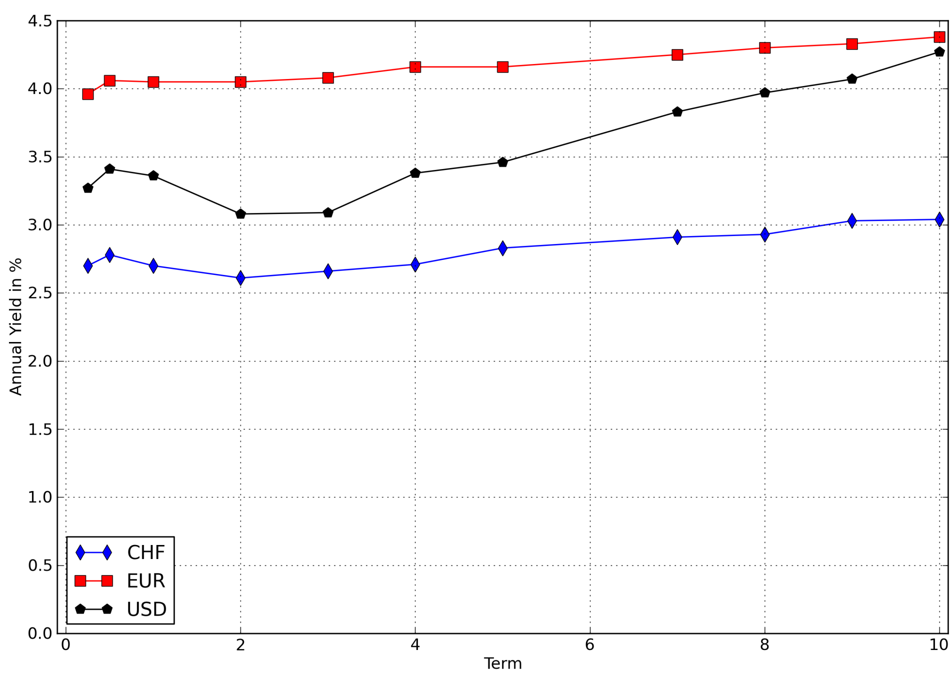

Figure 3.2: Yield curves as at 1.1.2008

This point of view provides a realistic evaluation of the interest rate. To utilize this we are going to define the so called zero coupon bond.

Definition 3.3.1 (Zero coupon bond). *Let $t \in \mathbb{R}$. Then the* zero coupon bond *with contract period t is defined by*

$$\mathcal{Z}_{(t)} = (\delta_{t,\tau})_{\tau \in \mathbb{R}^+}.$$

Thus the zero coupon bound is a security, which has the value 1 at time t.

Definition 3.3.2 (Price of a zero coupon bond). *Let $t \in \mathbb{R}$. Then the* price *of a zero coupon bond $\mathcal{Z}_{(s)}$ at time t with contract period s is denoted by*

$$\pi_t\left(\mathcal{Z}_{(s)}\right).$$

Based on these curves one can calculate the forward rates, i.e. the interest rate for year $n \rightsquigarrow n+1$, of the corresponding investment:

$$(1 + i_k) = \frac{\pi_t\left(\mathcal{Z}_{(k)}\right)}{\pi_t\left(\mathcal{Z}_{(k+1)}\right)}.$$

Here i_k is called the forward rate at time t for the contract period $[k, k+1[$, it is the *expected* rate for this time interval. Therefore the discount rate is given by

$$v_k = \frac{\pi_t\left(\mathcal{Z}_{(k+1)}\right)}{\pi_t\left(\mathcal{Z}_{(k)}\right)}.$$

Thus one can use a time dependent technical interest rate and adapt the necessary elements in the expectation based on the liabilities. This is especially useful for short term contracts with cash flows which are assessable. Consider for example the acquisition of a pension portfolio. In this case one takes over the obligation to pay the pensions of a given pension fund. The described method reduces the risks taken by fixing the technical interest rate.

The third method to determine the technical interest rate uses a stochastic interest rate model. This is interesting for practical and theoretical purposes. On the one hand this method is useful for policies which are tied to the performance of fonds and which provide guaranties. On the other hand it enables us to derive models which provide a tool to measure the risk of an insurance portfolio with respect to changes of the interest rate (see Chapter **??**). It turns out that for models with stochastic interest rate the corresponding risk does not vanish when the number of policies increases. This is a major difference to the deterministic model. Furthermore it means that the risk induced by the interest rate has a systemic, and thus dangerous, component. The construction of these models requires an analysis of the returns of several investment categories. Figure 3.3 shows the performance of two indices. These are measurements of the mean value of the return on investment in a given category. The indices in Figure 3.3 are the following

SPI	Swiss Performance Index: Swiss shares
SWISBGB	Swiss government bonds

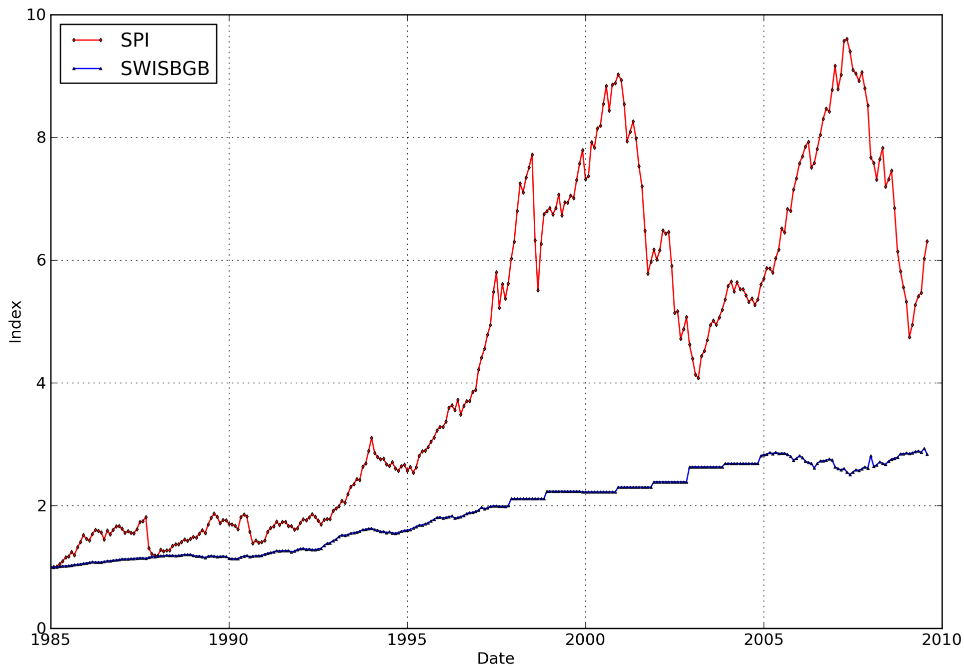

Figure 3.3: Performance of different indices

Observing the performances of these indices (Figure 3.3) we find a significant difference between shares and bonds. The expected return is larger for shares, but they also have a larger volatility (greater variance).

For a model with stochastic interest rate one has to model processes as depicted in the Figures 3.1, 3.2 and 3.3. The main difficulty in this context is the fact that there is no common standard model. Thus the actuary is responsible for selecting an appropriate model for the given problem.

In the following section we will describe some popular models. The reader interested in further details on the financial market, in particular interest rate models, is referred for example to [Hul12].

3.4 Stochastic interest rate

In the previous section we have seen several possibilities to determine the value of a cash flow based on the interest rate. Furthermore, the basics of a stochastic interest rate model were introduced. In this section we will present explicitly several stochastic interest rate models.

First of all one has to understand the difference between the stochastic behavior of the interest rate and the stochastic component of the mortality. Both create a risk for an insurance company. On the one hand there is the risk induced by the fluctuation of the interest rate and on the other hand there is risk based on the individual mortality. Changes of the interest rate affect all policies to the same degree. But the variation of the risk based on the individual mortality decreases when the number of policies increases. This is due to the law of large numbers and the independence of individual lifetimes.

Now we give a brief survey of stochastic interest rate models. We will concentrate on a description of these models without rating them. Nevertheless one should note that some models (e.g. the random walk model) are unsuitable to realistically describe an interest rate process.

3.4.1 Discrete time interest rate models

Random walk: Let $\mu \in \mathbb{R}_+$ and $t \geq 0$. The interest rate i_t is defined by

$$\begin{aligned} i_t &= \mu + X_t, \mu \in \mathbb{R}, \\ X_t &= X_{t-1} + Y_t, \\ Y_t &\sim \mathcal{N}(0, \sigma^2) \text{ i.i.d.} \end{aligned}$$

Note that this model is too simple to capture the real behavior.

AR(1)-model: In this model the interest rate is an autoregressive process of order 1:

$$\begin{aligned} i_t &= \mu + X_t, \\ X_t &= \phi\, X_{t-1} + Y_t, \text{ with } |\phi| < 1, \\ Y_t &\sim \mathcal{N}(0, \sigma^2). \end{aligned}$$

The model is mostly used by actuaries in England. The main idea is to start with an AR(1)-process as a model for the inflation. Then, in a second step, the models of the other economic values are based on this inflation. The constructed models have many parameters and are difficult to fit. References: [BP80] [Wil86] [Wil95].

3.4.2 Continuous time interest rate models

Brownian motion: $\delta_t = \delta + \sigma\, W_t$, where W_t is a standard Brownian motion.

Vasiček-model: The interest intensity is defined by the following stochastic differential equation

$$d\delta_t = -\alpha\,(\delta_t - \delta)\, dt + \sigma\, dW_t.$$

References: [Vas77].

Cox-Ingersoll-Ross: The interest intensity is defined by the following stochastic differential equation

$$d\delta_t = -\alpha\,(\delta_t - \delta)\, dt + \sigma\, \sqrt{\delta_t}\, dW_t.$$

References: [CIR85].

Markovian interest intensities: This model ([Nor95b]) uses a Markov chain $(X_t)_{t\geq 0}$ on a finite state space and a deterministic function $\delta_j(t)$ for each $j \in S$. Then the interest intensity is defined by

$$\delta_t = \sum_{j\in S} \chi_{\{X_t=j\}} \delta_j(t).$$

This means, that the interest intensity in a given state j at time t is determined by the corresponding deterministic function $\delta_j(.)$ evaluated at t. The model has the advantage, that it can be integrated into the Markov model. Furthermore it is very flexible due to its general state space. Therefore we will focus on this model in the following.

One should note that the Vasiček-model and the CIR-model feature mean reversion. Thus the interest intensity without stochastic noise (dW) converges in the long run toward the the mean intensity δ, since the differential equation without stochastic noise

$$d\delta_t = -\alpha\left(\delta_t - \delta\right) dt$$

has the solution

$$\delta_t = \gamma \times \exp(-\alpha\, t) + \delta.$$

The Vasiček-model and the Cox-Ingersoll-Ross-model are often used to model interest rate processes in applications. We will use these models in Chapter **??**.

Brownian motion and the Vasiček-model are problematic, since they allow negative interest rates with positive probability. In the Cox-Ingersoll-Ross model this can be prevented by an appropriate choice of the parameters.

In the following we assume that the presented stochastic differential equations have a solution.

We have seen above various models which are based on fundamentally different ideas. But in addition to the risk in the choice of the model there are further relevant systemic risks which affect the interest rate. These are:

The interest rate paid on an investment is not purely random. It also depends on political decisions. For example a monetary union causes the interest rates to converge, since in this case only one currency with one (random) interest rate exists (e.g. the European monetary union).

4 Cash flows and the mathematical reserve

4.1 Introduction

In the previous two chapters we introduced several types of insurances and their setup. Based on this we will now answer several fundamental questions.

First of all we will decide which general model we are going to use. Afterward we will explain how to value and price an insurance policy.

The present value of an insurance policy, the so called mathematical reserve, has to be determined by an insurance company on a yearly basis for the annual statement. This is necessary since the company has to reserve this value. The mathematical reserve is also important for the insured when he wants to cancel his policy before maturity.

In the remainder of the chapter the insurance model from Chapter 1 will be combined with the stochastic models of Chapter 2. Obviously Markov chains with a countable state space are not the only possible stochastic model, but we will focus on these. On the one hand they are general enough to model many phenomena. On the other hand the corresponding formulas are simple enough to perform explicit calculations.

4.2 Examples

In this section we present some examples which motivate the use of the Markov chain model for insurance policies.

Example 4.2.1 (Life insurance). Usually the state space of a permanent life insurance consists of the states “dead” and “alive”. Thus we use for the policy setup and for the stochastic process the state space $S = \{*, \dagger\}$, where $*$ denotes “alive” and $\dagger$ denotes “dead”. Based on the benefits of such a policy, as described in Chapter 1, one has to model the corresponding stochastic process. We will use the exemplary life insurance from Chapter 2. A typical sample path of the stochastic process corresponding to this policy is shown in Figure 4.1. It indicates that at the time of death (here at $x = 45$) the corresponding payment (e.g. 200,000 USD) is due. The mortality at that time is:

$$\begin{aligned} \mu_{*\dagger}(x)\big|_{x=45} &= \exp(-9.13275 + 0.08094x - 0.000011x^2)\big|_{x=45} \\ &= 0.00404. \end{aligned}$$

This means that on average 4 out of 1000 forty-five year old men die per year.

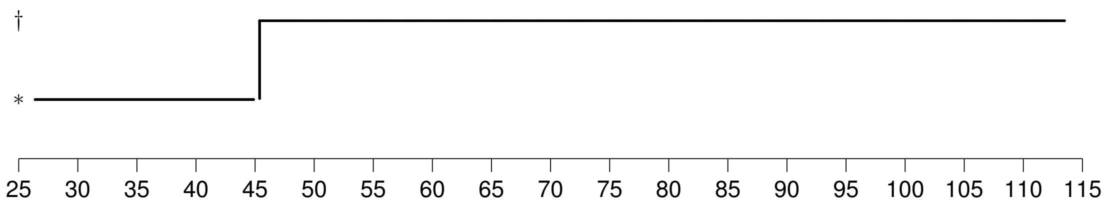

Figure 4.1: Trajectory of a mortality cover

Up to now we are not able to calculate the premiums for the policy in the example above. But we already notice the interplay between the payments and the stochastic processes.

Example 4.2.2 (Temporary disability pension). In this example we consider a policy of a disability pension which corresponds to the sample path in Figure 4.2. We want to record the various cash flows which it induces. For this we take the transition intensities from Example 2.4.2 with the additional assumption $\mu_{\diamond *}(x) = 0.05$. Then the sample path presented in Figure 4.2 causes the cash flows listed in Table 4.1.

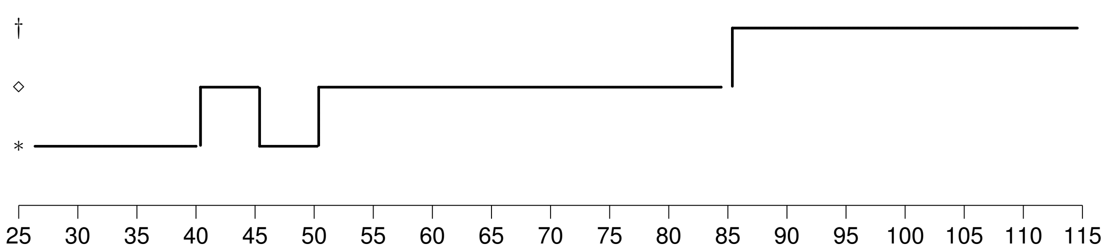

Figure 4.2: Trajectory of a disability cover

4.3 Fundamentals

In order to derive realistic models we have to know the fundamentals (underlying probabilities and biometric quantities). They are especially needed for the calculation of the premiums and mathematical reserves. As actuary one can look up the fundamentals in published tables. These list the probability of given events, for example the probability to die at a given age.

The tables used by insurance companies often incorporate a certain spread. For example one increases the probability of dying at a certain age if one calculates a life insurance. Conversely,

Table 4.1: Example of cash flows for a disability pension

time	**state**	**cash flow**	μ
$x \in \ [0,40[$	active $(*)$	premiums	
$x = 40$	becomes disabled	disability capital	$\mu_{*\diamond} = 0.00214$
$x \in \ ]40,45[$	disabled $(\diamond)$	disability pension	
$x = 45$	becomes active	—	$\mu_{\diamond *} = 0.05000$
$x \in \ [45,50[$	active $(*)$	premiums	
$x = 50$	becomes disabled	(maybe) disability capital	$\mu_{*\diamond} = 0.00387$
$x \in \ ]50,85[$	disabled $(\diamond)$	disability pension	
from 65		pension	
$x = 85$	dead	sum payable at death	$\mu_{\diamond \dagger} = 0.12932$

one increases the survival rate if one calculates a pension. This spread is used to decrease the risk of default and to cover possible demographic trends. That this is necessary illustrate for example the Tables 4.2 and 4.3. They list the average life expectancy for several generations, as given in the Swiss mortality tables. The average life expectancy is the number of years which a person of given age has on average still ahead. These tables clearly show that the average life expectancy increased during the last one hundred years. Therefore a spread is clearly necessary to cover this trend. Other western countries experience a similar increase of the average life expectancy.

Table 4.2: Average life expectancy based on Swiss mortality tables (male)

Alter	1881-88	1921-30	1939-44	1958-63	1978-83	1998-03	2008-13
1	51.8	61.3	64.8	69.4	72.1	76.6	79.5
20	39.6	45.2	47.9	51.5	53.8	58.0	60.7
40	25.1	28.3	30.4	32.8	35.1	39.0	41.3
60	12.4	13.8	14.8	16.2	17.9	21.1	23.0
80	4.2	4.3	4.8	5.5	6.3	7.5	8.3

Table 4.3: Average life expectancy based on Swiss mortality tables (female)

Alter	1881-88	1921-30	1939-44	1958-63	1978-83	1998-03	2008-13
1	52.8	63.8	68.5	74.5	78.6	82.2	83.8
20	41.0	47.6	51.3	56.2	60.1	63.4	64.9
40	26.7	30.9	33.4	37.0	40.7	43.8	45.3
60	12.7	15.1	16.7	19.2	22.4	25.2	26.4
80	4.2	4.9	5.3	6.1	7.8	9.3	10.0

We note that demographic quantities are constantly changing, like the average life expectancy. But where does this data actually come from and how where these tables calculated?

For the mortality tables one uses either the samples which are owned by the given insurance company or a collection of samples which is obtained jointly by several insurers. Then, to calculate the mortality rate, one counts the number of persons at risk and the number of died subjects for a given period of time (e.g. five years). The following example is based on data obtained by a large Swiss insurance company [PT93]. Figure 4.3 shows the number of alive and dead people at a given age. Figure 4.4 shows the raw mortality and the smoothed mortality.

The smoothed mortality is obtained by a smoothing algorithm. We are not going to discuss these algorithms. But we want to note, that there are various algorithms which greatly differ in their complexity.

On the raw curve one notices for example the accident-bump (i.e. the increased mortality) between 15 and 25 years. This is not visible in the smooth curve. Thus one has to adjust in this region the smoothed mortality.

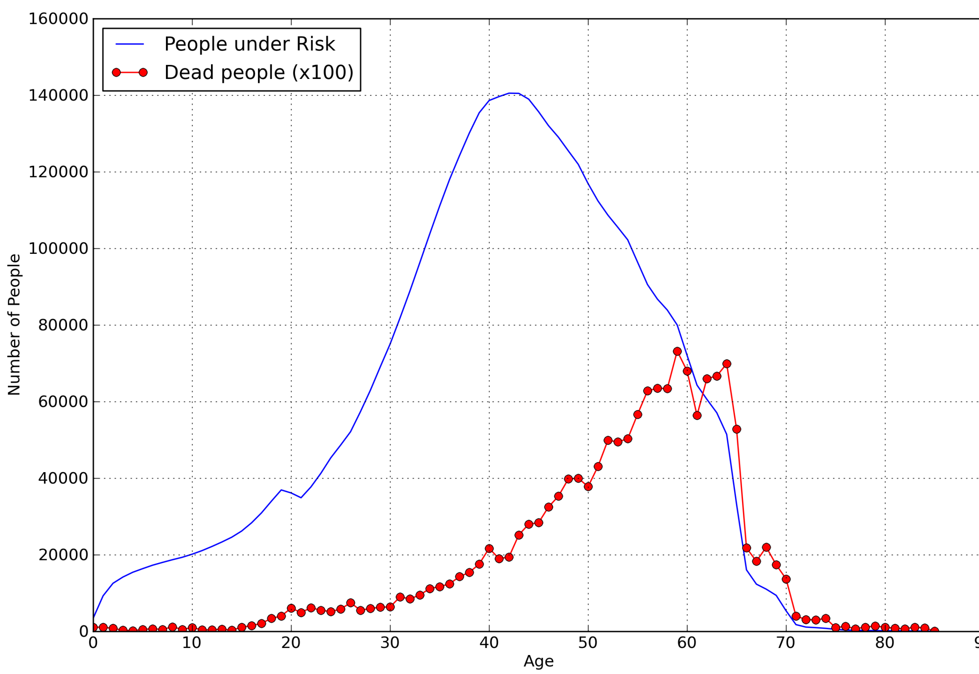

Figure 4.3: Inforce and number of dead people

In this example a polynomial of degree two was fitted to $\log(\mu_{*\dagger})$:

$$\mu_{*\dagger}(x) = \exp(-7.85785 + 0.01538 \cdot x + 5.77355 \cdot 10^{-4} \cdot x^2).$$

Analogous other demographic quantities which are relevant for the calculation of the premiums are obtained. Also for these a smoothed curve is obtained by an application a smoothing algorithm to the raw data.

The relevant probabilities and biometric quantities are collected in a catalog. Then, with the help of such a catalog, one can calculate the premiums and values of various products.

Table 4.4 lists the typically relevant quantities.

Table 4.4: Typical quantities for the calculation of premiums

variable	meaning
q_x	mortality, possibly separate for accidents and illness,
i_x	probability of becoming disabled, possibly separate for accidents and illness,
r_x	probability of becoming active again, possibly partitioned by the lengths of the disability period,
g_x	average degree of disability,
h_x	probability of being married at the time of death,
y_x	average age of the surviving marriage partner at the death of the insured.

Further details about the calculation of mortality tables and disability tables for the German and European market can be found in [DAV09].

Exercise 4.3.1. In this exercise we try to calculate the mortality in the Middle Ages. Below are the recorded birth and death dates for the adult royal family of Wales and the associated Marcher relations, beginning with Joanna (the daughter of King John of England) and Llywelyn Fawr (Llywelyn the Great, the Prince of Wales):

– Joanna: 1190-1237 (Daughter of King John of England; wife of Llywelyn Fawr) (47)

– Llywelyn Fawr: 1173-1240 (Prince of Wales) (67)

– Tangwystl: 1168-1206 (Mistress of Llywelyn Fawr) (38)

– Gwladys: 1206-1251 (Princess of Wales) (45)

– Ralph Mortimer 1198-1246 (Husband of Gladwys) (48)

– Gruffydd: 1196-1244 (Prince of Wales) (Fell from a rope while escaping the Tower of London) (48)

– Roger Mortimer: 1231-1282 (51)

– Maud de Braose: 1224-1300 (76)

– William de Braose: 1198-1230 (Hung by Llywelyn Fawr for sleeping with his wife, Joanna) (32)

– Eve Marshall: 1203-1246 (43)

– Dafydd ap Llywelyn: 1208-1246 (Prince of Wales) (42)

– Isabella de Braose: 1222-1248 (Wife of Dafydd) (26)

– Eleanor de Braose: 1226-1251 (25) (Childbirth)

– Humphrey de Bohun: 1225-1265 (40) (War)

– Edmund Mortimer: 1251-1304 (53)

– Margaret de Fiennes: 1269-1333 (64)

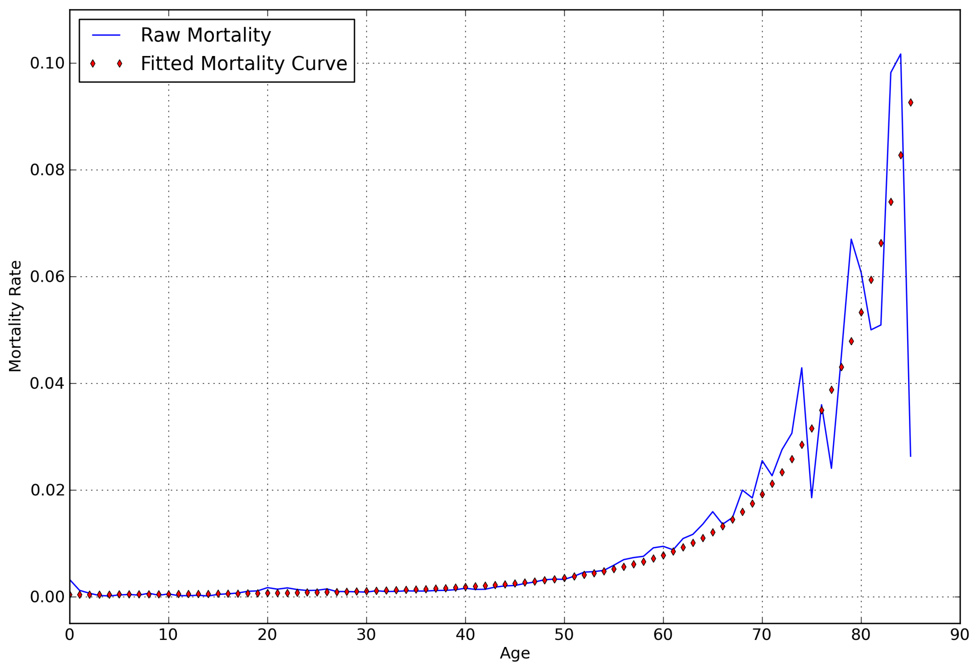

Figure 4.4: Mortality man

- Humphrey de Bohun: 1249-1298 (49)
- Maud de Fiennes: 1254-1296 (42)
- Llywelyn ap Gruffydd: 1225-1282 (57) (War)
- Elinor de Montfort: 1252-1282 (30) (Childbirth)

The task is to determine a suitable mortality density of the form

$$\mu_{*\dagger}(x) = \exp(a + b \cdot x + c \cdot x^2)$$

by means of a maximum likelihood estimator for the attained age at time of death. In order to do this we remark that

$$_t p_0 = (1 - q_0) \times (1 - q_1) \times \dots (1 - q_{t-1}).$$

Moreover it is worth noting that deaths of non-adults have been excluded from the list above. In order to allow for children which have died before adulthood we suggest to amend the list of attained ages by some virtual children. For our example we assume this list to be $(1, 1, 6, 9, 12, 14, 18)$. In order to check the reasonableness of the own results we have provided below the fitted parameters and the respective life expectancies which we have calculated for this exercise.

```
(a,b,c) = (-4.36, 1.01e-02, 4.08e-04)

Age  1 --> ex: 35.2   (x + ex = 36.2)
Age 20 --> ex: 24.7   (x + ex = 44.7)
Age 40 --> ex: 14.5   (x + ex = 54.5)
Age 60 --> ex:  6.5   (x + ex = 66.5)
Age 80 --> ex:  1.8   (x + ex = 81.8)
```

In case further help is needed:

$$\log(MLE(x_1, x_2, \dots x_n)) = \sum_{k=1}^{n} \log({}_{X_k}p_0).$$

Exercise 4.3.2. Compare the mortality densities as per example 4.2.1 and exercise 4.3.1 to determine the relative reduction of mortality since the Middle ages and interpret the implications on the respective life spans.

4.4 Deterministic cash flows

Definition 4.4.1 (Payout function). *A deterministic payout function A is a function*

$$A : T \to \mathbb{R}, t \mapsto A(t),$$

on $T \subset \mathbb{R}$ with the following properties:

1. *A is right continuous,*
2. *A is of bounded variation.*

The value $A(t)$ represents the total payments from the insurer to the insured up to time t. The payout functions are those functions in the policy setup which represent the benefits for the insured.

Example 4.4.2 (Disability insurance). We continue Example 4.2.2 by calculating the corresponding payout function. For this we assume that the policy does not contain a waiting period and that the disability pension is fixed to 20,000 USD per year (until the age of 65) with premiums of 2,500 USD per year until 65. Furthermore we suppose that the insurance was contracted at the age $x_0 = 25$. The payout function of this example is shown in Figure 4.5.

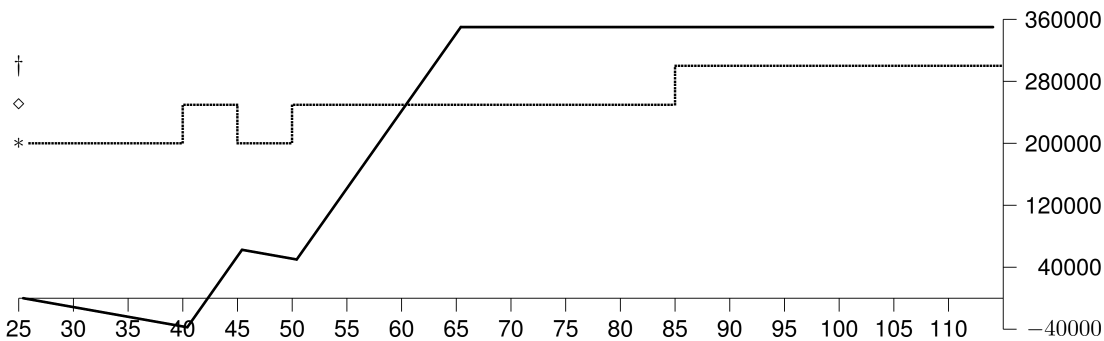

Figure 4.5: Cumulative payout of a disability annuity

Exercise 4.4.3. Derive the payout function for Example 4.4.2.

Remark 4.4.4 (Functions of bounded variation). Functions of bounded variation have the following properties [DS57]:

1. A function A of bounded variation corresponds to a measure on $\sigma(\mathbb{R})$. We will denote the measure again by A. This measure is called *Stieltjes measure*. In our setting it is also called payout measure.

2. Let A be a function of bounded variation on $\mathbb{R}$. Then there exist two positive, increasing and bounded functions B and C such that $A = B - C$. In the insurance model we can interpret B as inflow and C as outflow of cash. This representation is unique if one assumes that the measures corresponding to B and C have disjoint support. (Exercise: Calculate B and C for Example 4.4.2.)
3. Let A be the measure corresponding to a function of bounded variation on $\mathbb{R}$. Then A can be decomposed uniquely into a discrete measure μ and a continuous measure ψ. Furthermore one can decompose ψ into a part which is absolute continuous with respect to the Lebesgue measure and a remainder part. The support of μ is a countable set since A is finite on bounded sets.
4. Let A be a function of bounded variation and $T \in \sigma(\mathbb{R})$. Then $A \times \chi_T$ is also a function of bounded variation. (Here the function χ_T is the indicator function introduced in Definition 2.1.2.)

The above properties also hold for payout functions, since these are just functions of bounded variation. The decomposition of the Stieltjes measures will be useful later on. Thus we introduce the following notation:

Definition 4.4.5 (Decomposition of measures). *Let f be a function of bounded variation with corresponding Stieltjes measure A. Then we define*

$$\mu_f := A.$$

We know that we can decompose this measure uniquely into $A = B - C$, where B and C are positive measures with disjoint support. Therefore we define:

$$\begin{aligned} A^+ &:= B, \\ A^- &:= C. \end{aligned}$$

Furthermore the Stieltjes measure A can be decomposed uniquely into $A = D + E$, where D is discrete and E is continuous. Therefore we define:

$$\begin{aligned} A^{atom} &:= D, \\ A^{cont} &:= E. \end{aligned}$$

Furthermore, let μ be a measure which is absolute continuous with respect to the Lebesgue measure λ. Then we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodym density *of μ with respect to λ.*

Above we have seen the most important properties of deterministic cash flows. This enables us to define their values with the help of the discount rate. Recall, the discount rate is given by

$$v(t) = \exp(-\int_0^t \delta(\tau)\, d\tau).$$

Then the present value of a cash flow is defined as follows.

Definition 4.4.6 (Value of a cash flow). *Let A be a deterministic cash flow and $t \in \mathbb{R}$. We define:*

1. *The value of a cash flow A at time t is*

$$V(t,A) := \frac{1}{v(t)} \int_0^\infty v(\tau)\, dA(\tau).$$

2. *The value of the future cash flow is*

$$V^+(t,A) := V(t, A \times \chi_{]t,\infty]}).$$

It is also called prospective value of the cash flow *or* prospective reserve.

Concerning these definitions one should note:

Remark 4.4.7. 1. The idea of the prospective reserve is to calculate the present value of the future cash flows. Thus a payment of ζ which is due in two years contributes $v(2) \times \zeta$ to the present value. Initially the reserves are defined for deterministic cash flows. To define them also for random cash flows one uses the corresponding conditional expectations.

2. The definition implicitly requires that $v(t)$ is integrable with respect to the measure A, i.e. $v \in L^1(A)$.
3. The equation $A = A^{\text{atom}} + A^{\text{cont}}$ also implies $V(t,A) = V(t, A^{\text{atom}}) + V(t, A^{\text{cont}})$. This decomposition allows us to use different methods of proof for the discrete and the continuous part of the measure.

Example 4.4.8. We want to calculate $V^+(t,A)$ for the cash flow defined in Example 4.4.2 with $\delta(\tau) = \log(1.04)$. The first step is to calculate A^+ and A^-:

$$\begin{aligned} dA^+ &= 20000\,(\chi_{[40,45[} + \chi_{[50,65[})d\tau, \\ dA^- &= 2500\,(\chi_{[25,40[} + \chi_{[45,50[})d\tau. \end{aligned}$$

Then we get for $t \in [25,65[$

$$\begin{aligned} V^+(t,A) &= 20000 \int_t^{65} (1.04)^{-(\tau-t)}\,(\chi_{[40,45[} + \chi_{[50,65[})d\tau \\ &\quad -2500 \int_t^{65} (1.04)^{-(\tau-t)}\,(\chi_{[25,40[} + \chi_{[45,50[})d\tau. \end{aligned}$$

4.5 Stochastic cash flows

Definition 4.5.1 (Stochastic cash flow). *A* stochastic cash flow *or a* stochastic process of bounded variation *is a stochastic process $(X_t)_{t\in T}$ for which almost all sample paths are functions of bounded variation.*

Let A be a stochastic process of bounded variation such that $t \mapsto A_t(\omega)$ is right continuous and increasing for each $\omega \in \Omega$. Then it is possible to calculate the integral $\int f(\tau) d\mu_{A.(\omega)}(\tau)$

for a bounded Borel function f. Similarly, one can define P-almost everywhere the integral $\int f(\tau,\omega) d\mu_{A.(\omega)}(\tau)$ if $F_t = f(t,\omega)$ is a bounded function which is measurable with respect to the product sigma algebra. The construction of these integrals can be extended to general processes of bounded variation by decomposing the sample path, a function of bounded variation, into its positive (increasing) and negative (decreasing) part.

Definition 4.5.2. *Let $(A_t)_{t\in T}$ be a process of bounded variation on $(\Omega, \mathcal{A}, P)$ and $F : \mathbb{R}\times\Omega \to \mathbb{R}$ be a bounded and product measurable function. Then the description above yields the following definition*

$$(F \cdot A)_t(\omega) = \int_0^t F(\tau,\omega) dA\tau(\omega).$$

We also write this relation in the symbolic notation of stochastic differential equations:

$$d(F \cdot A) = F\, dA.$$

This definition allows us to give a precise definition of the stochastic cash flows in our insurance model.

Definition 4.5.3 (Policy cash flows). *We consider an insurance policy with state space S and payout functions $a_{ij}(t)$ and $a_i(t)$.*[1] *Based on Definition 2.1.8 we can define the stochastic cash flows corresponding to an insurance policy by*

$$\begin{aligned} dA_{ij}(t,\omega) &= a_{ij}(t)\, dN_{ij}(t,\omega), \\ dA_i(t,\omega) &= I_i(t,\omega)\, da_i(t), \\ dA &= \sum_{i\in S} dA_i + \sum_{(i,j)\in S\times S, i\neq j} dA_{ij}. \end{aligned}$$

The quantity $A_{ij}(t,\omega)$ is the sum of the random cash flows which are induced by transitions from state i to state j up to time t. Similarly, $A_i(t,\omega)$ represents the sum of the random cash flows up to time t which are pension payments for being in state i.

Remark 4.5.4.
1. The quantity $dA_{ij}(t,\omega)$ corresponds to the increase of the liabilities by a transition $i \rightsquigarrow j$. Therefore $A_{ij}(t,\omega)$ increases at time t by the capital benefit $a_{ij}(t)$ if at time t a transition $i \rightsquigarrow j$ takes place, i.e. if $N_{ij}(t)$ increases by 1. Similarly, $dA_i(t)$ corresponds to the increase of the liabilities caused by the insured being in state i.
2. The integrals appearing above are well defined since the corresponding processes are, by definition, of bounded variation. Moreover also the payout functions have the required regularity.
3. The quantities $(F \cdot A)_t$ are measurable for each t since F was assumed to be product measurable. Therefore also the expectation $E[(F \cdot A)_t]$ is well defined. Similarly, the conditional expectations $E[(F \cdot A)_t \,|\, \mathcal{F}_s]$ are well defined.
4. Thus one can apply Definition 4.4.6 (value of a cash flow) point wise (i.e. for each sample) to a stochastic cash flow. This yields the equation

[1] Thus the functions are of bounded variation and, in particular, bounded.

$$\begin{aligned} dV(t,A) &= v(t)\,dA(t) \\ &= v(t)\left[\sum_{i\in S} I_i(t)da_i(t) + \sum_{(i,j)\in S\times S, i\neq j} a_{ij}(t)dN_{ij}(t).\right] \end{aligned}$$

5. In the discrete Markov model at most two cash flows occur during a time interval $[t, t+1[$. Firstly, if the policy is in state i, $a_i^{\mathrm{Pre}}(t)$ is paid at the beginning of the interval. Secondly, if there is a transition $i \rightsquigarrow j$, $a_{ij}^{\mathrm{Post}}(t)$ is due at the end of the interval. Hence the following equations can be used to calculate the total cash flows:

$$\Delta A_{ij}(t,\omega) = \Delta N_{ij}(t,\omega) a_{ij}^{\mathrm{Post}}(t), \tag{4.1}$$

$$\Delta A_i(t,\omega) = I_i(t,\omega) a_i^{\mathrm{Pre}}(t), \tag{4.2}$$

$$\Delta A(t,\omega) = \sum_{i\in S} \Delta A_i(t,\omega) + \sum_{i,j\in S} \Delta A_{ij}(t,\omega), \tag{4.3}$$

where $\Delta A(t)$ (and similarely ΔA_{ij}, and ΔA_i, respectively) stands for the change of $A(t)$ from t to $t+1$, eg $\Delta A(t) := A(t+1) - A(t)$.

Definition 4.5.5. *Let A and (optionally) also v be stochastic processes on $(\Omega, \mathcal{A}, P)$ which are adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. In this case the* prospective reserve *is defined by:*

$$V_{\mathbb{F}}^{+}(t,A) = E\left[V^{+}(t,A) \,|\, \mathcal{F}_t\right].$$

One should note that also these reserves, like the usual expectations, might not exist, i.e., they might be infinite. In the following we will always assume that $V_{\mathbb{F}}^{+}(t,A)$ and the other quantities exist. This assumption is always satisfied in applications.

For a Markov chain the conditional expectation with respect to $\mathcal{F}_t$ depends only on the state at time t. Thus we additionally define

$$V_j^{+}(t,A) = E\left[V^{+}(t,A) \,|\, X_t = j\right].$$

The following definition fixes our assumptions on the regularity of an insurance model.

Definition 4.5.6 (Regular insurance model). *A regular insurance model consists of:*

1. *a regular Markov chain $(X_t)_{t\in T}$ with a state space S,*
2. *payout functions $a_{ij}(t)$ and $a_i(t)$,*
3. *right continuous interest intensities $\delta_i(t)$ of bounded variation.*

4.6 Mathematical reserve

The mathematical reserve is the amount of money an insurance company has to reserve for the expected liabilities in order to remain solvent. We assume that the interest intensity δ has the following structure: $\delta_t = \sum_{j\in S} I_j(t)\,\delta_j(t)$. Then the required reserves for the cash flows are defined by:

Definition 4.6.1 (Mathematical reserve). *The mathematical reserve for being in state $g \in S$ within a time interval $T \in \sigma(\mathbb{R})$ under the condition $X_t = j$ is defined by*

$$V_j(t, A_{gT}) = E\left[\frac{1}{v(t)} \times \int_T v(\tau)\, dA_g(\tau) \,|\, X_t = j\right].$$

Similarly, for transitions from g to $h \in S$, we define

$$V_j(t, A_{ghT}) = E\left[\frac{1}{v(t)} \times \int_T v(\tau)\, dA_{gh}(\tau) \,|\, X_t = j\right].$$

We use the notation $V_j(t, A_g)$ and $V_j(t, A_{gh})$ for $V_j(t, A_{g\mathbb{R}})$ and $V_j(t, A_{gh\mathbb{R}})$, respectively.

Remark 4.6.2. The definitions of the mathematical reserve can be translated to the discrete model. One just has to replace the integrals by the corresponding sums:

$$V_j(t, A_{gT}) = E\left[\frac{1}{v(t)} \times \sum_{\tau \in T} v(\tau)\, \Delta A_g(\tau) \,|\, X_t = j\right].$$

Analogous, for transitions from g to $h \in S$ we set:

$$V_j(t, A_{ghT}) = E\left[\frac{1}{v(t)} \times \sum_{\tau \in T} v(\tau + 1)\, \Delta A_{gh}(\tau) \,|\, X_t = j\right],$$

where we assumed that the payments always take place at time $\tau + 1$.

Therefore the total reserve (or mathematical reserve) for a given state j is

$$V_j(t, A) = \sum_{g \in S} V_j(t, A_g) + \sum_{g,h \in S, g \neq h} V_j(t, A_{gh})$$

for the continuous time model and

$$V_j(t, A) = \sum_{g \in S} V_j(t, A_g^{\text{Pre}}) + \sum_{g,h \in S} V_j(t, A_{gh}^{\text{Post}})$$

for the discrete time model. Thus we have defined the mathematical reserves. The next step is to calculate their values. Let us consider the relevant cash flows. On the one hand there are flows of the form $dA_1(t) = a(t) dN_{jk}(t)$ and on the other hand are flows of the form $dA_2(t) = I_j(t) dA(t)$.

The first step is to calculate the integrals $\int dA$ for the partial cash flows. Afterwards we will derive explicit formulas for the mathematical reserves.

Theorem 4.6.3. *Let $(X_t)_{t \in T}$ be a regular Markov chain on $(\Omega, \mathcal{A}, P)$ (cf. Def. 2.3.2). Furthermore let $i, j, k \in S$, $s < t$ and $T \in \sigma(\mathbb{R})$ where $T \subset [s, \infty]$. Then the following statements hold:*

1.
$$E\left[\int_T a(\tau)\, dN_{jk}(\tau) \,|\, X_s = i\right] = \int_T a(\tau)\, p_{ij}(s, \tau)\, \mu_{jk}(\tau) d\tau$$

for $a \in L^1(\mathbb{R})$.

2. *Let A be a function of bounded variation, then*

$$E\left[\int_T I_j(\tau)\, dA(\tau) \,|\, X_s = i\right] = \int_T p_{ij}(s,\tau)\, dA(\tau).$$

Proof. 1. The step functions are dense in L^1. Therefore it is enough to show the equality for functions of the form $\chi_{[a,b]}$. Moreover, the Borel σ-algebra is generated by the intervals in $\mathbb{R}_+$. Thus we can take $T = [c,d]$. Further we can set $c = a$ and $d = b$ without loss of generality, since the indicator function is equal to zero outside the interval $[a,b]$.

