Oxford University Mathematical Institute

Prelim Linear Algebra I, Michaelmas Term 2017

Exercise Sheet 2 (designed for a tutorial in Week 4):

More about matrices. Elementary row operations; echelon form of a matrix. The beginnings of vector space theory

1. Let J_n be the $n \times n$ matrix with all entries equal to 1. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ and $\alpha + n\beta \neq 0$. Show that the matrix $\alpha I_n + \beta J_n$ is invertible.

[Hint: note that $J_n^2 = nJ_n$; seek an inverse of $\alpha I_n + \beta J_n$ of the form $\lambda I_n + \mu J_n$ where $\lambda, \mu \in \mathbb{R}$.]

Find the inverse of $\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$.

2. Use EROs to reduce each of the following matrices to echelon form:

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix}; \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix}; \qquad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}.$$

3. For each $\alpha \in \mathbb{R}$, find an echelon form for the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 2 \\ 1 & 4 & -7 & 4 & 0 & \alpha \end{pmatrix}.$$

Use your result either to solve the following system of linear equations over $\mathbb{R}$, or to find values of α for which it has no solution:

$$\left\{ \begin{array}{rcl} x_1 + 2x_2 - 3x_3 - 2x_4 + 4x_5 & = & 1 \\ 2x_1 + 5x_2 - 8x_3 - x_4 + 6x_5 & = & 2 \\ x_1 + 4x_2 - 7x_3 + 4x_4 \phantom{{}+ 6x_5} & = & \alpha \end{array} \right. .$$

4. Use EROs to find the inverses of each of the following matrices

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

5. (a) Show that if the $m \times n$ matrices A, B can be reduced to the same matrix E in echelon form, then there is a sequence of EROs that changes A into B.

(b) Show that an $n \times n$ real matrix may be reduced to RRE form by a sequence of at most n^2 EROs.

6. (a) Prove from the vector space axioms (as presented in Definition 3.1 of the *Notes*) that if V is a vector space, $v, z \in V$ and $v + z = v$ then $z = 0_V$.

(b) Let $V := \mathbb{R} \times \mathbb{Z}$, the set of all pairs (x, k) where x is a real number and k is an integer. Define addition componentwise so that $(x, k) + (y, m) = (x + y, k + m)$, and define scalar multiplication by real numbers λ by the rule $\lambda(x, k) = (\lambda x, 0)$. Show that in this system conditions (VS1)–(VS7) are all satisfied but (VS8) fails.

Note: additional exercises and discussion points are to be found in the lecture notes posted on the course web-page

IIMN, Queen's: October 2017