CHAPTER 4

Section 4.1

1. (a) Using integration by parts twice, the integral can be written as

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C = 2x \sin x - \left(x^2 - 2\right) \cos x + C$$

(b) Making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\int \frac{x}{1+x^4}\, dx = \frac{1}{2} \int \frac{2x}{1+\left(x^2\right)^2}\, dx = \frac{1}{2} \int \frac{1}{1+u^2}\, du = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} x^2 + C$$

(c) Using partial fraction expansion, we can write

$$\int \frac{1}{(x-1)(x-2)}\, dx = \int \left(-\frac{1}{x-1} + \frac{1}{x-2}\right) dx = -\int \frac{dx}{x-1} + \int \frac{dx}{x-2}$$
$$= -\ln(x-1) + \ln(x-2) + C = \ln \frac{x-2}{x-1} + C$$

(d) Making the substitution $u = \sqrt{x-1}$ so that $2udu = dx$, the integral can be written as

$$\int \frac{1}{1+\sqrt{x-1}}\, dx = 2 \int \frac{u}{1+u}\, du = 2 \int \frac{-1+1+u}{1+u}\, du = 2 \int \left(-\frac{1}{1+u} + 1\right) du$$
$$= -2 \int \frac{du}{1+u} + 2 \int du = -2 \ln(1+u) + 2u + C$$
$$= 2\left[\sqrt{x-1} - \ln\left(1+\sqrt{x-1}\right)\right] + C$$

2. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_0^1 \cos^2\theta \, d\theta = \frac{1}{2} \int_0^1 (1 + \cos 2\theta)\, d\theta = \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\sin 2\theta}{4}\right|_0^{\pi/2}$$
$$= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\cos\theta \sin\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{4}$$

(b) Using the identity $\sin mx \sin nx = (1/2)\cos[(m-n)x] - (1/2)\cos[(m+n)x]$, the integral can be written as

$$\int_0^\pi \sin 2x \sin 3x \, dx = \frac{1}{2} \int_0^\pi (\cos x - \cos 5x)\, dx = \frac{1}{2} \int_0^\pi \cos x \, dx - \frac{1}{2} \int_0^\pi \cos 5x \, dx$$
$$= \left.\frac{\sin x}{2}\right|_0^\pi - \left.\frac{\sin 5x}{10}\right|_0^\pi = 0$$

(c) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int_0^1 \left(2x^2-3x+1\right)e^x\,dx &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - \int_0^1 (4x-3)\,e^x\,dx \\ &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - (4x-3)\,e^x\Big|_0^1 + \int_0^1 4e^x\,dx \\ &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - (4x-3)\,e^x\Big|_0^1 + 4e^x\Big|_0^1 = 3e-8\end{aligned}$$

(d) Using integration by parts, the fact that $(d/dx)\tan^{-1}x = 1/(1+x^2)$ and making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \\ &= x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{du}{1+u} = x\tan^{-1}x\Big|_0^1 - \frac{\ln(1+u)}{2}\Bigg|_0^1 \\ &= x\tan^{-1}x\Big|_0^1 - \frac{\ln(1+x^2)}{2}\Bigg|_0^1 = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\end{aligned}$$

3. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b\to 0^+}\int_b^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \lim_{b\to 0^+}\int_b^{\pi/2} d\theta \\ &= \lim_{b\to 0^+}\theta\Big|_b^{\pi/2} = \lim_{b\to 0^+}\left(\frac{\pi}{2}-b\right) = \frac{\pi}{2}\end{aligned}$$

(b) Making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to-\infty} -\int_0^b e^u\,du = \lim_{b\to-\infty} -e^u\Big|_0^b = \lim_{b\to-\infty}\left(-e^b+1\right) = 1$$

(c) Using integration by parts, the integral can be written as

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{b\to 0^+}\int_b^1 \ln x\,dx = \lim_{b\to 0^+} x\ln x\Big|_b^1 - \lim_{b\to 0^+}\int_b^1 dx = \lim_{b\to 0^+}\left(x\ln x - x\right)\Big|_b^1 \\ &= \lim_{b\to 0^+}\left(-1-b\ln b+b\right) = -1 - \lim_{b\to 0^+} b\ln b = -1\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} \overset{LH}{=} \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = \lim_{b\to 0^+} -b = 0$$

using L'Hopital's rule.

(d) Making the substitutions $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, $2u = \theta$ so that $2du = d\theta$, $v = \cos u$ so that $dv = -\sin u du$ and $w = \sin u$ so that $dw = \cos u du$ and the identities $1 + \tan^2\theta = \sec^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the integral can be written as

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{1+x^2}} = \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta \\
&= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{d\theta}{\sin\theta} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{2du}{\sin 2u} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{du}{\sin u\cos u} \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin^2 u + \cos^2 u}{\sin u\cos u}\,du \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin u}{\cos u}\,du + \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\cos u}{\sin u}\,du \\
&= \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2+\sqrt{2}}/2}^b -\frac{dv}{v} + \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2-\sqrt{2}}/2}^b \frac{dw}{w} \\
&= \lim_{b\to\sqrt{2}/2} -\ln v\Big|_{\sqrt{2+\sqrt{2}}/2}^b + \lim_{b\to\sqrt{2}/2} \ln w\Big|_{\sqrt{2-\sqrt{2}}/2}^b \\
&= \lim_{b\to\sqrt{2}/2} -\ln b + \ln\frac{\sqrt{2+\sqrt{2}}}{2} + \lim_{b\to\sqrt{2}/2} \ln b - \ln\frac{\sqrt{2-\sqrt{2}}}{2} \\
&= \ln\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)^2 = \ln\left(1+\sqrt{2}\right)
\end{aligned}$$

(e) Using integration by parts twice and making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx &= \lim_{b\to-\infty}\int_0^b -u^2 e^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty}\int_0^b 2ue^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty}\int_0^b 2e^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty} 2e^u\Big|_0^b \\
&= \lim_{b\to-\infty}\left(-b^2e^b + 2be^b - 2e^b + 2\right) = 2
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to-\infty} -b^2e^b = \lim_{b\to-\infty} -\frac{b^2}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty}\frac{2b}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty} -\frac{2}{e^{-b}} = 0$$

(f) Using integration by parts, the integral can be written as

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b + \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}$$

$$= \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b - \lim_{b\to\infty} \frac{1}{x}\Big|_1^b = \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to\infty} -\frac{\ln b}{b} \stackrel{LH}{=} \lim_{b\to\infty} -\frac{1/b}{1} = \lim_{b\to\infty} -\frac{1}{b} = 0$$

4. (a)

$$\int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{x^{1/3}} + \lim_{b\to 0^+} \int_b^1 \frac{dx}{x^{1/3}}$$

$$= \lim_{b\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^b + \lim_{b\to 0^+} \frac{3}{2}x^{2/3}\Big|_b^1$$

$$= \lim_{b\to 0^-} \frac{3}{2}\left(b^{2/3} - 1\right) + \lim_{b\to 0^+} \frac{3}{2}\left(1 - b^{2/3}\right) = 0$$

(b)

$$\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{x^3} + \lim_{b\to 0^+} \int_b^1 \frac{dx}{x^3}$$

$$= \lim_{b\to 0^-} -\frac{1}{4x^4}\Big|_{-1}^b + \lim_{b\to 0^+} \frac{1}{4x^4}\Big|_b^1$$

$$= \lim_{b\to 0^-} \frac{1}{4}\left(-b^{-4} + 1\right) + \lim_{b\to 0^+} \frac{1}{4}\left(1 - b^{-4}\right) = -\infty$$

Hence, the integral is divergent.

(c) Making the substitution $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, the integral can be written as

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\pi/2} \int_0^b \frac{\sec^2\theta}{1+\tan^2\theta}\, d\theta = \lim_{b\to\pi/2} \int_0^b d\theta = \lim_{b\to\pi/2} \theta\big|_0^b$$

$$= \lim_{b\to\pi/2} b = \frac{\pi}{2}$$

(d) Using a partial fraction expansion, the integral can be written as

$$\int_0^\infty \frac{x^2-x-1}{x\left(x^3+1\right)}\, dx = \lim_{b\to 0^+} \int_b^1 \frac{x^2-x-1}{x\left(x^3+1\right)}\, dx + \lim_{b\to\infty} \int_1^b \frac{x^2-x-1}{x\left(x^3+1\right)}\, dx$$

$$= \lim_{b\to 0^+} \int_b^1 \left[\frac{4x-2}{3\left(x^2-x+1\right)} - \frac{1}{3\left(x+1\right)} - \frac{1}{x}\right] dx + \ldots$$

$$= \lim_{b\to 0^+} \frac{1}{3}\int_b^1 \frac{4x-2}{x^2-x+1}\, dx - \lim_{b\to 0^+} \frac{1}{3}\int_b^1 \frac{dx}{x+1} - \lim_{b\to 0^+} \int_b^1 \frac{dx}{x} + \ldots$$

It is clear to see that the first two integrals (obtained by a partial fraction expansion) belonging to the first partial integral converge. However, the third diverges:

$$\lim_{b\to 0^+}\int_b^1 \frac{dx}{x} = \lim_{b\to 0^+} \ln x\Big|_b^1 = -\lim_{b\to 0^+}\ln b = \infty$$

Hence, since the first partial integral diverges, we conclude that the original integral is divergent.

(e)

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty} -\cos x\Big|_0^b = \lim_{b\to\infty}(-\cos b + 1)$$

Since $\lim_{b\to\infty}\cos b$ does not exist the integral is divergent.

(f) Making the substitution $u = \cosh x$ so that $du = \sinh x dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty (1-\tanh x)\,dx = \lim_{b\to\infty}\int_0^b (1-\tanh x)\,dx &= \lim_{b\to\infty}\int_0^b dx - \lim_{b\to\infty}\int_0^b \tanh x\,dx \\
&= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_0^b \frac{\sinh x}{\cosh x}\,dx \\
&= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_1^{\cosh b}\frac{du}{u} \\
&= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\ln u\Big|_1^{\cosh b} \\
&= \lim_{b\to\infty}(b - \ln\cosh b) = \ln 2
\end{aligned}$$

where the last step follows from the fact that

$$\begin{aligned}
\lim_{b\to\infty}(b-\ln\cosh b) = \lim_{b\to\infty}\left[\ln e^b - \ln\left(e^b + e^{-b}\right) + \ln 2\right] &= \lim_{b\to\infty}\left[\ln 2 - \ln\left(1+e^{-2b}\right)\right] \\
&= \ln 2
\end{aligned}$$

5. (a) The curves $y = 0$, $y = 1 - x^2$ intersect at the point $(-1,0)$, $(1,0)$. Hence, the area between the curves is given by

$$A = \int_{-1}^1 \left(1-x^2\right)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{4}{3}$$

(b) The curves $y = x^3$, $y = x^{1/3}$ intersect at the points $(-1,-1)$, $(0,0)$, $(1,1)$. Hence the area between the curves is given by

$$A = 2\int_0^1 \left(x^{1/3} - x^3\right)\,dx = \left[\frac{3x^{4/3}}{2} - \frac{x^4}{2}\right]_0^1 = 1$$

Note that we have used the fact that the intersection of the two curves is antisymmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1$ and multiply the result by two.

(c) The curves $y = 6\sin^{-1} x$, $y = \pi \sin \pi x$ intersect at the points $(-1/2, -\pi)$, $(0,0)$, $(1/2, \pi)$. Hence, the area between the curves is given by

$$\begin{aligned}
A &= 2\int_0^{1/2} \left(\pi \sin \pi x - 6\sin^{-1} x\right)\, dx \\
&= 2\pi \int_0^{1/2} \sin \pi x \, dx - 12\int_0^{1/2} \sin^{-1} x \, dx \\
&= -2\cos \pi x\Big|_0^{1/2} - 12x\sin^{-1} x\Big|_0^{1/2} + \int_0^{1/2} \frac{12x}{\sqrt{1-x^2}}\, dx \\
&= 2 - \pi - 6\int_1^{3/4} \frac{du}{\sqrt{u}} \\
&= 2 - \pi - 6\int_1^{3/4} u^{-1/2}\, du \\
&= 2 - \pi - 12\sqrt{u}\Big|_1^{3/4} = 14 - \pi - 6\sqrt{3}
\end{aligned}$$

where we have used integration by parts and the substitution $u = 1 - x^2$ so that $du = -2xdx$ in order to solve the second integral. Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1/2$ and multiply the result by two.

6. (a)

$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin x \, dx = -\frac{2\cos x}{\pi}\Bigg|_0^{\pi/2} = \frac{2}{\pi}$$

(b)

$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_{-\pi/2}^{0} \sin x \, dx = -\frac{2\cos x}{\pi}\Bigg|_{-\pi/2}^{0} = \frac{2\cos x}{\pi}\Bigg|_0^{-\pi/2} = -\frac{2}{\pi}$$

(c) Using the identities $\sin^2 x + \cos^2 x = 1$, $\cos 2x = 2\cos^2 x - 1$, we find

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 x \, dx &= \frac{2}{\pi}\int_0^{\pi/2} \left(1 - \cos^2 x\right)\, dx \\
&= \frac{1}{\pi}\int_0^{\pi/2} (1 - \cos 2x)\, dx \\
&= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}
\end{aligned}$$

(d)

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{x_2-x_1}\int_{x_1}^{x_2}(ax+b)\,dx = \frac{1}{x_2-x_1}\left[\frac{ax^2}{2}+bx\right]_{x_1}^{x_2}$$
$$= b+\frac{a}{2}(x_1+x_2)$$

7. Let $f(x)$ and $g(x)$ be continuous for $a \le x \le b$ and $|g(x)-f(x)| \le \epsilon$ for $a \le x \le b$. Defining $h(x)=g(x)-f(x)$ so that $|h(x)| \le \epsilon$ and using (4.6) we then find

$$\left|\int_a^b h(x)\,dx\right| = \left|\int_a^b [g(x)-f(x)]\,dx\right| = \left|\int_a^b g(x)\,dx - \int_a^b f(x)\,dx\right| \le \epsilon\,(b-a)$$

8. (a)

$$\int_0^1 \sin x^2\,dx \approxeq \int_0^1\left(x^2-\frac{x^6}{6}\right)dx = \left[\frac{x^3}{3}-\frac{x^7}{42}\right]_0^1 = \frac{13}{42} \approxeq 0.3095$$

The worst error is approximately 0.0081 at the point $x=1$.

(b)

$$\int_0^1 e^{-x^2}\,dx \approxeq \int_0^1\left(1-x^2+\frac{x^4}{2}\right)dx = \left[x-\frac{x^3}{3}+\frac{x^5}{10}\right]_0^1 = \frac{23}{30} \approxeq 0.7667$$

The worst error is approximately 0.1321 at the point $x=1$.

9. Let $f(x)$ be continuous for $0 \le x \le 1$. Then (4.20) may be used to approximate the integral of $f(x)$ numerically:

$$\int_0^1 f(x)\,dx \sim \frac{1}{2n}\left[f(0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(1)\right]$$

where $0 < x_1 < x_2 \cdots < x_{n-1} < 1$. If we then let $n \to \infty$ and choose $x_1 = 1/n$, $x_2 = 2/n, \ldots, x_{n-1} = (n-1)/n$, $x_n = n/n$ such that the endpoints converge to $x=0$ and $x=1$ respectively, while at the same time choosing an infinite number of equally spaced, but infinitely close interior points $x_1, x_2, \ldots, x_{n-1}$ the finite sum converges to:

$$\lim_{n\to\infty}\frac{1}{2n}\left[f(0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(1)\right] =$$
$$\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)+f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,dx$$

Note that the end points of the first and second limits differ by a factor of $1/2$. However, since $2\infty = \infty$ this difference is of no importance.

