Life Insurance Mathematics

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Changes:

Version	Date	Author	Changes
0.10	Sommer 1997	M. Koller	Handschriftliches Manuskript
0.91	Sommer 2013	M. Koller	Fehler korrigiert
1.10	Autumn 2025	M. Koller	English Version

Note: As of version 1.00E the script has undergone material reworking reflecting the fact that the lecture is now given within one semester as opposed to two semesters before. As a consequence the materials has been reordered with an emphasis on the Markov model. Also ancillary material has been removed and the script has been translated into English.

As such this script is now suitable for a one semester course and is complemented by worked examples.

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1 Introduction / Historical Notes

1.1 History of Life Insurance

Chronological Summary of Life Insurance History:

1. Ancient Roots (ca 1750 BCE – 200 CE) – Early risk-sharing in Babylonian Code of Hammurabi and Roman burial societies.
2. Medieval Guilds (5th–14th c.) – Guilds and confraternities provided mutual aid for members' families.
3. Renaissance Marine & Life Wagers (14th–16th) – Merchants experimented with contracts on lives and voyages.
4. First Recognised Life Policy (1583) – First modern-style life insurance contract in London on William Gibbons' life.
5. Early Life Assurance Societies (17th) – Formal pooled cover like Amicable Society (1706).
6. Actuarial Science Emerges (1693–1750s) – Halley's mortality table and Dodson's premium calculations.
7. Equitable Life Assurance Society (1762) – First to use age-based premiums and reserves.
8. 19th-Century Expansion – Industrialisation spurred growth in Britain, Europe, and the U.S.
9. 20th-Century Regulation & Innovation – Solvency rules, government oversight, and new product types.
10. 21st-Century Trends – Digital underwriting, Insurtech, and demographic shifts.

1. Ancient Roots (ca 1750 BCE – 200 CE)

1. Babylonian Code of Hammurabi contained clauses allowing debt cancellation or payment upon death, an early form of risk sharing.
2. In ancient Rome, collegia funeraticae (burial clubs) pooled member contributions to fund funeral expenses and provide for survivors, resembling basic life cover.

2. Medieval Guilds (5th–14th century)

1. In medieval Europe, craft guilds and religious confraternities collected dues to support members' families in cases of death.

2. These mutual-aid systems provided a social safety net, functioning as precursors to modern life insurance principles.

3. Renaissance Marine & Life Wagers (14th–16th century)

1. Italian city-states like Genoa and Florence saw merchants using marine insurance contracts that evolved to include human lives.
2. In England, by the 16th century, life wagersâ€"essentially betting on someone's survivalâ€"were common among traders, blending gambling with risk transfer.
3. These practices laid groundwork for structured life assurance in the early modern period.

1.2 Evolution of Life Tables and Life Insurance Mathematics

1. Early Mortality Observations (Ancient–1600s) – Rough demographic notes in ancient civilisations and parish records, but no formal mathematics for life expectancy.
2. Graunt's Mortality Analysis (1662) – John Graunt publishes Natural and Political Observations, analysing London Bills of Mortality to estimate life expectancy.
3. Halley's Mortality Table (1693) – Edmond Halley builds the first scientific life table using Breslau parish data, linking age-specific death probabilities to annuity pricing.
4. Bernoulli's Mathematical Framework (1725–1732) – Jakob Bernoulli and later Daniel Bernoulli apply probability theory to life annuities, establishing expected value methods for insurance.
5. De Moivre's Law of Mortality (1725) – Abraham de Moivre proposes a linear mortality model simplifying annuity and premium calculations.
6. Price & Dodson (1750s–1760s) – Richard Price refines life tables and advises Equitable Life; James Dodson pioneered using mortality data for premium setting.
7. 19th-Century Refinement – National censuses and statistical societies expand mortality data; actuaries develop graduation techniques and commutation functions.
8. 20th-Century Advances – Stochastic processes, Markov models, and computers enable dynamic life tables, select and ultimate tables, and multi-decrement models.
9. Modern Actuarial Science (Late 20th–Early 21st) – Use of generalised linear models, survival analysis, and credibility theory improves mortality forecasting and risk pricing.
10. Contemporary Innovations – Big data, machine learning, and longevity research drive dynamic cohort tables and updated actuarial practice.

1.3 Life Expectancy over the centuries

The following figures aim to estimate the life expectancy over the past centuries. The following approach has been chosen:

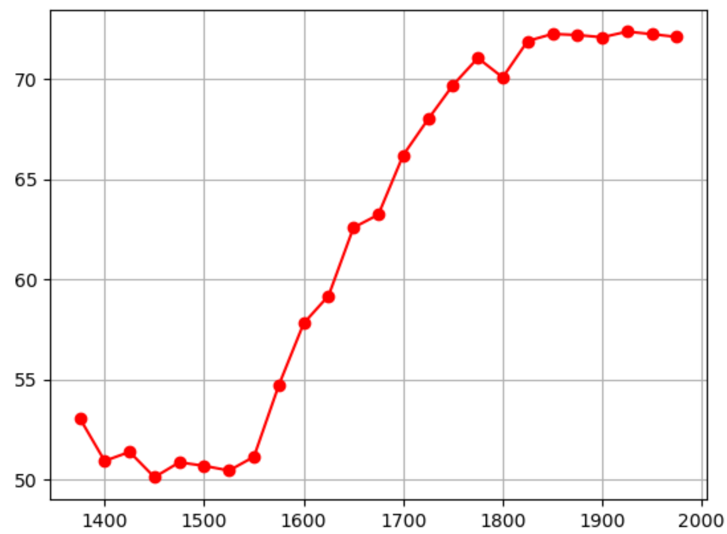
- About 100 people per century have been chosen and the attained age has been recorded. These are famous people where life records exist hence there might be a bias.

- For each time period a range has been chosen and life expectancy and cumulative survival probabilities have been estimated, also using Meier-Kaplan extimators)

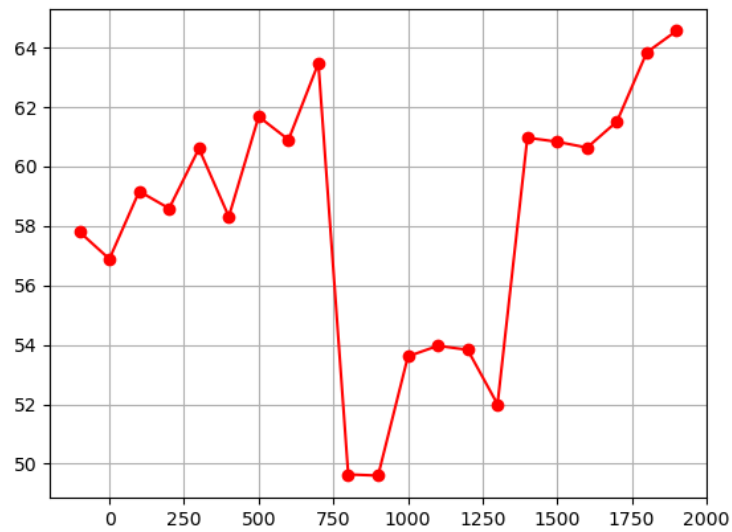
- There is a material bias since almost no juvenile deaths have been recorded. As such the figures should likely be interpreted to the people surviving at least to age 20.

Life Expectancy

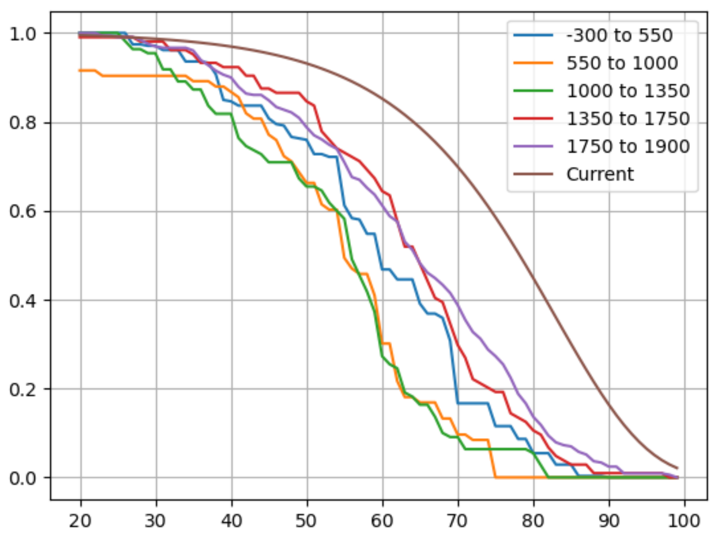
Big Data Set, 2M records



Data Set as per above



Survival Probabilities



2 Financial Mathematics

2.1 Present Value

2.1.1 Definitions and nomenclature

Loan and interest: A loan is the borrowing of a sum. e.g. bond, mortgage, debt, ..., an the interest is the cost for this loan for the borrower.

Interest rate i: The interest on a capital 1 during a unit period. It is paid out at the end of the period and is therefore called **in arrears**.

Present value: The value of capital at a given moment.

Initial and final capital, K_0, K_n: Present value at the beginning and end of a period with n time units.

Simple interest rate: A constant linear interest amount is assumed, i.e.

$$K_n = K_0(1+ni)\,. \tag{2.1}$$

This assumption is appropriate for a short period of time, e.g. interest calculation within one year, $K_t = K_0(1+\frac{t}{360}i)$. So e.g. CHF 10'000.-,$i = 5\%$, $t = 3$ months, then the interest for the three months is CHF 125.

Compound interest: The interest bears interest again in the next period, i.e. the interest amount increases,

$$K_n = K_0(1+i)^n\,. \tag{2.2}$$

$r := 1+i$ is called **accumulation factor**. The compounding factor r thus corresponds to the final value of capital 1 after one period.

To see the connection to the simple interest rate, you can make a series expansion: $K_n = K_0 \sum_{k=0}^{n} \binom{n}{k} i^k = K_0(1+in)+o(i)$.

Discounting: The calculation of the initial value from the final value, i.e. the inverse procedure to compounding:

$$K_0 = \frac{1}{1+in} K_n \text{ with simple interest} \tag{2.3}$$

$$K_0 = (1+i)^{-n} K_n \text{ with compound interest} \tag{2.4}$$

The coefficient $v := \frac{1}{1+i} = \frac{1}{r}$ is called **discount factor**.

Interest in advance d**:** You can also imagine paying out the interest at the beginning of a period. However, one would like to then that the final value of the interest in advance is equal to the interest in arrears, i.e. $d(1+i) = i$, i.e.

$$d = iv = 1 - v \tag{2.5}$$

Conversely, the discounted interest paid in arrears must be equal to the interest paid in advance.

2.1.2 General case with additional payments

K_t is again the present value at time t. The additional payments are denoted by r_t, i.e. it is

$$K_t = (1+i)K_{t-1} + r_t \tag{2.6}$$

respectively $K_t - (1+i)K_{t-1} = r_t$. If the formula is applied recursively, then you get for the final value

$$K_n = (1+i)^n K_0 + \sum_{k=1}^{n} (1+i)^{n-k} r_k$$

and for the initial value

$$\begin{aligned} K_0 &= (1+i)^{-n} K_n - \sum_{k=1}^{n} (1+i)^{-k} r_k \\ &= v^n K_n - \sum_{k=1}^{n} v^k r_k \end{aligned}$$

You can now consider what the difference $K_n - K_0$ is made up of. we deduce

$$K_t - K_{t-1} = iK_{t-1} + r_t\,,$$

therefore

$$\begin{aligned} K_n - K_0 &= (K_n - K_{n-1}) + (K_{n-1} - K_{n-2}) + \ldots + (K_1 - K_0) \\ &= (iK_{n-1} + r_n) + (iK_{n-2} + r_{n-1}) + \ldots + (iK_0 + r_1) \\ &= \sum_{k=0}^{n-1} iK_k + \sum_{k=1}^{n} r_k\,. \end{aligned}$$

This shows that the difference between the initial and final capital is made up of the interest on the capital and the the co-payments.

Example: Let $K_0 = 12000$, the additional payments $r_t = t \cdot 500$, the interest $i = 9\%$. Then the capital develops over 15 periods according to the figure below. ◇

Example: A 30-year-old person brings a deposit of CHF 25'000.- with them for the annuity fund. The annual salary amounts to CHF 50'000 (the be constant until retirement for the sake of simplicity). The interest rate is 4%. The after BVG mandatory contributions are shown in table 2.1. Then the following develops the capital according to the table 2.1 and figure 2.2. Note that contributions increase with age, from 7% to 18%.

◇

Alter	K_t	Zins	Beitrag relativ	Beitrag absolut	K_{t+1}
30	25000	1000	7%	3500	29500
31	29500	1180	7%	3500	34180
32	34180	1367	7%	3500	39047
33	39047	1562	7%	3500	44109
34	44109	1764	7%	3500	49373
35	49373	1975	10%	5000	56348
36	56348	2254	10%	5000	63602
37	63602	2544	10%	5000	71146
38	71146	2846	10%	5000	78992
39	78992	3160	10%	5000	87152
40	87152	3486	10%	5000	95638
41	95638	3826	10%	5000	104464
42	104464	4179	10%	5000	113642
43	113642	4546	10%	5000	123188
44	123188	4928	10%	5000	133115
45	133115	5325	15%	7500	145940
46	145940	5838	15%	7500	159278
47	159278	6371	15%	7500	173149
48	173149	6926	15%	7500	187575
49	187575	7503	15%	7500	202578
50	202578	8103	15%	7500	218181
51	218181	8727	15%	7500	234408
52	234408	9376	15%	7500	251284
53	251284	10051	15%	7500	268836
54	268836	10753	15%	7500	287089
55	287089	11484	18%	9000	307573
56	307573	12303	18%	9000	328875
57	328875	13155	18%	9000	351030
58	351030	14041	18%	9000	374072
59	374072	14963	18%	9000	398035
60	398035	15921	18%	9000	422956
61	422956	16918	18%	9000	448874
62	448874	17955	18%	9000	475829
63	475829	19033	18%	9000	503862
64	503862	20154	18%	9000	533017
65	533017				

Table 2.1: Example pension fund: The development of the pension fund assets

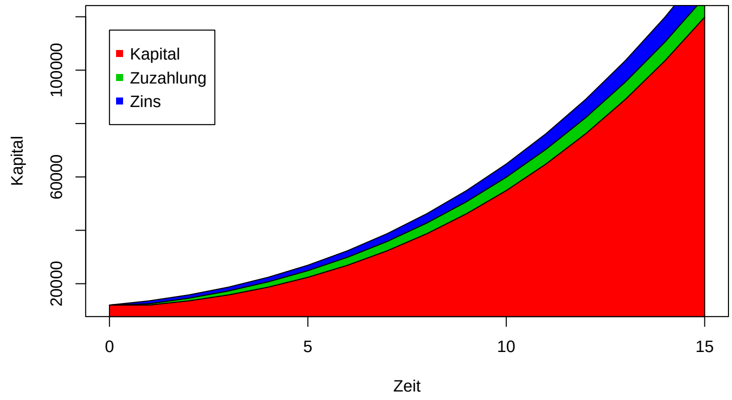


Figure 2.1: Example with co-payments

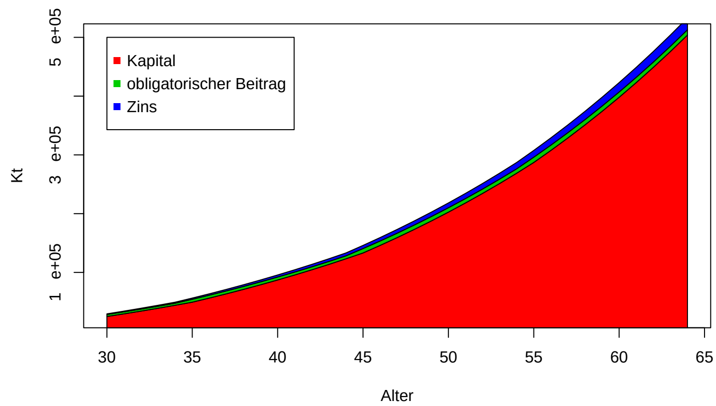


Figure 2.2: Example with co-payments

2.2 Present Value of an Annuity

Financial annuity: Recurring payments at regular intervals.

Annuity rates: These

- are paid in advance (at the beginning of the year) or in arrears (at the end of the year).
- can also be paid out during the year,
- are paid out on a temporary or permanent basis,
- are constant or variable,
- are paid out immediately or starting at a later date – deferred.

Present value of an annuity: The value of future annuity payments at a specific point in time (at a given interest rate). In most cases, the present value is the value at the beginning of the period and the final value at the end of the period.

2.2.1 Nomeclatura

In principle, present values are denoted by a and terminal values by s. In the following table the designations for the present value are noted. The same applies to the terminal value. Under the tick $_{\overline{\cdot}|}$ represents the run duration. Here, only one point $\cdot$ was used in each case instead of a fixed number.

in arrears	$a_{\overline{\cdot}\vert}$	in advance	$\ddot{a}_{\overline{\cdot}\vert}$
paid once a year (1/1)	$a_{\overline{\cdot}\vert}$	paid in m installments (1/m)	$a^{(m)}_{\overline{\cdot}\vert}$
temporary n years	$a_{\overline{n}\vert}$	perpetual	$a_{\overline{\infty}\vert}$
immediate	$a_{\overline{\cdot}\vert}$	deferred k years	${}_{k\vert}a_{\overline{\cdot}\vert}$
constant	$a_{\overline{\cdot}\vert}$	linearly increasing with parameter q linearly decreasing with parameter q	$(I^{(q)}a)_{\overline{\cdot}\vert}$ $(D^{(q)}a)_{\overline{\cdot}\vert}$
discrete time	$a_{\overline{\cdot}\vert}$	continuous time	$\bar{a}_{\overline{\cdot}\vert}$

Of course, combinations can also occur, e.g. a deferred (k periods), Temporary (n periods) annuity in advance during the year is denoted by ${}_{k|}\ddot{a}^{(m)}_{\overline{n}|}$.

2.2.2 Perpetual annuity

(a) Prenumerando annuities:

(1/1)-advance

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + v^3 + \ldots = \frac{1}{1-v} = \frac{1}{d}.$$

(1/m)-advance

$$\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{m} + \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \frac{1}{m}v^{\frac{3}{m}} + \ldots = \frac{1}{m}\frac{1}{1-v^{\frac{1}{m}}} = \frac{1}{d^{(m)}}, \tag{2.7}$$

where $d^{(m)}$ is the nominal advance interest rate.

(b) annuities in arrears:

(1/1)-arrears

$$a_{\overline{\infty}|} = v + v^2 + v^3 + \ldots = \frac{v}{1-v} = \frac{1}{i}.$$

(1/m)-arrears

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \frac{1}{m}v^{\frac{1}{m}} + \ldots = \frac{1}{m}\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}} = \frac{1}{m[(1+i)^{\frac{1}{m}}-1]} = \frac{1}{i^{(m)}}.$$

(c) Continuous annuities:

$$\bar{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t}\, dt = [-\frac{1}{\delta}e^{-\delta t}]_0^\infty = \frac{1}{\delta}$$

Remarks:

1. $\ddot{a}^{(m)}_{\overline{\infty}|} > \bar{a}_{\overline{\infty}|} > a^{(m)}_{\overline{\infty}|}$

2. $\lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{\infty|}} = \lim_{m\to\infty} a^{(m)}_{\overline{\infty|}}$

◇

(d) Increasing annuities:

Linearly increasing annuity rates are determined by two parameters, namely

- $m =$ Number of payments per year,
- $q =$ Number of increases per year.

Now let q be a factor of m, e.g. $q = 4$, $m = 12$ (see also figure 2.3)

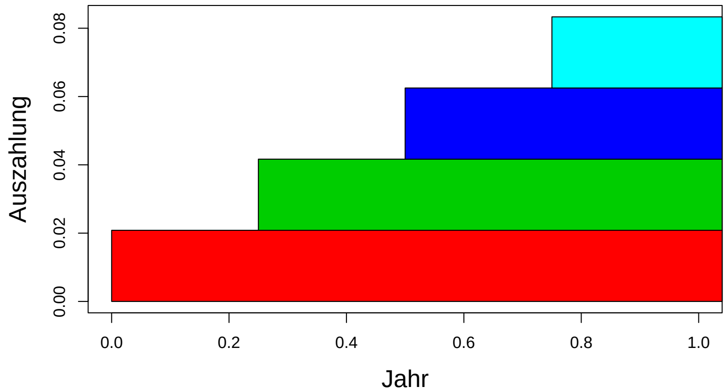


Figure 2.3: Linear increasing time annuity

Time				Payment
0	$\frac{1}{m}$	$\dots$	$\frac{1}{q}-\frac{1}{m}$	$\frac{1}{mq}$
$\frac{1}{q}$	$\frac{1}{q}+\frac{1}{m}$	$\dots$	$\frac{2}{q}-\frac{1}{m}$	$\frac{2}{mq}$
$\frac{2}{q}$	$\frac{2}{q}+\frac{1}{m}$	$\dots$	$\frac{3}{q}-\frac{1}{m}$	$\frac{3}{mq}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

For the calculation of the *advance* present value, the following can be seen from Fig. 2.3 it is easy to see that the following applies:

$$\begin{aligned}(I^q\ddot{a})^{(m)}_{\overline{\infty|}} &= \frac{1}{q}\ddot{a}^{(m)}_{\overline{\infty|}} + \frac{1}{q}v^{\frac{1}{q}}\ddot{a}^{(m)}_{\overline{\infty|}} + \frac{1}{q}v^{\frac{2}{q}}\ddot{a}^{(m)}_{\overline{\infty|}} \dots \\ &= \ddot{a}^{(m)}_{\overline{\infty|}}(\frac{1}{q}[1+v^{\frac{1}{q}}+v^{\frac{2}{q}}\dots]) \\ &= \ddot{a}^{(m)}_{\overline{\infty|}} \cdot \ddot{a}^{(q)}_{\overline{\infty|}} = \frac{1}{d^{(m)}}\frac{1}{d^{(q)}}.\end{aligned} \tag{2.8}$$

In particular, $(I\ddot{a})_{\overline{\infty|}} = \frac{1}{d^2}$ $(q = m = 1)$ now applies.

We get:

$$(I^q a)^{(m)}_{\overline{\infty|}} = a^{(m)}_{\overline{\infty|}} \cdot \ddot{a}^{(q)}_{\overline{\infty|}} = \frac{1}{i^{(m)}}\frac{1}{d^{(q)}}. \tag{2.9}$$

A constant perpetual annuity can be calculated by calculating the limit value $m \to \infty$ forms in (2.8) and (2.9)

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \int_0^\infty t e^{-\delta t}\, dt = \frac{1}{\delta^2}$$
$$(I\bar{a})_{\overline{\infty}|} = \int_0^\infty [t+1] e^{-\delta t}\, dt = \frac{1}{\delta}\frac{1}{d}$$

You can now choose any annuity with annuity instalments r_0,r_1,r_2, ... and calculate

$$\begin{aligned}\ddot{a}_{\overline{\infty}|} &= r_0 + v r_1 + v^2 r_2 + \dots \\ &= r_0 \frac{1}{d} + (r_1 - r_0) v \frac{1}{d} + (r_2 - r_1) v^2 \frac{1}{d} + \dots \\ &= \frac{1}{d} (r_0 + (r_1 - r_0) v + (r_2 - r_1) v^2 + \dots).\end{aligned}$$

Linear or exponentially growing annuities are common, e.g. $r_t = e^{\tau t}$, then you get with $v = e^{-\delta}$

$$\ddot{a}_{\overline{\infty}|} = 1 + e^{-(\delta - \tau)} + e^{-2(\delta - \tau)} + \dots = \frac{1}{1 - e^{-(\delta - \tau)}}$$

2.2.3 Temporary, immediately commencing annuities

The present values and terminal values of temporary annuities can be calculated in the same way as perpetual annuities, only that you are dealing with finite instead of infinite series.

1. **Barwerte**
   Prenumerando Present value of an annuity with duration over n periods:

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \dots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d} = \ddot{a}_{\overline{\infty}|} - v^n \ddot{a}_{\overline{\infty}|}$$

   It can therefore be seen that the present value of a temporary annuity is calculated from the Difference of a present value of a perpetuity and the present value of an n-times perpetual annuity can be calculated. Such differences of We will encounter present values more often.

   Similarly, the present value of a *in arrears* temporary annuity is obtained by

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = a_{\overline{\infty}|} - v^n a_{\overline{\infty}|} \tag{2.10}$$

   Present values in advance and in arrears are related as follows

$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

2. **Final Values**
   Final values can be easily calculated from the present values by simply compounded for n years, i.e.

$$\begin{aligned}s_{\overline{n}|} &= r^n a_{\overline{n}|} = \frac{r^n - 1}{i} \\ \ddot{s}_{\overline{n}|} &= r^n \ddot{a}_{\overline{n}|} = \frac{r^n - 1}{d}\end{aligned}$$

Present values and final values in arrears are related as follows

$$s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1$$

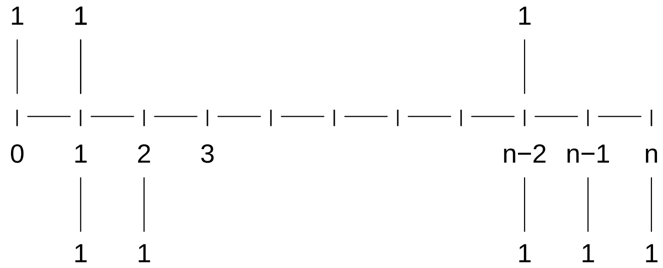


Figure 2.4: Schematic representation of the final values: On the one hand, payments of an advance $n-1$ temporary annuity, on the other hand payments of a n temporary annuity in arrears

3. **Temporary annuity instalments during the year**
   To calculate a temporary annuity during the year, we use the interest during the year and the fact that from the equilibrium condition it follows that $v = {v^{[m]}}^m$. The annuity 1 is now equalised in m. paid out in instalments during the year $\frac{1}{m}$ (during n years) with Interest rate $i^{[m]}$

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} a_{\overline{nm}|} = \frac{1}{m} \frac{1-(v^{[m]})^{mn}}{i^{[m]}} = \frac{1-v^n}{i^{(m)}}$$

   This present value is closely related to the full-year present value, namely

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|}$$

   The more the effective interest rate deviates from the nominal interest rate, the more the two present values differ.

   The following applies to the present value in advance during the year

$$\ddot{a}_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)} + \frac{1}{m}(1-v^n) = \frac{1-v^n}{d^{(m)}} \tag{2.11}$$

   In contrast to the present value in arrears, a present value is added at the beginning. annuity payment of $\frac{1}{m}$ added, at the end one is cancelled (discounted $\frac{1}{m}v^n$).

4. **Increasing annuity instalments**
   Again, let m be the number of payments per year and q the number of increases (linear) per year. Then the temporary (n periods) increasing present value as a linear combination of perpetual present values write, namely

$$\begin{aligned} (I^{(q)}\ddot{a})_{\overline{n}|}^{(m)} &= (I^{(q)}\ddot{a})_{\overline{\infty}|}^{(m)} - v^n (I^{(q)}\ddot{a})_{\overline{\infty}|}^{(m)} - n v^n \ddot{a}_{\overline{\infty}|}^{(m)} \\ &= \frac{\ddot{a}_{\overline{n}|}^{(q)} - n v^n}{d^{(m)}} \end{aligned}$$

   Analogue for present values in arrears

$$\begin{aligned} (I^{(q)}a)_{\overline{n}|}^{(m)} &= (I^{(q)}a)_{\overline{\infty}|}^{(m)} - v^n (I^{(q)}a)_{\overline{\infty}|}^{(m)} - n v^n a_{\overline{\infty}|}^{(m)} \\ &= \frac{\ddot{a}_{\overline{n}|}^{(q)} - n v^n}{i^{(m)}} \end{aligned}$$

5. **Deferred annuities**
   The present values of k periods of deferred annuities are simply the k times the discounted present values of the non-deferred present values.

$$
\begin{aligned}
{}_{k|}a_{\overline{n}|} &= v^k a_{\overline{n}|} \\
{}_{k|}\ddot{a}_{\overline{n}|} &= v^k \ddot{a}_{\overline{n}|} \\
\text{etc.}
\end{aligned}
$$

   It can also be written as the difference between two temporary present values

$$
\begin{aligned}
{}_{k|}a_{\overline{n}|} &= a_{\overline{n+k}|} - a_{\overline{k}|} \\
{}_{k|}\ddot{a}_{\overline{n}|} &= \ddot{a}_{\overline{n+k}|} - \ddot{a}_{\overline{k}|} \\
\text{etc.}
\end{aligned}
$$

6. **Savings insurance**
   The most common insurance in individual insurance is the Payment of a lump sum at the end of the insurance term or before in the event of death. This is the so-called **mixed insurance**. This insurance includes both the savings and the risk element.