Define the function

$$h(t) := E\left[N_{jk}(t) \,|\, X_s = i\right].$$

Based on this definition we get

$$\begin{aligned} h(t+\Delta t) - h(t) &= E\left[N_{jk}(t+\Delta t) - N_{jk}(t) \,|\, X_s = i\right] \\ &= \sum_{l\in S} E\left[\chi_{\{X_t = l\}}(N_{jk}(t+\Delta t) - N_{jk}(t)) \,|\, X_s = i\right] \\ &= \sum_{l\in S} E\left[N_{jk}(t+\Delta t) - N_{jk}(t) \,|\, X_t = l\right] \times p_{il}(s,t). \end{aligned}$$

Now we observe that all the terms where $j \neq l$ are of order $o(\Delta t)$. Thus we get

$$= p_{ij}(s,t) \times \mu_{jk}(t) \times \Delta t + o(\Delta t).$$

Therefore $h'(t) = p_{ij}(s,t)\, \mu_{jk}(t)$, and an integration of this equation with initial condition $h(0) = 0$ yields the first statement of the theorem.

2. For the second statement one has to interchange the order of integration, which is allowed by Fubini's theorem.

Remark 4.6.4. Also these statements can be translated easily to the discrete model. One gets the equations

$$E\left[\sum_{\tau\in T} a(\tau)\, \Delta N_{jk}(\tau) \,|\, X_s = i\right] = \sum_{\tau\in T} a(\tau)\, p_{ij}(s,\tau) p_{jk}(\tau,\tau+1)$$

and

$$E\left[\sum_{\tau\in T} I_j(\tau)\, \Delta A(\tau) \,|\, X_s = i\right] = \sum_{\tau\in T} p_{ij}(s,\tau)\, \Delta A(\tau).$$

Exercise 4.6.5. Complete the proof of Theorem 4.6.3.

An important consequence of Theorem 4.6.3 is the next theorem.

Theorem 4.6.6. *Let the assumptions of Theorem 4.6.3 be satisfied. Then*

$$dM_{ij}(t) := dN_{ij}(t) - I_i(t)\, \mu_{ij}(t) dt$$

is a martingale.

Proof. We have

$$N_{ij}(t) \in L^1(\Omega, \mathcal{A}, P)$$

and

$$\int_0^t I_j(\tau)\, \mu_{ij}(\tau) d\tau \in L^1(\Omega, \mathcal{A}, P),$$

which implies

$$M_{ij}(t) \in L^1(\Omega, \mathcal{A}, P).$$

Next, we have to prove the equality $E[M_{ij}(t) \,|\, \mathcal{F}_s] = M_{ij}(s)$ for $s < t$. But, since the processes M, N and I are all derived from $(X_t)_{t \in T}$, it is enough to prove $E[M_{ij}(t) \,|\, X_s = k] = M_{ij}(s)$. But this is true, since

$$\begin{aligned} E[M_{ij}(t) \,|\, X_s = k] - M_{ij}(s) &= E\left[\int_s^t dM_{ij}(\tau) \,|\, X_s = k\right] \\ &= E\left[\int_s^t dN_{ij}(t) - I_i(t)\, \mu_{ij}(t) dt \,|\, X_s = k\right] \\ &= 0, \end{aligned}$$

where we used Theorem 4.6.3 in the last step.

Another application of Theorem 4.6.3 yields the following equations for the mathematical reserves in our insurance model:

Theorem 4.6.7. *Let a_{ij} and a_i be payout functions and $(X_t)_{t \in T}$ be a regular Markov chain on $(\Omega, \mathcal{A}, P)$. Then the following equations hold for fixed interest intensities (i.e., $\delta_i = \delta$):*

$$\begin{aligned} & E[V(t, A_{jT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)} \int_T v(\tau)\, p_{ij}(s, \tau) da_j(\tau), \\ & E[V(t, A_{jkT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)} \int_T v(\tau)\, a_{jk}(\tau)\, p_{ij}(s, \tau)\, \mu_{jk}(\tau) d\tau, \\ & E[V(t, A_{jS}) V(t, A_{lT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)^2} \int_{T \times S} v(\theta) v(\tau) \Big\{ \chi_{\{\theta \le \tau\}} p_{ij}(s, \theta) p_{jl}(\theta, \tau) \\ & \qquad\qquad + \chi_{\{\theta > \tau\}} p_{il}(s, \tau) p_{lj}(\tau, \theta) \Big\} da_j(\theta) da_l(\tau), \\ & E[V(t, A_{jkS}) V(t, A_{lmT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)^2} \Bigg[\int_{T \times S} v(\theta) v(\tau) \Big\{ \chi_{\{\theta \le \tau\}} p_{ij}(s, \theta) p_{kl}(\theta, \tau) \\ & \quad + \chi_{\{\theta > \tau\}} p_{il}(s, \theta) p_{mj}(\theta, \tau) \Big\} \mu_{jk}(\theta) \mu_{lm}(\tau) a_{jk}(\theta) a_{lm}(\tau) d\theta d\tau \\ & \quad + \delta_{jk,lm} \int_{T \cap S} v(\tau)^2 p_{ij}(s, \tau) \mu_{jk}(\tau) a_{jk}^2 d\tau \Bigg], \\ & E[V(t, A_{jS}) V(t, A_{lmT}) |\, X_s = i] \\ & \quad = \frac{1}{v(t)^2} \int_{T \times S} v(\theta) v(\tau) \Big\{ \chi_{\{\theta \le \tau\}} p_{ij}(s, \theta) p_{jl}(\theta, \tau) \\ & \quad + \chi_{\{\theta > \tau\}} p_{il}(s, \tau) p_{mj}(\tau, \theta) \Big\} da_j(\theta) \mu_{lm}(\tau) a_{lm}(\tau) d\tau. \end{aligned}$$

Proof. The first two equations are a direct consequence of Theorem 4.6.3. For a proof of the remaining equations we refer to [Nor91].

Remark 4.6.8. Also this theorem can easily be translated to the discrete setting. The following equalities hold:

$$
\begin{aligned}
E[V(t, A_{jT})|\, X_s = i] &= \frac{1}{v(t)} \sum_{\tau \in T} v(\tau)\, p_{ij}(s,\tau) a_j^{\text{Pre}}(\tau), \\
E[V(t, A_{jkT})|\, X_s = i] &= \frac{1}{v(t)} \sum_{\tau \in T} v(\tau+1)\, p_{ij}(s,\tau)\, p_{jk}(\tau,\tau+1)\, a_{jk}^{\text{Post}}(\tau),
\end{aligned}
$$

where we used that, for a transition $j \rightsquigarrow k$, the payments $a_{jk}^{\text{Post}}(\tau)$ are made the end of the period.

Exercise 4.6.9. Complete the proof of Theorem 4.6.7.

Given the transition probabilities one can use Theorem 4.6.7 to calculate the expectations and variances of the prospective reserves for each cash flow. Then, based on these partial reserves, one can calculate the total prospective reserves by the following result.

Theorem 4.6.10. *Let a regular insurance model (Definition 4.5.6) with deterministic interest intensities be given. Then the prospective reserves are given by*

$$
\begin{aligned}
V_j^+(t) &= \frac{1}{v(t)} \int_{]t,\infty[} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \\
&\qquad \times \left\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau) \mu_{gh}(\tau) d\tau \right\}.
\end{aligned}
$$

Remark 4.6.11. The formula of the previous theorem is not very useful, since one has to calculate integrals based on the transition probabilities p_{ij}. This becomes even more complicated by the fact that in applications often only the μ_{ij} are given. In the next section we will find a more elegant way to calculate this quantity.

4.7 Recursion formulas for the mathematical reserves

In this section we will derive a recursion formula for the reserves based on their integral representation. This recursion can be used in two ways. On the one hand it can be used to prove Thiele's differential equation. On the other hand it can be applied to the discrete model. Thereby it provides a way to calculate the values for various types of insurances. We will see in the remaining sections that these recursion equations, difference equations and differential equations are a extremely useful tools for explicit calculations. We adapt our definition of mathematical reserves, in order to simplify the proofs:

Definition 4.7.1. *We define for a regluar insurance model (Definiton 4.5.6):*

$$W_j^+(t) \quad := \quad v(t)\, V_j^+(t).$$

The difference between W and the usual mathematical reserve V is only the discount factor. V resembles the value of the cash flow at time t, whereas W is the value of the cash flow at time 0. Thus W is only a new notation which will help to keep the proofs simple. Based on this we are now able to derive a recursion formula for the prospective reserve.

Lemma 4.7.2. *Let $j \in S$, $s < t < u$ and $(X_t)_{t\in T}$ be a regular insurance model in continuous time with deterministic interest intensities. Then the following equation holds:*

$$\begin{aligned} W_j^+(t) \quad &= \quad \sum_{g\in S} p_{jg}(t,u)\, W_g^+(u) \\ &\quad + \int_{]t,u]} v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\}. \end{aligned}$$

Proof. The proof is based on the Chapman-Kolmogorov equation. We get

$$\begin{aligned} W_j^+(t) \quad &= \quad \int_{]t,\infty]} v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \\ &= \quad \left(\int_{]t,u]} + \int_{]u,\infty]} \right) v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \\ &\qquad\qquad \times \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \\ &= \quad \int_{]t,u]} v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \\ &\quad + \int_{]u,\infty]} v(\tau) \sum_{g\in S} \left(\sum_{k\in S} p_{jk}(t,u) p_{kg}(u,\tau) \right) \\ &\qquad\qquad \times \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \\ &= \quad \int_{]t,u]} v(\tau) \sum_{g\in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \\ &\quad + \sum_{k\in S} p_{jk}(t,u) \Bigg(\int_{]u,\infty]} v(\tau) \sum_{g\in S} p_{kg}(u,\tau) \\ &\qquad\qquad \times \left\{ da_g(\tau) + \sum_{S\ni h\neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{g \in S} p_{jg}(t,u)\, W_g^+(u) \\
&\quad + \int_{]t,u]} v(\tau) \sum_{g \in S} p_{jg}(t,\tau) \left\{ da_g(\tau) + \sum_{S \ni h \neq g} a_{gh}(\tau)\mu_{gh}(\tau) d\tau \right\}.
\end{aligned}
$$

The recursion formula can also be translated to the discrete model. For this one assumes that the payments are done at discrete times rather than in continuous time. For example pensions are paid at the beginning of the interval and death capital is paid at the end of the interval. We denote the payments at the beginning of the year by $a_i^{Pre}(t)$ and those at the end of the year by $a_{ij}^{Post}(t)$. Thus, in particular we assume that a transition between states can only occur at the end of the year.

Setting $\Delta t = 1$ in the previous lemma yields the following recursion for the reserves in the discrete setting.

Theorem 4.7.3 (Thiele's difference equation). *For a discrete time Markov model the prospective reserve satisfies the following recursion:*

$$
V_i^+(t) = a_i^{Pre}(t) + \sum_{j \in S} v_t\, p_{ij}(t) \{ a_{ij}^{Post}(t) + V_j^+(t+1) \}.
$$

Remark 4.7.4.

- The formula shows that errors which are introduced by the discretization of time are due to payments between the discretization times.
- The recursion formula of the mathematical reserve is very important for applications, since it provides a way to calculate a single premium and yearly premiums. In fact it is the most important formula for explicit calculations.
- To solve a differential equation or a difference equation one needs a boundary condition. For example, if one calculates a pension, the boundary condition is given by the fact that the reserve has to be equal to zero at the final age ω.

4.8 Calculation of the premiums

In this section we are going to calculate single premiums and yearly premiums for several types of insurance policies. The calculations in the examples are based on the discrete recursion (Theorem 4.7.3). We start with an endowment policy.

Example 4.8.1 (Endowment policy in discrete time). We consider the insurance defined in Example 4.2.1. Thus there is a death benefit of 200,000 USD. Moreover we assume an endowment of 100,000 USD and a starting age of 30 with 65 as fixed age at maturity.

- How much is a single premium for this insurance, given a technical interest rate of 3.5%?
- How much are the corresponding yearly premium?

We use the mortality rates given by (2.14). First we calculate the single premium. The following payout functions are given:

$$a_{*\dagger}^{\text{Post}}(x) = \begin{cases} 200000, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_{**}^{\text{Post}}(x) = \begin{cases} 0, & \text{if} \quad x < 64, \\ 100000, & \text{if} \quad x = 64, \\ 0, & \text{otherwise.} \end{cases}$$

An application of Theorem 4.7.3 yields the results presented in Table 4.5. Here one has to note that the mathematical reserves for the case of survival and the case of death have to be calculated separately. The reserve in the case of survival is $V_*(t, A_{**\mathbb{R}})$ and the reserve in the case of death is $V_*(t, A_{*\dagger\mathbb{R}})$ (cf. Definition 4.6.1).

Table 4.5: Reserves for an endowment

age	q_x	**res. for endowment**	**res. for death benefit**	**sums reserve**
65	0.01988	**100000**	**0**	**100000**
64	0.01836	94844	3548	98392
63	0.01696	90083	6647	96730
62	0.01566	85674	9348	95022
61	0.01446	81579	11696	93275
60	0.01336	77768	13730	91498
55	0.00897	62086	20275	82360
50	0.00602	50444	22766	73210
45	0.00404	41470	22956	64426
40	0.00271	34362	21874	56236
35	0.00181	28624	20135	48759
30	0.00121	23928	18116	42044

The table of the mathematical reserves indicates that the recursion formula was used with the boundary condition $x = 65$. Figure 4.6 shows the necessary reserves for several values of the technical interest rate.

After the calculation of the single premium we will now consider the case of yearly premiums. The yearly payment of the premiums is modeled by the following payout function:

$$a_{*}^{\text{Pre}}(x) = \begin{cases} -P, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

P has to be calculated such that the value of the insurance is equal to zero at the beginning of the policy. (Equivalence principle: The expected value of the benefits provided by the insurer and the value of the expected premium payments by the insured coincide.) The simplest method to determine the value of P is to consider V_x^{payout} (given by the first two payout functions of the policy) and V_x^{premiums} (given by the third function of the policy, i.e. by the premiums paid in). The total mathematical reserve is then given by $V_x = V_x^{\text{payout}} + V_x^{\text{premiums}}$. But we know that $V_x^{\text{premiums}} = P \times V_x^{\text{premiums},P=1}$ holds. Thus we can calculate P by the formula

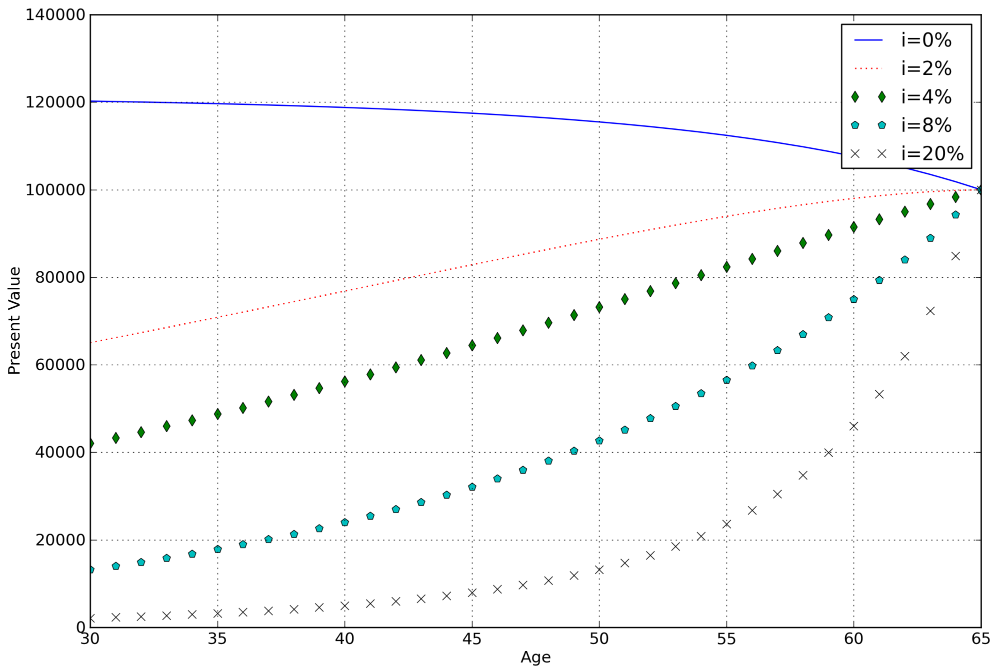

Figure 4.6: Mathematical reserves as a function of interest rates

$$P = -V_x^{\text{payout}} / V_x^{\text{premiums},P=1},$$

since V_x at inception of the policy is 0, as a consequence of the equivalence principle. For our example we get

$$P = 2.129, 15 \text{ USD per year}.$$

Table 4.6 lists the reserves for this insurance with yearly premiums. Figure 4.7 illustrates the same data in a graph.

Exercise 4.8.2. Do the calculation for the previous example.

In the next example we will consider the simple disability insurances model which we looked at earlier. We will show how to model the disability pension with and without an exemption from payment of premiums.

Example 4.8.3 (Disability insurance). We use the model for the disability insurance introduced in Example 2.4.2. Thus in particular we do not incorporate the possibility that insured becomes active again. Moreover we also do not model a waiting period.

– Calculate for a 30-year old man the present value of a (new) disability pension based on 65 as age at maturity and a technical interest rate of 4%.

Table 4.6: Reserves for an endowment with yearly premiums

age	**present value premiums**	**present value payout**	**reserve**
65	0	100000	100000
64	-2129	98392	96263
63	-4149	96730	92581
62	-6069	95022	88952
61	-7901	93275	85374
60	-9653	91498	81845
50	-23928	73210	49282
40	-34292	56236	21943
30	-42044	42044	0

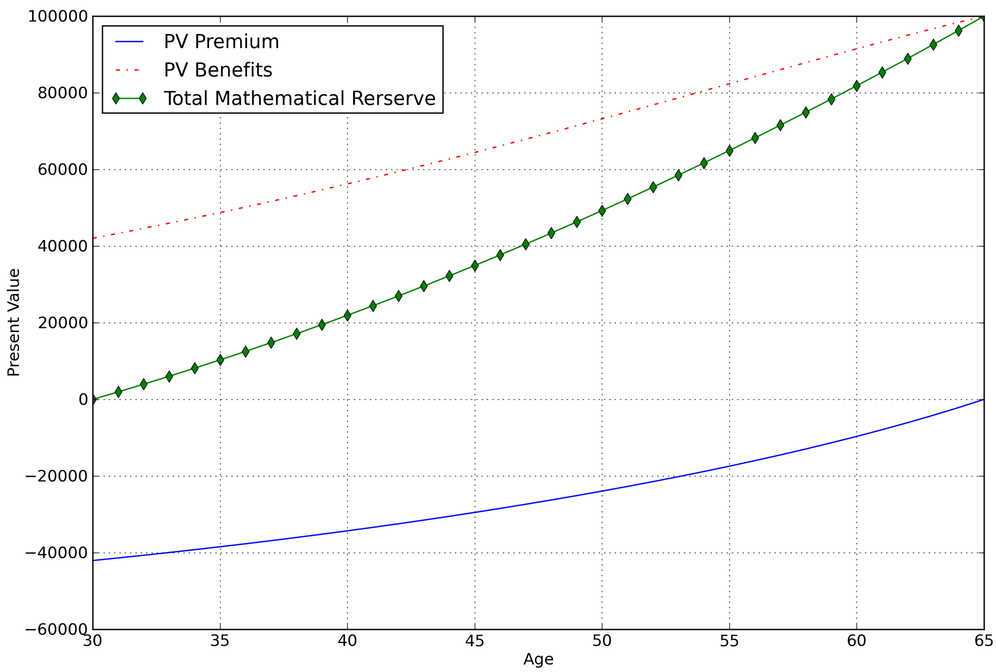

Figure 4.7: Endowment policy against regular premium payment

– Compare for the same person the present value of the premiums for a policy with exemption from payment of premiums and for a policy without this option.

First we calculate the present value of the payouts of the future disability pension. In this case the non-trivial functions which model the policy are: (where we assumed that the disability pension is payable in advance and has the value 1.)

$$a_{\diamond}^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

The boundary conditions are zero in this case, i.e., there is no payment when the age of maturity is reached. Table 4.7 lists the calculated values for this example. Thus one has to pay 4,396.8

USD as single premium for a future disability pension of 10,000 USD. Furthermore, the reserve is 170,790 USD for a disabled man of 35 years with the same disability pension as above.

Table 4.7: Reserves of a disability pension

age	$p_{*\dagger}$	$p_{*\diamond}$	$V_*(x)$	$V_\diamond(x)$
65	0.02289	0.02794	**0.00000**	**0.00000**
64	0.02101	0.02439	0.00000	1.00000
63	0.01929	0.02129	0.02047	1.94299
62	0.01772	0.01860	0.05372	2.83515
61	0.01628	0.01625	0.09427	3.68174
60	0.01495	0.01420	0.13828	4.48719
55	0.00983	0.00732	0.34176	8.01299
50	0.00653	0.00387	0.46175	10.90260
45	0.00439	0.00214	0.50531	13.31967
40	0.00301	0.00127	0.50178	15.35782
35	0.00212	0.00084	0.47493	17.07904
30	0.00155	0.00062	0.43968	18.53012

Next we consider the present values of the premiums. We have to treat the following two cases separately. On the one hand there is the present value of a policy without exemption from payment of premiums

$$a_*^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_\diamond^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand there is the present value of a policy with exemption from payment of premiums ("premium raider")

$$a_*^{\text{Pre}} = \begin{cases} 1, & \text{if} \quad x < 65, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_\diamond^{\text{Pre}} = \begin{cases} 0, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

The difference between these two policies is, that in the first case also with status "disabled" the insured has to pay the premiums. In both cases the boundary condition at the age of 65 is 0. An application of Thiele's difference equations yields the values listed in Table 4.8. We note that the value of the paid-in premiums is smaller in the case of an exemption of premiums. This can also be seen in the plot of these values in Figure 4.8.

Now we are able to calculate the yearly premium for a disability insurance of 10,000 USD with exemption from premiums

$$P = 4396.8/18.09044 = 243.05 \text{ USD per year.}$$

Exercise 4.8.4. 1. Do the calculations of the above example also for a model which includes the possibility of reactivation (cf. Example 4.2.2).

Table 4.8: Present value of the premiums

age	$V(x)$ without premium raider	$V(x)$ with premium raider
65	**0.00000**	**0.00000**
64	1.00000	1.00000
63	1.94299	1.92251
62	2.83515	2.78144
61	3.68174	3.58747
60	4.48719	4.34891
55	8.01299	7.67123
50	10.90260	10.44085
45	13.31967	12.81436
40	15.35782	14.85604
35	17.07904	16.60411
30	18.53012	18.09044

2. Extend the model by incorporating a waiting period of one year.

Next we consider an insurance on two lives. There are several possible states for which the policy could guarantee a pension.

Example 4.8.5 (Pension on two lives). We start with the calculation of a single premium for several types of an insurance on two lives. For this we assume that the two persons have the same mortality as given in Example 4.2.1 and that $x_1 = 30$ and $x_2 = 35$ are fixed. We set the technical interest rate to 3.5 %, and $\omega = 114$ be the maximal possible age of a living person.

There are three possible pensions: (we denote by x_1 the age of the first person)

state	**type**	**formula**
$**$	both persons are alive	$a_{**}^{\text{Pre}}(x) = \alpha_{**} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise,} \end{cases}$
$*\dagger$	second person is dead	$a_{*\dagger}^{\text{Pre}}(x) = \alpha_{*\dagger} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise,} \end{cases}$
$\dagger*$	first person is dead	$a_{\dagger*}^{\text{Pre}}(x) = \alpha_{\dagger*} \begin{cases} 0, & \text{if} \quad x_1 < 65, \\ 1, & \text{otherwise.} \end{cases}$

The definitions of the pensions above are particular, since the pension for the second life (i.e. for $\dagger*$) is paid at the age of 65. Usually this pension would be paid immediately after the death of the first person.

We set $x = (x_1, x_2)$ and suppose that the two insured die independently. In this case the recursion takes the form

$$\begin{aligned} V_{**}(x) &= a_{**}^{\text{Pre}}(x) + p_{x_1}\, p_{x_2}\, v\, V_{**}(x+1) + p_{x_1}\,(1-p_{x_2})\, v\, V_{*\dagger}(x+1) \\ &\quad +(1-p_{x_1})\, p_{x_2}\, v\, V_{\dagger*}(x+1), \end{aligned}$$

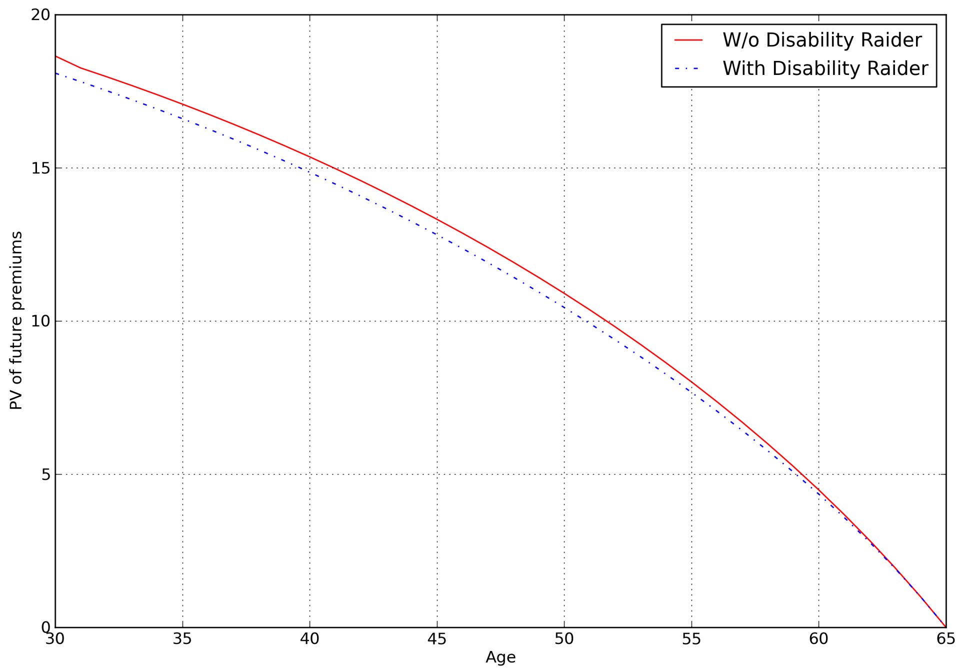

Figure 4.8: Present value of premiums

$$V_{*\dagger}(x) = a_{*\dagger}^{\text{Pre}}(x) + p_{x_1}\, v\, V_{*\dagger}(x+1),$$

$$V_{\dagger *}(x) = a_{\dagger *}^{\text{Pre}}(x) + p_{x_2}\, v\, V_{\dagger *}(x+1).$$

This recursion yields the values listed in Table 4.9.

Exercise 4.8.6. 1. Calculate the present values of the premiums for the insurance on two lives. Note that also in this calculation one has to consider three different cases.

2. Create a model for an orphan's pension. For this one has to consider three persons: farther, mother and child. Define the payout functions for a policy which pays 5,000 USD to the child if one of parent dies and 10,000 USD if both die. Assume that the policy matures if the child is 25 years old.

Table 4.9: Mathematical reserves (res.) for an insurance on two lives

Notations

$V_{*\cdot}$(R1)	res. for the pension of 1th life, independent of 2nd life
$V_{\cdot *}$(R2)	res. for the pension of 2nd life, independent of 1th life
V_{**}(R1)	res. for the pension of 1th life, if $X_t = (**)$,
V_{**}(R2)	res. for the pension of 2th life, if $X_t = (**)$,
V_{**}(R $(**)$)	res. for the pension of joint lives, if $X_t = (**)$,.

Alter 1	Alter 2	$V_{*\cdot}$(R1)	$V_{\cdot *}$(R2)	V_{**}(R1)	V_{**}(R2)	V_{**}(R $(**)$)
115	120	**0.00000**		**0.00000**	**0.00000**	**0.00000**
114	119	1.00000		0.00000	0.00000	1.00000
113	118	1.11505		0.11505	0.00000	1.00000
112	117	1.19991		0.19991	0.00000	1.00000
111	116	1.28640		0.28640	0.00000	1.00000
110	**115**	1.37771	**0.00000**	0.37771	0.00000	1.00000
109	114	1.47450	1.00000	0.45824	0.00000	1.01626
108	113	1.57710	1.11505	0.52973	0.06845	1.04736
90	95	4.64366	3.53796	2.04747	0.94178	2.59618
75	80	9.05696	7.43141	3.39065	1.76510	5.66631
65	70	12.54173	10.77780	3.88770	2.12377	8.65403
55	60	7.78663	6.26733	3.37939	1.86010	4.40724
40	45	4.30964	3.34208	2.13044	1.16288	2.17920
30	35	3.00101	2.30680	1.52353	0.82932	1.47748

5 Difference equations and differential equations

5.1 Introduction

In this chapter we focus on the Markov model in continuous time. The differential equations are the continuous counter part to the difference equations of the discret model.

These differential equations where first proved for simple insurance models by Thiele at the end of the 19th century. We are going to derive these equations for the Markov model. They are useful in two ways. On the one hand they help to deepen our understanding of the model. On the other hand they can be used to calculate the premiums for a policy.

5.2 Thiele's differential equations

In this section we are going to derive Thiele's differential equations for the mathematical reserve. For simplicity we consider in this chapter only reserves without jumps. Later on we will also allow jumps, but then the proofs become more involved.

Theorem 5.2.1 (Thiele's differential equation). *Let $(X_t)_{t\in T}$, a_{ij}, a_i and δ_t be a regular insurance model (Definition 4.5.6). Moreover, $da_g(t)$ be absolute continuous with respect to the Lebesgue measure λ, i.e. $da_g(t) = a_g(t)\,d\lambda$. (Thus the payout function $A_g(t)$ is continuous.) Then, assuming a deterministic interest intensity, the following statements hold:*

1. *$W_g^+(t)$ is continuous for all $g \in S$.*
2. $\frac{\partial}{\partial t} W_j^+(t) = -v(t)\left\{a_j(t) + \sum_{j\neq g\in S} \mu_{jg}(t) a_{jg}(t)\right\}$ $+\mu_j(t) W_j^+(t) - \sum_{j\neq g\in S} \mu_{jg}(t) W_g^+(t)$. *(Thiele's differential equation)*
3. $V_j^+(t) = \frac{1}{v(t)} \left[\begin{array}{c} \int_t^u v(\tau)\overline{p}_{jj}(t,\tau)\{a_j(\tau) + \sum\limits_{j\neq g\in S} \mu_{jg}(\tau)(a_{jg}(\tau) \\ +V_g^+(\tau))\}d\tau + v(u)\overline{p}_{jj}(t,u)V_j^+(u^-) \end{array} \right].$

Proof. The proof of the first statement is left as an exercise to the reader. To prove the second statement we fix $j \in S, t \in \mathbb{R}$ and $\Delta t > 0$. Then Lemma 4.7.2 implies

$$W_j^+(t) \quad = \quad v(t)\left\{a_j(t) + \sum_{j\neq g\in S} \mu_{jg}(t) a_{jg}(t)\right\}\Delta t$$

$$
\begin{aligned}
&+(1-\mu_j(t)\mathit{\Delta} t)W_j^+(t+\mathit{\Delta} t)\\
&+\sum_{j\neq g\in S}\mu_{jg}(t)W_g^+(t+\mathit{\Delta} t)\,\mathit{\Delta} t+o(\mathit{\Delta} t),
\end{aligned}
$$

where we used the following facts

$$
\begin{aligned}
p_{jj}(t,t+\mathit{\Delta} t) &= 1-\mathit{\Delta} t\,\mu_j(t)+o(\mathit{\Delta} t),\\
p_{jk}(t,t+\mathit{\Delta} t) &= \mathit{\Delta} t\,\mu_{jk}(t)+o(\mathit{\Delta} t).
\end{aligned}
$$

The above equation yields

$$
\begin{aligned}
\frac{W_j^+(t+\mathit{\Delta} t)-W_j^+(t)}{\mathit{\Delta} t} &= -v(t)\,\{a_j(t)+\sum_{j\neq g\in S}\mu_{jg}(t)a_{jg}(t)\}\\
&+\ \mu_j(t)\,W_j^+(t+\mathit{\Delta} t)\\
&-\ \sum_{g\neq j}\mu_{jg}(t)W_g^+(t+\mathit{\Delta} t)+\frac{o(\mathit{\Delta} t)}{\mathit{\Delta} t}.
\end{aligned}
$$

Letting $\mathit{\Delta} t\longrightarrow 0$ we get

$$
\begin{aligned}
\frac{\partial}{\partial t}W_j^+(t) &= -v(t)\left\{a_j(t)+\sum_{j\neq g\in S}\mu_{jg}(t)a_{jg}(t)\right\}+\mu_j(t)W_j^+(t)\\
&- \sum_{j\neq g\in S}\mu_{jg}(t)W_g^+(t).
\end{aligned}
$$

For the proof of the third statement we use Thiele's differential equation:

$$
\begin{aligned}
&\exp(-\int_o^t\mu_j(\tau)d\tau)\Bigg(-v(t)\{a_j(t)+\sum_{j\neq g\in S}\mu_{jg}(t)a_{jg}(t)\}\\
&\qquad\qquad -\sum_{j\neq g\in S}\mu_{jg}(t)W_g^+(t)\Bigg)\\
&= \exp(-\int_o^t\mu_j(\tau)d\tau)\left(\frac{\partial}{\partial t}W_j^+(t)-\mu_j(t)W_j^+(t)\right)\\
&= \frac{\partial}{\partial t}\left(\exp(-\int_o^t\mu_j(\tau)d\tau)W_j^+(t)\right).
\end{aligned}
$$

An integration $\int_t^u$ of both sides yields

$$
\exp(-\int_o^t\mu_j(\tau)d\tau)\left(\exp(-\int_t^u\mu_j(\tau)d\tau)W_j^+(u)-W_j^+(t)\right)
$$

$$
\begin{aligned}
= \int_t^u \exp(-\int_o^t \mu_j(\xi) d\xi) \exp(-\int_t^\tau \mu_j(\xi) d\xi) \\
\times \Big[- v(\tau) \Big\{ a_j(\tau) + \sum_{j \neq g \in S} \mu_{jg}(\tau) a_{jg}(\tau) \Big\} \\
- \sum_{j \neq g \in S} \mu_{jg}(\tau) W_g^+(\tau) \Big] d\tau.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
V_j^+(t) &= \frac{1}{v(t)} \Big[\int_t^u v(\tau) \overline{p}_{jj}(t,\tau) \{ a_j(\tau) + \sum_{j \neq g \in S} \mu_{jg}(\tau) \\
&\quad \times (a_{jg}(\tau) + V_g^+(\tau)) \} d\tau + v(u) \overline{p}_{jj}(t,u) V_j^+(u^-) \Big],
\end{aligned}
$$

where we used that $\overline{p}_{jj}(t,\tau) = \exp(-\int_t^\tau \mu_j(\xi) d\xi)$.

Remark 5.2.2. We derived the following integral equation from Thiele's differential equation:

$$
V_j^+(t) = \frac{1}{v(t)} \left[\begin{array}{c} \int_t^u v(\tau) \overline{p}_{jj}(t,\tau) \{ \underbrace{a_j(\tau)}_{I} + \sum_{j \neq g \in S} \overbrace{\underbrace{\mu_{jg}(\tau)(a_{jg}(\tau)}_{IIa} + V_g^+(\tau))}^{IIb} \} d\tau \\ + \underbrace{v(u) \overline{p}_{jj}(t,u) V_j^+(u^-)}_{III} \end{array} \right].
$$

This formula shows the structure of the reserve. The components of the reserve are:

I)		reserve for payments in state j (pensions and premiums),
II)		reserves for state transitions composed of
	IIa)	transition cost (e.g. death benefit) and
	IIb)	necessary reserves in the new state,
III)		reserve, for the case that the insured is still in j after $[t, u]$.

5.3 Examples - Thiele's differential equation

In this section we look at examples related to those in the discrete setting. In the first example the differential equations have an explicit solution.

Example 5.3.1 (Term life insurance). We consider a term life insurance with death benefit b, which is financed by a premium of size c. In this situation the differential equations take the following form:

$$
\begin{aligned}
\frac{\partial}{\partial t} W_*(t) &= v^t (c - \mu_{x+t}\, b) + \mu_{x+t} W_*(t) - \mu_{x+t} W_\dagger(t), \\
\frac{\partial}{\partial t} W_\dagger(t) &= 0
\end{aligned}
$$

with the boundary condition $W_*(s-x) = W_\dagger(s-x) = 0$, where s denotes the age of maturity of the policy.

Next, we are going to calculate the mathematical reserve. The above equations obviously imply $W_\dagger(t) \equiv 0$. Thus we only have to calculate $W_*(t)$. The homogeneous part of the equation satisfies

$$\frac{dW_*(t)}{W_*(t)} = \mu_{x+t} dt$$

and therefore

$$L_h(t) = A \times \exp\Big(\int_0^t \mu_{x+\tau}\, d\tau\Big).$$

By variation of constants we get

$$\begin{aligned}
L_p(t) &= A(t) \times L_h(t), \\
\frac{d}{dt} L_p &= A' \times L + A \times L' \\
&= A' \times L + A \times L \\
&= A' \times L + L_p, \\
A' \times L &= v^t (c - \mu_{x+t}\, b), \\
A' &= v^t (c - \mu_{x+t}\, b) \exp\Big(-\int \mu_{x+\tau}\, d\tau\Big) \\
&= v^t (c - \mu_{x+t} b)\, {}_t p_x, \\
A(t) &= \int_0^t v^\tau (c - \mu_{x+\tau} b)\, {}_\tau p_x d\tau.
\end{aligned}$$

Finally, the boundary condition $W_*(s-x) = 0$ yields

$$\begin{aligned}
W_*(s-x) &= A(s-x) \times L(s-x) \\
&= \left[\int_0^{s-x} v^\tau (c - \mu_{x+\tau} b)\, {}_\tau p_x d\tau\right] \times \left[\exp\Big(\int_0^{s-x} \mu_{x+\tau} d\tau\Big)\right], \\
c &= b\, \frac{\int_0^{s-x} v^\tau\, {}_\tau p_x\, \mu_{x+\tau} d\tau}{\int_0^{s-x} v^\tau\, {}_\tau p_x d\tau}.
\end{aligned}$$

Example 5.3.2 (Endowment policy). We consider the endowment policy defined in Example 4.8.1. Thus it contains a death benefit of 200,000 USD and an endowment of 100,000 USD. We consider a 30 year old man and 65 as the age of maturity of the policy.

– How much is a single premium for this insurance if the technical interest rate is 3.5%?

– How do these results compare to the values in the corresponding example in discrete time?

We use the mortality rates given by (2.14). For the single premium the following payout function defines the policy:

$$a_{*\dagger}(x) = \begin{cases} 200000, & \text{if} \quad x < 65, \\ 0, & \text{otherwise.} \end{cases}$$

Now Thiele's differential equations are

$$\begin{aligned} \frac{\partial}{\partial t} W_*(t) &= v^t (c - \mu_{x+t}\, a_{*\dagger}(x+t)) + \mu_{x+t}\, W_*(t) - \mu_{x+t}\, W_\dagger(t), \\ \frac{\partial}{\partial t} W_\dagger(t) &= 0, \end{aligned}$$

with the boundary conditions $W_*(s-x) = 100000 \times v(s)$ and $W_\dagger(s-x) = 0$. Then Theorem 5.2.1 yields the results listed in Table 5.1.

Table 5.1: Discretization error for an endowment policy

age	$\mu_{*\dagger}(x)$	**reserve discrete model**	**reserve cont. model**	**diff. in %**
65	0.01988	**100000**	**100000**	
64	0.01836	98392	98512	0.12
63	0.01696	96730	96955	0.23
62	0.01566	95022	95341	0.34
61	0.01446	93275	93678	0.43
60	0.01336	91498	91975	0.52
55	0.00897	82360	83092	0.89
50	0.00602	73210	74059	1.16
45	0.00404	64426	65308	1.37
40	0.00271	56236	57096	1.53
35	0.00181	48759	49566	1.65
30	0.00121	42044	42782	1.75

Note that the difference of the reserves in the discrete and continuous model have always the same sign. This is caused by the fact, that people only die at the end of the year in the discrete model. Therefore the required single premium is smaller than in the continuous model.

Exercise 5.3.3. – Calculate yearly premiums for the previous example.

- What happens to the discretization error, if we suppose that people die in the discrete model only at the middle of the year?
- What happens, if we suppose that the interest rates drops linearly from 6% at the age of 30 to 3% at the age of 65?

Exercise 5.3.4. Calculate with the continuous model the premiums for the policy defined in Example 4.8.3.

Example 5.3.5 (Pensions on two lives). 1. Derive Thiele's differential equation for pensions on two lives.

2. Calculate the premiums for the pensions based on the assumption that the husband and his wife die independently.
3. What happens if the the mortality rate for the state $(**)$ (or $(*\dagger) \cup (\dagger *)$) decreases (or increases) by 15 % for each life? (Empirical studies show that the mortality rate of widows and widowers is increased in comparison to the rest of the population.)

For this example we only derive Thiele's differential equations and present the results in a figure. The calculations are done in the setting of Example 4.8.5. (Δt denotes the difference in age of the husband and his wife.)

$$
\begin{aligned}
\frac{\partial W_{**}^{+}(t)}{\partial t} &= -v^t\, a_{**}(t) + (\mu_{x+t}^{husband} + \mu_{x+t+\Delta t}^{wife}) W_{**}^{+}(t) \\
&\qquad - \mu_{x+t}^{husband} W_{\dagger *}^{+}(t) - \mu_{x+t+\Delta t}^{wife} W_{*\dagger}^{+}(t), \\
\frac{\partial W_{*\dagger}^{+}(t)}{\partial t} &= -v^t\, a_{*\dagger}(t) + \mu_{x+t}^{wife} \left(W_{*\dagger}^{+}(t) - W_{\dagger\dagger}^{+}(t) \right), \\
\frac{\partial W_{\dagger *}^{+}(t)}{\partial t} &= -v^t\, a_{\dagger *}(t) + \mu_{x+t+\Delta t}^{wife} \left(W_{\dagger *}^{+}(t) - W_{\dagger\dagger}^{+}(t) \right), \\
\frac{\partial W_{\dagger\dagger}^{+}(t)}{\partial t} &= 0.
\end{aligned}
$$

Figure 5.1 shows the relation of the present values of the benefits for a change in mortality by $\pm 15\%$. The results are what we expected: a pension on the joint lives becomes more expensive, whereas a pension on the 2nd life becomes cheaper.

Exercise 5.3.6. Complete the previous example.

Exercise 5.3.7. 1. Calculate the present value of the premiums for the insurance on two lives. (Also in the continuous setting one has to treat three cases separately.)

2. Create a model for an orphan's pension. In the model one has to consider three persons: farther, mother and child. Define the payout functions for a policy which pays 5,000 USD to the child if one of parent dies and 10,000 USD if both die. Assume that the policy matures if the child is 25 years old.

