

CHAPTER 6

Section 6.4

1. (a)

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{(1/n^2) + 1/n^3}{1 + 1/n^3} = \frac{0}{1} = 0$$

(b)

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{d(\ln n)/dn}{d(n)/dn} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

(c)

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$$

(d)

$$\begin{aligned} \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n)}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{d[\ln(1 + 1/n)]/dn}{d(n^{-1})/dn} \\ &= \lim_{n \rightarrow \infty} \frac{1/(n^2 + n)}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} \\ &= \frac{1}{1 + 0} = 1 \end{aligned}$$

(e)

$$\lim_{n \rightarrow \infty} s_n = 1 \text{ for } n = 1, 2, 3, \dots \implies \lim_{n \rightarrow \infty} s_n = 1$$

2. (a)

$$\overline{\lim}_{n \rightarrow \infty} \cos n\pi = 1 \qquad \underline{\lim}_{n \rightarrow \infty} \cos n\pi = -1 \qquad (1)$$

(b)

$$\overline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx 0.951 \qquad \underline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx -0.951$$

(c)

$$\overline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = \infty \qquad \underline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = -\infty$$

3. (a) A sequence

$$s_n = 1 + \cos n\pi$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 2 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = 0$$

(b) A sequence

$$s_n = -n^2 \sin^2 \left(\frac{1}{2} n \pi \right)$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 0 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = \underline{\lim}_{n \rightarrow \infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left| \left(1 + \frac{1}{n} \right)^n - e \right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

$$e \approx \left(1 + \frac{1}{164} \right)^{164} \approx 2.71$$

6.

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } |x| > 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = \pm 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1\end{aligned}$$

$$\begin{aligned}\underline{\lim}_{n \rightarrow \infty} x^n &= -\infty && \text{for } x < -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= -1 && \text{for } x = -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } x > 1\end{aligned}$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin \frac{\theta}{2} \times AE \cos \frac{\theta}{2} = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \rightarrow \infty$:

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n}$$

Now using the fact that $\lim_{x \rightarrow 0} \sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n} = \lim_{x \rightarrow 0} \pi \frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π , it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n \rightarrow \infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the n th term test $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$ diverges.

- (b) Since

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^3} = \lim_{n \rightarrow \infty} \frac{2^{n-1}}{3n} = \lim_{n \rightarrow \infty} \frac{(n-1)2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the n th term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3 - 1} = \left| \frac{1}{n^3 - 1} \right| < \frac{1}{n - 1}$$

for $n = 2, 3, \dots$. Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n - 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 - (1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3 - 1)$ is absolutely convergent.

- (b) Since $|\sin n| < 1$ for $n \geq 1$ it follows that

$$\left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}$$

for $n = 1, 2, \dots$. Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a) Since $n + 5 > n$ and $n^2 - 3n - 5 < n^2$ for $n \geq 1$ it follows that

$$\frac{n + 5}{n^2 - 3n - 5} > \frac{n}{n^2} = \frac{1}{n}$$

for $n = 1, 2, \dots$. Now since $\sum_{n=1}^{\infty} 1/n$ is the *harmonic series*, which diverges, it follows by the comparison test for divergence that $\sum_{n=1}^{\infty} (n + 5)/(n^2 - 3n - 5)$ diverges as well.

(b) Since $\sqrt{n} \ln n < n \ln n$ for $n \geq 2$ it follows that

$$\frac{1}{\sqrt{n} \ln n} > \frac{1}{n \ln n}$$

for $n = 2, 3, \dots$. Using the inequality $\ln(1+x) \leq x$ we may continue to write

$$\frac{1}{n \ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln \left(1 + \frac{\ln(1+1/n)}{\ln n} \right) \geq \ln \frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n} \ln n} > \ln \frac{\ln(1+n)}{\ln n} = \ln \ln(1+n) - \ln \ln n$$

Now let us consider the series

$$\sum_{n=2}^N \ln \ln(1+n) - \ln \ln n = \ln \ln(1+N) - \ln \ln 2$$

Hence, when $N \rightarrow \infty$

$$\sum_{n=2}^{\infty} \ln \ln(1+n) - \ln \ln n = \lim_{N \rightarrow \infty} \ln \ln(1+N) - \ln \ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that $\sum_{n=2}^{\infty} 1/(\sqrt{n} \ln n)$ diverges as well.