10. (a)

$$\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}+\frac{n}{n}\right) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

(b)

$$\begin{aligned}\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} &= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2+\left(\frac{n}{n}\right)^2\right]\\ &= \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}\end{aligned}$$

(c) Provided that $P\geq 0$

$$\begin{aligned}\lim_{n\to\infty}\frac{1^P+2^P+\cdots+n^P}{n^{P+1}} &= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^P+\left(\frac{2}{n}\right)^P+\cdots+\left(\frac{n-1}{n}\right)^P+\left(\frac{n}{n}\right)^P\right]\\ &= \int_0^1 x^P\,dx = \left.\frac{x^{P+1}}{P+1}\right|_0^1 = \frac{1}{P+1}\end{aligned}$$

(d) Taking the natural log of both sides of the equation gives $\ln(4/e)=\ln 4-1$ and

$$\ln\left\{\lim_{n\to\infty}\frac{1}{n}\left[(n+1)(n+2)\ldots(2n)\right]^{1/n}\right\}=\mathfrak{L}$$

Then, manipulating the left-hand side further, we find

$$\begin{aligned}\mathfrak{L} &= \lim_{n\to\infty}\ln\left\{\frac{1}{n}\left[(n+1)(n+2)\ldots(2n)\right]^{1/n}\right\} = \lim_{n\to\infty}\frac{1}{n}\ln\left[\frac{1}{n^n}(n+1)(n+2)\ldots(2n)\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n+1}{n}\frac{n+2}{n}+\cdots+\frac{2n}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{n+1}{n}+\ln\frac{n+2}{n}+\cdots+\ln\frac{n+n}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n-1}{n}\right)+\ln\left(1+\frac{n}{n}\right)\right]\\ &= \int_0^1\ln(1+x)\,dx = x\ln(1+x)\,|_0^1-\int_0^1\frac{x}{1+x}\,dx = \ln 2-\int_1^2\frac{u-1}{u}\,du\\ &= \ln 2-\int_1^2\left(1-\frac{1}{u}\right)du = \ln 2-[u-\ln u]_1^2 = \ln 4-1 = \ln\frac{4}{e}\end{aligned}$$

where we have used integration by parts to solve the first integral and the substitution $u=1+x$ so that $du=dx$ to solve the second integral.

11. Let $f(x)$ be a continuous function for $a\leq x\leq b$ and let it be a given fact that

$$\int_{a_1}^{b_1} f(x)\,dx = 0$$

for every interval $a_1 \le x \le b_1$ contained in the interval $a \le x \le b$. Next, let us choose a fixed point x_0 such that $a_1 \le x_0$, $x_0 + \delta \le b_1$, where $\delta > 0$. Then by (4.13) we find

$$\int_{x_0}^{x_0+\delta} f(x)\ dx = f(x^*)\,\delta = 0 \qquad \text{for } x_0 \le x^* \le x_0 + \delta$$

Now if we let $\delta \to 0$ we get

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{x_0}^{x_0+\delta} f(x)\ dx = f(x_0) = 0$$

As x_0 was chosen arbitrarily within $a_1 \le x \le b_1$, we conclude that $f(x) \equiv 0$.

12. Let $f(x)$ be a continuous function for $a \le x \le b$, $f(x) \ge 0$ on the interval and

$$\int_a^b f(x)\ dx = 0$$

Next, let c be such that $a < c < b$. Then the integrals $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ are either positive or zero. However, since their sum must be zero, the only option is that in fact they both are zero. The interval of each partial integral thus obtained can in turn be subdivided into smaller intervals over which to individually integrate $f(x)$ and since again $f(x)$ is either positive or zero on this new sub interval, but the total integral over $a \le x \le b$ must be zero, we conclude that each partial integral must be zero over the relevant sub interval. We can continue to apply this argument indefinitely for every smaller sub interval obtained from a larger sub interval and so we conclude that $\int_{a_1}^{b_1} f(x)\,dx = 0$ for every choice a_1, b_1 on the interval $a \le x \le b$. Hence, by Problem 11, $f(x) \equiv 0$.

Section 4.2

1. (a) Let $f(x) = x$ and $F(x) = \int_0^x x\,dx$. Then

x	x	$\int_x^{x+1} t\,dt$	$\int_0^x t\,dt = F(x)$
0	0	0.5	0.0
1	1	1.5	0.5
2	2	2.5	2.0
3	3	3.5	4.5
4	4	4.5	8.0
5	5	5.5	12.5
6	6	6.5	18.0
7	7	7.5	24.5
8	8	8.5	32.0
9	9	9.5	40.5
10	10		50.0

(b) Let $f(x) = e^{-x^2}$ and $F(x) = \int_0^1 e^{-x^2}\,dx$. Then

x	e^{-x^2}	$\int_x^{x+0.1} e^{-t^2}\,dt$	$\int_0^x e^{-t^2}\,dt = F(x)$
0	1.0	0.100	0.00
0.1	0.99	0.098	0.100
0.2	0.96	0.094	0.197
0.3	0.91	0.088	0.291
0.4	0.85	0.082	0.379
0.5	0.78	0.074	0.460
0.6	0.70	0.066	0.534
0.7	0.61	0.057	0.600
0.8	0.53	0.049	0.657
0.9	0.44	0.041	0.705
1.0	0.37		0.746

(c) Let $f(x) = \cos x$ and $F(x) = \int_0^1 \cos x\,dx$. Then

x	$\cos x$	$\int_x^{x+0.1} \cos t\,dt$	$\int_0^x \cos t\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.099	0.100
0.2	0.98	0.097	0.199
0.3	0.95	0.094	0.295
0.4	0.92	0.090	0.389
0.5	0.88	0.085	0.479
0.6	0.83	0.080	0.564
0.7	0.76	0.073	0.644
0.8	0.70	0.066	0.717
0.9	0.62	0.058	0.783
1.0	0.54		0.841

(d) Let $f(x) = 1/(1+x^3)$ and $F(x) = \int_0^1 dx/(1+x^3)$. Then

x	$1/(1+x^3)$	$\int_x^{x+0.1} dt/(1+t^3)$	$\int_0^x dt/(1+t^3) = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	0.99	0.098	0.200
0.3	0.97	0.096	0.298
0.4	0.94	0.091	0.393
0.5	0.89	0.086	0.485
0.6	0.82	0.078	0.570
0.7	0.75	0.070	0.649
0.8	0.66	0.062	0.719
0.9	0.58	0.054	0.781
1.0	0.50		0.835

(e) Let $f(x) = \sqrt{1-x^3}$ and $F(x) = \int_0^0 .5\sqrt{1-x^3}\,dx$. Then

x	$\sqrt{1-x^3}$	$\int_x^{x+0.1} \sqrt{1-t^3}\,dt$	$\int_0^x \sqrt{1-t^3}\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	1.00	0.099	0.200
0.3	0.99	0.098	0.299
0.4	0.97	0.095	0.397
0.5	0.04		0.492

2. Let $f(x)$ be continuous for $a \le x \le b$

(a) By (4.15) and (4.9) we find

$$\frac{d}{dx}\int_x^b f(t)\;dt = \frac{d}{dx}\left[-\int_b^x f(t)\;dt\right] = \frac{d}{dx}\left[-F(x)\right] = -\frac{dF}{dx} = -f(x)$$

(b) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\frac{d}{dx}\int_a^{x^2} f(t)\;dt = \frac{d}{dx}\int_a^u f(t)\;dt = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = 2xf(u) = 2xf\left(x^2\right)$$

(c) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^b f(t)\;dt = \frac{d}{dx}\left[-\int_b^{x^2} f(t)\;dt\right] = \frac{d}{dx}\left[-\int_b^u f(t)\;dt\right] &= \frac{d}{dx}\left[-F(u)\right] \\ &= -\frac{dF}{du}\frac{du}{dx} \\ &= -2xf(u) = -2xf\left(x^2\right)\end{aligned}$$

(d) Let us make the substitutions $u = x^2$, $v = x^3$ and note that $a \le u, v \le b$. Furthermore, let c be a fixed point $x^2 \le c \le x^3$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3} f(t)\;dt &= \frac{d}{dx}\left(\int_c^{x^3} f(t)\;dt + \int_{x^2}^c f(t)\;dt\right) \\ &= \frac{d}{dx}\int_c^v f(t)\;dt + \frac{d}{dx}\int_u^c f(t)\;dt \\ &= \frac{d}{dx}\int_c^v f(t)\;dt - \frac{d}{dx}\int_c^u f(t)\;dt \\ &= \frac{dF}{dv}\frac{dv}{dx} - \frac{dF}{du}\frac{du}{dx} = 3x^2 f\left(x^3\right) - 2xf\left(x^2\right)\end{aligned}$$

3. (a) By (4.20) we find approximately

$$\ln 1 = \int_1^1 \frac{dt}{t} \sim \frac{1-1}{4}[1+1] = 0$$

$$\ln 2 = \int_1^2 \frac{dt}{t} \sim \frac{1}{20}(1 + 1.818 + 1.667 + 1.538 + \cdots + 1.053 + 0.5) \approxeq 0.694$$

$$\ln 0.5 = \int_1^{1/2} \frac{dt}{t} = -\int_{1/2}^1 \frac{dt}{t} \sim -\frac{1}{40}(2 + 3.636 + 3.333 + \cdots + 2.105 + 1) \approxeq -0.694$$

(b) Using the definition

$$\ln x = \int_1^x \frac{dt}{t} \qquad x > 0$$

let $F(x) = \ln x$. Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x} \qquad x > 0$$

Hence, the first derivative of $\ln x$, $x > 0$ exists and as such, $\ln x$ is defined and continuous for $0 < x < \infty$.

(c) Let $F(x) = \ln ax - \ln x$ for $a > 0$ and $x > 0$. Next, let us make the substitution $u = ax$ so that $du = adx$. Then by (b) we find

$$\begin{aligned}\frac{dF}{dx} = \frac{d}{dx}(\ln ax - \ln x) = a\frac{d}{du}\ln u - \frac{d}{dx}\ln x &= a\frac{d}{du}\int_1^u \frac{dt}{t} - \frac{d}{dx}\int_1^x \frac{dt}{t} \\ &= \frac{a}{u} - \frac{1}{x} = \frac{a}{ax} - \frac{1}{x} = 0\end{aligned}$$

Hence, $F'(x) \equiv 0$ so that $F(x) \equiv \text{const} = \ln a$. And so

$$F(x) = \ln a = \ln ax - \ln x \iff \ln ax = \ln a + \ln x \quad \text{for } a, x > 0$$

4. Let an ellipse be given by the parametric equations: $x = a\cos\phi$, $y = b\sin\phi$, $b > a > 0$. Then by (3.53) the element of arc ds on the curve traced out by the ellipse is defined as $ds^2 = dx^2 + dy^2$. Hence, the length of arc from $\phi = 0$ to $\phi = \alpha$ is given by

$$\begin{aligned}s = \int_0^\alpha ds = \int_0^\alpha \sqrt{dx^2 + dy^2} &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\, d\phi \\ &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\left(1 - \sin^2\phi\right)}\, d\phi \\ &= \int_0^\alpha \sqrt{b^2 - (b^2 - a^2)\sin^2\phi}\, d\phi \\ &= \int_0^\alpha b\sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\phi}\, d\phi = b\int_0^\alpha \sqrt{1 - k^2\sin^2\phi}\, d\phi\end{aligned}$$

5. (a) Let $F(x)$ be as in (4.24). Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{1}{\sqrt{1-k^2\sin^2 x}}$$

Hence, the first derivative of $F(x)$, $0 < k^2 < 1$ exists for all x and as such, $F(x)$ is defined and continuous for all x.

(b) Let $x_2 > x_1$. Then since $F'(x) > 0$ for $0 < k^2 < 1$ it follows from the very definition of the derivative that $F(x_2) > F(x_1)$. Hence, we conclude that as x increases, $F(x)$ increases.

(c) Let $F(x)$ be as in (4.24). Then to show that $F(x+\pi) - F(x) = \text{const}$ we use (4.10) to find

$$\begin{aligned}
\frac{d}{dx}\left[F\left(x+\pi\right) - F\left(x\right)\right] &= \frac{d}{dx}\int_0^{x+\pi} \frac{dt}{\sqrt{1-k^2\sin^2 t}} - \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= \frac{1}{\sqrt{1-k^2\sin^2\left(x+\pi\right)}} - \frac{1}{\sqrt{1-k^2\sin^2 x}} \\
&= \frac{1}{\sqrt{1-k^2\left(-\sin x\right)^2}} - \frac{1}{\sqrt{1-k^2\sin^2 x}} = 0
\end{aligned}$$

Hence, since $F'(x+\pi) - F'(x) \equiv 0$, we conclude that the quantity $F(x+\pi) - F(x) = 2K$, where $K > 0$ is some positive constant. The fact that K must be positive and non-zero follows from (b).

(d) We know from (b) that as x increases, $F(x)$ increases. Furthermore, since $F(x) \geq 0$ for $0 < k^2 < 1$ it then follows that $\lim_{x\to\infty} F(x) = \infty$. Next, to show that $\lim_{x\to-\infty} F(x) = -\infty$ we write

$$\begin{aligned}
\lim_{x\to-\infty} F\left(x\right) = \lim_{x\to-\infty}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} &= \lim_{x\to-\infty}\int_x^0 -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= \lim_{x\to\infty}\int_0^x -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= -\lim_{x\to\infty} F\left(x\right) = -\infty
\end{aligned}$$

6. Let $x = am(y)$ be the inverse of the function $y = F(x)$ of (4.24).

(a) Let $y_2 > y_1$. Furthermore, from (a) and (b) of Problem 5 we know that $F'(x) > 0$ for $0 < k^2 < 1$ so that $F(x_2) > F(x_1)$ for $x_2 > x_1$. Since $y = F(x)$ this implies that $y_2 = F(x_2)$, $y_1 = F(x_1)$. Then noting that $x = am(y)$ is defined as the inverse of $y = F(x)$ we can write $am(y_2) = x_2$, $am(y_1) = x_1$. Now since $x_2 > x_1$ we conclude that $am(y_2) > am(y_1)$ for $y_2 > y_1$.

(b) From (c) of Problem 5 we know that $F(x+\pi) = F(x) + 2K = y + 2K$. Using the fact that $x = am(y)$ is defined as the inverse of $y = F(x)$ we then find

$$am\,(y + 2K) = am\,[F\,(x + \pi)] = x + \pi = am\,(y) + \pi$$

(c) From (a) of Problem 5 we know that $dF/dx = dy/dx = 1/\sqrt{1 - k^2 \sin^2 x}$. Since $(dy/dx)(dx/dy) \equiv 1$ we thus conclude that $am'(y) = dx/dy = \sqrt{1 - k^2 \sin^2 x}$. Hence, the first derivative of $am(y)$, $0 < k^2 < 1$ exists for all y and as such, $am(y)$ is defined and continuous for all y.

7. Let the functions $sn(y)$, $cn(y)$, $dn(y)$ be defined in terms of the function of Problem 6:

$$sn\,(y) = \sin\,[am\,(y)] \qquad cn\,(y) = \cos\,[am\,(y)] \qquad dn\,(y) = \sqrt{1 - k^2 \sin^2 y}$$

Then

(a)

$$sn^2\,(y) + cn^2\,(y) = \sin^2\,[am\,(y)] + \cos^2\,[am\,(y)] = \sin^2 x + \cos^2 x = 1$$

(b)

$$\begin{aligned}\frac{d}{dy} sn\,(y) = \frac{d}{dy} \sin\,[am\,(y)] = \cos\,[am\,(y)]\, \frac{d}{dy} am\,(y) &= cn\,(y)\, \frac{dx}{dy} \\ &= cn\,(y) \sqrt{1 - k^2 \sin^2 x} \\ &= cn\,(y)\, dn\,(y)\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dy} cn\,(y) = \frac{d}{dy} \cos\,[am\,(y)] = -\sin\,[am\,(y)]\, \frac{d}{dy} am\,(y) &= -sn\,(y)\, \frac{dx}{dy} \\ &= -sn\,(y) \sqrt{1 - k^2 \sin^2 x} \\ &= -sn\,(y)\, dn\,(y)\end{aligned}$$

(d)

$$sn\,(y + 4K) = \sin\,[am\,(y + 4K)] = \sin\,[am\,(y) + 2\pi] = \sin\,[am\,(y)] = sn\,(y)$$

(e)

$$cn\,(y + 4K) = \cos\,[am\,(y + 4K)] = \cos\,[am\,(y) + 2\pi] = \cos\,[am\,(y)] = cs\,(y)$$

(f)

$$\begin{aligned}
dn\,(y+2K) = \sqrt{1-k^2 sn^2\,(y+2K)} &= \sqrt{1-k^2\sin^2\left[am\,(y+2K)\right]} \\
&= \sqrt{1-k^2\sin^2\left[am\,(y)+\pi\right]} \\
&= \sqrt{1-k^2\left(-\sin\left[am\,(y)\right]\right)^2} \\
&= \sqrt{1-k^2\sin^2\left[am\,(y)\right]} \\
&= \sqrt{1-k^2 sn^2\,(y)} = dn\,(y)
\end{aligned}$$

8. Let the error function $y = erf(x)$ be defined by the equation

$$y = erf\,(x) = \int_0^x e^{-t^2}\,dt$$

(a) Using the definition of $y = erf(x)$ form above, then by (4.10) we find

$$\frac{dy}{dx} = \frac{d}{dx} erf\,(x) = \frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}$$

Hence, the first derivative of $y = erf(x)$ exists for all x and as such, $y = erf(x)$ is defined and continuous for all x.