5.4 Differential equations for moments of higher order

Thiele's differential equations characterize the mathematical reserve of an insurance policy. In this section we look at the moments of the mathematical reserve. This will enable us for example to calculate the variance of the reserve, which is a measure for its variation. Furthermore, it can be used to analyze the risk structure of an insurance policy.

We start with the difference equations corresponding to the discrete model. The payout functions in the discrete Markov model have the following form:

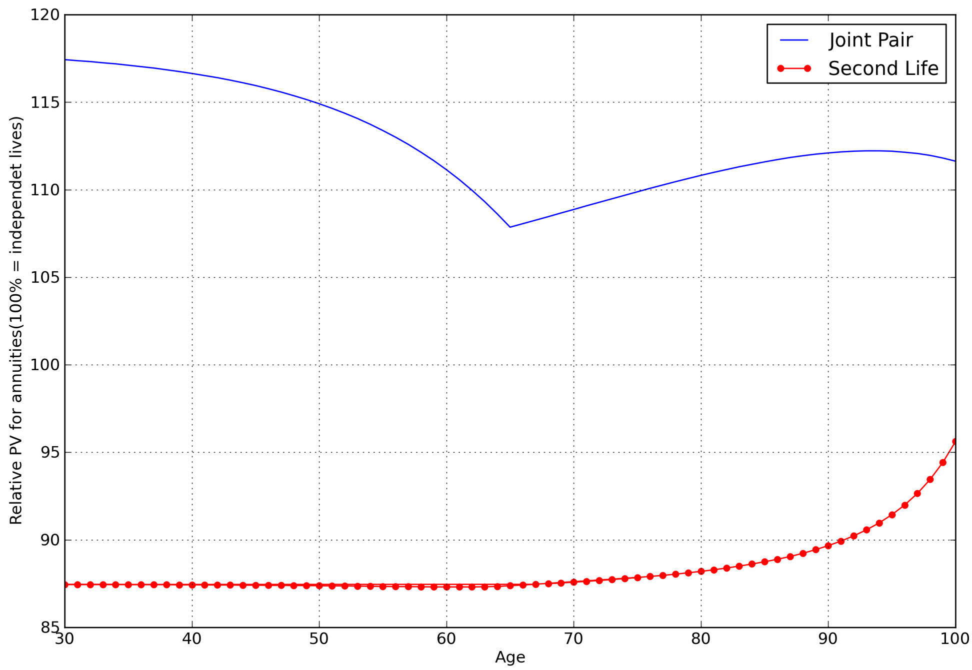

Figure 5.1: Ratio between the present values of benefits for annuities on two lives (100% = independent mortality probabilities).

$$\Delta B_t = \sum_{j \in J} I_j(t) a_j^{Pre}(t) + \sum_{j,k \in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t),$$

where v_t is the yearly discount from $t+1$ to t, with $v_t = \sum_{j \in J} I_j(t) v_j(t)$.

The prospective reserves are

$$\begin{aligned} V_t^+ &= \sum_{\xi=t}^{\infty} \Big(\prod_{k=t}^{k<\xi} v_k\Big)\, \Delta B_\xi \\ &= \sum_{\xi=t}^{\infty} \Big(\prod_{k=t}^{k<\xi} v_k\Big) \left\{ \sum_{j \in J} I_j(\xi) a_j^{Pre}(\xi) + \sum_{j,l \in J} \Delta N_{jl}(\xi)\, v_\xi\, a_{jl}^{Post}(\xi) \right\}. \end{aligned}$$

Now our aim is to calculate the expectation of the p-th power of the mathematical reserve $((V_t^+)^p)$ conditioned on $\mathcal{F}_t$. The linearity of the integral yields the difference equation

$$V_t^+ = v_t \sum_{j \in J} I_j(t+1) V_{t+1}^+ + \sum_{j \in J} I_j(t) a_j^{Pre}(t) + \sum_{j,k \in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t).$$

This formula indicates that the future reserve is composed of the payments in the period $]t, t+1]$, the payments at time $\{t\}$ and the payments in the period $]t+1, \infty[$. To keep the calculations simple we assume that there is no payment at time $\{t\}$. Furthermore we will use the notation

$$L_t = \sum_{j,k \in J} \Delta N_{jk}(t)\, v_t\, a_{jk}^{Post}(t).$$

Therefore we can simplify the recursion to

$$V_t^+ = v_t \sum_{j \in J} I_j(t+1) V_{t+1}^+ + L_t = \sum_{j \in J} I_j(t+1) \left(v_t\, V_{t+1}^+ + L_t \right).$$

Now the p-th moment is given by

$$\begin{aligned}
(V_t^+)^p &= \left(\sum_{j \in J} I_j(t+1)(v_t\, V_{t+1}^+ + L_t) \right)^p \\
&= \sum_{k=0}^{p} \binom{p}{k} \left(\sum_{j \in J} I_j(t+1) v_t\, V_{t+1}^+ \right)^k L_t^{p-k} \\
&= \sum_{k=0}^{p} \binom{p}{k} \sum_{j \in J} I_j(t+1)(v_t\, V_{t+1}^+)^k L_t^{p-k},
\end{aligned}$$

where we used the fact that $I_\alpha(t+1) I_\beta(t+1) = \delta_{\alpha\beta} I_\alpha(t+1)$. Next, we can use $P[A \cap B|C] = P[A|B \cap C] \times P[B|C]$ to simplify the recursion for the expectation. We get

$$\begin{aligned}
&E\left[(V_t^+)^p \mid X_t = i \right] \\
&\quad = E\left[\sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j \in J} I_j(t+1)(V_{t+1}^+)^k L_t^{p-k} \mid X_t = i \right] \\
&\quad = \sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j \in J} E\left[I_j(t+1)(V_{t+1}^+)^k L_t^{p-k} \mid X_t = i \right] \\
&\quad = \sum_{k=0}^{p} \binom{p}{k} (v_t^i)^k \sum_{j \in J} E\left[I_j(t+1)(V_{t+1}^+)^k \left(v_t\, a_{ij}^{Post}(t) \right)^{p-k} \mid X_t = i \right] \\
&\quad = (v_t^i)^p \sum_{j \in J} p_{ij}(t,t+1) \sum_{k=0}^{p} \binom{p}{k} \left(a_{ij}^{Post}(t) \right)^{p-k} E\left[(V_{t+1}^+)^k \mid X_{t+1} = j \right].
\end{aligned}$$

This is the difference equation for the higher order moments of the reserve, if there are no payments in advance. Note that the integration of the payments in advance $a_j^{Pre}(t)$ is not complicated. Nevertheless, to keep the presentation clear we continue without them. We summarize our findings in the following theorem.

Theorem 5.4.1 (Differential equation for moments of higher order). *Under the above assumptions the higher order moments of the reserve satisfy the recursion*

$$\begin{aligned}
&E\left[(V_t^+)^p \mid X_t = i \right] \\
&\quad = (v_t^i)^p \sum_{j \in J} p_{ij}(t,t+1) \sum_{k=0}^{p} \binom{p}{k} \left(a_{ij}^{Post}(t) \right)^{p-k} E\left[(V_{t+1}^+)^k \mid X_{t+1} = j \right].
\end{aligned}$$

Exercise 5.4.2. Derive the above formula for a model which includes payments in advance.

After treating the discrete case we are now going to derive the analog statements for the continuous setting. The proofs will become more involved, since the model is more general. Before stating the theorem we recall some definitions:

$$\begin{aligned} dB &= \sum_{j \in S} I_j(t) dB_j + \sum_{j \neq k} dB_{jk}, \\ dv_t &= -v_t \times \delta_t \, dt, \\ \delta_t &= \sum_{j \in S} I_j(t) \delta_j(t). \end{aligned}$$

The p-th moment of the prospective reserve is defined by

$$\begin{aligned} V_j^{(p)}(t) &:= E\left[(V_t^+)^p \mid X_t = j\right] \\ &= E\left[\left(\frac{1}{v_t}\int_t^\infty v \, dB\right)^p \mid X_t = j\right], \end{aligned}$$

where we implicitly assumed that $V_t^+ \in L^p(\Omega, \mathcal{A}, P)$ and that the functions δ, a_i and a_{jk}, μ_{jk} are piecewise continuous. Then the following theorem holds.

Theorem 5.4.3 (Differential equations for moments of higher order). *Under the above assumptions the functions $V_j^{(p)}(t)$ satisfy the differential equation*

$$\begin{aligned} \frac{\partial}{\partial t} V_j^{(p)}(t) &= \Big(p\,\delta_j(t) + \sum_{S \ni k \neq j} \mu_{jk}(t)\Big) V_j^{(p)}(t) - p\, a_j(t) V_j^{(p-1)}(t) \\ &- \sum_{j \neq k \in S} \mu_{jk}(t) \sum_{k=0}^{p} \binom{p}{k} (a_{jk}(t))^{p-k} \, V_j^{(k)}(t) \end{aligned}$$

for all $t \in]0,n[\setminus \mathcal{D}$ with the boundary condition

$$V_j^{(p)}(t^-) = \sum_{k=0}^{p} \binom{p}{k} (\Delta a_j(t))^{p-k} \, V_j^{(k)}(t)$$

for all $t \in \mathcal{D}$. Here $\mathcal{D}$ is the set of discontinuities of the payout function B.

Remark 5.4.4.

- The differential equation given above also holds at points where the functions $V_j^{(p)}(t)$ are not differentiable. In these points one gets a valid interpretation by considering the differentials which are given by a formal multiplication with the factor dt.
- The idea of the proof is to represent a suitable martingale in two different ways. These representations will then be used to find a stochastic differential equation for the martingale. Then, since the drift term of the differential equation is zero for a martingale, one obtains an ordinary differential equation.

Proof. It turns out to be more convenient to show the differential equation for $W_j^{(p)}(t) = v_t^p\, V_j^{(p)}(t)$. We have

$$\begin{aligned} dW_j^{(p)}(t) &= d(v_t^p)\, V_j^{(p)}(t) + v_t^p\, dV_j^{(p)}(t) && (5.1)\\ &= -pv_t^p \sum_{j \in S} I_j(t)\delta_j(t)dt\, V_j^{(p)}(t) + v_t^p\, dV_j^{(p)}(t). && (5.2)\end{aligned}$$

Now we define the martingale

$$\begin{aligned} M^{(p)}(t) &:= E\left[\left(\int_0^\infty v\, dB\right)^p \mid \mathcal{F}_t\right] \\ &= E\left[\left\{\left(\int_0^t + \int_t^\infty\right) v\, dB\right\}^p \mid \mathcal{F}_t\right] \end{aligned}$$

and the function

$$U_t = \int_0^t v\, dB.$$

The Markov property implies

$$E\left[\left\{\int_t^\infty v\, dB\right\}^{p-k} \mid \mathcal{F}_t\right] = \sum_{j \in S} I_j(t) W_j^{(p-k)}(t),$$

and using the Binomial Theorem we get

$$M^{(p)}(t) = \sum_{k=0}^{p} \binom{p}{k} \sum_{j \in S} U_t^k I_j(t) W_j^{(p-k)}(t).$$

By choosing a right continuous modification of $M^{(p)}$ we can ensure that U and $I_j(t)$ are right continuous.

Now we want to simplify the differential form

$$dM^{(p)}(t) = \sum_{k=0}^{p} \binom{p}{k} \sum_{j \in S} d\left(U_t^k I_j(t) W_j^{(p-k)}(t)\right).$$

Recall that for a function of bounded variation A we denote by A^{cont} the continuous part and by A^{atom} the discontinuous part. An application of Itô's formula yields

$$\begin{aligned} & d\left(U_t^{(k)} I_j(t) W_j^{(p-k)}(t)\right) \\ &\quad = kU_t^{(k-1)} dU_t^{cont} I_j(t) W_j^{(p-k)}(t) \\ &\quad + U_t^k dI_j^{cont}(t) W_j^{(p-k)}(t) + U_t^k I_j(t) dW_j^{(p-k),cont}(t) \\ &\quad + \left\{U_t^k I_j(t) W_j^{(p-k)}(t) - U_{t^-}^k I_j(t^-) W_j^{(p-k)}(t^-)\right\}. && (5.3)\end{aligned}$$

To simplify this formula we use for the first line the identities

$$\begin{aligned} dU_t^{cont} &= v_t \sum_{l\in S} I_l(t) a_l(t), \\ I_\alpha(t) I_\beta(t) &= \delta_{\alpha\beta} I_\alpha(t). \end{aligned}$$

For the second line of (5.3) note that the continuous part of $I_\alpha(t)$ vanishes. Finally we have to deal with the jump part in the third line. The jumps have two possible origins. On the one hand they might be caused by a transition:

$$\sum_{j\neq l\in S} \left(\{U_{t^-} + v_t a_{jl}(t)\}^k \, W_l^{(p-k)}(t) - U_{t^-}^k \, W_l^{(p-k)}(t^-) \right) dN_{jl}(t).$$

On the other hand they might be due to a jump in a pension:

$$\sum_{l\in S} I_l(t) \left(\{U_{t^-} + v_t \Delta a_l(t)\}^k \, W_l^{(p-k)}(t) - U_{t^-}^k \, W_l^{(p-k)}(t^-) \right).$$

Moreover, we know that jumps can only occur on the set $\mathcal{D}$ and on this set one can replace $I_l(t)$ by $I_l(t^-)$, since these coincide with probability 1. We also know that a simultaneous jump of both components occurs with probability 0. Finally, $W_l^{(p-k)}(t)$ is continuous and does not induce any jumps, since $\int_t^n v \, dB$ is almost surely continuous on $t \notin \mathcal{D}$. Therefore we get

$$\begin{aligned} & d\left(U_t^{(k)} I_j(t) W_j^{(p-k)}(t) \right) \\ & \quad = \sum_{l\in S} I_l(t) \left(k U_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t) + U_t^k dW_j^{(p-k),cont}(t) \right) \\ & \quad + \sum_{j\neq l\in S} \left(\{U_{t^-} + v_t a_{jl}(t)\}^k \, W_l^{(p-k)}(t) - U_{t^-}^k \, W_l^{(p-k)}(t^-) \right) dN_{jl}(t) \\ & \quad + \sum_{l\in S} I_l(t^-) \left(\{U_{t^-} + v_t \Delta a_l(t)\}^k \, W_l^{(p-k)}(t) - U_{t^-}^k \, W_l^{(p-k)}(t^-) \right). \end{aligned}$$

Applying the fact $X_{t^-} dt = X_t \, dt$ and the previous formula to (5.3) we derive

$$\begin{aligned} & dM^{(p)}(t) - \sum_{j\neq l\in S} \sum_{k=0}^{p} \binom{p}{k} \Big(\{U_{t^-} + v_t a_{jl}(t)\}^k \, W_l^{(p-k)}(t) \\ & \qquad\qquad - U_{t^-}^k \, W_l^{(p-k)}(t^-) \Big) dM_{jl}(t) \\ & = \sum_{j\in S} I_j(t) \sum_{k=0}^{p} \binom{p}{k} \Bigg[k U_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t) dt + U_t^k dW_j^{(p-k),cont}(t) \\ & + \sum_{j\neq l\in S} \left(\{U_t + v_t a_{jl}(t)\}^k \, W_l^{(p-k)}(t) - U_t^k W_j^{(p-k)}(t) \right) \mu_{jl}(t) dt \Bigg] \\ & + \sum_{j\in S} I_l(t^-) \sum_{k=0}^{p} \binom{p}{k} \left(\{U_{t^-} + v_t \Delta a_j(t)\}^k \, W_j^{(p-k)}(t) - U_{t^-}^k \, W_j^{(p-k)}(t^-) \right), \end{aligned} \tag{5.4}$$

where we used the identity

$$dM_{ij}(t) \quad = \quad dN_{ij}(t) - I_i(t)\,\mu_{ij}(t).$$

Note that

$$dW_j^{(p-k),cont}(t) \quad = \quad -(p-k)v_t^{(p-k)}\delta_j(t)\,dt\,V_j^{(p-k)}(t) + v_t^{(p-k)}dV_j^{(p-k),cont}(t). \tag{5.5}$$

The left hand side of (5.4) is the differential of a sum of martingales. Thus also the right hand side is the differential of a martingale. Now this has to be constant, since it is previsible and of bounded variation. Therefore the increments of the continuous and the discrete part have to be equal to zero. But this is only possible if

$$\begin{aligned} 0 \quad = \quad & \sum_{k=1}^{p} \binom{p}{k} kU_t^{(k-1)} v_t a_j(t) W_j^{(p-k)}(t)dt \\ & + \sum_{k=0}^{p} \binom{p}{k} U_t^k W_l^{(p-k)}(t) \\ & + \sum_{k=0}^{p} \binom{p}{k} \sum_{j \neq l \in S} \sum_{r=0}^{k} \binom{k}{r} U_t^{(r)} \left\{v_t a_{jl}(t)\right\}^{k-r} W_l^{(p-k)}(t)\mu_{jl}(t)dt \\ & - \sum_{k=0}^{p} \binom{p}{k} U_t^k W_j^{(p-k)}(t) \left(\sum_{j \neq l \in S} \mu_{jl}(t) \right) dt \end{aligned} \tag{5.6}$$

holds for all $j \in S$ and all $t \in]0, n[\setminus \mathcal{D}$. For $x \in \mathcal{D}$ we get

$$0 = \sum_{k=0}^{p} \binom{p}{k} \left(\sum_{r=0}^{k} \binom{k}{r} U_{t^-}^{(r)} \left\{v_t \Delta a_l(t)\right\}^{k-r} W_j^{(p-k)}(t) - U_{t^-}^{(k)} W_j^{(p-k)}(t^-) \right).$$

Using the identity

$$\binom{p}{k} k = \binom{p}{k-1} (p-(k-1))$$

we can transform the first line of Equation (5.6) into

$$\begin{aligned} & \sum_{k=1}^{p} \binom{p}{k-1} (p-(k-1)) U_t^{(k-1)} v_t a_j(t) W_j^{(p-1-(k-1))}(t)dt \\ & \qquad = \sum_{k=0}^{p} \binom{p}{k} (p-k) U_t^{(k)} v_t a_j(t) W_j^{(p-1-k)}(t)dt, \end{aligned}$$

where we set $W_j^{(-1)} \equiv 0$. Hence the third line of (5.6) becomes

$$\sum_{k=0}^{p} U_t^{(r)} \sum_{r=0}^{k} \binom{p}{k} \binom{k}{r} \sum_{j \neq l \in S} \left\{v_t a_{jl}(t)\right\}^{k-r} W_l^{(p-k)}(t)\mu_{jl}(t)dt.$$

This can be transformed by the identity

$$\binom{p}{k}\binom{k}{r} = \binom{p}{r}\binom{p-r}{k-r}$$

into the representation

$$\sum_{r=0}^{p}\binom{p}{r}U_t^{(r)}\sum_{k=r}^{p}\binom{p-r}{k-r}\sum_{j\neq l\in S}\{v_t a_{jl}(t)\}^r\, W_l^{(p-k-r)}(t)\mu_{jl}(t)dt,$$

i.e.,

$$\sum_{k=0}^{p}\binom{p}{k}U_t^{(k)}\sum_{r=0}^{p-k}\binom{p-k}{r}\sum_{j\neq l\in S}\{v_t a_{jl}(t)\}^r\, W_l^{(p-k-r)}(t)\mu_{jl}(t)dt.$$

If we now gather the powers of U_t we get

$$0 = \sum_{k=0}^{p}\binom{p}{k}U_t^{(k)}dQ_j^{(p-k)}(t),$$

where

$$\begin{aligned}
dQ_j^{(q)}(t) &= q v_t a_j(t) W_j^{(q-1)}(t)dt + dW_j^{(q),cont}(t)\\
&\quad + \sum_{k=0}^{q}\binom{q}{k}\sum_{j\neq l\in S}\{v_t a_{jl}(t)\}^k\, W_l^{(q-k)}(t)\mu_{jl}(t)dt\\
&\quad - W_j^{(q)}(t)\sum_{j\neq l\in S}\mu_{jl}(t)\,dt.
\end{aligned}$$

This equation implies $dQ_j^{(0)}(t) \equiv 0$ and thus $dQ_j^{(q)}(t) \equiv 0$ by induction. Finally, the formula above and Equation (5.1) and (5.5) imply the result.

5.5 The distribution of the mathematical reserve

The distribution function can be used to answer questions which only depend on the tail of the distribution. Thus it is important for the estimation of extremal risks.

This section has the same structure as the previous section. In the beginning, we solve the problem for the discrete time set. Afterward we treat the continuous time model.

Recall that the cash flows in the discrete Markov model are given by

$$\Delta B_t = \sum_{j\in J} I_j(t)a_j^{Pre}(t) + \sum_{j,k\in J}\Delta N_{jk}(t)\,v_t\,a_{jk}^{Post}(t).$$

We want to calculate the distribution function of the discounted future cash flows

$$P_i(t,u) = P\left[\sum_{j=t}^{\infty}(\prod_{k=t}^{k<j} v_k)\, \Delta B_j < u | X_t = i\right].$$

The following identities hold:

$$\begin{aligned}
P_i(t,u) &= P\left[\sum_{j=t}^{\infty} D_{t,j}\, \Delta B_j < u | X_t = i\right] \\
&= \sum_{l\in J} p_{il}(t)\, P\left[\sum_{j=t}^{\infty} D_{t,j}\, \Delta B_j < u | X_t = i, X_{t+1} = l\right] \\
&= \sum_{l\in J} p_{il}(t) P\Bigg[v_{i,t} \sum_{j=t+1}^{\infty} D_{t,j}\, \Delta B_j < u - a_i^{Pre}(t) \\
&\qquad\qquad - v_{i,t}\, a_{il}^{Post}(t) | X_{t+1} = l\Bigg] \\
&= \sum_{l\in J} p_{il}(t)\, P_l(t+1, v_{i,t}^{-1}(u - a_i^{Pre}(t)) - a_{il}^{Post}(t)), \text{ where} \\
D_{t,j} &= \prod_{k=t}^{k<j} v_k.
\end{aligned}$$

These relations are summarized in the following theorem.

Theorem 5.5.1 (Distribution of the reserves). *The distribution function of the reserves satisfy the recursion*

$$P_i(t,u) = \sum_{j\in J} p_{ij}(t)\, P_j(t+1, v_{i,t}^{-1}(u - a_i^{Pre}(t)) - a_{ij}^{Post}(t)).$$

Besides the recursion formula also boundary conditions are required. These are, in contrast to the previous problems, now given in form of distributions rather than fixed values. For an insurance whose mathematical reserve is equal to zero at maturity the boundary condition is for example given by: (Here ω denotes the maximal age at which insured persons are alive.)

$$P_i(\omega+1, u) = \begin{cases} 0, & \text{if } u \le 0, \\ 1, & \text{if } u > 0. \end{cases}$$

The distribution function of a pension in the discrete model is shown in Figure 5.2. The jumps, which are caused by the discrete model, are clearly visible.

In the previous sections we have seen that one can use differential equations to find the moments of the reserves. For cash flows of sufficient regularity one could also prove that the moments are differentiable.

Analogous one could try to find differential equations for the distribution functions. But the following example shows, that for distribution functions this is not an easy task.

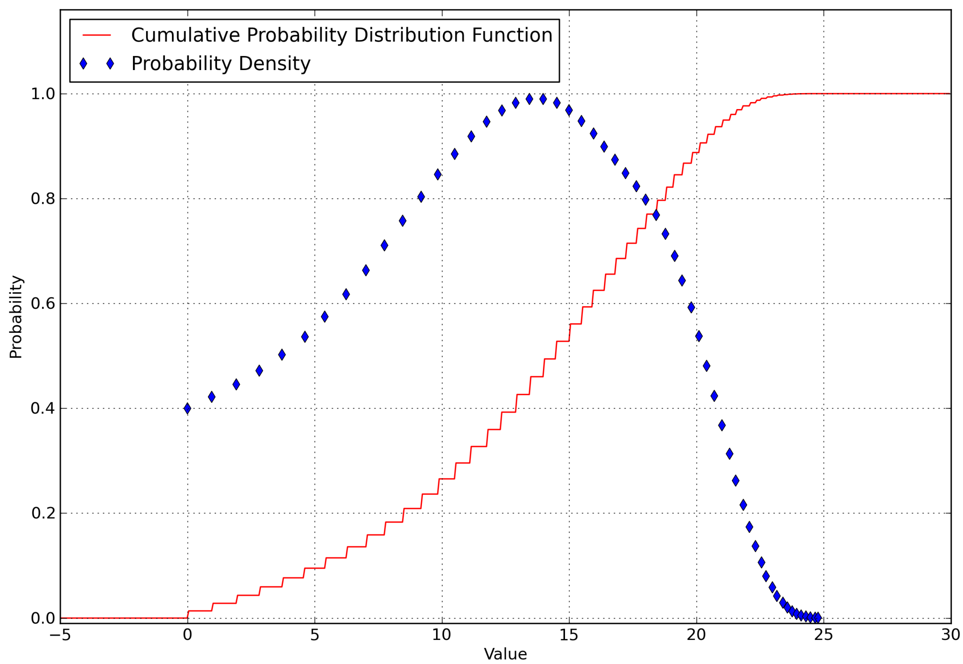

Figure 5.2: Probability distribution function for the present value of an immediate payout annuity (x=65)

Example 5.5.2. This example will illustrate that the distribution function can be discontinuous even for relatively simple insurance policies. We consider an endowment policy, with a death benefit of 100,000 USD and an endowment of 200,000 USD. Now we want to calculate the reserve for an insured of 30 years of age. Here T_x will denote the future life span. The following equations hold:

$$V_{30} = \begin{cases} 100000 \times v^{T_x}, & \text{if} \quad T_x < 35, \\ 200000 \times v^{65-30}, & \text{if} \quad T_x \geq 35. \end{cases}$$

Now assume a technical interest rate of 1.5 %. Then the following equation holds for $0 < \alpha \leq 100000$:

$$\begin{aligned} P[V_{30} < \alpha] &= P[100000 v^{T_x} < \alpha,\, T_x < 35] \\ &= P[35 > T_X > \log(\alpha/100000)/\log(v)] \\ &= {}_{\gamma}p_x - {}_{35}p_x, \text{where} \\ \gamma &= \log(\alpha/100000)/\log(v). \end{aligned}$$

This calculation shows that the distribution function has a jump of size ${}_{35}p_x$ at $200000 v^{35}$, and thus it is discontinuous.

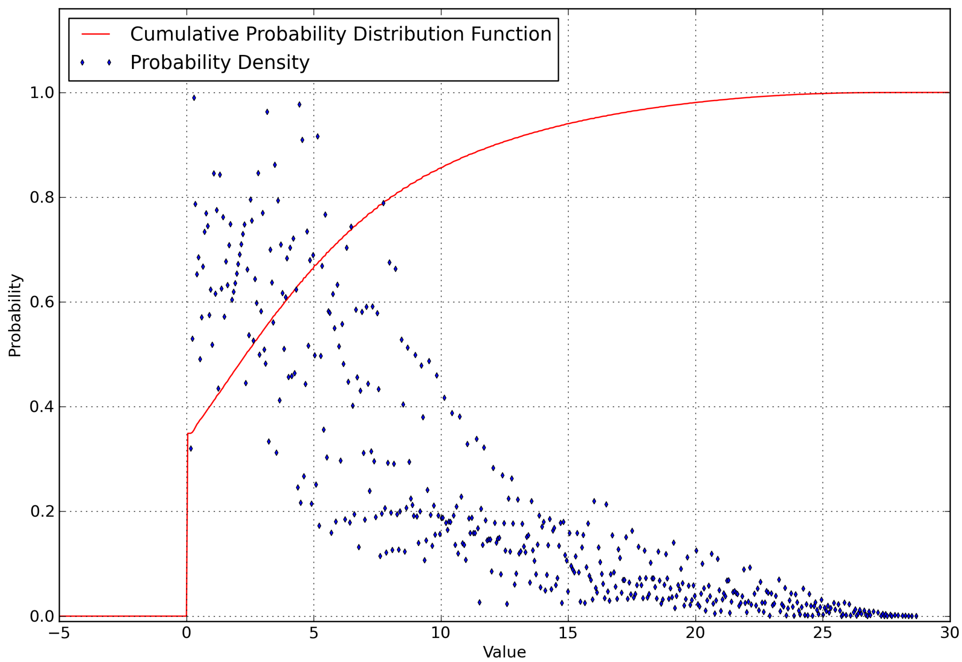

Figure 5.3: Probability distribution function of the present value of a deferred widows pension (x=65)

The next theorem shows that the distribution functions satisfy an integral equation. Note that the recursion in discrete time provides an approximation to this integral equation.

Theorem 5.5.3. *The conditional distribution functions of the reserves*

$$P_j(t,u) = P\left[\int_t^\infty \exp\left(-\int_t^\xi \delta_\tau d\tau\right) dB(\xi) \leq u | X_t = j\right]$$

satisfy the integral equation

$$\begin{aligned} P_j(t,u) = &\sum_{k\neq j} \int_t^\infty \exp\left(-\int_t^\xi \sum_{l\neq j} \mu_{jl}(\tau) d\tau\right) \mu_{jk}(\xi) \\ &\times P_k\left(\xi, \exp(\delta_j(\xi-t))u - \int_t^\xi \exp(\delta_j(\xi-\tau)) dB_j(\tau) - a_{jk}(s)\right) d\xi \\ &+ \exp\left(-\int_t^\infty \sum_{l\neq j} \mu_{jl}(\tau) d\tau\right) \chi_{\left[\int_t^n \exp(-\delta_j(\tau-t)) dB_j(\tau) \leq u\right]}. \end{aligned} \tag{5.7}$$

Proof. The proof is analogous to the discrete setting. One considers

$$A = \left\{\int_t^\infty \exp\left(-\int_t^\xi \delta\right) dB(\xi) \leq u\right\}$$

and treats, as in the discrete setting, the various cases separately. We leave the proof to the reader and refer to [HN96].

Exercise 5.5.4. Complete the proof of the previous theorem. [HN96]

After deriving the integral equations for the distribution function, we will modify these slightly. The equations are still hard to handle, since the right hand side depends on t. To overcome this problem we define

$$Q_j(t,u) := P_j\left(t, \exp(\delta_j t)\left(u - \int_0^t \exp(-\delta_j\,\tau)dB_j(\tau)\right)\right).$$

The mapping from P to Q can be inverted by

$$P_j(t,u) = Q_j\left(t, \exp(-\delta_j t)u + \int_0^t \exp(-\delta_j\,\tau)dB_j(\tau)\right).$$

Using Q_j one can easily derive the equation

$$\begin{aligned}
\exp\left(-\int_0^t \mu_j\right) Q_j\,(t,u) &= \int_t^n \exp\left(-\int_0^s \mu_j\right) \sum_{k\neq j} \mu_{jk}(s) \\
&\times Q_k\Bigg(s, \exp((\delta_j-\delta_k)s)u + \int_0^s \exp(-\delta_k\tau)dB_k(\tau) \\
&- \exp((\delta_j-\delta_k)s)\int_0^s \exp(-\delta_j\tau)dB_j(\tau) - \exp(-\delta_k s)a_{jk}(s)\Bigg)ds \\
&+ \exp\left(-\int_0^n \mu_j\right)\chi_{\left[\int_0^n \exp(-\delta_j(\tau))dB_j(\tau)\leq u\right]}.
\end{aligned}$$

Theorem 5.5.5. *The functions Q satisfy (in the sence of Stieltje's differentials) the following differential equations*

$$\begin{aligned}
d_t Q_j(t,u) &= \mu_j\,dt\,Q_j(t,u) - \sum_{k\neq j} \mu_{jk}\,dt \\
&\times\; Q_k(t, \exp((\delta_j-\delta_k)t)\,u + \int_0^t \exp(-\delta_k\,\tau)dB_k(\tau) \\
&-\; \exp((\delta_j-\delta_k)t)\int_0^t \exp(-\delta_j\,\tau)dB_j(\tau) - \exp(-\delta_k\,t)\,a_{jk}(t)),
\end{aligned}$$

with the boundary conditions

$$Q_j(n,u) \;=\; \chi_{\left[\int_0^n \exp(-\delta_j\,\tau)dB_j(\tau)\leq u\right]}.$$

Remark 5.5.6. Theorem 5.5.5 proves useful, since the equations given therein are easier to solve by numerical methods. The main idea is to derive Q in a first step and then calculate P based on Q.

6 Abstract Valuation

In this chapter we look at abstract valuation methods. These methods will be useful, when looking at variable annuities in detail. We also explain the concept of a deflator. During this chapter, we assume that the reader is familiar with the basic concepts of functional analysis such as Banach and Hilbert spaces. For the interested reader we suggest as further reading: [Ped89] for the analytical basics such as integration theory, Banach- and Hilberspaces.

6.1 Framework: Valuation Portfolios

Definition 6.1.1 (Stochastic Cash Flows). *A* Stochastic Cash Flow *is a sequence* $x = (x_k)_{k\in\mathbb{N}} \in L^2(\Omega, \mathcal{A}, P)^{\mathbb{N}}$*, which is* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *adapted.*

Definition 6.1.2 (Regular Stochastic Cash Flows). *A* Regular Stochastic Cash Flow x with respect to $(\alpha_k)_{k\in\mathbb{N}}$*, with* $\alpha_k > 0 \forall k$ *is a stochastic cash flow such that*

$$Y := \sum_{k\in\mathbb{N}} \alpha_k \, X_k \in L^2(\Omega, \mathcal{A}, P).$$

We denote the vector space of all regular cash flows by $\mathcal{X}$*.*

Remark 6.1.3. 1. We note that for all $n \in \mathbb{N}_0$ the image of

$$\psi : L^2(\Omega, \mathcal{A}, P)^n \to \mathcal{X}, (x_k)_{k=0,\dots n} \mapsto (x_0, x_1, \dots, x_n, 0, 0 \dots)$$

is a sub-space of $\mathcal{X}$.

2. $\mathcal{X}$ has been defined this way in order to capture cash flow streams where the sum of the cash flows is infinite with a finite present value. In this set up α_k can be interpreted as a majorant of the price of the payment 1 at time k.

Theorem 6.1.4. *1. For* $x, y \in \mathcal{X}$*, we define the scalar product as follows:*

$$\begin{aligned} < x, y > &= \sum_{k\in\mathbb{N}} < \alpha_k \, x_k, \alpha_k \, y_k > \\ &= E[\sum_{k\in\mathbb{N}} \alpha_k^2 \, x_k \, y_k], \end{aligned}$$

and remark that the scalar product exists as a consequence of the Cauchy-Schwartz inequality .

2. *$\mathcal{X}$ equipped with the above defined scalar product is a Hilbert space with norm $||x|| = \sqrt{< x, x >}$.*

Proof. We leave the proof of this proposition to the reader.

In a next step we introduce the concept of a positive valuation functional and we closely follow [Büh95].

Definition 6.1.5 (Positivity). *1. $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ is called* positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$. In this case we write $x \geq 0$.*

2. *$x = (x_k)_{k\in\mathbb{N}} \in \mathcal{X}$ is called* strictly positive *if $x_k > 0$ P-a.e. for all $k \in \mathbb{N}$ and there exists a $k \in \mathbb{N}$, such that $x_k > 0$ with a positive probability. In this case we write $x > 0$.*

Definition 6.1.6 (Positive functionals). *$Q : \mathcal{X} \to \mathbb{R}$ is called a* positive, continuous and linear functional *if the following hold true:*

1. *If $x > 0$, we have $Q[x] > 0$.*
2. *If $x = \lim_{n\to\infty} x_n$, for $x_n \in \mathcal{X}$ we have $Q[x] = \lim_{n\to\infty} Q[x_n]$.*
3. *For $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{R}$ we have $Q[\alpha x + \beta y] = \alpha Q[x] + \beta Q[y]$.*

Theorem 6.1.7 (Riesz representation theorem). *For Q a positive, linear functional as defined before, there exists $\phi \in \mathcal{X}$, such that*

$$Q[y] \quad = \quad < \phi, y > \forall y \in \mathcal{X}.$$

Proof. This is a direct consequence of Riesz representation theorem of continuous linear functionals of Hilbert spaces.

Definition 6.1.8 (Deflator). *The $\phi \in \mathcal{X}$ generating $Q[\bullet]$ is called* deflator.

Theorem 6.1.9. *For a positive functional $Q : \mathcal{X} \to \mathbb{R}$, with deflator $\psi \in \mathcal{X}$ we have the following:*

1. *$\phi_k > 0$ for all $k \in \mathbb{N}$.*
2. *ϕ is unique.*

Proof. 1. Assume $\phi_k = 0$. In this case we have $Q[(\delta_{kn})_{n\in\mathbb{N}}] = 0$ which is a contradiction.

2. Assume $Q[y] =< \phi, y >=< \phi^*, y >$ for all $y \in \mathcal{X}$. In this case we have $< \phi - \phi^*, y >= 0$, in particular for $y = \phi - \phi^*$. Hence we have $||\phi - \phi^*|| = 0$.

Definition 6.1.10 (Projections). *For $k \in \mathbb{N}$ we define the following projections:*

1. *$p_k : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\delta_{kn}\, x_n)_{n\in\mathbb{N}}$, the projection on the k-th coordinate.*

2. $p_k^+ : \mathcal{X} \to L^2(\Omega, \mathcal{A}, P), x = (x_n)_{n\in\mathbb{N}} \mapsto (\chi_{k\leq n}\, x_n)_{n\in\mathbb{N}}$, *the projection starting on the k-th coordinate.*

Definition 6.1.11 (Valuation at time t, pricing functionals). *For $t \in \mathbb{N}$ we define the valuation of $x \in \mathcal{X}$ at time t by*

$$Q_t[x] \quad = \quad Q[x|\mathcal{F}_t] = \frac{1}{\phi_t}\, E[\sum_{k=0}^{\infty} \phi_k\, x_k | \mathcal{F}_t]$$

In the same sense as for mathematical reserves we define the value of the future cash flows at time t by

$$Q_t^+[x] \quad = \quad Q[p_t^+(x)].$$

The operators Q_t and Q_t^+ are called pricing functionals.

Definition 6.1.12 (Zero Coupon Bonds). *The Zero Coupon Bond $\mathcal{Z}_{(k)} = (\delta_{kn})_{n\in\mathbb{N}}$ is an element of $\mathcal{X}$. We remark that*

$$\pi_0\left(\mathcal{Z}_{(t)}\right) \quad = \quad Q[\mathcal{Z}_{(t)}] = E[\phi_t].$$

Theorem 6.1.13. *The cash flow $x = (x_k)_{k\in\mathbb{N}}$ in the discrete Markov model (cf. proposition 4.7.3) on a finite time interval $T \subset \mathbb{N}$ is given by:*

$$x_k \quad = \quad \sum_{(i,j)\in S^2} \Delta N_{ij}(k-1) a_{ij}^{Post}(k-1) + \sum_{i\in S} I_i(k) a_i^{Pre}(k),$$

where we assume that $\Delta N_{ij}(-1) = 0$.

Proof. The form of the cash flow follows from the calculations of the earlier chapters. It remains to show that $(x_k)_{k\in\mathbb{N}}$ is in L^2. This is however easy, since the benefit functions and the state space are finite. Given the fact that also the time considered for a life insurance is finite, the required property follows.

Theorem 6.1.14. *For $x \in \mathcal{X}$, as defined above we have the following:*

1. $E[\Delta N_{ij}(s)|X_t = k] = p_{ki}(t,s) p_{ij}(s, s+1)$,
2. $E[I_i(s)|X_t = k] = p_{ki}(t,s)$,
3. $E[x_s|X_t = k] =$

$$\sum_{(i,j)\in S^2} p_{ki}(t, s-1) p_{ij}(s-1, s) a_{ij}^{Post}(s-1) + \sum_{i\in S} p_{ki}(t,s) a_i^{Pre}(s),$$

and we assume that $p_{ki}(t, s-1) = 0$ if $t = s$.

Proof. We leave the proof of this proposition to the reader as an exercise.

Definition 6.1.15. *The abstract vector space of financial instruments we denote by $\mathcal{Y}$. Elements of this vector space are for example all zero coupon bonds, shares, options on shares etc.*

Remark 6.1.16.
- Link to the arbitrage free pricing theory: If we assume that Q does not allow arbitrage we are in the set up of chapter 9. In proposition 9.2.15 we have seen that $\pi(X) = E^Q[\beta_T X]$, where β_T denotes the risk free discount rate. In the context of the above, we would have $\pi_0(x) = Q[x] = E^P[\phi_T x]$. Hence we can identify $\phi_T = \frac{dQ}{dP}\beta_T$. In consequence we can interpret a deflator as a discounted Radon-Nikodym density with respect to the two measures P and Q.
- In the same sense the concepts of definition 6.1.11 have a lot in common with the definition of the present values of a cash flow stream as defined in chapter 4.6.
- For the interest rate model in section **??** we know that

$$\begin{aligned} B_t(s) &= \pi_t\left(\mathcal{Z}_{(s)}\right) \\ &= E\left[\exp(-\int_t^s r_u du) \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du) \mid \mathcal{G}_t\right], \end{aligned}$$

  and hence we have actually calculated the corresponding deflators as follows:

$$\phi_t(s) = \exp(-\int_t^s r_u du) \times \exp(-\int_t^s \lambda^T dW - \frac{1}{2}\int_t^s \lambda^T \lambda\, du).$$

Theorem 6.1.17. *Let Q be a positive, continuous functional $Q : \mathcal{X} \to \mathbb{R}$, and assume $Q[\bullet] =< \phi, \bullet >$, with $\phi = (\phi_t)_{t\in\mathbb{N}}$ $\mathbb{F}$ – adapted. In this case $(\phi_t\, Q_t[x])_{t\in\mathbb{N}}$ is an $\mathbb{F}$-martingale over P.*

Proof. Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ and the projection property of the conditional expectation we have

$$\begin{aligned} E^P[\phi_{t+1}\, Q_{t+1}[x]|\mathcal{F}_t] &= E^P[E^P[\sum_{k\in\mathbb{N}} \phi_k x_k\, Q_{t+1}[x]|\mathcal{F}_{t+1}]|\mathcal{F}_t] \\ &= E^P[\sum_{k\in\mathbb{N}} \phi_k x_k\, Q_{t+1}[x]|\mathcal{F}_t] \\ &= \phi_t\, Q_t[x]. \end{aligned}$$

Example 6.1.18 (Replicating Portfolio Mortality). In this first example we consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. For sake of simplicity we assume that the yearly mortality is $(1 + \frac{x-50}{10} \times 0.1)\%$. We assume that the death benefit amounts to 100,000 USD and we assume that the premium has been determined with an interest rate $i = 2\%$. In this case the premium amounts to $P = 1394.29$. The replicating portfolio in the sense of expected cash flows at inception is therefore given as follows (cf proposition 6.1.14). We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively, and remark the the use of arbitrary yield curves does not imply additional complexity.

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
51	$\mathcal{Z}_{(1)}$	1000.00	-1380.34	-380.34	-372.88	-365.71
52	$\mathcal{Z}_{(2)}$	1089.00	-1365.16	-276.16	-265.43	-255.32
53	$\mathcal{Z}_{(3)}$	1174.93	-1348.77	-173.84	-163.81	-154.54
54	$\mathcal{Z}_{(4)}$	1257.56	-1331.24	-73.67	-68.06	-62.97
55	$\mathcal{Z}_{(5)}$	1336.69	-1312.60	24.09	21.82	19.80
56	$\mathcal{Z}_{(6)}$	1412.12	-1292.91	119.20	105.85	94.21
57	$\mathcal{Z}_{(7)}$	1483.67	-1272.23	211.44	184.07	160.67
58	$\mathcal{Z}_{(8)}$	1551.18	-1250.60	300.57	256.54	219.62
59	$\mathcal{Z}_{(9)}$	1614.50	-1228.09	386.41	323.33	271.48
60	$\mathcal{Z}_{(10)}$	1673.52	–	1673.52	1372.87	1130.57
Total					**0.00**	**-336.47**

Exercise 6.1.19 (Replicating Portfolio Disability). Consider a disability cover and calculate the replicating portfolios for a deferred disability annuity and a disability in payment.