   You can also look at the savings process on its own (BVG retirement credits, pure savings process).

   Question: After n years, capital 1 should be available. How large must be the annual savings contribution?

   The savings contribution is paid in advance and with $P_{\overline{n}|}$ labelled. So

$$P_{\overline{n}|}(r^n + r^{n-1} + \ldots + r) = P_{\overline{n}|} \cdot \ddot{s}_{\overline{n}|} = 1.$$

   This means that the savings contribution is equal to:

$$P_{\overline{n}|} = \frac{1}{\ddot{s}_{\overline{n}|}}$$

   After t years, the savings capital has the value

$$P_{\overline{n}|}\ddot{s}_{\overline{t}|} = \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}$$

2.3 Bonds

Let S be the value of a debt at the time 0, which is represented by the payments $r_1, \ldots, r_n$ is repaid at the end of each year $k = 1, 2, \ldots, n$. S is therefore the value of the payments at time 0:

$$S = vr_1 + v^2 r_2 + \ldots + v^n r_n \tag{2.12}$$

Let S_k be the remaining debt after r_k has been paid. You consists of the debt of the previous year less the last Payment

$$S_k = (1+i)S_{k-1} - r_k \tag{2.13}$$

This equation can be transformed to

$$r_k = iS_{k-1} + (S_{k-1} - S_k) \tag{2.14}$$

From this it can be seen that the amortisation in the period k, the so-called **annuity**, consists of two components, the interest on the remaining debt and a **amortisation**. If you multiply the equation (2.13) on both sides with $(1+i)^{h-k}$ and summed over all values of k, then almost removes all terms in S. and it remains

$$S_h = (1+i)^h S - \sum_{k=1}^{h} (1+i)^{h-k} r_h$$

This is a **retrospective** view of the remaining debt. On the other hand, the residual debt can of course also be regarded as the The present value of the outstanding annuities is to be regarded as the present value:

$$S_k = v r_{k+1} + v^2 r_{k+2} + \ldots + v^{n-k} r_n$$

This is then a **prospective** consideration.

There are various repayment options:

1. No fixed end date for repayment, i.e. perpetual annuity:
   Here, the term $(S_{k-1} - S_k)$ in equation 2.14 becomes zero The interest rate on the bond is therefore only paid.

2. bond on fixed maturity, i.e. on a specific date (period n) is fixed.
   For example, if you have made a loan of CHF 50 with a fixed interest rate of 5%, the first $n-1$ annuities consist only of the interest rate of CHF 2.5 on the bond, with the last (nth) annuity you amortise in addition to the Interest of CHF 2.5 all plus the 50.

3. periodically through amortisation:
   If the annuities are constant, i.e. $r_k = r$, you can calculate formula (2.12) can also be expressed as a present value. $r \cdot a_{\overline{n|}} = S$ and therefore

$$r = \frac{S}{a_{\overline{n|}}}$$

Until now we have assumed that there is a constant interest rate i which governs the pricing of bonds. What one can observe is that typically interest rates for bonds are dependent on the term of the bond. In the following we want to explore this. In order to do so we introduce the concept of *zero coupon bonds* ("ZCB"). Assume for the moment that payments are made at integer times $n \in \mathbb{N}_+$. An n-year ZCB is the instrument with the payout pattern, where only a payment of the amount 1 is made at time n. Hence we could define:

$$Z^{(n)} = (\chi_{k=n})_{k \in \mathbb{N}_+}.$$

Until now we did not put a value on this instrument. Similarly we could for example consider a bond with a term of m years with an annual interest rate payable for i_m. Formally we can relate the two instruments as follows:

$$\mathscr{B} = \sum_{k=1}^{m} i_m \times Z^{(k)} + Z^{(m)}.$$

In a next step we can look at the prices π of the traded bonds and we can for each bond at each time determine $\tilde{i}_m$. The vector $(\tilde{i}_m)_{m\in\mathbb{N}_+}$ is called yield curve.

Given that the pricing functional π is linear, one can now determine the price of the zero coupon bonds by so called boot-strapping, relating in a first step $\mathscr{B}(1)$ to $Z^{(1)}$, and in subsequent steps the set $\{\mathscr{B}(k) : k < n\}$ with the set $\{Z^{(k)} : k < n\}$. The prices of the ZCB and the respective relationships can be used to both price financial instruments and also to price insurance liabilities.

In the following we use the following notation

$$P(t,t+\tau,k) = \text{Price } (\pi) \text{ at time } t+\tau \text{ for } Z^{(k)} \text{ given the economics at time } t.$$

We note that the following formula holds using an easy *absence of arbitrage* argument:

$$P(t,t+\tau,t+T) = \frac{P(t,t,t+T)}{P(t,t,t+\tau)}.$$

Before we have seen the concept of a discount rate v which is used to calculate the value at the beginning of the period of a payment ξ at the end of a period, by discounting - ie $v\times\xi$. In a economic environment with a constant interest rate environment (equally with a flat (constant) yield curve), the discount rate v is also constant. Things changes when using a non-constant yield curve. In this case v becomes dependent on time, hence formally

$$v_k = \frac{P(t,t,t+k+1)}{P(t,t,t+k)}.$$

We wantr tyo illustrate the above by an example and we assume that the following table represents the universe of the tradable bonds today:

Term of Bond n	Coupon in % $i_c(n)$	Price in% b_n
1	2.0	97
2	2.5	99
3	3.0	100
4	3.5	105
5	4.0	110

The task is to calculate the zero coupon bond prices. As per above we can in a first step identify the coupon bearing bonds with the respective zero coupn bonds, such as

$$\begin{aligned}
\mathscr{B}_1 &= 1.020Z^{(1)} \\
\mathscr{B}_2 &= 0.025Z^{(1)} + 1.025Z^{(2)} \\
\mathscr{B}_3 &= 0.030Z^{(1)} + 0.030Z^{(2)} + 1.030Z^{(3)} \\
&\dots
\end{aligned}$$

When applying the pricing functional we can rewrite the above linear equation system in concrete terms as follows:

$$\begin{aligned}
0.97 = \pi(\mathscr{B}_1) &= 1.020\pi(Z^{(1)}) \\
0.99 = \pi(\mathscr{B}_2) &= 0.025\pi(Z^{(1)}) + 1.025\pi(Z^{(2)}) \\
1.00 = \pi(\mathscr{B}_3) &= 0.030\pi(Z^{(1)}) + 0.030\pi(Z^{(2)}) + 1.030\pi(Z^{(3)}) \\
&\dots
\end{aligned}$$

Hence we have to solve the equation

$$b = A\,\pi((Z^{(k)})_{k\in\{1,2,3,4,5\}}$$

with

$$A = \begin{bmatrix} 1.020 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.025 & 1.025 & 0.000 & 0.000 & 0.000 \\ 0.030 & 0.030 & 1.030 & 0.000 & 0.000 \\ 0.035 & 0.035 & 0.035 & 1.035 & 0.000 \\ 0.040 & 0.040 & 0.040 & 0.040 & 1.040 \end{bmatrix}$$

Hence we get

$$\pi((Z^{(k)})_{k\in\{1,2,3,4,5\}} = A^{-1}b$$

This results in

Term	Price	Interest Rate	Discount (v_k)
1	0.950980	5.15 %	0.950980
2	0.942659	3.00 %	0.991250
3	0.915719	2.98 %	0.971422
4	0.919490	2.12 %	1.004118
5	0.914275	1.81 %	0.994328

Now one can determine the same rates for varying economic environments, for example if all bonds trade at par (100%):

Term	Price	Interest Rate	Discount (v_k)
1	0.980392	2.00 %	0.980392
2	0.951698	2.51 %	0.970732
3	0.914599	3.02 %	0.961019
4	0.869919	3.55 %	0.951147
5	0.818592	4.08 %	0.940998

Please note that the above interest rate is the one in respect to zero coupon bonds, and hence the whole ensemble represents the zero coupon bond yield curve. In the same same sense the discount rates represents the one year forwards discount rates, which are used for the reserve recursion in the following sections.

2.4 Internal Rate of Return

An investor makes an investment at a price of P and receives Payments $r_1, \dots, r_n$ at the times $\tau_1, \dots, \tau_n$. The current value of the cash flow received by the investor, is a function of the interest intensity δ.

$$a(\delta) := \sum_{k=1}^{n} e^{-\delta \tau_k} r_k \quad \left(= \sum_{k=1}^{n} (1+i)^{-\tau_k} r_k\right)$$

Let t be the solution of $a(t) = P$. Then t is called **internal interest rate** (internal rate of return). With its help it is possible to compare different cash flows.

3 The future lifespan of a x-year-old person

As the premiums of a life insurance policy depend to a large extent on which the insured person can still expect to live for the rest of his or her life. In this chapter, we look at the survival and mortality probabilities of policyholders.

3.1 The Model

3.1.1 Definitions

A person with age x is considered. His **future lifespan** is denoted by T. It is actually $T(x)$,but the index x is usually omitted. The **cumilative probability distribution function** of T is given by

$$G(t) = P[T \leq t].$$

(Here too, one should actually denote it by $G_x(t)$). It is assumed that the distribution function of T is absolutely continuous and thus has a **density**

$$g(t)dt = P[t < T < t + dt]. \tag{3.1}$$

This is the infinitesimal probability of dying between t and $t + dt$. In principle, you could now use $G(t)$ and $g(t)$ for all grids that are of interest to us. But there are international conventions, which will also be used in this text going forwards. The most important definitions and conventions are:

$$\begin{aligned} {}_tq_x &:= G(t) \\ {}_tp_x &:= 1 - G(t) \\ {}_{s|t}q_x &:= P[s < T < s + t] \\ &= G(t + s) - G(s) = {}_{s+t}q_x - {}_sq_x \\ \overset{\circ}{e}_x &:= E[T(x)] = \int_0^\infty t g(t) dt = \int_0^\infty (1 - G(t)) dt = \int_0^\infty {}_tp_x dt \end{aligned} \tag{3.2}$$

$\overset{\circ}{e}_x$ is the expected value of the remaining lifespan of an x-year-old person.

Remark: For one-year probabilities of death and survival, the index 1 is usually omitted, i.e. $q_x := {}_1q_x$ and $p_x := {}_1p_x$. ◇

From the above definitions, the following obvious principles can be derived

$$
\begin{aligned}
{}_t q_{x+s} &= G_{x+s}(t) = P[T(x+s) < t] = P[T(x) \leq s+t | T(x) > s] = \frac{G(s+t) - G(s)}{1 - G(s)} && (3.3) \\
{}_t p_{x+s} &= P[T \geq s+t | T > s] = \frac{1 - G(s+t)}{1 - G(s)} \\
{}_{s+t} p_x &= 1 - G(s+t) = (1 - G(s)) \frac{1 - G(s+t)}{1 - G(s)} = {}_s p_x \, {}_t p_{x+s} && (3.4) \\
{}_{s|t} q_x &= G(s+t) - G(s) = (1 - G(s)) \frac{G(s+t) - G(s)}{1 - G(s)} = {}_s p_x \, {}_t q_{x+s}
\end{aligned}
$$

3.1.2 Mortality Density

Again, consider an x-year-old person. The **Mortality Density** (or also **hazard rate**) is defined as the current mortality rate at age $x+t$, given that the person has already survived the t years. So

$$
\begin{aligned}
\mu_{x+t} &:= \lim_{\triangle t \to 0^+} \frac{P[t \leq T < t + \triangle t | T \geq t]}{\triangle t} && (3.5) \\
&= \frac{g(t)}{1 - G(t)} = -\frac{d}{dt} \ln(1 - G(t))
\end{aligned}
$$

From (3.5) and (3.3) it follows that $\mu_{x+t} \triangle t$ approximate ${}_{\triangle t} q_{x+t}$ is.

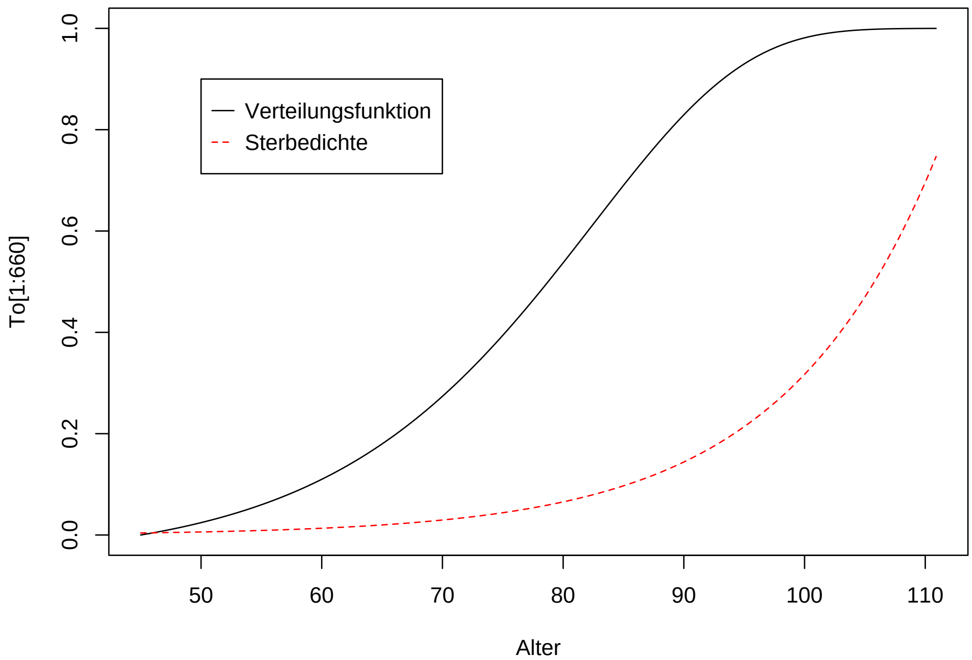


Figure 3.1: Distribution of lifespan and mortality density

3.2 Mortality Tables

In the last section the probabilities of death and survival of a x-year-old person have been considered. From equation (3.4) it follows that

$$ {}_t p_x = \prod_{k=0}^{k<t} p_{x+k} = \prod_{k=0}^{k<t} (1 - q_{x+k}) \text{ for } t \in \mathbb{N} $$

This means that the one-year probabilities can be used to determine the conclude that the probabilities of survival are several years higher.

3.2.1 Types of life tables

There are typically two different types of mortality tables: **simple** tables depending typically on age and **double graduated** mortality tables, where there is an additional parameter used.

Simple graded mortality tables

They typically have the following attributes:

Observation time: important because of the mortality trend.

Gender: Previously, it was mainly men who were insured. To simplify administration, only one mortality table was used at the time. Since 1970, this has been used in Switzerland. This is no longer the case because, on the one hand, the differences in mortality between men and women are becoming ever greater, secondly because More and more women are also taking out risk insurance (i.e. insurance with low savings element).

General population versus insured people: The general population mortality rate is higher. One has a selection effect here: in the case of individual insurance Persons only admitted after a medical certificate, in the collective the effect is somewhat smaller, but after all, the fact that the insured persons are capable of are to work that they are in relatively good health are.

Individual versus group insurance: Separate bases for the two types of insurance.

Capital and annuity insurance: Only those who are healthy buy themselves a annuity.

Occupations: Not every occupation burdens or promotes health in equal measure. In addition, a profession often also says a lot about the standard of living of a person, which in turn has an influence on the state of health.

Marital status: There is some research that single, divorced or widowed persons do not have the same life expectancy. There are also studies that claim that These differences in men and women Women, on the other hand, are very different.

Place/country: Here you will find the largest differences. Swiss mortality tables are not transferable to another country.

Double graded panels/ selection panels

These tables take into account the selection effect mentioned above for the Underwriting:

- Capital insurance: insured persons with medical examination have significantly lower mortality rates in the first few years than others.
- Annuity insurance: own choice.
- The mortality rate of disabled persons depends on the duration of the Disability and on the age.

If selection tables are used, a special selection table is used. Notation:

$$q_{[x]+t}: \text{ Mortality at age } x+t \text{ at entry into the labour market Insurance with age } x$$

$q_{[x]+t} \leq q_{[x-1]+t+1} \leq q_{[x-2]+t+2} \leq \cdots$
If one assumes that the selection effect lasts r years, then $q_{[x]+t}$ after r years over in q_{x+t}.

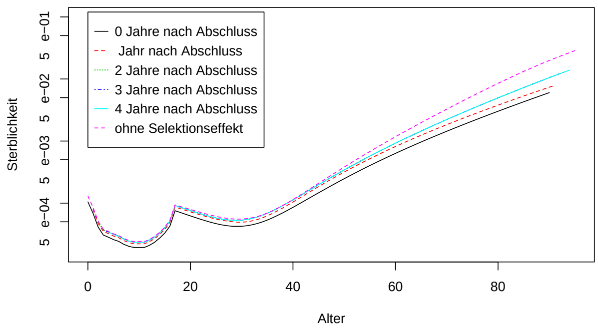


Figure 3.2: Selection table

Generation tables

These tables take into account the expected Mortality trend. Hence the consider the mortality by age and generation (cohort with the same year of birth for example). This means that these boards are in principle Graded by age and calendar year. Snapshots are measured for periods of time. One makes a model and extrapolates (which is always a bit tricky, of course).

3.3 The Number of full Years lived

In section 3.1.1 we used $T(x)$ to calculate the future lifetime of an x-yearly person. $T(x)$ assumes values in $\mathbb{R}^+$.

Now the **number of fully years survived** is denoted by

$$K := [T]. \tag{3.6}$$

Example: $T = 12.75 \Longrightarrow K = 12$ ◇

K is therefore an integer random variable. The pricing system relies on this random variable, since the formulas become a little easier. For this, you make a small mistake in.

$$\begin{aligned} P[K=k] &= P[k \leq T < k+1] \\ &= {}_k p_x \cdot q_{x+k} \end{aligned}$$

We use e_x to denote the expected value $E[K]$,

$$e_x = \sum_k^{\infty} k P[K=k] = \sum_k^{\infty} k\,{}_k p_x \cdot q_{x+k}$$

The advantage of working with K instead of T is that the formulas become much simpler. The price is that you have to pay a small makes an error. The „remaining“ survival time is denoted by S, i.e.

$$T = K + S \tag{3.7}$$

where $S \in [0,1[$. If it is also assumed that S is uniformly distributed over $[0,1[$, then the expected value is of T, $\overset{\circ}{e}_x = \mathrm{E}[T] = \mathrm{E}[K] + \mathrm{E}[S] = e_x + \frac{1}{2}$.

3.4 Mortality for the Part of a Year

Normally q_x, i.e. ${}_1 q_x$ are known by a Estimation. If you consider the influence of dying during the year you should know ${}_s q_x$ for $s < 1$.

1. Possibility: ${}_s q_x$ is linear for $s < 1$

$$\begin{aligned} {}_0 q_x &= 0 \\ {}_1 q_x &= q_x \end{aligned}$$

It follows that

$$\begin{aligned} {}_s q_x &= s \cdot q_x \\ {}_s p_x &= 1 - s \cdot q_x \end{aligned}$$

and therefore

$$\mu_{x+s} = -\frac{d}{dt} \ln({}_t p_x)|_{t=s} = \frac{q_x}{1 - s \cdot q_x}$$

Remark: If K and S are independent and S is equally distributed on $[0,1]$, This results in the above situation. ◇

2. Possibility: μ_{x+s} is constant for $0 < s < 1$.
We denote the constant value with

$$\mu_{x+\frac{1}{2}} := \mu_{x+s} = -\frac{d}{dt} \ln({}_t p_x)|_{t=s}$$

Then applies

$$ {}_1p_x = e^{-\int_0^1 \mu_{x+t}\,dt} = e^{-\mu_{x+\frac{1}{2}}} $$

and also

$$ {}_sp_x = e^{-\int_0^s \mu_{x+t}\,dt} = e^{-s\cdot\mu_{x+\frac{1}{2}}} = (p_x)^s $$

Further

$$ P[S \leq s | K = k] = \frac{1 - p_{x+k}^s}{1 - p_{x+k}} $$

It follows that the conditional distribution of S given $K = k$ is a is a truncated exponential distribution. This means that S and K are not independent.

3. Possibility: Linearity of ${}_{1-s}q_{x+s}$.
This assumption, known as the Balducci assumption, is very popular in the USA. known:

$$ {}_{1-s}q_{x+s} = (1-s)\cdot q_x $$

therefore applies

$$ {}_sp_x = \frac{p_x}{{}_{1-s}p_{x+s}} = \frac{1-q_x}{1-(1-s)q_x} $$

is further

$$ \mu_{x+s} = \frac{q_x}{1-(1-s)\cdot q_x} $$

and

$$ P[S \leq s | K = k] = \frac{s}{1-(1-s)\cdot q_{x+k}} $$

4 Types of Life Insurance

4.1 Capital Insurance

In the following, we want to consider both the traditional and the near-market valuation of insurance products. For this purpose, we use $Z^{(t)}$ to denote the zero coupon bond with maturity t. For simplicity, we use π to denote the price of $Z^{(t)}$ instead of using the term $\pi(Z^{(k)}) = P(t,t,k)$ etc.

An endowment insurance policy is characterised by the fact that the insurance is usually makes only *one payment*. This chapter looks at the 3 basic types of such endowment insurance policies:

Death insurance: In the event of the death of the insured person a capital payment is due.

Endowment case: When surviving a certain age, a Capital due.

Mixed insurance: Mixture of the two types above: It a lump sum becomes due on death or on reaching a certain age.

In this chapter, the costs are omitted for the time being, i.e. everything is considered once „net". The benefit that the policyholder pays at the start of the insurance is called **(net) single premium**. How large must this payment be?

The **present value** of the capital to be paid out is denoted by Z. This present value now depends on the type of insurance, it depends on when someone dies or whether they reach a certain survival date and is thus a *random variable*. The *fundamental principle of life insurance* means that the expected present value of insured person's benefit should be the same as the expected present value of the payments (premium) from the insured to the insurance company. Of course, this cannot be the case for each individual policyholder, but for the entire cohort of policyholders. This therefore means that the Expected value of the present value, $E[Z]$, (what the insurance company pays on average to a policyholder) should be equal to the single premium. The above principle is called *Equivalence Principle*.

4.1.1 Commutation functions

Since the present value $\mathrm{E}[Z]$ depends on the one hand on the *mortality* and on the other hand on the *interest*, it is useful to consider the following auxiliary variables that significantly simplify the formulas for the present values. Previously, all of these quantities was also tabulated so that the present values could be calculated so quickly, today this is actually no longer the case. necessary, as the computer does this work. First of all, the so-called **discounted numbers of the Living**:

$$\begin{aligned} D_x &:= v^x l_x \\ N_x &:= \sum_{j=x}^{\infty} D_j \\ S_x &:= \sum_{j=x}^{\infty} N_j. \end{aligned}$$

Further the **discounted numbers of the dead**

$$\begin{aligned} C_x &:= v^{x+1} d_x \\ M_x &:= \sum_{j=x}^{\infty} C_j \\ R_x &:= \sum_{j=x}^{\infty} M_j. \end{aligned}$$

The following relationships apply:

$$\begin{aligned} C_x &= v^{x+1}(l_x - l_{x+1}) = vD_x - D_{x+1} \\ M_x &= \sum_{j=x}^{\infty} (vD_j - D_{j+1}) = \ldots = D_x - dN_x \\ R_x &= \ldots = N_x - dS_x \end{aligned}$$

4.1.2 Death insurance

It is assumed that a death always occurs at the end of the year. **Single premium** (=expected present values) of Death benefit insurance policies are generally denoted by A_x, where x denotes the age of the policyholder at the time of taking out the insurance.

Whole of life insurance

On death, a payment of 1 is due. Then

$$Z := v^{K+1}, \tag{4.1}$$

where $K = 0, 1, 2, \ldots$.. In the case of a near-market valuation, Z is as follows:

$$Z := \pi(Z^{(K+1)}), \tag{4.2}$$

Example: $x = 50$, death at 77 years and 9 months. So $K = 27$ years. With an insurance benefit of CHF $100,000$ and an interest rate of 4% results in a present value of $Z(\omega) = (1+0.04)^{-28} \cdot 100'000 = 33347.74$ ◇

Z therefore assumes the values $v, v^2, v^3, \ldots$ and $P[Z = v^{k+1}] = P[K = k] = {}_k p_x q_{x+k}$. This means that

$$A_x = \mathrm{E}[Z] = \mathrm{E}\left[v^{K+1}\right] = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{x+k} \tag{4.3}$$

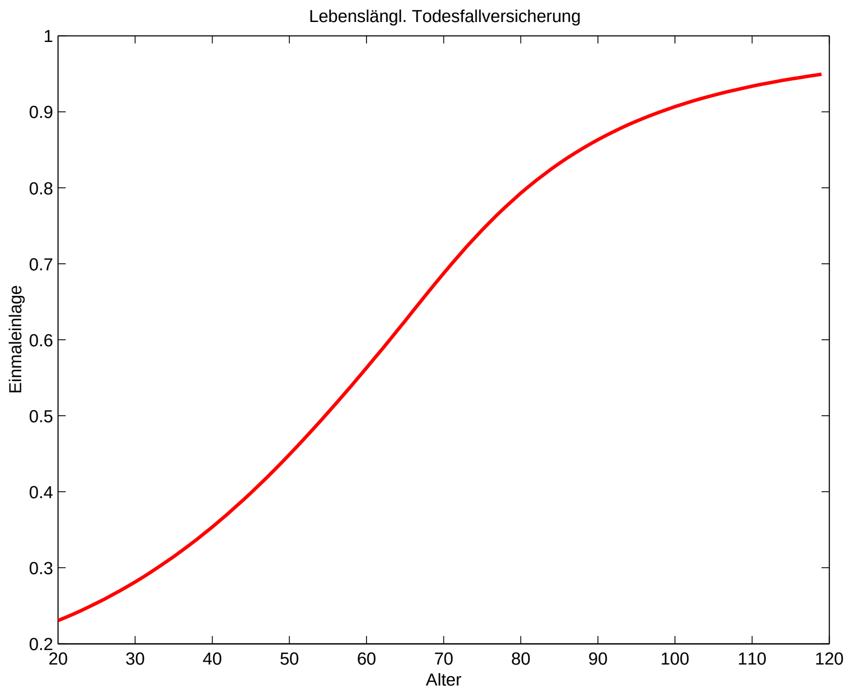


Figure 4.1: Single premium for a Life Annuity Insurance Policy

or for the valuation close to the market:

$$A_x = \mathrm{E}[Z] = \mathrm{E}\left[\pi(Z^{(K+1)})\right] = \sum_{k=0}^{\infty} \pi(Z^{(k)})_k p_x q_{x+k} \tag{4.4}$$

If we now set estimates for the probabilities in (4.3), you get

$$\begin{aligned} A_x &= \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k}}{l_x} \frac{d_{x+k}}{l_{x+k}} \\ &= \sum_{k=0}^{\infty} \frac{v^{x+k+1} d_{x+k}}{v^x l_x} = \frac{1}{v^x l_x} \sum_{k=0}^{\infty} v^{x+k+1} d_{x+k} = \frac{1}{D_x} \sum_{k=0}^{\infty} C_{x+k} = \frac{M_x}{D_x} \end{aligned} \tag{4.5}$$

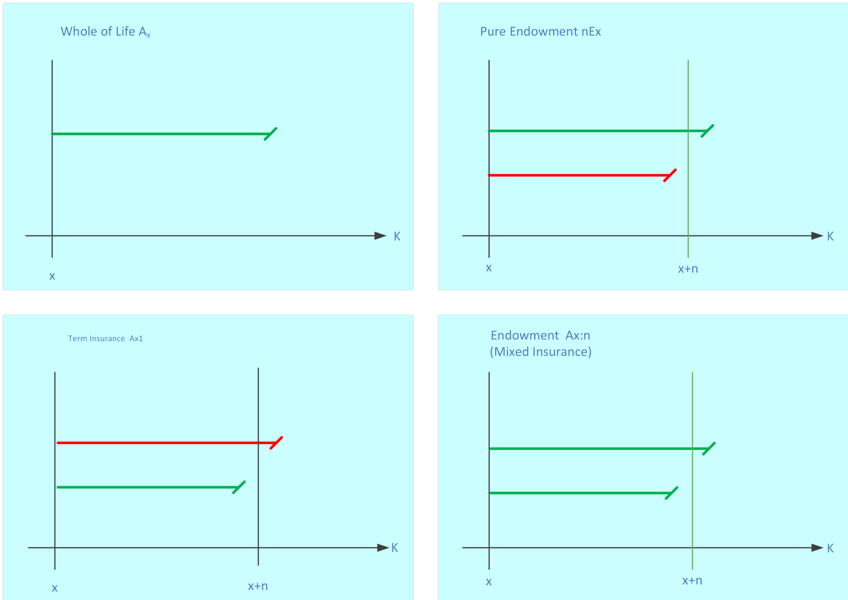
The variance of Z is

$$\mathrm{Var}[Z] = \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2$$

and $\mathrm{E}\left[Z^2\right] = \sum_{k=0}^{\infty} v^{2(k+1)} \cdot P[K=k] = \mathrm{E}\left[v^{2(K+1)}\right]$. The variance can therefore be calculated by that A_x (with $v \rightsquigarrow v^2$) is calculated.

$$\mathrm{Var}[Z] = A_x(\text{discount } v^2) - A_x(\text{discount } v)^2$$

The following figure provides a graphical overview of the standard types of capital insurance on one life. We note that the red realisation of $K(\omega)$ refers to the case where no payment is due, and similarly green reflect the case where an claim is due.