(b)

$$erf\,(-x) = \int_0^{-x} e^{-t^2}\,dt = -\int_{-x}^0 e^{-t^2}\,dt = -\int_0^x e^{-t^2}\,dt = -erf\,(x)$$

(c) Let us first consider the case where $x \geq 0$. Then we find

$$\begin{aligned}
erf\,(x) = \int_0^x e^{-t^2}\,dt \leq \int_0^1 dt + \int_1^x te^{-t^2}\,dt &= 1 - \frac{1}{2}\int_1^x \frac{d}{dt}e^{-t^2}\,dt \\
&= 1 - \frac{1}{2}\frac{d}{dt}\int_1^x e^{-t^2}\,dt \\
&= 1 - \frac{e^{-x^2}}{2} + \frac{e^{-1}}{2} < 1
\end{aligned}$$

Note that we can move the differentiation operation outside of the integral, since the limits of integration are assumed constant relative to the integration variable t. From (b) it then follows immediately that $-1 < erf(x) < 1$.

Section 4.5

1. (a) If R is a triangle with vertices $(0,0)$, $(1,0)\,(1,1)$, so that $0 \le x \le 1$, $0 \le y \le x$, then

$$\iint_R \left(x^2+y^2\right)\,dx\,dy = \int_0^1 \int_0^x \left(x^2+y^2\right)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^x dx$$
$$= \int_0^1 \left(x^3 + \frac{x^3}{3}\right) dx$$
$$= \left[\frac{x^4}{4} + \frac{x^4}{12}\right]_0^1 = \frac{1}{3}$$

(b) Let R be the region: $u^2+v^2 \le 1$, $0 \le w \le 1$, so that $-1 \le x \le 1$, $-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$, $0 \le w \le 1$. Then

$$\iiint_R u^2 v^2 w\,du\,dv\,dw = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^1 u^2 v^2 w\,dw\,dv\,du$$
$$= 4\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^1 u^2 v^2 w\,dw\,dv\,du$$
$$= 2\int_0^1 \int_0^{\sqrt{1-u^2}} u^2 v^2 w^2\Big|_0^1\,dv\,du = 2\int_0^1 \int_0^{\sqrt{1-u^2}} u^2 v^2\,dv\,du$$
$$= \frac{2}{3}\int_0^1 u^2 v^3\Big|_0^{\sqrt{1-u^2}}\,du = \frac{2}{3}\int_0^1 u^2\left(1-u^2\right)^{3/2}\,du$$
$$= \frac{2}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta = \frac{1}{12}\int_0^{\pi/2} \sin^2 2\theta\,(1+2\cos 2\theta)\,d\theta$$
$$= \frac{1}{48}\int_0^{\pi/2} (2+\cos 2\theta - 2\cos 4\theta - \cos 6\theta)\,d\theta$$
$$= \frac{1}{48}\left[2\theta + \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{2} - \frac{\sin 6\theta}{6}\right]_0^{\pi/2} = \frac{\pi}{48}$$

where we have used the identities: $2\sin^2\theta = 1-\cos 2\theta$, $2\cos^2\theta = 1+\cos 2\theta$, $2\cos A\cos B = \cos(A+B)+\cos(A-B)$.

(c) If R is the region: $1 \le r \le 2$, $(\pi/4) \le \theta \le \pi$, then

$$\iint_R r^3\cos\theta\,dr\,d\theta = \int_1^2 \int_{\pi/4}^{\pi} r^3\cos\theta\,d\theta\,dr = \int_1^2 r^3\sin\theta\Big|_{\pi/4}^{\pi}\,dr = -\int_1^2 \frac{\sqrt{2}}{2}r^3\,dr$$
$$= -\frac{\sqrt{2}}{8}r^4\Big|_1^2 = -\frac{15\sqrt{2}}{8}$$

(d) Let R be a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$. To determine the x and y limits we can use the triangle in the xy-plane with vertices

$(0,0)$, $(1,0)$, $(0,2)$, since this is the projection of the tetrahedron onto the xy-plane. Hence, we find $0 \le x \le 1$, $0 \le y \le 2-2x$. To find the z limit we need to find the equation for the plane passing through the points $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ as this gives the top surface of the tetrahedron. To this end, we first form the two planar vectors $\mathbf{u} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{k}$. Next, to find the normal to the plane we compute $\mathbf{u} \times \mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Then by (1.23) the equation for the plane is given by $z = 3 - 3x - (3/2)y$. Hence,

$$\begin{aligned}
\iiint_R (x+z)\ dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-(3/2)y} (x+z)\ dz\,dy\,dx \\
&= \int_0^1 \int_0^{2-2x} \left[xz + \frac{z^2}{2} \right]_0^{3-3x-(3/2)y} dy\,dx \\
&= \frac{1}{8} \int_0^1 \int_0^{2-2x} \left(12x^2 + 24xy - 48x + 9y^2 - 36y + 36 \right)\ dy\,dx \\
&= \frac{1}{8} \int_0^1 \left[12x^2 y + 12xy^2 - 48xy + 3y^3 - 18y^2 + 36y \right]_0^{2-2x}\ dx \\
&= 3 \int_0^1 (-x+1)^2\ dx = 1
\end{aligned}$$

2. (a) Let $z = f(x,y) = e^x \cos y$ and $0 \le x \le 1$, $0 \le y \le \pi/2$. Then

$$\begin{aligned}
V = \iint_R f(x,y)\ dx\,dy = \int_0^1 \int_0^{\pi/2} e^x \cos y\,dy\,dx = \int_0^1 e^x \sin y \Big|_0^{\pi/2} dx = \int_0^1 e^x\ dx \\
= e^x \Big|_0^1 = e - 1
\end{aligned}$$

(b) Let $z = f(x,y) = x^2 e^{-x-y}$ and $0 \le x \le 1$, $0 \le y \le 2$. Next, let $u = e^{-x}$ so that $-u^{-1} du = dx$. Then

$$\begin{aligned}
V = \iint_R f(x,y)\ dx\,dy &= \int_0^1 \int_0^2 x^2 e^{-x-y}\,dy\,dx = -\int_0^1 x^2 e^{-x-y} \Big|_0^2 dx \\
&= \int_0^1 x^2 \left(e^{-x} - e^{-x-2} \right)\ dx = \left(e^{-2} - 1 \right) \int_1^{e^{-1}} \ln^2 u\,du \\
&= \left(e^{-2} - 1 \right) \left[u \ln^2 u \Big|_1^{e^{-1}} - 2 \int_1^{e^{-1}} \ln u\,du \right] \\
&= \left(e^{-2} - 1 \right) \left(e^{-1} - 2u \ln u \Big|_1^{e^{-1}} + 2 \int_1^{e^{-1}} du \right) \\
&= \left(e^{-2} - 1 \right) \left(3e^{-1} + 2u \Big|_1^{e^{-1}} \right) = \left(1 - e^{-2} \right) \left(2 - 5e^{-1} \right)
\end{aligned}$$

(c) Let $z = f(x,y) = x^2 y$ and $0 \le x \le 1$, $x+1 \le y \le x+2$. Then

$$V = \iint_R f(x,y)\;dx\,dy = \int_0^1 \int_{x+1}^{x+2} x^2 y\,dy\,dx = \frac{1}{2}\int_0^1 x^2 y^2\Big|_{x+1}^{x+2} dx$$
$$= \frac{1}{2}\int_0^1 \left(2x^3 + 3x^2\right)\,dx$$
$$= \frac{1}{4}\left[x^4 + 2x^3\right]_0^1 = \frac{3}{4}$$

(d) Let $z = f(x,y) = \sqrt{x^2 - y^2}$ and $x^2 - y^2 \ge 0$, $0 \le x \le 1$. Next, let $\sin u = y/x$ so that $\cos u\,du = dy/x$. Then

$$V = \iint_R f(x,y)\;dx\,dy = \int_0^1 \int_{-x}^{x} \sqrt{x^2-y^2}\,dy\,dx$$
$$= \int_0^1 \int_{-x}^{x} x^2\cos^2 u\,du\,dx = \frac{1}{2}\int_0^1 x^2 \int_{-x}^{x} (1+\cos 2u)\;du\,dx$$
$$= \frac{1}{2}\int_0^1 x^2 \left[u + \frac{\sin 2u}{2}\right]_{-x}^{x} dx$$
$$= \frac{1}{2}\int_0^1 x^2 \left[\sin^{-1}\frac{y}{x} + \frac{1}{2}\sin\left(2\sin^{-1}\frac{y}{x}\right)\right]_{-x}^{x} = \frac{\pi}{2}\int_0^1 x^2\,dx$$
$$= \frac{\pi}{6}x^2\Big|_0^1 = \frac{\pi}{6}$$

3. (a) Let $1 \le x \le 2$, $1 - x \le y \le 1 + x$. Then

$$\iint_R f(x,y)\;dA = \int_1^2 \int_{1-x}^{1+x} f(x,)\;dy\,dx =$$

(b) Let $y^2 + x(x-1) \le 0$. Then

$$\iint_R f(x,y)\;dA = \int_0^1 \int_{-\sqrt{x(1-x)}}^{\sqrt{x(1-x)}} f(x,y)\;dy\,dx$$

4. (a) Let R be the cube of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$,

$(0,1,1)$, $(1,1,1)$.

$$\iiint_R f(x,y,z)\,dV = \int_0^1\int_0^1\int_0^1 \sqrt{x+y+z}\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \underbrace{\sqrt{u}}_{u=x+y+z}\,du\,dy\,dx$$

$$= \frac{2}{3}\int_0^1\int_0^1 (x+y+z)^{3/2}\Big|_0^1\,dy\,dx$$

$$= \frac{2}{3}\int_0^1\int_0^1 \left[(1+x+y)^{3/2}-(x+y)^{3/2}\right]\,dy\,dx$$

$$= \frac{4}{15}\int_0^1 \left[(1+x+y)^{5/2}-(x+y)^{5/2}\right]_0^1\,dx$$

$$= \frac{4}{15}\int_0^1 \left[(2+x)^{5/2}-2\,(1+x)^{5/2}+x^{5/2}\right]\,dx$$

$$= \frac{8}{105}\left[(2+x)^{7/2}-2\,(1+x)^{7/2}+x^{7/2}\right]_0^1 = \frac{8}{35}\left(9\sqrt{3}-8\sqrt{2}+1\right)$$

(b) Let R be the pyramid of vertices $(\pm1,\pm1,0)$ and $(0,0,1)$. In order to determine the limits of integration we will use the cross-section method. Let $0\le z\le 1$. Then a plane perpendicular to the z-axis whose boundaries satisfy the inequalities $-1+z\le y\le 1-z$, $-1+z\le x\le 1-z$ will represents the cross-sectional area of the pyramid for a given value of z. Hence,

$$\iiint_R f(x,y,z)\,dV = \int_0^1\int_{-1+z}^{1-z}\int_{-1+z}^{1-z}\left(x^2+z^2\right)\,dx\,dy\,dz$$

$$= \int_0^1\int_{-1+z}^{1-z}\left[\frac{x^3}{3}+xz^2\right]_{-1+z}^{1-z}\,dy\,dz$$

$$= \frac{2}{3}\int_0^1\left(1-3z+6z^2-4z^3\right)\int_{-1+z}^{1-z}\,dy\,dz$$

$$= \frac{2}{3}\int_0^1\left(1-3z+6z^2-4z^3\right)y\Big|_{-1+z}^{1-z}\,dz$$

$$= \frac{4}{3}\int_0^1\left(1-4z+9z^2-10z^3+4z^4\right)\,dz$$

$$= \frac{4}{3}\left[z-2z^2+3z^3-\frac{5}{2}z^4+\frac{4}{5}z^5\right]_0^1 = \frac{2}{5}$$

5. (a) Let the integral

$$\int_{1/2}^1\int_0^{1-x} f(x,y)\,dy\,dx$$

be given. It then follows that $1/2\le x\le 1$, $0\le y\le 1-x$ and so the region R is the triangle with vertices $(1/2,0)$, $(1,0)$, $(1/2,1/2)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^{1/2} \int_{1/2}^{1-y} f(x, y)\; dx\, dy$$

(b) Let the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y)\; dy\, dx$$

be given. It then follows that $0 \leq x \leq 1$, $0 \leq y \leq \sqrt{1-x^2}$ and so the region R is the quarter circle in the positive quadrant of the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y)\; dx\, dy$$

(c) Let the integral

$$\int_0^1 \int_{y-1}^{0} f(x, y)\; dx\, dy$$

be given. It then follows that $y - 1 \leq x \leq 0$, $0 \leq y \leq 1$ and so the region R is the triangle with vertices $(0, 0)$, $(-1, 0)$, $(0, 1)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_{-1}^0 \int_0^{x+1} f(x, y)\; dy\, dx$$

(d) Let the integral

$$\int_0^1 \int_{1-x}^{1+x} f(x, y)\; dy\, dx$$

be given. It then follows that $0 \leq x \leq 1$, $1-x \leq y \leq 1+x$ and so the region R is the triangle with vertices $(1, 0)$, $(0, 1)$, $(1, 1)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^2 \int_{|y-1|}^{1} f(x, y)\; dx\, dy$$

6. (a) First of all, let us define a diametral plane of a sphere as a plane that cuts the sphere into two equal halves, hence, containing the midpoint of the sphere. Next, let $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$, where $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, be a unit vector normal to the diametral plane and let $P = (x_0, y_0, z_0)$ be an arbitrary point of the plane so that by (1.23) the equation for the plane can be written as $Ax + By + Cz + D = 0$, where $D = -Ax_0 - By_0 - Cz_0$ and x, y, z are expected to be on the place. Then the distance of the point P to another arbitrary point $Q = (x_1, y_1, z_1)$ is the length of

the projection of the vector $\overrightarrow{PQ}$ onto the normal unit vector $\mathbf{n}$, or in other words by (1.12), as the component of $\overrightarrow{PQ}$ in the direction of $\mathbf{n}$:

$$d = \text{comp}_{\mathbf{n}}\overrightarrow{PQ} = |\overrightarrow{PQ}|\cos\alpha = |\overrightarrow{PQ}\cdot\mathbf{n}| = \frac{|A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}}$$
$$= \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

Recognizing that the point Q was chosen arbitrarily then by (2.84) the distance of any point to the diametral plane can be written as

$$d(\rho,\phi,\theta) = \frac{|A\rho\sin\phi\cos\theta + B\rho\sin\phi\sin\theta + C\rho\cos\phi + D|}{\sqrt{A^2+B^2+C^2}}$$

in spherical coordinates. Hence, the mass of a sphere whose density is proportional to the distance from one diametral plane is given by

$$M = k\iiint_R d(\rho,\phi,\theta)\;dV = k\int_0^{2\pi}\int_0^{\pi}\int_0^{R} d(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \kappa\int_0^{2\pi}\int_0^{\pi}\int_0^{R} |A\rho\sin\phi\cos\theta + B\rho\sin\phi\sin\theta + C\rho\cos\phi + D|\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

where R is the radius of the sphere, $\kappa = k/\sqrt{A^2+B^2+C^2}$, k is the constant of proportionality and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

(b) By (4.54), the coordinates of the center of mass of the sphere of part (a) are given by

$$\bar{x} = \frac{k}{M}\iiint_R xd(\rho,\phi,\theta)\;dV = \frac{k}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} d(\rho,\phi,\theta)\,\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$
$$= \frac{\kappa}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} |A\rho\sin\phi\cos\theta + B\rho\sin\phi\sin\theta + C\rho\cos\phi + D|\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$
$$\bar{y} = \frac{k}{M}\iiint_R yd(\rho,\phi,\theta)\;dV = \frac{k}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} d(\rho,\phi,\theta)\,\rho^3\sin^2\phi\sin\theta\,d\rho\,d\phi\,d\theta$$
$$= \frac{\kappa}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} |A\rho\sin\phi\cos\theta + B\rho\sin\phi\sin\theta + C\rho\cos\phi + D|\rho^3\sin^2\phi\sin\theta\,d\rho\,d\phi\,d\theta$$

$$\bar{z} = \frac{k}{M}\iiint_R zd(\rho,\phi,\theta)\;dV = \frac{k}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} d(\rho,\phi,\theta)\,\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$
$$= \frac{\kappa}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} |A\rho\sin\phi\cos\theta + B\rho\sin\phi\sin\theta + C\rho\cos\phi + D|\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

where again R is the radius of the sphere, $\kappa = k/\sqrt{A^2+B^2+C^2}$, k is the constant of proportionality and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

(c) The moment of inertia about the x-axis of a solid filling the region $0 \le z \le 1 - x^2 - y^2$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and having density proportional to xy is, using (4.55), given by

$$I_x = \iiint_R \left(y^2 + z^2\right) f\left(x, y, z\right)\, dx\, dy\, dz = k \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} xy\left(y^2 + z^2\right)\, dz\, dy\, dx$$

where k is the constant of proportionality.