6.2 Cost of Capital

In section 6.1 we have seen how to abstractly valuate $x \in \mathcal{X}$ by means of a pricing functional Q . For some financial instruments $y \in \mathcal{Y}^\star$ we can directly observe $Q[y]$ such as for a lot of zero coupons bonds $\mathcal{Z}_{(\bullet)}$. On the other hand this is not always possible.

Definition 6.2.1. *We denote by $\mathcal{Y}^\star$ the set of all stochastic cash flows in $x \in \mathcal{Y}$ such that $Q[x]$ is observable. With $\tilde{\mathcal{Y}} = span < \mathcal{Y}^\star >$ we denote the vector space generated by $\mathcal{Y}^\star$ and we define:*

1. *$x \in \mathcal{Y}^\star$ is called of level 1.*
2. *$x \in \tilde{\mathcal{Y}}$ is called of level 2.*
3. *$x \in \mathcal{Y} \setminus \tilde{\mathcal{Y}}$ is called of level 3.*

Remark 6.2.2. It is clear that the model uncertainty and the difficulties to value assets or liabilities increases from level 1 to level 3. Since we are interested in market values only the valuation of level 1 assets and liabilities are really reliable. For level 2 assets and liabilities on has to find a sequence of $x_n = \sum_{k=1}^n \alpha_k \, e_k$ with $e_k \in \mathcal{Y}^\star$ such that $x = \lim_{n\to\infty} x_n$. Since we <u>assume</u> that Q is linear and continuous we can calculate

$$\begin{aligned} Q[x] &= \lim_{n\to\infty} Q[x_n] \\ &= \lim_{n\to\infty} \sum_{k=1}^n Q[\alpha_k \, e_k] \\ &= \lim_{n\to\infty} \sum_{k=1}^n \alpha_k \, Q[e_k]. \end{aligned}$$

For level 3 assets and liabilities the situation is even more difficult, since there is no obvious way to do it. The best, which we can be done is to define $\tilde{Q}[x]$ such that $\tilde{Q}[x] = Q[x] \forall x \in \mathcal{Y}$ and hope that $\tilde{Q}[x] \approx Q[x]$ for the $x \in \mathcal{Y}$ we want to valuate. In most cases such $\tilde{Q}[\bullet]$ are based on first economic principles. In the following we want to see how the *Cost of Capital* concept works for insurance liabilities and how we can concretely implement it.

Definition 6.2.3 (Utility Assumption). *If we have $x, y \in L^2(\Omega, \mathcal{A}, P)^+$, with $x = E[y]$. A rational investor would normally prefer x, since there is less uncertainty. The way to understand this, is by using utility functions. For $x \in L^2(\Omega, \mathcal{A}, P)^+$ and u a concave function, the utility of x is defined as $E[u(x)]$. The idea behind utilities is that the first 10,000 USD are higher valued than the one 10,000 USD from 100,000 USD to 110,000 USD. Hence the increase of utility per fixed amount decreases if amounts increase. As a consequence of the Jensen's inequality, we see that the utility of a constant amount is higher than the utility of a random payout with the same expected value.*

Definition 6.2.4. *Let $x = (x_k)_{k \in \mathbb{N}} \in \mathcal{X}$ be an insurance cash flow, for example generated by a Markov model.*

1. *In this case we define the expected cash flows by*
$$CF(x) \quad = \quad (E[x_k])_{k \in \mathbb{N}}.$$
2. *The corresponding portfolio of financial instruments in the vector space $\mathcal{Y}$ we define by*
$$VaPo^{CF}(x) \quad = \quad \sum_{k \in \mathbb{N}} CF(x)_k \mathcal{Z}_{(k)} \in \mathcal{Y}$$
3. *By $R(x)$ we denote the residual risk portfolio given by*
$$\begin{aligned} R(x) &= x - VaPo^{CF}(x) \\ &= \sum_{k \in \mathbb{N}} (x_k - CF(x)_k)\, \mathcal{Z}_{(k)} \in \mathcal{Y} \end{aligned}$$
4. *For a given $x \in \mathcal{X}$ we denote by $VaPo^*(x)$ an approximation $y \in \tilde{\mathcal{Y}}$ of x, such that $||x - VaPo^*(x)|| \leq ||x - VaPo^{CF}(x)||$.*

Since we are sometimes interested in conditional expectations, we will also use the following notations for $A \in \mathcal{A}$:

$$\begin{aligned} CF(x \,|\, A) &= (E[x_k \,|\, A])_{k \in \mathbb{N}}, \\ VaPo^{CF}(x \,|\, A) &= \sum_{k \in \mathbb{N}} CF(x \,|\, A)_k \mathcal{Z}_{(k)} \in \mathcal{Y}, \end{aligned}$$

Theorem 6.2.5. *The value of $x \in \mathcal{X}$ can be decomposed in*

$$Q[x] = Q[VaPo^{CF}(x)] + Q[R(x)],$$

and we have

$$Q[VaPo^{CF}(x)] \geq Q[x]$$

if we use the utility assumption.

Remark 6.2.6. 1. We will denote $x \in \mathcal{X}$ with $x \leq 0$ as a liability. Proposition 6.2.5 hence tells us that we need to reserve more than $Q[VaPo^{CF}(x)]$ for this liability as a consequence of the corresponding uncertainty.

2. A risk measure is a functional (not necessarily linear) $\psi : \mathcal{X} \to \mathbb{R}$ which aims to measure the capital needs in an adverse scenario. There are two risk measures, which are commonly used the *Value at Risk* and the *Expected Shortfall* to a given quantile $\alpha \in \mathbb{R}$. The value at risk (VaR) is defined as the corresponding quantile minus the expected value. The expected shortfall is the conditional expectation of the random variable given a loss bigger than the corresponding loss, again minus the expected value. We can hence speak about a 99.5% VaR or a 99% expected shortfall. It is worthwhile to remark that these two concepts are normally applied to losses. Hence in the context introduced above one would strictly speaking calculating the VaR$(-x)$, when considering $x \in \mathcal{X}$. Furthermore in a lot of applications, such as Solvency II, we assume that there is a Dirac measure (aka stress scenario), which just represents the corresponding VaR-level for example. So concretely the stress scenarios, which are used under Solvency II should in principle represent the corresponding point (Dirac) measures at to the confidence level 99.5 %. In the concrete set up, one would for example assume that $q_x(\omega) \in L^2(\Omega, \mathcal{A}, P)$ is a stochastic mortality and one would define the $A, B \in \mathcal{A}$, as the corresponding probabilities in the average and in the tail. In consequence for a policy $x \in \mathcal{X}$, we would have two replicating portfolios, namely $VaPo^{CF}(x \,|\, A)$ for the average and $VaPo^{CF}(x \,|\, B)$ for the stressed event according to the risk measure chosen. The corresponding required risk capital is then given (in present value terms) by $Q[VaPo^{CF}(x \,|\, B) - VaPo^{CF}(x \,|\, A)]$.

Definition 6.2.7 (Required Risk Capital). *For a risk measure ψ_α such as VaR or expected shortfall to a security level α we define the required risk capital at time $t \in \mathbb{N}$ by*

$$RC_t(x) \quad = \quad \psi_\alpha(p_k(x - VaPo^{CF}(x))).$$

Remark 6.2.8. 1. If we use $VaR_{99.5\%}$ the required risk capital at time t corresponds to the capital needed to withstand a 1 in 200 year event.

2. The definition above could apply to individual insurance policies, but is normally applied to insurance portfolios $\tilde{x} = \sum_{k=1}^{n} x_k$, where x_k are the individual insurance policies. As we have seen in section **??** the pure diversifiable risk disappears for $n \to \infty$.

3. What is more material than the diversifiable risk is the risk, which affects all of the individual insurance policies at the same time, such as a pandemic event, where the overall mortality could increase by 1 % in a certain year such as 1918.

Definition 6.2.9 (Cost of Capital). *For a unit cost of capital $\beta \in \mathbb{R}^+$ and an insurance portfolio $\tilde{x} \in \mathcal{X}$, we define:*

1. The present value of the required risk capital by

$$PVC(\tilde{x}) \quad = \quad Q[\sum_{k \in \mathbb{N}} RC_t(\tilde{x}) \mathcal{Z}_{(k)}].$$

2. The cost of capital $CoC(\tilde{x})$ is given by:

$$CoC(\tilde{x}) \quad = \quad \beta \times PVC(\tilde{x}),$$

and $\tilde{Q}$ *is defined by* $\tilde{Q}[\tilde{x}] = Q[VaPo^{CF}(\tilde{x})] + \beta PVC(\tilde{x})$.

Remark 6.2.10. 1. The concept as defined before is somewhat simplified, since one normally assumes that the required capital C from the shareholder is $\alpha \times C$ after tax and investment income on capital, Assume a tax-rate κ and a risk-free yield of i. In this case we have

$$\alpha \times C = i \times (1 - \kappa) \times C + \beta \times C,$$

and hence $\beta = \alpha - i \times (1 - \kappa)$. In reality the calculation can still become more complex since we discount future capital requirements risk-free and because of the fact that the interest rate i is not constant. In order to avoid these technicalities, we will assume for this book that i is constant.

2. We remark $\tilde{Q}[\tilde{x}]$ is not uniquely determined, but depends on a lot of assumptions such as $\psi\alpha$, α, β, ...

3. For the moment we did not yet see how to actually model $\tilde{x}$ and we remark that one is normally focusing on the non-diversifiable part of the risks within $\tilde{x}$.

Example 6.2.11. We continue with example 6.1.18 and we assume that the risk capital is given by a pandemic event where $\Delta q_x = 1\%$ for all ages. This roughly corresponds to the increase in mortality of 1918 as a consequence of the Spanish flu pandemic. The aim of this example is to calculate the required risk capital and the market value of this policy based on the cost of capital method using $\beta = 6\%$. The required risk capital in this context can be calculated as $\Delta q_x \times 100,000$ and we get the following results:

Age	Unit	Units for Risk Capital	Units for Benefits	Total Units	$-\tilde{Q}[x]$ $i = 2\%$	$-\tilde{Q}[x]$ $i = 4\%$
50	$\mathcal{Z}_{(0)}$	1000.00	-1394.28	-1334.28	-1334.28	-1334.28
51	$\mathcal{Z}_{(1)}$	990.00	-380.34	-320.94	-314.65	-308.60
52	$\mathcal{Z}_{(2)}$	979.11	-276.16	-217.41	-208.97	-201.01
53	$\mathcal{Z}_{(3)}$	967.36	-173.84	-115.80	-109.12	-102.95
54	$\mathcal{Z}_{(4)}$	954.78	-73.67	-16.38	-15.14	-14.00
55	$\mathcal{Z}_{(5)}$	941.41	24.09	80.57	72.98	66.22
56	$\mathcal{Z}_{(6)}$	927.29	119.20	174.84	155.25	138.18
57	$\mathcal{Z}_{(7)}$	912.45	211.44	266.19	231.73	202.28
58	$\mathcal{Z}_{(8)}$	896.94	300.57	354.39	302.47	258.95
59	$\mathcal{Z}_{(9)}$	880.80	386.41	439.26	367.55	308.61
60	$\mathcal{Z}_{(10)}$	–	1673.52	1673.52	1372.87	1130.57
Total					**520.69**	**143.98**

We remark that the value of the policy at inception becomes positive, which means nothing else, that the insurance company does need equity capital to cover the economic loss. It is obvious that this is the case for $i = 2\%$, since the premium principle did not allow for a compensation of the risk capital. More interestingly even at the higher interest rate the compensating effect is not big enough to turn this policy into profitability.

Exercise 6.2.12. In the same sense as for the mortality example calculate the respective risk capitals and the $\tilde{Q}$ for a disability cover.

6.3 Inclusion in the Markov Model

In this section we want to have a look how we could concretely use the recursion technique for the calculation of the cost of capital in a Markov chain similar environment. In order to do that we look at an insurance policy with a term of one year.

We assume that we have a mortality of q_x in case of a "normal" year with a probability of $(1-\alpha)$ and an excess mortality of Δq_x in an extreme year with probability α. We denote with $\Gamma = \frac{q_x + \Delta q_x}{q_x}$. Furthermore we assume a mortality benefit of 100,000. In this case we get the following by some simple calculations:

$$\begin{aligned}
VaPo^{CF}(x) &= (\delta_{1k}(q_x + \alpha(\Gamma - 1)q_x \times 100000))_{k\in\mathbb{N}} \\
RC_1(x) &= (\delta_{1k}(1-\alpha)(\Gamma - 1)q_x \times 100000)_{k\in\mathbb{N}} \\
\tilde{Q}[x] &= Q[(\delta_{1k}(q_x + \alpha(\Gamma - 1)q_x \times 100000 + \\
&+ \beta(1-\alpha)(\Gamma - 1) \times 100000))_{k\in\mathbb{N}}]
\end{aligned}$$

We see that the price of this insurance policy with only payments at time 1 can be decomposed into a part representing best estimate mortality:

$$\delta_{1k}\{q_x\,(1+\alpha\,(\Gamma-1))\},$$

where we can arguably say that this $\tilde{q}_x = q_x\,(1+\alpha\,(\Gamma-1))$ is our actual best-estimate mortality. On top of that we get a charge for the excess mortality Δq_x with an additional cost of β. Hence we get the following:

1. There is a contribution to the reserve from the people surviving the year with a probability p_x.
2. There is a contribution to the reserve from the people dying in normal years with probability q_x and the defined benefit $a_{*\dagger}^{\text{post}}$, and
3. There is finally a contribution of the people dying in extreme years with probability Δq_x and the additional cost of defined benefit of $\beta \times a_{*\dagger}^{\text{post}}$.

The interesting fact is that we can actually use the same recursion of the reserves for the Markov chain model as in proposition 4.7.3 with the exception that now the "transition probabilities" do not fulfil anymore the requirement that their sum equals 1. However this method provides a pragmatic way to implement the cost of capital in legacy admin systems.

The main problem for the determining of the corresponding Markov chain model is the underlying stochastic mortality model. For the QIS 5 longevity model a similar calculation can be used. In this model it is assumed that the mortality drops by 25 % in an extreme scenario. Hence the calculation goes along the following process:

1. Determine $x_1 = VaPo^{CF}(\tilde{x})$.
2. Determine $x_2 = VaPo^{CF}(\tilde{x})$ for stressed mortality.
3. $\tilde{Q}[x] = Q[x_1] + \beta\, Q[x_2 - x_1]$

Example 6.3.1. In this example we want to revisit the exercise 6.1.18 and we want again to calculate the market value of the insurance liability, but this time with the recursion. We get the following results:

Age	Benefit Normal	Benefit Premium	Excess Risk	Math Res. $i = 2\%$	Value $i = 2\%$	Value $i = 4\%$
50	100000	-1394.28	6000	**0.00**	**520.69**	**143.98**
51	100000	-1394.28	6000	426.43	901.09	542.82
52	100000	-1394.28	6000	765.56	1193.21	861.67
53	100000	-1394.28	6000	1015.22	1394.79	1096.96
54	100000	-1394.28	6000	1172.95	1503.20	1244.68
55	100000	-1394.28	6000	1235.88	1515.45	1300.33
56	100000	-1394.28	6000	1200.79	1428.16	1258.89
57	100000	-1394.28	6000	1064.00	1237.49	1114.74
58	100000	-1394.28	6000	821.42	939.18	861.64
59	100000	-1394.28	6000	468.45	528.45	492.63
60				0	0	0

We remark that this calculation was much faster to calculate since it is based on Thiele's difference equation for the mathematical reserves, and we get at the same time the corresponding results for the classical case and also for the case using the cost of capital approach.

As seen in the calculation above there is a small second order effect, which we can detect, when looking more closely. The results below correspond to the 2% valuation:

```
Direct Method  520.698380872792
Recursion      520.698380872793
```

Exercise 6.3.2. Perform the corresponding calculation for the disability example.

6.4 Asset Liability Management

Until now we have looked only at insurance liabilities as an $x \in \mathcal{X}$. An insurance company needs to cover its insurance liabilities $l = \sum x_\iota \in \mathcal{X}$ with corresponding assets, which are also elements in $\mathcal{X}$.

Definition 6.4.1 (Assets and Liabilities). *An $x \in \mathcal{X}$ with a valuation functional Q is called*

1. *an asset if $Q[x] \geq 0$ and*
2. *a liability if $Q[x] \leq 0$.*

Definition 6.4.2 (Insurance balance sheet). *An insurance balance sheet consists of a set of assets $(a_i)_{i \in I}$ and a set of liabilities $(l_j)_{j \in J}$. The equity of an insurance balance sheet is defined as*

$$e = \sum_{i \in I} a_i + \sum_{j \in J} l_j.$$

The insurance entity is called bankrupt if $Q[e] < 0$.

Definition 6.4.3. *In an insurance market, each insurance company is required to hold an adequate amount of risk capital in order to absorb shocks. In order to do that, the regulator defines a risk measure $\psi\alpha$ to a security level α. In this context an insurance company is called solvent if:*

$$Q[e] \geq \psi\alpha(e).$$

Remark 6.4.4. Note that an insurance regulator may not want to use a market consistent approach. Never the less the above definition can be used, be suitably adjust ψ.

Definition 6.4.5 (Asset Liability Management). *Under Asset Liability Management we understand the process of analysing $(l_j)_{j \in J}$ and the (dynamic) management of $(a_i)_{i \in I}$ in order to achieve certain targets , such as remaining solvent.*

Definition 6.4.6. *For an insurance liability $l \in \mathcal{X}$ an asset portfolio $(a_i)_{i \in I}$ is called:*

1. matching *if $\sum_{i \in I} a_i + l = 0$, and*
2. cash flow matching *if $\sum_{i \in I} a_i + VaPo^{CF}(l) = 0$.*

Remark 6.4.7. We remark that is normally not feasible to do a perfect matching, and hence one normally uses a cash flow matching to a achieve a proxy for a perfect match. We also remark that in this case the shareholder equity needs still be able to absorb the basis risk $l - VaPo^{CF}(l)$.

Definition 6.4.8 (Duration). *The duration for an $x \in \mathcal{X}$ with $x = \sum_{i \in \mathbb{N}} \alpha_i \mathcal{Z}_{(i)}$ and $\alpha_i \geq 0$ is defined by*

$$d(x) = \frac{Q[\sum_{i \in \mathbb{N}} \alpha_i \times i \times \mathcal{Z}_{(i)}]}{Q[\sum_{i \in \mathbb{N}} \alpha_i \times \mathcal{Z}_{(i)}]}$$

We say that an asset portfolio $(a_i)_{i \in I}$ is duration matching a liability l if the following two conditions are fulfilled:

1. $Q[\sum_{i \in I} a_i + l] = 0$, *and*
2. $d(\sum_{i \in I} a_i) = d(-l)$.

Example 6.4.9. In this example we want to further elaborate on the example 6.1.18 and we want to see how the replicating scenario changes in case a pandemic occurs in year three, with an excess mortality of 1 %. We want also to have a look on what risk is implied in this, assuming that the pandemic at the same time leads to a reduction of interest rates down from 2% to 0.5 %. Finally we want to see an example how we could do a perfect cash flow matching portfolio and duration matched portfolio.

Definitions We assume that $A \in \mathcal{A}$ represents the information that we have going to have average mortality after year 3 and three and that the person survived until then (year 2). In the same sense we assume that $B \in \mathcal{A}$ represents the same as A but with the exception that we assume a pandemic event in the year 3 with an average excess mortality of 1%. For simplicity reasons (to avoid notation) we use $x, y \in \mathcal{X}$ as abbreviations for the corresponding conditional random variables.

Calculation of the Replicating Portfolios In a first step we will calculate the replicating portfolios (starting at time 2) with respect to both A and B. Doing this we get the following results for case A:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
52	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
53	$\mathcal{Z}_{(1)}$	1200.00	-1377.55	-177.55	-174.07	-170.72
54	$\mathcal{Z}_{(2)}$	1284.40	-1359.64	-75.24	-72.32	-69.57
55	$\mathcal{Z}_{(3)}$	1365.21	-1340.61	24.60	23.18	21.87
56	$\mathcal{Z}_{(4)}$	1442.25	-1320.50	121.75	112.48	104.07
57	$\mathcal{Z}_{(5)}$	1515.33	-1299.37	215.95	195.59	177.49
58	$\mathcal{Z}_{(6)}$	1584.27	-1277.28	306.99	272.59	242.61
59	$\mathcal{Z}_{(7)}$	1648.95	-1254.29	394.65	343.57	299.90
60	$\mathcal{Z}_{(8)}$	1709.23	–	1709.23	1458.81	1248.91
Total					**765.56**	**460.30**

For case B we get:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
52	$\mathcal{Z}_{(0)}$	–	-1394.28	-1394.28	-1394.28	-1394.28
53	$\mathcal{Z}_{(1)}$	1200.00	-1377.55	-177.55	-174.07	-170.72
54	$\mathcal{Z}_{(2)}$	**2272.40**	-1345.87	**926.52**	890.54	856.62
55	$\mathcal{Z}_{(3)}$	1351.38	-1327.03	24.35	22.95	21.65
56	$\mathcal{Z}_{(4)}$	1427.64	-1307.12	120.51	111.34	103.01
57	$\mathcal{Z}_{(5)}$	1499.97	-1286.21	213.76	193.61	175.70
58	$\mathcal{Z}_{(6)}$	1568.22	-1264.34	303.88	269.83	240.16
59	$\mathcal{Z}_{(7)}$	1632.24	-1241.58	390.65	340.09	296.86
60	$\mathcal{Z}_{(8)}$	1691.91	–	1691.91	1444.03	1236.26
Total					**1704.05**	**1365.28**

We note two things:

- The pandemic happens when the person is aged 53 and we see the impact in $\mathcal{Z}_{(2)}$ at age 54. This has to do with the convention that we assume that the deaths occur at the end on the year, hence just before the person gets 54.
- We see that the difference in reserves amounts to $1704.05 - 765.56 = 938.49$ which represents the economic loss as a consequence of the pandemic. The biggest contributor to this loss is the increased death benefit, e.g. $926.52 - 1284.40 = 962.87$.

Matching asset portfolios Based on the above it is now easy to calculate the cash flow matching portfolio, by just investing the different amounts of liabilities into the corresponding assets, such as buying $24.60\mathcal{Z}_{(3)}$. We remark that consequently we would have to sell $-177.55\mathcal{Z}_{(1)}$. In normal circumstances for mature businesses this will not occur, since it is a consequence that we consider a term insurance policy and not for example an endowment.

Mismatch in case of a pandemic The table below finally shows the cash flow mismatch as a consequence of the pandemic and we see that in this case the present values do not have a big impact since the main difference is at time 1.

Age	Unit	Units Normal	Units Stress	Difference Units	Value $i=2\%$	Value $i=0\%$
52	$\mathcal{Z}_{(0)}$	-1394.28	-1394.28	0.00	0.00	0.00
53	$\mathcal{Z}_{(1)}$	-177.55	-177.55	0.00	0.00	0.00
54	$\mathcal{Z}_{(2)}$	-75.24	926.52	1001.77	962.87	1001.77
55	$\mathcal{Z}_{(3)}$	24.60	24.35	-0.24	-0.23	-0.24
56	$\mathcal{Z}_{(4)}$	121.75	120.51	-1.23	-1.13	-1.23
57	$\mathcal{Z}_{(5)}$	215.95	213.76	-2.18	-1.98	-2.18
58	$\mathcal{Z}_{(6)}$	306.99	303.88	-3.11	-2.76	-3.11
59	$\mathcal{Z}_{(7)}$	394.65	390.65	-3.99	-3.48	-3.99
60	$\mathcal{Z}_{(8)}$	1709.23	1691.91	-17.31	-14.78	-17.31
Total					**938.49**	**973.67**

Example 6.4.10 (Lapses). In this example we want to see how lapses can influence the replicating portfolios. In order to do that we have to change the example 6.1.18 a little bit, as follows:

- We consider a term insurance, for a 50 year old man with a term of 10 years, and we assume that this policy is financed with a regular premium payment. Hence there are actually two different payment streams, namely the premium payment stream and the benefits payment stream. For sake of simplicity we assume that the yearly mortality is $(1 + \frac{x-50}{10} \times 0.1)\%$. We assume that the benefit amounts to 100.000 USD and we assume that the premium has been determined with an interest rate $i = 2\%$.
- In this case the premium amounts to $P = 9562.20$.
- In addition the policyholder can surrender the policy at any time and gets back 98 % of the expected future cash flows valued at the pricing interest rate of 2%. We remark here that this is a risk since the surrenders can happen in case the market value of the corresponding units is below the surrender value.
- We remark that the units have been valued with two (flat) yield curves with interest rates of 2% and 4% respectively.

In order to calculate this example we will perform the following steps:

1. Calculation of the cash flow matching portfolio in case of no surrenders.
2. Calculation of the cash flow including lapses with an average lapse rate of 7 %

3. Calculation of the cash flows at time 2, assuming average lapses, lapses at 25 % at time 2.

Calculation of the cash flow matching portfolio in case of no surrenders:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1000.00	-9466.57	-8466.57	-8300.56	-8140.94
52	$\mathcal{Z}_{(2)}$	1089.00	-9362.44	-8273.44	-7952.17	-7649.26
53	$\mathcal{Z}_{(3)}$	1174.93	-9250.09	-8075.16	-7609.40	-7178.79
54	$\mathcal{Z}_{(4)}$	1257.56	-9129.84	-7872.27	-7272.76	-6729.25
55	$\mathcal{Z}_{(5)}$	1336.69	-9002.02	-7665.32	-6942.72	-6300.34
56	$\mathcal{Z}_{(6)}$	1412.12	-8866.99	-7454.87	-6619.71	-5891.69
57	$\mathcal{Z}_{(7)}$	1483.67	-8725.12	-7241.45	-6304.11	-5502.90
58	$\mathcal{Z}_{(8)}$	1551.18	-8576.79	-7025.61	-5996.29	-5133.54
59	$\mathcal{Z}_{(9)}$	1614.50	-8422.41	-6807.90	-5696.55	-4783.14
60	$\mathcal{Z}_{(10)}$	–	88080.30	88080.30	72256.53	59503.90
Total					**0**	**-7368.19**

We remark that the there is considerable value in the policy if we assume no lapses, in case we earn a higher interest rate, such as 4 %.

Calculation of the cash flow matching portfolio in case of 7% surrenders:

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1019.67	-8797.22	-7777.54	-7625.04	-7478.40
52	$\mathcal{Z}_{(2)}$	1594.01	-8084.65	-6490.63	-6238.59	-6000.95
53	$\mathcal{Z}_{(3)}$	2066.98	-7421.70	-5354.72	-5045.87	-4760.33
54	$\mathcal{Z}_{(4)}$	2449.23	-6805.70	-4356.47	-4024.71	-3723.93
55	$\mathcal{Z}_{(5)}$	2750.73	-6234.02	-3483.29	-3154.92	-2863.01
56	$\mathcal{Z}_{(6)}$	2980.77	-5704.13	-2723.35	-2418.26	-2152.31
57	$\mathcal{Z}_{(7)}$	3147.94	-5213.57	-2065.63	-1798.25	-1569.71
58	$\mathcal{Z}_{(8)}$	3260.18	-4759.99	-1499.81	-1280.07	-1095.89
59	$\mathcal{Z}_{(9)}$	3324.77	-4341.11	-1016.34	-850.42	-714.06
60	$\mathcal{Z}_{(10)}$	5486.26	42159.27	47645.53	39085.93	32187.61
Total					**-2912.44**	**-7733.21**

We remark that at that time, the company makes still some additional gains as a consequence of the 2% surrender penalty.

Calculation of the cash flow matching portfolio in case of high surrenders: We assume that there has been observed an exceptional lapse rate at time 2 of 25% of the portfolio.

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i=2\%$	Value $i=4\%$
50	$\mathcal{Z}_{(0)}$	–	-9562.20	-9562.20	-9562.20	-9562.20
51	$\mathcal{Z}_{(1)}$	1019.67	-8797.22	-7777.54	-7625.04	-7478.40
52	$\mathcal{Z}_{(2)}$	1594.01	-8084.65	-6490.63	-6238.59	-6000.95
53	$\mathcal{Z}_{(3)}$	4773.16	-5966.47	-1193.30	-1124.48	-1060.84
54	$\mathcal{Z}_{(4)}$	1968.98	-5471.25	-3502.26	-3235.55	-2993.75
55	$\mathcal{Z}_{(5)}$	2211.37	-5011.66	-2800.29	-2536.31	-2301.63
56	$\mathcal{Z}_{(6)}$	2396.30	-4585.67	-2189.36	-1944.09	-1730.28
57	$\mathcal{Z}_{(7)}$	2530.70	-4191.30	-1660.60	-1445.65	-1261.92
58	$\mathcal{Z}_{(8)}$	2620.93	-3826.66	-1205.73	-1029.08	-881.01
59	$\mathcal{Z}_{(9)}$	2672.85	-3489.91	-817.05	-683.67	-574.05
60	$\mathcal{Z}_{(10)}$	4410.52	33892.74	38303.27	31422.02	25876.31
Total					**-4002.67**	**-7968.76**

ALM Risk of mass lapses Finally we want to look what happens when we have mass lapses as indicated before, but if we have invested in the cash flow matching portfolio according to average 7 % lapses. Hence we have to calculate the assets according to 7 % lapses and the liabilities according 25 % lapses.

Age	Unit	Units for Assets	Units for Liability	Total Units	Value $i=2\%$	Value $i=4\%$
52	$\mathcal{Z}_{(0)}$	-6490.63	6490.63	0	0	0
53	$\mathcal{Z}_{(1)}$	-5354.72	1193.30	-4161.42	-4079.82	-4001.36
54	$\mathcal{Z}_{(2)}$	-4356.47	3502.26	-854.21	-821.04	-789.76
55	$\mathcal{Z}_{(3)}$	-3483.29	2800.29	-682.99	-643.60	-607.18
56	$\mathcal{Z}_{(4)}$	-2723.35	2189.36	-533.99	-493.32	-456.45
57	$\mathcal{Z}_{(5)}$	-2065.63	1660.60	-405.02	-366.84	-332.90
58	$\mathcal{Z}_{(6)}$	-1499.81	1205.73	-294.08	-261.13	-232.41
59	$\mathcal{Z}_{(7)}$	-1016.34	817.05	-199.28	-173.48	-151.43
60	$\mathcal{Z}_{(8)}$	47645.53	-38303.27	9342.26	7973.53	6826.29
Total					**1134.26**	**254.76**

Now we see that the lapses induce quite a big risk for the company since they lose in case of mass lapses almost 1 % of the face value of the policy, more concretely $1134.26 - 254.76 = 879.50$.

The above example shows very clearly how the behaviour of the policyholders can change the cash flow matching portfolio and in consequence induces a risk. As a consequence the risk minimising portfolio in the sense of $VaPo^*(x)$ for an insurance portfolio $x \in \mathcal{X}$ does also consist of additional assets offsetting the corresponding risks. In the above example the corresponding asset would be a (complex) put option, which allows to sell the bond portfolio at the predefined (book-) values. So in reality insurance companies aim to model these risk in order to determine the corresponding assets to manage and reduce the undesired risk.

In the example above we have assumed that at a given year 25% of the policies in force lapse. In practise one models the dynamic lapse behaviours. Eg the lapse rate is a function of the interest differential between market and book yields. Normally the corresponding lapse rates stay below 1, which is interesting. Assuming a market efficient behaviour, one would expect that there is a binary decision of the policyholders to stick to the contract or to lapse as a function of the before mentioned interest differential. In consequence the underlying theory how to model such policyholder behaviour is not as crisp and transparent as with the arbitrage free pricing theory, since market efficient behaviours is normally not observed. As a corollary there is a lot of model risk intrinsic to these calculations and it is important to test the results from the models with different scenarios.

Remark 6.4.11. At the end of this section a remark on how to determine a $VaPo^*(x)$ for an $x \in \mathcal{X}$: One normally models an $l \in \mathcal{X}$ and simulates $l(\omega)$ together with some test assets $D \subset \mathcal{Y}$ observable prices and cash flows. We denote $D = \{d_1, \ldots d_n\}$. Hence at the end of this process we have a vector

$$\mathcal{W} := (l(\omega_i), d_1(\omega_i), \ldots d_n(\omega_i))_{i \in I}.$$

Now the process is quite canonical:

1. We define a distance between two $x, y \in \mathcal{X}$, for example by means of $||x||$ as defined.
2. We solve the numerical optimisation problem, for minimising the distance between l and the target $y \in \text{span} < \mathcal{D} >$.

We note two things:

– The numerical procedures to determine y can sometimes prove to be difficult since the corresponding design matrix can be near to a singular matrix, and hence additional care is needed.

– In case of the $|| \bullet ||$ defined before, we remark that it has been deducted from the Hilbert space $\mathcal{X}$. Hence what we actually doing is to use the projection $\tilde{p} : \mathcal{X} \to \text{span} < \mathcal{D} >$, which can be expressed by means of $< \bullet, \bullet >$. We remark that $y = \tilde{p}(x)$.

7 Withprofit Policies, Natural Cash Flows and Bonus Mechanisms

Withprofit policies are the typical policies sold in Europe and America over the last century. The guiding principle is that there is a guaranteed part which is conservatively priced (at inception) and where policyholder receive a part of the difference to best estimate as a policyholder participation ("Policyholder Bonus"). The typical requirement is that the guarantee rates at inception should not exceed 60% of the long term government yields for commensurate durations. If for example the average government yields were at 5%, the guaranteed interest rate ("technical interest rate") was expected not to exceed 3% in this example. At the beginning of the 21st century after the financial crisis of 2008, interest rates began to sink, as a consequence of quantitative easing to historical lows and the guarantees began biting. Later on around 2022/23, interest rates rose again materially.

The main question with regards to these policies is:

- How to protect and immunise the written guarantees, and
- How to invest the remainder in order to give the policy- and shareholders a decent return.

Conceptually withprofit policies are a mixture between cash flow guaranteed policies ("nonprofit policies") and unit linked polices with guarantees (for pure withprofit part). It is worth noting that there are still material differences between unit linked products with guarantees and withprofit policies. Hence the aim of this chapter is to understand these differences and how to approach this type of product.

In order to understand the natural cash flows of an insurance policy, one needs in a first instance to understand the different options the policyholder has and the respective consequences. The following three aspects are particularly important and will be discussed below:

- Stop paying regular premiums
- Surrender
- Policyholder Bonuses and Legal Quotes

Before we start doing the respective analysis and examples, it makes sense to remind ourselves the principles how we model life insurance. The implicit assumption is that mortality benefits are paid at the end of the respective year, hence if a person for argument sake dies with an age of 56y6m, the death benefit is paid at 57y0m in the model. Hence considering the entire year in the model means: premium payment at 56y0m, death benefit (if person dies) at 57y0m, and premium payment (subsequent year if person survives) at 57y0m. Hence there are to some extent payments just before 57y (death benefit, call it 57^-) and payments just after 57y (premium, call it 57^+). The notation we are using is the mathematical correct on, as one would expect. Now where this is relevant is when we calculate reserves. In the classical set up one normally calculates reserves at $x = x^-$, which means that the reserve is zero at inception in line with equivalence principle. On the other hand, the stochastic model (including the Markov model above) stipulates reserves at x^+. Conclusion - we need to be careful when doing ALM and accrued interest rates. Hence for the following examples we explicitly try to distinguish between the cash flows at times x and x^+.

7.1 Stop Paying Premium

We have seen before that premiums are determined by the equivalence principle, which means that at inception the present value of the benefits equals the present value of the premium be paid by the policyholder. This is closely linked the concept of mathematical reserve ${}_nV_x$ which is the value within a policy after n years. In consequence the mathematical reserve is 0 before paing the regular premium at inception and equals the premium at 0^+. Similarly the mathematical reserve is equal to the single premium for a single premium contract, when ignoring all costs, in particular distribution costs. Now most policies allow the policyholder to stop paying premium after having paid the first premium. Now what does this mean for the policy?

Ultimately the the mathematical reserve at the end of the respective period is used as a single premium for the policy. This means that the benefits are then materially reduced. Assume that we consider a whole of life policy with a benefit level L, which is financed by a regular premium P, which is to be paid until the death of the person. In this case the premium is calculated as follows:

$$\begin{aligned} L \times A_x &= P \times \ddot{a}_x, \text{ and} \\ P &= L \frac{A_x}{\ddot{a}_x}. \end{aligned}$$

Now assume that the policyholder decides to stop after the first premium at age $x+1$. At this time the mathematical reserve for people surviving the first year equals

$$V_{x+1} = L \times A_{x+1} - P \times \ddot{a}_{x+1}.$$

This mathematical reserve does now serve as single premium for the same contract with a reduced benefit $\tilde{L}$, which can be calculated as follows:

$$\begin{aligned} \tilde{L} &= \frac{V_{x+1}}{A_x} \\ &= \frac{L \times A_{x+1} - P \times \ddot{a}_{x+1}}{A_{x+1}}. \end{aligned}$$

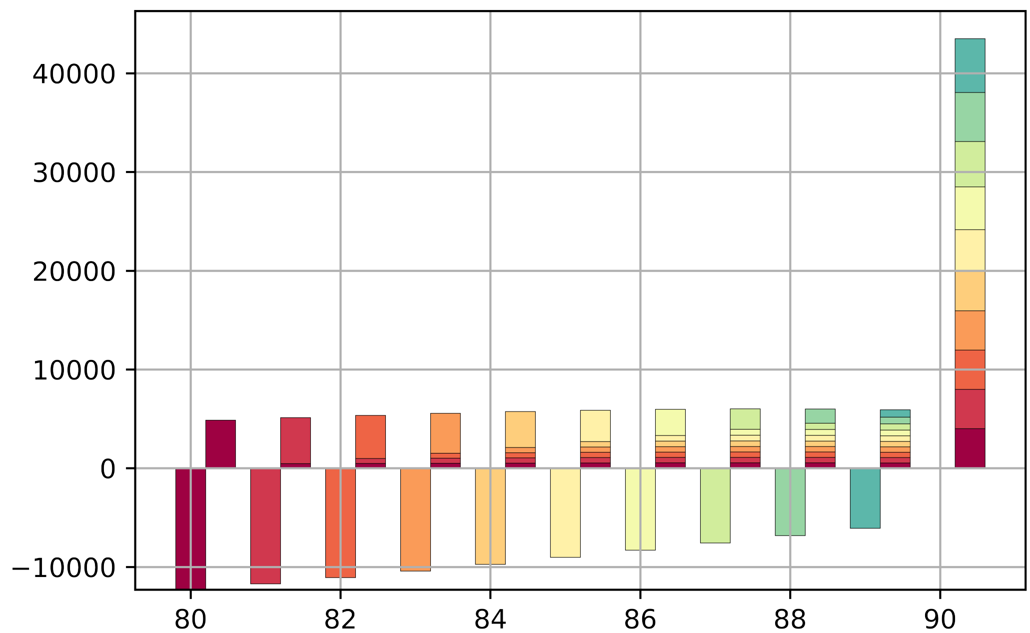

Figure 7.1: Natural decomposition of Premium

In order to risk minimise this canonical insurance feature, it is important to *not* consider future premium as given forward payments, but rather interpret the first premium to be used for:

1. First year risk premium (ie $L\,v\,q_x$), and
2. The cash flows induced by the premium assuming that no further premium are being paid.

Hence by repeating this approach, one generates the risk minimising replicating portfolio for such policies. We note that there are some additional complexities, which play in and which need to be considered in real life:

– The fact that the mathematical reserve is at inception being chosen in a conservative way. Regulation stipulates that the technical interest rate (ie the interest rate to calculate the mathematical reserve) should in normal cases not exceed 60% of government bonds (think of a 10 year average of 10 year government bonds). Hence if the prevailing interest rate is for example 3%, the technical interest rate should note exceed 1.8%.
– Bonus rate requirements of the various regulator normally ensure that the policyholder gets back a material part of this conservatism, in our example an excess return of 1.2%. This will be discussed in further detail in section 7.3.
– The demographic bases are typically chosen conservatively and are not best estimates.

Figure 7.1 illustrates how the sequence of premiums is used in a natural way to build up the entire guarantee, and it is easy to see the the company would assume quite some risk, when assuming the future premium as a guaranteed forward contract. More concretely each colour represents a premium and the corresponding resulting liability cash flows. This is even more valid in a raising interest rate environment where the incentive for the policyholder to stop paying premium is higher.

Finally it is worth noting that there are different approaches, what happens if the policyholder decides after some years to start paying the premium again. We note that there have been cases where this further option has been used to the detriment of the insurance company.

The following tables provide the replicating portfolios as per the example in figure 7.1 :

The following table shows the replicating portfolio assuming that all premiums will be paid:

Cash flow induced by all Premiums

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{Z}_{(0^+)}$	0.00	-12302.98	-12302.98	-12302.98	-12302.98
80	$\mathcal{Z}_{(1^-)}$	4880.07	0.00	4880.07	4784.39	4692.38
81	$\mathcal{Z}_{(1^+)}$	0.00	-11702.58	-11702.58	-11473.12	-11252.48
81	$\mathcal{Z}_{(2^-)}$	5125.73	0.00	5125.73	4926.69	4739.02
82	$\mathcal{Z}_{(2^+)}$	0.00	-11071.96	-11071.96	-10642.03	-10236.65
82	$\mathcal{Z}_{(3^-)}$	5355.77	0.00	5355.77	5046.86	4761.26
83	$\mathcal{Z}_{(3^+)}$	0.00	-10413.05	-10413.05	-9812.45	-9257.16
83	$\mathcal{Z}_{(4^-)}$	5563.69	0.00	5563.69	5139.99	4755.86
84	$\mathcal{Z}_{(4^+)}$	0.00	-9728.55	-9728.55	-8987.67	-8316.00
84	$\mathcal{Z}_{(5^-)}$	5742.31	0.00	5742.31	5200.98	4719.76
85	$\mathcal{Z}_{(5^+)}$	0.00	-9022.07	-9022.07	-8171.57	-7415.49
85	$\mathcal{Z}_{(6^-)}$	5883.87	0.00	5883.87	5224.71	4650.11
86	$\mathcal{Z}_{(6^+)}$	0.00	-8298.18	-8298.18	-7368.55	-6558.17
86	$\mathcal{Z}_{(7^-)}$	5980.31	0.00	5980.31	5206.22	4544.54
87	$\mathcal{Z}_{(7^+)}$	0.00	-7562.42	-7562.42	-6583.55	-5746.82
87	$\mathcal{Z}_{(8^-)}$	6023.53	0.00	6023.53	5141.02	4401.33
88	$\mathcal{Z}_{(8^+)}$	0.00	-6821.35	-6821.35	-5821.96	-4984.29
88	$\mathcal{Z}_{(9^-)}$	6005.85	0.00	6005.85	5025.43	4219.63
89	$\mathcal{Z}_{(9^+)}$	0.00	-6082.45	-6082.45	-5089.52	-4273.45
89	$\mathcal{Z}_{(10^-)}$	5920.55	0.00	5920.55	4856.92	3999.71
90	$\mathcal{Z}_{(10^+)}$	43518.32	0.00	43518.32	35700.18	29399.42
Total					**0.00**	**-5460.47**

When relaxing the assumption that all premiums are paid, we get the following natural replicating portfolios for each premium payment.

