

# CHAPTER 6

## Section 6.4

1. (a)

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{(1/n^2) + 1/n^3}{1 + 1/n^3} = \frac{0}{1} = 0$$

(b)

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{d(\ln n)/dn}{d(n)/dn} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

(c)

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$$

(d)

$$\begin{aligned} \lim_{n \rightarrow \infty} n \ln \left( 1 + \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n)}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{d[\ln(1 + 1/n)]/dn}{d(n^{-1})/dn} \\ &= \lim_{n \rightarrow \infty} \frac{1/(n^2 + n)}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} \\ &= \frac{1}{1 + 0} = 1 \end{aligned}$$

(e)

$$\lim_{n \rightarrow \infty} s_n = 1 \text{ for } n = 1, 2, 3, \dots \implies \lim_{n \rightarrow \infty} s_n = 1$$

2. (a)

$$\overline{\lim}_{n \rightarrow \infty} \cos n\pi = 1 \qquad \underline{\lim}_{n \rightarrow \infty} \cos n\pi = -1 \qquad (1)$$

(b)

$$\overline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx 0.951 \qquad \underline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx -0.951$$

(c)

$$\overline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = \infty \qquad \underline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = -\infty$$

3. (a) A sequence

$$s_n = 1 + \cos n\pi$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 2 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = 0$$

(b) A sequence

$$s_n = -n^2 \sin^2 \left( \frac{1}{2} n \pi \right)$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 0 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = \underline{\lim}_{n \rightarrow \infty} s_n = \infty$$

4. Let a sequence  $s_n = 1/n$  be given. Now this sequence converges, since

$$s = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Hence, for every  $\epsilon > 0$  an  $N$  can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all  $n > N$ . Hence, for all  $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define  $e$  to 2 decimal places from its definition

$$e = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n$$

we let  $\epsilon = 0.00828$  in order to find a value  $N$  such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left| \left( 1 + \frac{1}{n} \right)^n - e \right| < 0.00828$$

which is satisfied for  $n = 164$ . Hence,

$$e \approx \left( 1 + \frac{1}{164} \right)^{164} \approx 2.71$$

6.

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } |x| > 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = \pm 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1\end{aligned}$$

$$\begin{aligned}\underline{\lim}_{n \rightarrow \infty} x^n &= -\infty && \text{for } x < -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= -1 && \text{for } x = -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } x > 1\end{aligned}$$

7.



Assuming the figure above represents the unit circle, it follows that  $AE = BE = 1$  and that the area of the polygon  $AEB$  is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin \frac{\theta}{2} \times AE \cos \frac{\theta}{2} = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of  $n$  such polygons in the limit  $n \rightarrow \infty$ :

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n}$$

Now using the fact that  $\lim_{x \rightarrow 0} \sin(x)/x = 1$  and setting  $x = 2\pi/n$  we find

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n} = \lim_{x \rightarrow 0} \pi \frac{\sin x}{x} = \pi$$

Hence, since the sequence  $s_n$  is bounded and has limit  $\pi$ , it is monotone increasing.

## Section 6.7

1. (a) Since

$$\overline{\lim}_{n \rightarrow \infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the  $n$ th term test  $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$  diverges.

- (b) Since

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^3} = \lim_{n \rightarrow \infty} \frac{2^{n-1}}{3n} = \lim_{n \rightarrow \infty} \frac{(n-1)2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the  $n$ th term test  $\sum_{n=1}^{\infty} 2^n/n^3$  diverges.

2. (a) Since  $n^3 > n$  for  $n > 0$  it follows that

$$\frac{1}{n^3 - 1} = \left| \frac{1}{n^3 - 1} \right| < \frac{1}{n - 1}$$

for  $n = 2, 3, \dots$ . Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n - 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 - (1/n)} = 0$$

then  $\sum_{n=2}^{\infty} 1/(n-1)$  converges and hence, by the comparison test for convergence  $\sum_{n=2}^{\infty} 1/(n^3 - 1)$  is absolutely convergent.

- (b) Since  $|\sin n| < 1$  for  $n \geq 1$  it follows that

$$\left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}$$

for  $n = 1, 2, \dots$ . Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

then  $\sum_{n=1}^{\infty} 1/n^2$  converges and hence, by the comparison test for convergence  $\sum_{n=1}^{\infty} \sin(n)/n^2$  is absolutely convergent.

3. (a) Since  $n + 5 > n$  and  $n^2 - 3n - 5 < n^2$  for  $n \geq 1$  it follows that

$$\frac{n + 5}{n^2 - 3n - 5} > \frac{n}{n^2} = \frac{1}{n}$$

for  $n = 1, 2, \dots$ . Now since  $\sum_{n=1}^{\infty} 1/n$  is the *harmonic series*, which diverges, it follows by the comparison test for divergence that  $\sum_{n=1}^{\infty} (n + 5)/(n^2 - 3n - 5)$  diverges as well.