7. The moment of inertia of a solid about an arbitrary line L is defined as

$$I_L = \iiint\limits_R d^2 f\left(x, y, z\right)\, dx\, dy\, dz$$

where f is density and d is the distance from a general point (x, y, z) of the solid to the line L. Next, let $\bar{L}$ be a line parallel to L that coincides with the z-axis, such that the center of mass is located at the origin. Since $\bar{L}$ and L are parallel, we can define the distance between $\bar{L}$ and L by the constant h. We may assume, without loss of generality, that in a Cartesian coordinate system the distance between the lines L and $\bar{L}$ lies along the x-axis. Hence, the square of the distance from a general point $(x, y, z$ of the solid to the line L may be written as $d^2 = (x + h)^2 + y^2$. Substituting for this in the equation for I_L then gives

$$\begin{aligned} I_L &= \iiint\limits_R \left[\left(x + h\right)^2 + y^2\right] f\left(x, y, z\right)\, dx\, dy\, dz \\ &= \iiint\limits_R \left(x^2 + y^2 + h^2 + 2xh\right) f\left(x, y, z\right)\, dx\, dy\, dz \\ &= \underbrace{\iiint\limits_R \left(x^2 + y^2\right) f\left(x, y, z\right)\, dx\, dy\, dz}_{(4.55)} + h^2 \underbrace{\iiint\limits_R f\left(x, y, z\right)\, dx\, dy\, dz}_{(4.52)} \\ &\quad + 2h \underbrace{\iiint\limits_R x f\left(x, y, z\right)\, dx\, dy\, dz}_{(4.54)} = I_{\bar{L}} + Mh^2 + 2Mh\bar{x} = I_{\bar{L}} + Mh^2 \end{aligned}$$

Note that the term $2Mh\bar{x} = 0$, since this is a multiple of the x-coordinate of the center of mass, which is located at the origin.

8. Let L be a line through the origin O with directions cosines l, m, n. Then we can define the vector $\mathbf{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ as a unit vector parallel to the line L. Furthermore, we can also define the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as the position vector of a point in space. The shortest distance from an arbitrary point in space P to the line L is then

given by the length of the perpendicular line from P to the line L, which may be expressed as $d = |\mathbf{r}|\sin\theta$, where $0 \le \theta \le \pi$ is the angle between the vectors $\mathbf{n}$ and $\mathbf{r}$. Since $\mathbf{n}$ is a unit vector (i.e. a vector with magnitude 1), we note that by (1.16) $|\mathbf{r}|\sin\theta = |\mathbf{r}||\mathbf{n}|\sin\theta = |\mathbf{r}\times\mathbf{n}|$. The moment of inertia of a solid about the line L can then be defined as

$$\begin{aligned}
I_L &= \iiint\limits_R d^2 f(x,y,z)\ dx\,dy\,dz = \iiint\limits_R |\mathbf{r}\times\mathbf{n}|^2 f(x,y,z)\ dx\,dy\,dz \\
&= \iiint\limits_R \underbrace{\left[(ny-mz)^2 + (lz-nx)^2 + (mx-ly)^2\right]}_{(1.20)}\ dx\,dy\,dz \\
&= l^2 \iiint\limits_R \left(y^2+z^2\right)\ dx\,dy\,dz + m^2 \iiint\limits_R \left(x^2+z^2\right)\ dx\,dy\,dz + n^2 \iiint\limits_R \left(x^2+y^2\right)\ dx\,dy\,dz \\
&\quad - 2lm \iiint\limits_R xy\,dx\,dy\,dz - 2mn \iiint\limits_R yz\,dx\,dy\,dz - 2ln \iiint\limits_R xz\,dx\,dy\,dz \\
&= I_x l^2 + I_y m^2 + I_z n^2 - 2I_{xy}lm - 2I_{yz}mn - 2I_{xz}ln
\end{aligned}$$

9. Let us consider a tetrahedron whose base (i.e. one of its sides) is in the xy-plane such that the centroid of its base is located at the origin O and let the three vertices of the base be given by the points (x_1, y_1), (x_2, y_2), (x_3, y_3). As such, the centroid of the triangular base satisfies the equations

$$\frac{x_1+x_2+x_3}{3} = 0 \qquad\qquad \frac{y_1+y_2+y_3}{3} = 0$$

Next, the z-coordinate of the only vertex that is *not* in the xy-plane can be defined as the constant $z = h$. Then the coordinates of the tetrahedron are given by $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, $(x_3, y_3, 0)$, (x_4, y_4, h). Next, let us consider the line L passing through the origin and the point (x_4, y_4, h). The centroid of the tetrahedron thus is given by

$$\bar{x} = \frac{x_1+x_2+x_3+x_4}{4} \qquad \bar{y} = \frac{y_1+y_2+y_3+y_4}{4} \qquad \bar{z} = \frac{z_1+z_2+z_3+z_4}{4}$$

Using the fact that $x_1+x_2+x_3 = y_1+y_2+y_3 = z_1+z_2+z_3 = 0$ the three equations above reduce to $\bar{x} = x_4/4$, $\bar{y} = y_4/4$, $\bar{z} = h/4$. Denoting the vertex not in the xy-plane by $P = (x_4, y_4, h)$ and the centroid of the tetrahedron by $G = (1/4)(x_4, y_4, h)$ we then find that

$$P - G = (x_4, y_4, h) - \frac{1}{4}(x_4, y_4, h) = \frac{3}{4}(x_4, y_4, h) = \frac{3}{4}(P - O)$$

10. (a) Let $\mathbf{F}(t) = t^2\mathbf{i} - e^t\mathbf{j} + \mathbf{k}/(1+t)$. Then

$$\int_0^1 \mathbf{F}(t)\ dt = \int_0^1 t^2\,dt\mathbf{i} - \int_0^1 e^t\,dt\mathbf{j} + \int_0^1 \frac{dt}{1+t}\mathbf{k} = \left.\frac{t^3}{3}\right|_0^1 \mathbf{i} - \left.e^t\right|_0^1\mathbf{j} + \left.\ln(1+t)\right|_0^1\mathbf{k}$$
$$= \frac{1}{3}\mathbf{i} + (1-e)\,\mathbf{j} + \ln 2\mathbf{k}$$

(b) Let R be the triangular region enclosed by the triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$ and $\mathbf{F}(x,y) = x^2y\mathbf{i} + xy^2\mathbf{j}$. Then

$$\begin{aligned}
\iint_R \mathbf{F}(x,y)\ dA &= \int_0^1\int_0^{1-x} x^2y\,dy\,dx\mathbf{i} + \int_0^1\int_0^{1-x} xy^2\,dy\,dx\mathbf{j} \\
&= \frac{1}{2}\int_0^1 \left.x^2y^2\right|_0^{1-x} dx\mathbf{i} + \frac{1}{3}\int_0^1 \left.xy^3\right|_0^{1-x} dx\mathbf{j} \\
&= \frac{1}{2}\int_0^1 x^2(1-x)^2\ dx\mathbf{i} + \frac{1}{3}\int_0^1 x(1-x)^3\ dx\mathbf{j} \\
&= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1\mathbf{i} + \frac{1}{3}\left[\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right]_0^1\mathbf{j} = \frac{1}{60}(\mathbf{i}+\mathbf{j})
\end{aligned}$$

11. Let $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be continuous for $a \le t \le b$ and let $\mathbf{q}$ be a constant vector. Then

(a)

$$\begin{aligned}
\int_a^b \mathbf{q}\cdot\mathbf{F}(t)\ dt &= \int_a^b (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})\cdot[f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\ dt \\
&= \int_a^b [q_xf(t) + q_yg(t) + q_zh(t)]\ dt \\
&= \int_a^b q_xf(t)\ dt + \int_a^b q_yg(t)\ dt + \int_a^b q_zh(t)\ dt \\
&= q_x\int_a^b f(t)\ dt + q_y\int_a^b g(t)\ dt + q_z\int_a^b h(t)\ dt \\
&= (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})\cdot\underbrace{\left[\int_a^b f(t)\ dt\mathbf{i} + \int_a^b g(t)\ dt\mathbf{j} + \int_a^b h(t)\ dt\mathbf{k}\right]}_{(4.57)} \\
&= \mathbf{q}\cdot\int_a^b \mathbf{F}(t)\ dt
\end{aligned}$$

(b)

$$\begin{aligned}
\int_a^b \mathbf{q} \times \mathbf{F}(t)\, dt &= \int_a^b (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \times [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\, dt \\
&= \int_a^b [(q_y h(t) - q_z g(t))\,\mathbf{i} + (q_z f(t) - q_x h(t))\,\mathbf{j} + (q_x g(t) - q_y f(t))\,\mathbf{k}]\, dt \\
&= \int_a^b [q_y h(t) - q_z g(t)]\, dt\mathbf{i} + \int_a^b [q_z f(t) - q_x h(t)]\, dt\mathbf{j} \\
&\quad + \int_a^b [q_x g(t) - q_y f(t)]\, dt\mathbf{k} \\
&= \left[q_y \int_a^b h(t)\, dt - q_z \int_a^b g(t)\, dt\right]\mathbf{i} + \left[q_z \int_a^b f(t)\, dt - q_x \int_a^b h(t)\, dt\right]\mathbf{j} \\
&\quad + \left[q_x \int_a^b g(t)\, dt - q_y \int_a^b f(t)\, dt\right]\mathbf{k} \\
&= (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \times \left[\int_a^b f(t)\, dt\mathbf{i} + \int_a^b g(t)\, dt\mathbf{j} + \int_a^b h(t)\, dt\mathbf{k}\right] \\
&= \mathbf{q} \times \int_a^b \mathbf{F}(t)\, dt
\end{aligned}$$

Section 4.6

1. (a) Let $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Using the identity $2\cos^2\theta = 1 + \cos 2\theta$, then

$$\begin{aligned}
\int_0^1 \left(1 - x^2\right)^{3/2} dx = \int_0^{\pi/2} \cos^4\theta\, d\theta &= \frac{1}{4}\int_0^{\pi/2} (1 + \cos 2\theta)^2\, d\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \cos^2\theta\right) d\theta \\
&= \frac{1}{8}\int_0^{\pi/2} (3 + 4\cos 2\theta + \cos 4\theta)\, d\theta \\
&= \frac{1}{32}\left[12\theta + 8\sin 2\theta + \sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{16}
\end{aligned}$$

(b) Let $x = u^2 - 1$ so that $dx = 2u\, du$ and $v = 1 + u$ so that $dv = du$. Then

$$\begin{aligned}
\int_0^1 \frac{dx}{1 + \sqrt{1 + x}} = 2\int_1^{\sqrt{2}} \frac{u}{1 + u}\, du = 2\int_2^{1+\sqrt{2}} \frac{v - 1}{v}\, dv &= 2\left[v - \ln v\right]_2^{1+\sqrt{2}} \\
&= 2\sqrt{2} - 2 + 2\ln\left(2\sqrt{2} - 2\right)
\end{aligned}$$

(c) Let $t = \tan(x/2)$ so that $dt = (1/2)\sec^2 x\, dx$ and $\sqrt{2}u = t + 1$ so that $\sqrt{2}du = dt$.

Then

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sin x + \cos x + 2} &= \int_0^1 \frac{2\cos^2(x/2)}{\sin x + \cos x + 2}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2) + 2\cos^2(x/2) + 1}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\tan(x/2)\cos^2(x/2) + 2\cos^2(x/2) + 1}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\tan(x/2)\cos^2(x/2) + 3\cos^2(x/2) + \sin^2(x/2)}\, dt \\
&= \int_0^1 \frac{2}{\tan^2(x/2) + 2\tan(x/2) + 3}\, dt = 2\int_0^1 \frac{dt}{t^2 + 2t + 3} \\
&= 2\int_0^1 \frac{dt}{(t+1)^2 + 2} = \sqrt{2}\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{du}{u^2 + 1} = \sqrt{2}\tan^{-1} u\Big|_{\sqrt{2}/2}^{\sqrt{2}} \\
&= \sqrt{2}\left(\tan^{-1}\sqrt{2} - \tan^{-1}\frac{\sqrt{2}}{2}\right)
\end{aligned}$$

(d) Let $t = 1 + x\cos x$ so that $dt = (\cos x - x\sin x)\, dx$. Then

$$\int_0^{\pi/4} \frac{x\cos x\,(x\sin x - \cos x)}{1 + x\cos x}\, dx = \int_1^{1+c} \left(\frac{1}{t} - 1\right) dt = [\ln t - t]_1^{1+c} = \ln(1+c) - c$$

where $c = \pi/(4\sqrt{2})$.

2. Let $\phi(u)$ be a function of u. Next, we consider the integral

$$\int_{u_1}^{u_2} \phi'(u)\, du = \int_{u_1}^{u_2} \frac{d\phi}{du}\, du = \int_{u_1}^{u_2} d\phi = \phi(u_2) - \phi(u_1)$$

which is equal to (4.60) when setting $f(x) = f[x(u)] \equiv 1$.

3. (a) Let $f(x)$ be continuous for $x_1 \le x \le x_2$. Next, let us choose a u_0 and $x_0 = x(u_0)$ such that $u_1 < u_0 < u_2$. Replacing u_2 for u_0 in (4.60) then gives

$$\int_{x_1}^{x_0} f(x)\, dx = \int_{u_1}^{u_0} f[x(u)]\frac{dx}{du}\, du$$

which remains to hold. Then if we let $u_0 \to u_2$ we get

$$\begin{aligned}
\lim_{u_0 \to u_2} \int_{u_1}^{u_0} f[x(u)]\frac{dx}{du}\, du = \lim_{u_0 \to u_2} (F[x(u_0)] - F[x(u_1)]) &= F[x(u_2)] - F[x(u_1)] \\
&= F(x_2) - F(x_1)
\end{aligned}$$

which is the same as the value of the left-hand side of (4.60). Hence, if the right-hand side of (4.60) has a limit, then the other side has a limit also and the two limits are equal.