Temporary death benefit insurance (Term Insurance)

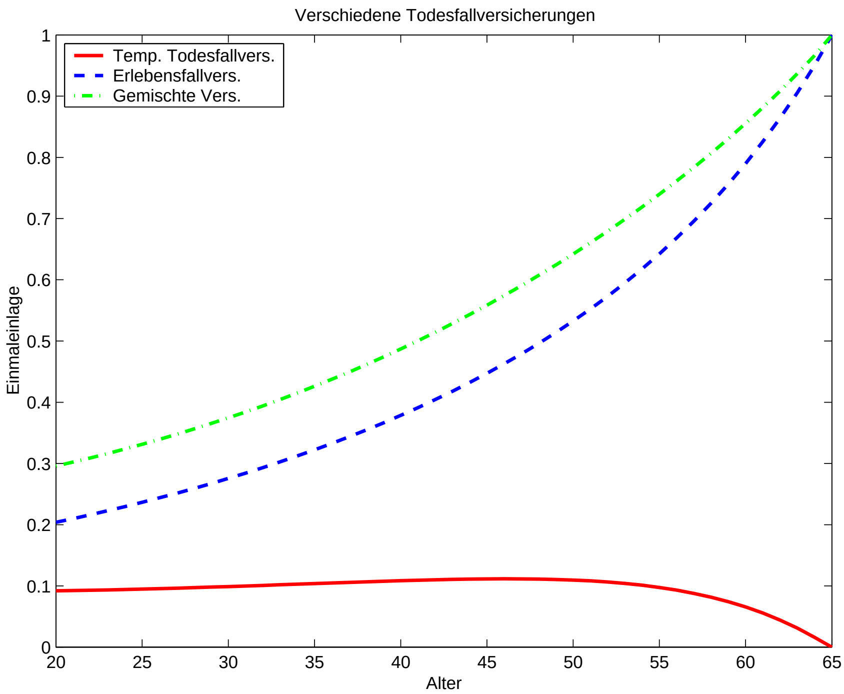


Figure 4.2: Single Premium for Term Insurance

Until the expiry of the insurance after n years, a Death benefit due. The designation for the single premium is $A^1_{x:\overline{n}|}$. The present value of the Payment is

$$Z = \begin{cases} v^{K+1}, & \text{falls } K = 0, \dots, n-1 \\ 0, & \text{other} \end{cases} \tag{4.6}$$

and therefore

$$\begin{aligned} A^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} v^{k+1}\,{}_k p_x q_{x+k} \\ &= \sum_{k=0}^{n-1} \frac{v^{x+k+1}}{v^x} \frac{l_{x+k}}{l_x} \frac{d_{x+k}}{l_{x+k}} \\ &= \frac{1}{v^x l_x} \sum_{k=0}^{n-1} v^{x+k+1} d_{x+k} = \frac{1}{D_x} \sum_{k=0}^{n-1} C_{x+k} = \frac{1}{D_x}(M_x - M_{x+n}) \end{aligned} \tag{4.7}$$

The same applies to the market-based valuation:

$$Z = \begin{cases} \pi(Z^{(K+1)}), & \text{falls } K = 0, \dots, n-1 \\ 0, & \text{other} \end{cases} \tag{4.8}$$

and therefore

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \pi(Z^{(k+1)})\,{}_k p_x q_{x+k} \tag{4.9}$$

The above illustration shows that the translation from traditional actuarial mathematics to near-market valuation is that v^k is replaced by the price of the corresponding zero coupon bond with a maturity in k years, i.e. by $\pi(Z^{(k+1)})$. As this principle is valid, we will not duplicate the corresponding formulas below.

In (4.7) the probabilities were again estimated via l_x and d_x.

The variance of the temporary death benefit insurance is calculated analogously for life annuity insurance:

$$\mathrm{Var}[Z] = \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = \mathrm{E}\left[Z^2\right] - (A^{1}_{x:\overline{n}|})^2$$

whereby

$$Z^2 = \begin{cases} e^{-2\delta(K+1)} = v^{2(K+1)}, & \text{if } K = 0, \dots, n-1 \\ 0, & \text{other} \end{cases}$$

Deferred death benefit insurance (for m years)

With this insurance, the death benefit is only paid in m years „active “. The single premium is calculated with ${}_{m|}A_x$ is denoted. The present value amounts to

$$Z = \begin{cases} 0, & \text{falls } K = 0, 1, \dots m-1 \\ v^{K+1}, & \text{if } K = m, m+1, \dots \end{cases}$$

and the expected value is

$$\begin{aligned}
\mathrm{E}[Z] &= \sum_{k=m}^{\infty} v^{k+1} P[K=k] = \sum_{k=m}^{\infty} {}_k p_x q_{x+k} v^{k+1} \\
&= \sum_{k=m}^{\infty} v^{k+1} \frac{l_{x+k}}{l_x} \frac{d_{x+k}}{l_{x+k}} = v^m \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+m+k}}{l_x} \frac{d_{x+m+k}}{l_{x+m+k}} \\
&= v^m \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+m+k}}{l_{x+m}} \frac{l_{x+m}}{l_x} \frac{d_{x+m+k}}{l_{x+m+k}} \\
&= v^m {}_m p_x A_{x+m} = \frac{D_{x+m}}{D_x} \frac{M_{x+m}}{D_{x+m}} = \frac{M_{x+m}}{D_x}
\end{aligned}$$

4.1.3 Pure endowment insurance

In the case of a pure endowment insurance with a duration of n, the insured sum is due if the policyholder is not insured at the end of the insurance term is still alive. The single premium is denoted by $A_{x:\overline{n}|}^{\;\;1}$ and the present value of the payment amounts to

$$Z = \begin{cases} 0, & \text{if } K = 0, 1, \dots, n-1 \\ v^n, & \text{if } K = n, n+1, \dots \end{cases}$$

Therefore

$$\begin{aligned}
A_{x:\overline{n}|}^{\;\;1} &= \sum_{k=0}^{\infty} Z(k) P[K=k] \\
&= \sum_{k=n}^{\infty} v^n P[K=k] = v^n P[K \geq n] = v^n (1 - P[K<n]) = v^n (1 - {}_n q_x) = v^n {}_n p_x \\
&= \frac{v^{x+n}}{v^x} \frac{l_{x+n}}{l_x} = \frac{D_{x+n}}{D_x}
\end{aligned}$$

The variance of pure endowment insurance can be calculated either directly or using a Bernoulli variable can be derived. The following applies $\mathrm{E}\left[Z^2\right] = v^{2n} {}_n p_x$ (as above) and therefore

$$\begin{aligned}
\mathrm{Var}[Z] &= \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = v^{2n} ({}_n p_x - {}_n p_x{}^2) \\
&= v^{2n} {}_n p_x \cdot {}_n q_x
\end{aligned}$$

4.1.4 Endowment insurance

This is the combination of a death benefit and a Endowment insurance. In the case of a mixed insurance policy of duration n, the insured Sum payable either on the death of the policyholder, if dies before the expiry of the insurance, or otherwise in the case of Expiry of the insurance term. The designation for the single premium is $A_{x:\overline{n}|}$. The present value of the payment amounts to

$$Z = \begin{cases} v^{K+1}, & \text{if } K = 0, 1, \dots, n-1 \\ v^n, & \text{if } K = n, n+1, \dots \end{cases} \tag{4.10}$$

If we use Z_1 to denote the value of the temporary death benefit and Z_2 is the value of the cash flow of the pure endowment insurance, always with duration n, so applies

$$Z = Z_1 + Z_2$$

so

$$\begin{aligned} A_{x:\overline{n}|} &= \mathrm{E}[Z] = \mathrm{E}[Z_1] + \mathrm{E}[Z_2] = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} \\ &= \sum_{k=0}^{n-1} v^{k+1}{}_k p_x q_{x+k} + v^n{}_n p_x \\ &= \frac{1}{D_x}(M_x - M_{x+n} + D_{x+n}) \end{aligned}$$

The following also applies

$$\mathrm{Var}[Z] = \mathrm{Var}[Z_1 + Z_2] = \mathrm{Var}[Z_1] + \mathrm{Var}[Z_2] + 2\mathrm{Cov}[Z_1, Z_2]$$

Since $\mathrm{Cov}[Z_1, Z_2] = \mathrm{E}[Z_1 Z_2] - \mathrm{E}[Z_1]\mathrm{E}[Z_2]$, and $Z_1 \cdot Z_2 = 0$ is $\mathrm{Cov}[Z_1, Z_2] = -\mathrm{E}[Z_1]\mathrm{E}[Z_2] = -A^1_{x:\overline{n}|} \cdot A_{x:\overline{n}|}^{\ \ 1}$. This in turn means that

$$\begin{aligned} \mathrm{Var}[Z] &= \mathrm{Var}[Z_1] + \mathrm{Var}[Z_2] - 2 \cdot A^1_{x:\overline{n}|} \cdot A_{x:\overline{n}|}^{\ \ 1} \\ &\leq \mathrm{Var}[Z_1] + \mathrm{Var}[Z_2] \end{aligned}$$

This can now be interpreted as follows: The risk of person to sell a mixed insurance policy is less than two individual persons the individual sub-insurances (i.e. $A^1_{x:\overline{n}|}$ and $A_{x:\overline{n}|}^{\ \ 1}$).

Mixed insurance

In the case of mixed insurance, it is possible to obtain an insurance policy in a similar way. result. However, the modification of the single premium only takes place on the part of the temporary death benefit insurance.

$$\begin{aligned} \bar{A}_{x:\overline{n}|} &= \bar{A}^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} = \frac{i}{\delta} A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} \\ &= A_{x:\overline{n}|} + (\frac{i}{\delta} - 1) A^1_{x:\overline{n}|} \end{aligned}$$

4.1.5 General types of life insurance

It is quite possible that someone who has a death benefit policy not have a fixed amount paid out on his or her death but get more or less depending on the policy year rights. (For example, should the money be used for the children's education? If it is to be used, it needs to survive every year, less money as the children get closer and closer to the end of their education).

The payment in policy year j is now denoted by c_j for in discrete time, or with $c(t)$ in continuous time. In discrete time is the present value

$$Z = c_K v^{K+1}$$

and the h-th moment of the present value amounts to

$$\mathrm{E}\left[Z^h\right] = \sum_{k=0}^{\infty} c_k^h v^{h(k+1)}{}_k p_x q_{x+k}$$

The single premium is calculated from a sum of Deferred death benefits

$$\mathrm{E}[Z] = c_0 A_x + (c_1 - c_0)_1|A_x + (c_2 - c_1)_2|A_x + \dots$$

The following applies analogously to continuous time

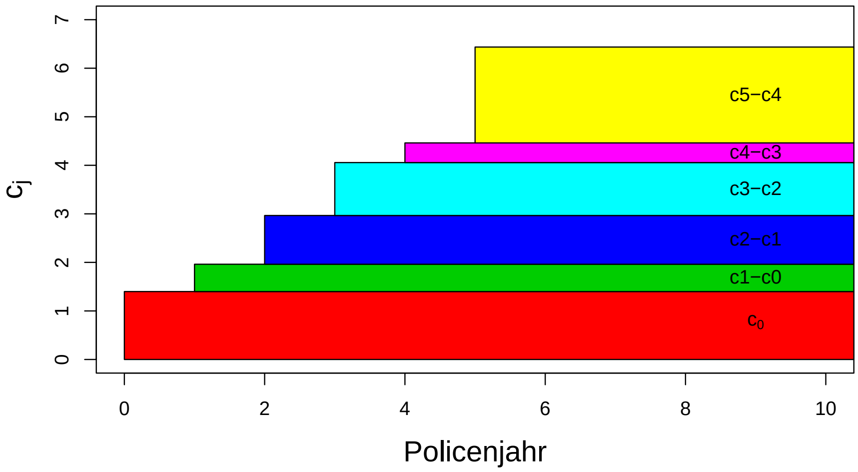


Figure 4.3: General Death Benefit Insurance

$$Z = c(T) v^T$$

and therefore

$$\mathrm{E}[Z] = \int_0^\infty c(t) v^t {}_t p_x \mu_{x+t} dt$$

4.1.6 Standard types of death benefit insurance

Linear death benefit

Whole of life

$$\begin{aligned}
(IA)_x &= A_x + {}_1|A_x + {}_2|A_x + {}_3|A_x + \dots \\
&= \frac{M_x}{D_x} + \frac{M_{x+1}}{D_x} + \frac{M_{x+2}}{D_x} + \dots \\
&= \frac{R_x}{D_x}
\end{aligned}$$

Temporary This insurance can also be put together, as can be seen from Fig. 4.1.6 can be recognised.

$$\begin{aligned}
(IA)^1_{x:\overline{n}|} &= n A^1_{x:\overline{n}|} - A^1_{x:\overline{n-1}|} - A^1_{x:\overline{n-2}|} - \dots - A^1_{x:\overline{1}|} \\
&= \frac{R_x - R_{x+n} - n M_{x+n}}{D_x}
\end{aligned}$$

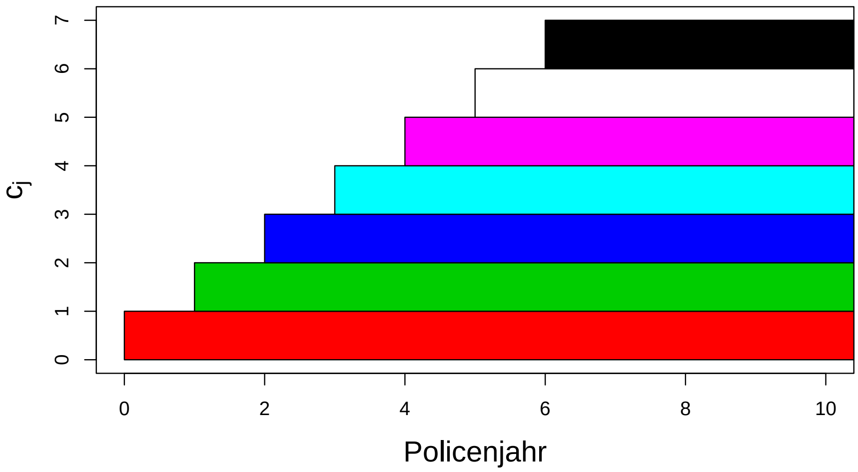


Figure 4.4: Linear increasing Term Insurance

Linear falling death benefit

The single premium is calculated on the same basis as with the linearly increasing death benefit.

$$
\begin{aligned}
(DA)^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} (n-k) v^{k+1} {}_k p_x q_{x+k} \\
&= A^{1}_{x:\overline{n}|} + A^{1}_{x:\overline{n-1}|} + \ldots + A^{1}_{x:\overline{2}|} + A^{1}_{x:\overline{1}|} \\
&= \frac{M_x - M_{x+n}}{D_x} + \frac{M_x - M_{x+n-1}}{D_x} + \ldots + \frac{M_x - M_{x+1}}{D_x} \\
&= \frac{1}{D_x} (R_{x+1} - R_{x+n+1} + n M_x)
\end{aligned}
$$

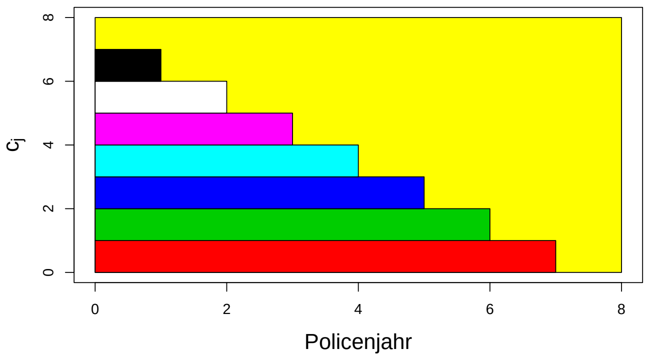


Figure 4.5: Linear falling Term Insurance for the duration n=8.

4.1.7 Recursion Formulas

For any death benefit insurance with death benefit $c_j = 1$, the following applies due to recursion

$$ {}_k p_x = p_x \cdot {}_{k-1} p_{x+1} $$

$$
\begin{aligned}
A_x &= \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{x+k} \\
&= v q_x + \sum_{k=1}^{\infty} v^{k+1} p_x \, {}_{k-1} p_{x+1} \, q_{x+k} \\
&= v q_x + v p_x \sum_{k=1}^{\infty} v^{k} \, {}_{k-1} p_{x+1} \, q_{x+k} \\
&= v q_x + v p_x \sum_{k=0}^{\infty} v^{k+1} \, {}_{k} p_{x+1} \, q_{x+k+1} \\
&= v q_x + v p_x A_{x+1} \qquad (4.11)
\end{aligned}
$$

This means that the single premium of the death benefit insurance for the age x changes The annuity consists of the discounted payment in the event of death plus the discounted single premium for age $x+1$ in the event of survival. The recursion formula can also be used differently if they are paraphrased as follows

$$ A_x = v A_{x+1} + v(1 - A_{x+1}) q_x \qquad (4.12) $$

The amount A_{x+1} is reserved in the event of death and survival, In addition, the amount $1 - A_{x+1}$ is provided for the Death benefit. For a general death benefit insurance policy with death benefit C_x, the recursion is as follows follows

$$ A_x^{allg} = C_x \cdot v \cdot q_x + v \cdot p_x A_{x+1}^{allg} $$

respectively

$$ A_x^{allg} = v A_{x+1}^{allg} + v(C_x - A_{x+1}^{allg}) q_x $$

At this point, we would like to take another look at near-market valuation. What do the above recursion formulae look like in a near-market valuation? It is important to bear in mind what v is. It is the one-year discount factor in the future. Let us assume that we are today at time t at age x_0, and that we want to use the recursion for age $x = x_0 + k$. In this case, we use the corresponding forward discount for v, i.e:

$$ v = \frac{P(t,t,k+1)}{P(t,t,k)} $$

This means that the recursion for the general death benefit insurance in the case of a valuation close to the market is as follows:

$$ A_{x+k}^{allg} = C_{x+k} \cdot \frac{P(t,t,k+1)}{P(t,t,k)} \cdot q_{x+k} + \frac{P(t,t,k+1)}{P(t,t,k)} \cdot p_{x+k} A_{x+k+1}^{allg} $$

respectively

$$ A_{x+k}^{allg} = \frac{P(t,t,k+1)}{P(t,t,k)} A_{x+k+1}^{allg} + \frac{P(t,t,k+1)}{P(t,t,k)} (C_{x+k} - A_{x+k+1}^{allg}) q_{x+k} $$

4.2 Annuity Insurance

4.2.1 Introduction

In contrast to endowment insurance, which consists of a single payment exists, in the case of annuity insurance *several payments* takes place.

An annuity can also be calculated as a sequence of survival benefits. and use the corresponding results. If one agrees a premium payment, this can be treated as an annuity with negative payments. The insured person therefore pays the insurance company a annuity.

4.2.2 Elementary types of annuity insurance

Life-long annuity

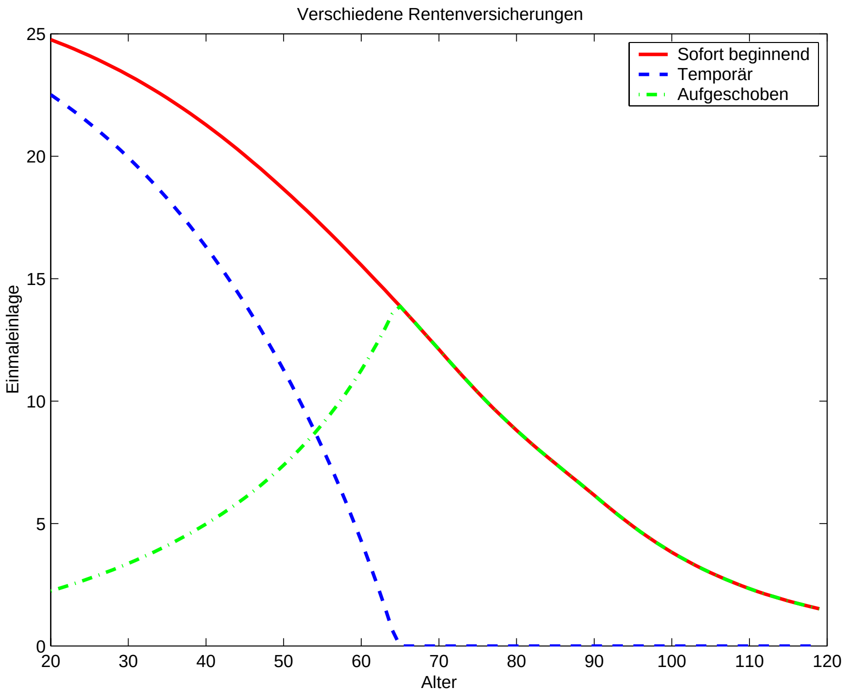


Figure 4.6: Single Premium for an Annuity Insurance

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An immediate (1/1) annuity payable in advance pays per year an annuity of amount 1. The single premium of this life annuity is paid with denotes $\ddot{a}_x$. (Not to be confused with the time rents $\ddot{a}_{\overline{n}|}$). Payments are made at the following times $0, 1, 2, \ldots, K(\omega)$. The present value therefore amounts to

$$Y = 1 + v + v^2 + \ldots + v^K = \ddot{a}_{\overline{K+1}|} \tag{4.13}$$

and $P[Y = \ddot{a}_{\overline{k+1}|}] = P[K = k] = {}_k p_x q_{x+k}$. Thus calculated the necessary single contribution through

$$\ddot{a}_x = \mathrm{E}[Y] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|\,k} p_x q_{x+k} \tag{4.14}$$

You can also represent Y as follows:

$$Y = \sum_{k=0}^{\infty} v^k \mathbb{I}_{\{K \geq k\}} \tag{4.15}$$

and therefore

$$\begin{aligned} \ddot{a}_x &= \mathrm{E}[Y] = \sum_{k=0}^{\infty} v^k P[K \geq k] = \sum_{k=0}^{\infty} v^k {}_k p_x = \sum_{k=0}^{\infty} v^k \frac{l_{x+k}}{l_x} \\ &= \frac{1}{v^x l_x} \sum_{k=0}^{\infty} v^{x+k} l_{x+k} = \frac{1}{D_x} \sum_{k=0}^{\infty} D_{x+k} = \frac{N_x}{D_x} \end{aligned} \tag{4.16}$$

The difference between the two types of calculation is as follows, that with (4.14) the annuity is calculated as a whole, with (4.16) as sum of the individual annuity payments in case the policyholder is alive at this age. The next step is to establish the the relationship with death benefit amounts.

The present value of the annuity amounts to

$$Y = 1 + v + v^2 + \ldots + v^K = \frac{1 - v^{K+1}}{1 - v} = \frac{1 - v^{K+1}}{d}$$

On the right-hand side is a derived term of a Death benefit. So

$$\begin{aligned} \ddot{a}_x &= \mathrm{E}[Y] = \mathrm{E}\left[\frac{1-Z}{d}\right] = \frac{1}{d} - \frac{\mathrm{E}[Z]}{d} \\ &= \frac{1 - A_x}{d} \end{aligned}$$

or

$$1 = d\ddot{a}_x + A_x \tag{4.17}$$

This equation is mainly used to calculate the variance of Y interesting

$$\mathrm{Var}[Y] = \mathrm{Var}\left[\frac{1-Z}{d}\right] = \frac{1}{d^2} \mathrm{Var}[Z]$$

Temporary annuity

An immediately commencing (1/1) payable in advance for n years temporary annuity entitles the holder to draw an annual annuity until death, but a maximum of n years. The single premium is calculated with $\ddot{a}_{x:\overline{n}|}$ labelled. The present value amounts to

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{if } K = 0, 1, \ldots, n-1 \\ \ddot{a}_{\overline{n}|}, & \text{other .} \end{cases}$$

And the single premium then amounts to

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_k p_x q_{x+k} + \ddot{a}_{\overline{n}|}\, {}_n p_x$$

Or rather

$$\begin{aligned} \ddot{a}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} v^k {}_k p_x = \frac{\sum_{k=0}^{n-1} D_{x+k}}{D_x} \\ &= \frac{N_x - N_{x+n}}{D_x} \end{aligned}$$

The equation $Y = \frac{1-Z}{d}$ also applies here, where Z is the Random variable of a endowment policy Thus the following now applies

$$\ddot{a}_{x:\overline{n}|} = \frac{1}{d}(1 - A_{x:\overline{n}|}) \tag{4.18}$$

respectively

$$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$$

For the variance, the following applies again analogously to the whole of life insurance

$$\mathrm{Var}[Y] = \frac{1}{d^2}\mathrm{Var}[Z]$$

Lifetime annuity

The single premium of the lifelong annuity is calculated as a_x is denoted. The present value is $Y = v + v^2 + \ldots + v^K = a_{\overline{K}|}$. If you compare this with (4.13), you can see that the present value of the pre-annuity is equal to the present value increased by 1 of the deferred annuity and for the single premium therefore applies

$$a_x = \ddot{a}_x - 1$$

The following identities therefore apply:

$$a_x = \sum_{k=0}^{\infty} v^k {}_k p_x - 1 = \frac{N_x - D_x}{D_x}$$

Here, too, there is a connection with the corresponding Death benefit. according (2.10) $1 = ia_{\overline{n}|} + v^n$ therefore applies also

$$1 = ia_{\overline{K}|} + (1+i)v^{K+1}$$

and if you take the expected value of this, then you get

$$1 = ia_x + (1+i)A_x$$

Deferred annuities

In this case, the annuity only begins after a deferral period of m years. run. The annuity payable in advance and deferred for m years is denoted by ${}_m|\ddot{a}_x$. The present value amounts to

$$Y = \begin{cases} 0, & \text{if } K = 0, 1, \ldots, m-1 \\ v^m + v^{m+1} + \ldots + v^K, & \text{other} \end{cases}$$

and the single premium can be calculated as follows:

$$\begin{aligned}
{}_m|\ddot{a}_x &= \sum_{k=0}^{\infty} P[K=k]Y(K=k) \\
&= v^m \sum_{k=m}^{\infty} {}_k p_x q_{x+k} v^{k-m} \\
&= v^m \sum_{k=0}^{\infty} {}_{k+m} p_x q_{x+m+k} v^k \\
&= v^m \sum_{k=0}^{\infty} v^k {}_k p_{x+m} {}_m p_x q_{x+m+k} \\
&= v^m {}_m p_x \ddot{a}_{x+m}
\end{aligned}$$

Another way leads to the intuitive formula

$$_{m|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{m}|}$$

This means that the present value of a deferred life annuity is calculated as Difference between an immediate annuity and a temporary annuity life annuity.