(b) Since  $\sqrt{n} \ln n < n \ln n$  for  $n \geq 2$  it follows that

$$\frac{1}{\sqrt{n} \ln n} > \frac{1}{n \ln n}$$

for  $n = 2, 3, \dots$ . Using the inequality  $\ln(1+x) \leq x$  we may continue to write

$$\frac{1}{n \ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln \left( 1 + \frac{\ln(1+1/n)}{\ln n} \right) \geq \ln \frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n} \ln n} > \ln \frac{\ln(1+n)}{\ln n} = \ln \ln(1+n) - \ln \ln n$$

Now let us consider the series

$$\sum_{n=2}^N \ln \ln(1+n) - \ln \ln n = \ln \ln(1+N) - \ln \ln 2$$

Hence, when  $N \rightarrow \infty$

$$\sum_{n=2}^{\infty} \ln \ln(1+n) - \ln \ln n = \lim_{N \rightarrow \infty} \ln \ln(1+N) - \ln \ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that  $\sum_{n=2}^{\infty} 1/(\sqrt{n} \ln n)$  diverges as well.

4. (a) Let  $y = f(x) = 1/(x^2 + 1)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^{\infty} f(x) dx$  with  $c = 1$  then evaluates to

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^2 + 1} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2 + 1} = \lim_{b \rightarrow \infty} \int_{\pi/4}^{\tan^{-1} b} du = \lim_{b \rightarrow \infty} u \Big|_{\pi/4}^{\tan^{-1} b} = \lim_{b \rightarrow \infty} \tan^{-1} b - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

where we have used the substitution  $x = \tan u$ . Hence, by the integral test, since the improper integral  $\int_1^{\infty} f(x) dx$  converges so will the series  $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2 + 1)$ .

- (b) let  $y = f(x) = 1/(x \ln^2 x)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^{\infty} f(x) dx$  with  $c = 2$  then evaluates to

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln^2 x} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln^2 x} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b \rightarrow \infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \lim_{b \rightarrow \infty} \frac{1}{\ln b} \\ &= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2} \end{aligned}$$

where we have used the substitution  $u = \ln x$ . Hence, by the integral test, since the improper integral  $\int_2^\infty f(x) dx$  converges so will the series  $a_n = f(n) = \sum_{n=2}^\infty 1/(n \ln^2 n)$ .

5. (a) Let  $y = f(x) = x/(x^2 + 1)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $(f(x))$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^\infty f(x) dx$  with  $c = 1$  then evaluates to

$$\begin{aligned} \int_1^\infty \frac{x}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2 + 1} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_2^{b^2+1} \frac{du}{u} = \lim_{b \rightarrow \infty} \left. \frac{\ln u}{2} \right|_2^{b^2+1} \\ &= \lim_{b \rightarrow \infty} \frac{\ln |b^2 + 1| - \ln 2}{2} \\ &= \infty - \frac{\ln 2}{2} = \infty \end{aligned}$$

where we have used the substitution  $u = x^2 + 1$ . Hence, by the integral test, since the improper integral  $\int_1^\infty f(x) dx$  diverges so will the series  $a_n = f(n) = \sum_{n=1}^\infty n/(n^2 + 1)$ .

- (b) Let  $y = f(x) = 1/(x \ln x \ln \ln x)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^\infty f(x) dx$  with  $c = 10$  then evaluates to

$$\begin{aligned} \int_{10}^\infty \frac{dx}{x \ln x \ln \ln x} &= \lim_{b \rightarrow \infty} \int_{10}^b \frac{dx}{x \ln x \ln \ln x} = \lim_{b \rightarrow \infty} \int_{\ln 10}^{\ln b} \frac{du}{u \ln u} = \lim_{b \rightarrow \infty} \int_{\ln \ln 10}^{\ln \ln b} \frac{dv}{v} \\ &= \lim_{b \rightarrow \infty} \ln v \Big|_{\ln \ln 10}^{\ln \ln b} \\ &= \lim_{b \rightarrow \infty} \ln \ln \ln b - \ln \ln \ln 10 \\ &= \infty - \ln \ln \ln 10 = \infty \end{aligned}$$

where we have used the substitutions  $u = \ln x$  and  $v = \ln u$ . Hence, by the integral test, since the improper integral  $\int_{10}^\infty f(x) dx$  diverges so will the series  $a_n = f(n) = \sum_{n=10}^\infty 1/(n \ln n \ln \ln n)$ .

6. (a)