(b) Let $u = 1/x$ so that $du = -dx/x^2$. Then

$$\int_1^\infty \frac{1}{x^2} \sinh \frac{1}{x}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2} \sinh \frac{1}{x}\, dx = \int_0^1 \sinh u\, du = \cosh u\Big|_0^1 = \cosh 1 - 1$$

(c) Let $u = \tanh x$ so that $du = (1 - \tanh^2 x)\, dx$. Then

$$\int_0^\infty (1 - \tanh x)\ dx = \lim_{b\to\infty} \int_0^b (1 - \tanh x)\ dx = \int_0^1 \frac{du}{1+u} = \ln(1+u)\Big|_0^1 = \ln 2$$

4. (a) Let R_{xy} be the region $x^2 + y^2 \le 1$ and $x = r\cos\theta$, $y = r\sin\theta$. Then by (4.64) we find

$$\begin{aligned}\iint_{R_{xy}} \left(1 - x^2 - y^2\right)\ dx\, dy &= \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right)\ dy\, dx \\ &= \int_0^1 \int_0^{2\pi} \left(1 - r^2\right) r\, d\theta\, dr = \int_0^1 \left(1 - r^2\right) r\theta\Big|_0^{2\pi} dr \\ &= 2\pi \int_0^1 \left(1 - r^2\right) r\, dr = \pi \left[r^2 - \frac{r^4}{2}\right]_0^1 = \frac{\pi}{2}\end{aligned}$$

(b) Let R be the region $1 \le x \le 2$, $0 \le y \le x$ and $x = r\cos\theta$, $y = r\sin\theta$. Furthermore, let $u = \sec^3\theta$ so that $du = 3\sec^3\theta\tan\theta\, d\theta$. Then by (4.64) we find

$$\begin{aligned}\iint_R \frac{y\sqrt{x^2+y^2}}{x}\, dx\, dy &= \int_1^2 \int_0^x \frac{y\sqrt{x^2+y^2}}{x}\, dy\, dx = \int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} r^2 \tan\theta\, dr\, d\theta \\ &= \frac{1}{3} \int_0^{\pi/4} r^3 \tan\theta\Big|_{\sec\theta}^{2\sec\theta} d\theta = \frac{7}{9} \int_0^{\pi/4} \sec^3\theta \tan\theta\, d\theta \\ &= \frac{7}{9} \int_1^{2\sqrt{2}} du = \frac{7}{9} u\Big|_1^{2\sqrt{2}} = \frac{14\sqrt{2} - 7}{9}\end{aligned}$$

(c) Let R_{xy} be the parallelogram with successive vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$ and $u = x - y$, $v = x + y$ so that R_{uv} is the square region with vertices (π, π), $(\pi, 3\pi)$, $(-\pi, 3\pi)$, $(-\pi, \pi)$. Then by (4.61)

$$\begin{aligned}\iint_{R_{xy}} (x-y)^2 \sin^2(x+y)\ dx\, dy &= \int_0^{2\pi} \int_{|\pi - x|}^{2\pi - |\pi - x|} (x-y)^2 \sin^2(x+y)\ dy\, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} u^2 \sin^2 v\, dv\, du \\ &= \frac{1}{4} \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} u^2 (1 - \cos 2v)\ dv\, du \\ &= \frac{1}{8} \int_{-\pi}^{\pi} u^2 \left[2v - \sin 2v\right]_{\pi}^{3\pi}\ du = \frac{\pi}{2} \int_{-\pi}^{\pi} u^2\, du \\ &= \frac{\pi u^3}{6}\Big|_{-\pi}^{\pi} = \frac{\pi^4}{3}\end{aligned}$$

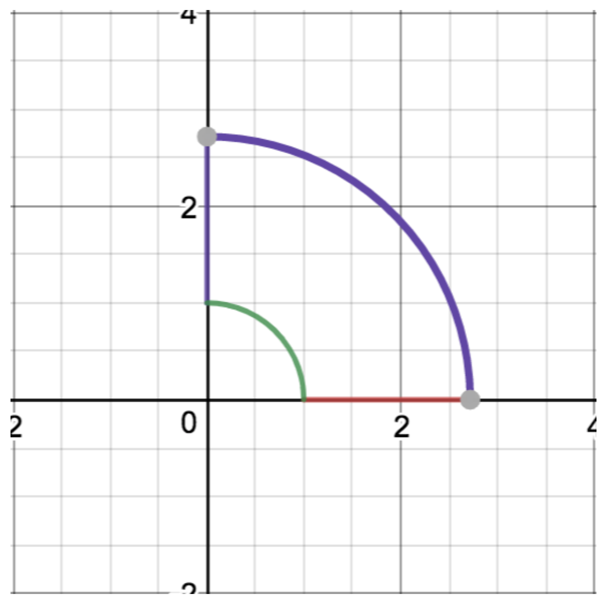
(d) Let R be the trapezoidal region bounded by the lines $x+y=1$, $x+y=2$ in the first quadrant and $u=1+x+y$, $v=x-y$. Then by (4.61)

$$\begin{aligned}
\iint_R \frac{(x-y)^2}{1+x+y}\,dx\,dy &= \int_0^1 \int_{1-x}^{2-x} \frac{(x-y)^2}{1+x+y}\,dy\,dx + \int_1^2 \int_0^{2-x} \frac{(x-y)^2}{1+x+y}\,dy\,dx \\
&= \frac{1}{2}\int_2^3 \int_{1-u}^{-1+u} \frac{v^2}{u}\,dv\,du = \frac{1}{6}\int_2^3 \left.\frac{v^3}{u}\right|_{1-u}^{-1+u} du \\
&= \frac{1}{3}\int_2^3 \left(u^2-3u+3-u^{-1}\right)\,du = \frac{1}{3}\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]_2^3 \\
&= \frac{11}{18}+\frac{1}{3}\ln\frac{2}{3}
\end{aligned}$$

(e) Let R be the region bounded by the ellipse $5x^2+2xy+2y^2=1$ and $x=u+v$, $y=-2u+v$. Then by (4.61)

$$\begin{aligned}
\iint_R \sqrt{5x^2+2xy+2y^2}\,dx\,dy &= \int_{-\sqrt{2}/3}^{\sqrt{2}/3} \int_{-\left(\sqrt{2-9x^2}-x\right)/2}^{\left(\sqrt{2-9x^2}-x\right)/2} \sqrt{5x^2+2xy+2y^2}\,dy\,dx \\
&= 9\int_{-1/3}^{1/3} \int_{-\sqrt{(1/9)-u^2}}^{\sqrt{(1/9)-u^2}} \sqrt{u^2+v^2}\,dv\,du \\
&= 9\int_0^{2\pi} \int_0^{1/3} r^2\,dr\,d\theta = 3\int_0^{2\pi} \left. r^3\right|_0^{1/3} d\theta \\
&= \frac{3}{27}\int_0^{2\pi} d\theta = \frac{2\pi}{9}
\end{aligned}$$

5. According to Section 2.7, the transformation $u=e^x\cos y$, $v=e^x\sin y$ can be regarded as a mapping from the xy-plane to the uv-plane. Since the inverse mapping exists and is given by $x=(1/2)\ln(u^2+v^2)$, $y=\tan^{-1}(v/u)$, hence, taking each point of a region in the uv-plane to a unique point in the xy-plane, the mapping is one-to-one. Now let us consider the rectangle R_{xy} : $0\le x\le 1$, $0\le y\le \pi/2$. Then u and v are defined and continuous for each point $(x,y)\in R_{xy}$ (i.e. their derivatives exist). Hence, the given transformation defines a one-to-one mapping of the rectangle R_{xy} onto a region of the uv-plane. To find the boundaries of the region in the uv-plane we consider each of the sides of the rectangle R_{xy} in turn. For the bottom side of the rectangle given by $0\le x\le 1$, $y=0$ we find $1\le u\le e$, $v=0$. For the top side of the rectangle given by $0\le x\le 1$, $y=\pi/2$ we find $u=0$, $1\le v\le e$. The left side of the rectangle given by $x=0$, $0\le y\le\pi/2$ transforms as $u=\cos y$, $v=\sin y$, whereas the right side of the rectangle given by $x=1$, $0\le y\le\pi/2$ transforms as $u=e\cos y$, $v=e\sin y$. Hence, the rectangle R_{xy} transforms to the ring R_{uv} as shown in the figure below



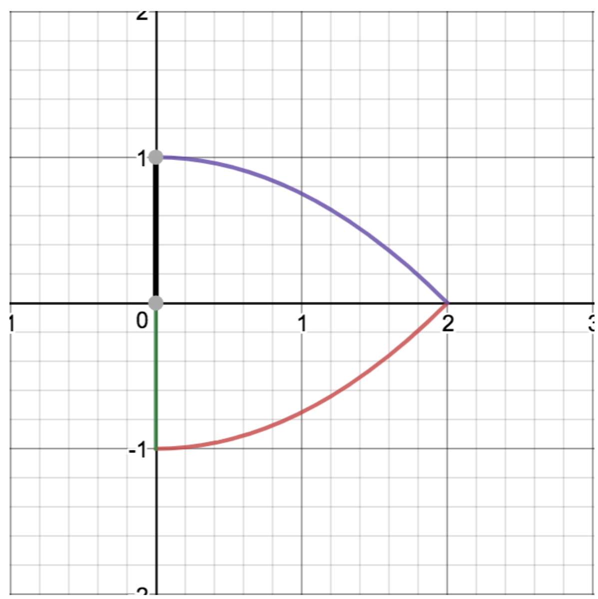
As such, by (4.61) we find that

$$\iint\limits_{R_{xy}} \frac{e^{2x}}{1+e^{4x}\cos^2 y \sin^2 y}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \frac{e^{2x}}{1+e^{4x}\cos^2 y\sin^2 y}\,dx\,dy$$
$$= \int_0^1 \int_{\sqrt{1-u^2}}^{\sqrt{e^2-u^2}} \frac{dv\,du}{1+u^2v^2} + \int_1^e \int_0^{\sqrt{e^2-u^2}} \frac{dv\,du}{1+u^2v^2}$$
$$= \iint\limits_{R_{uv}} \frac{du\,dv}{1+u^2v^2}$$

6. The inverse mapping is given by

$$x = \pm\sqrt{(1/2)(v \pm \sqrt{u^2+v^2})} \qquad y = \pm\sqrt{(1/2)(-v \pm \sqrt{u^2+v^2})}$$

Since the inverse mapping is well-defined, the mapping from the xy-plane to the $uv-$plane is one-to-one. Now we consider the square region $0 \le x \le 1$, $0 \le y \le 1$. Then u and v are defined and continuous for each point (x,y) of the square. Hence, the given transformation defines a one-to-one mapping of the square $0 \le x \le 1$, $0 \le y \le 1$ onto a region of the uv-plane. To find the boundaries of the region in the uv-plane we consider each of the sides of the square in the xy-plane in turn. For the bottom side of the square given by $0 \le x \le 1$, $y = 0$ we find $u = 0$, $v = x^2$. Hence, the bottom side of the square in the xy-plane maps to the vertical line segment with endpoints $(0,0)$, $(0,1)$ in the uv-plane. For the top side of the square given by $0 \le x \le 1$, $y = 1$ we find $u = 2x$, $v = x^2 - 1$. Treating this as a parametric equation in the variable t this becomes $u = 2t$, $v = t^2 - 1$ where $0 \le t \le 1$. Then solving for v gives $v = (1/4)u^2 - 1$, $0 \le u \le 2$. Hence, the top side of the square in the xy-plane maps to the line segment given by the aforementioned equation with endpoints $(0,-1)$, $(2,0)$ in the uv-plane. For the left side of the square given by $x = 0$, $0 \leq y \leq 1$ we find $u = 0$, $v = -y^2$. Hence, the left side of the square in the xy-plane maps to the vertical line segment with endpoints $(0, 0)$, $(0, -1)$ in the uv-plane. Lastly, for the right side of the square given by $x = 1$, $0 \leq y \leq 1$ we find $u = 2y$, $v = 1 - y^2$. Treating this as a parametric equation in t this becomes $u = 2t$, $v = 1 - t^2$ where $0 \leq t \leq 1$. Solving for v then gives $v = 1 - (1/4)u^2$, $0 \leq u \leq 2$. Hence, the right side of the square in the xy-plane maps to the line segment given by the aforementioned equation, having endpoints $(0, 1)$, $(2, 0)$ in the uv-plane. To summarize, the figure below shows the transformed region in the uv-plane.



As such, by (4.61) we find that

$$\iint\limits_{R_{xy}} \sqrt[3]{x^4 - 6x^2y^2 + y^4}\, dx\, dy = \int_0^1 \int_0^1 \sqrt[3]{x^4 - 6x^2y^2 + y^4}\, dx\, dy$$

$$= \frac{1}{4} \int_0^2 \int_{-1+(1/4)u^2}^{1-(1/4)u^2} \frac{\sqrt[3]{v^2 - u^2}}{\sqrt{v^2 + u^2}}\, dv\, du = \frac{1}{4} \iint\limits_{R_{uv}} \frac{\sqrt[3]{v^2 - u^2}}{\sqrt{v^2 + u^2}}\, du\, dv$$

7. (a) Let R_{xy} be the triangular region bounded by the lines $y = x$, $0 \leq x \leq 1$ and $x = 1$, $0 \leq y \leq 1$ in the first quadrant of the xy-plane. And let the transformation $x = u + v$, $y = u - v$ be given. Then the inverse transformation is given by

$$u = \frac{1}{2}(x + y) \qquad\qquad v = \frac{1}{2}(x - y)$$

so that it defines a one-to-one mapping of the triangle R_{xy} onto the triangle R_{uv}, which boundaries are given by the two lines $v = u$, $0 \leq u \leq 1/2$ and $v = 1 - u$, $1/2 \leq u \leq 1$ in the first quadrant of the uv-plane. As such, by (4.61) we find

$$\iint\limits_{Rxy} \ln\left(1+x^2+y^2\right)\, dx\, dy = \int_0^1 \int_0^x \ln\left(1+x^2+y^2\right)\, dy\, dx$$
$$= 2\int_0^{1/2} \int_v^{1-v} \ln\left(1+2u^2+2v^2\right)\, du\, dv$$
$$= 2\iint\limits_{R_{uv}} \ln\left(1+2u^2+2v^2\right)\, du\, dv$$

(b) Let R_{xy} be the triangular region bounded by the lines $y = 1-x$, $y = 1+x$, $0 \leq x \leq 1$ and $x = 1$, $0 \leq y \leq 2$ in the first quadrant of the xy-plane. And let the transformation $x = u$, $y = u+v$ be given. Then the inverse transformation is given by

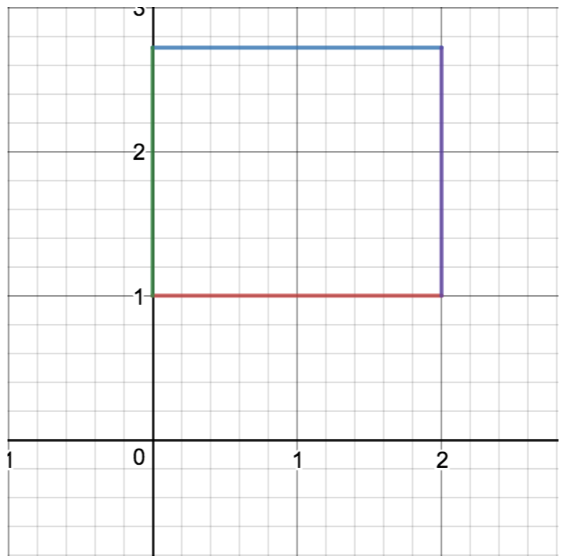
$$u = x \qquad\qquad v = y - x$$

so that it defines a one-to-one mapping of the triangle R_{xy} onto the triangle R_{uv}, which boundaries are given by the lines $v = 1-2u$, $0 \leq u \leq 1$, $v = 1+u$, $0 \leq u \leq 1$ and $u = 1$, $-1 \leq v \leq 1$ in the first and fourth quadrants of the uv-plane. As such, by (4.61) we find

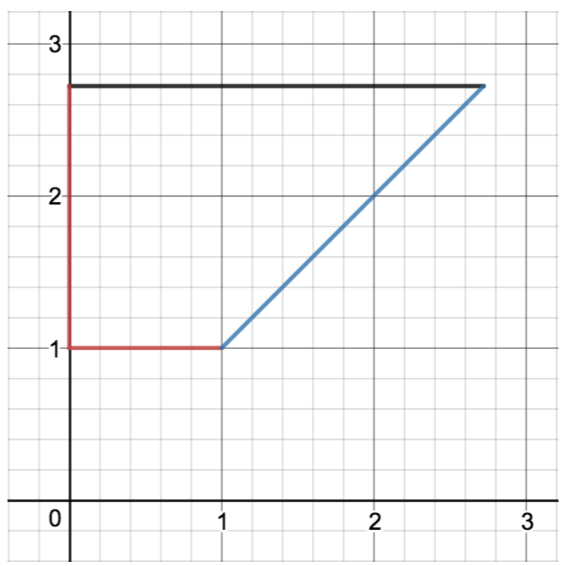
$$\iint\limits_{R_{xy}} \sqrt{1+x^2y^2}\, dx\, dy = \int_0^1 \int_{1-x}^{1+x} \sqrt{1+x^2y^2}\, dy\, dx = \int_0^1 \int_{1-2u}^{1} \sqrt{1+u^2\left(u+v\right)^2}\, dv\, du$$
$$= \iint\limits_{R_{uv}} \sqrt{1+u^2\left(u+v\right)^2}\, du\, dv$$

8. Let R_{uv} be the square $0 \leq u \leq 1$, $0 \leq v \leq 1$.

(a) Let the transformation $x = u + u^2$, $y = e^v$ be given. It then defines a one-to-one mapping of the square R_{uv} onto the rectangle R_{xy}. The bottom side of the square given by $0 \leq u \leq 1$, $v = 0$ maps to the line segment $y = 1$, $0 \leq x \leq 2$ forming the bottom side of the rectangle. The top side of the square given by $0 \leq u \leq 1$, $v = 1$ maps to the line segment $y = e$, $0 \leq x \leq 2$ forming the top side of the rectangle. The left side of the square given by $u = 0$, $0 \leq v \leq 1$ maps to the line segment $x = 0$, $1 \leq y \leq e$ forming the left side of the rectangle. Finally, the right side of the square given by $u = 1$, $0 \leq v \leq 1$ maps to the line segment $x = 2$, $1 \leq y \leq e$ forming the right side of the rectangle. The below figure shows the graph for the rectangle R_{xy}.