4.2.3 Annuities with variable annuity levels

The payments for a annuity do not necessarily have to be constant, but can, for example, be the values $r_0, r_1, r_2, \ldots$ at the times be $k = 0, 1, 2, \ldots$. The present value then amounts to

$$Y = \sum_{k=0}^{\infty} v^k r_k \mathbb{I}_{\{}K \geq k\}$$

and the expected value is then

$$\mathrm{E}[Y] = \sum_{k=0}^{\infty} v^k r_k\, {}_k p_x$$

4.2.4 Standard types of annuities

Guaranteed annuities

The annuity is guaranteed for g years. The present value consists of a guaranteed portion plus the present value of a deferred life annuity. So

$$\mathrm{E}[Y] = \ddot{a}_{\overline{g}|} + {}_{g|}\,\ddot{a}_x = \frac{v^g - 1}{v - 1} + \frac{N_{x+g}}{D_x}$$

Linear increasing annuity

The annuity payments grow linearly. The term for these annuity type is

$$\begin{aligned} (I\ddot{a})_x &= \ddot{a}_x + v\,{}_1p_x \ddot{a}_{x+1} + v^2\,{}_2p_x \ddot{a}_{x+2} + \ldots \\ &= \frac{N_x}{D_x} + \frac{D_{x+1}}{D_x}\frac{N_{x+1}}{D_{x+1}} + \frac{D_{x+2}}{D_x}\frac{N_{x+2}}{D_{x+2}} + \ldots \\ &= \frac{S_x}{D_x} \end{aligned}$$

Exponentially growing annuities

Let $r_k = (1+\alpha)^k$. To calculate the expected present value, the present value is converted into a perpetual annuity, with the discount $\frac{1+\alpha}{r}$ instead of v. In concrete terms, this means

$$Y = \sum_{k=0}^{\infty} v^k r_k \mathbb{I}_{\{K \geq k\}}$$

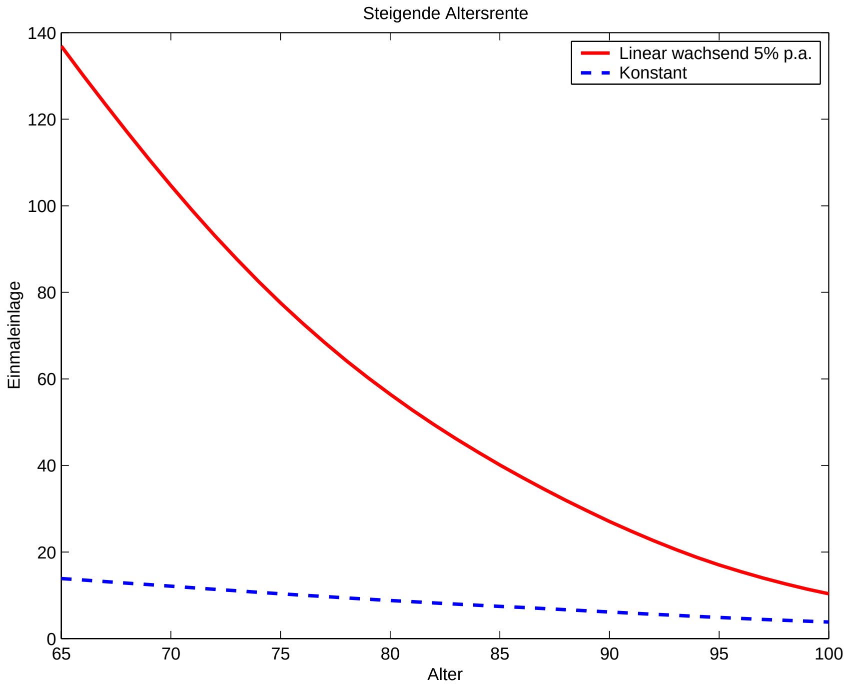

Figure 4.7: Linear increasing Annuity

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and the present value amounts to

$$\begin{aligned} \mathrm{E}[Y] &= \sum_{k=0}^{\infty} v^k (1+\alpha)^k {}_k p_x \\ &= \sum_{k=0}^{\infty} \left(\frac{1+\alpha}{r}\right)^k {}_k p_x \\ &= \ddot{a}_x\left(v \rightsquigarrow \frac{1+\alpha}{r}\right) \end{aligned}$$

4.2.5 Recursion formulas

Simple case

$$\begin{aligned} \ddot{a}_x &= \sum_{k=0}^{\infty} v^k {}_k p_x \\ &= 1 + \sum_{k=1}^{\infty} v^k {}_k p_x \\ &= 1 + v \sum_{k=0}^{\infty} v^k {}_1 p_x {}_k p_{x+1} \\ &= 1 + v p_x \sum_{k=0}^{\infty} v^k {}_k p_{x+1} \\ &= 1 + v p_x \ddot{a}_{x+1} \end{aligned} \tag{4.19}$$

General case

If unequal annuity payments r_x are made at age x. The following applies analogously to (4.19)

$$\ddot{a}_x^{\text{allg.}} = r_x + v p_x \ddot{a}_{x+1}^{\text{allg.}}$$

These recursion formulas are useful for the calculation of single premiums. They also apply in more general models. In the continuous case, they are replaced by differential equations.

At this point, we would like to take another look at near-market valuation. What do the above recursion formulae look like in a near-market valuation? It is important to bear in mind what v is. It is the one-year discount factor in the future. Let us assume that we are today at time t at age x_0, and that we want to use the recursion for age $x = x_0 + k$. In this case, we use the corresponding forward discount for v, i.e:

$$v = \frac{P(t,t,k+1)}{P(t,t,k)}$$

This means that the recursion for the general death benefit insurance in the case of a valuation close to the market is as follows:

$$\ddot{a}_{x+k}^{\text{allg.}} = r_{x+k} + \frac{P(t,t,k+1)}{P(t,t,k)} p_{x+k} \ddot{a}_{x+k+1}^{\text{allg.}}$$

4.2.6 Annuities that do not start at integer ages

In the mortality tables, the mortality probabilities for integer age x and calculates present values and single deposits then also always for integer ages. This naturally leads to a small error. If you now want to calculate the present value of $\ddot{a}_{x+u}$ where x is again an integer and $u \in [0,1[$, then you can use the equality (3.4)

$${}_u p_x \cdot {}_k p_{x+u} = {}_{k+u} p_x = {}_k p_x \cdot {}_u p_{x+k}$$

For the interpolation of the mortality during the year, we again use the 1st variant in the previous section, (i.e. ${}_u q_x = u \cdot q_x$) then the following applies

$$(1 - u \cdot q_x) \cdot {}_k p_{x+u} = {}_k p_x (1 - u \cdot q_{x+k})$$

and thus by multiplying by v^k and adding over all values you get from k

$$(1 - u \cdot q_x) \cdot \ddot{a}_{x+u} = \ddot{a}_x - u(1+i) A_x$$

Because of (4.17) follows

$$\begin{aligned}
\ddot{a}_{x+u} &= \frac{1}{1 - u \cdot q_x} (\ddot{a}_x - u(1+i) A_x) \\
&= \frac{1}{1 - u \cdot q_x} (\ddot{a}_x - u(1+i)(1 - d\ddot{a}_x)) \\
&= \frac{(1+ui)\ddot{a}_x - u(1+i)}{1 - u q_x} \\
&= \frac{(1-u)\ddot{a}_x}{1 - u q_x} + \frac{u(1+i)}{1 - u q_x} (\ddot{a}_x - 1)
\end{aligned}$$

Because of the recursion (4.19) then applies

$$\ddot{a}_{x+u} = \frac{1-u}{1-uq_x}\ddot{a}_x + \frac{u(1-q_x)}{1-uq_x}\ddot{a}_{x+1}$$

This means that the present value is calculated using *linear interpolation*. can be used. In practice, the following *approximation* is commonly used

$$\ddot{a}_{x+u} = (1-u)\ddot{a}_x + u\ddot{a}_{x+1}$$

This approximation is good if q_x is small. Similarly, for payments of less than one year and constant age, you can also use a approximation:

$$\ddot{a}^{(m)}_{x+u} = \frac{1-u}{1-uq_x}\ddot{a}^{(m)}_x + \frac{u(1-q_x)}{1-uq_x}\ddot{a}^{(m)}_{x+1}$$

In practice, present values of less than one year are often calculated using linear interpolation is calculated. The interpolation formula found has Of course, the validity of death benefit insurance policies is also subject to (4.17) are connected to each other.

5 Net Premium

5.1 Introduction

After we have determined the value of the insurance company's liabilities, we need to ask ourselves how these insurance policies can be financed. The following three types are in the foreground

- financing with single premium.
- financing with premiums of a constant amount.
- financing with premiums of variable amounts.

Normally, the premiums are used in a defined form and paid in advance.

In order to be able to calculate the premiums, one usually uses the **Equivalence principle**, which states that the present value of the premiums correspond to the present value of the obligations.

If the single premium is defined as the present value of the obligations, the above principle is fulfilled.

The *equivalence principle* can also be defined by the fact that the **loss** L is 0 in the expected value. One speaks of a **net premium** if

$$\mathrm{E}[L] = 0$$

5.2 An Example

We consider a temporary death benefit insurance with $x = 40$ and $n = 10$. The insured sum is C and at the end of the year of death is due. The annual price corresponds to π.

The **loss** L (for the insurer) can now be calculated as follows become:

$$L = \begin{cases} Cv^{K+1} - \Pi \ddot{a}_{\overline{K+1}|}, & \text{falls } K = 0, \dots, 9 \\ -\Pi \ddot{a}_{\overline{10}|}, & \text{other} \end{cases}$$

We first calculate the annual net premium. In order to be able to calculate the expected value of L, we have to enter in The first step is to calculate the corresponding probabilities:

$$\begin{aligned} P[L = Cv^{k+1} - \Pi \cdot \ddot{a}_{\overline{k+1}|}] &= P[K = k] = {}_k p_{40} q_{40+k} \\ P[L = -\Pi \cdot \ddot{a}_{\overline{10}|}] &= P[K \geq 10] = {}_{10} p_{40} \end{aligned}$$

and we obtain the following identity by $\mathrm{E}[L] = 0$:

$$C \cdot A^{1}_{40:\overline{10}|} - \Pi \ddot{a}_{40:\overline{10}|} = 0$$

or

$$\Pi = C\frac{A^{1}_{40:\overline{10}|}}{\ddot{a}_{40:\overline{10}|}} = \frac{\text{pv of benefits}}{\text{pv of premium}} \cdot \text{benefit level}$$

To make the example easy to calculate, we start from de Moivre's law of death in section 3 with $a = 100$ and $i = 4\%$. You then obtain

$$A^{1}_{40:\overline{10}|} = \frac{1}{60}(v + v^2 + \ldots + v^{10}) = \frac{1}{60} a_{\overline{10}|} = 0.1352$$

Furthermore, $A_{40:\overset{1}{\overline{10}|}} = \frac{5}{6}v^{10} = 0.5630$. And therefore $A_{40:\overline{10}|} = A^{1}_{40:\overline{10}|} + A_{40:\overset{1}{\overline{10}|}} = 0.6981$ und $\ddot{a}_{40:\overline{10}|} = \frac{1-A_{40:\overline{10}|}}{d} = 7.848$ The annual net premium is therefore $\Pi = 0.0172 \cdot C$

The above premium principle is called the **expected value principle**. The question is whether this principle is justified. Given that no one can be excluded and high sums assured require a commensurate underpinning capital from the shareholder, an appropriate safety margin for the risk is necessary. In the following, a possibility is shown on how to use premiums that take account of the risk.

The premiums are to be determined using a **benefit function** $u(\cdot)$. Such utility functions can come from different places:

- Missing capacity
- Economy of scale
- Non-linear taxes

A utility function has the following characteristics Properties:

- it is growing, i.e. $u' \geq 0$
- it is concave, i.e. $u'' < 0$

An example of such a function would be $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$, where $\alpha > 0$ is the risk aversion of the insurance company.

The exponential principle (exponential principle) is now called

$$\mathrm{E}[u(-L)] = u(0)$$

i.e. the premiums should be fair with regard to the benefit. One therefore obtains $\mathrm{E}\left[e^{\alpha L}\right] = 1$ and thus also ${}_kp_{40}q_{40+k} = \frac{1}{60}$, ${}_{10}p_{40} = \frac{5}{6}$ and the premiumn principle for the above example means

$$\frac{1}{60}\sum_{k=0}^{9} e^{\alpha \cdot C \cdot v^{k+1} - \alpha \Pi \ddot{a}_{\overline{k+1}|}} + \frac{5}{6} e^{-\alpha \Pi \ddot{a}_{\overline{10}|}} = 1$$

for $\alpha = 10^{-6}$ now gives the following picture:

C	annual Π	premium in per cent of the net premium
100'000	1'790	104%
500'000	10'600	123%
1'000'000	26'400	153%
5'000'000	1'073'600	1248 %

Remarks: Obviously, the premium is no longer as in the case of the net premium is proportional to the insured capital, but increases progressively with C. ◇

5.3 Elementary Forms of Insurance

5.3.1 Lifetime and temporary death benefit insurance

Whole of life insurance

The procedure is now always based on the same principle. First, the loss function is calculated, then the expected loss equals be set to zero. The price is denoted by P_x. The loss amounts to

$$L = v^{K+1} - P_x \cdot \ddot{a}_{\overline{K+1}|}$$

If the expected value is taken on both sides, the result is

$$A_x = P_x \cdot \ddot{a}_x$$

and the premium is calculated as

$$P_x = \frac{A_x}{\ddot{a}_x}$$

From (4.5) and (4.16) we immediately obtain $P_x = \frac{M_x}{N_x}$. If we define the premiums as the difference between two perpetual annuities (one starting at 0 and a start at $K+1$), we get

$$L = (1 + \frac{P_x}{d}) v^{K+1} - \frac{P_x}{d}$$

and so we can now calculate the variance of the loss:

$$\mathrm{Var}[L] = (1 + \frac{P_x}{d})^2 \mathrm{Var}\left[v^{K+1}\right]$$

(In the case of single premium insurance, $\mathrm{Var}[L] = \mathrm{Var}\left[v^{K+1}\right]$). This means that the sale of single premiums harbours a greater risk than the sale of single premiums.

Temporary death benefit insurance policies of duration n

The term for the premium is $P^1_{x:\overline{n}|}$. The loss amounts to then

$$L = \begin{cases} v^{K+1} - P^1_{x:\overline{n}|} \ddot{a}_{\overline{K+1}|}, & \text{if } K = 0, \dots, n-1 \\ -P^1_{x:\overline{n}|} \ddot{a}_{\overline{n}|}, & \text{if } K \geq n \end{cases}$$

respectively $L = -P^1_{x:\overline{n}|} \ddot{a}_{\overline{n}|} + (1 + P^1_{x:\overline{n}|} \ddot{a}_{\overline{n-K+1}|}) v^{K+1} \mathbb{I}_{\{K<n\}}$. This results in

$$P^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}$$

5.3.2 Pure endowment insurance and mixed insurance

Experience case

The premia is denoted by $P_{x:\overline{n}|}{}^{1}$. The loss amounts to

$$L = \begin{cases} -P_{x:\overline{n}|}{}^{1} \ddot{a}_{\overline{K+1}|}, & \text{if } K = 0, \dots, n-1 \\ v^n - P_{x:\overline{n}|}{}^{1} \ddot{a}_{\overline{n}|}, & \text{if } K \geq n \end{cases}$$

and we get

$$P_{x:\overline{n}|}{}^{1} = \frac{A_{x:\overline{n}|}{}^{1}}{\ddot{a}_{x:\overline{n}|}} = \frac{D_{x+n}}{N_x - N_{x+n}}$$

Mixed insurance

The premium is denoted by $P_{x:\overline{n}|}$ and is calculated from

$$P_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{A_{x:\overline{n}|}{}^{1}}{\ddot{a}_{x:\overline{n}|}} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$$

5.4 Individual Premium Payment

Analogous to the previous designations, the transition from $P \rightsquigarrow P^{(m)}$ the transition from annual premium payments to annual premium payments. The analogue calculations can be as in the case of annuities and increases, for example

$$P^{(m)}_{x:\overline{n}|} = \frac{P_{x:\overline{n}|}}{\frac{d}{d^{(m)}} - \beta(m) P^1_{x:\overline{n}|}} \tag{5.1}$$

To see this, use the following two equations

$$\begin{aligned} A_{x:\overline{n}|} &= P_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{n}|} \\ \ddot{a}^{(m)}_{x:\overline{n}|} &= \frac{d}{d^{(m)}} \ddot{a}_{x:\overline{n}|} - \beta(m) A^1_{x:\overline{n}|} \end{aligned}$$

The second equation is obtained using the equation (4.18) materialised:

$$\begin{aligned} \ddot{a}^{(m)}_{x:\overline{n}|} &= \frac{di}{d^{(m)} i^{(m)}} \ddot{a}_{x:\overline{n}|} - \frac{i - i^{(m)}}{d^{(m)} i^{(m)}} (1 - {}_n p_x v^n) \\ &= -\frac{i}{d^{(m)} i^{(m)}} A^1_{x:\overline{n}|} + \frac{d}{d^{(m)}} \frac{1 - A_{x:\overline{n}|}{}^{1}}{d} - \frac{d}{d^{(m)}} \frac{A^1_{x:\overline{n}|}}{d} + \frac{A^1_{x:\overline{n}|}}{d^{(m)}} \\ &= \frac{d}{d^{(m)}} \ddot{a}_{x:\overline{n}|} - \beta(m) A^1_{x:\overline{n}|} \end{aligned}$$

If you reshape (5.1) slightly, then you get

$$P_{x:\overline{n}|} = \ddot{a}^{(m)}_{\overline{1}|} P^{(m)}_{x:\overline{n}|} - \beta(m) P^{(m)}_{x:\overline{n}|} P^1_{x:\overline{n}|}$$

Because $i > i^{(m)}$, it can be seen that

$$P_{x:\overline{n}|} \leq P^{(m)}_{x:\overline{n}|}$$

For the other types, the following formulas apply, which are not used here can be derived:

$$\begin{aligned} P_x &= \ddot{a}^{(m)}_{\overline{1}|} \cdot P^{(m)}_x - \beta(m) P^{(m)}_x P_x \\ P^1_{x:\overline{n}|} &= \ddot{a}^{(m)}_{\overline{1}|} P^1_{x:\overline{n}|}{}^{(m)} - \beta(m) P^1_{x:\overline{n}|}{}^{(m)} P^1_{x:\overline{n}|} \\ P_{x:\overline{n}|}{}^{1} &= \ddot{a}^{(m)}_{\overline{1}|} P_{x:\overline{n}|}{}^{1(m)} - \beta(m) P_{x:\overline{n}|}{}^{1(m)} P^1_{x:\overline{n}|} \end{aligned}$$

5.5 General life insurance

For a general life insurance policy with premiums $\Pi_0, \Pi_1, \ldots, \Pi_K$ and death benefit $C_1, C_2, \ldots$ the total loss is calculated by

$$L = C_{K+1} v^{K+1} - \sum_{k=0}^{K} \Pi_k v^k$$

The following therefore applies to the fair price

$$\sum_{k=0}^{\infty} C_{k+1} v^{k+1} {}_k p_x q_{x+k} = \sum_{k=0}^{\infty} \Pi_k v^k {}_k p_x$$

With this formula, all general life insurance policies can be be calculated.

5.6 Life insurance with Return Guarantee

The life insurance policies with a reinsurance policy are Products for which the death benefit depends on the premiums paid. In principle, such insurance policies are used for the endowment insurance and annuities. For the Endowment insurance, as we call it so far are all premiums “lost” if you die before the survival date. In the case of endowment insurance policies with reinsurance, one (or the surviving dependants) in the event of death, the assets held up to that point in time paid-in premiums back (with/without) interest.

annuities are also of the above type before the retirement age or the full return of the annuity, in which, after the annuity has been paid out the annuity sum is reduced by the annuity.

The normal return weight is now calculated below. The Death benefit at time t amounts to $t \cdot P_x$. So the following applies

$$P_x \ddot{a}_{x:\overline{n}|} = P_x (IA)^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

and therefore

$$P_x = \frac{A_{x:\overline{n}|}^{\;\;1}}{\ddot{a}_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|}}$$

The procedure for the other types of return watch is very similar. similar. The only problem with the complete recovery lies in the special form of death benefit insurance. For the normal type of The following applies to old-age annuities

$$P_x = \frac{{}_{n|}\ddot{a}_x^{(m)}}{\ddot{a}_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|}}$$

6 Markov Chains

6.1 Traditional Pricing

The process of traditional pricing can be described as follows:

1. We start with mortality probabilities and then calculate a mortality table starting with e.g. 100,000 people over the age of 20.

   Denote l_x the number of living persons of age x.

$$\begin{aligned} l_0 &= \text{100'000} \\ l_{x+1} &= l_x(1-q_x) \\ d_x &= l_x - l_{x+1} \end{aligned}$$

2. The various commutation functions are then calculated:

$$D_x = l_x v^x, C_x = d_x v^{x+1}, \text{ etc.}$$

   These figures depend on

   - the number of living persons and
   - from the technical interest rate.

3. Using this formalism, we can now calculate present values, premiums, etc:

$$\begin{aligned} \ddot{a}_x &= \frac{N_x}{D_x} \\ A_x &= \frac{M_x}{D_x} \end{aligned}$$

Almost all premiums can be calculated by adding and multiplying commutation functions. This This procedure is particularly useful if no computer is available.

The *Markov model* allows us to rate life insurance policies *without commutation functions*. It starts with the calculation of the actuarial reserve via recursion and directly uses the different probabilities.

Example: ${}_n p_x$ denotes the probability that an x-year-old person will survive n years. The following apply

$$\ddot{a}_x = \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = \cdots = \sum_{k=0}^{\infty} v^k \cdot {}_k p_x$$

and

$$ {}_{k+1}p_x = {}_kp_x \cdot p_{x+k} \, . $$

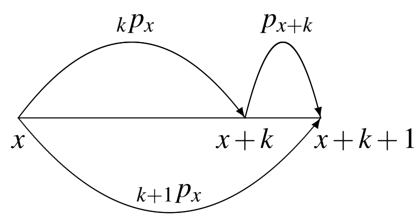

Figure 6.1: Transition Probabilities

$\ddot{a}_x$ corresponds to the actuarial reserve V_x for an annuity of 1.

$$ V_x = \ddot{a}_x = 1 + \sum_{k=1}^{\infty} v^k \cdot {}_kp_x = 1 + v p_x \sum_{k=1}^{\infty} v^{k-1} \cdot {}_{k-1}p_{x+1} = 1 + v p_x V_{x+1} \, . $$

The actuarial reserve at age x can therefore be divided into two components, namely

1. an annuity payment and
2. the necessary actuarial reserve at age $x+1$.

We have therefore found a recursion for the actuarial reserve. We can therefore simply calculate the premium recursively. To do this, however, we need an initial condition, namely $\ddot{a}_\omega = V_\omega = 0$. ◇

Remark: We do not need any commutation functions, only the survival probability p_x and the Discount factor v. ◇

6.2 Life insurance as random Cash Flows

In the following, let $(\Omega, \mathscr{A}, P)$ be a probability space and $(S, \mathscr{B})$ a measurable Room.

Definition 6.2.1 (Markov chain)**:** A stochastic process $X = (X_t)_{t \in \mathbb{N}}$ on $(\Omega, \mathscr{A}, P)$ with values in $(S, \mathscr{B})$ is called *Markov chain* if P-a.s.

$$ P[X_{t+1} \in B \mid X_0, \ldots, X_t] = P[X_{t+1} \in B \mid X_t] $$

applies to every $t \in \mathbb{N}$ and every $B \in \mathscr{B}$. The range of values $(S, \mathscr{B})$ is called *state space of* X. ◇

Remarks: For *countable* state spaces where every single-point quantity $\{k\}$ is an elementary event then X is a Markov chain if and only if

$$ P[X_{t+1} = k \mid X_0 = l_0, \ldots, X_t = l_t] = P[X_{t+1} = k \mid X_t = l_t] $$

applies for every $t \in \mathbb{N}$, for every sequence $(l_0, \ldots, l_t)$ in S and every $k \in S$, for for which the left-hand side is well-defined, i.e. for which $P[X_0 = l_0, \ldots, X_t = l_t] > 0$ holds. Note $P[X_t = l_t] \geq P[X_0 = l_0, \ldots, X_t = l_t]$, which means that the right-hand side is also well-defined.

For more general state spaces, a more general definition of conditional probabilities, which will not be discussed here.

In fact, in practice - especially in life insurance - you usually even have to deal with finite state spaces.

The probability for $X_{t+1} = k$ depends only on the state at time t, but not on the path $X_0, \dots, X_t$; it is also said that such processes have no memory. ◇

To represent a life insurance policy, you need three things, namely

1. a Markov chain $X = (X_t)_{t \in \mathbb{N}}$,
2. a one-year discount factor v and
3. Contract functions.

The starting point of the Markov model is a set S of states whose elements represent the various possible conditions of an insured person.

Example: $S = \{$'living', 'dead'$\}$. ◇

In the following, we only consider finite state spaces. Since we only consider models in discrete time payments (benefits, bonuses, etc.) are only possible at integer times $t = 0, 1, \dots$. Payments can be made either at the beginning or at the end of a period $[t, t+1]$. Payments at The beginning of a period $[t, t+1]$ depends only on the state i at time t, while payments at the end of a period $[t, t+1[$ period may also depend on the 'final state' j at time $t+1$ (e.g. death benefit at transition 'living' $\rightarrow$ 'dead' from t to $t+1$). We use the following designations:

Definition 6.2.2 (Generalised annuities and lump-sum benefits)**:**

- *generalised annuities* $a_i^{Pre}(t)$: Due at the beginning of the time interval $[t, t+1]$ if $X_t = i$. The $a_i^{Pre}(t)$ are called *generalised annuities which are payable at time t, given that the person is in state i at time t.*
- *generalised capital payments* $a_{ij}^{Post}(t)$: Due at the end of the time interval $[t, t+1]$ at the transition $i \rightarrow j$ from t to $t+1$. The $a_{ij}^{post}(t)$ are called *generalised capital benefits at time* $t+1$*, given that the insured person from t changes from state i to state j after* $t+1$ *.*

◇

The $a_{ij}^{Post}(t)$ must therefore first be discounted with v in order to be comparable with the $a_i^{Pre}(t)$. to be.

Up to now, we have defined sums that are due on the occurrence of a certain insured event. become. Now we need a probability law to quantify the various transitions.