Cash flow induced by Premium at age 80

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{Z}_{(0^+)}$	–	-12302.98	-12302.98	-12302.98	-12302.98
80	$\mathcal{Z}_{(1^-)}$	4880.07	–	4880.07	4784.39	4692.38
81	$\mathcal{Z}_{(2^-)}$	473.04	–	473.04	454.67	437.35
82	$\mathcal{Z}_{(3^-)}$	494.27	–	494.27	465.76	439.41
83	$\mathcal{Z}_{(4^-)}$	513.46	–	513.46	474.36	438.91
84	$\mathcal{Z}_{(5^-)}$	529.94	–	529.94	479.99	435.58
85	$\mathcal{Z}_{(6^-)}$	543.01	–	543.01	482.18	429.15
86	$\mathcal{Z}_{(7^-)}$	551.91	–	551.91	480.47	419.41
87	$\mathcal{Z}_{(8^-)}$	555.90	–	555.90	474.45	406.19
88	$\mathcal{Z}_{(9^-)}$	554.27	–	554.27	463.79	389.42
89	$\mathcal{Z}_{(10^-)}$	546.39	–	546.39	448.23	369.12
90	$\mathcal{Z}_{(10^+)}$	4016.21	–	4016.21	3294.69	2713.21
Total					**0.00**	**-1132.86**

Cash flow induced by Premium at age 81

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
81	$\mathcal{Z}_{(1^+)}$	–	-11702.58	-11702.58	-11473.12	-11252.48
81	$\mathcal{Z}_{(2^-)}$	4652.69	–	4652.69	4472.02	4301.67
82	$\mathcal{Z}_{(3^-)}$	489.88	–	489.88	461.62	435.50
83	$\mathcal{Z}_{(4^-)}$	508.89	–	508.89	470.14	435.01
84	$\mathcal{Z}_{(5^-)}$	525.23	–	525.23	475.72	431.70
85	$\mathcal{Z}_{(6^-)}$	538.18	–	538.18	477.89	425.33
86	$\mathcal{Z}_{(7^-)}$	547.00	–	547.00	476.20	415.68
87	$\mathcal{Z}_{(8^-)}$	550.95	–	550.95	470.23	402.58
88	$\mathcal{Z}_{(9^-)}$	549.34	–	549.34	459.66	385.96
89	$\mathcal{Z}_{(10^-)}$	541.54	–	541.54	444.25	365.84
90	$\mathcal{Z}_{(10^+)}$	3980.49	–	3980.49	3265.39	2689.08
Total					**0.00**	**-964.14**

Cash flow induced by Premium at age 82

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
82	$\mathcal{Z}_{(2^+)}$	–	-11071.96	-11071.96	-10642.03	-10236.65
82	$\mathcal{Z}_{(3^-)}$	4371.62	–	4371.62	4119.47	3886.35
83	$\mathcal{Z}_{(4^-)}$	507.58	–	507.58	468.92	433.88
84	$\mathcal{Z}_{(5^-)}$	523.87	–	523.87	474.49	430.59
85	$\mathcal{Z}_{(6^-)}$	536.79	–	536.79	476.65	424.23
86	$\mathcal{Z}_{(7^-)}$	545.59	–	545.59	474.97	414.60
87	$\mathcal{Z}_{(8^-)}$	549.53	–	549.53	469.02	401.53
88	$\mathcal{Z}_{(9^-)}$	547.92	–	547.92	458.47	384.96
89	$\mathcal{Z}_{(10^-)}$	540.13	–	540.13	443.10	364.90
90	$\mathcal{Z}_{(10^+)}$	3970.19	–	3970.19	3256.94	2682.12
Total					**0.00**	**-813.50**

Cash flow induced by Premium at age 83

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
83	$\mathcal{Z}_{(3^+)}$	–	-10413.05	-10413.05	-9812.45	-9257.16
83	$\mathcal{Z}_{(4^-)}$	4033.76	–	4033.76	3726.57	3448.07
84	$\mathcal{Z}_{(5^-)}$	526.66	–	526.66	477.02	432.88
85	$\mathcal{Z}_{(6^-)}$	539.65	–	539.65	479.19	426.49
86	$\mathcal{Z}_{(7^-)}$	548.49	–	548.49	477.50	416.81
87	$\mathcal{Z}_{(8^-)}$	552.46	–	552.46	471.52	403.67
88	$\mathcal{Z}_{(9^-)}$	550.83	–	550.83	460.91	387.01
89	$\mathcal{Z}_{(10^-)}$	543.01	–	543.01	445.46	366.84
90	$\mathcal{Z}_{(10^+)}$	3991.34	–	3991.34	3274.29	2696.41
Total					**0.00**	**-678.98**

Cash flow induced by Premium at age 84

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
84	$\mathcal{Z}_{(4^+)}$	–	-9728.55	-9728.55	-8987.67	-8316.00
84	$\mathcal{Z}_{(5^-)}$	3636.59	–	3636.59	3293.78	2989.02
85	$\mathcal{Z}_{(6^-)}$	547.83	–	547.83	486.46	432.96
86	$\mathcal{Z}_{(7^-)}$	556.81	–	556.81	484.73	423.13
87	$\mathcal{Z}_{(8^-)}$	560.83	–	560.83	478.66	409.79
88	$\mathcal{Z}_{(9^-)}$	559.19	–	559.19	467.90	392.88
89	$\mathcal{Z}_{(10^-)}$	551.24	–	551.24	452.21	372.40
90	$\mathcal{Z}_{(10^+)}$	4051.85	–	4051.85	3323.93	2737.29
Total					**0.00**	**-558.55**

Cash flow induced by Premium at age 85

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
85	$\mathcal{Z}_{(5+)}$	–	-9022.07	-9022.07	-8171.57	-7415.49
85	$\mathcal{Z}_{(6-)}$	3178.42	–	3178.42	2822.35	2511.95
86	$\mathcal{Z}_{(7-)}$	571.97	–	571.97	497.93	434.65
87	$\mathcal{Z}_{(8-)}$	576.10	–	576.10	491.70	420.95
88	$\mathcal{Z}_{(9-)}$	574.41	–	574.41	480.64	403.57
89	$\mathcal{Z}_{(10-)}$	566.25	–	566.25	464.52	382.54
90	$\mathcal{Z}_{(10+)}$	4162.17	–	4162.17	3414.43	2811.81
Total					**0.00**	**-450.01**

Cash flow induced by Premium at age 86

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
86	$\mathcal{Z}_{(6+)}$	–	-8298.18	-8298.18	-7368.55	-6558.17
86	$\mathcal{Z}_{(7-)}$	2658.55	–	2658.55	2314.42	2020.28
87	$\mathcal{Z}_{(8-)}$	600.19	–	600.19	512.25	438.55
88	$\mathcal{Z}_{(9-)}$	598.43	–	598.43	500.74	420.45
89	$\mathcal{Z}_{(10-)}$	589.93	–	589.93	483.95	398.53
90	$\mathcal{Z}_{(10+)}$	4336.19	–	4336.19	3557.19	2929.37
Total					**0.00**	**-350.99**

Cash flow induced by Premium at age 87

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
87	$\mathcal{Z}_{(7+)}$	–	-7562.42	-7562.42	-6583.55	-5746.82
87	$\mathcal{Z}_{(8-)}$	2077.57	–	2077.57	1773.19	1518.06
88	$\mathcal{Z}_{(9-)}$	633.80	–	633.80	530.34	445.30
89	$\mathcal{Z}_{(10-)}$	624.80	–	624.80	512.55	422.09
90	$\mathcal{Z}_{(10+)}$	4592.52	–	4592.52	3767.47	3102.54
Total					**0.00**	**-258.82**

Cash flow induced by Premium at age 88

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
88	$\mathcal{Z}_{(8+)}$	–	-6821.35	-6821.35	-5821.96	-4984.29
88	$\mathcal{Z}_{(9-)}$	1437.67	–	1437.67	1202.98	1010.09
89	$\mathcal{Z}_{(10-)}$	674.28	–	674.28	553.15	455.52
90	$\mathcal{Z}_{(10+)}$	4956.23	–	4956.23	4065.83	3348.25
Total					**0.00**	**-170.44**

Cash flow induced by Premium at age 89

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i=2\%$	Value $i=4\%$
89	$\mathcal{Z}_{(9^+)}$	–	-6082.45	-6082.45	-5089.52	-4273.45
89	$\mathcal{Z}_{(10^-)}$	742.97	–	742.97	609.50	501.93
90	$\mathcal{Z}_{(10^+)}$	5461.13	–	5461.13	4480.03	3689.34
Total					**0.00**	**-82.18**

7.2 Surrender

Policy lapse or surrender is the process where a policyholder does not only stop paying premium, but wants to cancel the insurance contract. The consequences of such action depend on the policy terms and condition and the can grosso modo be categorised as follows:

Not possible: There are certain policies where surrender is not possible, for example for tax reasons, or in case of annuities in payment. For annuities the reason is quite easy to understand. Assume I would know that my future life span is just under one year, because I am terminally ill and I furthermore know that the Mathematical Reserve is far higher. Then surrendering the policy would clearly qualify as arbitrage against the insurer and the remaining collective of policyholders.

Possible, but no surrender value: This is typically the case for pure risk products, where no surrender value is foreseen by the respective terms and conditions.

Possible with a positive surrender value: This is quite common and the starting point for the surrender value is typically the mathematical reserve minus some penalty.

From the table above we have seen that case three is the most interesting one from an economic point of view, in terms of the definition of the penalty. Some very old policies or policies from less sophisticated market players do not foresee a penalty. Lets consider this case. We can for example take 7.1 as an example and consider that after 5 premiums paid, the policyholder can surrender for getting the Mathematical Reserves. Assume for arguments sake that after the fifth premium we have residual expected cash flows $cf_0, \dots cf_5$, noting that we assume a natural investment strategy. In such case the present mathematical reserve equals the present value of the expected cash flows:

$$V_{x+5} = \sum_{k=0}^{5} cf_k \times (1 + i_{techn})^{-k}.$$

if interest rates have not moved this is not a great deal, if the company has an invested portfolio $\{cf_k \, \mathcal{Z}_{(k)} : k = 0, \dots 5\}$. In case the interest rate has moved up to r (and we use flat yield curves to keep things simple), we see that insurer makes a loss equalling to

$$PnL = \sum_{k=0}^{5} cf_k \times (1 + r)^{-k} - V_{x+5}$$

$$= \sum_{k=0}^{5} cf_k \times \{(1+r)^{-k} - (1+i_{techn})^{-k}\}.$$

It is clear that this effect is an interest option which the insurer has written. In an economic balance sheet this would be reflected under the position ”options and guarantees”. One way to mitigate this risk is to introduce market value adjustments in case of lapse. In such case the surrender value approximates the value of the risk minimising portfolio at time of surrender, or the marketvalue adjustment equals the corresponding on the matched asset portfolio.

7.3 Policyholder Bonus

In the section above we have seen that insurance companies are required to price conservatively with respect to technical interest rates and that by design, the asset yield should be bigger than the guaranteed interest rate. This approach has worked in Europa until interest rates began to sink to historic lows, meaning that it became difficult for the insurance companies to earn enough money to serve the crediting of the interest rates. In times where interest rate are relatively 'normal' with respect to guaranteed interest rates, the policyholder is entitled to get the excess return back.

In some countries the minimal allocation to the policyholder is defined via a minimal policyholder / shareholder split. With profits funds in the UK are typically 90/10 or 100/0, which means that in the former case 90% of the excess return is allocated to the policyholder or in the latter case even 100%. Similar concepts with minimal policyholder shares between 85% and 100 % are legislated in Germany, France and Italy. As an example: assume the return on the Portfolio is 6% with a guarantee level of 2%. In this case the excess return is 4%, of which 3.6% is allocated to the policyholder and the remainder 0.4% is allocated to the shareholder.

In case of an insufficient return, say 1% in the above case, the *entire* loss is born by the shareholder. Hence one can consider the interest guarantee as a written put option from the shareholder to the policyholder in case the return of the portfolio does not reach the required technical interest rate level. It is in this context important to understand which levers the shareholder has to avoid such losses: he determines the asset allocation by means of ALM. Hence a sensible strategy avoids the zone where such losses occur. More precisely in most countries the respective regulation refer to statutory accounting, which is book value based. A consequence of this is that the minimal statutory return is an important boundary condition when doing ALM.

During the past 10 year interest rates were at historical lows and in consequence policyholder participation was minimal, which means that only guarantees have been paid out and an insurance policy could (al least for single premium business) be considered as a portfolio of zero coupon bonds. With raising interest rates, policyholder allocation will raise again and in consequence will the policyholder induced cash flows morph away for pure zero coupon bonds, which makes ALM more complex. Figure 7.2 shows such a set of cash flows – orange premium, green – guaranteed benefits and cyan – cash flows induced by policyholder bonuses. The circles on the x-axis show the respective duration, We note that we have added some virtual asset portfolio in red. One can see clearly that the duration of the total cash flows including bonus (cyan) is longer than considering only guaranteed cash flows (green). The example consists of a portfolio

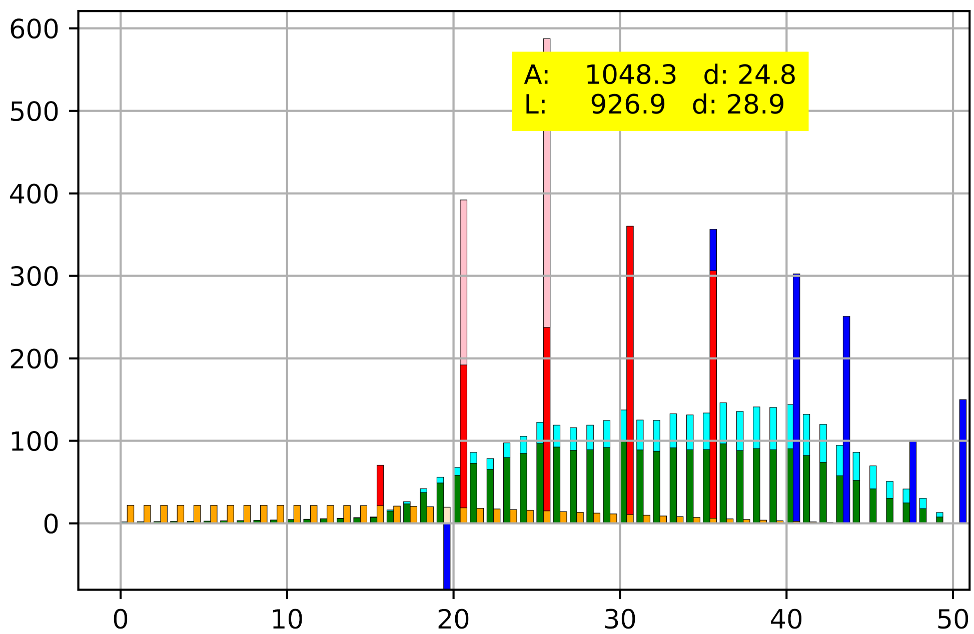

Figure 7.2: Cash flow including policyholder bonus

of endowments with regular premium payment. The liability duration for such a portfolio is quite typical.

In a next step we want to see how this excess return is treated. The different steps to determine and allocate the policyholder bonus are as follows:

1. **Year End Closing:** During the year end closing the gross yield of the investments is determined and the required interest rates plus eventual strengthening of reserves is determined. This yield in the gross profit before policyholder and shareholder split.
2. **Determining Policyholder part and high level allocation:** In this step a the gross profit is split into policyholder and shareholder part, adhering to minimal policyholder allocation rules. The part allocated to the shareholder goes in to the profit calculation of the insurance company. On the level of the policyholder the allocated policyholder bonus is further segregated in a "free part" which is not used during the year and can possibly serve as a buffer in case of bad asset returns and a part which is directly allocated to all policyholders together.
3. **Allocation to the individual policyholders:** in this step the bonus allocated for the year is distributed to the individual policyholders according to objective criteria, such as interest rate guarantee of the client, mathematical reserve etc. Also during this step the local regulatory specificities need to be observed.

4. **Use of the allocated bonus for each policyholder:** Each policyholder has signed up when acquiring the policy to rules what would happen to the bonus. Essentially there are three possibilities how the bonus can be used: a) for reduction of the regular premium; b) for increasing the benefit, and c) putting the yearly bonus in a type of bank account. In all these three cases the allocated bonus and the past accruals are guaranteed, sometimes also with a mandatory crediting rate. Below we show the mechanism how version b) works. We note that sometimes, the company has sold products with terminal bonus, where the company has more flexibility in case how to value it.

The increase in benefit level can be thought as buying some additional cover by means of paying as single premium. Assume as before a whole of life with a benefit L and a premium P. At the beginning of the year the mathematical reserve was V_x and at the end of the year $V_{x+1} = L\ A_{x+1} - P\ \ddot{a}_{x+1}$ assume for the moment that the bonus of the year was B. With this amount on can by $\Delta L = \frac{B}{A_{x+1}}$, and hence the new benefit is $\tilde{L} = L + \Delta L$.

The interaction between the asset allocation strategy and the policyholder bonus is complex.Assume for example a term for the insurance of 30 years and a guaranteed rate of 2% with market rate of 3%. For an terminal value of 100'000, the required single premium is c 55'000 but for the replication one needs only 42'000. The question is what to with the CHF 13'000? Lets assume two strategies: a) invest all in a $\mathcal{Z}_{(30)}$, and b) invest all in a $\mathcal{Z}_{(5)}$ and in 5 years all in a $\mathcal{Z}_{(25)}$. We will assume that in 5 year interest rates can be 1%, 3% or 5%. The table below shows the different maturity benefits:

	1%	3%	5%
Strategy a	132'004	132'004	132'004
Strategy b	120'826	132'004	154'993

This example clearly shows the complexity of the ALM and also that there is a risk to early lock in all gains in a low interest rate environment. We note that the same calculation can easily be made for insurance products with financing by both single and regular premium. Figure 7.3 shows a whole of life policy with a premium payment until death. The bars in blue show the premium payment and the guaranteed benefit cash flows, the bars in red the cash flows induced by the policyholder bonuses (which are used to increase the benefit level). Finally the green bars refer to the full mortality benefits to paid after a certain premium has been paid. We can easily see, that the duration of the benefits lengthens when allowing for bonus rates.

In order to formalise a ALM strategy for with profits business it makes sense to consider the following schematic balance sheet (figure 7.4). Context wise we see that the policyholder liabilities are split into three different parts, the guarantees, the options and guarantees and finally the present value of future policyholder bonus. The most difficult concept to understand is the cost of options and guarantees, and hence there is merit to quickly explain what this means. In Italy for example the withprofit funds (ie the three positions mentioned above) can be considered like a sealed box of money for the policyholder with some additional rules, such as that the book value return over a given year exceeds the technical interest rate. In case this is not met, the shareholder has to make good for it. The return of the policyholder money box is obviously dependent on the asset allocation of the respective matching assets. Hence the Cost of options and guarantees can be considered as the total return put option (over all future years) on the assets with the strike price of the minimally required return (technical interest rate). We will consider an example of this effect in section 8.3.

It is worth to note that figure 7.3 can serve different purposes, besides explanatory ones: Firstly one can segregate the ALM strategy and the corresponding intended revenue streams by category, and secondly one can manage the performance according these categories. In the latter case one needs to be careful to define when and how cash flows cross boundaries. Consider for example the cash flow matching part. Here one would prima facie not expect material gains and losses. If there were however to occur material losses one would have to transfer money in from for example the shareholder pot. Contrarily excess return typically need to be split between policyholders and shareholders and hence there would be a transfer into the corresponding pots. Ultimately the question is when and how often such transfers should take place.

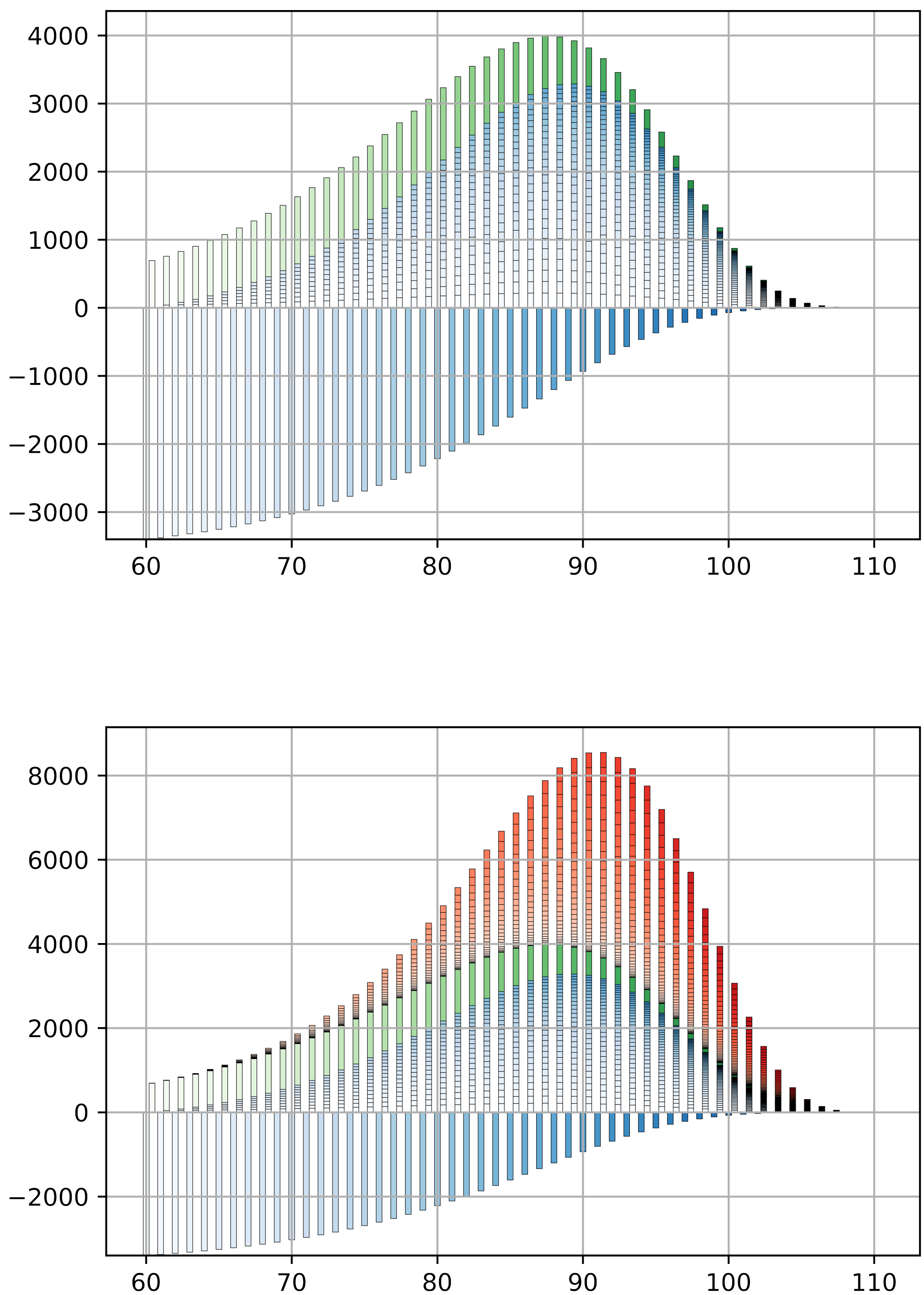

Figure 7.3: Cash flow including policyholder bonus by vintage

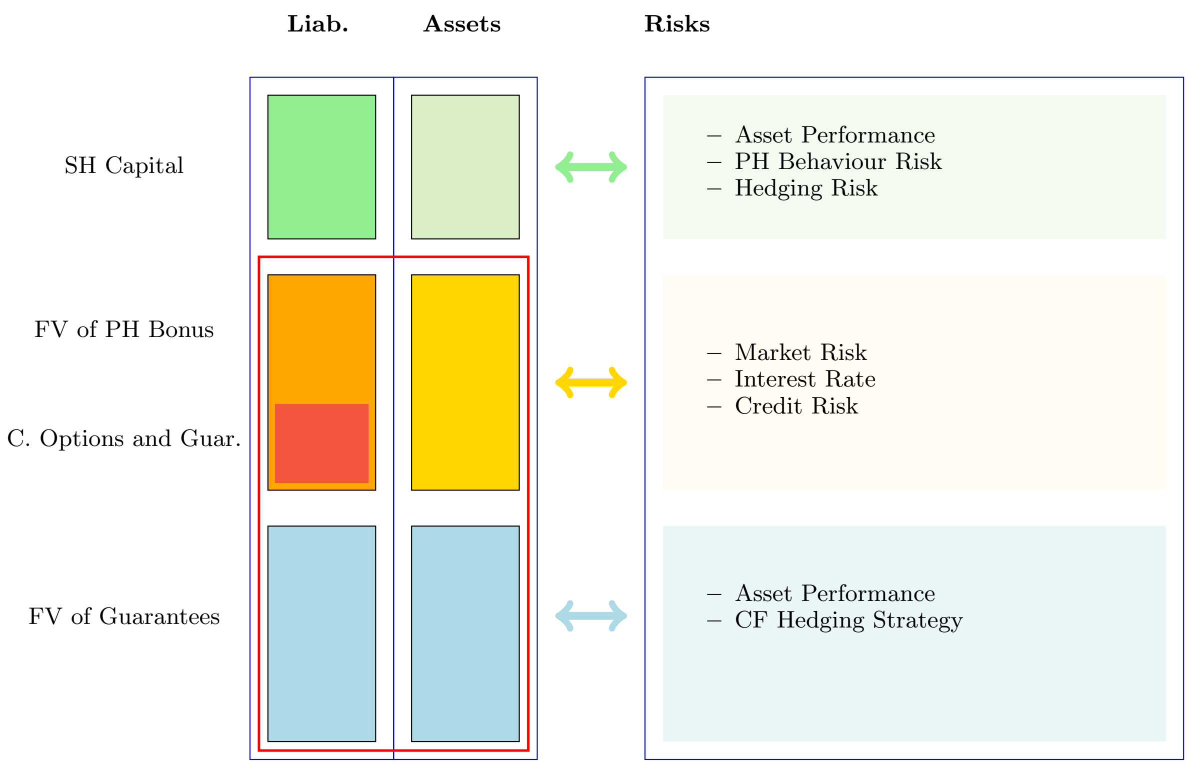

Figure 7.4: Balance sheet of an insurance company for with profits business

Finally we want to continue our example from above. We will look at the first age, where we have the following risk minimising portfolio:

Cash flow induced by Premium at age 80

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{Z}_{(0^+)}$	0.00	-12302.98	-12302.98	-12302.98	-12302.98
80	$\mathcal{Z}_{(1^-)}$	4880.07	0.00	4880.07	4784.39	4692.38
81	$\mathcal{Z}_{(2^-)}$	473.04	0.00	473.04	454.67	437.35
82	$\mathcal{Z}_{(3^-)}$	494.27	0.00	494.27	465.76	439.41
83	$\mathcal{Z}_{(4^-)}$	513.46	0.00	513.46	474.36	438.91
84	$\mathcal{Z}_{(5^-)}$	529.94	0.00	529.94	479.99	435.58
85	$\mathcal{Z}_{(6^-)}$	543.01	0.00	543.01	482.18	429.15
86	$\mathcal{Z}_{(7^-)}$	551.91	0.00	551.91	480.47	419.41
87	$\mathcal{Z}_{(8^-)}$	555.90	0.00	555.90	474.45	406.19
88	$\mathcal{Z}_{(9^-)}$	554.27	0.00	554.27	463.79	389.42
89	$\mathcal{Z}_{(10^-)}$	546.39	0.00	546.39	448.23	369.12
90	$\mathcal{Z}_{(10^+)}$	4016.21	0.00	4016.21	3294.69	2713.21
Total					**0.00**	**-1132.86**

What we see in particular is that the present value at the prevailing interest rate (4% in the table above) is -1132. Which means that, assuming a legal quote of 100%, the policyholder has a right to receive an additional 1132. ALM from a policyholder point of view determines how to invest this money. Assume we invest in a funds $\mathcal{H}$. In this case the schema above becomes:

Cash flow including Bonus Fonds

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{Z}_{(0^+)}$	0.00	-12302.98	-12302.98	-12302.98	-12302.98
80	$\mathcal{Z}_{(1^-)}$	4880.07	0.00	4880.07	4784.39	4692.38
81	$\mathcal{Z}_{(2^-)}$	473.04	0.00	473.04	454.67	437.35
82	$\mathcal{Z}_{(3^-)}$	494.27	0.00	494.27	465.76	439.41
83	$\mathcal{Z}_{(4^-)}$	513.46	0.00	513.46	474.36	438.91
84	$\mathcal{Z}_{(5^-)}$	529.94	0.00	529.94	479.99	435.58
85	$\mathcal{Z}_{(6^-)}$	543.01	0.00	543.01	482.18	429.15
86	$\mathcal{Z}_{(7^-)}$	551.91	0.00	551.91	480.47	419.41
87	$\mathcal{Z}_{(8^-)}$	555.90	0.00	555.90	474.45	406.19
88	$\mathcal{Z}_{(9^-)}$	554.27	0.00	554.27	463.79	389.42
89	$\mathcal{Z}_{(10^-)}$	546.39	0.00	546.39	448.23	369.12
90	$\mathcal{Z}_{(10^+)}$	4016.21	0.00	4016.21	3294.69	2713.21
80	$\mathcal{H}$	–	–	1132.86	–	1132.86
Total					**0.00**	**0.00**

7.4 Policyholder Bonus Strategy

In this section we will look concretely at three different policyholder bonus strategies and analyse the respective issues and merits:

- Invest the surplus ($\mathcal{H}$) into zero coupon bonds which mature according to the benefit pattern.
- Invest in a polling portfolio of credit paper and sell appropriate amount when benefit fall due.
- Invest in shares protected by put options according to the benefit pattern.

What we hence need to do is to define a investment strategy for the natural occurring bonus parts ($\mathcal{H}$) as per below. We note that the the whole mechanism is similar for all different policyholder parts, we will limit ourselves to analyse the part stemming from $\mathcal{H}_{(0)}$.

Policyholder Bonus Parts

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{H}_{(0)}$	0.00	0.00	0.00	0.00	-1132.86
81	$\mathcal{H}_{(1)}$	0.00	0.00	0.00	0.00	-964.14
82	$\mathcal{H}_{(2)}$	0.00	0.00	0.00	0.00	-813.50
83	$\mathcal{H}_{(3)}$	0.00	0.00	0.00	0.00	-678.98
84	$\mathcal{H}_{(4)}$	0.00	0.00	0.00	0.00	-558.55
85	$\mathcal{H}_{(5)}$	0.00	0.00	0.00	0.00	-450.01
86	$\mathcal{H}_{(6)}$	0.00	0.00	0.00	0.00	-350.99
87	$\mathcal{H}_{(7)}$	0.00	0.00	0.00	0.00	-258.82
88	$\mathcal{H}_{(8)}$	0.00	0.00	0.00	0.00	-170.44
89	$\mathcal{H}_{(9)}$	0.00	0.00	0.00	0.00	-82.18
Total					**0.00**	**-5460.47**

7.4.1 Zero Coupon Strategy

This is a very simple strategy, where one looks in the current rates and one is exposed to interest rate moves, as per the example as per above. For a given time t the natural allocation is according to the respective mortality or survival probabilities, ie if for $t < 8$ the proportion is ${}_{t-1}p_x \, q_{x+t}$ and for the last year ($t = 9$), we have ${}_tp_x$, noting that from age 89 to 90, some of the people will die and other survive, which is treated the same way in respect to bonus allocation. This results for the allocation of $\mathcal{H}_{(0)}$ with a value of 1132.86, in the following replicating portfolio.

Age	Unit	Part for Pay Out	Units for Bonus	Total Units	Value $i = 2\%$	Value $i = 4\%$
90	$\mathcal{Z}_{(1)}$	0.04880	73.83	73.83	72.38	70.99
90	$\mathcal{Z}_{(2)}$	0.05126	77.54	77.54	74.53	71.69
90	$\mathcal{Z}_{(3)}$	0.05356	81.02	81.02	76.35	72.03
90	$\mathcal{Z}_{(4)}$	0.05564	84.17	84.17	77.76	71.95
90	$\mathcal{Z}_{(5)}$	0.05742	86.87	86.87	78.68	71.40
90	$\mathcal{Z}_{(6)}$	0.05884	89.01	89.01	79.04	70.35
90	$\mathcal{Z}_{(7)}$	0.05980	90.47	90.47	78.76	68.75
90	$\mathcal{Z}_{(8)}$	0.06024	91.13	91.13	77.78	66.59
90	$\mathcal{Z}_{(9)}$	0.06006	90.86	90.86	76.03	63.84
90	$\mathcal{Z}_{(10)}$	0.05921	89.57	89.57	73.48	60.51
90	$\mathcal{Z}_{(10)}$	0.43518	658.36	658.36	540.09	444.77
Total					**1304.88**	**1132.86**

The advantages of the above strategy is that there is only very limited downside for both policy- and shareholders, but at the same time, there is neither upside.

7.4.2 Rolling Bond Strategy

The idea of this strategy is to have a rolling portfolio of bonds with durations $1, 2, \ldots 5$ for example and roll them over in case they mature. Hence formally one splits the 1132.86 into five equal pots and invests accordingly. The outcome is very similar to strategy b) above and has the advantage for the policyholder to outperform the zero coupon based strategy in case of raising interest rates. On the downside the result could be worse in case interest rate were low for the entirety of the term of the policy.

When investing in credit paper, the respective risk and opportunities will more pronounced, and in contrast to the rolling government bond strategy also the risk and opportunities for the shareholder will increase.

What we did not discuss yet is how to allocate the 1132.86 to the various cohorts of policyholders. Ultimately this will be done in the same way as per the example above with respect to the natural occurrence of claims. In case of such an instance happening, the respective part of the portfolio will be sold and the corresponding value being attributed to the policyholder. We note that depending on the regulatory regime and the way bonuses are defined in the contract this might also result in an additional risk for either the remaining policyholders or the shareholder. This risk materialises in case the respective part of the investment is sold below pari and the policyholder is entitled to receive the money at par.

We note that the strategy shown in this section is a dynamic strategy in the sense that the portfolio is regularly rebalanced. In a lot of simulation engines of life insurance polices dynamic portfolio strategies are used and the portfolio is rebalanced depending on the current state of the economy, the solvency ratio of the company at that point in time, etc.

7.4.3 Equities plus Put Option Strategy

The idea of this strategy is to invest the 1132.86 in shares and to allocate the respective value to the policyholder. One complexity with this strategy is that there might well be cases where the entire collective of policyholders receive less that 1132.86, which is sometime not allowed or deemed unacceptable. Hence one needs to protect the share value by respective put options according to the maturity profile of the insurance portfolio. Hence the 1132.86 need to be split into a part invested in a share and the reminder into the respective put option portfolio. Essentially this is the example in section **??**, which also establishes the link between some sorts of bonus strategies to variable annuities.

7.5 Portfolio Calculations

When doing ALM it is normally important to use efficient calculation processes since a lot of simulations are needed. Let's look at the moment at an insurance portfolio with 1 million polices. During the year end calculation, one needs normally to calculate the mathematical reserves for the past, the current and the next year, in order to save these values in the data base to be able to interpolate them for a possible policy surrender.

Assume for the moment that your tariff engine is performing such a calculation in 0.01 seconds. Hence the whole year end calculation takes you 8h 20min. When doing ALM this is obviously too long when requiring 10000 simulations, since this would result in about 3500 years run-time on the same infrastructure. Some acceleration can be gained by using a grid, but even then it seems to make sense to look for faster methods to do the above task.

In this section we will look at such approaches, which can concretely be implemented. If we consider a simple set up we have a set of policies $\mathcal{P}$, and each policy can be characterised by its Markov representation, eg the state space S_i, the discount factors (which is normally the same for all polices), the transition probabilities (which are normally structurally similar per different state space) and finally the benefits vectors $a_{ij}(t)$ and $a_i(t)$. In a lot of cases one can also restrict the state space to a common one, which we call S.

In order to be concrete we want to have a look at the set of all insurance policies on one life, be it a lump sum or an annuity. In this case we choose $S = \{*, \dagger, \ddagger\}$ as corresponding state space for a person with age ξ, where $\ddagger$ represents the state of a surrendered policy. In this set up we introduce the linear vector space of all insurance policies for this person ξ by

$$\mathcal{F}_\xi = \{x_\xi = (a_{ij}(t), a_i(t)) : i, j \in S \text{ and } t \in \mathbb{N}\}.$$

We now remark that both the mathematical reserve and the expected cash flow operators

$$\begin{aligned}
\Phi_{t,j}(x_\xi) &: \mathcal{F}_\xi \to \mathbb{R}, x_\xi \mapsto V_j(t)[x_\xi] \\
\Psi_{t,j}(x_\xi) &: \mathcal{F}_\xi \to \mathbb{R}, x_\xi \mapsto \mathbb{E}[CF(s)[x_\xi] | X_{t_0} = j]
\end{aligned}$$

are linear (continuous) functionals from $\mathcal{F}_\xi \to \mathbb{R}$, where ξ denotes the policy considered, t and s the respective times and $j \in S$ a state. When recognising this fact we can now construct the space of all insurance policies for a given portfolio $\mathcal{P} = \{\xi_1, \xi_2, \dots \xi_n\}$ by defining the respective Cartesian product such as:

$$\begin{aligned} S &= \prod_{i \in \mathcal{P}} S_i, \text{ and} \\ \mathcal{F} &= \prod_{i \in \mathcal{P}} \mathcal{F}_i. \end{aligned}$$

In the same sense we can now define the mathematical reserve and expected cash flow operator of the whole insurance portfolio as the corresponding sum:

$$\Phi_t(x) : \mathcal{F} \to \mathbb{R}, x((\xi_i)_{i \in \mathcal{P}}) \mapsto \sum_{i \in \mathcal{P}} V_{j(\xi_i)}(t)[x_{\xi_i}] \text{ and}$$
$$\Psi_t(x) : \mathcal{F} \to \mathbb{R}, x((\xi_i)_{i \in \mathcal{P}}) \mapsto \sum_{i \in \mathcal{P}} \mathbb{E}[CF(s)[x_{\xi_i}] | X_{t_0} = j(\xi_i)].$$

Until now we have gained nothing except for a more complex representation of what we already know. The way we can now make the whole thing much more efficient is to use the given structure and the fact that the two operators defined above are linear. In the concrete set up where each policy is characterised by the three states above we can define a new "pseudo" state space

$$\tilde{S} = \{x0, x1, \dots, x120, y0, y1, \dots, \dagger, \ddagger\},$$

where the states $x0, \dots, x120$ stand for males which are alive and have the respective age at time t_0. Similarly $y0, \dots, y120$ stand for the respective females. There is only a need to map the respective $x_\xi \in \mathcal{F}_\xi$ into $\tilde{\mathcal{F}}$.

Assume that $\mathcal{R}50 \subset \mathcal{P}$ is a homogeneous set of policies representing the males ages 50. In this case we have the following benefit functions:

$$\begin{aligned} a_{x50}^{Pre} &= \sum_{\xi \in \mathcal{R}50} a_*^{Pre}(\xi) \\ a_{x50,x50}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,*}^{Post}(\xi) \\ a_{x50,\dagger}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,\dagger}^{Post}(\xi) \\ a_{x50,\ddagger}^{Post} &= \sum_{\xi \in \mathcal{R}50} a_{*,\ddagger}^{Post}(\xi) \end{aligned}$$

Please note that in order to implement the approach described above, one needs to be careful with respect to the definition of time. In the usual Markov model the time t is calculated with respect to the age of each policyholder in a certain year. For the above purpose, it is useful to enumerate by the number of years into the future, starting at $t = 0$. Hence one needs to correspondingly adjust the time of the benefit functions. We remark that for all the other states in $\tilde{S} \setminus \{\dagger, \ddagger\}$ the same approach is used. It is also worth noting that this "pseudo" Markov chains can be interpreted as "normal" Markov chains, where the initial state of the portfolio is given by a probability distribution, over which one integrates.

After having done so, we can now perform a lot of the calculations much faster. After having applied the above mapping into $\tilde{\mathcal{F}}$, the calculations do not depend anymore on the actual size of the portfolio. By this we gain considerable amounts of times when doing the actual calculations. Looking at the sample portfolio above we would possibly use 1h calculation time to perform the mapping on $\tilde{\mathcal{F}}$, including the data base queries. Once this is done the calculation of the mathematical value and expected cash flow operator take some 1 to 2 seconds on a common laptop. Hence doing ALM in a lot of cases result in run-times of several minutes. In the same sense stress scenarios can be calculated much faster.

Finally I would like to mention that this approach has been applied concretely for the examples in section 7.6 using 402 states for $\tilde{\mathcal{F}}$ by also splitting annuities from capital insurance. Using this approach one can also map deferred widows pension using a collective approach and hence one can cover the vast majority of traditional life insurance covers sold by a life insurance company. Since structurally the method is a slight variation of the Markov recursion, this method can actually be implemented using the same core Markov calculation objects.

7.6 Portfolio Dynamics and ALM

The aim of this section is to look at the portfolio dynamics induced by the relationship between assets and liabilities and the corresponding asset liability management (ALM). To this end we fix $(\Omega, \mathcal{A}, P)$, together with a filtration $\mathcal{G} = (\mathcal{G}_t)_{t\in\mathbb{N}}$. We assume that we have represented the benefits and premiums in a suitable ("pseudo") Markov model with state space $\tilde{S}$ and benefits vector space $\tilde{F}$.