4. (a) Let $y = f(x) = 1/(x^2 + 1)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x \rightarrow \infty} f(x) = 0$. The improper integral $\int_c^{\infty} f(x) dx$ with $c = 1$ then evaluates to

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^2 + 1} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2 + 1} = \lim_{b \rightarrow \infty} \int_{\pi/4}^{\tan^{-1} b} du = \lim_{b \rightarrow \infty} u \Big|_{\pi/4}^{\tan^{-1} b} = \lim_{b \rightarrow \infty} \tan^{-1} b - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

where we have used the substitution $x = \tan u$. Hence, by the integral test, since the improper integral $\int_1^{\infty} f(x) dx$ converges so will the series $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2 + 1)$.

- (b) let $y = f(x) = 1/(x \ln^2 x)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x \rightarrow \infty} f(x) = 0$. The improper integral $\int_c^{\infty} f(x) dx$ with $c = 2$ then evaluates to

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln^2 x} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln^2 x} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b \rightarrow \infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \lim_{b \rightarrow \infty} \frac{1}{\ln b} \\ &= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2} \end{aligned}$$

where we have used the substitution $u = \ln x$. Hence, by the integral test, since the improper integral $\int_2^\infty f(x) dx$ converges so will the series $a_n = f(n) = \sum_{n=2}^\infty 1/(n \ln^2 n)$.

5. (a) Let $y = f(x) = x/(x^2 + 1)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $(f(x))$ decreases as x increases and $\lim_{x \rightarrow \infty} f(x) = 0$. The improper integral $\int_c^\infty f(x) dx$ with $c = 1$ then evaluates to

$$\begin{aligned} \int_1^\infty \frac{x}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2 + 1} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_2^{b^2+1} \frac{du}{u} = \lim_{b \rightarrow \infty} \left. \frac{\ln u}{2} \right|_2^{b^2+1} \\ &= \lim_{b \rightarrow \infty} \frac{\ln |b^2 + 1| - \ln 2}{2} \\ &= \infty - \frac{\ln 2}{2} = \infty \end{aligned}$$

where we have used the substitution $u = x^2 + 1$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x) dx$ diverges so will the series $a_n = f(n) = \sum_{n=1}^\infty n/(n^2 + 1)$.

- (b) Let $y = f(x) = 1/(x \ln x \ln \ln x)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x \rightarrow \infty} f(x) = 0$. The improper integral $\int_c^\infty f(x) dx$ with $c = 10$ then evaluates to

$$\begin{aligned} \int_{10}^\infty \frac{dx}{x \ln x \ln \ln x} &= \lim_{b \rightarrow \infty} \int_{10}^b \frac{dx}{x \ln x \ln \ln x} = \lim_{b \rightarrow \infty} \int_{\ln 10}^{\ln b} \frac{du}{u \ln u} \\ &= \lim_{b \rightarrow \infty} \int_{\ln \ln 10}^{\ln \ln b} \frac{dv}{v} \\ &= \lim_{b \rightarrow \infty} \left. \ln v \right|_{\ln \ln 10}^{\ln \ln b} \\ &= \lim_{b \rightarrow \infty} \ln \ln \ln b - \ln \ln \ln 10 \\ &= \infty - \ln \ln \ln 10 = \infty \end{aligned}$$

where we have used the substitutions $u = \ln x$ and $v = \ln u$. Hence, by the integral test, since the improper integral $\int_{10}^\infty f(x) dx$ diverges so will the series $a_n = f(n) = \sum_{n=10}^\infty 1/(n \ln n \ln \ln n)$.

6. (a) Let $a_n = (-1)^n/n!$. As such we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = \lim_{n \rightarrow \infty} \left| \frac{n!}{(-1)^n} \frac{(-1)^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| -\frac{1}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence, $L < 1$ and so according to the ratio test the series $\sum_{n=1}^\infty (-1)^n/n!$ is absolutely convergent.