Definition 6.2.3 (transition probabilities)**:** For all states $i, j \in S$ and times $s, t \in \mathbb{N}$, the probabilities are called

$$p_{ij}(s,t) := P[X_t = j \mid X_s = i]$$

Transition probabilities for the transition from state i at time s to state j at the time t.

The matrix $P(s,t)$ with entries $p_{ij}(s,t)$ is called *transition matrix from s to t*.

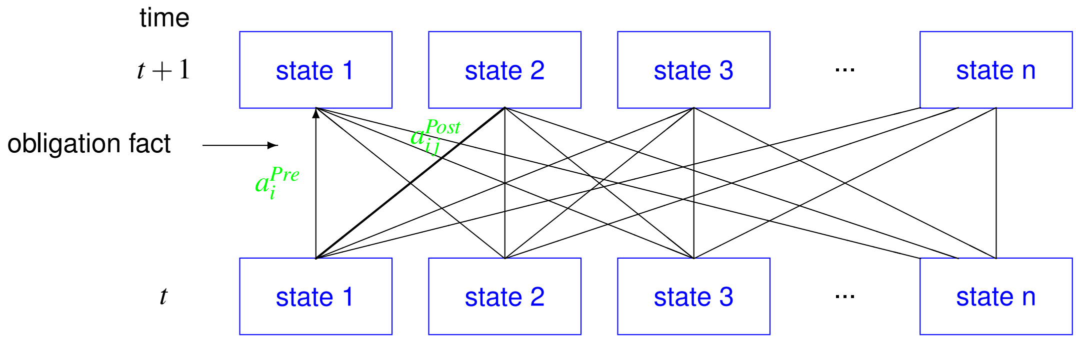

Figure 6.2: Transition Probabilities and contractual Functions

The transition probabilities for a time step are calculated with

$$p_{ij}(t) := p_{ij}(t, t+1)$$

and the corresponding transition matrices with

$$P(t) := P(t, t+1)\,.$$

◇

The most important property of these transition probabilities is characterised by

Theorem 6.2.4 (Chapman-Kolmogorov equation)**:** *For any integer* $0 \leq r \leq s \leq t$ *applies*

$$P(r,t) = P(r,s) \cdot P(s,t)\,,$$

or in component notation

$$p_{ik}(r,t) = \sum_{j \in S} p_{jk}(s,t) \cdot p_{ij}(r,s)\,.$$

◇

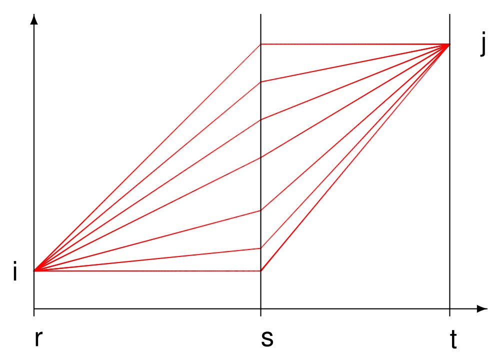

Figure 6.3: Interpretation of the Chapman-Kolmogorov Equation

Next, let's look at some examples. In general, the symbols $*$ ('living') and † ('dead') is used.

Example (Retirement annuity starting immediately): All contract functions except $a_*^{Pre}(t)$ are equal to zero. In one period 1 monetary unit is paid out in advance, i.e. $a_*^{Pre}(t) = 1$. If the insured person has time $t = 0$ the age x, the following applies to the transition probabilities

$$
\begin{aligned}
p_{**}(t) &= 1 - q_{x+t}\,, \\
p_{*\dagger}(t) &= q_{x+t}\,, \\
p_{\dagger *}(t) &= 0\,, \\
p_{\dagger\dagger}(t) &= 1\,.
\end{aligned}
$$

Figure 6.4 schematically shows the model used. ◇

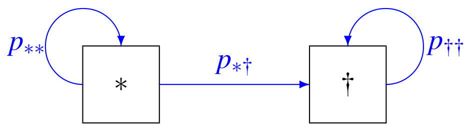

Figure 6.4: Markov Model for an old-age Annuity starting immediately

Example (Insurance for one life): With this Markov model, almost all insurance policies on *one life* can be modelled including of arbitrary yield curves, e.g. $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$, $_{m|}\ddot{a}_x$, A_x and $A^1_{x:\overline{n}|}$.

For example, we can calculate $\ddot{a}_x$ under the assumption

$$a_*^{Pre}(t) = 1, \quad a_{*\dagger}^{Post}(t) = a_{**}^{Post}(t) = 0$$

(constant annuity), or for a linearly increasing annuity with an annual increase of 5%, i.e.

$$a_*^{Pre}(t) = 1 + 0.05 \cdot t, \quad a_{*\dagger}^{Post}(t) = a_{**}^{Post}(t) = 0\,.$$

For mixed insurance, i.e. to calculate $A^1_{x:\overline{n}|}$, the model looks as follows from:

$$
\begin{aligned}
a_{*\dagger}^{Pre}(t) &= \begin{cases} 1 & \text{if } t < x+n \\ 0 & \text{other,} \end{cases} \\
a_{**}^{Pre}(t) &= \begin{cases} 1 & \text{if } t = x+n-1 \\ 0 & \text{other} \end{cases}
\end{aligned}
$$

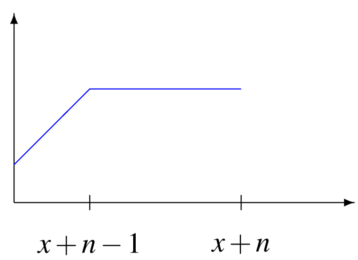

Figure 6.5: Performance history of a mixed insurance policy

◇

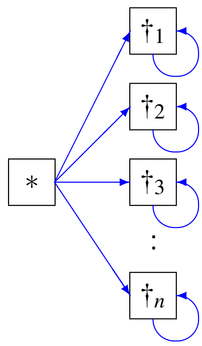

Figure 6.6: Markov Model for multiple Causes of Elimination

Example (Multiple causes of elimination): The model for multiple causes of elimination (e.g. 'death by illness', 'death by accident', ...) is shown schematically in Figure 6.6. $\diamond$

Example (Multiple lives): As an example, let us consider a couple, man and woman. The state of the couple is now characterised by the ordered pairs $(*,*)$ = 'both alive', $(*,\dagger)$ = 'man alive, woman dead', etc. are shown. The possible transitions between the states are shown in Figure 6.7 is displayed.

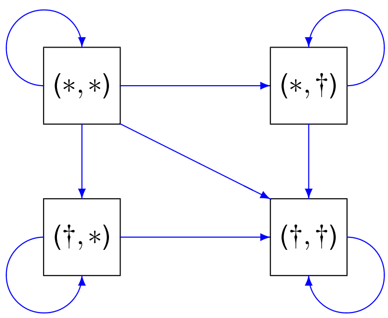

Figure 6.7: Markov Model for Widow's Annuity

Examples are

- *Connection annuity (present value $\ddot{a}_{xy}$)*: Here is

$$\ddot{a}^{Pre}_{(*,*)}(t) = 1$$

  for all t, all other contract functions are zero.

- *Annuity on the last life (present value $\ddot{a}_{\overline{xy}}$)*: Here are

$$\begin{aligned}\ddot{a}^{Pre}_{(*,*)}(t) &= 1\,,\\ \ddot{a}^{Pre}_{(*,\dagger)}(t) &= 1 \text{ and}\\ \ddot{a}^{Pre}_{(\dagger,*)}(t) &= 1\end{aligned}$$

  for all t, all other contract functions are zero.

$\diamond$

6.3 Capital Cover, Recusion and Premiums

A central task in life insurance is the determination of the actuarial reserve, the amount of money which needs to be reserved for each policy in order to cover all future benefits.

Let A_t denote the payments that are due for a policy (or a portfolio) at a time t. are. Here, $A = (A_t)_{t\in\mathbb{N}}$ is a stochastic process. With the definition 6.2.2 introduced notation we have at a time t on the one hand the generalised annuity $a_i^{Pre}(t)$, which is determined by the state $X_t = i$ at time t, and the generalised capital output $a_{ji}^{Post}(t-1)$, which is determined by the transition j to i from $t-1$ is due after t. So we define (pointwise on Ω)

$$A_t := a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t-1)\,; \quad t\in\mathbb{N},$$

where $a_{ij}^{Post}(-1) := 0$ is to be set for all $i, j \in S$.

Remark: For analytical purposes, it is useful to use the representations

$$a_{X_t}^{Pre}(t) = \sum_{i\in S} \chi_{[X_t=i]} \cdot a_i^{Pre}(t)$$

and

$$a_{X_{t-1}X_t}^{Post}(t-1) = \sum_{j,i\in S} \chi_{[X_{t-1}=j, X_t=i]} \cdot a_{ji}^{Post}(t-1)$$

to use. $\diamond$

Definition 6.3.1 (prospective actuarial reserve)**:** The present value of the future cash flow (A), determined on the basis of today's information: $\diamond$

Definition 6.3.2 (Concept of prospective actuarial reserve)**:** The current value of the future actuarial reserve cash flow (A), determined on the basis of today's information:

$$V_i(t) = E[A|X_t = i]$$

$\diamond$

This notation shows the strong dependency of the actuarial reserve of the condition of the policy. (Image 6.3)

The direct calculation of the necessary reserves for the various conditions is not so easy if we want to create a generalised continuous Markov model. One advantage of this model is the existence of an often useful backward recursion.

The following formula allows the recursive calculation of the necessary reserves and thus the necessary individual premiums (**Thiele's difference equation**):

$$V_i(t) = a_i^{Pre}(t) + \sum_{j\in J} p_{ij}(t,t+1)\cdot v\cdot (a_{ij}^{Post}(t) + V_j(t+1))$$

In the case of a near-market valuation, the recursion for the age $x = x_0 + k$ at time $t + k$ is as follows:

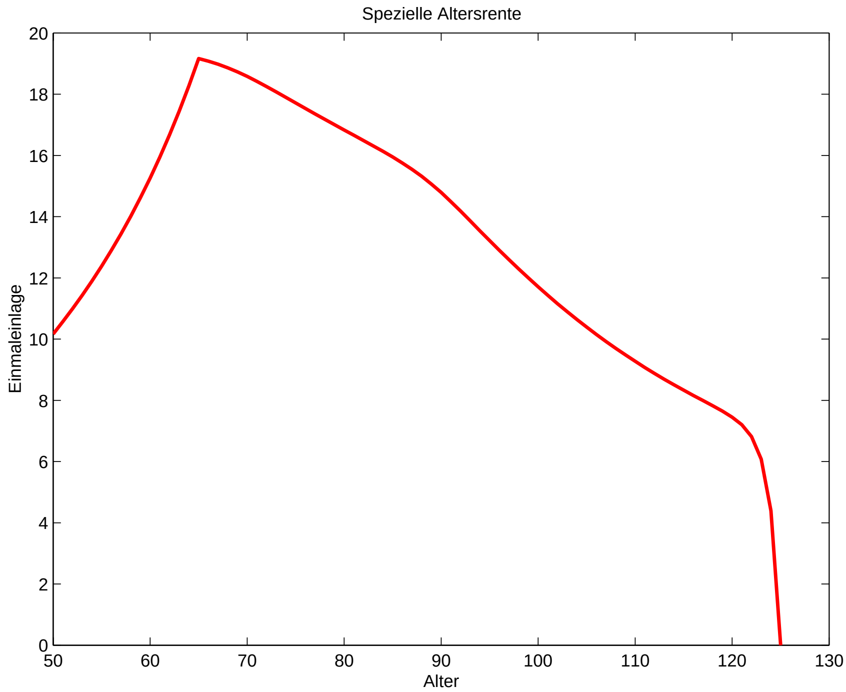

Figure 6.8: Increasing Annuity

$$V_i(t+k) = a_i^{Pre}(t+k) + \sum_{j \in J} p_{ij}(t+k, t+k+1) \cdot \frac{P(t,t,k+1)}{P(t,t,k)} \cdot (a_{ij}^{Post}(t+k) + V_j(t+k+1))$$

This formula only uses the various services, the probabilities and the discount factor.

Interpretation: The current reserve consists of:

- payments due to the various possible transitions.
- the discounted values of the future necessary reserves. ◇

Applications:

- $\ddot{a}_x = 1 + p_x \cdot v \cdot (0 + \ddot{a}_{x+1})$
- $A_x = q_x \cdot v \cdot 1 + p_x \cdot v \cdot A_{x+1}$ ◇

To determine the reserves for a certain age, we need to calculate backwards, starting at Expiry date of the policy. For annuities, this is usually the age when everyone has died. To calculate the reserves we need boundary conditions that depend on the cash flow on the expiry date. We usually set the Limit conditions equal zero for all reserves. We note that the vector of these reserves are calculated at the same time.

After calculating the various reserves, we can calculate the corresponding premiums required using Determine equivalence principle.

To determine the fair annual premium, we have to also apply the above principle. In the In most cases (exception: return year), we can act analogue to the traditional procedure.

We introduce the following variables:

- V_j^B: Reserve, generated by services
- V_j^P: Reserve, generated by premiums

The total actuarial reserve is (as usual) $V_j(t) = V_j^B(t) - V_j^P(t)$.

The premium can be calculated via the equivalence principle ($V_{j_o}(t_0) \equiv 0$).

6.4 Concrete Problems

So far, we have dealt with the definitions of the various elements required for the Markov model. Now we turn to specific problems. The following examples have their own little difficulties:

- Annuity payment during the year,
- Annuity with guaranteed payment period,
- annuities on one and two lives with return of the actuarial reserve.

6.4.1 Annuity payment during the year

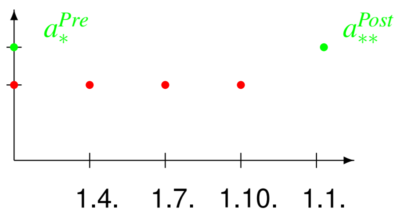

Figure 6.9: Example for Annuity Payments during the Year

In Switzerland, most annuities are paid quarterly. We therefore have to take the current value of the annuities. The simplest option is to set the time interval under consideration as three months.

The approximation normally used is known. For example, we know the following formula:

$$\ddot{a}^{(4)}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 0.375(1 - A_{x:\overline{n}|}{}^{1})$$

Now we want to do the same in our Markov model. To solve this problem, we set the adjustment term for $n = 1$. The above situation can be created in this case with the following two corrections to the cash flow if such a annuity is paid at time t for a transition $* \to *$:

- $a_{*}^{Pre} = 1 - \frac{3}{8}$
- $a_{**}^{Post} = 0 + \frac{3}{8}$

If we split the payment in two, we set $a_{*t}^{Post} = 0 + \frac{3}{8}$!

Example: Geometrically increasing annuity (3%), during the year ($\frac{4}{4}$), for a man aged t, starting at age $x+m$.

$$
\begin{aligned}
{}_{m|}\ddot{a}_x^{(4)} \to a_{*}^{pre}(t) &= \begin{cases} 0, & \text{falls} \quad t < x+m \\ 1.03^{t-(x+m)}(1-0.375), & \text{falls} \quad t \geq x+m \end{cases} \\
a_{**}^{Post}(t) &= \begin{cases} 0, & \text{falls} \quad t < x+m \\ 1.03^{t-(x+m)}(0.375), & \text{falls} \quad t \geq x+m \end{cases}
\end{aligned}
$$

◇

Remark: This method is not as error-prone as the usual approach with commutation functions. The difference between a temporary annuity of the above type and the normal one consists in the different starting times of the Backward recursion. ◇

6.4.2 Guaranteed annuity

Such a annuity is also paid after the death of the insured person. paid out for a maximum of m years. Therefore we carry in the Markov model:

- death without guarantee.
- death with guarantee.

three states

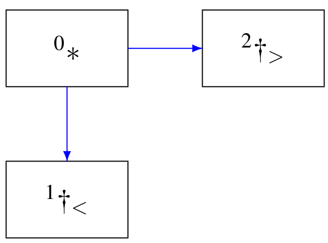

Figure 6.10: State space for Guaranteed Annuities

Example: Future annuity, starting at 65, 10-year guarantee.

assumption: annuity ends at age 85.

annuity: man	$s = 65$	$g = 10$	$t = 20$
state:	$0 =$ alive	$1 = \dagger$ before 65	$2 = \dagger$ after 65

To calculate the annuities, we use 85 as the start time of the recursion with boundary condition 0. We obtain the following non-trivial contract functions:

$$a_0^{Pre}(t) = \begin{cases} (1-0.375), & \text{if} \quad 65 \le t < 85 \\ 0, & \text{otherwise} \end{cases}$$

$$a_{00}^{Post}(t) = \begin{cases} 0.375, & \text{if} \quad 65 \le t < 85 \\ 0, & \text{otherwise} \end{cases}$$

$$a_{02}^{Post}(t) = \begin{cases} 0.375, & \text{falls} \quad 65 \le t \le 85 \\ 0, & \text{sonst} \end{cases}$$

$$a_2^{Pre}(t) = \begin{cases} (1-0.375), & \text{if} \quad 65 \le t < 75 \\ 0, & \text{otherwise} \end{cases}$$

$$a_{22}^{Post}(t) = \begin{cases} 0.375, & \text{if} \quad 65 \le t < 75 \\ 0, & \text{otherwise} \end{cases}$$

To $a_{00}^{post}(t)$: We must not forget adjust the term due to the annuity payment during the year.

Image 6.4.2 shows the behaviour of the required reserves. ◇

Example: Finally, let's look at an example with more than three different states. We are interested in the premium for the following insurance on two lives:

$(*,*)$	$\rightarrow$	1.2	nach $x = 65$
$(*,t)$	$\rightarrow$	0.8	nach $x = 65$, 10 years guaranteed
$(t,*)$	$\rightarrow$	0.6	after the death of the husband
Premium		$(1/1)$	

We have five different states to consider:

1. State 0: $(*,*)$
2. State 1: $(*,t)$
3. State 2: Woman dies before man, man dies after 65
4. State 3: $(t,*)$
5. State 4: All Other

◇

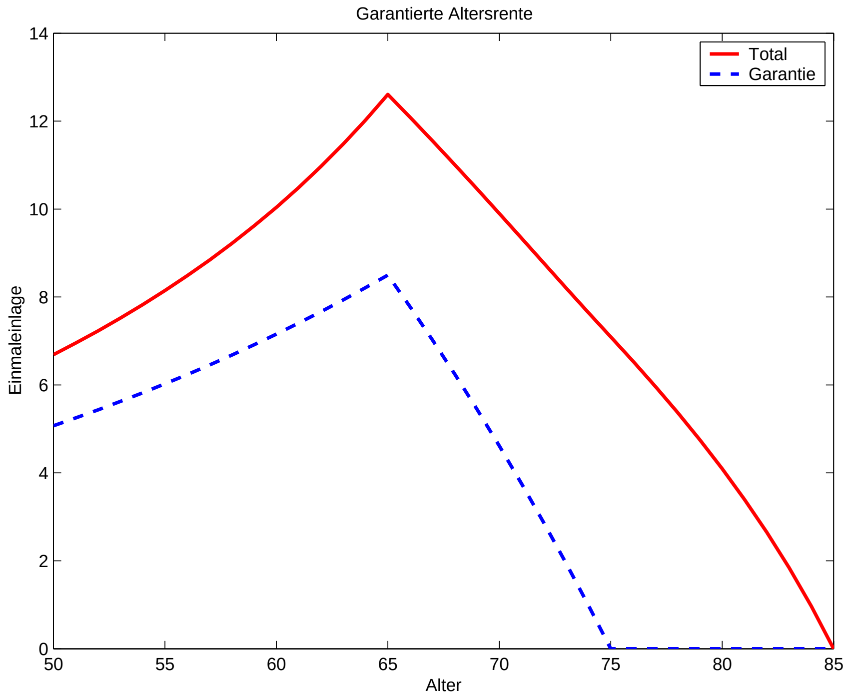

Figure 6.11: Guaranteed, temporary annuity

6.4.3 Return of Mathematical Reserve

A special class of insurance contracts are contracts with Warranty in the event of premature death. The different types include the risk of return:

1) repayment of the premiums paid in the event of death prior to the annuity payment.

2) Return of the paid premiums in relation to the paid premiums annuities.

3) return of the actuarial reserve.

We can regard cases 1) and 2) as additional death premiums. They are already used for a long time.

Let us now look at case 3). For this type of return, there are also many different forms. We will get to know some of them. Before we start, let's look at the following simple situation:

In the event of death, you pay the current actuarial reserve. So you have we use the following recursion:

$$V_0(t) = 1 + p_{00}(t) \cdot v \cdot V_0(t+1) + p_{01}(t) V_0(t)$$

(In this example, we assume that we carry out the return at the beginning of the period).

By transforming we get:

$$V_0(t) = \frac{1 + p_{00}(t) \cdot v \cdot V_0(t+1)}{1 - p_{01}(t)}$$

In principle, this solves the above problem in the case of the single deposit. Now we still have to consider the case of regular annual premiums. Of course, we can no longer use the term Use $V_j^B(t)/V_j^P(t)$.

The simplest (but probably not the most elegant) solution is to use an iterative approximation for to find the annual premium.

Example: Let's look at an example of an *annuity with return of the actuarial reserve* before the start of the payment ($s = 65$). In this case, we have the following values:

Age	Reserve (line)	ATH101 ($\diamond$)	ATW ($\otimes$)
40	5.869210	5.506800	5.586737
45	6.970781	6.640532	6.786052
50	8.279101	8.011918	8.279320
55	9.832975	9.659149	10.156673
60	11.678489	11.611176	12.551519
65	13.870382	13.870381	15.692769

The table above shows the results of the following The following types of return are listed:

1. Reserve: Return of the actuarial reserve before s
2. ATH101: Return of the paid premiums before s
3. ATW: Return of the paid premiums paid for Services.

It is obvious that 3) the most expensive type of return of premium is. 1) and 2) are approximately equal.

Analogue calculations for the different types can also be performed in the case of regular premiums (see image 6.4.3). $\diamond$

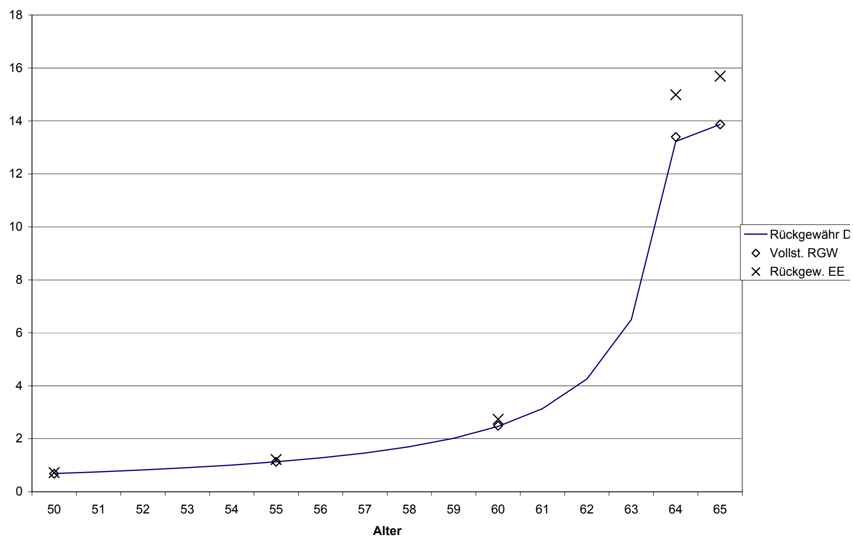

Figure 6.12: Refund of Premimum

Finally, a somewhat exotic example:

A widow's annuity is insured with repayment of the actuarial reserve if the woman dies before or dies in the same year as the husband. In this case, we have the following recursion:

$$V_0(t) = v(p_{00}(t)V_0(t+1) + p_{02}(t)V_2(t+1))/(1 - p_{01}(t) - p_{02}(t)).$$

◇

6.5 Markov Chain: Proofs

$(X_t)_{t\in\mathbb{N}} : (\Omega, \mathscr{A}, P) \to \{1, 2, \dots n\}$ is called *Markov Chain*
iff

$$P[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \dots X_{t_m} = i_m] \; = \; P[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for

$$t_1 < t_2 < \dots < t_m < t_{m+1}$$

and

$$i_1, i_2, \dots, i_m, i_{m+1} \in \{1, 2, \dots n\}$$

We *define*

$$p_{ij}(s,t) \; := \; P[X_t = j | X_s = i]$$

Question: How does $p_{ij}(s,t)$, $p_{jk}(t,u)$ and $p_{ik}(s,u)$ relate?

$\mathscr{S} := \{1, 2, \dots, n\}$

6.5.1 Chapman-Kolmogorov Theorem

For $s < t < u$ and $i, k \in \mathscr{S}$ we have the following

$$\begin{aligned} p_{ik}(s,u) &= \sum_{j\in\mathscr{S}} p_{ij}(s,t)\, p_{jk}(t,u) \\ P(s,u) &= P(s,t)P(t,u) \end{aligned}$$

Proof

$$\begin{aligned} p_{ik}(s,u) &= P[X_u = k | X_s = i] \\ &= P[X_u = k \cap \Omega | X_s = i] \\ &= P[X_u = k \cap \cup_{j\in\mathscr{S}} \{X_t = j\} | X_s = i] \\ &= \sum_{j\in\mathscr{S}} P[X_u = k, X_t = j | X_s = i] \end{aligned}$$

Now lets look at $P[X_u = k, X - t = j | X_s = i]$

$$\begin{aligned}
P[X_u = k, X_t = j | X_s = i] &= \frac{P[X_u = k, X_t = j, X_s = i]}{P[X_s = i]} \\
&= \frac{P[X_u = k, X_t = j, X_s = i]}{P[X_s = i]} \times \frac{P[X_t = j, X_s = i]}{P[X_t = j, X_s = i]} \\
&= P[X_t = j | X_s = i] \times P[X_u = k | X_t = j, X_s = i] \\
&= P[X_t = j | X_s = i] \times P[X_u = k | X_t = j] \\
&= p_{ij}(s,t)\, p_{jk}(t,u)
\end{aligned}$$

$$\begin{aligned}
p_{ik}(s,u) &= P[X_u = k | X_s = i] \\
&= P[X_u = k \cap \Omega | X_s = i] \\
&= P[X_u = k \cap \cup_{j \in \mathscr{S}} \{X_t = j\} | X_s = i] \\
&= \sum_{j \in \mathscr{S}} P[X_u = k, X_t = j | X_s = i] \\
&= p_{ij}(s,t)\, p_{jk}(t,u)
\end{aligned}$$

6.5.2 Markov Model

We have been given $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$, and a Markov Chain X_t. We assume a constant discount factor $v = \frac{1}{1+i}$.

What is the Cash flow induced?

We define $I_i(t) = \chi_{X_t = i}$. Which mean that you would get the following annuity at time t: $\sum_{i \in S} a_i^{pre}(t) \times I_i(t)$. Which transition (death) benefit would you get at time t? You get $a_{ij}^{post}(t)$ if you are in state i at time t and in state j at $t+1$? ie $a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1)$. In sum for the death benefit $\sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1)$.

So the cash flow at times t can be calculated as

$$\begin{aligned}
A(t) &= \sum_{i \in S} a_i^{pre}(t) \times I_i(t) + \sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1) \\
\tilde{A}(t) &= \sum_{i \in S} a_i^{pre}(t) \times I_i(t) + \sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1) \times v
\end{aligned}$$

What is the value of this insurance cash flow?