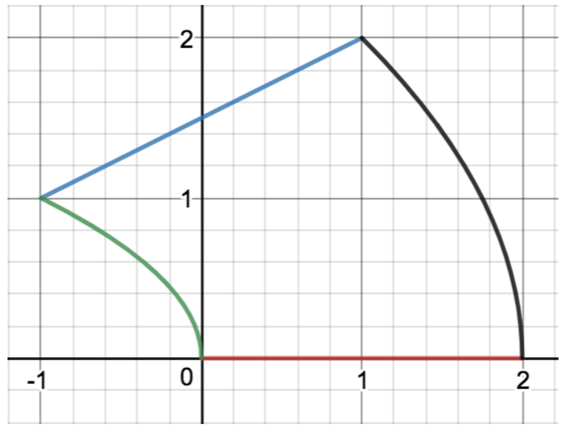


(b) Let the transformation $x = ue^v$, $y = e^v$ be given. The bottom side of the square given by $0 \le u \le 1$, $v = 0$ maps to the line segment $0 \le x \le 1$, $y = 1$. The top side of the square given by $0 \le u \le 1$, $v = 1$ maps to the line segment $0 \le x \le e$, $y = e$. The left side of the square given by $u = 0$, $0 \le v \le 1$ maps to the line segment $x = 0$, $1 \le y \le e$. Finally, the right side of the square given by $u = 1$, $0 \le v \le 1$ maps to the line segment $x = y$, $1 \le x \le e$. The below figure shows the graph for the region R_{xy} in the xy-plane.

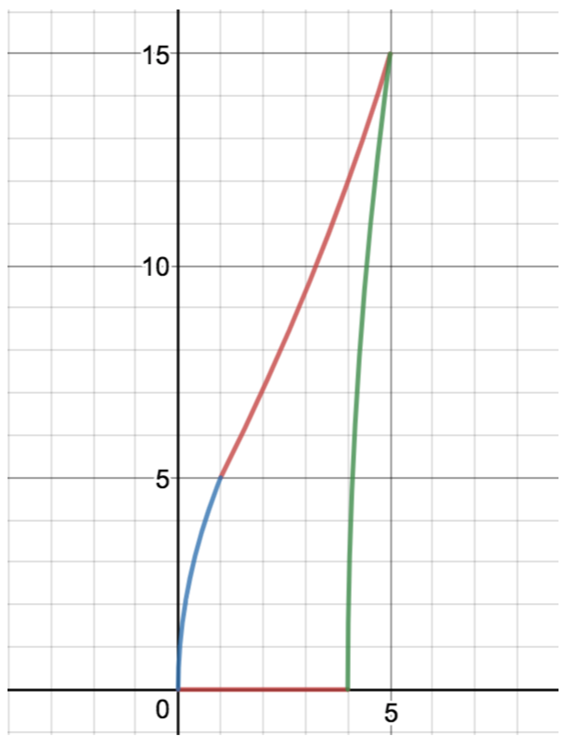


(c) Let the transformation $x = 2u - v^2$, $y = v + uv$ be given. The bottom side of the square maps to the line segment $0 \le x \le 2$, $y = 0$. The top side of the square maps to the line segment $y = (1/2)(3 + x)$, $-1 \le x \le 1$. The left side of the square maps to the line segment $x = -y^2$, $0 \le y \le 1$. Finally, the right side of the square maps to the line segment $x = (1/4)(8 - y^2)$, $1 \le$

$x \leq 2$. The below figure shows the graph for the region R_{xy} in the xy-plane.



(d) Let the transformation $x = 5u - u^2 + v^2$, $y = 5v + 10uv$ be given. The bottom side of the square maps to the line segment $0 \leq x \leq 4$, $y = 0$. The top side of the square maps to the line segment $x = -(1/100)y^2 + (3/5)y - (7/4)$, $5 \leq y \leq 15$. The left side of the square maps to the line segment $x = (1/25)y^2$, $0 \leq y \leq 5$. Finally, the right side of the square maps to the line segment $x = 4 + (1/225)y^2$, $0 \leq y \leq 15$. The below figure shows the graph for the given region R_{xy} in the xy-plane.



9. Cylindrical coordinates are defined by the transformation $x = r\cos\theta$, $y = r\sin\theta$, $z =$

z. Hence, we find

$$J = \left| \frac{\partial (x, y, z)}{\partial (r, \theta, z)} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

And so by (4.66)

$$\iiint\limits_{R_{xyz}} f(x, y, z)\; dx\, dy\, dz = \iiint\limits_{R_{r\theta z}} F(r, \theta, z) \left| \frac{\partial (x, y, z)}{\partial (r, \theta, z)} \right| dr\, d\theta\, dz = \iiint\limits_{R_{r\theta z}} F(r, \theta, z)\, r\, dr\, d\theta\, dz$$

The element of volume is approximately a rectangular box with sides $r\Delta\theta$, Δr and Δz: $\Delta V \sim r\Delta\theta\Delta r\Delta z$.

10. Spherical coordinates are defined by the transformation $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Hence, we find

$$J = \left| \frac{\partial (x, y, z)}{\partial (\rho, \phi, \theta)} \right| = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi$$

And so by (4.66)

$$\begin{aligned} \iiint\limits_{R_{xyz}} f(x, y, z)\; dx\, dy\, dz &= \iiint\limits_{R_{\rho\phi\theta}} F(\rho, \phi, \theta) \left| \frac{\partial (x, y, z)}{\partial (\rho, \phi, \theta)} \right| d\rho\, d\phi\, d\theta \\ &= \iiint\limits_{R_{\rho\phi\theta}} F(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{aligned}$$

The element of volume is approximately a rectangular box with sides $\rho\Delta\phi$, $\Delta\rho$ and $\rho\sin\phi\Delta\theta$: $V \sim \rho^2\sin\phi\Delta\phi\Delta\rho\Delta\theta$.

11. Let cylindrical coordinates be given: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

    (a)

$$\begin{aligned} \iiint\limits_{R_{xyz}} x^2 y\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 y\, dx\, dy\, dz &= \int_0^1 \int_0^{2\pi} \int_0^1 r^4 \cos^2\theta \sin\theta\, dr\, d\theta\, dz \\ &= \iiint\limits_{R_{r\theta z}} r^4 \cos^2\theta \sin\theta\, dr\, d\theta\, dz \end{aligned}$$

    where R_{xyz} is the region $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

(b)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1+x+y} \left(x^2 - y^2\right) \, dz \, dy \, dx = \int_0^{\pi/2} \int_0^1 \int_0^{1+r(\cos\theta+\sin\theta)} r^3 \cos 2\theta \, dz \, dr \, d\theta$$

12. Let spherical coordinates be given by $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$.

(a)

$$\iiint\limits_{R_{xyz}} x^2 y \, dx \, dy \, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^5 \sin^4\phi \cos^2\theta \sin\theta \, d\rho \, d\phi \, d\theta$$

where R_{xyz} is the spherical region $x^2 + y^2 + z^2 \leq a^2$.

(b) From the bounds $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ follows $0 \leq \theta \leq 2\pi$. From the bound $\sqrt{x^2+y^2} \leq z \leq 1$ follows $0 \leq \phi \leq \pi/4$, $0 \leq \rho \leq \sec\phi$. Hence,

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \left(x^2 + y^2 + z^2\right) \, dz \, dy \, dx = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

Section 4.7

1. (a) Let $x = a\cos\theta$, $y = a\sin\theta$. Then by (4.71) the length of the circumference of the circle is given by

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = a \int_0^{2\pi} d\theta = 2\pi a$$

(b) Let $x = a(1-t^2)/(1+t^2)$, $y = 2at/(1+t^2)$. Then by (4.70) the length of the circumference of the circle is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = 2\int_{-1}^1 \sqrt{\left[-\frac{4at}{(1+t^2)^2}\right]^2 + \left[\frac{2a\,(1-t^2)}{(1+t^2)^2}\right]^2} \, dt$$
$$= 4a \int_{-1}^1 \frac{1}{1+t^2} \, dt = 4a \tan^{-1} \Big|_{-1}^1 = 2\pi a$$

2. (a) Let the equation $z = \pm\sqrt{a^2 - x^2 - y^2}$ be given. Furthermore let us use the substitution $x = r\cos\theta$, $y = r\sin\theta$ so that $J = r$. Then by (4.72) and (4.64) the

area of the surface of a sphere is given by

$$S = \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= 2\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\, dx\, dy$$

$$= 2a\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy = 2a\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 4\pi a\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr = 2\pi a\int_0^{a^2} u^{-1/2}\, du = 4\pi a\sqrt{u}\Big|_0^{a^2} = 4\pi a^2$$

where for the last step we have made use of the substitution $u = a^2 - r^2$.

(b) Let the parametric equations $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ be given. Then by (4.74) and (4.75) the area of the surface of a sphere is given by

$$S = \iint_{R_{\phi\theta}} \sqrt{EG - F^2}\, d\phi\, d\theta = a^2\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = -a^2\int_0^{2\pi} \cos\phi\Big|_0^{\pi}\, d\theta$$

$$= 2a^2\int_0^{2\pi} d\theta = 4\pi a^2$$

3. (a) Let a surface in space be given by the parametric equations $x = (b + a\cos v)\cos u$, $y = (b + a\cos v)\sin u$, $z = a\sin v$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$, where a and b are constants, $0 < a < b$ (torus). Then by (4.74) and (4.75) the surface area is given by

$$S = \iint_{R_{uv}} \sqrt{EG - F^2}\, du\, dv = a\int_0^{2\pi}\int_0^{2\pi} (b + a\cos v)\, du\, dv$$

$$= a\int_0^{2\pi} (b + a\cos v)\, u\Big|_0^{2\pi}\, dv = 2\pi a\int_0^{2\pi} (b + a\cos v)\, dv$$

$$= 2\pi a\left[bv + a\sin v\right]_0^{2\pi} = 4\pi^2 ab$$

(b) Let a surface in space be given by the parametric equations $x = u\cos v$, $y = u\sin v$, $z = u^2\sin 2v$, $0 \le u \le 1$, $0 \le v \le \pi/2$ (portion of saddle surface $z = 2xy$). Then by (4.74) and (4.75) the surface area is given by

$$S = \iint_{R_{uv}} \sqrt{EG - F^2}\, du\, dv = \int_0^1\int_0^{\pi/2} u\sqrt{1 + 4u^2}\, dv\, du = \int_0^1 uv\sqrt{1 + 4u^2}\Big|_0^{\pi/2} du$$

$$= \frac{\pi}{2}\int_0^1 u\sqrt{1 + 4u^2}\, du = \frac{\pi}{16}\int_1^5 \sqrt{w}\, dw = \frac{\pi}{24}w^{3/2}\Big|_1^5$$

$$= \frac{\pi\left(5\sqrt{5} - 1\right)}{24}$$

4. Let us use the parametric equations $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ to approximate the surface area of earth. The area of the United States then may be approximated by the rectangle bounded by the parallels 30°N and 47°N and meridians 75°W and 122°W. Furthermore, we take the radius of the earth to be 4000 miles, such that $a = 4000$. Hence, by (4.74) we find for the area of the United States

$$A = \iint\limits_{R_{\phi\theta}} \sqrt{EG - F^2}\, d\phi\, d\theta = 4000^2 \int_{75^\circ}^{122^\circ} \int_{30^\circ}^{47^\circ} \sin\phi\, d\phi\, d\theta = -\left(4000\right)^2 \int_{75^\circ}^{122^\circ} \cos\phi\big|_{30^\circ}^{47^\circ}\, d\theta$$
$$= -\left(4000\right)^2 \left(\cos 47^\circ - \cos 30^\circ\right) \int_{75^\circ}^{122^\circ} d\theta = \left(4000\right)^2 \left(\cos 47^\circ - \cos 30^\circ\right) \left(\frac{47^\circ\pi}{180^\circ}\right)$$
$$\approx 2415332\,\text{mi}^2$$

5. (a) Let the vectors **a** and **b** be the sides of a parallelogram in space. Then by (1.35), the area of the parallelogram is given by $A = |\mathbf{a} \times \mathbf{b}|$, where by (1.20) the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**. Next, let the vector **c** be a unit vector (i.e. $|\mathbf{c}| = 1$) perpendicular to an arbitrary plane C in space. Hence, by (1.9) and (1.12) the dot product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = |\mathbf{a} \times \mathbf{b}| \cos\theta = A\cos\theta = \text{comp}_c\, \mathbf{a} \times \mathbf{b}$, where $\theta = \sphericalangle(\mathbf{a} \times \mathbf{b}, \mathbf{c})$, $0 \le \theta \le \pi$. As such, we conclude that this dot product equals plus or minus the area of the projection of the parallelogram on the plane C.

   (b) As already discussed for part (a), the area of the parallelogram is given by $S = A = |\mathbf{a} \times \mathbf{b}|$ and the angle between the vectors $\mathbf{a} \times \mathbf{b}$ and **c** as $\gamma = \theta = \sphericalangle(\mathbf{a} \times \mathbf{b}, \mathbf{c})$, $0 \le \gamma \le \pi$. Hence, by (1.9) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = S\cos\gamma$.

   (c) Let $S_{yz} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{i}$, $S_{zx} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{j}$, $S_{xy} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{k}$ be the areas of the projections of the parallelogram on the yz-plane, zx-plane and xy-plane respectively. Then by (1.13) we find

$$\sqrt{S_{yz}^2 + S_{zx}^2 + S_{xy}^2} = \sqrt{\left([\mathbf{a} \times \mathbf{b}]_x\right)^2 + \left([\mathbf{a} \times \mathbf{b}]_y\right)^2 + \left([\mathbf{a} \times \mathbf{b}]_z\right)^2} = |\mathbf{a} \times \mathbf{b}| = S$$

6. Let a surface of revolution be given by rotating a curve $z = f(x)$, $y = 0$ in the xz-plane about the z-axis.

   (a) Next, we consider the transformation $x = r\cos\theta$, $y = r\sin\theta$, thus introducing curvilinear, polar coordinates in the xy-plane. As such, we find that $y = r\sin\theta = 0 \implies \theta = 0$. And so $x = r\cos 0 = r$. Hence, by rotating $z = f(x) = f(r)$ around the z-axis one will obtain the same surface of revolution.

   (b) By (4.64) and (4.72), the surface area then is given by

$$S = \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \iint\limits_{R_{r\theta}} \sqrt{1 + f'(r)^2} r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_a^b \sqrt{1 + f'(r)^2} r\, dr\, d\theta = \int_a^b \sqrt{1 + f'(r)^2} r\theta\big|_0^{2\pi}\, dr = 2\pi \int_a^b \sqrt{1 + f'(r)^2} r\, dr$$

where $0 \leq \theta \leq 2\pi$, since the curve is rotated around the z-axis once fully to obtain the revolution of the surface and $a \leq r \leq b$, since the radius r will attain both an identical minimum value a and maximum value b in each plane perpendicular to the plane of rotation (e.g. the xz-plane).

7. By (4.72) the area of a surface $z = f(x, y)$ is given by

$$S = \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Now as (4.73) states, a surface in space can be represented parametrically by the equations

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

where u and v vary in a region R_{uv} of the uv-plane. Furthermore, since

$$J = \frac{\partial (x, y)}{\partial (u, v)}$$

it follows that (see the solution to Problem 4 following section 2.12)

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v} \qquad \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v} \qquad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u} \qquad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}$$

Next, by (4.61) and using the chain rule it follows that

$$\begin{aligned}
S &= \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \\
&= \iint\limits_{R_{uv}} \sqrt{1 + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)^2} \left|\frac{\partial (x, y)}{\partial (u, v)}\right| du\, dv \\
&= \iint\limits_{R_{uv}} \sqrt{1 + \frac{1}{J^2}\left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \frac{1}{J^2}\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)^2} \left|\frac{\partial (x, y)}{\partial (u, v)}\right| du\, dv \\
&= \iint\limits_{R_{uv}} \sqrt{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)^2}\, du\, dv \\
&= \iint\limits_{R_{uv}} \sqrt{EG - F^2}\, du\, dv
\end{aligned}$$

where E, F, G are given by (4.75).