In this case we can represent the benefits for the entire portfolio by $x = ((a_{i,j}^{Post}(t))_{i,j\in\tilde{S}}, (a_i^{Pre}(t))_{i\in\tilde{S}})_{t\in\mathbb{N}} \in \tilde{F}$, and we know that we can decompose this into

$$x = x^{Benefits} + x^{Premium},$$

where $x^{Benefits}$ and $x^{Premium}$ represent the corresponding benefit vectors in $\tilde{\mathcal{F}}$ for benefits and premiums, respectively. At this point the bonus concept enters. If for example the benefits are increased, the corresponding $x^{Benefits}$ is increased accordingly. Formally one introduces hence a random vector α_t which represents the relative benefit level. This means that $\alpha_0 = 1$ and also that α is previsible. For traditional bonus promises, the bonus allocated to the individual policy become a guarantee, means that α is increasing in t for each trajectory and hence we define the new benefit representation of the policy at time t as follows:

$$\hat{x}^{Benefits} = ((\alpha_t \times a_{i,j}^{Post}(t))_{i,j\in\tilde{S}}, (\alpha_t \times a_i^{Pre}(t))_{i\in\tilde{S}})_{t\in\mathbb{N}},$$

remarking that this is now a random quantity, since α is a random vector. In consequence the entire portfolio after bonus allocation is given by

$$\hat{x} = \hat{x}^{Benefits} + x^{Premium}.$$

The mechanism to increase α is performed by using the allocated bonus as a single premium. In a next step we want have a look at a concrete example. The first step is to define a stochastic model, which generates the corresponding states of the world. In the concrete set up we are assuming a world with a constant interest rate with one risky asset (say a share) with has a non constant, stochastic volatility. We are using the Heston model, which is given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t^1 \\ dS_t &= \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^2. \end{aligned}$$

This model is given by two stochastic differential equations, where the first equation describes the volatility of the shares. This volatility process has a structural similarity with the interest rate models we have seen before, in the sense that the volatility process is also mean reverting. In order to solve this stochastic differential equation system we can use a numerical method such as the Milstein scheme (see for example [KP92]). After having done this, it is important to understand how the simulation works. In principle one does first a loop over the different simulation and calculates the quantities, which need to be analysed. The following code performs the corresponding task:

```
1  # 4. Stepper
2  # ---------------------------
3  for i in 1... n:
4      sim.vNewTrajectory()
5     (pl, cf, ... ) = CalcPV(sim, lvp, lvl)

    ... keep and analyse results for run i
```

Hence first one generates a new trajectory of the world (line 4, in this case based on the Heston model), and secondly on calculates the development of the insurance portfolio, as outlined above in the subroutine `CalcPV` (line 5). Hence it makes sense to look at this part of the code in order to understand what happens:

```
for i in range(1, MaxTime+1):
        TargetSurplus = ... target surplus level required
        perf = ((1+iRF) * (1-EquityProportion) +
              EquityProportion* sim.dGetValue(1,float(i)) / sim.dGetValue(1,float(i
        cf[i] = psi[i-1] * lvl.dGetCF(i) + lvp.dGetCF(i)
        l[i] = psi[i-1] * lvl.dGetDKTilde(i) + lvp.dGetDKTilde(i) - cf[i]
        a[i] = a[i-1] * perf - cf[i]
        ExAssets = a[i] - l[i]
      # Calculate Excess Assets and Pl, afterwards adjust assets by pl for SH
        if ExAssets > 0.:
             ExAssets = max(0., ExAssets - TargetSurplus)
```

```
        Bonus = ExAssets * RelBonusAllocation * LegalQuote
        pl[i-1] = ExAssets * RelBonusAllocation * (1-LegalQuote)
    else:
        Bonus = 0.
    pl[i-1] = ExAssets
    a[i] -= pl[i-1]
    e[i] += pl[i-1]
    psi[i] = psi[i-1] + Bonus / (psi[i-1] * (lvl.dGetDKTilde(i) - lvl.dGetCF(i
```

What this code does, is the following:

1. In a first step, the minimal required surplus of the policyholder funds (line 1) and the performance of the assets (line 2) is calculated. We see that in the concrete set up we have a mixture of assets. A part of them yielding risk free and the reminder, the equity portion having an equity yield.
2. In lines 4,5 and 6 the cash-flows and the assets and liabilities are calculated at the end of the period. We see that

$$\begin{aligned} \Psi_i(\hat{x}^{Benefits}) &= \texttt{psi[i-1] * lvl.dGetCF(i)}, \text{ and} \\ \Psi_i(x^{Premium}) &= \texttt{lvp.dGetCF(i)}. \end{aligned}$$

3. In a next (lines 7 to 15) step the excess assets are calculated in order to determine, whether there is a bonus in the corresponding year. If there are excess assets (line 9), the bonus is calculated.
4. Finally the benefit level for the subsequent year is calculated (line 18)

We remark that the initialisation of the code plus the analysis have been left in order to focus on the essential parts of the calculation.

After having done this, we want to have a look at some sample output of the program. Figure 7.5 and 7.6 show the mathematical reserves and the expected cash flows for the benefits and the premiums respectively. Moreover figure 7.7 show the quantiles of the profit and loss over time for the 5%, 10%, 33%, 50 %, 67%, 90% and 95% quantiles for both the profit and loss account and the corresponding dividends. We remark that this figure shows that the underlying portfolio is insufficiently financed and suffers considerable losses between the time 5 and 15, and also that the losses start earlier in cases where we observe an adverse equity performance.

The output of the program looks as follows:

```
Input Parameter / Main Results
------------------------------

Simulations                                  5000
>>   PV Benefits ..... .....DK 1   =  22,786,998,031
>>   PV Premium. ..... .....DK 0   =  -4,091,249,171
>>   Mathematical Reserves..DK     =  18,695,748,860
>>   Underlying Assets .....A0     =  22,635,000,000
```

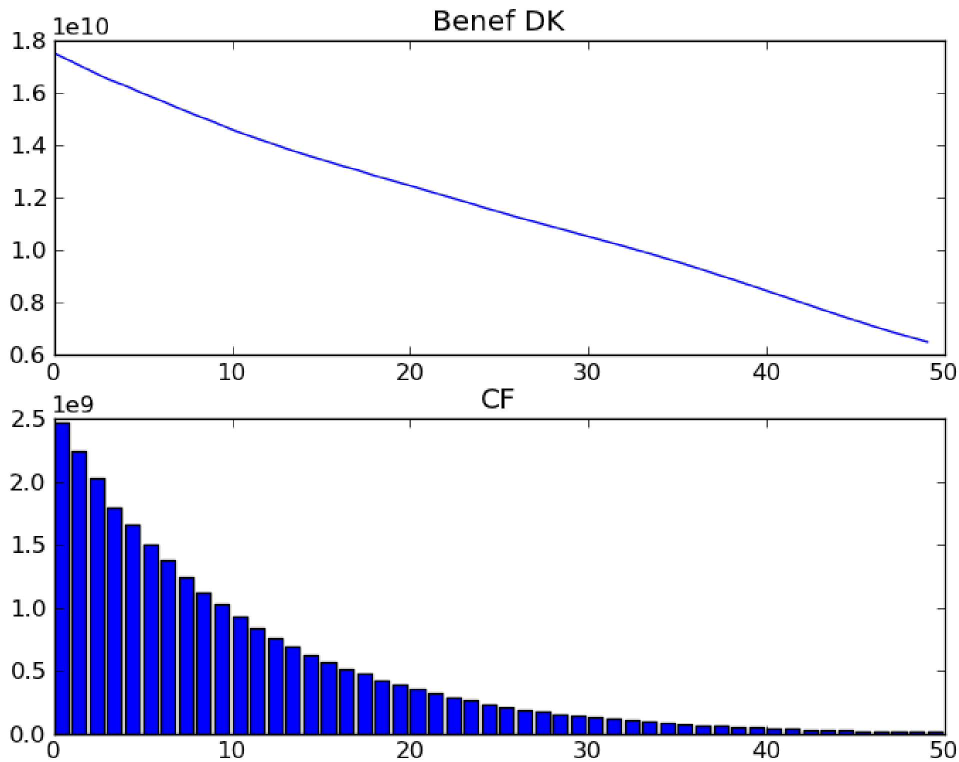

Figure 7.5: Mathematical reserve and expected cash flows of portfolio (benefits only)

```
>>   Shareholder Equity.....E0     =   1,584,450,000

Distribution of Economic Profit
-------------------------------

Min Return                                    0.0962
Max Return                                    0.2473
Average                                       0.1792
 0.5%  Quantile                               0.1288
 1.0%  Quantile                               0.1324
 5.0%  Quantile                               0.1452
10.0%  Quantile                               0.1529
25.0%  Quantile                               0.1654
99.5%  Quantile                               0.2289


Time used
```

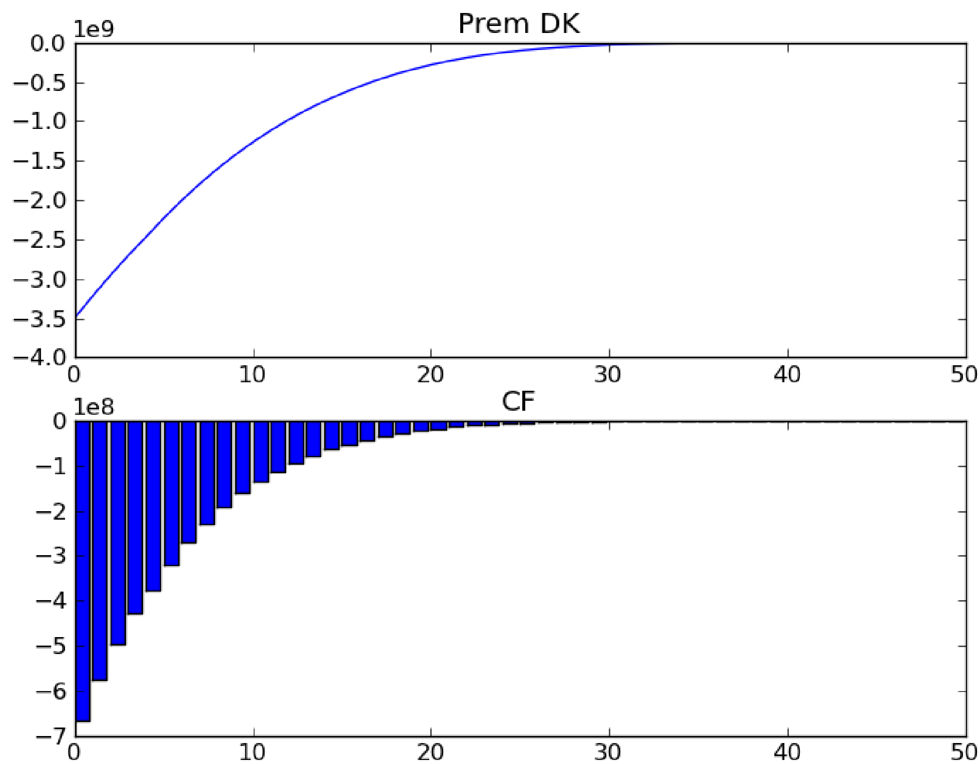

Figure 7.6: Mathematical reserve and expected cash flows of portfolio (premiums only)

```
---------

Step              Calc EP Time used     2.3515 s
Step              Calc PF Time used     3.6497 s
Step              Calc PF Time used     3.2491 s
Step            A0 and E0 Time used     0.0190 s
Step   Simulation Stepper Time used    81.8312 s

Time used for preparation               9.2 s
Time used for simulations              81.8 s

Time per simulation                     0.0163 s
```

We see in particular that the implementation via the "pseudo" Markov model results in an extremely fast calculation of the portfolio in about 9 seconds. Also the simulation is performed very fast, despite the fact that a relatively slow laptop was used. The output of the above run is summarised also in figure 7.8.

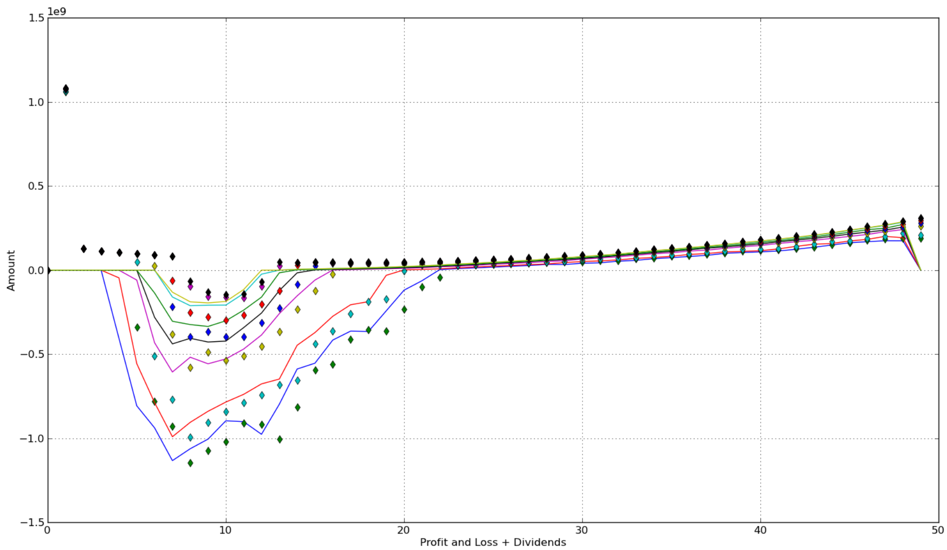

Figure 7.7: Quantiles for profits and losses

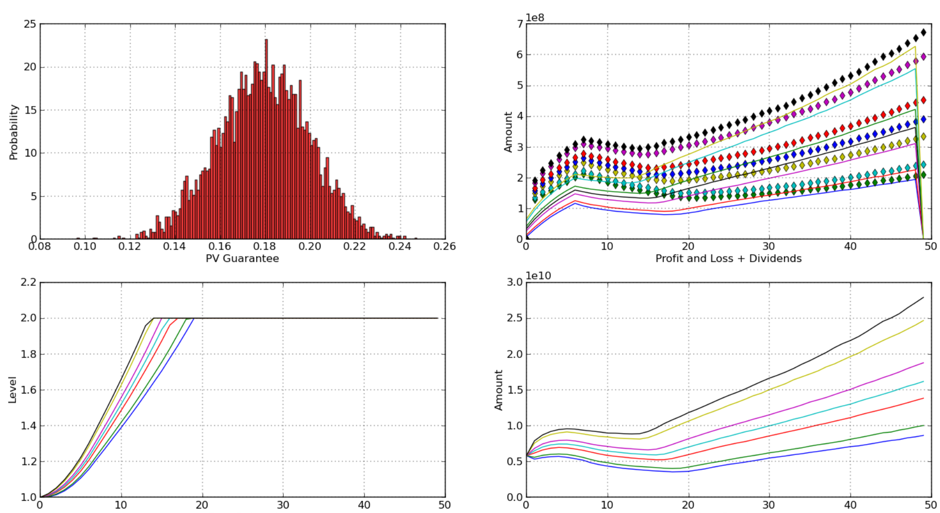

Figure 7.8: Summary of ALM analysis

8 Optimisation Targets and Boundary Condition

8.1 What is ALM?

We have seen that ALM is a process where the overarching aim is to determine Assets and Liabilities in an "optimal" way. Given that the change in liability profile with exception of reinsurance is a slow process – ie you wait until policies mature, create a new sub-portfolio or sell a (sub-) portfolio, it is normally boiling down to the question of asset allocation, together with the respective trading strategies on the asset side and the determining of bonus sharing strategies and bonus allocation of the liability side. This is also why investment in life insurance companies is sometimes characterised as liability driven investment.

There are various complexities when doing ALM, such as:

What does mean optimal? What are the dimensions you want to be optimal? There are many things which you can optimise for, which are sometimes not aligned. Do we want for example to optimise for Policyholder or Shareholder outcome? Do we want to optimise for remittences, capital – which type of profit and loss approach etc. One of the important learnings of ALM is that is advisable to define ex ante metrics which define the relevant dimensions which you want to opimize for. Typically it is also advisable to have a balanced approach with respect to stakeholders and also with respect to metrics chosen. Ultimately a mathematical optimisation needs a leading quantity for which on is optimising for, but one needs at the end look what the chosen strategy means for the other competing priorities.

How to define strategies over a long time horizon? One of the main challenges when doing ALM is the time horizon one is considering. In particular when optimising statutory returns, given the path dependency of the results it is more difficult to do ALM over multiple periods. It is important to recognise the different characteristics of Premium, Guaranteed Benefits and Policyholder bonus. The former two are more deterministic, than the later one, which could necessitate to bifurcate the ALM exercise along this lines.

How to handle model complexities and the respective model risk? In the same sense as per above when projecting dynamic asset allocation strategies the is a material amount of model risk, mainly because it is very difficult to define reliable management action ex ante and as consequence of the accumulation of errors, the uncertainly for later years is materially higher. As a consequence it is important to consider multiple strategies and ensure that the results of the more complex ones are commensurate to the similar, simpler strategy. This later effect is closely linked to policyholder behaviour and the corresponding induced changes to cash flows. Ultimately some policyholder behaviour requires material amount of judgment.

In order to compensate for this uncertainty it is advisable to run corresponding scenarios with the aim that the chosen strategy is rather robust with respect to these scenarios.

How to define a process which equally represents the various key stakeholders? ALM is an optimisation where one can optimise for one stakeholder only. The risk with this is that it puts ad disadvantage the other stakeholder, which ultimately can result in unwanted behaviour. Consider for example optimising shareholder return and for example fixing the bonus rates at inception. In this case policyholder would receive less policyholder bonus in a rising interest rate environment. This could ultimately lead to higher (possibly avoidable) lapses. As a consequence of this it is important to ex ante define the respective weight of the respective stakeholder to eventually avoid an extreme optimisation stance.

How to ensure that the key stakeholders speak the same language and have a similar core understanding? When doing ALM in practise one of the most difficult areas is the way the different functions work together (eg CFO, CRO, Actuaries and Investment Management). Normally it not the willingness to collaborate, but the fact that the different parties have an insufficient understanding of the other parties and do not speak the same language. As a consequence it is advisable to train people up in such a way that this issue is mitigated, resulting in an overall better outcome.

8.2 Optimisation Target and Boundary conditions

In order to determine an SAA and doing ALM the idea is to optimise one or more target variables with some additional boundary conditions. The below table provides an example:

Dimension	**Type**	**Remarks and addl. mitigation possibilities**
Economic Value Creation	Primary Optimisation	S2 available capital as a proxy
GAAP P&L and GAAP Equity	Secondary Optimisation (as per ¿0 and max loss absorption as per Statutory Equity)	– Book Value based Assets and Liabilities with exception of Derivatives witch are MTM. – Hedge Accounting for Derivatives.
Solvency Capital Level	Constraint – $>150\%$ and – $>110\%$ after a 1:20 event.	Can use reinsurance to partially hedge.
Policyholder Returns / PH Bonus	Secondary Optimisation, Constraint – As a consequence of legal quote: "book return – technical interest rate – additional requirements for reserve strengthening should be >0". – Expected PH returns should be achieved in 4 out of 5 years.	– Note mainly book value based on both asset and liability side. – Legal quote combined with statutory loss economically value destructing from SH point of view. – Need to have adequate PH Bonus returns to remain competitive and avoid undue losses.

8.3 Brave New World

In the not so distant past ALM consisted mainly considering statutory balance sheets. By now it is best practise to consider and potentially optimise for economic value creation. In this section we want to explain how to address this problem concretely and to see what such an approach means re potential boundary conditions. We note that we do some simplifications to focus on the main critical aspects. For the purpose of explaining this effect, we use the same example as per above, with the only modification that we finance the policy by means of a single premium of 86253.38. The table below show the respective replicating portfolio, and we see that amount which the policyholder is entitles to as present value of future bonus (including any other expenses like cost of options and guarantees is 11370.35 (symbol $\mathcal{H}$). For sake of simplicity we assume that there is a 100/0 policyholder/ shareholder split, which means that the entire difference (ie 11370.35) belongs to the policyholder.

Age	Unit	Units for Mortality	Units for Premium	Total Units	Value $i = 2\%$	Value $i = 4\%$
80	$\mathcal{Z}_{(0^+)}$	0.00	-86253.38	-86253.38	-86253.38	-86253.38
80	$\mathcal{Z}_{(1^-)}$	4880.07	0.00	4880.07	4784.39	4692.38
81	$\mathcal{Z}_{(2^-)}$	5125.73	0.00	5125.73	4926.69	4739.02
82	$\mathcal{Z}_{(3^-)}$	5355.77	0.00	5355.77	5046.86	4761.26
83	$\mathcal{Z}_{(4^-)}$	5563.69	0.00	5563.69	5139.99	4755.86
84	$\mathcal{Z}_{(5^-)}$	5742.31	0.00	5742.31	5200.98	4719.76
85	$\mathcal{Z}_{(6^-)}$	5883.87	0.00	5883.87	5224.71	4650.11
86	$\mathcal{Z}_{(7^-)}$	5980.31	0.00	5980.31	5206.22	4544.54
87	$\mathcal{Z}_{(8^-)}$	6023.53	0.00	6023.53	5141.02	4401.33
88	$\mathcal{Z}_{(9^-)}$	6005.85	0.00	6005.85	5025.43	4219.63
89	$\mathcal{Z}_{(10^-)}$	5920.55	0.00	5920.55	4856.92	3999.71
90	$\mathcal{Z}_{(10^+)}$	43518.32	0.00	43518.32	35700.18	29399.42
80	$\mathcal{H}$	0.00	0.00	11370.35	–	11370.35
Total					**0.00**	**0.00**

If we were to consider only an economic balance sheet one could consider no additional boundary condition. In most regulatory regimes, there is however an additional boundary condition induced by the statutory (conservative accounting regime). To this end we need to remind ourselves what happens on the statutory balance sheet. Here we need to look at the valuation rate of ($i = 2\%$), and we see that the mathematical reserve (${}_0V_x$) is 0. The statutory requirement is that at a given point in time the assets covering the liabilities exceed said mathematical reserves and that the yield on the assets exceeds the required technical interest rate - ie in the first year one needs to earn at least $i \times ({}_0V_x) + P$), where P denotes the single premium as per above.

In order to formulate the boundary condition we need to know the respective mathematical reserves, which for this example are as follows:

```
Age            Math. Reserve
80 ---> Vx:       0.00
81 ---> Vx:   87361.69
```

```
82 ---> Vx:  88488.61
83 ---> Vx:  89641.95
84 ---> Vx:  90832.14
85 ---> Vx:  92073.15
86 ---> Vx:  93383.75
87 ---> Vx:  94789.43
88 ---> Vx:  96325.10
89 ---> Vx:  98039.21
90 ---> Vx: 100000.00
```

To determine the boundary condition at time 1 (ie ${}_1V_{80} = 87361.69$) we need to decompose our liability and in particular $\mathcal{H}$ further. We have 11370.35 units of $\mathcal{H}$ to split into the people dying within the first year ($q_{80} = 0.04880$), hence the surviving cohort will get the proportion of $1 - q_{80}$ - lets call this part $\mathcal{H}_1$. Furthermore, we need to be cognisant, that the inforce has split into people surviving and hence all nominal amounts of zero coupon bonds will increase by a factor of $\frac{1}{1-q_{80}}$. This will lead to the following ALM schema. We note to make effects more pronounced we have have assumed that assets are valued at $t = 2$ with forward interest rate of 3%.

Lia. Instr.	Lia. Units	Assets Instr.	Assets Units	Lia. Value	Assets Value	Difference Value
$\mathcal{H}_1$	10815.47	$\mathcal{X}$	10815.48		x	x
$\mathcal{Z}_{(1)}$	5388.70	$\mathcal{Z}_{(1)}$	5388.70	5283.04	5231.75	-51.29
$\mathcal{Z}_{(2)}$	5630.54	$\mathcal{Z}_{(2)}$	5630.54	5411.90	5307.32	-104.58
$\mathcal{Z}_{(3)}$	5849.13	$\mathcal{Z}_{(3)}$	5849.13	5511.76	5352.78	-158.98
$\mathcal{Z}_{(4)}$	6036.91	$\mathcal{Z}_{(4)}$	6036.91	5577.17	5363.72	-213.46
$\mathcal{Z}_{(5)}$	6185.73	$\mathcal{Z}_{(5)}$	6185.73	5602.61	5335.87	-266.74
$\mathcal{Z}_{(6)}$	6287.12	$\mathcal{Z}_{(6)}$	6287.12	5582.78	5265.37	-317.42
$\mathcal{Z}_{(7)}$	6332.56	$\mathcal{Z}_{(7)}$	6332.56	5512.87	5148.95	-363.92
$\mathcal{Z}_{(8)}$	6313.97	$\mathcal{Z}_{(8)}$	6313.97	5388.91	4984.31	-404.61
$\mathcal{Z}_{(9)}$	6224.30	$\mathcal{Z}_{(9)}$	6224.30	5208.21	4770.40	-437.81
$\mathcal{Z}_{(9)}$	45750.97	$\mathcal{Z}_{(9)}$	45750.97	38282.36	35064.31	-2318.8
Total				**87361.63**	**81824.77 + x**	Δ

What you can see from the above schema is the following:

- The value of the liabilities discounted at the technical interest rate matches the mathematical reserves; and
- One needs a part of $\mathcal{X}$ to match the mathematical reserves, given the present value of the replicating portfolio (ie $\mathcal{Z}_{(x)}$) is too small.

The equation which needs to be valid at the beginning of year two is:

$$87361.63 \quad \leq \quad 81824.77 + \pi\,(\mathcal{X})\,.$$

If one solves this equation, one get that the price of $\mathcal{X}$ needs to be bigger than c51% of its price at the beginning. This is where we get the link back to the cost of options and guarantees and also to variable annuities. Typically the company wants to protect itself against such a "burn through" of the withprofit funds. Hence at time $t = 0$ having an excess value of 10815, one would like to invest in ξ shares ($\mathcal{S}$) and the same amount of put options ($\mathcal{P}$) with a strike price to avoid burn through, which matches the respective price. Hence ξ is determined by the following equation:

$$\xi \times (\pi(\mathcal{S}) + \pi(\mathcal{P})) = 11370.35,$$

assuming we want to have the same cover also for the first period. We note that is in this case easy to calculate the various parts, using for example the Black-Scholes-Merton formula. Assume for the moment that $\pi(\mathcal{P}) = 0.2\,\pi(\mathcal{S})$. In this case we have $\xi = 9475.29$. This leads us finally to the following ALM schema, which adheres to the mentioned boundary condition:

Lia. Instr.	Lia. Units	Assets Instr.	Assets Units	Lia. Value	Assets Value	Difference Value
$\mathcal{Z}_{(0^+)}$	-86253.38	$\mathcal{Z}_{(0)}$				
$\mathcal{Z}_{(1^-)}$	4880.07	$\mathcal{Z}_{(1)}$	4880.07	4784.38	4692.38	-92.01
$\mathcal{Z}_{(2^-)}$	5125.73	$\mathcal{Z}_{(2)}$	5125.73	4926.69	4739.03	-187.67
$\mathcal{Z}_{(3^-)}$	5355.77	$\mathcal{Z}_{(3)}$	5355.77	5046.86	4761.26	-285.60
$\mathcal{Z}_{(4^-)}$	5563.69	$\mathcal{Z}_{(4)}$	5563.69	5139.99	4755.87	-384.12
$\mathcal{Z}_{(5^-)}$	5742.31	$\mathcal{Z}_{(5)}$	5742.31	5200.99	4719.76	-481.23
$\mathcal{Z}_{(6^-)}$	5883.87	$\mathcal{Z}_{(6)}$	5883.87	5224.71	4650.11	-574.60
$\mathcal{Z}_{(7^-)}$	5980.31	$\mathcal{Z}_{(7)}$	5980.31	5206.22	4544.54	-661.68
$\mathcal{Z}_{(8^-)}$	6023.53	$\mathcal{Z}_{(8)}$	6023.53	5141.02	4401.33	-739.69
$\mathcal{Z}_{(9^-)}$	6005.85	$\mathcal{Z}_{(9)}$	6005.85	5025.43	4219.63	-805.80
$\mathcal{Z}_{(10^-)}$	5920.55	$\mathcal{Z}_{(10)}$	5920.55	4856.91	3999.71	-857.20
$\mathcal{Z}_{(10^+)}$	43518.32	$\mathcal{Z}_{(10)}$	43518.32	35700.18	29399.42	-6300.76
$\mathcal{H}$	11370.35			–		–
		$\mathcal{S}$	9475.29		9096.28	9096.28
		$\mathcal{P}$	9475.29		2274.07	2274.07
Total				**86253.38**	**86253.38**	**0.00**

We conclude this section by remarking that the above example is very simplistic to the extent, that we consider only one boundary condition and also only one time period. Furthermore we assume that lapse is at market value, hence that lapse does not give raise to addition optionality on the asset side. When addressing the above issues one, normally would consider the additional arbitrage opportunities as additional options on the asset side, which means that the free part of the future bonus (aka $\mathcal{S}$) in the above example gets smaller and smaller. Furthermore in real world, it is a too heroic ambition to explicitly calculate the options, particularly since we have to look at much more complex options - type of rainbow options, or options involving multiple asset categories - such as total return swaps, and in consequence often simulation methods are applied which makes the analysis of the results more challenging.

9 Unit-linked policies

9.1 Introduction

Up to now we have mostly considered models with deterministic interest rates or with an interest rate given by a Markov chain on a finite state space. This helped us to keep the calculations simple. In this chapter we have a look at variable annuities. We consider models for policies whose actual value depends on the performance of an underlying unit (usually a funds). Since we do not need at the beginning the entire complexity of a Markov model for the underlying demographic process, we will start using the simpler traditional approach. In case the reader is not aware of this we will formally introduce this below.

Definition 9.1.1. *A classical life insurance model consists of a Markov model with a state space $S = \{\star, \dagger\}$, where $\star$ and $\dagger$ represent the states of being alive or death, respectively. We assume that the state $\dagger$ is absorbing (ie $p_{\dagger,\star} \equiv 0$. If we assume that the person is alive at age x (ie $X_x = \star$), we can define the future life span $T_x$of a person aged x as*

$$T_x \quad = \quad min\{\xi > x | X_\xi = \star\} - x,$$

and we use the following (common) notations (for $\Delta_x > 0$, $t > 0$):

$$\begin{aligned}
{}_tq_{x+\Delta_x} &= P(T_x \leq t + \Delta_x | T_x > \Delta_x) \\
{}_tp_{x+\Delta_x} &= P(T_x > t + \Delta_x | T_x > \Delta_x) \\
\mu_{x+\Delta_x} &= \lim_{\Delta \to 0} \frac{{}_\Delta q_{x+\Delta_x}}{\Delta}
\end{aligned}$$

Remark 9.1.2. In the same sense we can allow for a slightly refined insurance model allowing for lapse (or also called surrender). In this case the state space consists of $S = \{\star, \dagger \ddagger\}$, where $\ddagger$ represents the additional state of lapse. It is worth noting that also this state is absorbing. Furthermore, we have in this case (as also in the above case)

$$\begin{aligned}
{}_tp_x &= p_{\star,\star}(x, x+t) \\
&= \bar{p}_{\star,\star}(x, x+t).
\end{aligned}$$

We begin with a look at policies whose value is tied to a bond or a funds. The payout of these so called "unit-linked policies" usually consists of a certain number of shares of a funds (the underlying unit) to the insured, in case of an occurrence of the insured event.

These policies have the characteristic feature that the level of the benefits (endowments or death benefits) are not deterministic, but random, depending on the underlying funds. A unit-linked policy is usually financed by a single premium. This type of financing is preferred due to the management of these policies. Note that this is in contrast to traditional policies. Moreover one has to note that the value at risk is constant for a traditional policy, but for a unit-linked policy it depends on the underlying funds.

To analyse unit-linked policies, we introduce the following notation:

$N(t)$	number of shares at time t,
$S(t)$	value of a share at time t.

We assume that $N(t)$ is deterministic. The relevant quantities for a life insurance in this setting and in the traditional setting are summarised in the following table:

	traditional	(pure) unit-linked
death benefit	$C(t) = 1$	$C(t) = S(t)$
value (time 0)	$\pi_0(t) = \exp(-\delta t)$	$\pi_0(t) = S(0)$ (assuming a "normal" economy)
single premium	$E\left[\int_0^T \pi_0(t) d(\chi_{T_x \leq t})\right]$ $= \int_0^T \exp(-\delta t)\, {}_tp_x \mu_{x+t} dt$	$E\left[\int_0^T \pi_0(t) d(\chi_{T_x \leq t})\right]$ $= S(0) \int_0^T {}_tp_x \mu_{x+t} dt$ $= (1 - {}_Tp_x) S(0)$

The above terms indicate that the financial risk taken by the insurer is smaller for a unit-linked product than for a traditional product with a fixed technical interest rate[1]. Furthermore, note that in the calculation of the single premium we implicitly assumed that the mean of the discounted (to time 0) value of the funds at time t coincides with the value of the funds at time 0. This means, that we have to start with a discussion of the value or price of a funds.

Also the model did not include any guarantees. But generally one would like to add a guarantee (e.g. a refund guarantee for the paid in premiums) to the policy. In this case we have a unit linked insurance with a guarantee also known as variable annuity. For example, the guarantee could be of the form

$$G(t) = \int_0^t \bar{p}(s) ds,$$

where $\bar{p}(s)$ denotes the density of the premiums at time s. More generally one could be interested in a refund guarantee of the paid in premiums with an additional interest at a fixed rate:

[1] Actually, this is not true in general. Here we implicitly assumed that the capital market risks of a unit-linked policy are minimised by an appropriate trading strategy. For a classical insurance such a trading strategy replicates the cash flows by zero coupon bonds with the corresponding maturities.

$$G(t) = \int_0^t \exp(r(t-s))\bar{p}(s)ds.$$

In these examples the payout function would be

$$C(t) = \max(S(t), G(t)).$$

Let us assume that the value of the funds is given by a stochastic process (with distribution P). What is, in this setting, the value of the discounted payment $C(t)$ at time t? A first guess might be

$$\pi_0\left(C(t)\right) = E^P\left[\max(S(t), G(t))\right].$$

But it is not that simple! If this would be the value of the payment, there would be the possibility to make a profit without risk (arbitrage). In order to prevent this possibility one has to change the measure P. In mathematical finance it is proved that an equivalent martingale measure exists, such that there is no arbitrage. Then in a "fair" market we have

$$\pi_0\left(C(t)\right) = E^Q\left[\max(S(t), G(t))\right],$$

where Q is a measure equivalent to P such that the discounted value of the underlying funds is a martingale.

Furthermore, in mathematical finance payments like $C(t)$ are called the payouts of an option. To determine the price of an option, one uses the arbitrage free pricing theory. A quick introduction to this theory will be given in the next section.

9.2 Pricing theory

In this section we look at modern financial mathematics. It is not our aim to give a comprehensive exposition with proofs of every detail, which would easily fill a whole book. We only want to give a brief survey which illustrates the theory. The reader interested in more details is referred to [Pli97], [HK79], [HP81] and [Duf92]. In the same sense we suggest [Oks03] for further reading regarding stochastic integrals and stochastic differential equations.

In this context we clearly also have to mention the paper of Black and Scholes [BS73] with their famous formula for the pricing of options.

9.2.1 Definitions

First we start with an example which illustrates the use of the pricing theory. The price of a share, modelled by a geometric Brownian motion $(S_t(\omega))$, might develop as shown in Figure 9.1.

A European call option for a certain share is the right to buy these shares at a fixed price c (strike price) at a fixed time T. The value of this right at time T is

$$H = \max\left(S_T - c, 0\right).$$

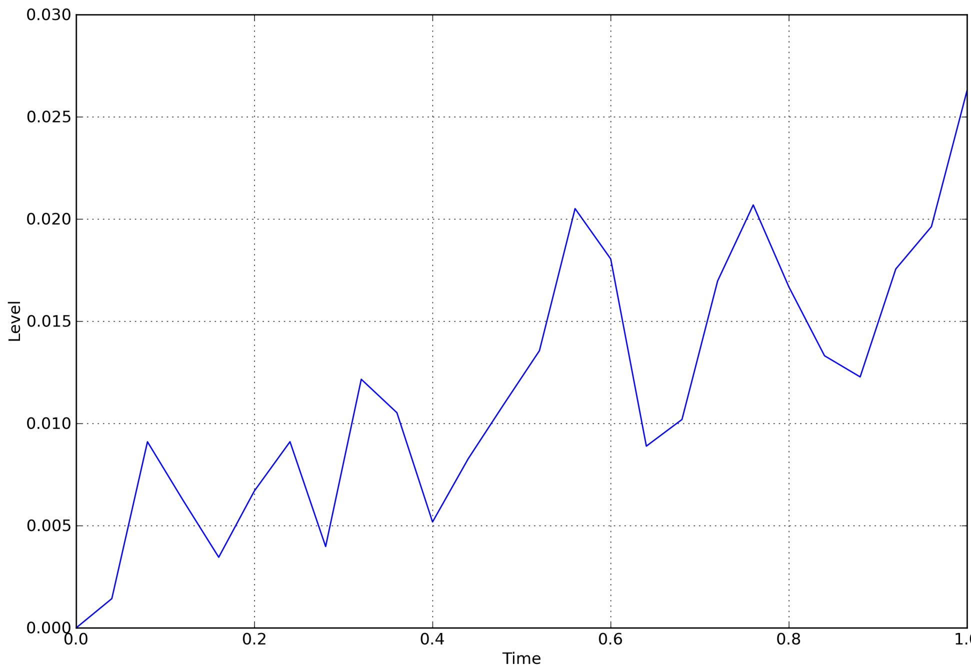

Figure 9.1: Movement of a share price

Now a bank would like to know the value (i.e., the fair price) of this option at time 0. As noted in the previous section, taking the expectation would systematically yield the wrong values. In many cases this would provide the possibility to make profits without risk. Thus there would be arbitrage opportunities.

To simplify the exposition we will consider the most simple models of the economy, i.e. finite models. In particular the time set will be discrete. The reader interested in the corresponding theorems for continuous time can for example find these in [HP81]. The ideas and concepts of the pricing theory are presented, as follows:

Let $(\Omega, \mathcal{A}, P)$ be a probability space where Ω is a finite set. Moreover, we assume that $P(\omega) > 0$ holds for all $\omega \in \Omega$.

We also fix a finite time T, as the time at which all trading is finished. The σ-algebra of the observable events at time t is denoted by $\mathcal{F}_t$, and the shares are traded at the times $\{0, 1, 2, \dots, T\}$. We note that $(\mathcal{F}_t)_{t \in T}$ is a filtration, ie $\mathcal{F}_t \subseteq \mathcal{F}_s$, for $t \leq s$.

We suppose that there are $k < \infty$ stochastic processes, which represent the prices of the shares $1, \dots, k$, i.e.,

$$S = \{S_t, t = 0, 1, 2, \dots, T\} \text{ with components } S^0, S^1, \dots S^k.$$

As usual, we assume that each S^j is adapted to $(\mathcal{F}_t)_t$. Here S_t^j can be understood as the price of the jth share at time t. The fact that the price process has to be adapted reflects the necessity that one has to know at time t the previous price of S. The share S^0 plays a special role. We suppose that $S_t^0 = (1+r)^t$, i.e., we have the possibility to make risk free investments which provide interest rate r. The risk free discount factor is defined by

$$\beta_t = \frac{1}{S_t^0}.$$

Next, we are going to define what is meant by a trading strategy.

Definition 9.2.1. *A* trading strategy *is a previsible ($\phi_t \in \mathcal{F}_{t-1}$) process $\Phi = \{\phi_t, t = 1, 2, \ldots, T\}$ with components ϕ_t^k.*

We understand ϕ_t^k as the number of shares of type k which we own during the time interval $[t-1, t[$. Therefore ϕ_t is called the portfolio at time $t-1$.

Notation 9.2.2. *Let X, Y be vector valued stochastic processes. Then we use the notations:*

$$\begin{aligned} < X_s, Y_t > &= X_s \cdot Y_t = \sum_{k=0}^{n} X_s^k \times Y_t^k, \\ \Delta X_t &= X_t - X_{t-1}. \end{aligned}$$

Next, we want to determine the value of the portfolio at time t:

time	value of the portfolio
$t-1$	$\phi_t \cdot S_{t-1}$
t^-	$\phi_t \cdot S_t$

Thus, the return in the interval $[t-1, t[$ is $\phi_t \cdot \Delta S_t$, and hence the total return is

$$G_t(\phi) = \sum_{\tau=1}^{t} \phi_\tau \cdot \Delta S_\tau.$$

We fix $G_0(\phi) = 0$, and $(G_t)_{t \geq 0}$ is called return process.

Theorem 9.2.3. *G is an adapted and real valued stochastic process.*

Proof. The proof is left as an exercise to the reader.

Definition 9.2.4. *A trading strategy is* self financing, *if*

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \qquad \forall t = 1, 2, \ldots, T-1.$$

A self financing trading strategy is just a trading strategy where at no time further money is added to or deduced from the portfolio.

Definition 9.2.5. *A trading strategy is* admissible*, if it is self financing and*

$$V_t(\phi) := \begin{cases} \phi_t \cdot S_t, & \text{if} \quad t = 1, 2, \ldots, T, \\ \phi_1 \cdot S_0, & \text{if} \quad t = 0 \end{cases}$$

is non negative. (In other words, one is not allowed to become bankrupt.) The set of admissible trading strategies is denoted by Φ.

Remark 9.2.6. The idea of admissible trading strategies is to only consider portfolios which neither lead to bankruptcy nor allow an addition or deduction of money. This also indicates that the value of the trading strategies remains constant when the portfolio is rearranged. Thus a trading strategy, which generates the same cash flow as an option, can be used to determine the value of the option.

Definition 9.2.7. *A contingent claim is a positive random variable X. The set of all contingent claims is denoted by $\mathcal{X}$.*

A random variable X is attainable, if there exists an admissible trading strategy $\phi \in \Phi$ which replicates it, i.e.

$$V_T(\phi) = X.$$

In this case one says "ϕ replicates X".

Definition 9.2.8. *The price of an attainable contingent claim, which is replicated by ϕ, is denoted by*

$$\pi = V_0(\phi)$$

(We will see later, that this price is not necessarily unique. It coincides with the initial value of the portfolio.)

9.2.2 Arbitrage

We say, the model offers arbitrage opportunities, if there exists

$$\phi \in \Phi \text{ with } V_0(\phi) = 0 \text{ and } V_T(\phi) \text{ positive and } P[V_T(\phi) > 0] > 0,$$

i.e., money is generated out of nothing. If such a strategy exists, one can make a profit without taking any risks. One of the axioms of modern economy says that there are no arbitrage opportunities. This is fundamental for some important facts in the option pricing theory.

Now, we are going to define what is meant by a price system.

Definition 9.2.9. *A mapping*

$$\pi : \quad \mathcal{X} \to [0, \infty[, \quad X \mapsto \pi(X)$$

is called price system *if and only if the following conditions hold:*

– $\pi(X) = 0 \iff X = 0$,
– π *is linear.*

A price system is consistent*, if*

$$\pi(V_T(\phi)) = V_0(\phi) \qquad \text{for all } \phi \in \Phi.$$

The set of all consistent price systems is denoted by Π, and $\mathbb{P}$ denotes the set

$$\mathbb{P} = \{Q \text{ is a measure equivalent to } P, \text{ s.th. } \beta \times S \text{ is a martingale w.r.t. } Q\},$$

where β is the discount factor from time t to 0. The measures $\mu \in \mathbb{P}$ are called equivalent martingale measures.

Theorem 9.2.10. *There is a bijection between the consistent price systems $\pi \in \Pi$ and the measures $Q \in \mathbb{P}$. It is given by*

1. $\pi(X) = E^Q[\beta_T X]$.
2. $Q(A) = \pi(S_T^0 \chi_A)$ *for all* $A \in \mathcal{A}$.

Proof. Let $Q \in \mathbb{P}$. We define $\pi(X) = E^Q[\beta_T X]$. Then π is a price system, since P is strictly positive on Ω and Q is equivalent to P. Thus it remains to show, that π is consistent. For $\phi \in \Phi$ we get

$$\begin{aligned}\beta_T V_T(\phi) &= \beta_T \phi_T S_T + \sum_{i=1}^{T-1} (\phi_i - \phi_{i+1}) \beta_i S_i \\ &= \beta_1 \phi_1 S_1 + \sum_{i=2}^{T} \phi_i \left(\beta_i S_i - \beta_{i-1} S_{i-1}\right),\end{aligned}$$

where we used that ϕ is self financing. This yields

$$\begin{aligned}\pi\left(V_T(\phi)\right) &= E^Q\left[\beta_T V_T(\phi)\right] \\ &= E^Q\left[\beta_1 \phi_1 S_1\right] + E^Q\left[\sum_{i=2}^{T} \phi_i \left(\beta_i S_i - \beta_{i-1} S_{i-1}\right)\right] \\ &= E^Q\left[\beta_1 \phi_1 S_1\right] + \sum_{i=2}^{T} E^Q\left[\phi_i E^Q\left[\left(\beta_i S_i - \beta_{i-1} S_{i-1}\right) | \mathcal{F}_{i-1}\right]\right] \\ &= \phi_1 E^Q\left[\beta_1 S_1\right] \\ &= \phi_1 \beta_0 S_0,\end{aligned}$$

since ϕ is previsible and βS is a martingale with respect to Q.