We define the *Mathematical Reserve* as

$$\begin{aligned}
V_j(t) &= \mathbb{E}[PV\, of\, future CF\, | X_t = j] \\
&= \mathbb{E}[\sum_{\tau=0}^{\infty} v^{\tau} \tilde{A}(t+\tau) | X_t = j]
\end{aligned}$$

In order to calculate the mathematical reserve you can substitute $\tilde{A}$ in the formula below and ultimately what you need to calculate (keeping in mind the linearity of the $\mathbb{E}$ functional as the following quantities:

$$\begin{aligned} \mathbb{E}[I_i(t+\tau)|X_t=j] &= p_{ji}(t,t+\tau) \\ \mathbb{E}[I_i(t+\tau)\times I_k(t+\tau+1)|X_t=j] &= ? \end{aligned}$$

How do we do this?

$$\mathbb{E}[I_i(t+\tau)\times I_k(t+\tau+1)|X_t=j] = P[X_{t+\tau+1}=k, X_{t+\tau}=i|X_t=j]$$

Now we do the same as in the proof of the Chapman-Kolmogorov-Equation]

$$\begin{gathered} \mathbb{E}[I_i(t+\tau)\times I_k(t+\tau+1)|X_t=j] = P[X_{t+\tau+1}=k, X_{t+\tau}=i|X_t=j] \\ = \frac{P[X_{t+\tau+1}=k, X_{t+\tau}=i, X_t=j]}{P[X_t=j]} \\ = \frac{P[X_{t+\tau+1}=k, X_{t+\tau}=i, X_t=j]}{P[X_t=j]} \times \frac{P[X_t=j, X_{t+tau}=i]}{P[X_t=j, X_{t+tau}=i]} \\ = P[X_{t+\tau}=i|X_t=j]\times P[X_{t+\tau+1}=k|X_t=j, X_{t+\tau}=i] \\ = p_{ji}(t,t+\tau)\times p_{ik}(t+\tau,t+\tau+1) \end{gathered}$$

If we put now all things together we can can calcuate the mathematical reserves as follows

$$\begin{gathered} V_j(t) = \mathbb{E}[\textstyle\sum_{\tau=0}^{\infty} v^{\tau}\tilde{A}(t+\tau)|X_t=j] \\ = \textstyle\sum_{\tau=0}^{\infty} v^{\tau}\left(\sum_{i\in S} a_i^{pre}(t+\tau)\times p_{ji}(t,t+\tau) + \sum_{i,j\in S^2} a_{ij}^{post}(t+\tau)p_{ji}(t,t+\tau)\times p_{ik}(t+\tau,t+\tau+1)\times v\right) \end{gathered}$$

Remark: With this formula we can also calculate the expected cash flows at time t as follows:

$$\mathbb{E}[A(t+\tau)\,|X_t=j] = \sum_{i\in S} a_i^{pre}(t+\tau)\times p_{ji}(t,t+\tau) + \sum_{i,j\in S^2} a_{ij}^{post}(t+\tau)p_{ji}(t,t+\tau)\times p_{ik}(t+\tau,t+\tau+1)$$

6.5.3 Thiele Difference Equation

This is the relationship between the mathematical reserves between times t and $t+1$. The relationship is as follows:

$$V_j(t) = a_j^{pre}(t) + v\sum_{k\in S} p_{jk}(t,t+1)\times\left(a_{jk}^{post}(t) + V_k(t+1)\right)$$

Proof

To prove this equation we split the time-sum into $\tau = 0$ and the rest. For $\tau = 0$ we get

$$a_j^{pre}(t) + v \sum_{k \in S} p_{jk}(t, t+1) \times a_{jk}^{post}(t)$$

as per above.

In a second step we need to consider

$$\begin{aligned} &\textstyle\sum_{\tau=1}^{\infty} v^{\tau} \left(\sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{(i,k) \in S^2} a_{ik}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \textstyle\sum_{\tau=1}^{\infty} v^{\tau} \sum_{i \in S} p_{ji}(t, t+\tau) \times \left(a_i^{pre}(t+\tau) + \sum_{k \in S} a_{ik}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \end{aligned}$$

We can now calculate the quantity $p_{ji}(t, t+\tau)$ as follows, by means of the Chapman-Kolmorgorov equation

$$p_{ji}(t, t+\tau) = \sum_{l \in S} p_{jl}(t, t+1) \times p_{li}(t+1, t+\tau)$$

$$\begin{aligned} &\textstyle\sum_{\tau=1}^{\infty} v^{\tau} \left(\sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \textstyle\sum_{\tau=1}^{\infty} v^{\tau} p_{ji}(t, t+\tau) \times \left(\sum_{i \in S} a_i^{pre}(t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \textstyle\sum_{\tau=1}^{\infty} v^{\tau} \left(\sum_{l \in S} p_{jl}(t, t+1) \times p_{li}(t+1, t+\tau) \right) \times \left(\sum_{i \in S} a_i^{pre}(t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \textstyle\sum_{l \in S} p_{jl}(t, t+1) \times v \times \left(\sum_{\tau=0}^{\infty} v^{\tau} p_{li}(t+1, t+1+\tau) \times \left(\sum_{i \in S} a_i^{pre}(t+1+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+1+\tau) p_{ik}(t+1+\tau, t+1+\tau+1) \times v \right) \right) \\ &= \textstyle\sum_{l \in S} p_{jl}(t, t+1) \times v \times V_l(t+1) \end{aligned}$$

Remarks:

1) For one life we have a recursion of reals $A_x = q_x \times v + p_x \times v \times A_{x+1}$. In case of MR of a Markov model we have a recursion of vectors.

2) To solve it one needs boundary conditions as per before with $V_j(\omega) = 0 \forall j \in S$

3) Thiele Difference Equations leads to the same results as for the classical life insurance we have seen.

7 Mathematical Reserves

Mathematical Reserves are closely related to the equivalence principle, and the value of an insurance after inception. They are defined at a given point in time t as the present value of future benefits minus the present value of future premiums. As such this linke to the present value calculation which we have seen so far and we note that the equivalence principle stipulates the equality of the present value of future benefits and the present value of future premiums. Hence the mathematical reseve at inception using the equivalence principle are zero.

We also note that the Markov model focuses on the mathematical reserves directly and on the respective recursion formulas.

7.1 Mathematical Reserves in the classical model (Deckungskapital)

In summary we have

- The concept to determine the value of an insurance after inception;
- Used to determine the the amount of assets needed to safeguard the insurance;
- Needed to calculate a change in insurance (the MR is used as a single premium);
- The mathematical reserve is used also to decompose the premium into risk and savings premium.

Definition

The mathematical reserve at a given time is defined as the PV(future benefits) - PV(future premium). We denote it as ${}_kV_x$, where x age at inception and k policy years, noting that ${}_0V_x = 0$ by equivalence principle.

Example explicit calculation

We consider the following example: $A_{x:n}$ mixed endowment with Premium Π according to $\ddot{a}_{x:n}$, and we calculate the premiums as follows:

$$\begin{aligned}
PV(Benefits) &= \frac{M_x - M_{x+n} + D_{x+n}}{D_x};\\
PV(Premium) &= \Pi \times \frac{N_x - N_{x+n}}{D_x}\\
\Pi &= \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}.
\end{aligned}$$

Now we will calculate ${}_kV_x$:

$$\begin{aligned} PV(FutureBenefits) &= \frac{M_{x+k} - M_{x+n} + D_{x+n}}{D_{x+k}}; \\ PV(FuturePremium) &= \Pi \times \frac{N_{x+k} - N_{x+n}}{D_{x+k}}. \end{aligned}$$

We can do the same using commutation functions:

$${}_kV_x = \frac{M_{x+k} - M_{x+n} + D_{x+n} - \Pi \times (N_{x+k} - N_{x+n})}{D_{x+k}}.$$

Given we have recursions for both the present value of benefits $A_{x:n}$ and for the present value of premiums $\ddot{a}_{x:n}$, we can get the following recursion of the mathematical reserves:

$$\begin{aligned} A_{x,n} &= q_x \times v + p_x \times v \times A_{x+1,n-1}; \\ \ddot{a}_{x,n} &= 1 + p_x \times v \times \ddot{a}_{x+1,n-1}. \end{aligned}$$

- If one deducts the second formuls (times Π) from the first one;
- One gets the following recursion for the reserve:

$$\begin{aligned} {}_kV_x &= A_{x,n} - \Pi \times \ddot{a}_{x,n} \\ &= q_x \times v + p_x \times v \times A_{x+1,n-1} \\ &\quad -\Pi \times (1 + p_x \times v \times \ddot{a}_{x+1,n-1}) \\ &= -\Pi + q_x \times v + p_x \times v \times {}_{k+1}V_x. \end{aligned}$$

$$\frac{{}_kV_x - (-\Pi + q_x \times v)}{p_x \times v} = {}_{k+1}V_x.$$

This approach can be generalised. We use the following set up:

- Benefit Vector: C_k death benefit at policy year k; and
- Premium Vector: Π_k at the begining of policy year $k+1$.

Now the recursion reads as follows:

$$ {}_kV_x = -\Pi_k + q_x \times v \times C_{k+1} + p_x \times v \times {}_{k+1}V_x. $$

or equivalent

$$ {}_kV_x + \Pi_k = q_x \times v \times C_{k+1} + p_x \times v \times {}_{k+1}V_x. $$

Be applying this formulae and transforming them we finally get the following:

$$
\begin{aligned}
{}_kV_x + \Pi_k &= q_x \times v \times C_{k+1} + p_x \times v \times {}_{k+1}V_x \\
{}_kV_x + \Pi_k &= q_x \times v \times C_{k+1} + (1-q_x) \times v \times {}_{k+1}V_x \\
({}_kV_x - v \times {}_{k+1}V_x) + \Pi_k &= q_x \times v \times (C_{k+1} - {}_{k+1}V_x).
\end{aligned}
$$

7.2 Savings and Risk Premium

The above recursion for the mathematical reserves can be used to decompose the premiums into a risk premium and a savings premium. This decomposition is helpful when determining the sources of profit, ie profit from the investment process and profit from the risk process. We note that the decomposition of the premium into a savings and risk part works also for the Markov model (see below).

In a first step we define savings and risk premiums as follows:

Savings Premium: $\Pi^s = v \times {}_{k+1}V_x - {}_kV_x$; and

Risk Premium: $\Pi^r = q_x \times v \times (C_{k+1} - {}_{k+1}V_x)$.

In a next step we show that the sum of savings and risk premium equals the premium:

$$
\begin{aligned}
({}_kV_x - v \times {}_{k+1}V_x) + \Pi_k &= q_x \times v \times (C_{k+1} - {}_{k+1}V_x) \\
-\Pi_k^s + \Pi_k &= \Pi_k^r \\
\Pi_k &= \Pi_k^r + \Pi_k^s
\end{aligned}
$$

Based on this decomposition we can now determine the technical profit.

- As a consequence of the equivalence principle the expected profit and loss per period is zero.
- In reality a mortality table with best estimates is called a second order basis.
- For pricing one uses a first order motality table. For annuities mortality is generally lowered and for capital insurance increased.

- For example ERM95 has about a margin of 10-15%, which means if $q_{67} = 1.15\%$, the effective mortality is c1%.
- Assume that we have n people with an age of 67 and I_n is a RV which is 1 if person n dies and 0 else.

Now we can calculate the loss of this portfolio as follows:

$$\begin{aligned} L(\omega) &= \sum_{j=1}^{n} I_j(\omega) \times v \times (C_{k+1} - {}_{k+1}V_x). \\ \mathbb{E}[L(\omega)] &= \sum_{j=1}^{n} \mathbb{E}[I_j(\omega)] \times v \times (C_{k+1} - {}_{k+1}V_x) \\ &= q_{67}^2 \sum_{j=1}^{n} v \times (C_{k+1} - {}_{k+1}V_x). \end{aligned}$$

We now get the following for the technical profit:

$$PnL^{risk} = (q_{67}^1 - q_{67}^2) \times v \times (C_{k+1}^j - {}_{k+1}V_x^j).$$

This easy calculation is applied as follows in practise:

1. Calculate the sum of the risk premium according the underlying assumptions $RP_{tot} \sum_k \Pi^r(k)$;
2. Calculate the respective losses $L(\omega) = \sum_k I_k(\omega) \times v \times (C(k) - V(k))$;
3. Determine the technical Profit Loss as $RP_{tot} - L(\omega)$; and
4. Consider $\frac{L(\omega)}{RP_{tot}}$ which is the loss ratio.

Example: Assume a risk premium in a portfolio of CHF 120M and a loss of CHF 90M. In this case the loss ratio is $\frac{q_x^2}{q_x} = \frac{90}{120} = 0.75$

7.3 Savings and Risk Premium for the Markov Model

In order to do the technical analysis one has to define the normal subsequent state for each state $j \in S$. The definition of the subsequent state depends on the model. This state describes the transitions without incurred losses. In the setting of a classical life insurance with state space $S = \{*, \dagger\}$ the subsequent state of $*$ is usually defined to be $*$.

Normal subsequent state:

The normal subsequent state is defined by a function

$$\phi : S \to S, i \mapsto \phi(i),$$

which assigns to each state i, a state which is subsequent to it and for which, according to the policy setup, no payout to the insured is due.

One can derive the following quantities based on the normal subsequent state.

Savings premium: The savings premium for state i during the time interval $]t,t+1]$ is denoted by $\Pi_i^{(s)}$. It can be calculated by the following formula: $\Pi_i^{(s)}(t) = v_t^i V_{\phi(i)}(t+1) - V_i(t)$.The savings premium is the amount of money, which one has to add to the mathematical reserve at time t in order to have the necessary mathematical reserve in the subsequent state $\phi(i)$ at time $t+1$.

Technical pension dept: The regular cash flow or the technical pension dept is $\Pi_i^{(tr)}(t) = a_i^{Pre}(t) + v_t^i a_{i\phi(i)}^{Post}(t)$. We note that for a policy with premiums the technical pension dept $\Pi_i^{(tr)}(t)$ and the premiums coincide.

Value at risk and risk premium: Let $i \in S$ and $j \neq \phi(i)$. Then the value at risk $R_{ij}(t)$ is given by $R_{ij}(t) = V_j(t+1) + a_{ij}^{Post}(t) - (V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t))$. It corresponds to the loss, which incurs for a transition $i \rightsquigarrow j$. The risk premium corresponding to the transition $i \rightsquigarrow j$ is given by $\Pi_{ij}^{(r)}(t) = p_{ij}(t)\, v_t^i R_{ij}(t)$. Furthermore, $\Pi_i^{(r)}(t) = \sum_{j\neq\phi(i)} \Pi_{ij}^{(r)}(t)$ denotes the total risk premium.

Based on these quantities one can perform a technical analysis by comparing the loss, which effectively incurred in the year, and the risk premium for the corresponding transition. Usually one also considers the loss quotient $\frac{\text{loss}(i\rightsquigarrow j)}{\Pi_{ij}^{(r)}}$. We have seen that one can use the risk premium and the losses to check if the underlying best estimate assumptions fit to the current situation.

The following theorem relates the two kinds of premiums.

Theorem: The regular cash flow is related to the savings premium and the risk premium by the following equation:

$$-\Pi_i^{(tr)}(t) = \Pi_i^{(r)}(t) + \Pi_i^{(s)}(t).$$

This decomposition of the premiums is called technical decomposition.

The statement is proved by a direct calculation:

$$\begin{aligned}
&\Pi_i^{(r)}(t) + \Pi_i^{(s)}(t) \\
&= \sum_{j\in S} v_t\, p_{ij}(t) \left(V_j(t+1) + a_{ij}^{Post}(t) - V_{\phi(i)}(t+1) - a_{i\phi(i)}^{Post}(t) \right) \\
&+ (v_t V_{\phi(i)}(t+1) - V_i(t)) \\
&= -v_t \left(V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t) \right) \times p_{i\phi(i)}(t) - v_t \left(V_{\phi(i)}(t+1) + a_{i\phi(i)}^{Post}(t) \right) \\
&\times (1 - p_{i\phi(i)}(t)) + v_t V_{\phi(i)}(t+1) - a_i^{Pre}(t) \\
&= -\Pi_i^{(tr)}(t),
\end{aligned}$$

where we used Thiele's difference equations.

The theorem above states that the premium can be decomposed into a savings premium and a risk premium. This holds for net premiums without any administration charge. The gross premium is composed of three parts: savings premium, risk premium, cost premiums.

8 Multiple Decrements

8.1 The Model

So far in each case the transition from the state „life“ to the state „death“ since this transition is considered the payout (e.g death benefit insurance) or the termination of a benefit (e.g old-age annuity). However, other conditions can also be considered (e.g. the invalidity), which can cause a change in the cash flow.

This problem can be tackled in a general way by analysing the Survival time T in the „normal“ state yet another random variable J, the *reason for elimination*, defined.

Examples:

- In disability insurance, the original state is „Active“ and the possible separation regulations are „disability“ or „death“.
- Or a distinction is made between death due to „Accident“ or „Disease“. $\diamond$

Let $\mathscr{J}$ be the set of elimination criteria (e.g. $\mathscr{J} = \{$ Death by accident, death by illness$\}$. $J(\omega)$ is the random variable, the values from the set of elimination causes $\mathscr{J}$ is assumed. With $g_j(t)$ $j = 1, \ldots, m$ the *density function*regarding the joint distribution of T and J.

$$g_j(t)dt = P[t < T < t + dt, J = j]$$

and

$$g(t) = g_1(t) + \ldots + g_m(t)$$

If the retirement takes place at time t, then the conditional Probability of j given t

$$P[J = j | T = t] = \frac{g_j(t)}{g(t)}$$

New notation:

$$\begin{aligned} {}_tq_{j,x} &:= P[T < t, J = j] \\ {}_tq_{j,x+s} &:= P[T < s + t, J = j | T \geq s] \end{aligned}$$

$\diamond$

This last probability can be calculated as follows:

$${}_tq_{j,x+s} = \int_s^{s+t} \frac{g_j(z)dz}{1 - G(s)}$$

8.2 Mortality Densities

For an age x you can again analogue to definition 3.5 define a *mortality density* or *hazard rate*:

$$\mu_{j,x+t} = \frac{g_j(t)}{1-G(t)} = \frac{g_j(t)}{{}_t p_x}$$

And the *total precipitation density* can be calculated from this as

$$\mu_{x+t} = \mu_{1,x+t} + \ldots + \mu_{m,x+t} \tag{8.1}$$

It also applies

$$P[t < T < t+dt, J = j] = {}_t p_x \mu_{j,x+t} dt$$

respectively

$$P[J = j | T = t] = \frac{\mu_{j,x+t}}{\mu_{x+t}}$$

8.3 Years completely survived

Before the survival time was converted into a integer part K and divided into a remainder S . This around the to make policy calculations easier. The same principle is used here:

$$q_{j,x+k} = P[T < k+1, J = j | T \geq k]$$

and

$$q_{x+k} = q_{1,x+k} + \ldots + q_{m,x+k}$$

This results in

$$P[K = k, J = j] = {}_k p_x q_{j,x+k}$$

In order to model the mortality during the year, there are again Analogue procedure to the chapter 3.4. Again, it is often assumed that ${}_u q_{j,x+k}$ *linear* is in u for $0 < u < 1$, i.e.

$${}_u q_{j,x+k} = u q_{j,x+k}$$

From this equation follows

$$\begin{aligned} g_j(k+u) &= u \cdot {}_k p_x q_{j,x+k} \\ {}_{k+u} p_x &= {}_k p_x (1 - u \cdot q_{x+k}) \end{aligned}$$

and

$$\mu_{j,x+k+u} = \frac{u \cdot q_{j,x+k}}{1 - u \cdot q_{x+k}}$$

The above equation shows in particular

$$P[J = j | K = k, S = u] = \frac{q_{j,x+k}}{q_{x+k}}$$

can be derived, since $P[J = j | T = t] = \frac{\mu_{j,x+t}}{\mu_{x+t}}$

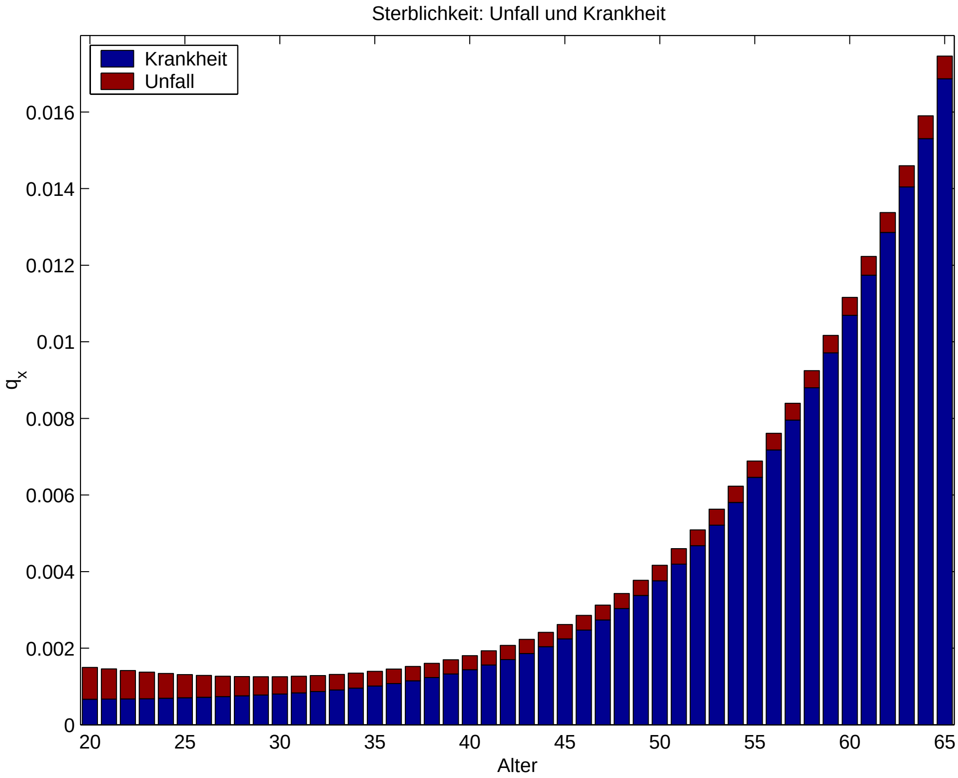

Figure 8.1: Single Premium for one-year death benefit insurance policies with Decrements illness and accident

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8.4 General Types of Endowment Insurance

Previously, C_k was insured as a death benefit in the event of death has been calculated. Now a death sum $C_{j,k}$ is calculated for each cause of elimination. insured.

$$Z = C_{J,K+1} v^{K+1}$$

whereby

$J : \Omega \longrightarrow \{1, \ldots, m\}$ the *cause of elimination* labelled and $K : \Omega \longrightarrow \mathbb{N}$ the *separation time*.

The necessary single premium is calculated as the expected value

$$E[Z] = \sum_{j=1}^{m} \sum_{k=0}^{\infty} v^{k+1} C_{j,k+1\,k} p_x q_{j,x+k}$$

In the case of an insurance policy payable on the exact date of death

$$Z = C_J(T) v^T$$

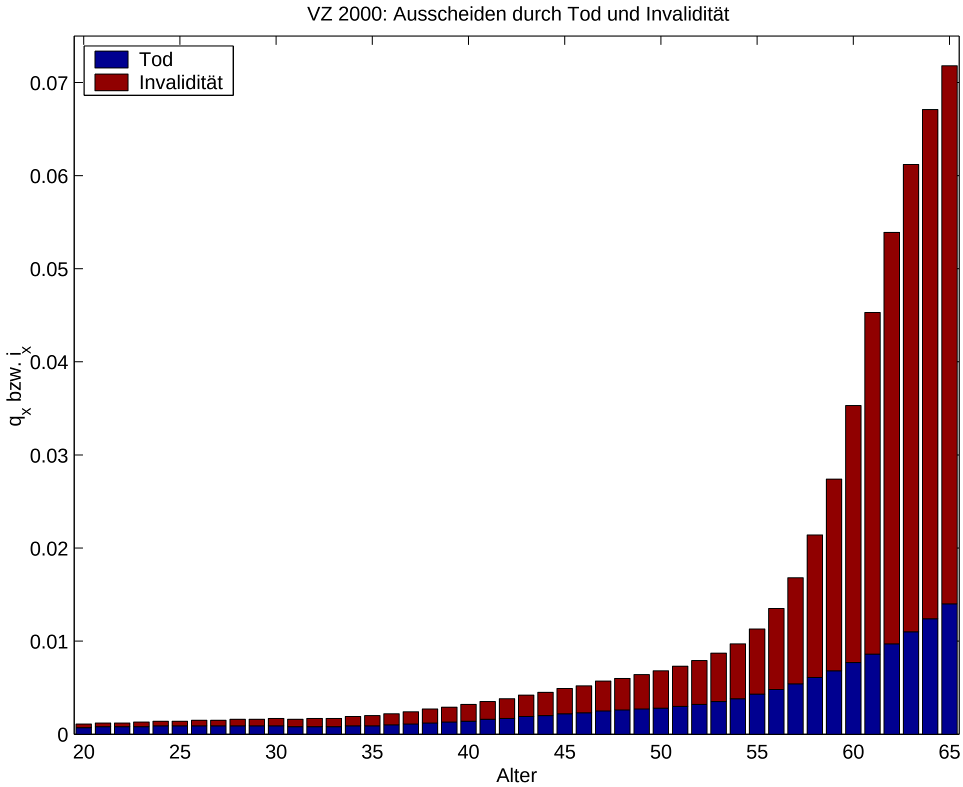

Figure 8.2: Probabilities of elimination

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The necessary single premium is now calculated as follows

$$\begin{aligned} E[Z] &= \sum_{j=1}^{m} \int_0^{\infty} C_j(t) v^t g_j(t) dt \\ &= \sum_{j=1}^{m} \int_0^{\infty} C_j(t) v^t {}_t p_x \mu_{j,x+t} dt \end{aligned}$$

Once the discrete and continuous models have been defined, you can in turn try to merge the one into the other.

$$\begin{aligned} E[Z]^{stetig} &= \sum_{j=1}^{m} \int_0^{\infty} C_j(t) v^t {}_t p_x \, \mu_{j,x+t} \, dt \\ &= \sum_{j=1}^{m} \int_0^{\infty} C_j(t) v^t g_{j,x+t} \, dt \\ &= \sum_{j=1}^{m} \sum_{k=0}^{\infty} v^k {}_k p_x \int_0^1 {}_u p_{x+k} \, \mu_{j,x+k+u} \, v^u C_j(k+u) du \end{aligned}$$

Now $\mu_{j,x+k+u} = \frac{q_{j,x+k}}{1-uq_{x+k}}$ and ${}_u p_{x+k} = 1 - uq_{x+k}$

So equation (8.2) is equal to

$$\sum_{j=1}^{m} \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{j,x+k} \underbrace{\int_0^1 v^{u-1} C_j(k+u)}_{=:C_{j,k+1}} \, du$$

for concretecases results in $C_{j,k+1} \sim C_j(k+\frac{1}{2})(1+i)^{\frac{1}{2}}$ is often a good approximation.