8. If u, v are curvilinear coordinates in a plane area R_{xy} then this implies that $\partial z/\partial u = \partial z/\partial v \equiv 0$. Hence, we see from the second equality of the previous problem that (4.74) reduces to

$$\begin{aligned}
S &= \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \\
&= \iint\limits_{R_{uv}} \sqrt{1 + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)^2} \left|\frac{\partial (x,y)}{\partial (u,v)}\right| du\, dv \\
&= \iint\limits_{R_{uv}} \left|\frac{\partial (x,y)}{\partial (u,v)}\right| du\, dv = \iint\limits_{R_{xy}} dx\, dy
\end{aligned}$$

9. Let a surface $z = f(x,y)$ be given in implicit form $F(x,y,z) = 0$. Then by (2.59) we find that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

And so (4.72) becomes

$$\begin{aligned}
S = \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy &= \iint\limits_{R_{xy}} \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dx\, dy \\
&= \iint\limits_{R_{xy}} \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dx\, dy
\end{aligned}$$

Section 4.8

1. As stated in the text, the error integral is given by

$$\int_0^\infty e^{-x^2}\, dx$$

One way to evaluate this is to use the equations

$$\left(\int_0^\infty e^{-x^2}\, dx\right)^2 = \int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2}\, dy = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dx\, dy = \iint\limits_{R} e^{-x^2-y^2}\, dx\, dy$$

where R is the unbounded closed region in the positive quadrant of the xy-plane. Next, we note that $f(x,y) = e^{-x^2-y^2} > 0$ and is continuous outside and on a circle

$x^2+y^2=a^2$. Hence, the double integral above can be expressed as the sum of two double integrals:

$$\iint\limits_{R} e^{-x^2-y^2}\,dx\,dy = \iint\limits_{R_1} e^{-x^2-y^2}\,dx\,dy + \iint\limits_{R_2} e^{-x^2-y^2}\,dx\,dy$$

Here the two regions R_1, R_2 overlap only on the boundary $x^2+y^2=a^2$ and the region $R_2=R_k$ is the region $a^2\le x^2+y^2\le k^2$ with $k\to\infty$. Introducing polar coordinates and noting that $|J|=r$, then by (4.61) we can evaluate the square of the error integral as

$$\begin{aligned}
\left(\int_0^\infty e^{-x^2}\,dx\right)^2 &= \iint\limits_{R} e^{-x^2-y^2}\,dx\,dy = \iint\limits_{R_1} e^{-x^2-y^2}\,dx\,dy + \iint\limits_{R_2} e^{-x^2-y^2}\,dx\,dy \\
&= \int_0^{\pi/2}\int_0^a e^{-r^2} r\,dr\,d\theta + \lim_{k\to\infty}\int_0^{\pi/2}\int_a^k e^{-r^2} r\,dr\,d\theta \\
&= \int_0^{\pi/2} -\frac{1}{2}e^{-r^2}\Big|_0^a\,d\theta + \int_0^{\pi/2}\lim_{k\to\infty} -\frac{1}{2}e^{-r^2}\Big|_a^k\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2}\left(1-e^{-a^2}\right)d\theta + \frac{1}{2}\int_0^{\pi/2}\lim_{k\to\infty}\left(e^{-a^2}-e^{-k^2}\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi/2}\left(1-\lim_{k\to\infty}e^{-k^2}\right)d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4}
\end{aligned}$$

where we have used the substitution $u=-r^2$, so that $du=-2r\,dr$. From this it follows that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

2. Let the integral

$$\iint\limits_{R} \ln\sqrt{x^2+y^2}\,dx\,dy$$

where R is the region $x^2+y^2\le 1$ be given. Now $f(x,y)=\ln\sqrt{x^2+y^2}$ is continuous in R, except at the point $(0,0)$. As such, we isolate the point $(0,0)$ by integrating up to a small circle of radius h about $(0,0)$ and then letting h approach zero. Employing polar coordinates, integration by parts and (4.61) we thus find

$$\begin{aligned}
\iint_R \ln\sqrt{x^2+y^2}\,dx\,dy &= \lim_{h\to 0}\int_0^{2\pi}\int_h^1 r\ln r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\lim_{h\to 0}\left[r^2\ln(r)-\frac{r^2}{2}\right]_h^1 d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\lim_{h\to 0}\left(-\frac{1}{2}-h^2\ln(h)+\frac{h^2}{2}\right)d\theta = -\frac{1}{4}\int_0^{2\pi}d\theta = -\frac{\pi}{2}
\end{aligned}$$

where the limit $\lim_{h\to 0}h^2\ln h$ is evaluated using L'Hospital's rule:

$$\lim_{h\to 0}h^2\ln h = \lim_{h\to 0}\frac{\ln h}{1/h^2} \overset{H}{=} \lim_{h\to 0}\frac{1/h}{-2/h^3} = \lim_{h\to 0}-\frac{h^2}{2} = 0$$

3. (a) Let the integral

$$\iiint\limits_R \frac{1}{r^p}\,dx\,dy\,dz \qquad r = \sqrt{x^2+y^2+z^2},\ p > 0$$

where R is the spherical region $x^2+y^2+z^2 \leq 1$ be given. As for the previous problem, we integrate up to a small circle of radius h about $(0,0,0)$ and then let h approach zero. Employing spherical coordinates and (4.61) we thus find

$$\begin{aligned}
\lim_{h\to 0}\iiint\limits_R \frac{1}{r^p}\,dx\,dy\,dz &= \lim_{h\to 0}\int_0^{2\pi}\int_0^{\pi}\int_h^1 \frac{1}{r^p} r^2\sin\phi\,dr\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\lim_{h\to 0}\left.\frac{r^{3-p}}{3-p}\right|_h^1 \sin\phi\,d\phi\,d\theta \\
&= \lim_{h\to 0}\frac{1-h^{3-p}}{3-p}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta \\
&= \lim_{h\to 0}\frac{1-h^{3-p}}{3-p}\int_0^{2\pi} -\cos\phi\big|_0^{\pi}\,d\theta = \lim_{h\to 0}\frac{1-h^{3-p}}{3-p}\int_0^{2\pi} 2\,d\theta \\
&= \lim_{h\to 0}\frac{4\pi\left(1-h^{3-p}\right)}{3-p}
\end{aligned}$$

From this it follows that the integral converges to the value $4\pi/(3-p)$ for $p < 3$ and diverges for $p \geq 3$.

(b) For the integral of part (a), let R be the exterior region $x^2+y^2+z^2 \geq 1$, thus forming an unbounded closed region. Since $f(x,y,z) = 1/r^p \geq 1$ (i.e. is positive for the region given), the integral over R can be defined as the limit

$$\begin{aligned}
\lim_{k\to\infty}\iiint\limits_R \frac{1}{r^p}\,dx\,dy\,dz &= \lim_{k\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_1^k \frac{1}{r^p} r^2\sin\phi\,dr\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\lim_{k\to\infty}\left.\frac{r^{3-p}}{3-p}\right|_1^k \sin\phi\,d\phi\,d\theta \\
&= \lim_{k\to\infty}\frac{k^{3-p}-1}{3-p}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta \\
&= \lim_{k\to\infty}\frac{k^{3-p}-1}{3-p}\int_0^{2\pi} -\cos\phi\big|_0^{\pi}\,d\theta = \lim_{k\to\infty}\frac{k^{3-p}-1}{3-p}\int_0^{2\pi} 2\,d\theta \\
&= \lim_{k\to\infty}\frac{4\pi\left(k^{3-p}-1\right)}{3-p}
\end{aligned}$$

From this it follows that the integral converges to the value $4\pi/(3-p)$ for $p > 3$.

4. (a) Let the integral

$$\iint\limits_R \frac{1}{x^2+y^2}\,dx\,dy$$

where R is the square $|x| < 1$, $|y| < 1$ be given. Since R is an unbounded open region the integral will diverge.

(b) Let the integral

$$\iint\limits_R \frac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}\, dx\, dy$$

where R is the circle $x^2 + y^2 \leq 1$ be given. Here the region R is a unbounded closed region and $f(x,y) = \ln(x^2+y^2)/\sqrt{x^2+y^2} \leq 0$ and continuous in R. As such, employing polar coordinates, integration by parts and (4.61) the integral over R can be defined as the limit

$$\begin{aligned}\lim_{h\to 0} \iint\limits_R \frac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}\, dx\, dy &= \lim_{h\to 0} \int_0^{2\pi} \int_h^1 \ln r^2\, dr\, d\theta = \int_0^{2\pi} \lim_{h\to 0} \left[r\ln r^2 - 2r\right]_h^1\, d\theta \\ &= 2\pi \lim_{h\to 0} \left(2h - h\ln h^2 - 2\right) = -4\pi\end{aligned}$$

where the limit $\lim_{h\to 0} h\ln h^2$ is evaluated using L'Hospital's rule:

$$\lim_{h\to 0} h\ln h^2 = \lim_{h\to 0} \frac{\ln h^2}{1/h} \overset{H}{=} \lim_{h\to 0} \frac{2/h}{-1/h^2} = \lim_{h\to 0} -2h = 0$$

Hence, the integral converges.

(c) Let the integral

$$\iint\limits_R \ln\left(x^2+y^2\right)\, dx\, dy$$

where R is the region $x^2+y^2 \geq 1$ be given. Here R is an unbounded closed region and $f(x,y) = \ln(x^2+y^2) \geq 0$ and continuous in R. Again, employing polar coordinates, integration by parts and (4.61) the integral over R can be defined as the limit

$$\begin{aligned}\lim_{k\to\infty} \iint\limits_R \ln\left(x^2+y^2\right)\, dx\, dy &= \lim_{k\to\infty} \int_0^{2\pi} \int_1^k r\ln r^2\, dr\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \lim_{k\to\infty} \left[r^2\ln r^2 - r^2\right]_1^k\, d\theta \\ &= \pi \lim_{k\to\infty} \left(k^2\ln k^2 - k^2 + 1\right) = -\infty\end{aligned}$$

where the limit $\lim_{k\to\infty} k^2\ln k^2$ is evaluated using L'Hospital's rule:

$$\lim_{k\to\infty} k^2\ln k^2 = \lim_{k\to\infty} \frac{\ln k^2}{1/k^2} = \lim_{k\to\infty} \frac{2/k}{-2/k^3} = \lim_{k\to\infty} -k^2 = -\infty$$

Hence, the integral diverges.

(d) Let the integral

$$\iint\limits_R \frac{\sqrt{x^2+xy+y^2}}{x^2+y^2}\,dx\,dy$$

where R is the region $x^2+y^2 \leq 1$ be given. Here the region R is a unbounded closed region and $f(x,y) = \sqrt{x^2+xy+y^2}/(x^2+y^2) \geq 0$ and continuous in R. Again, employing polar coordinates and (4.61) the integral over R can be defined as the limit

$$\begin{aligned}
\lim_{h\to 0}\iint\limits_R \frac{\sqrt{x^2+xy+y^2}}{x^2+y^2}\,dx\,dy &= \lim_{h\to 0}\int_0^{2\pi}\int_h^1 \sqrt{1+\sin\theta\cos\theta}\,dr\,d\theta \\
&= \int_0^{2\pi} \lim_{h\to 0}\sqrt{1+\sin\theta\cos\theta}\,r\Big|_h^1\,d\theta \\
&= \int_0^{2\pi} \sqrt{1+\sin\theta\cos\theta}\lim_{h\to 0}\,(1+h)\,d\theta \\
&= \int_0^{2\pi} \sqrt{1+\sin\theta\cos\theta}\,d\theta
\end{aligned}$$

The resulting ordinary integral in θ has a finite value, hence, the original double integral converges.

(e) Let the integral

$$\iiint\limits_R \ln\left(x^2+y^2+z^2\right)\,dx\,dy\,dz$$

where R is the solid $x^2+y^2+z^2 \leq 1$ be given. Here R is an unbounded closed region and $f(x,y,z) = \ln(x^2+y^2+z^2) \leq 0$ and continuous in R. Employing spherical coordinates, integration by parts and (4.66) the integral over R can be defined as the limit

$$\begin{aligned}
\lim_{h\to 0}\iiint\limits_R \ln\left(x^2+y^2+z^2\right)\,dx\,dy\,dz &= \lim_{h\to 0}\int_0^{2\pi}\int_0^{\pi}\int_h^1 \rho^2\sin\phi\left(\ln\rho^2\right)\,d\rho\,d\phi\,d\theta \\
&= \frac{2}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\lim_{h\to 0}\left[\rho^3\ln\rho - \frac{\rho^3}{3}\right]_h^1\,d\phi\,d\theta \\
&= \frac{2}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\lim_{h\to 0}\left(\frac{h^3}{3} - h^3\ln h - \frac{1}{3}\right)\,d\phi\,d\theta \\
&= -\frac{2}{9}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{2}{9}\int_0^{2\pi}\cos\Big|_0^{\pi}\,d\theta \\
&= -\frac{4}{9}\int_0^{2\pi}\,d\theta = -\frac{8\pi}{9}
\end{aligned}$$

Hence, the integral converges.

Section 4.9

1. (a)
$$\frac{d}{dt}\int_{\pi/2}^{\pi}\frac{\cos xt}{x}\,dx=\int_{\pi/2}^{\pi}\frac{\partial}{\partial t}\frac{\cos xt}{x}\,dx=-\int_{\pi/2}^{\pi}\sin xt\,dx$$

(b)
$$\frac{d}{dt}\int_{1}^{2}\frac{x^2}{(1-tx)^2}\,dx=\int_{1}^{2}\frac{\partial}{\partial t}\frac{x^2}{(1-tx)^2}\,dx=2\int_{1}^{2}\frac{x^3}{(1-tx)^3}\,dx$$

(c)
$$\frac{d}{du}\int_{1}^{2}\ln(xu)\,dx=\int_{1}^{2}\frac{\partial}{\partial u}\ln(xu)\,dx=\int_{1}^{2}\frac{dx}{u}$$

(d)
$$\frac{d^n}{dy^n}\int_{1}^{2}\frac{\sin x}{x-y}\,dx=\int_{1}^{2}\frac{\partial^n}{\partial y^n}\frac{\sin x}{x-y}\,dx=n!\int_{1}^{2}\frac{\sin x}{(x-y)^{n+1}}\,dx$$

2. (a) By (4.10)
$$\frac{d}{dx}\int_{1}^{x}t^2\,dt=x^2$$

(b) By (4.10) and the first term of (4.95)
$$\frac{d}{dt}\int_{1}^{t^2}\sin x^2\,dx=\frac{d}{dt}\int_{1}^{b(t)}\sin x^2\,dx=\sin\left[b(t)\right]^2 b'(t)=2t\sin t^4$$

(c) By (4.10) and the first term of (4.95)
$$\begin{aligned}\frac{d}{dt}\int_{t^3}^{2}\ln\left(1+x^2\right)\,dx=-\frac{d}{dt}\int_{2}^{t^3}\ln\left(1+x^2\right)\,dx&=-\frac{d}{dt}\int_{2}^{b(t)}\ln\left(1+x^2\right)\,dx\\&=-3t^2\ln\left(1+t^6\right)\end{aligned}$$

(d) By (4.10) and the first term of (4.95)
$$\begin{aligned}\frac{d}{dx}\int_{x}^{\tan x}e^{-t^2}\,dt&=\frac{d}{dx}\int_{x}^{a}e^{-t^2}\,dt+\frac{d}{dx}\int_{a}^{\tan x}e^{-t^2}\,dt\\&=-\frac{d}{dx}\int_{a}^{x}e^{-t^2}\,dt+\frac{d}{dx}\int_{a}^{\tan x}e^{-t^2}\,dt\\&=e^{-\tan^2 x}\sec^2 x-e^{-x^2}\end{aligned}$$

3. (a) By (4.95)

$$\begin{aligned}\frac{d}{d\alpha}\int_{\sin\alpha}^{\cos\alpha}\ln(x+\alpha)\;dx &= -\sin\alpha\ln(\cos\alpha+\alpha)-\cos\alpha\ln(\sin\alpha+\alpha)\\ &\quad+\int_{\sin\alpha}^{\cos\alpha}\frac{\partial}{\partial\alpha}\ln(x+\alpha)\;dx\\ &= -\sin\alpha\ln(\cos\alpha+\alpha)-\cos\alpha\ln(\sin\alpha+\alpha)+\int_{\sin\alpha}^{\cos\alpha}\frac{dx}{x+\alpha}\\ &= -\sin\alpha\ln(\cos\alpha+\alpha)-\cos\alpha\ln(\sin\alpha+\alpha)+\ln(x+\alpha)\Big|_{\sin\alpha}^{\cos\alpha}\\ &= \ln\frac{\cos\alpha+\alpha}{\sin\alpha+\alpha}-[\sin\alpha\ln(\cos\alpha+\alpha)+\cos\alpha\ln(\sin\alpha+\alpha)]\end{aligned}$$