Thus, π is a consistent price system.

Now let $\pi \in \Pi$ be a consistent price system and Q be defined as above. Then $Q(\omega) = \pi(S_t^0 \chi_{\{\omega\}}) > 0$ holds for all $\omega \in \Omega$, since $S_t^0 \chi_{\{\omega\}} \neq 0$. Moreover, we have $\pi(X) = 0 \iff X = 0$ and therefore Q is absolutely continuous with respect to P.

In the next step, we are going to show that Q is a probability measure. We define

$$\phi^0 = 1 \qquad \text{and} \qquad \phi^k = 0 \quad \forall k \neq 0.$$

Hence, by the consistency of π, we get

$$\begin{aligned} 1 &= V_0(\phi) \\ &= \pi(V_T(\phi)) \\ &= \pi(S_T^0 \cdot 1) \\ &= Q(\Omega). \end{aligned}$$

The prices of positive contingent claims are positive and Q is additive. Therefore, Kolmogorov's axioms are satisfied, since Ω is finite. We have $Q(\omega) = \pi(S_T^0 \cdot \chi_{\{\omega\}})$ by definition. Hence, also

$$E[f] = \sum_\omega \pi(S_T^0 \cdot \chi_{\{\omega\}}) \cdot f(\omega) = \pi(S_T^0 \cdot \sum_\omega f(\omega)).$$

Thus, with $f = \beta_t X$, we have

$$E^Q[\beta_T X] = \pi(S_T^0 \cdot \beta_T \cdot X) = \pi(X).$$

Now we still have to show that $\beta_T S_T^k$ is a martingale for all k. Let k be a coordinate and τ be a stopping time, and set

$$\begin{aligned} \phi_t^k &= \chi_{\{t \leq \tau\}}, \\ \phi_t^0 &= \left(S_\tau^k / S_\tau^0\right) \chi_{\{t > \tau\}}. \end{aligned}$$

(We keep the share k up to time τ, then it is sold and the money is used for a risk free investment.) It is easy to show, that the strategy ϕ is previsible and self financing. Finally, for an arbitrary stopping time τ,

$$\begin{aligned} V_0(\phi) &= S_0^k, \\ V_T(\phi) &= \left(S_\tau^k / S_\tau^0\right) S_T^0 \\ \text{and} \\ S_0^k &= \pi(S_T^0 \cdot \beta_\tau \cdot S_\tau^k) \\ &= E^Q\left[\beta_\tau \cdot S_\tau^k\right]. \end{aligned}$$

Thus $\beta_T S_T^k$ is a martingale with respect to Q.

Above we have proved one of the main theorems in the option pricing theory. Next, we will present further statements without proofs. They all can be found for example in [HP81].

Theorem 9.2.11. *The following statements are equivalent*

1. *There is no arbitrage opportunity,*
2. $\mathbb{P} \neq \emptyset$,

3. $\Pi \neq \emptyset$.

Lemma 9.2.12. *Suppose there exists a self financing strategy* $\phi \in \Phi$ *such that*

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0,\ E[V_T(\phi)] > 0.$$

Then there exists an arbitrage opportunity.

Example 9.2.13. We are going to calculate the price of an option for a simple example. Consider a market with two shares $Z = (Z_1, Z_2)$ which are traded at the times $t = 0$, $t = 1$ and $t = 2$. Figure 9.2 shows the possible behaviour of these shares in form of a tree. To calculate the price of the option we suppose that all nine possibilities have the same probability.

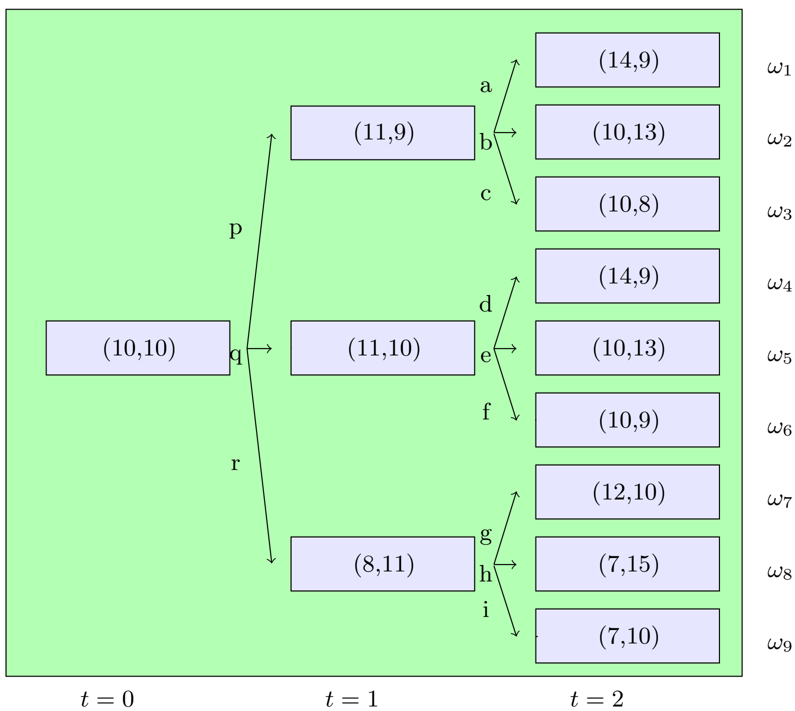

Figure 9.2: Example calculation of an option price

We want to calculate the price of a complex option given by

$$X = \left\{2Z_1(2) + Z_2(2) - [14 + 2\min\left(\min\left\{Z_1(t), Z_2(t)\right\}, 0 \leq t \leq 2\right)]\right\}^+.$$

First of all, we have to find an equivalent martingale measure. Thus we have to solve for the times $t = 0$ and $t = 1$ the following equations:

$$\begin{aligned}
10 &= 11p + 11q + 8r, && \text{(martingale condition for } Z_1\text{)}\\
10 &= 9p + 10q + 11r, && \text{(martingale condition for } Z_2\text{)}\\
1 &= p + q + r.
\end{aligned}$$

The solution to these equations is $p = q = r = \frac{1}{3}$.

Here we can see explicitly which circumstances imply the existence and uniqueness of a martingale measure. In this example the martingale measure is, from a geometric point of view, defined as the intersection of three hyper-planes. Depending on their orientation, there is either one or there are many or there is none equivalent martingale measure.

Next, we can derive the xsequations for the times $t = 1$ and $t = 2$. These are

$$\begin{aligned} 11 &= 14a + 10b + 10c, \\ 9 &= 9a + 13b + 8c, \\ 1 &= a + b + c, \\[1ex] 11 &= 14d + 10e + 10f, \\ 10 &= 9d + 13e + 9f, \\ 1 &= d + e + f, \\[1ex] 8 &= 12g + 7h + 7i, \\ 11 &= 10g + 15h + 10i, \\ 1 &= g + h + i, \end{aligned}$$

and they are solved by

$$\begin{aligned} (a, b, c) &= (\frac{1}{4}, \frac{3}{20}, \frac{3}{5}) \\ (d, e, f) &= (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \\ (g, h, i) &= (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}) \end{aligned}$$

Now we know the transition probabilities with respect to the martingale measure, which enables us to calculate the martingale measure Q itself. The results of these calculations are summarised in the following table:

state	$X(\omega_i)$	$Q(\omega_i)$
ω_1	5	1/12
ω_2	1	1/20
ω_3	0	1/5
ω_4	5	1/12
ω_5	0	1/12
ω_6	0	1/6
ω_7	4	1/15
ω_8	1	1/15
ω_9	0	1/5

Finally, we can calculate the price of the option as expectation with respect to Q. The result is $\frac{73}{60}$. This calculation can also be done recursively as per figure 9.3. Here one calculates the conditional expectations backwards. This in the end results in the same result – $\frac{73}{60}$. In a next step we want to determine the replicating portfolio from $t = 0 \rightsquigarrow t = 1$.

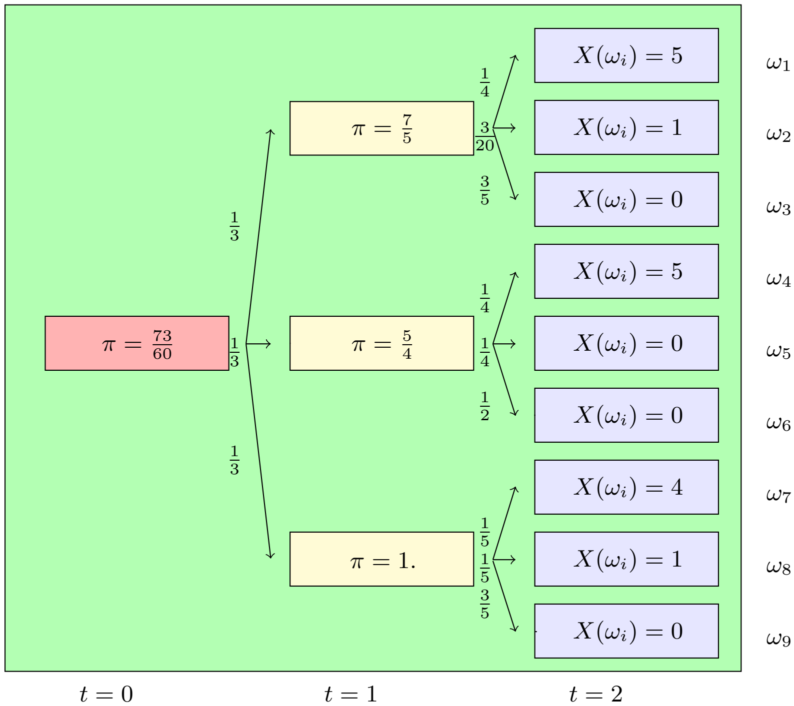

Figure 9.3: Calculation of price

Assume that we hold at time $t = 0$ a portfolio of α cash and β (resp. γ) units of security Z_1 (resp. Z_2). Then the following equation holds:

$$\begin{pmatrix} 1 & 11 & 9 \\ 1 & 11 & 10 \\ 1 & 8 & 11 \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{5}{4} \\ 1 \end{pmatrix}$$

Since the matrix is invertible the solution for the equation is unique:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{60} \begin{pmatrix} 143 \\ 2 \\ -9 \end{pmatrix}$$

Hence we need to hold at time $t = 0$ cash in the amount of 143, 2 shares Z_1 and -9 shares Z_2. We note that the value of the replicating portfolio at time $t = 0$

$$\frac{143 + 2 \times 10 - 9 \times 10}{60} = \frac{73}{60}$$

equals again the value of the option. This calculation can be performed for the entire tree resulting in the respective replicating portfolios (exercise). Figure 9.4 shows the corresponding replicating portfolios per state.

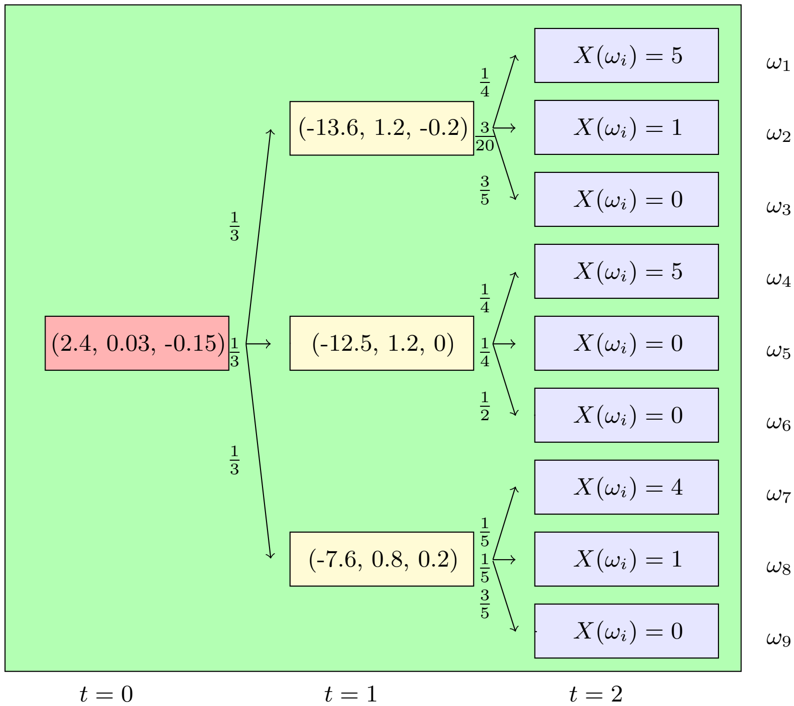

Figure 9.4: Calculation of replicating portfolios

9.2.3 Continuous time models

For models in continuous time we restrict our exposition to the statements, the proofs can be found in the references previously mentioned. A major difference between the discrete and the continuous setting is that we are going to *assume* that $\mathbb{P} \neq \emptyset$ holds for the continuous time model.

We start with some basic definitions.

Definition 9.2.14. – *A trading strategy ϕ is a locally bounded, previsible process.*

- *The value process corresponding to a trading strategy ϕ is defined by*

$$V : \mathit{\Pi} \to \mathbb{R}, \phi \mapsto V(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{k} \phi_t^k \cdot S_t^k.$$

- *The return process G is defined by*

$$G : \mathit{\Pi} \to \mathbb{R}, \phi \mapsto G(\phi) = \int_0^{\tau} \phi \, dS = \int_0^{\tau} \sum_{i=0}^{k} \phi^k dS^k.$$

- *ϕ is self financing, if $V_t(\phi) = V_0(\phi) + G_t(\phi)$.*
- *To define admissible trading strategies we use the notation:*

$$\begin{aligned} Z_t^i &= \beta_t \cdot S_t^i, && \textit{discounted value of share } i \\ G^*(\phi) &= \int \sum_{i=1}^k \phi^i \, dZ^i, && \textit{discounted return} \\ V^*(\phi) &= \beta \, V(\phi) = \phi^0 + \sum_{i=1}^k \phi^i \, Z^i. \end{aligned}$$

A trading strategy is called admissible, if it has the following three properties:

1. $V^*(\phi) \geq 0$,
2. $V^*(\phi) = V^*(\phi)_0 + G^*(\phi)$,
3. $V^*(\phi)$ *is a martingale with respect to* Q.

Theorem 9.2.15. *1. The price of a contingent claim* X *is given by* $\pi(X) = E^Q[\beta_T \, X]$.

2. A contingent claim is attainable $\iff V^* = V_0^* + \int H dZ$ *for all* H.

Definition 9.2.16. *The market is called complete, if every integrable contingent claim is attainable.*

Although this theory is very important, we only gave a brief sketch of the main ideas. It is therefore recommended that the reader extends his knowledge of financial mathematics by consulting the references.

9.3 The Black-Scholes Model and the Itô-Formula

As we have seen in the previous sections, we need an underlying economic model to calculate the price of an option. In principle one can use various different economic models. Exemplary, we are going to consider the most common model: geometric Brownian motion.

The following references are a good source for various aspects of the economic model: [Dot90], [Duf88], [Duf92], [CHB89], [Per94], [Pli97].

Convention 9.3.1 (General conventions). *For the remainder of this chapter we will use the following notations and conventions:*

- T_x *denotes the future lifespan of an* x *year old person.*
- *The* σ*-algebras generated by* T_x *are denoted by* $\mathcal{H}_t = \sigma\left(\{T > s\}, 0 \leq s \leq t\right)$.
- *We assume, that the values of the shares in the portfolio are given by standard Brownian motions* W. *(Compare with Figure 9.5.).*
- $\mathcal{G}_t$ *denotes the* σ*-algebra generated by* W *augmented by the* P*-null sets.*

Convention 9.3.2 (Independence of the financial variables). – *We assume that* $\mathcal{G}_t$ *and* $\mathcal{H}_t$ *are stochastically independent. This means, that the financial variables are independent of the future lifespan.*

Figure 9.5: 5 Simulations of a Brownian motion

– *$\mathcal{F}_t = \sigma(\mathcal{G}_t, \mathcal{H}_t)$ denotes the σ-algebra generated by $\mathcal{G}_t$ and $\mathcal{H}_t$.*

Next step we look at specific semimartingales which are of particular interest for stochastic finance and variable annuities. For these the Itô-calculus is simpler. First we define the corresponding class of stochastic processes:

Definition 9.3.3 (Money Market Account; Risk Free Bond). *We assume that an investor can invest in a money market account B_t which earns interest rate δ, ie*

$$\begin{aligned} B_{\tau+\Delta\tau} &= \exp(\Delta\tau\,\delta)\, B_\tau, \text{ or equivalently} \\ dB_\tau &= \delta\, B_\tau\, d\tau. \end{aligned}$$

We remark that this investment is risk free and that we do not assume any defaults.

Definition 9.3.4 (Forward Price of a Zero Coupon Bond). *For $t < T$, we denote by $B_0(t,T)$ the expected price of a $\mathcal{Z}_{(T)}$ as at time 0 , ie*

$$B_0(t,T) = E\left[\pi_t(\mathcal{Z}_{(T)})\middle|\, \mathcal{F}_0\right].$$

Theorem 9.3.5. *For $t < T$ the following formula holds:*

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

In the context of the model of Definition 9.3.3 this leads to

$$\begin{aligned} B(t,T) &= \exp\left(-\delta\,(T-t)\right) \text{ and },\\ dB(\tau,T) &= \delta\, B(\tau,T)\, d\tau. \end{aligned}$$

Proof. In an arbitrage free environment we can compare the following two strategies:

- Invest at $\tau = 0$ in one unit of $\mathcal{Z}_{(t)}$ and invest at $\tau = t$ the proceeds into $\mathcal{Z}_{(T)}$. At time $\tau = T$ we get the following proceeds:
$$\frac{1}{\pi_0(\mathcal{Z}_{(t)})}\,\frac{1}{\pi_t(\mathcal{Z}_{(T)})}.$$
- Alternatively we can invest the same value (ie $\pi_0(\mathcal{Z}_{(t)})$ into a T-year zero coupon bon ($\mathcal{Z}_{(T)}$), which will yield 1.

Since $\mathcal{Z}_{(t)}$ is risk free and because prices are linear, under the assumption of absence of arbitrage, this yields to

$$B(t,T) = \frac{B(0,T)}{B(0,t)}.$$

Definition 9.3.6 (Itô process (1-dim)). *Let $(W_t)_{t\in\mathbb{R}^+}$ be a 1-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$. A (1-dimensional) Itô process or stochastic integral is a stochastic process $(X_t)_{t\in\mathbb{R}^+}$ on $(\Omega, \mathcal{A}, P)$ of the form:*

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dW_s,$$

where $v \in \mathcal{W}_{\mathcal{H}}$ (see [Oks03] chapter 4 for the precise meaning), such that

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty \forall t \geq 0\right] = 1.$$

We assume that u is $\mathcal{H}_t$-adapted and

$$P\left[\int_0^t |u(s,\omega)| ds < \infty \forall t \geq 0\right] = 1.$$

Remark 9.3.7. Note that writing

$$dX = u(s)ds + v(s)dW_s$$

is an abbreviation for:

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_o^t v(s,\omega) dW_s.$$

Definition 9.3.8 (Share Price Process). *We assume that the share price S_t follows the following stochastic differential equation:*

$$dS_t = \eta\, S_t\, dt + \sigma\, S_t\, dW,$$

where $\eta \in \mathbb{R}$ is the real world equity return drift term. $\sigma \geq 0$ represents the equity volatility and $(W_t)_{t \in \mathbb{R}_0^+}$ is a one-dimensional standard Brownian motion on $(\Omega, \mathcal{A}, P)$.

Remark 9.3.9. – When considering

$$\frac{dS}{S} = \eta\, dt + \sigma\, dW,$$

we see that the Black-Scholes-Merton Model is based on a geometric Brownian motion, where we denote with W the Wiener measure or Brownian motion.

– The equity drift term η means that the value of the equity price would follow:

$$S_{t+\Delta t} = S_t\, \exp\left(\eta\, \Delta t\right)$$

in absence of volatility (ie for $\sigma = 0$). At the same time the risk free account value would yield

$$B_{t+\Delta t} = B_t\, \exp\left(\delta\, \Delta t\right).$$

In consequence the difference $\eta - \delta$ is called equity risk premium. It is typically positive and represents the additional return one can expect when investing in shares by assuming the additional risk induced by the volatility.

– The stochastic differential equation above is to be understood in sense of a stochastic integral (see appendix **??**) and the Itô-calculus.

– We note that $(W_t)_{t \in \mathbb{R}_0^+}$ has P-a.e. continuous sample paths and is nowhere differentiable. As a consequence of this the stochastic differential equation is to be understood as an integral equation, ie

$$dS_t = \eta\, S_t\, dt + \sigma\, S_t\, dW$$

is a short form for

$$S_t = S_0 + \int_0^t \eta\, S_\tau\, d\tau + \int_0^t \sigma\, S_\tau\, dW_\tau,$$

where the integral is akin to a Riemann integral, eg one can define

$$\int_0^t X_\tau dW_\tau = \lim_{n \to \infty} \sum_{j=0}^n X_{t_j^n} \left(W_{t_{j+1}^n} - W_{t_j^n} \right),$$

with limit taken over arbitrary partitions of the interval [O,T], into n pieces

$$0 = t_0^n < t_1^n < t_2^n \ldots < t_n^n = T \qquad \text{with}$$
$$\lim_{n \to \infty} \max\{|t_j^n - t_{j-1}^n| : j = 1, \ldots n\} = 0.$$

Definition 9.3.10 (Black-Scholes-Merton model). *This economic model consists of two investment options:*

$B(t) = \exp(\delta\, t)$		*risk free investment.*
$S(t) = S(0) \exp\left[\left(\eta - \frac{1}{2}\sigma^2\right) t + \sigma\, W(t)\right]$		*shares, modeled by a geometric Brownian motion (cf. Figure 9.6).*

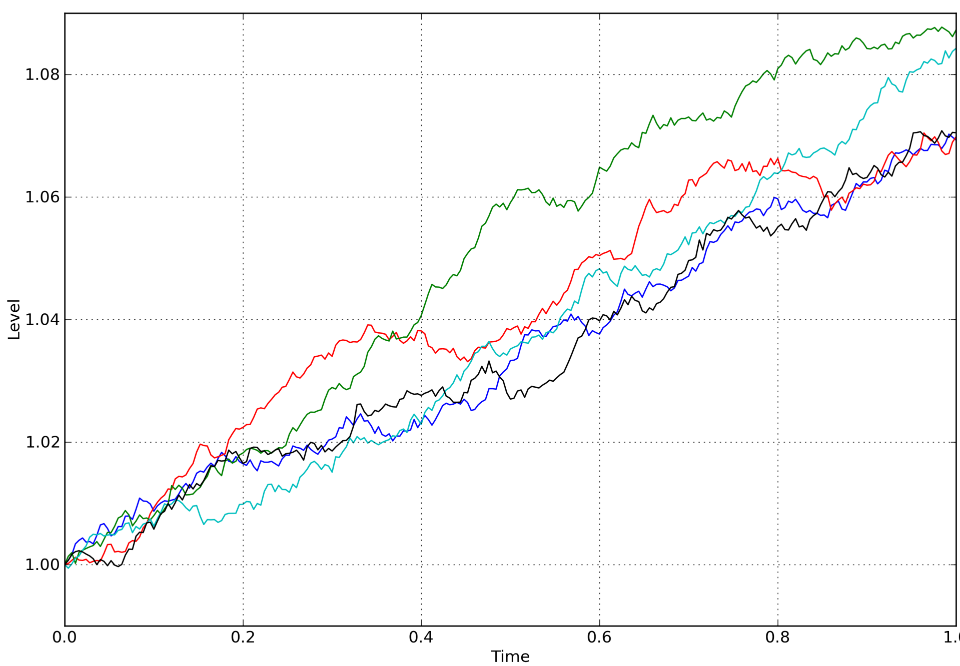

Figure 9.6: 5 Simulations of a geometric Brownian motion

S is the solution to the following stochastic differential equation:

$$dS = \eta\, S\, dt + \sigma\, S\, dW.$$

Exercise 9.3.11. Prove that S solves the stochastic differential equation given above.

Next we will calculate the discounted values of B and S:

$$\begin{aligned} B^*(t) = \frac{B(t)}{B(t)} &= 1, \\ S^*(t) = \frac{S(t)}{B(t)} &= S(0) \exp\left[\left(\eta - \delta - \tfrac{1}{2}\sigma^2\right) t + \sigma\, W(t)\right]. \end{aligned}$$

Thus we have defined the investment options. To calculate the option prices we need to find an equivalent martingale measure.

Note that we interpret dX in the sense of a stochastic integral, to which we can apply Itô's formula:

Theorem 9.3.12 (Itô). *Assume X being an Itô process with*

$$dX = a\,dt + b\,dW$$

and let $g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, (x,t) \mapsto g(x,t)$ be a function, where the following partial derivatives are continuous: $\frac{\partial}{\partial x} g$, $\frac{\partial^2}{\partial x^2} g$, and $\frac{\partial}{\partial t} g$. In this case $Y_t = g(X_t, t)$ is also an Itô process with the following stochastic differential equation:

$$dY = (\frac{\partial g}{\partial x}\, a + \frac{\partial g}{\partial t} + \frac{1}{2}\,\frac{\partial^2 g}{\partial x^2}\, b^2)\,dt + \frac{\partial g}{\partial x}\, b\,dW$$

Proof. For the proof we refer to [Oks03], or also [Pro90] or [IW81].

We can now apply Itô's lemma to the geometric Brownian motion, as follows:

Theorem 9.3.13. *Let a stock price S be modelled by a geometric Brownian motion*

$$dS = \eta\, S dt + \sigma\, S\, dW$$

and let

$$V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}, (s,t) \mapsto V(x,t)$$

be a function fulfilling the regularity criteria of the function g of the Itô lemma. In this case $V := V(S,t)$ is also an Itô process with the following stochastic differential equation:

$$dV = (\frac{\partial V}{\partial s}\, \eta\, S + \frac{\partial V}{\partial t} + \frac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\, \sigma^2\, S^2)\,dt + \frac{\partial V}{\partial S}\, \sigma\, S\, dW.$$

Proof. This follows immediately from the Itô formula keeping in mind that

$$\begin{aligned} a &= \eta\, S, \text{and} \\ b &= \sigma\, S. \end{aligned}$$

In order to simulate a geometric Brownian motion, the following corollary is helpful:

Theorem 9.3.14. *For a geometric Brownian motion*

$$dS = \eta\, S\, dt + \sigma\, S\, dW$$

we define $Y = \log(S)$. Then we have the following:

1. *$dY = (\eta - \frac{1}{2}\sigma^2)\,dt + \sigma\, dW$ is the unique solution to the above stochastic differential equation.*

2. *We can calculate S_t by:*

$$S_t = S_0 \exp\left(\left(\eta - \frac{1}{2}\sigma^2\right) \times t + \sigma W_t\right)$$

3. *For a series of times $t_0 = 0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots$, and $(X_k)_{k\in\mathbb{N}}$ standard normally distributed independent variables we can simulate $(S(t_k))_{k\in\mathbb{N}}$ with $S(0) = 1$ by the following recursion $\forall n \in \mathbb{N}$:*

$$S(t_n) = S(t_{n-1}) \times \exp\left[(\eta - \frac{\sigma^2}{2}) \times (t_n - t_{n-1}) + \sqrt{t_n - t_{n-1}} \times \sigma \times X_n\right].$$

Proof. Define

$$dX_t = (\eta - \frac{1}{2}\sigma^2)\, dt + \sigma\, dW,$$

with $X_0 = 0$. We know that $\tilde{S}_t := g(X_t)$, with

$$g : \mathbb{R} \to \mathbb{R}, x \mapsto g(x) = \exp(x),$$

and that $g = \frac{d}{dx} g = \frac{d^2}{dx^2} g$. As a consequence of the Itô-formula we get

$$\begin{aligned} d\tilde{S}_t &= dg(X_t) \\ &= g'(X_t)\,(\eta - \frac{1}{2}\sigma^2)\, dt + g'(X_t)\,\sigma\, dW_t \\ &\quad + \frac{1}{2} g''(X_t)\sigma^2\, dt \\ &= g(X_t)\ (\eta\, dt + \sigma\, dW_t) \\ &= \tilde{S}_t\ (\eta\, dt + \sigma\, dW_t)\,. \end{aligned}$$

Hence both S and $\tilde{S}$ fulfill the same stochastic differential equation with identical boundary condition. The proposition follows as a consequence of a general result for stochastic differential equations an the Itô-calculus, stating the uniqueness if a stochastic differential equation if the coefficients of the stochastic differential equations are Lipschitz continuous.

Theorem 9.3.15. *Under the natural filtration $\mathcal{F}$ induced by $(W_t)_{t\in\mathbb{R}_0^+}$ the following hold:*

1. *The Brownian motion is a martingale, and*
2. *$M_t = \exp\left(\sigma\, W_t - \frac{1}{2}\sigma^2\, t\right)$ is a martingale.*

Proof. Let $Z \sim \mathcal{N}(0,1)$ and let $u \leq t$. For the first equation we have

$$\begin{aligned} E^P\left[W_t | \mathcal{F}_u\right] &= E^P\left[W_t - W_u | \mathcal{F}_u\right] + E\left[W_u | \mathcal{F}_u\right] \\ &= E\left[(t-u)\, Z\right] + W_u \\ &= W_u. \end{aligned}$$

For the second equation we have:

$$\begin{aligned}
E^P\left[M_t|\mathcal{F}_u\right] &= E^P\left[\exp\left(\sigma\,W_t-\frac{1}{2}\sigma^2\,t\right)|\mathcal{F}_u\right]\\
&= E^P\left[\exp\left(\sigma\left((W_t-W_u)+W_u\right)-\frac{1}{2}\sigma^2\left((t-u)+u\right)\right)|\mathcal{F}_u\right]\\
&= \exp\left(\sigma\,W_u-\frac{1}{2}\sigma^2\,u\right)\times E^P\left[\exp\left(\sigma\left(W_t-W_u\right)-\frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right]\\
&= M_u\times E^P\left[\exp\left(\sigma\left(W_t-W_u\right)-\frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right]\\
&= M_u\times E^P\left[\exp\left(\sigma\,\sqrt{t-u}\,Z-\frac{1}{2}\sigma^2\left(t-u\right)\right)|\mathcal{F}_u\right]\\
&= M_u\times E^P\left[\exp\left(\sigma\,\sqrt{t-u}\,Z-\frac{1}{2}\sigma^2\left(t-u\right)\right)\right]\\
&= M_u
\end{aligned}$$

where we used the defining charcateristics of the Brownian motion and where $E^P\left[\exp\left\{\sigma\,\sqrt{t-u}\,Z\right\}\right]=\exp\left(\frac{1}{2}\,\sigma^2\,(t-u)\right)$ can be shown by elementary calculus (exercise).

Corollary 9.3.16. $(S_t)_{t\in\mathbb{R}_0^+}$ *follows a martingale under* P *if and only if* $\eta=0$.

Remark 9.3.17. We note that S_t is a time continuous Markov-process under P with respect to $\mathcal{F}$ and we have

$$\begin{aligned}
p_S(u,x,t,y) &:= P\left[S_t=y|S_u=s\right]\\
&= \frac{1}{\sqrt{2\pi(t-u)}\,\sigma\,y}\\
&\quad\times\exp\left\{-\frac{\left(\ln\left(\frac{y}{x}\right)-\eta\,(t-u)+\frac{1}{2}\sigma^2\,(t-u)\right)^2}{2\sigma^2\,(t-u)}\right\}
\end{aligned}$$

Definition 9.3.18 (Trading Strategy). *A trading strategy in the Black-Scholes-Merton framework is a pair* $\Phi=(\Phi^1,\Phi^2)$ *of* $\mathcal{F}$*- previsible, progressively measurable stochastic processes. on* $(\Omega,\mathcal{A},P)$.

Definition 9.3.19. *A trading strategy* $\Phi=(\Phi^1,\Phi^2)$ *over* $[0,T]$ *is self-financing if the corresponding value process (also wealth process)* $(V_t(\Phi))_{t\in[0,t]}$,

$$V_t(\Phi) \;=\; \Phi_t^1\,S_t+\Phi_t^2\,B_t,(\forall t\in[0,T])$$

satisfies the following condition:

$$V_t(\Phi) \;=\; V_0(\Phi)+\int_0^t\Phi_\tau^1\,dS_\tau+\int_0^t\Phi_\tau^2 dB_\tau.$$

Remark 9.3.20. – As usual, we implicitly assume the existence of an integral if we write $\int X dW$. The following two conditions are sufficient that the integral in Definition 9.3.18 exists:

$$\begin{aligned} P\left[\int_0^T (\Phi_\tau^1)^2 d\tau < \infty\right] &= 1, \text{ and} \\ P\left[\int_0^T |\Phi_\tau^2| d\tau < \infty\right] &= 1. \end{aligned}$$

– Note that we will denote by * discounted quantities, with respect to the money market account, eg

$$S_t^* = \frac{S_t}{B_t}.$$

– Since $B_t = \exp(t\,\delta)\,B_0$, we get

$$S_t^* = S_0^* \exp\left(\left(\eta - \delta - \frac{1}{2}\sigma^2\right) t + \sigma\, W_t\right),$$

or equivalently

$$dS_t^* = (\eta - \delta) S_t^*\, dt + \sigma S_t^* dW_t,$$

with $S_0^* = S_0$.

Corollary 9.3.21. *The discounted stock price process $(S_t^*)_{t\in[0,T]}$ is a martingale under P if and only if $\eta = \delta$.*

Definition 9.3.22 (Martingale Measure). *A probability measure Q on $(\Omega, \mathcal{A})$ equivalent to P is called martingale measure for S^* if $(S_t^*)_{t\in[0,T]}$ is a local martingale under Q.*

Definition 9.3.23 (Spot Martingale Measure). *A probability measure P^* on $(\Omega, \mathcal{A}, P)$ on $(\Omega, \mathcal{A}, P)$ equivalent to P is called spot martingale measure, if the discounted value process $V_t^*(\Phi) = \frac{V_t(\Phi)}{B_t}$ of any self-financing trading strategy is a local martingale under P^*.*

Lemma 9.3.24. *A probability measure is a spot martingale measure if and only if it is a martingale measure for the discounted stock price S^*.*

Proof. Let Φ be a self-financing strategy and denote $V^* := V^*(\Phi)$. Using Itô's product rule and the fact that $dB_t^{-1} = -\delta\, B_t^{-1}\, dt$ we the following:

$$\begin{aligned} dV_t^* &= d(V_t\, B_t^{-1}) \\ &= V_t\, dB_t^{-1} + B_t^{-1}\, dV_t \\ &= \left(\Phi_t^1\, S_t + \Phi_t^2 B_t\right) dB_t^{-1} + B_t^{-1}\, \left(\Phi_t^1\, dS_t + \Phi_t^2 dB_t\right) \\ &= \Phi_t^1\, \left(B_t^{-1}\, dS_t + S_t\, dB_t^{-1}\right) \\ &= \Phi_t^1\, dS_t^*. \end{aligned}$$

Hence we get

$$V_t^*(\Phi) \;=\; V_0^*(\Phi) + \int_0^t \Phi_\tau^1 \, dS_\tau^*$$

Therefore the lemma follows

Lemma 9.3.25. *The unique martingale measure Q for the discounted stock price process S^* is given by the Radon-Nikodym density*

$$\begin{aligned} \xi_t &= \frac{dQ}{dP} \\ &= \exp\left(-\frac{1}{2}\left(\frac{\eta-\delta}{\sigma}\right)^2 t - \frac{\eta-\delta}{\sigma} W(t)\right) \qquad \textit{for all } t \in [0,T]. \end{aligned}$$

The discounted stock price S^* satisfies under Q the following:

$$dS_t^* \;=\; \sigma \, S_t^* \, dW_t^*,$$

and the continuous $\mathcal{F}$-adapted process W^* is given by

$$W_t^* \;=\; W_t - \frac{\delta-\eta}{\sigma} \, t \;\forall t \in [0,T].$$

– **Exercise 9.3.26.** Prove the following statements:

1. $E[\xi_t] = 1$,
2. $Var[\xi_t] = \exp\left(\left(\frac{\eta-\delta}{\sigma}\right)^2 t\right) - 1$,
3. $\xi_t > 0$.

(Hint: $W(t) \sim \mathcal{N}(0,t)$.)

Proof. An application of Girsanov's theorem – a theorem in the theory of stochastic integration (e.g. [Pro90] Theorem 3.6.21) – shows that

$$\hat{W}_t = W(t) + \frac{\eta-\delta}{\sigma} \, t$$

is a Brownian motion with respect to $Q = \xi \cdot P$.

Naturally, after this transformation we want to prove that

$$S^*(t) = S(0) \, \exp\left(-\frac{1}{2}\sigma^2 \, t + \sigma \, \hat{W}(t)\right)$$

is a martingale with respect to Q. (Then the price of the option is given by its expectation with respect to Q.)

We have to show the equality

$$E^Q \left[S^*(u) | \mathcal{F}_t\right] = S^*(t)$$

for $t, u \in \mathbb{R}, u > t$. With the notation $u = t + \Delta t$, $W_u = W_t + \Delta W$ and $Z \sim \mathcal{N}(0,1)$ we have

$$
\begin{aligned}
&E^Q\left[S^*(u)|\mathcal{F}_t\right] \\
&\quad = E^Q\left[S(0)\,\exp\left(-\frac{1}{2}\sigma^2\,t + \sigma\,\hat{W}(t) + (-\frac{1}{2}\sigma^2\,\Delta t + \sigma\,\Delta\hat{W})\right)|\mathcal{F}_t\right] \\
&\quad = S(0)\,\exp\left(-\frac{1}{2}\sigma^2\,t + \sigma\,W(t)\right)\,E^Q\left[\exp\left(-\frac{1}{2}\,\sigma^2\Delta t + \sigma\sqrt{\Delta t}Z\right)|\mathcal{F}_t\right] \\
&\quad = S^*(t).
\end{aligned}
$$

Therefore the measure Q is equivalent to P, and S^* is a martingale with respect to Q. An economist would say, "it exists (at least) one consistent price system". For the uniqueness of Q we refer to [MR07] lemma 3.1.3.

As a next step we want to understand the Black-Scholes-Merton partial differential equation from a theoretical aspect. The following theorem sheds light into this question. We note that this partial differential equation will be useful, when we will look at Δ-hedging.

Theorem 9.3.27 (Black-Scholes-Merton Differential Equation). *Let δ be the risk-free interest rate, η the equity drift rate and σ the volatility of a asset S_t in the Black-Scholes-Merton framework (eg the share price satisfying the following stochastic differential equation $dS = \eta\,S\,dt + \sigma\,S\,dW$). Suppose we have a contingent claim with expiry date T and underlying asset S_t, with value function:*

$$v : \mathbb{R}^+ \times]0,T[, \to \mathbb{R}, (x,t) \mapsto v(x,t),$$

where $v \in C^{2,1}(\mathbb{R}^+ \times [0,T])$. Then the following partial differential equation holds:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} = \delta v.$$

Moreover the portfolio $\mathcal{P} = (V - \Delta \times S, \Delta)$ consisting $V - \Delta \times S$ units cash (valued at 1) and $\Delta = \frac{\partial v}{\partial S}$ shares replicates the contingency claim.

Proof. Since this theorem is very important we offer in a first step an *outline the proof.* The proof is an application of Itô's lemma. We have the following:

$$
\begin{aligned}
dS &= \eta\,S\,dt + \sigma\,S\,dW, \text{ and} \\
dv &= (\frac{\partial v}{\partial S}\,\eta\,S + \frac{\partial v}{\partial t} + \frac{1}{2}\,\frac{\partial^2 v}{\partial S^2}\,\sigma^2\,S^2)\,dt + \frac{\partial v}{\partial S}\,\sigma\,S\,dW
\end{aligned}
$$

We define $\Pi := -v + \Delta\,S$ and we see that for $\Delta = \frac{\partial v}{\partial S}$ the stochastic part of the SDE cancels out and we get:

$$d\Pi = \left\{-\frac{\partial v}{\partial t} - \frac{1}{2}\,\frac{\partial^2 v}{\partial S^2}\,\sigma^2\,S^2\right\}\,dt$$

On the other hand Π can be interpreted as the value of an investment portfolio, shorting $-v$ of cash and buying Δ shares. One could also invest the respective value in cash, by means of absence of arbitrage. This needs to result in the same value and in the same increments, hence:

$$d\Pi \quad = \quad \delta\, \Pi\, dt$$

When now using the definition of Π for the right hand side and using the above formula for the right hand side, this results in:

$$\left\{ -\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial^2 v}{\partial S^2}\,\sigma^2\,S^2 \right\} dt \quad = \quad \delta\, \left\{ -v + \frac{\partial v}{\partial S}\,S \right\} dt.$$

In consequence we get the Black-Scholes-Merton differential equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \delta S \frac{\partial v}{\partial S} \quad = \quad \delta v$$

Proof. In a next step we offer a more formal proof. It follows essentially [MR07] theorem 3.1.1 where further details can be found. We will proof this theorem by directly determining the replicating strategy and note that the price v_t of the contingency claim is a function $v_t = v(S_t, t)$. We may assume that the replicating strategy Φ has the form:

$$\Phi_t \quad = \quad (\Phi_t^{(1)}, \Phi_t^{(2)}) = (h(S_t,t), g(S_t,t)),$$

for $t \in [0,T]$ and $g, h : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ unknown functions. Note that $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ represent the amount of cash and the number for shares which we hold at time t. Since Φ is assumed self-financing, its wealth process $V(\Phi)$ given by

$$V_t(\Phi) \quad = \quad g(S_t,t)S_t + h(S_t,t)B_t = v(S_t,t), \tag{9.1}$$

needs to satisfy the following:

$$dV_t(\phi) = g(S_t,t)dS_t + h(S_t,t)dB_t. \tag{9.2}$$

Given that we operate in the Black-Scholes-Merton framework, we can conclude from (9.2) that

$$\begin{aligned} dV_t(\Phi) \quad &= \quad g(S_t,t)(\eta S_t dt + \sigma S_t dW_t) + h(S_t,t)\delta B_t dt \\ &= \quad (\eta - \delta)S_t g(S_t,t)dt + \sigma S_t g(S_t,t)dW_t + \delta S_t g(S_t,t)dt + \delta h(S_t,t)B_t dt \\ &= \quad (\eta - \delta)S_t g(S_t,t)dt + \sigma S_t g(S_t,t)dW_t + \delta S_t v(S_t,t)dt. \end{aligned} \tag{9.3}$$

The application of the Itô lemma to v results in

$$dv(S_t,t) \quad = \quad \left(\frac{\partial v}{\partial t}(S_t,t) + \eta S_t \frac{\partial v}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t,t)dW_t.$$

Combining the above expression with (9.2), results in the following expression for the Itô differential of the process $Y_t = v(S_t,t) - V_t(\Phi)$:

$$\begin{aligned} dY_t \quad = \quad & \left(\frac{\partial v}{\partial t}(S_t,t) + \eta S_t \frac{\partial v}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial v}{\partial S}(S_t,t)dW_t \\ & + (\delta - \eta)S_t g(S_t,t)dt - \sigma S_t g(S_t,t)dW_t - \delta v(S_t,t)dt \end{aligned}$$

On the other hand, in view (9.1), the process Y vanishes identically (i.e. $Y \equiv 0$), and thus $dY_t = 0$. By virtue of the uniqueness of the canonical decomposition of continuous semimartingales, the diffusion term in the above decomposition of Y vanishes. In our case, this means that we have, for every $t \in [0,T]$,

$$\int_0^t \sigma S_\tau \left(g(S_\tau, \tau) - \frac{\partial v}{\partial S}(S_\tau, \tau) \right) dW\tau = 0 \qquad P - a.s.$$

In view of the properties of the Itô integral (isometry used for the construction of the Itô integral), this is equivalent to:

$$\int_0^T S_\tau^2 \left(g(S_\tau, \tau) - \frac{\partial v}{\partial S}(S_\tau, \tau) \right)^2 d\tau = 0. \tag{9.4}$$

For (9.4) to hold, it is sufficient and necessary that g satisfies

$$g(s,t) = \frac{\partial v}{\partial S}(s,t) \qquad \forall (s,t) \in \mathbb{R}^+ \times [0,T] \tag{9.5}$$

Strictly speaking, the equality above should hold $P \otimes \lambda-$almost surely, where λ is the Lebesgue measure. Using (9.5), results in still another equation of Y:

$$Y_t = \int_0^t \left\{ \frac{\partial v}{\partial \tau}(S_\tau, \tau) + \frac{1}{2}\sigma^2 S_\tau^2 \frac{\partial^2 v}{\partial S^2}(S_\tau, \tau) + \delta S_\tau \frac{\partial v}{\partial S}(S_\tau, \tau) - \delta v(S_\tau, \tau) \right\} d\tau$$

It is thus apparent that $Y \equiv 0$ whenever v satisfies the following partial differential equation, referred to as Black-Scholes PDE:

$$\frac{\partial v}{\partial t}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}(S_t, t) + \delta S_t \frac{\partial v}{\partial S}(S_t, t) - \delta v(S_t, t) = 0 \tag{9.6}$$

This terminates the first part of the proof. It thus remains to check that Φ is an admissible trading strategy.