8.5 Capital Insurance and Recursion

We assume that the insurance policies defined in advance with premiumn $\Pi_0, \Pi_1, \ldots$ have been financed. Then the Actuarial reserve defined by

$${}_k V_x = \sum_{j=1}^{m} \sum_{h=0}^{\infty} C_{j,k+h+1} v^{h+1} {}_h p_{x+k} q_{j,x+k+h} - \sum_{h=0}^{\infty} \Pi_{k+h} v^h {}_h p_{x+k}$$

From this formula you can simply derive the following recursion

$${}_k V_x + \Pi_k = \underbrace{{}_{k+1} V_x \, v p_{x+k}}_{\text{reserve for future}} + \underbrace{\sum_{j=1}^{m} C_{j,k+1} v q_{j,x+k}}_{\text{costs for the individual coverages}}$$

Once the recursion for the actuarial reserve has been derived, the the technical decomposition can now be carried out.

$${}_k V_x + \Pi_k = v \cdot {}_{k+1} V_x + \sum_{j=1}^{m} (C_{j,k+1} - {}_{k+1} V_x) v q_{j,x+k}$$

We define the *savings premium* as

$$\Pi_k^s = v \cdot {}_{k+1} V_x - {}_k V_x$$

and the *risk premium* for segregation order *j* as

$$\Pi_k^{j,r} = (C_{j,k+1} - {}_{k+1} V_x) v \cdot q_{j,x+k}$$

The *total risk premium* is defined by

$$\Pi_k^r = \sum_j \Pi_k^{j,r}$$

In consequence we get

$$L = \sum_{j \in J} I_{\{J=j\}} C_{j,K+1} \, v^{K+1} - \sum_{k=0}^{K} \Pi_k v^k$$

The *total loss* can in turn be allocated to the individual policy years. by considering the following equation:

$$L = \sum_{k=0}^{\infty} \Lambda_k v^k$$

whereby

$$\Lambda_k = \begin{cases} 0, & \text{falls } K \leq k-1 \\ -\Pi_k^r + (C_{J,k+1} - {}_{k+1}V_x)v, & \text{falls } K = k \\ -\Pi_k^r, & \text{falls } K \geq k+1 \end{cases}$$

The loss and technical analysis can also be applied to the apply different separation rules.

$$\Lambda_k = \sum_{j=1}^{m} \Lambda_{j,k}$$

whereby

$$\Lambda_{j,k} = \begin{cases} 0, & \text{falls } K \leq k-1 \\ -\Pi_k^{j,r} + (C_{j,k+1} - {}_{k+1}V_x)v, & \text{if } K = k \text{ and } J = j \\ -\Pi_k^{j,r}, & \text{falls } K \geq k+1 \\ & \text{or } K = k \text{ and } J \neq j \end{cases}$$

Finally, with this method it is also possible to result at the end of the year.

$$G_{k+1} = \begin{cases} ({}_kV_x + \Pi_k)(1+i') - C_{J,k+1}, & \text{falls } K = k \\ ({}_kV_x + \Pi_k)(1+i') - {}_{k+1}V_x, & \text{if } K \geq k+1 \end{cases}$$

where i' is the effectively realised interest rate.

Remark: The same considerations can also be applied to the continuous model. $\diamond$

9 Insurance on multiple Lives

We consider m insured persons with ageing $x_1, \dots, x_m$. The future lifetime of the kth person will be denoted by T_k. Starting from these m individuals we will define a state u and with $T(u)$ the future lifetime of the status u. Accordingly, ${}_t p_u$ denotes the Probability that u is still intact at time t. One can also consider annuities. For example, $\ddot{a}_u$ denotes the Net single premium for payments of 1 each made on the dates $0, 1, 2, \dots$ as long as the state u is intact A_u denotes the net single contribution for a Death insurance on leaving the state u.

9.1 The State of connected Lives

The condition

$$u = x_1 : x_2 : \dots : x_m$$

is defined as *intact* as long as all components are alive, and *erased* with the first death:

$$T(u) = \min\{T_i | i = 1, \dots, m\}$$

This state is called “joint life status”. In In the following, it is assumed that the random variables are *independent*.

$$\begin{aligned}
{}_t p_{x_1:\dots:x_m} &= P[T(u) \geq t] \\
&= P[T_1 \geq t, T_2 \geq t, \dots, T_m \geq t] \\
&= \prod_{k=1}^{m} P[T_k \geq t] = \prod_{k=1}^{m} {}_t p_{x_k}
\end{aligned}$$

The corresponding *mortality desity* then results analogously to (3.5) from:

$$\begin{aligned}
\mu_{u+t} &= -\frac{d}{dt} ln({}_t p_u) \\
&= -\frac{d}{dt} ln(\prod_{k=1}^{m} {}_t p_{x_k}) \\
&= -\frac{d}{dt} \sum_{k=1}^{m} ln({}_t p_{x_k}) \\
&= \sum_{k=1}^{m} \mu_{x_k+t}
\end{aligned} \tag{9.1}$$

The *single premium* for a first death insurance policy can be calculated as follows:

$$A_{x_1:\ldots:x_m} = \sum_{k=0}^{\infty} v^{k+1}{}_k p_{x_1:\ldots:x_m} q_{x_1+k:\ldots:x_m+k}$$

and the single premium is calculated for the so-called link annuities itself as

$$\ddot{a}_{x_1:\ldots:x_m} = \sum_{k=0}^{\infty} v^k{}_k p_{x_1:\ldots:x_m}$$

The identities in equation (4.17) also apply:

$$1 = d\ddot{a}_{x_1:\ldots:x_m} + A_{x_1:\ldots:x_m}$$

9.2 Simplifications

If the *Gompertz's law of death* is assumed for all lives, i.e.

$$\mu_{x_k+t} = Bc^{x_k+t}$$

this results in a considerable simplification. We define w like this, that

$$c^{x_1} + c^{x_2} + \ldots + c^{x_m} = c^w$$

Then applies

$$\mu_{u+t} = \mu_{w+t}$$

and therefore

$$\begin{aligned} A_{x_1:x_2:\ldots:x_m} &= A_w \\ \ddot{a}_{x_1:x_2:\ldots:x_m} &= \ddot{a}_w \end{aligned}$$

In the past, this simplification was important, but no longer so today.

9.3 State of the Last Survivor

$$u := \overline{x_1 : x_2 : \ldots : x_m}$$

is *intact* as long as at least one of the m persons lives and dies with the last death

$$T(u) = \max\{T_1, \ldots, T_m\}$$

To calculate the probability for $T(u)$ we remember the formula

$$P[B_1 \cup B_2 \cup \ldots \cup B_m] = S_1 - S_2 + S_3 - \ldots + (-1)^{m-1} S_m$$

whereby

$$S_k = \sum_{j_1 \neq j_2 \ldots} P[B_{j_1} \cap B_{j_2} \ldots \cap B_{j_k}]$$

where the summation over all $\binom{m}{k}$ possibilities extends.

If we consider $B_k = \{T_{x_k} > t\}$, we get

$$_t p_{\overline{x_1:x_2:\ldots:x_m}} = S_1^t - S_2^t + S_3^t - \ldots + (-1)^{m-1} S_m^t$$

whereby

$$S_k^t = \sum_{j_1 \neq j_2 \neq \ldots \neq j_m} {}_t p_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}}$$

If you now multiply the above equation by v^t and sum over all t then you get

$$\ddot{a}_{\overline{x_1:x_2:\ldots:x_m}} = S_1^{\ddot{a}} - S_2^{\ddot{a}} + S_3^{\ddot{a}} - \ldots + (-1)^{m-1} S_m^{\ddot{a}}$$

whereby

$$S_k^{\ddot{a}} = \sum_{j_1 \neq j_2 \ldots \neq j_k} \ddot{a}_{x_{j_1}:x_{j_2}:\ldots:x_{j_m}}$$

Finally, the connection

$$A_{\overline{x_1:x_2:\ldots:x_n}} = 1 - d\ddot{a}_{\overline{x_1:x_2:\ldots:x_n}}$$

example: $\ddot{a}_{\overline{x:y:z}} = S_1^{\ddot{a}} - S_2^{\ddot{a}} + S_3^{\ddot{a}}$ and

$$\begin{aligned} S_1^{\ddot{a}} &= \ddot{a}_x + \ddot{a}_y + \ddot{a}_z \\ S_2^{\ddot{a}} &= \ddot{a}_{xy} + \ddot{a}_{xz} + \ddot{a}_{yz} \\ S_3^{\ddot{a}} &= \ddot{a}_{xyz} \end{aligned}$$

9.4 Application in Practise

Now that we have more or less finalised the general forms of insurance have been dealt with in detail, we would now like to take a look at some special forms that are often used in practice.

Typical examples:

- Connection annuities: Payment as long as the couple lives
- Survivors' annuities: Payment after the death of the partner, e.g. widow's annuity
- Death benefit on the first and second life ◇

The aim is to derive the formulae so that formulae can be understood. can be used.

Definitions:

$l_{xy} =$ Number of persons of age (x/y).

This can be calculated via recursion from

$$l_{x+1,y+1} = (1-\hat{q}_x)(1-\hat{q}_y) l_{x,y}$$

From this, the raw *survival probabilities* can be calculated again. than

$$_t\hat{p}_{xy} = \frac{l_{x+t,y+t}}{l_{xy}} \; (= {}_t\hat{p}_x \cdot {}_t\hat{p}_y)$$

Analogue applies

$${}_t\hat{q}_{xy} = 1 - {}_t\hat{p}_{xy}$$

The probability that at least one person is still alive then according to the addition theorem

$${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy} \tag{9.2}$$

The probability that neither of the two people is still alive is complementary to (9.2):

$$\begin{aligned}
{}_tq_{\overline{xy}} &= (1 - {}_tp_x) \cdot (1 - {}_tp_y) \\
&= 1 - {}_tp_x + 1 - {}_tp_y - (1 - {}_tp_{x\,t}p_y) \\
&= 1 - {}_tp_x + 1 - {}_tp_y - (1 - {}_tp_{xy}) \\
&= {}_tq_x + {}_tq_y - {}_tq_{xy} && (9.3)
\end{aligned}$$

Commutation functions

There are various ways of forming communication figures:

$$\begin{aligned}
1. D_{xy} &= l_{xy} \cdot v^x && (9.4) \\
2. D_{xy} &= l_{xy} \cdot v^y && (9.5) \\
3. D_{xy} &= l_{xy} \cdot v^{\frac{x+y}{2}} && (9.6)
\end{aligned}$$

The formulas (9.4) and (9.5) are easier to calculate, the Formula (9.6) is therefore symmetrical.

$$\begin{aligned}
N_{xy} &= \sum_{t=0}^{\infty} D_{x+t,y+t} \\
S_{xy} &= \sum_{t=0}^{\infty} N_{x+t,y+t}
\end{aligned}$$

d_{xy} is the number of resolved pairs

$$d_{xy} = l_{xy} - l_{x+1,y+1}$$

Here, too, there are 3 options, such as

$$\begin{aligned}
C_{xy} &= d_{xy} v^{\frac{x+y}{2}+1} \\
M_{xy} &= \sum_{t=0}^{\infty} C_{x+t,y+t} \\
R_{xy} &= \sum_{t=0}^{\infty} M_{x+t,y+t}
\end{aligned}$$

9.5 Present Values for typical Insurance Policies on two Lives

9.5.1 Capital insurance in the event of survival

- The survival benefit is due if the couple survives n years. The single premium is then calculated as

$$A_{\overset{1}{xy:\overline{n}|}} = v^n {}_np_{xy} = \frac{D_{x+n,y+n}}{D_{x,y}}$$

Remark: Sometimes you can also find a different notation, ${}_nE_{xy}$ statt $A_{xy:\overline{n}|}^{\;\;\,1}$. ◇

- The endowment benefit is payable if at least 1 person is n years old. survived.

$$A_{\overline{xy}:\overline{n}|}^{\;\;\,1} = v^n(1 - {}_nq_{\overline{xy}}) = v^n\,{}_np_{\overline{xy}}$$

und ${}_np_{\overline{xy}} = {}_np_x + {}_np_y - {}_np_{xy}$. The following therefore applies

$$A_{\overline{xy}:\overline{n}|}^{\;\;\,1} = A_{x:\overline{n}|}^{\;\;\,1} + A_{y:\overline{n}|}^{\;\;\,1} - A_{xy:\overline{n}|}^{\;\;\,1} \tag{9.7}$$

Remark: Here, too, the notation ${}_nE_{\overline{xy}}$ is used from time to time instead of $A_{\overline{xy}:\overline{n}|}^{\;\;\,1}$). ◇

9.5.2 Connection annuities

- annuity starting immediately, until the couple is dissolved.

$$\begin{aligned}\ddot{a}_{xy} &= \sum_{k=0}^{\infty} {}_kp_{xy}v^k \\ &= 1 + \frac{D_{x+1,y+1}}{D_{xy}} + \frac{D_{x+2,y+2}}{D_{xy}} + \frac{D_{x+3,y+3}}{D_{xy}} + \ldots \\ &= \frac{N_{xy}}{D_{xy}}\end{aligned}$$

- Analogue you get

$$\begin{aligned}{}_{m|}\ddot{a}_{xy} &= \frac{N_{x+m,y+m}}{D_{xy}} \\ \ddot{a}_{xy:\overline{m}|} &= \frac{N_{x,y} - N_{x+m,y+m}}{D_{xy}}\end{aligned}$$

- Annuity commencing immediately, payable as long as at least one person is on the Life is.

$$\ddot{a}_{\overline{xy}} = \sum_{t=0}^{\infty} {}_tp_{\overline{xy}}v^t$$

Again ${}_np_{\overline{xy}} = {}_np_x + {}_np_x - {}_np_{xy}$, so

$$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$$

9.5.3 Survivors' annuities

- **One-sided: Widows and Widowers pension**
  In the event of the death of x, a payment is made to y. The Single premium is calculated as

$$\ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{xy}$$

and analogue for the temporary case

$$\ddot{a}_{x|y:\overline{n}|} = \ddot{a}_{y:\overline{n}|} - \ddot{a}_{xy:\overline{n}|}$$

- **Two-sided/symetrical widows pension**

  In the event of the death of x, a payment is made to y and vice versa.

$$\begin{aligned} \ddot{a}^{1}_{xy} &= \ddot{a}_{x|y} + \ddot{a}_{y|x} \\ &= \ddot{a}_x + \ddot{a}_y - 2\ddot{a}_{xy} \\ &= \ddot{a}_{\overline{xy}} - \ddot{a}_{xy} \end{aligned}$$

comments:

- Individual widow's and orphan's annuities fall under the unilateral survivor's annuities.
- The mortality rate has so far been determined independently of the condition are considered. But widow and widower mortality is higher.
- Widow's annuities have a second cause of cancellation! When If a widow remarries, the widow's marriage is cancelled. Widow's annuity. Remarriage is not exactly taken into account.
- The orphan's annuity often takes into account the mortality of the children. neglected, i.e. instead of

$$\ddot{a}_{x|z:\overline{g-z}|} = \ddot{a}_{z:\overline{g-z}|} - \ddot{a}_{xz:\overline{g-z}|}$$

  is used as the present value

$$\ddot{a}_{x|\overline{g-z}|} = \ddot{a}_{\overline{g-z}|} - \ddot{a}_{x:\overline{g-z}|}$$

  where g is the final age.
- The orphan's annuity is multiplied in the event of the death of the mother and Father. ◇

9.5.4 Widow's annuity, collective method

The entitlement is only dependent on the man, the age of the The woman remains unconsidered. Moreover, it is not even considered whether a husband is married or not. Only at the time of death is whether there is a wife or not. This wife acquires an entitlement to a widow's annuity.

To be able to calculate the present value of the annuity for this case, you need two more quantities:

h_x: Probability that a man is aged x is a married at the time of death. h_x is strongly dependent on the social structure of the of the insurance portfolio under review.

y_x: the avergae age of the surviving widow at the time a x year old husband dies. The mean age difference between the husband and women is dependent on age. It increases with age because of subsequent marriages in later years.

As can be seen from the following table, the probabilities of being married relatively quickly.

x	$h_{x+\frac{1}{2}}$ Collective tariff 80	$h_{x+\frac{1}{2}}$ Collective tariff 95	$y_{x+\frac{1}{2}}$ Collective tariff 80	$y_{x+\frac{1}{2}}$ Collective tariff 95
20	0.498	0.100	20	19
30	0.750	0.435	29	29
40	0.890	0.728	38	38
50	0.93	0.804	47	48
60	0.880	0.838	57	57
70	0.750	0.816	66	67
80	0.552	0.705	74	75
90	0.296	0.472	82	84

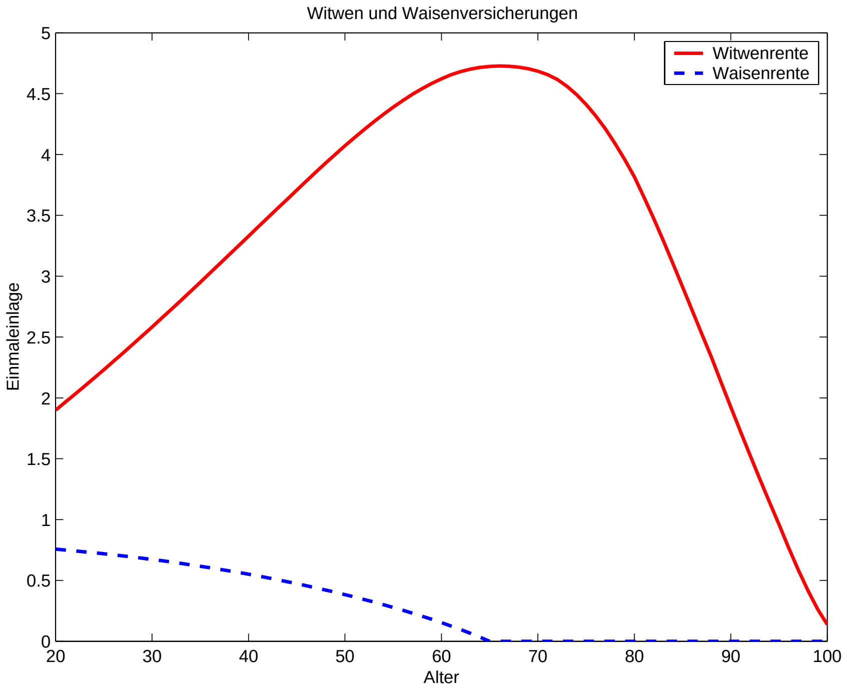


Figure 9.1: Single Premium for Survivors' Annuities

.

The single premium is calculated as

$$\begin{aligned} \ddot{a}_x^w &= \sum_{t=0}^{\omega-x} {}_t p_x q_{x+t} h_{x+t} v^{t+1} \ddot{a}_{y_{x+t+1}} \\ &= \sum_{t=0}^{\omega-x} \frac{v^{x+t+1}}{v^x} \frac{l_{x+t}}{l_x} h_{x+t} \frac{d_{x+t}}{l_{x+t}} \ddot{a}_{y_{x+t+1}} \\ &= \sum_{t=0}^{\omega-x} \frac{C_{x+t}}{D_{x+t}} h_{x+t} \ddot{a}_{y_{x+t+1}} \\ &=: \sum_{t=0}^{\omega-x} \frac{C_{x+t}^w}{D_{x+t}} \end{aligned}$$

where ω is the closing age.

9.5.5 Demographic bases according collective tariff 95

Remarriage of widows and widowers

Remarriage is recognised for both immediately commencing and prospective marriages. annuities due to reduction factors for annuities commencing immediately recorded. The factors depend on the age of the surviving spouse.

Widows

$$H_y^v = \begin{cases} \frac{y+40}{80}, & y < 40 \\ 1, & y \geq 40 \end{cases}$$

Widower

$$H_x^v = \begin{cases} \frac{x+40}{80}, & x < 40 \\ 1, & x \geq 40 \end{cases}$$

The reduction factor applies to all widows and widowers with or without Lump-sum settlement in the event of remarriage before the age of 45, However, the severance payment may not exceed three annual annuities.

Probability of being married at death

Probability h_u at death (i.e. between u and $u+1$) to be married. See figures 9.2 and 9.3.

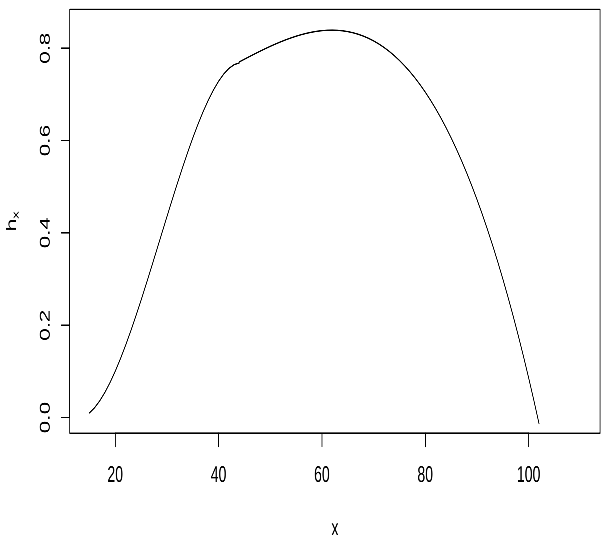


Figure 9.2: Probability h_u of being married at death at an age between u and $u+1$. Model from KT 95 (h_x).

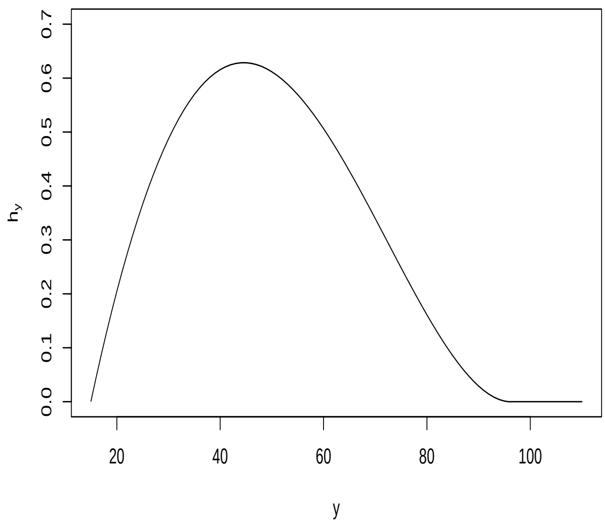


Figure 9.3: Probability h_u of being married at death at an age between u and $u+1$. Model from KT 95 (h_y).

Average age of thesurviving spouse at death of the insured person

Average age y_u or x_u of thesurviving spouse at the time of death. Death of the insured person (i.e. between the ages u and $u+1$): See figures 9.4 and 9.5.

9.5.6 Orphan's annuity, collective method

Here, too, two new invoice sizes are required:

k_x: The mean number of children who reach the final age g have not yet reached, on the death of a x-year-old man.

z_x: The average age of these children.

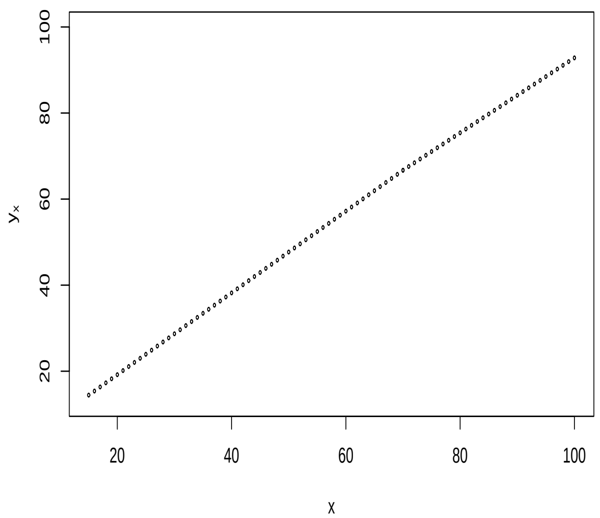


Figure 9.4: Average age y_u of the surviving spouse at the death of the insured person in the Age between u and $u+1$ (woman: y_x)

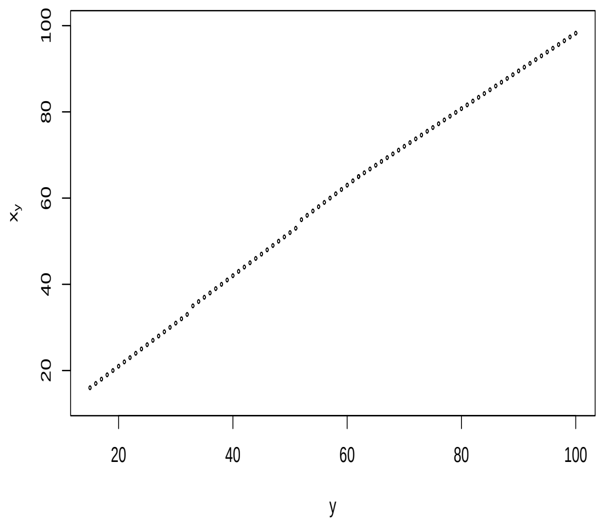


Figure 9.5: Average age x_u of the surviving spouse at the death of the insured person in the Age between u and $u+1$ (man: x_y)

Both of these reasons are in turn strongly dependent on the social structure of the of the insurance portfolio under review.

x	$k_{x+\frac{1}{2}}$ Collective tariff 80	$k_{x+\frac{1}{2}}$ Collective tariff 90	$z_{x+\frac{1}{2}}$ Collective tariff 80	$z_{x+\frac{1}{2}}$ Collective tariff 90
20	0.02	0.02	0.5	0.8
30	1.19	0.53	4.0	4.3
40	1.91	1.30	9.5	10.6
50	1.10	0.94	14.5	15.2
60	0.36	0.24	18.0	17.4

The single premium is calculated as

$$
\begin{aligned}
\ddot{a}^{k}_{x:g} &= \sum_{t=0}^{\omega-x} {}_t p_x q_{x+t} k_{x+t+\frac{1}{2}} v^{t+\frac{1}{2}} \ddot{a}_{\overline{g-z_{x+t-1/2}}|} \\
&= \sum_{t=0}^{\omega-x} \frac{v^{x+t+1}}{v^x} \frac{l_{x+t}}{l_x} k_{x+t+\frac{1}{2}} \frac{d_{x+t}}{l_{x+t}} v^{-\frac{1}{2}} \ddot{a}_{\overline{g-z_{x+t-1/2}}|} \\
&= \sum_{t=0}^{\omega-x} \frac{C_{x+t}}{D_x} k_{x+t+\frac{1}{2}} \ddot{a}_{\overline{g-z_{x+t-1/2}}|} v^{-\frac{1}{2}} \\
&=: \sum_{t=0}^{\omega-x} \frac{C^k_{x+t}}{D_x}
\end{aligned}
$$

where ω is the final age of the father, g is the final age of the children.

9.5.7 Death and mixed insurance

Such endowment insurance policies are generally symmetrical. This means that the payment of the sum insured in the Death (first or second death) is independent of the time of death.

However, it is also possible to take out unilateral death benefit insurance. define where the sum is paid out on the death of one person, if the other person is still alive. (e.g. widow's lump sum insurance).

Insurance on first death

A life insurance policy is taken out in the same way as a life insurance policy. Death benefit insurance on one life in equation (4.5) derived.

$$A_{xy} = \frac{M_{xy}}{D_{xy}}$$

or also

$$A_{xy} = 1 - d\ddot{a}_{xy}$$

A *temporary death benefit insurance* is required until the expiry of the insurance becomes due after n years, is calculated as

$$A^{1}_{xy:\overline{n}|} = \frac{M_{xy} - M_{x+n,y+n}}{D_{xy}}$$

A *mixed insurance* on the first life is calculated as

$$A_{xy:\overline{n}|} = A^{1}_{xy:\overline{n}|} + A_{xy:\overset{1}{\overline{n}|}} = 1 - d\ddot{a}_{xy:\overline{n}|}$$

Insurance on the second death

The probability that no one is still alive is according per above and for the single contribution of a *life insurance cover* results in

$$\begin{aligned} A_{\overline{xy}} &= \sum_t {}_tp_x q_{x+t} v^{t+1} + {}_tp_y q_{y+t} v^{t+1} \\ &\quad - {}_tp_{xy} q_{x+t,y+t} v^{t+1} \\ &= \sum_t \frac{C_{x+t}}{D_x} + \frac{C_{y+t}}{D_y} - \frac{C_{x+t,y+t}}{D_{xy}} \\ &= A_x + A_y - A_{xy} \end{aligned}$$

The relation also applies again

$$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}}$$

In the case of a mixed insurance on the second death, the following also applies in addition to the lump-sum death benefit Endowment benefit payable if at least one person is still on the is life. The single premium of the mixed insurance therefore amounts to second life

$$\begin{aligned} A_{\overline{xy}:\overline{n}|} &= A_{x:\overline{n}|} + A_{y:\overline{n}|} - A_{xy:\overline{n}|} \\ &= 1 - d(\ddot{a}_{x:\overline{n}|} + \ddot{a}_{y:\overline{n}|} - \ddot{a}_{xy:\overline{n}|}) \end{aligned}$$

and

$$A_{\overline{xy}:\overline{n}|} = 1 - d\ddot{a}_{\overline{xy}:\overline{n}|}$$

9.5.8 Ways to pay premium

There are different variants:

- The premiums are paid for as long as the couple lives. Then we can consider the premium payment as temporary connection annuity. This means that the relevant present value need to be be divided by $\ddot{a}_{xy:\overline{n}|}$.
- The premiums are only paid for as long as a particular insured person is insured. lives. This is particularly the case for survivors' and surviving dependants' annuities. Case. Then we divide by $\ddot{a}_{x:\overline{n}|}$ as before or $\ddot{a}_{y:\overline{n}|}$ respectively. Collectivisation is often taken so far in the case of orphans' annuities, that a constant rate of premium is used, regardless of age of the parents and children.