(b) Let $a_n = 2^n + 1/(3^n + n)$. As such we find

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= L = \lim_{n \rightarrow \infty} \left| \frac{3^n + n}{2^n + 1} \frac{2^{n+1} + 1}{3^{n+1} + n + 1} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3^n (1 + n/3^n)}{2^n (1 + 1/2^n)} \frac{2^{n+1} (1 + 1/2^{n+1})}{3^{n+1} [1 + (n/3^{n+1}) + (1/3^{n+1})]} \right| \\ &= \frac{2}{3} \lim_{n \rightarrow \infty} \left| \frac{(1 + n/3^n) (1 + 1/2^{n+1})}{(1 + 1/2^n) (1 + n/3^{n+1} + 1/3^{n+1})} \right| \\ &= \frac{2}{3} \frac{(1 + 0) \cdot (1 + 0)}{(1 + 0) \cdot (1 + 0 + 0)} = \frac{2}{3}\end{aligned}$$

where we have used the fact that

$$\lim_{x \rightarrow \infty} \frac{x}{a^x} = \lim_{x \rightarrow \infty} \frac{1}{x a^{x-1}} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^{\infty} 2^n + 1/(3^n + n)$ is absolutely convergent.

7. (a) Let $a_n = 1/\ln n$. Then for $2 \leq n < \infty$ we find

$$a_n = \frac{(-1)^n}{\ln n} = \frac{1}{\ln n}$$

Now since $\ln n$ is monotonically increasing for $2 \leq n < \infty$ we may conclude that $a_n = 1/\ln n$ is monotonically decreasing for $2 \leq n < \infty$ and so $a_{n+1} \leq a_n$. Furthermore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0$$

provided $n \geq 2$ and so by the alternating series test we may conclude that the series $\sum_{n=2}^{\infty} (-1)^n / \ln n$ converges.

(b) Let $f(x) = \ln x/x$. Hence,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{\ln x}{x} = \frac{1 - \ln x}{x^2}$$

Note that the derivative of $f(x)$ becomes negative when $x > e \approx 2.71828$ and hence, that $f(x)$ becomes monotonically decreasing when $e < x < \infty$. As such, the terms of the sequence $a_n = f(n) = \ln n/n$ are decreasing (i.e. $a_{n+1} \leq a_n$) when $3 \leq n < \infty$. Furthermore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. As such, by the alternating series test we may conclude that the series $\sum_{n=3}^{\infty} (-1)^n \ln n/n$ converges.

8. (a) Let $a_n = 1/n^n$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R = \lim_{n \rightarrow \infty} \sqrt[n]{n^{-n}} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} 1/n^n$ is absolutely convergent.

- (b) Let $a_n = [n/(n+1)]^{n^2}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} [n/(n+1)]^{n^2}$ is absolutely convergent.

9. (a) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{2}{3} - \frac{1}{2} + \frac{3}{4} - \frac{2}{3} + \cdots + \frac{n+1}{n+2} - \frac{n}{n+1} = -\frac{1}{2} + \frac{n+1}{n+2}$$

Taking the limit of S_n as $n \rightarrow \infty$ then gives

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} - \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{1+1/n}{1+2/n} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence, the series converges.

- (b) Let the series

$$\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{n+1}{2^{n+1}} - \frac{n}{2^n} \right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{1}{2} - \frac{1}{2} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} - \frac{3}{8} + \cdots + \frac{n+1}{2^{n+1}} = -\frac{1}{2} + \frac{n+1}{2^{n+1}}$$

Taking the limit of S_n as $n \rightarrow \infty$ then gives

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n+1}{2^{n+1}} - \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)2^n} - \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$$

using *L'Hospital's rule*. Hence, the series converges.

10. Let $y = f(x)$ satisfy the following conditions:

- (a) $f(x)$ is defined and continuous for $c \leq x < \infty$
- (b) $f(x)$ decreases as x increases and $\lim_{x \rightarrow \infty} f(x) = 0$
- (c) $f(n) = a_n$

Let us suppose the improper integral $\int_c^\infty f(x) dx$ diverges. Assumptions (b) and (c) imply that $a_n > 0$ for n sufficiently large. Hence, by Theorem 7 of Section 6.5 the series $\sum a_n$ is either convergent or properly divergent. Let the integer m be chosen so that $m > c$. Then, since $f(x)$ is decreasing

$$\int_n^{n+1} f(x) dx \leq f(n) = a_n \quad \text{for } n \geq m$$

Hence, $a_m + \cdots + a_{m+p} \geq \int_m^{m+p+1} f(x) dx$. However, since $\int_c^\infty f(x) dx$ diverges it follows that $\lim_{p \rightarrow \infty} \int_m^{m+p+1} f(x) dx$ diverges, which thus ultimately implies that the series $\sum_m^\infty a_n$ must be divergent as well.

11.