If the contingent claim is attainable, it thus remains to check that the replicating strategy Φ, given by the formula

$$\begin{aligned} \Phi^1 &= g(S,t) = \frac{\partial}{\partial S} v(S_t, t), \\ \Phi^2 &= h(S,t) = B_t^{-1} \left(v(S_t,t) - g(S_t,t) S_t \right), \end{aligned}$$

is admissible. Let us first check that Φ is self-financing. We need to check that

$$dV_t(\Phi) = \Phi_t^1 \, dS_t + \Phi_t^2 \, dB_t.$$

Since $V_t(\Phi) = \Phi_t^1 \, S_t + \Phi_t^2 \, B_t = v(S_t, t)$, by applying Itô's formula, we get

$$dV_t(\Phi) \;=\; \frac{\partial}{\partial S} v(S_t,t)\, dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S^2} v(S_t,t)\, dt + \frac{\partial}{\partial t} v(S_t,t)\, dt.$$

In view of (9.6), the last equality can also be given the following form

$$dV_t(\Phi) = \frac{\partial}{\partial S} v(S_t, t)\, dS_t + \delta\, v(S_t, t)\, dt - \delta\, S_t \frac{\partial}{\partial S} v(S_t, t)\, dt,$$

and thus

$$\begin{aligned} dV_t(\Phi) &= \frac{\partial}{\partial S} v(S_t, t) dS_t + \delta v(S_t, t) dt - \delta S_t \frac{\partial}{\partial S} v(S_t, t) dt \\ &= \Phi_t^1\, dS_t + \delta\, B_t \frac{\frac{\partial}{\partial t} v(S_t, t) - \Phi_t^1\, S_t}{B_t} \\ &= \Phi_t^1\, dS_t + \Phi_t^2\, dB_t. \end{aligned}$$

This ends the verification of the self-financing property.

Note that the Δ-portfolio is not permanently risk free, but only *instantaneously*. For the interested reader we suggest as further reading: [Ped89] for the analytical basics such as integration theory, Banach and Hilbert spaces etc, and [Per94] and [Oks03] for stochastic integration and stochastic differential equations. Finally we would also suggest [Hul12] and [MR07] as general valuable reference.

Remark 9.3.28. The proof of the above theorem is very helpful for understanding the concept of dynamical hedging. Assume a given contingency claim in the Black-Scholes framework which has been sold by an insurance company or a bank. Theorem 9.3.27 shows a way to calculate the value V via solving the corresponding partial differential equation. More importantly it helps us also to understand the intrinsic risk to this contract in case of changes in equity prices (S). In many instances the bank or insurance company is not willing to take this risk on its balance sheet (eg letting fluctuate the shareholder equity) and tries to mitigate this risk by a suitable hedging strategy.

One classical hedging strategy is to buy at each point of time assets with the same partial derivative as V with respect to S. Such a strategy is called δ-hedging. Figure 9.7 shows this. Note that the δ-hedging strategy has this name, because the first partial derivative of V with respect of S is called Δ, similarly Γ is the second partial derivative of V with respect to S, ie. we have the following

$$\begin{aligned} \Delta &= \frac{\partial}{\partial S} V \\ \Gamma &= \frac{\partial^2}{\partial S^2} V \end{aligned}$$

Turning back to the proof of theorem 9.3.27, we see that for a pure δ strategy the value process of the hedge portfolio becomes deterministic, following

$$d\Pi = \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \frac{\partial^2 V}{\partial S^2}\, \sigma^2\, S^2 \right\} dt.$$

This however means that a pure δ strategy leads to a deterministic difference in price as a consequence of the Γ term. This effect is known under the name γ-bleed. Figure 9.8 shows the effect of using a pure δ-strategy vs using a $\delta - \gamma$ strategy for a simulated trajectory. The red line shows the true value of V, the blue one using a *delta*-hedge and the green one a $\delta - \gamma$-hedge.

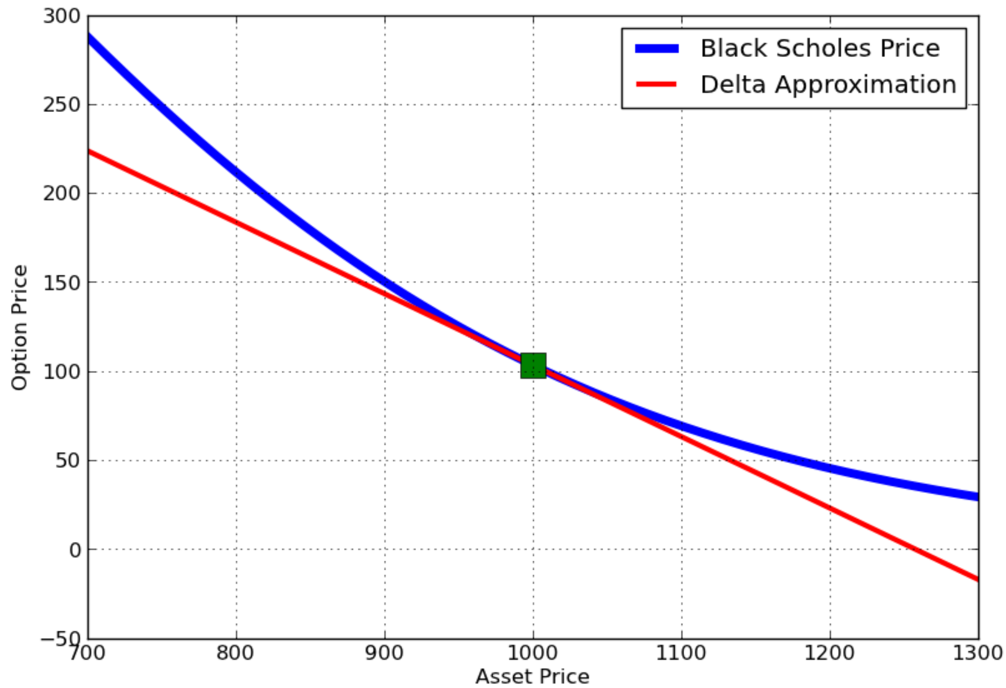

Figure 9.7: Delta Hedge

Theorem 9.3.29. *Let the economic model defined above be given, i.e. it is defined by* $(\Omega, \mathcal{A}, P)$, S *and* B. *Then at time* t *the price of a policy with death benefit* $C(T)$ *is*

$$\pi_t(T) = E^Q \left[\exp\left(-\delta(T-t)\right) C(T) | \mathcal{F}_t\right].$$

Remark 9.3.30. The main difference of this model in comparison to the classical model is that one has to calculate the expectation with respect to Q and not with respect to P. Moreover one should note, that we have not proved the uniqueness of the price system.

The following two formulas are an important consequence of the previous considerations.

Theorem 9.3.31. *A single premium for a policy based on the economic model defined above is given by the following formulas.*

Endowment policy:

$$V(0) = E^Q \left[\exp(-\delta T)\, C(T)\right] \cdot {}_T p_x.$$

Term life insurance:

$$V(0) = \int_0^T E^Q \left[\exp(-\delta t)\, C(t)\right] {}_t p_x \, \mu(x+t) dt.$$

9.4 Calculation of single premiums

Up to now the calculations have been relatively simple, since we did not include any guarantees in our policy model. Next, we will consider a unit-linked policy with an additional guarantee. We recall some of the notations from the previous sections:

$N(\tau)$	Number of shares at time τ,
$S(\tau)$	value of a share at time τ,
$G(\tau)$	guaranteed benefits at time τ,
$C(\tau) = \max\{N(\tau)S(\tau), G(\tau)\}$	value of the insurance at time τ.

9.4.1 Pure endowment policy

Theorem 9.4.1. *Let the Black-Scholes model be given. Then the single premium for a pure endowment policy with payout*

$$C(T) = \max\{N(T)S(T), G(T)\}$$

is given by

$${}_T G_x = {}_T p_x \left[G(T) \exp(-\delta T) \Phi(-d_2^0(T)) + S(0)N(T)\Phi(d_1^0(T))\right],$$

where

$$\begin{aligned}
\Phi(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{x^2}{2}) dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s > t), \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, (s > t).
\end{aligned}$$

Proof. In the following we denote by J^* the discounted value of a random variable J. The value of the pure endowment policy at time zero is $E^Q[C^*(T)]$. We set $Z = S^*(T)$. Then the following equations hold

$${}_T G_x = {}_T p_x \, E^Q \left[\max\{N(T)Z, G^*(T)\}\right]$$

and

$$Z = S(0) \exp\left(-\frac{1}{2}\sigma^2 T + \sigma \hat{W}(T)\right) \qquad \text{where} \qquad \hat{W}(T) \sim \mathcal{N}(0, T).$$

Thus we get

$$\begin{aligned}
{}_T G_x &= {}_T p_x \int_{-\infty}^{\infty} \max\left[N(T)S(0)\exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi), G^*(T)\right] f(\xi) d\xi, \\
f(\xi) &= \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}\xi^2\right).
\end{aligned}$$

Next we define $\bar{\xi} = \frac{1}{\sigma}\left[\ln\left(\frac{G^*(T)}{N(T)S(0)}\right) + \frac{1}{2}\sigma^2 T\right]$ and note that $\xi > \bar{\xi}$ implies $N(T)Z > G^*(T)$. Therefore, the single premium is given by

$$\begin{aligned}
{}_TG_x &= {}_Tp_x\Big(G^*(T)\int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\
&\qquad + N(T)S(0)\int_{\bar{\xi}}^{\infty} \exp(-\frac{1}{2}\sigma^2 T + \sigma\,\xi) f(\xi)d\xi\Big) \\
&= {}_Tp_x\Big(G^*(T)\int_{-\infty}^{\bar{\xi}} f(\xi)d\xi \\
&\qquad + N(T)S(0)\int_{\bar{\xi}}^{\infty} \frac{1}{\sqrt{2\pi T}}\exp(-\frac{1}{2T}(\xi - \sigma T)^2 d\xi\Big).
\end{aligned}$$

This equation, with adapted notation, yields the statement of the theorem.

9.4.2 Term life insurance

Theorem 9.4.2. *Let the Black-Scholes model be given. Then the single premium for a term life insurance with death benefit*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

is given by

$$G^1_{x:T} = \int_0^T \left(G(t)\exp(-\delta t)\Phi(-d_2^0(t)) + S(0)N(t)\Phi(d_1^0(t))\right) {}_tp_x \mu_{x+t}\, dt,$$

where

$$\begin{aligned}
\Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})dx, \\
d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\
d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},
\end{aligned}$$

for $s > t$.

Exercise 9.4.3. Prove the previous theorem by the same methods which we used for the pure endowment policy.

Remark 9.4.4. For the calculations above we have defined the guarantee for both temporary death benefit ("GMDB") and for the pure endowment ("GMAB") by

$$C(T) = \max\{N(T)S(T), G(T)\}.$$

This means that we have value before the total product, eg the value of the underlying funds plus the corresponding variable annuity guarantee. In practise the total value is often split into the value of the underlying funds ($N(T)S(T)$) and the value of the variable annuity guarantee or also called variable annuity rider. Most of the examples in the following will actually calculate the value of the variable annuity rider. Note that the split of the entire value of the variable annuity (including the value of the underlying funds) is simple as a consequence of the linearity of the expected value under the measure Q. Moreover, one can define the guaranteed part of the insurance benefit as follows:

$$\begin{aligned} C^G(T) &= \max\{N(T)S(T), G(T)\} - N(T)S(T) \\ &= \max\{0, G(T) - N(T)S(T)\}. \end{aligned}$$

In order to compare the value of the variable annuity rider on often compares it with the value of the underlying funds. Depending on lapse assumptions the value of the guarantee can often exceed 10% of the underlying funds value.

9.5 Thiele's differential equation

Now we want to derive Thiele's differential equation. For this we need to determine premiums for the policies. We introduce the notation $\bar{p}(t)$ for the density of the premiums at time t. Then the equivalence principle yields the following two equations:

$$ {}_TG_x = \int_0^T \bar{p}(t) \exp(-\delta\, t)\, {}_tp_x dt $$

and

$$ G^1_{x:T} = \int_0^T \bar{p}(t) \exp(-\delta\, t)\, {}_tp_x dt. $$

Also in this section the pure endowment policy and the term life insurance will be considered separately. The mathematical reserve for these policies is given by:

$$\begin{aligned} \text{Pure endowment:} \quad V(t) = \;& {}_{T-t}p_{x+t}\pi_t(T) \\ & - \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))\, {}_{\xi-t}p_{x+t}\, d\xi. \\ \text{Life insurance:} \quad V(t) = \;& \int_t^T (\pi_t(\xi)\mu_{x+\xi} - \bar{p}(\xi) \exp(-\delta(\xi - t))) \\ & \times {}_{\xi-t}p_{x+t}\, d\xi, \end{aligned}$$

where

$$\begin{aligned} \pi_t(s) = \;& G(s) \exp(-\delta(s-t)) \Phi(-d_2^t(s)) \\ & + N(s)S(t)\Phi(d_1^t(s)), \\ d_1^t(s) =\;& \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\ d_2^t(s) =\;& \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \end{aligned}$$

for $s > t$.

Remark 9.5.1. – In the classical setting the reserves were deterministic, but here they depend on the underlying share S.

– Note that we are beyond the deterministic theory of differential equations. In particular we have to use Itô's formula, which takes the following form for the purely continuous case of a standard Brownian motion W:

$$df(W) = f' \, dW + \frac{1}{2} f'' \, ds.$$

For the policies defined above we have the following theorem.

Theorem 9.5.2. *1. The differential equation for the price of a pure endowment policy is:*

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta) \, V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta \, S(t) \frac{\partial V}{\partial S}.$$

2. The differential equation for the price of a term life insurance is:

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta) \, V(t) - C(t)\mu_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta \, S(t) \frac{\partial V}{\partial S}.$$

Before we prove this theorem, we want to make some comments on the formulas.

Remark 9.5.3. 1. One obtains Black-Scholes formula by setting $\mu_{x+t} = \bar{p}(t) = 0 \, \forall t$.

2. The first terms in the differential equations in the theorem above coincide with the classical case (see section 5.3), i.e. the dependence of the values on the premiums, on the mortality and on the interest rate. Due to the shares in the model a further term appears: $-\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta \, S(t) \frac{\partial V}{\partial S}$. It represents the fluctuations of the underlying shares.

Proof. We have

$$\pi_t^*(T) = \exp(-\delta t)\pi_t(T).$$

Hence, by the definition of V, we get

$$V(t) = {}_{T-t}p_{x+t} \pi_t^*(T) \exp(\delta t) - \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t} p_{x+t} d\xi$$

and

$$\pi_t^*(T) = \Psi(t) \left[V(t) + \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t} p_{x+t} d\xi \right],$$

where

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

Now we can apply Itô's formula to the function $\pi_t^*(t, S)$, since π_t^* is a function of S and t. We get

$$\begin{aligned} dY_t &= U(t+dt, X_t + dX_t) - U(t, X_t) \\ &= \left(U_t dt + \frac{1}{2} U_{xx} b^2 dt\right) + U_x dX_t \\ &= \left(U_t + \frac{1}{2} U_{xx} b^2\right) dt + U_x \, b \, dB_t \end{aligned}$$

and

$$d\pi^* = \left(\frac{\partial \pi^*}{\partial t} + \frac{\partial \pi^*}{\partial S} \, a + \frac{1}{2} \, \frac{\partial^2 \pi^*}{\partial S^2} \, b^2\right) dt + \frac{\partial \pi^*}{\partial S} b \, d\hat{W}.$$

Furthermore we know that

$$dS = \delta S(t) dt + \sigma S(t) d\hat{W},$$

and thus we have $a = \delta S(t)$ and $b = \sigma S(t)$. In the next step we want to determine the two terms:

$$\begin{aligned} \frac{\partial \pi_t^*}{\partial S} &= \Psi(t) \, \frac{\partial V}{\partial S}, \\ \frac{\partial^2 \pi_t^*}{\partial S^2} &= \Psi(t) \, \frac{\partial^2 V}{\partial S^2}. \end{aligned}$$

To get $\frac{\partial \pi^*}{\partial t}$, we start with

$$\begin{aligned} \frac{\partial}{\partial t} {}_{\xi-t}p_{x+t} &= \mu_{x+t} \, {}_{\xi-t}p_{x+t}, \\ \frac{\partial}{\partial t} \Psi(t) &= \left(\frac{A}{B}\right)' = \frac{A'}{B} - \frac{A}{B^2} \, B' \\ &= -\left(\mu_{x+t} + \delta\right) \Psi(t). \end{aligned}$$

Now, with the formula from above we get

$$\begin{aligned} \frac{\partial \pi^*}{\partial t} &= \frac{\partial \Psi}{\partial t} \left(V(t) + \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt\right) \\ &+ \Psi(t) \left(\frac{\partial V}{\partial t} + \frac{\partial}{\partial t} \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt\right) \\ &= \Psi(t) \left(\frac{\partial V}{\partial t} - \left(\mu_{x+t} + \delta\right) V(t) - \bar{p}(t)\right), \end{aligned}$$

where we applied the chain rule to

$$\frac{\partial}{\partial t} \int_t^T \bar{p}(\xi) \exp(-\delta(\xi - t))_{\xi-t}p_{x+t} dt.$$

Thus we get

$$\begin{aligned} \pi_s^*(T) &= \pi_t^*(T) + \int_t^s \Psi(\xi) \frac{\partial V}{\partial S} \sigma S d\hat{W}(\xi) \\ &+ \int_t^s \Psi(\xi) \left[\frac{\partial V}{\partial S} \delta S + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (\mu_{x+\xi} + \delta) V(\xi) \right. \\ &\qquad \left. + \frac{\partial V}{\partial t}(\xi) - \bar{p}(\xi)\right] d\xi. \end{aligned}$$

Now the drift term is equal to zero, since $\pi^*_\cdot(T)$ is a martingale. Therefore we finally get

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)\, V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.$$

Exercise 9.5.4. Prove the second part of the theorem above.

Remark 9.5.5. Thiele's differential equation for pure endowments and term insurance (Theorem 9.5.2), has been stated assuming no lapses. In reality a lot of the corresponding GMDB and GMAB policies are lapse supported. Depending on the context the surrender value of the policy varies. If we assume that the surrender value in case of lapse is given by $f(x)$ for a function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$$

it is easy to show that the mathematical reserves for the model including lapsed with a lapse density $\mu^l(x)$ fulfill the following partial differential equations:

1. The differential equation for the price of a pure endowment policy is:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \bar{p}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - f(x)\mu^l_{x+t} \\ &\quad -\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}. \end{aligned}$$

2. The differential equation for the price of a term life insurance is:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \bar{p}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} - f(x)\mu^l_{x+t} \\ &\quad -\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}. \end{aligned}$$

It is worth mentioning that in many cases we have $f \equiv 0$, meaning that the option value of the variable annuity guarantee is lost in case of a surrender.

Exercise 9.5.6. Prove the modified partial differential equations as per Remark 9.5.5. Note that in this context one has to replace

$$\Psi(t) = \frac{\exp(-\delta t)}{{}_{T-t}p_{x+t}}.$$

by

$$\Psi(t) = \frac{\exp(-\delta t)}{\bar{p}_{**}(x+t, x+T)}$$

(see definition 2.3.6). By theorem 2.3.7 we can use the following equation:

$$\bar{p}_{**}(s,t) \quad = \quad \exp\left(- \sum_{k \neq *} \int_s^t \mu_{*k}(\tau) d\tau \right). \tag{9.7}$$

Remark 9.5.7. Until now we have mainly considered insurance covers which pay a lump sum in case a person survives a certain number of years (endowment) or in case of death (term insurance). The same methodology shown above can be used to model unit-linked annuities (GMWB variable annuities). Since an annuity can be considered as a negative insurance premium, we have actually already included this case in Theorem 9.5.2 and in Remark 9.5.5. We will use the same notation as in Remark 9.5.5. Furthermore we denote by $\bar{r}(t)$ the annuity density being paid to the policyholder. Keeping in mind that $\bar{p}(t)$ in Theorem 9.5.2 needs to be replaced by $\bar{p}(t) - \bar{r}(t)$ we get the following Thiele partial differential equation for a GMWB / GMDB combined insurance product:

$$\begin{aligned}\frac{\partial V}{\partial t} &= \bar{p}(t) - \bar{r}(t) + \left(\mu_{x+t} + \mu^l_{x+t} + \delta\right) V(t) - C(t)\mu_{x+t} \\ &\quad -f(x)\mu^l_{x+t} - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta\, S(t) \frac{\partial V}{\partial S}.\end{aligned}$$

By using a Taylor expansion we get the following movement of $V(t)$ during the interval $[t, t+\Delta t]$.

$$\begin{aligned}V(t) + \bar{p}(t)\Delta t &= \bar{r}(t)\,\Delta t + \sum_i \left(\mu^i_{x+t}\, C_i(t)\right)\,\Delta t \\ &\quad - \left(V(t) - S(t)\,\frac{\partial}{\partial S} V(t)\right)\,\delta\Delta t \\ &\quad - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2}\Delta t \\ &\quad + \left(1 - \left(\sum_i \mu^i_{x+t}\right)\Delta t\right)\, V(t+\Delta t)\end{aligned}$$

We will see in the following that the above roll forward of $V(t)$ is useful when introducing the concept of a hedge P&L.

9.6 Example

In this section we consider a concrete example for which the corresponding calculations can be explicitly performed. We consider the following steps:

1. Description of the product,
2. Valuation of the product, and

The value of a variable annuity from a policyholder point of view consists of two parts:

1. The value of the underlying fund investment, and
2. the value of the variable annuity rider (eg GMDB, GMAB, GMIB and GM(L)WB).

From an insurer's point of view the fund investment is normally classified as a separate account and rather easy to value. The focus on valuation of variable annuities regards the actual variable annuity riders, on which we will focus. Figure 9.9 shows the different parts of a balance sheet of an insurance company which sells unit linked products with guarantees (aka "variable annuities").

In order to value such variable annuities there are different ways to determine the value of the underlying guarantee:

- Explicit formula or recursion (only for insurance valuation and very simple variable annuities),
- Solution of Black-Scholes-Merton differential equation / Thiele's differential equation (different methods including tree method),
- Monte Carlo Simulation (this is the approach most often used for variable annuities).

Monte Carlo is most commonly used for variable annuities since it is very versatile and can also cope with very complex option structures, such as ratchets. For this example, we will however limit ourselves to a product which can be valued explicitly. In order to keep things simple we look in a first step at a discrete time version of Theorems 9.4.1 and 9.4.2.

Theorem 9.6.1. *Assume that the economy follows the Black-Scholes-Merton model and assume that deaths occur at time $K = \lfloor T \rfloor$. For*

$$C(t) = \max\{N(t)S(t), G(t)\}$$

we can calculate the single premiums as follows:

$$\begin{aligned} {}_T G_x &= {}_T p_x \, \pi_0^*(T), \\ G^1_{x:T} &= \sum_{k=0}^{T-1} {}_k p_x \, q_{x+k} \, \pi_0^*(k), \end{aligned}$$

where

$$\begin{aligned} \Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx, \\ d_1^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \\ d_2^t(s) &= \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}, \end{aligned}$$

for $s > t$.

Exercise 9.6.2. Prove the previous theorem.

9.6.1 Definition of the product

In the following we consider a very simple variable annuity product consisting of a term assurance ("Guaranteed Minimal Death Benefit (GMDB)") and of a pure endowment policy ("Guaranteed Minimal Accumulation Benefit (GMAB)"). This is exactly the set up which we have considered in section 9.4.

- Variable annuity for a 30 year old man consisting of GMDB and GMAB for a term of 25 years.
- Variable annuity guarantee level according to a "doubler", eg
  - At maturity guarantees twice the initial fund value,
  - In case of death, at least double the value after 10 years.
- Single premium for both funds value and guarantee premium.

In order to illustrate things clearer we will base our calculations on the following decrement table ($l_{35} = 100000$, $l_{x+1} = l_x\,(1 - q_x)$, and $d_x = l_x\, q_x$):

x	l_x	d_x
35	100000.0	147.7
36	99852.3	157.5
37	99694.8	169.5
38	99525.4	183.8
39	99341.5	200.8
40	99140.8	220.3
...		
60	86622.9	1394.0

We note that one can calculate

$$\begin{aligned} {}_tp_x &= \frac{l_{x+t}}{l_x}, \text{ and,} \\ {}_tp_x\, q_{x+t} &= \frac{d_{x+t}}{l_x}. \end{aligned}$$

In a next step we want to do this example also allowing for lapses. For sake of simplicity, we have assumed a level lapse rate of 4% with an exception for year 10 where the assumed lapse rate is 12%. These uneven lapse rates are typical – after 10 years (or so, product dependent), the policyholder has the ability to lapse without surrender penalty, leading to higher lapses in this year.

The following table shows the decrement table including lapses:

x	l_x	d_x	l_x incl. lapse	d_x incl. lapse
35	100000.0	147.7	100000.0	147.7
36	99852.3	157.5	95852.3	151.2
37	99694.8	169.5	91860.7	156.2
38	99525.4	183.8	88016.8	162.6
...				
44	98114.7	326.8	67604.5	225.2
45	97787.9	360.9	60517.2	223.4
46	97426.9	398.4	57735.6	236.1
...				
60	86622.9	1394.0	23716.1	381.7

Note that also in this context we can calculate ${}_tp_x$ and ${}_tp_x\, q_{x+t}$ by means of l_x and d_x. Based on the above table we see that the valuation of a variable annuities heavily depends on lapses and also that the guarantee gets cheaper when considering lapses, since less people profit from it. In consequence most variable annuity products are lapse supported. "Lapse supported" means that the pricing of the product relies on the fact that some of the policyholders will lapse early.

9.6.2 Valuation of the Product / Replicating of a Variable Annuity

Based on the example introduced before we now determine the value or price for this variable annuity guarantee. In order to do this, three steps are involved:

Determine the number of people which benefit: Since only a tiny percentage of the whole inforce dies within a given year, one only needs to provide the respective GMDB cover only to them. Similarly the GMAB cover is paid only to the people surviving the entire term of the policy. Hence we need to determine the respective percentages. This is done by means of life decrement tables, as introduced before.

Calculate what these people receive: We need to know what the respective policyholders are entitled to. Assume for example the people dying at age 40. They are entitled to get a GMDB at a certain level. Hence we need to determine the number of the corresponding units of guarantees. For our 40 year old dying person, this would be put options at a strike price.

Calculate the value: We know the valuation portfolio of guarantees representing the variable annuity guarantee (eg number of instruments and their characteristics). We now need to value them. For our example this is done via the Black-Scholes formula.

Note: only in this simple example we can easily distinguish between steps 2 and 3. Normally one performs 2 and 3 together using a Monte Carlo simulation. For our concrete example the table below provides the portfolio of guarantees at inception.

Instrument	Strike	Amount
Fund		100000
Put Fund value t=0 at	100000	0.1 %
Put Fund value t=1 at	107177	0.2 %
Put Fund value t=2 at	114870	0.2 %
Put Fund value t=3 at	123114	0.2 %
Put Fund value t=4 at	131951	0.2 %
Put Fund value t=5 at	141421	0.2 %
...		
Put Fund value t=9 at	186607	0.3 %
Put Fund value t=10 at	200000	0.4 %
...		
Put Fund value t=25 at	200000	86.6 %

Over time, as the policy matures more and more of these instruments are used to pay the guarantees of the corresponding period. Deviations from this modeled guarantee portfolio result in a profit or loss.

Based on the valuation portfolio we can now calculate the various metrics for the policy. In particular we can value the variable annuity guarantee of the policy by valuing each individual instrument:

	Instrument	Strike	Amount %age	**Value**
0	Put Fund	100000	0.1 %	8.7
1	Put Fund	107177	0.2 %	17.7
2	Put Fund	114870	0.2 %	27.7
3	Put Fund	123114	0.2 %	39.5
4	Put Fund	131951	0.2 %	53.6
5	Put Fund	141421	0.2 %	70.6
...				
9	Put Fund	186607	0.3 %	146.0
10	Put Fund	200000	0.4 %	181.8
...				
25	Put Fund	200000	86.6 %	34789.4
	Total			**40311.7**

We need to look at the consequence of different market shocks at inception, such as lower equity prices, lower interest rates and higher volatility. Note that the valuation portfolio does not change and remains (in terms of respective types of guarantees and amounts) the same! Concretely we look at the following three shocks:

- Equity Drop by 10%,
- Interest Lower by 1%,
- Volatility up by 1%.

The following table summarises the corresponding results. We can observe the high dependency of the value on the market variables.

	Instrument	Value Normal	Value Equity −10%	Value Interest −1%	Value Volatility +1%
0	Put Fund	8.7	16.5	9.5	14.5
1	Put Fund	17.7	26.5	19.6	26.5
2	Put Fund	27.7	37.6	31.3	39.4
3	Put Fund	39.5	50.6	45.3	54.1
4	Put Fund	53.6	66.0	62.4	71.4
5	Put Fund	70.6	84.6	83.4	92.0
...					
9	Put Fund	146.0	165.8	178.2	180.8
10	Put Fund	181.8	204.0	224.2	222.3
...					
25	Put Fund	34789.4	38092.7	54782.4	50112.1
	Total	**40311.7**	**44239.3**	**62445.2**	**57454.8**

Finally we look how the value of the valuation portfolio changes over time. There are two effects which affect its value, namely that, over time, parts of it are used to finance the claims which have occurred in the past and also as a consequence of the market movement of the underlying fund. Figure 9.10 shows how the guarantee of the hedge liability moves over time:

- Upper figure shows movement in fund value. The lower figure shows the changing value of the underlying guarantee.
- Note that at inception the value of the guarantee equals the value determined above (40311).

- Over time the valuation portfolio becomes smaller, since part of its instruments are used to finance the claims which have occurred in the past.
- Moreover the valuation portfolio changes its value as a consequence of changing fund levels, interest rates and volatilities.
- For this example interest rates and volatilities have been keep constant.
- It becomes obvious that the value of the guarantee increases each time fund value decreases and vice versa.

It is important to understand that none, except the most simple variable annuity guarantee structures can be calculated explicitly by the Black-Scholes formula. The examples in this text have been designed in such a manner that they still allow to use the Black-Scholes formula.

9.6.3 Value of a variable annuity as a function of equity level

Finally, the following table shows the dependency of the value (π) of the variable annuity guarantee as a function of the equity level and is called a trading grid. The lower the equity level, the more valuable the variable annuity guarantee.

Equity Level	π $V(S)$	δ $\frac{\partial}{\partial S} V$	γ $\frac{\partial^2}{\partial S^2} V$	ρ $\frac{\partial}{\partial r} V$	ν $\frac{\partial}{\partial \sigma} V$
-50 %	65777	-33818	39673	-2341692	86714
-40 %	59397	-36051	42961	-2241519	111455
-30 %	53734	-37319	44743	-2136639	133559
-20 %	48710	-37826	45431	-2030460	152596
-10 %	44254	-37744	45168	-1925345	168470
-5 %	42219	-37531	44705	-1873714	175247
0 %	40301	-37230	44052	-1822905	181284
+5 %	38494	-36854	43243	-1773034	186616
+10 %	36789	-36419	42315	-1724192	191283
+20 %	33662	-35417	40235	-1629845	198789
+30 %	30871	-34295	38026	-1540199	204141
+40 %	28372	-33102	35808	-1455379	207655
+50 %	26130	-31872	33640	-1375369	209618

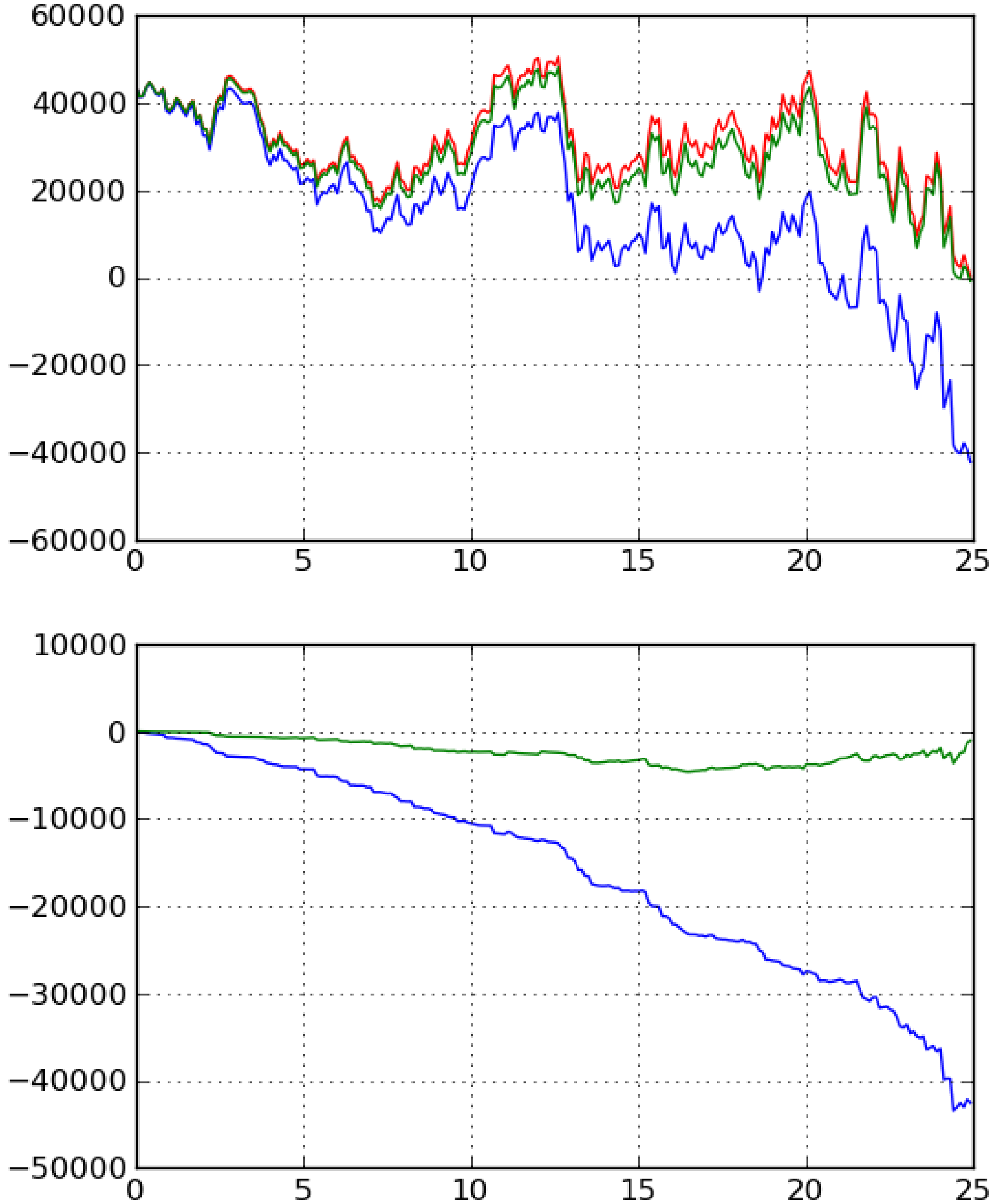

Figure 9.8: Delta Hedge

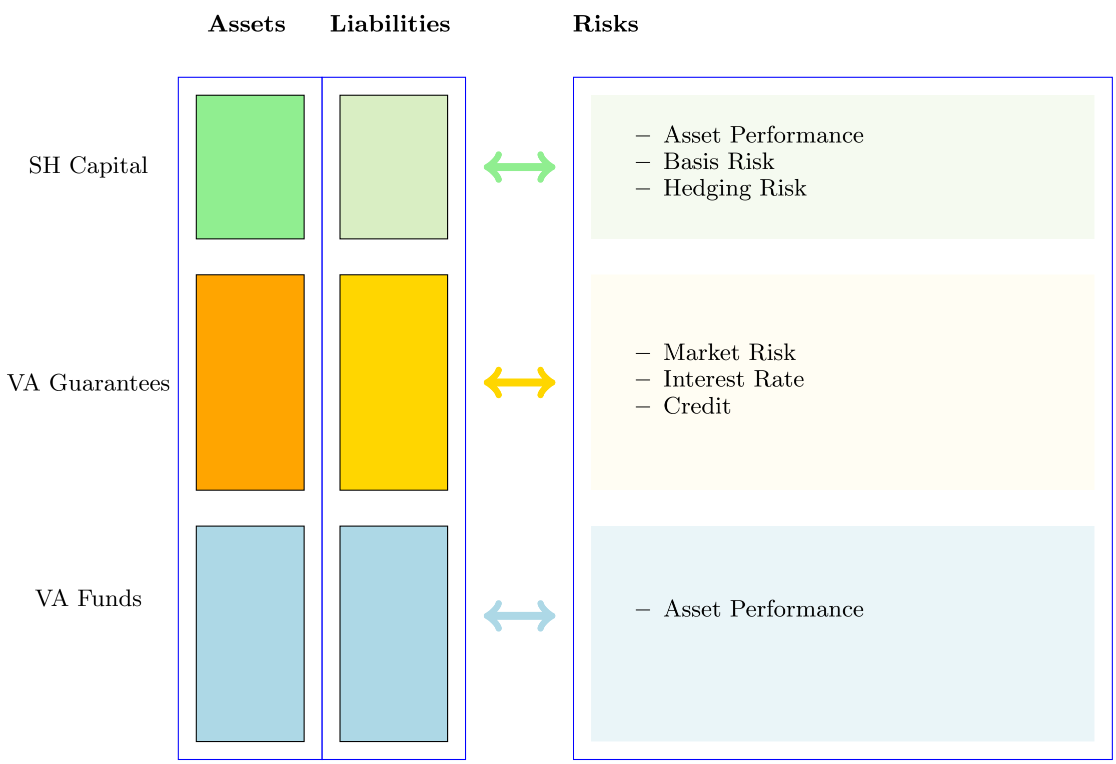

Figure 9.9: Balance sheet of an insurance company

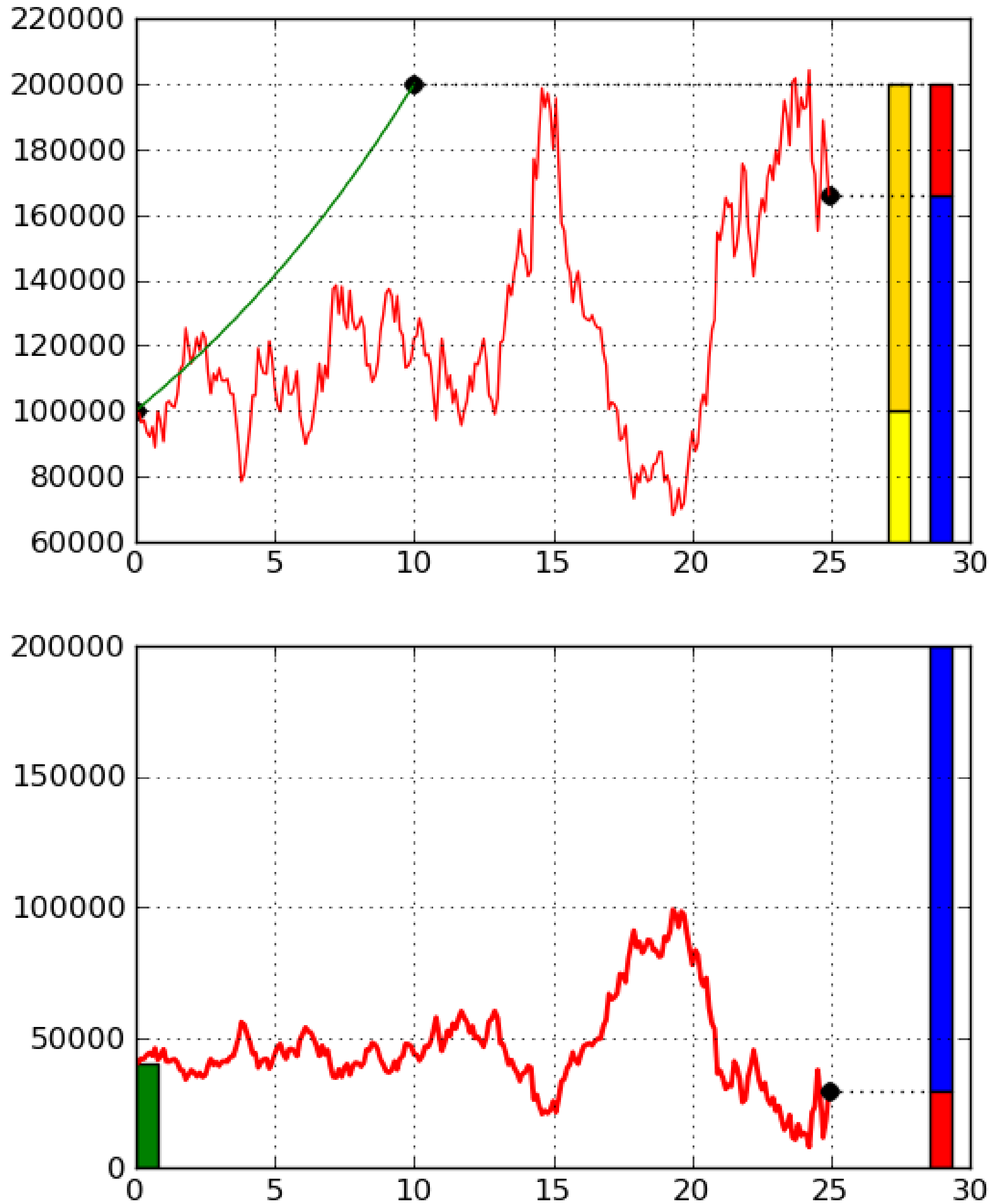

Figure 9.10: Value of Hedge Liability over time

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