10 Disability Insurance

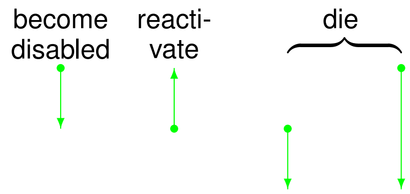
10.1 Introduction

We consider a model with *two decrements*:

- Death
- Disability

We have *three differnt states*:

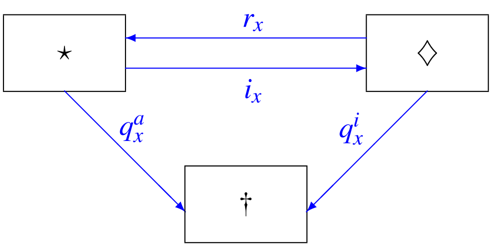
- Active/healthy
- Disabled
- Death



We have the following cohorts

Die Gesamtheit der Personen teilt sich auf in

- (Main) cohort of the actives l_x^a
- Cohort of the diabled l_x^i
- Cohort of the death d_x



i_x: Disability probability
r_x: Reactiveation probability
q_x^a: Mortality of active people
q_x^i: Mortality of disabled people

Figure 10.1: States and Transitions

10.2 Concept of Disability

To follow / covered in lectures orally.

- History
- Physical vs economic definition

10.3 Risks within Disability Insurance

To follow / covered in lectures orally.

- Cofactors of incidence rates (eg salary)
- Overinsurance
- Impcat of economic environment on disability
- Replacement ratio and impact of incidence rates

10.4 Traditional Disability Models

1. **General Model**

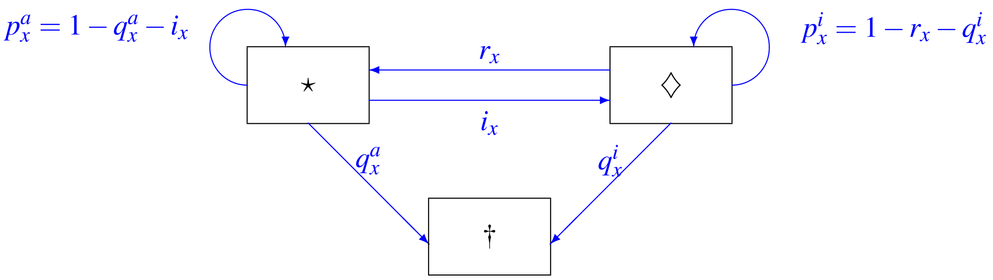


Figure 10.2: Standard model for Disability

$$l_x^a : \; l_{x+1}^a = l_x^a (1 - q_x^a - i_x) + l_x^i \cdot r_x$$

$$l_x^i : \; l_{x+1}^i = l_x^i (1 - q_x^i - r_x) + l_x^a \cdot i_x$$

$$d_x : d_{x+1} = l_x^a \cdot q_x^a + l_x^i \cdot q_x^i$$

2. **EVK 90**

Alter	q_x^a	q_x^i
20	0.00116	0.00200
40	0.00114	0.00200
60	0.00774	0.02670

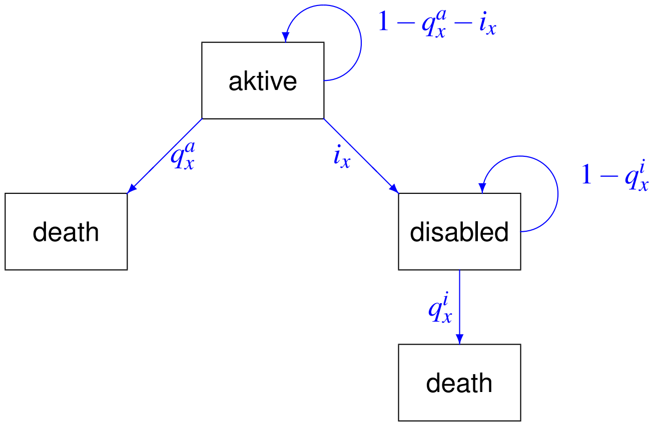


Figure 10.3: Typical Disability Model for pension insurance in Switzerland

These models typically ignore reactivation in the model.

$$\left.\begin{array}{l} q_{1,x} = q_x^a \\ q_{2,x} = i_x \end{array}\right\} \qquad \begin{array}{l} C_{1,x} = \text{Death Benefit} \\ C_{2,x} = \text{Disability Benefit} \end{array}$$

3. Modell KT 95

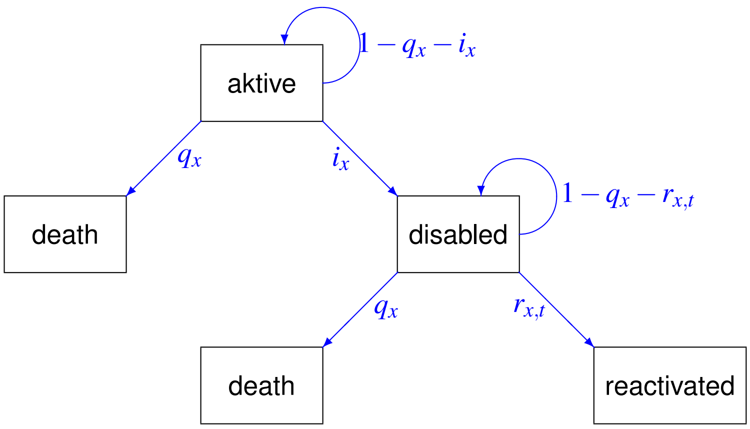
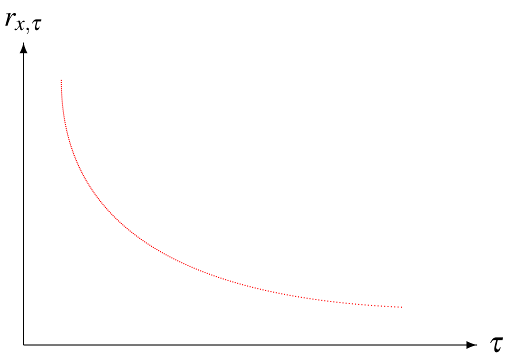


Figure 10.4: Disability Model of Swiss Insurers (zB KT 1995)

Reactivation probabilities depend on time being disabled.



Comparision of Mortlity and Disability rates for waiting period of 3 months in per mille:

	VPL 81/86		VZ 90	KT 95
	i_x	q_x	i_x	i_x
30	2.0	0.9	0.65	5.0
40	3.4	1.5	1.56	6.6
50	7.9	4.4	3.7	11.8
	Best Estimate			*Reserving*

Change in i_x over time (men in per mille, source VPL)

	1981/84	76/80	71/75	66/70	61/65	56/60
15-34	2.0	1.6	1.3	0.9	0.8	1.7
35-49	3.7	3.6	3.1	2.4	1.9	1.8
50-64	14.4	15.0	13.3	11.1	8.9	8.7
alle	5.8	5.8	4.8	4.0	3.4	3.7

Reactivation

Exponetial dependence of r_x on time t:

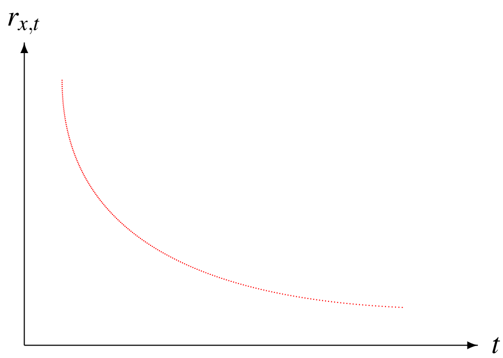


Illustration:

Reactivation probabilities in %

		Time disabled in years		
		0	2 e	8
15-34	q_x^i	1	3	1
	r_x	37	8	1
35-49	q_x^i	4	3	3
	r_x	27	7	0.5
50-64	q_x^i	4	4	4
	r_x	12	3	0.5
ϕ	q_x^i	4	4	3
	r_x	19	4	0.5

Model for $r_{x,t}$

Illustration: KT 95

$$\begin{array}{rcc}
\begin{array}{r} s_{x,t} = \\ s_{y,t} = \end{array} &
\underbrace{\begin{array}{c} q_{x+t} + 0.008 \\ q_{y+t} + 0.007 \end{array}}_{\substack{q_x^i \text{ resp. } q_y^i \\ \text{addl.} \\ \text{mortality} \\ \text{of 0.8 resp. 0.7\%}}} &
\begin{array}{c} + \\ + \end{array}
\underbrace{\begin{array}{c} e^{-0.94t}(c_1 - c_2 x) \\ e^{-1.25t}(c_3 - c_4 y) \end{array}}_{\substack{r_{x,t} \\ \\ \text{exponential} \\ \text{decay}}}
\end{array}$$

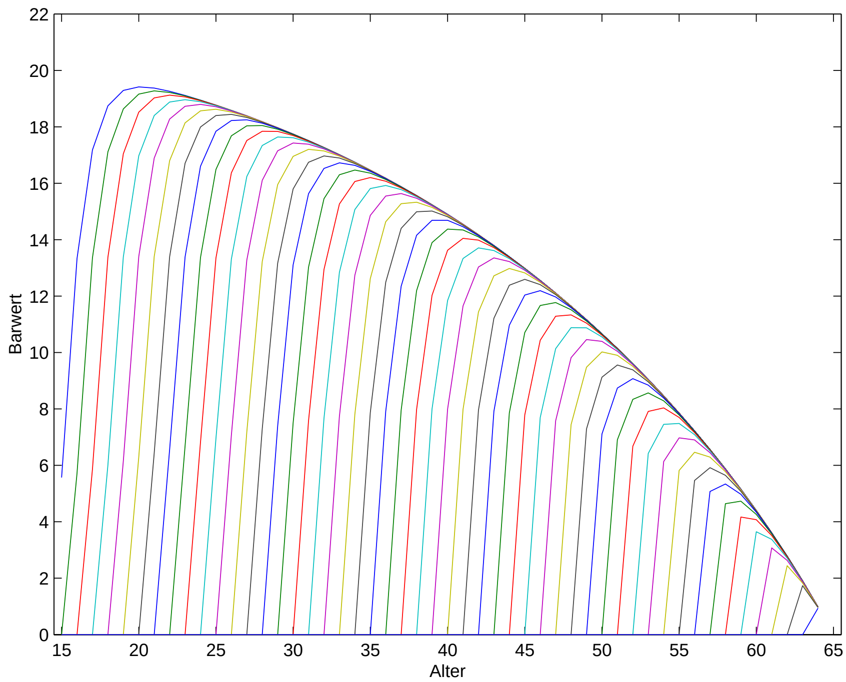


Figure 10.5: Present Value for an disability annuity in payment KT 1995.

10.4.1 Markovmodell for Disability

We consider the following model:

$$\begin{tikzcd}
\star \arrow[rr, shift right, "i_x"'] \arrow[dr, "q_x^a"'] & & \diamondsuit \arrow[ll, shift right, "r_x"'] \arrow[dl, "q_x^i"] \\
& \dagger &
\end{tikzcd}$$

Model including time dependency of $r(x,t)$:

($\diamondsuit_n$ is an pseudo absorbing state with respect to reactivation.)

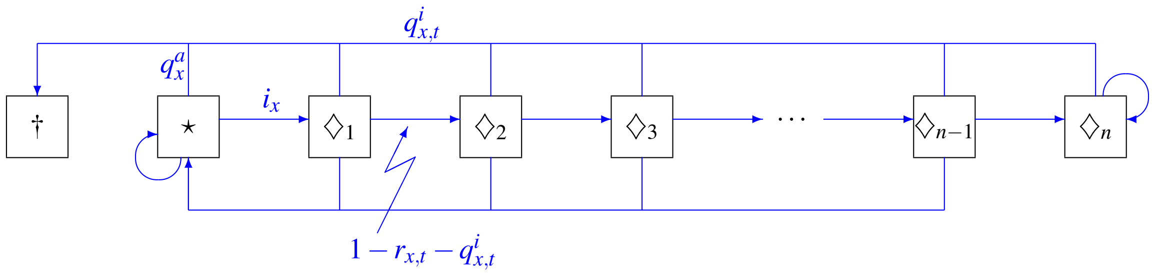


Figure 10.6: Markov modell for Disability Insurance

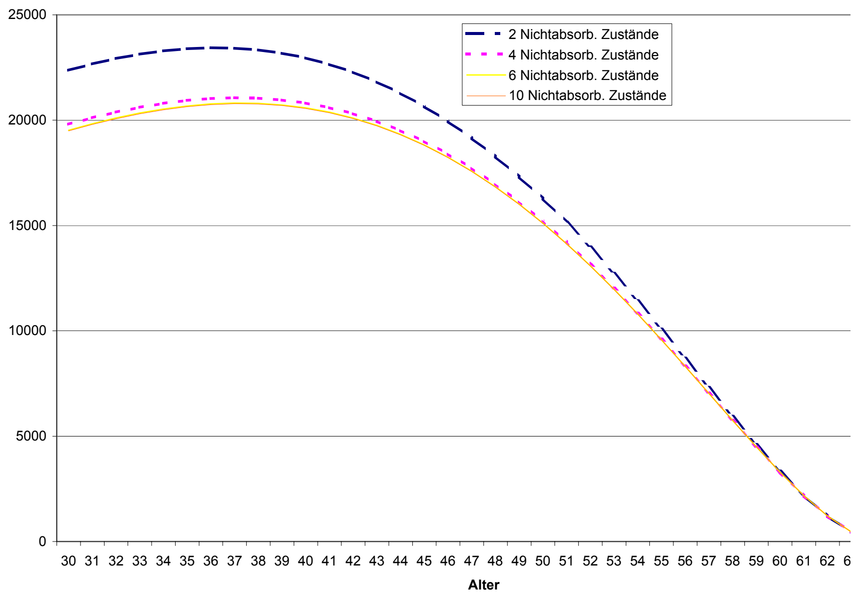


Figure 10.7: Present value of an disability annuity in payment.

A Schweizer Volkssterbetafeln

Sterbewahscheinlichkeit gemäss Schweizer Volkssterbetafeln 1988/93:

A.1 Männer

Alter	Ledig	Verheiratet	Verwitwet	Geschieden
1	2	3	4	5
18	0.001 128	0.006 055	-	-
19	0.001 414	0.002 360	-	-
20	0.001 530	0.001 410	0.014 641	0.011 985
21	0.001 575	0.001 138	0.035 403	0.010 034
22	0.001 621	0.001 123	0.053 526	0.008 393
23	0.001 684	0.001 217	0.036 550	0.007 221
24	0.001 764	0.001 298	0.031 296	0.006 315
25	0.001 861	0.001 256	0.026 818	0.005 606
26	0.001 974	0.001 125	0.022 998	0.005 048
27	0.002 102	0.000 977	0.020 349	0.004 607
28	0.002 245	0.000 861	0.018 237	0.004 262
29	0.002 399	0.000 798	0.016 521	0.003 995
30	0.002 563	0.000 777	0.015 095	0.003 793
31	0.002 736	0.000 785	0.013 875	0.003 646
32	0.002 916	0.000 814	0.012 802	0.003 548
33	0.003 102	0.000 855	0.011 827	0.003 493
34	0.003 293	0.000 901	0.010 918	0.003 477
35	0.003 490	0.000 953	0.010 060	0.003 496
36	0.003 695	0.001 010	0.009 243	0.003 546
37	0.003 909	0.001 073	0.008 460	0.003 627
38	0.004 137	0.001 145	0.007 705	0.003 736
39	0.004 379	0.001 225	0.006 947	0.003 873
40	0.004 640	0.001 316	0.006 267	0.004 037
41	0.004 923	0.001 420	0.005 593	0.004 227
42	0.005 231	0.001 539	0.005 009	0.004 448
43	0.005 570	0.001 676	0.004 558	0.004 703
44	0.005 945	0.001 834	0.004 267	0.005 000
45	0.006 359	0.002 017	0.004 159	0.005 345
46	0.006 814	0.002 227	0.004 259	0.005 747
47	0.007 315	0.002 468	0.004 558	0.006 211
48	0.007 864	0.002 742	0.005 053	0.006 743
49	0.008 466	0.003 055	0.005 753	0.007 350
50	0.009 125	0.003 410	0.006 667	0.008 037
51	0.009 844	0.003 811	0.007 796	0.008 811
52	0.010 629	0.004 263	0.009 120	0.009 677
53	0.011 485	0.004 770	0.010 579	0.010 642
54	0.012 415	0.005 336	0.012 087	0.011 711
55	0.013 427	0.005 966	0.013 603	0.012 890
56	0.014 529	0.006 668	0.015 110	0.014 184
57	0.015 729	0.007 447	0.016 598	0.015 599
58	0.017 036	0.008 311	0.018 069	0.017 138
59	0.018 460	0.009 266	0.019 529	0.018 803

Age	Célibataire	Marié	Veuf	Divorcé
1	2	3	4	5
60	0.020 012	0.010 321	0.020 998	0.020 594
61	0.021 705	0.011 483	0.022 506	0.022 508
62	0.023 552	0.012 760	0.024 091	0.024 538
63	0.025 567	0.014 158	0.025 805	0.026 675
64	0.027 765	0.015 688	0.027 704	0.028 910
65	0.030 159	0.017 360	0.029 810	0.031 252
66	0.032 759	0.019 190	0.032 140	0.033 720
67	0.035 574	0.021 193	0.034 710	0.036 340
68	0.038 614	0.023 390	0.037 533	0.039 141
69	0.041 885	0.025 801	0.040 627	0.042 164
70	0.045 392	0.028 454	0.044 004	0.045 453
71	0.049 137	0.031 377	0.047 677	0.049 068
72	0.053 121	0.034 606	0.051 656	0.053 077
73	0.057 337	0.038 182	0.055 948	0.057 568
74	0.061 788	0.042 150	0.060 563	0.062 632
75	0.066 513	0.046 550	0.065 535	0.068 314
76	0.071 573	0.051 423	0.070 915	0.074 645
77	0.077 042	0.056 815	0.076 758	0.081 643
78	0.083 016	0.062 770	0.083 136	0.089 318
79	0.089 609	0.069 337	0.090 129	0.097 658
80	0.096 962	0.076 566	0.097 834	0.106 602
81	0.105 249	0.084 508	0.106 369	0.116 122
82	0.114 685	0.093 214	0.115 873	0.126 131
83	0.125 538	0.102 737	0.126 511	0.136 507
84	0.138 059	0.113 122	0.138 443	0.147 136
85	0.152 175	0.124 391	0.151 677	0.158 087
86	0.167 632	0.136 549	0.166 132	0.169 516
87	0.183 997	0.149 582	0.181 609	0.181 474
88	0.200 642	0.163 451	0.197 969	0.194 544
89	0.216 727	0.178 080	0.214 824	0.208 894
90	0.231 206	0.193 394	0.231 725	0.224 938
91	0.243 232	0.209 086	0.248 263	0.243 194
92	0.253 389	0.224 057	0.264 600	0.264 315
93	0.262 862	0.236 751	0.281 162	0.289 130
94	0.273 068	0.245 392	0.298 513	0.318 708
95	0.285 658	0.248 199	0.317 368	0.354 442
96	0.302 611	0.246 575	0.338 616	0.398 178
97	0.326 447	0.253 864	0.363 367	0.452 388
98	0.360 629	0.309 682	0.393 034	0.520 445
99	0.410 258	0.377 884	0.429 450	0.607 003

A.2 Frauen

Alter	Ledig	Verheiratet	Verwitwet	Geschieden
1	2	3	4	5
17	0.000 328	0.000 721	-	-
18	0.000 388	0.000 561	-	-
19	0.000 441	0.000 478	-	-
20	0.000 453	0.000 424	0.020 666	0.002 840
21	0.000 451	0.000 382	0.017 142	0.002 252
22	0.000 479	0.000 353	0.014 314	0.001 790
23	0.000 531	0.000 333	0.011 996	0.001 426
24	0.000 603	0.000 322	0.008 481	0.001 138
25	0.000 686	0.000 317	0.006 747	0.001 070
26	0.000 767	0.000 317	0.005 588	0.001 125
27	0.000 841	0.000 323	0.004 632	0.001 258
28	0.000 906	0.000 333	0.003 840	0.001 418
29	0.000 963	0.000 347	0.003 307	0.001 548
30	0.001 015	0.000 366	0.002 901	0.001 630
31	0.001 063	0.000 389	0.002 590	0.001 666
32	0.001 110	0.000 416	0.002 351	0.001 664
33	0.001 160	0.000 446	0.002 170	0.001 635
34	0.001 215	0.000 481	0.002 037	0.001 591
35	0.001 276	0.000 519	0.001 946	0.001 542
36	0.001 345	0.000 562	0.001 893	0.001 500
37	0.001 423	0.000 609	0.001 875	0.001 473
38	0.001 510	0.000 663	0.001 890	0.001 471
39	0.001 606	0.000 722	0.001 939	0.001 502
40	0.001 712	0.000 788	0.002 023	0.001 577
41	0.001 829	0.000 862	0.002 144	0.001 690
42	0.001 958	0.000 944	0.002 297	0.001 836
43	0.002 102	0.001 036	0.002 475	0.002 005
44	0.002 261	0.001 140	0.002 669	0.002 185
45	0.002 437	0.001 255	0.002 866	0.002 376
46	0.002 629	0.001 383	0.003 051	0.002 577
47	0.002 838	0.001 524	0.003 223	0.002 789
48	0.003 061	0.001 680	0.003 386	0.003 015
49	0.003 298	0.001 849	0.003 542	0.003 254
50	0.003 548	0.002 034	0.003 699	0.003 510
51	0.003 809	0.002 233	0.003 863	0.003 785
52	0.004 079	0.002 446	0.004 041	0.004 081
53	0.004 353	0.002 672	0.004 242	0.004 402
54	0.004 630	0.002 910	0.004 477	0.004 752
55	0.004 912	0.003 162	0.004 749	0.005 135
56	0.005 202	0.003 430	0.005 061	0.005 553
57	0.005 507	0.003 716	0.005 416	0.006 011
58	0.005 832	0.004 025	0.005 818	0.006 513
59	0.006 186	0.004 362	0.006 271	0.007 063

Age	Célibataire	Marié	Veuf	Divorcé
1	2	3	4	5
60	0.006 579	0.004 731	0.006 779	0.007 665
61	0.007 022	0.005 141	0.007 346	0.008 326
62	0.007 531	0.005 601	0.007 978	0.009 051
63	0.008 123	0.006 121	0.008 677	0.009 847
64	0.008 820	0.006 716	0.009 451	0.010 722
65	0.009 636	0.007 396	0.010 308	0.011 687
66	0.010 585	0.008 173	0.011 261	0.012 755
67	0.011 685	0.009 061	0.012 326	0.013 942
68	0.012 952	0.010 075	0.013 518	0.015 267
69	0.014 407	0.011 233	0.014 860	0.016 754
70	0.016 072	0.012 554	0.016 375	0.018 430
71	0.017 968	0.014 062	0.018 093	0.020 327
72	0.020 119	0.015 781	0.020 051	0.022 485
73	0.022 546	0.017 740	0.022 289	0.024 952
74	0.025 275	0.019 971	0.024 858	0.027 783
75	0.028 344	0.022 513	0.027 809	0.031 036
76	0.031 803	0.025 413	0.031 198	0.034 772
77	0.035 710	0.028 723	0.035 092	0.039 062
78	0.040 131	0.032 506	0.039 565	0.043 990
79	0.045 148	0.036 835	0.044 702	0.049 647
80	0.050 852	0.041 794	0.050 599	0.056 135
81	0.057 358	0.047 482	0.057 368	0.063 581
82	0.064 796	0.054 016	0.065 130	0.072 117
83	0.073 325	0.061 534	0.074 026	0.081 892
84	0.083 107	0.070 165	0.084 191	0.093 079
85	0.094 215	0.079 881	0.095 685	0.105 899
86	0.106 648	0.090 513	0.108 498	0.120 614
87	0.120 340	0.101 736	0.122 548	0.137 536
88	0.135 130	0.113 035	0.137 660	0.157 030
89	0.150 746	0.124 050	0.153 545	0.179 533
90	0.166 786	0.134 099	0.169 787	0.205 560
91	0.182 852	0.142 395	0.185 995	0.235 727
92	0.199 087	0.148 065	0.202 420	0.270 769
93	0.215 927	0.150 285	0.219 670	0.311 564
94	0.233 995	0.152 549	0.238 596	0.359 168
95	0.254 134	0.154 857	0.260 339	0.414 850
96	0.277 454	0.157 210	0.286 422	0.480 140
97	0.305 427	0.159 611	0.318 917	0.556 893
98	0.340 042	0.162 060	0.360 711	0.647 356
99	0.384 045	0.164 560	0.415 969	0.754 265

B Notebooks

The link below allows to download the supporting jupyter notebooks as zip File: Please save the file as Notebooks.zip as opposed to Notebooks.abc

Bibliography

[Bowetal] Bowers, Gerber, Hickman, Jones, Nesbitt, Actuarial Mathematics, Society of Actuaries' Textbook, 1986.

[Braun] Braun, Geschichte der Lebensversicherung und der Lebensversicherungstechnik, Duncker & Humblot, Berlin, 1963.

[Dobson] Dobson, A. J., An Introduction to Generalized Linear Models, Statistics Texts from Chapman and Hall, Chapman &Hall London,1990

[FMW88] Forfar, D. O., McCutcheon, M. A. und Wilkie, A. D., On Graduation by Mathematical Formula. ,J. Inst. Act., 115 (1988), I, 1–149.

[Gerber] Gerber, H. U., Life Insurance Mathematics, Springer-Verlag, 3rd Edition, 1997.

[Hagelschuer] Hagelschuer, Lebensversicherung, Gabler, Wiesbaden, 1987.

[Jordan] Jordan, Life Contingencies, Society of Actuaries, Textbook, 1975.

[Koller] Koller, Stochastische Modelle in der Lebensversicherung, Springer-Verlag, 2000

[Nol59] Nolfi, Die BerÃ¼cksichtigung der Sterblichkeitsverbesserung in der Rentenversicherung nach der Optimalmethode der Spieltheorie. MVSVM 1959

[McCullagh] McCullagh, P., Nelder, J. A.,Generalized Linear Models,Chapmann& Hall London, 2nd edn,1999

[Panjer] Panjer H.H. et al.: With applications to investments, insurance and pensions. USA 1998 (The Actuarial Foundation).

[Ren92] Renshaw, A. E., Joint Modelling for Actuarial Graduation and Duplicate Policies. , J. Inst. Act., 119 (1992), I, 69–85.

[RHH96] Renshaw, A. E., Haberman, S. und Hatzopoulos, P. ,The Modelling of Recent Mortality Trends in United Kingdom Male Assured Lives., Br. Act. J., 2, II (1996), 449–447.

[Saxer] Saxer, Versicherungsmathematik 1. Teil, Springer, Berlin, 1955.

[Wolff] Wolff, Versicherungsmathematik, Springer, Wien, 1970.