(b) By (4.94) and (4.95) and using integration by parts

$$\begin{aligned}\frac{d}{du}\int_0^{\pi/(2u)}u\sin ux\,dx &= -\frac{\pi}{2u}+\int_0^{\pi/(2u)}\frac{\partial}{\partial u}u\sin ux\,dx\\ &= -\frac{\pi}{2u}+\int_0^{\pi/(2u)}(\sin ux+ux\cos ux)\;dx\\ &= -\frac{\pi}{2u}-\frac{\cos ux}{u}\bigg|_0^{\pi/(2u)}+x\sin ux\Big|_0^{\pi/(2u)}+\frac{\cos ux}{u}\bigg|_0^{\pi/(2u)}=0\end{aligned}$$

(c) By (4.95)

$$\begin{aligned}\frac{d}{dy}\int_y^{y^2}e^{-x^2y^2}\;dx &= 2ye^{-y^6}-e^{-y^4}+\int_y^{y^2}\frac{\partial}{\partial y}e^{-x^2y^2}\;dx\\ &= 2ye^{-y^6}-e^{-y^4}-2y\int_y^{y^2}x^2e^{-x^2y^2}\;dx\end{aligned}$$

4. (a) Using the fact that when $f(x)=a^x$ then $f'(x)=a^x\ln a$ and that $\int_0^1 x^n\,dx = 1/(n+1)$, $n>-1$ we find

$$\begin{aligned}\frac{d}{dn}\int_0^1 x^n\,dx &= \frac{d}{dn}\frac{1}{n+1}\\ \int_0^1\frac{d}{dn}x^n\,dx &= -\frac{1}{(n+1)^2}\\ \int_0^1 x^n\ln x\,dx &= -\frac{1}{(n+1)^2}\end{aligned}$$

(b) Assuming $a>0$ we firstly note that

$$\int_0^\infty e^{-ax}\,dx=\lim_{b\to\infty}\int_0^b e^{-ax}\,dx=-\frac{1}{a}\lim_{b\to\infty}e^{-ax}\Big|_0^b=-\frac{1}{a}\lim_{b\to\infty}\left(e^{-ab}-1\right)=\frac{1}{a}$$

and hence,

$$\frac{d^n}{da^n}\int_0^\infty e^{-ax}\,dx = \int_0^\infty \frac{\partial^n}{\partial a^n}e^{-ax}\,dx = \frac{d^n}{da^n}\frac{1}{a}$$
$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

(c) Using the substitution $u = x^2$, we firstly note that (assuming $x > 0$ and so $u > 0$)

$$\begin{aligned}\int_0^\infty \frac{dy}{u+y^2} = \lim_{b\to\infty}\int_0^b \frac{dy}{u+y^2} = \frac{1}{x^2}\lim_{b\to\infty}\int_0^b \frac{dy}{1+y^2/u} &= \frac{1}{\sqrt{u}}\lim_{b\to\infty}\tan^{-1}\frac{y}{\sqrt{u}}\bigg|_0^b \\ &= \frac{1}{\sqrt{u}}\lim_{b\to\infty}\tan^{-1}\frac{b}{\sqrt{u}} \\ &= \frac{\pi}{2}\frac{1}{\sqrt{u}}\end{aligned}$$

Next, we find

$$\frac{d^{n-1}}{du^{n-1}}\int_0^\infty \frac{dy}{u+y^2} = \int_0^\infty \frac{\partial^{n-1}}{\partial u^{n-1}}\frac{dy}{u+y^2} = (-1)^{n-1}(n-1)!\int_0^\infty \frac{dy}{(u+y^2)^n}$$

for $n = 1, 2, \ldots$. Also

$$\frac{d^{n-1}}{du^{n-1}}\frac{\pi}{2}\frac{1}{\sqrt{u}} = (-1)^{n-1}\frac{\pi}{2}\frac{(n-1)!\,[1\times 3\times\cdots\times(2n-3)]}{2\times 4\times\cdots\times(2n-2)}\frac{1}{\sqrt{u^{2n-1}}}$$

Equating both terms and substituting back for x then finally gives

$$\int_0^\infty \frac{dy}{(x^2+y^2)^n} = \frac{\pi}{2}\frac{1\times 3\times\cdots\times(2n-3)}{2\times 4\times\cdots\times(2n-2)}\frac{1}{x^{2n-1}}$$

5. (a) Firstly, we note that (assuming $a > 0$)

$$\int \frac{dx}{x^2+a} = \int \frac{dx}{1+x^2/a} = \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + C_1$$

Hence,

$$\begin{aligned}\frac{d^{n-1}}{da^{n-1}}\int \frac{dx}{x^2+a} = \int \frac{\partial^{n-1}}{\partial a^{n-1}}\frac{dx}{x^2+a} - C &= \frac{\partial^{n-1}}{\partial a^{n-1}}\left(\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + C_1\right) \\ (-1)^{n-1}(n-1)!\int \frac{dx}{(x^2+a)^n} &= \frac{\partial^{n-1}}{\partial a^{n-1}}\left(\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right) + C \\ \int \frac{dx}{(x^2+a)^n} &= \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial a^{n-1}}\left(\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right) + C\end{aligned}$$

(b) Firstly, we note that

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C_1$$

Hence,

$$\frac{d^n}{da^n}\int \cos ax\, dx = \int \frac{\partial^n}{\partial a^n}\cos ax\, dx - C = \frac{\partial^n}{\partial a^n}\left(\frac{\sin ax}{a} + C_1\right)$$
$$\int a^n \cos ax\, dx = \frac{\partial^n}{\partial a^n}\frac{\sin ax}{a} + C$$

for $n = 4, 8, 12, \ldots$.

(c) Let $\int f(x,t)\, dx = F(x,t) + C$, so that $\partial F/\partial x = f(x,t)$. Substituting for $f(x,t)$ in Eq. (a) then gives

$$\begin{aligned}
\frac{\partial}{\partial t}\int f\left(x,t\right)\, dx + C &= \int \frac{\partial}{\partial t} f\left(x,t\right)\, dx \\
\frac{\partial}{\partial t}\int \frac{\partial F}{\partial x}\, dx + C &= \int \frac{\partial}{\partial x}\frac{\partial F}{\partial t}\, dx \\
\int \frac{\partial}{\partial t}\frac{\partial F}{\partial x}\, dx + C &= \int \frac{\partial}{\partial x}\frac{\partial F}{\partial t}\, dx \\
\int \frac{\partial^2 F}{\partial t \partial x}\, dx + C &= \int \frac{\partial^2 F}{\partial x \partial t}\, dx
\end{aligned}$$

This last form is equivalent to the indefinite integral of

$$\frac{\partial^2 F}{\partial t \partial x} = \frac{\partial^2 F}{\partial x \partial t}$$

6. The integral

$$\int_0^{2\pi} \frac{\cos\theta}{1 - a\cos\theta}\, d\theta$$

where a is a constant, $0 < a < 1$, may be re-written as

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos\theta}{1 - a\cos\theta}\, d\theta &= \frac{1}{a}\int_0^{2\pi} \frac{d\theta}{1 - a\cos\theta} - \frac{1}{a}\int_0^{2\pi} d\theta \\
&= \frac{1}{a}\int_0^{2\pi} \frac{d\theta}{1 - a\cos\theta} - \frac{2\pi}{a}
\end{aligned}$$

In order to solve the remaining integral we make the substitution $u = \tan\theta/2 \iff \theta = 2\tan^{-1} u$ such that $d\theta = 2\, du/(1+u^2)$ and

$$\cos\theta = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{1 - u^2}{1 + u^2}$$

Hence,

$$\int \frac{d\theta}{1 - a\cos\theta} = \int \frac{2\,du}{(1+u^2)\left(1 - a\frac{1-u^2}{1+u^2}\right)} = \int \frac{2\,du}{(1+a)\,u^2 + 1 - a} = \frac{2}{1-a}\int \frac{du}{\frac{1+a}{1-a}u^2 + 1}$$

Since $0 < a < 1$ we can make the second substitution $v = \sqrt{(1+a)/(1-a)}u \implies dv = \sqrt{(1+a)/(1-a)}\,du$ so that

$$\frac{2}{1-a}\int \frac{du}{\frac{1+a}{1-a}u^2 + 1} = \frac{2}{\sqrt{(1-a)\,(1+a)}}\int \frac{dv}{v^2+1} = \frac{2}{\sqrt{1-a^2}}\tan^{-1} v + C$$

A such, we find

$$\frac{1}{a}\int_0^{2\pi} \frac{d\theta}{1 - a\cos\theta} = \frac{2}{a\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\tan\frac{\theta}{2}\right)\Bigg|_0^{2\pi} = \frac{2\pi}{a\sqrt{1-a^2}}$$

and so we finally may conclude that

$$\int_0^{2\pi} \frac{\cos\theta}{1 - a\cos\theta}\,d\theta = 2\pi\frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}}$$

Next, let $g(a)$ be defined as $g(a) = \int_0^{2\pi} \ln(1 - a\cos\theta)\,d\theta$. Then $g'(a) = -\int_0^{2\pi} \cos\theta/(1 - a\cos\theta)\,d\theta$. Hence, using the substitution $u = \sqrt{1-a^2}$ so that $du = -a(1-a^2)^{-1/2}\,da$:

$$\begin{aligned} g\left(a\right) = \int g'\left(a\right)\,da = -\int\int_0^{2\pi} \frac{\cos\theta}{1 - a\cos\theta}\,d\theta\,da = -2\pi\int \frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}}\,da \\ = 2\pi\int \frac{da}{a} - 2\pi\int \frac{da}{a\sqrt{1-a^2}} = 2\pi\ln a + C_1 - 2\pi\int \frac{du}{1-u^2} \\ = 2\pi\ln a + C_1 - \pi\int \frac{du}{1-u} - \pi\int \frac{du}{1+u} \\ = 2\pi\ln a - \pi\ln\left(1 - \sqrt{1-a^2}\right) + \pi\ln\left(1 + \sqrt{1-a^2}\right) + C \\ = 2\pi\ln\left(1 + \sqrt{1-a^2}\right) + C \end{aligned}$$

Using the continuity of g for $a = 0$ and $g(0) = 0$ it follows that $C = -2\pi\ln 2$. As such, we find

$$\int_0^{2\pi} \ln\left(1 - a\cos\theta\right)\,d\theta = 2\pi\ln\left(1 + \sqrt{1-a^2}\right) - 2\pi\ln 2 = 2\pi\ln\frac{1+\sqrt{1-a^2}}{2}$$

7. Let $f(x,t)$ be a scalar associated with the flow of a fluid along the x-axis. The Stokes derivative then is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t} = v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}$$

As the problem states, a piece of the fluid occupying an interval $a_0 \leq x \leq b_0$ when $t = 0$ will occupy an interval $a(t) \leq x \leq b(t)$ at time t, where $da/dt = v(a,t)$, $db/dt = v(b,t)$. The integral

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\, dx$$

is then an integral of f over a definite piece of the fluid, whose position varies with time. Using (4.60), this integral can be written as an integral over the region $a_0 \leq x \leq b_0$ as

$$\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a_0}^{b_0} f(x_0,t)\, \frac{dx}{dx_0}\, dx_0$$

Differentiating both sides of this equation and applying Leibnitz's rule and the chain rule (since $f(x,t) = f[\phi(t),t]$ so that $df/dt = \partial f/\partial t + (\partial f/\partial \phi)(d\phi/dt)$) then gives

$$\begin{aligned}
\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx &= \frac{d}{dt}\int_{a_0}^{b_0} f(x_0,t)\, \frac{dx}{dx_0}\, dx_0 = \int_{a(t)}^{b(t)} \frac{d}{dt}\left(f(x_0,t)\, \frac{dx}{dx_0}\right) dx_0 \\
&= \int_{a_0}^{b_0} \left(\frac{\partial f}{\partial t}(x_0,t)\, \frac{dx}{dx_0} + \frac{\partial f}{\partial x_0}\frac{dx_0}{dt}\frac{dx}{dx_0} + f(x_0,t)\, \frac{d}{dt}\frac{dx}{dx_0}\right) dx_0 \\
&= \int_{a_0}^{b_0} \left(\frac{\partial f}{\partial t}(x_0,t)\, \frac{dx}{dx_0} + v\frac{\partial f}{\partial x_0} + f(x_0,t)\, \frac{dv}{dx_0}\right) dx_0 \\
&= \int_{a_0}^{b_0} \left[\frac{\partial f}{\partial t}(x_0,t)\, \frac{dx}{dx_0} + \frac{\partial}{\partial x_0}(fv)\right] dx_0 \\
&= \int_{a(t)}^{b(t)} \left[\frac{\partial f}{\partial t}(x,t) + \frac{\partial}{\partial x_0}(fv)\, \frac{dx_0}{dx}\right] dx = \int_{a(t)}^{b(t)} \left[\frac{\partial f}{\partial t}(x,t) + \frac{\partial}{\partial x}(fv)\right] dx \\
&= \int_{a(t)}^{b(t)} \left(\frac{Df}{Dt} + f\frac{dv}{dx}\right) dx
\end{aligned}$$

8. Let $f(\alpha)$ be continuous for $0 \leq \alpha \leq 2\pi$ and let

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \alpha)}\, d\alpha$$

for $r < 1$, r and θ being polar coordinates. Next, let $w = 1 + r^2 - 2r\cos(\theta - \alpha)$ and $v(r,\theta,\alpha) = (1 - r^2)w^{-1}$. Applying Leibnitz's rule to the right-hand side of the integral equation above and using (2.138) we find

$$\begin{aligned}
\nabla^2 u = \frac{1}{2\pi}\nabla^2 \int_0^{2\pi} f(\alpha)\, v\, d\alpha &= \frac{1}{2\pi}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\int_0^{2\pi} f(\alpha)\, v\, d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) v\, d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, \nabla^2 v\, d\alpha
\end{aligned}$$

Next, we compute all required partial derivatives:

$$\begin{aligned}
v_r &= -2rw^{-1} - \left(1-r^2\right) w^{-2} w_r \\
v_{rr} &= -2w^{-1} + 2rw^{-2}w_r + 2rw^{-2}w_r + 2\left(1-r^2\right) w^{-3} w_r^2 - \left(1-r^2\right) w^{-2} w_{rr} \\
v_{\theta\theta} &= 2\left(1-r^2\right) w^{-3} w_\theta^2 - \left(1-r^2\right) w^{-2} w_{\theta\theta}
\end{aligned}$$

which gives

$$\begin{aligned}
\nabla^2 v = v_{rr} + \frac{v_{\theta\theta}}{r^2} + \frac{v_r}{r} &= -4w^{-1} + \left(5r - r^{-1}\right) w^{-2} w_r \\
&\quad + \left(r^2 - 1\right)\left(w^{-2}w_{rr} - 2w^{-3}w_r^2 + r^{-2}w^{-2}w_{\theta\theta} - 2r^{-2}w^{-3}w_\theta^2\right)
\end{aligned}$$

Multiplying both sides by $r^2 w^3$ and substituting for w, $w_r \ldots$ then gives

$$\begin{aligned}
r^2 w^3 \nabla^2 v &= -4r^2w^2 + \left(5r^2 - 1\right) rww_r + \left(r^2 - 1\right)\left(r^2 ww_{rr} - 2r^2 w_r^2 + ww_{\theta\theta} - 2w_\theta^2\right) \\
&= (-4 + 10 + 2 - 8)\, r^6 + (16 - 30 - 4 + 16 + 2)\, r^5 \cos(\theta - \alpha) \\
&\quad + (-8 - 8 + 8 + 8)\, r^4 + (16 - 4 + 4 - 16)\, r^3 \cos(\theta - \alpha) + (-4 - 2 - 2 + 8)\, r^2 \\
&\quad + (2 - 2)\, r \cos(\theta - \alpha) \\
&= 0
\end{aligned}$$

In conclusion, since $r^2 w^3 \nabla^2 v = 0$ so will $\nabla^2 v = 0$.

Section 4.11

1. (a) Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in V^n. Then we have

$$\begin{aligned}
|\mathbf{u}| = |\mathbf{u} - \mathbf{v} + \mathbf{v}| \le |\mathbf{u} - \mathbf{v}| + |\mathbf{v}| &\implies |\mathbf{u}| - |\mathbf{v}| \le |\mathbf{u} - \mathbf{v}| \\
|\mathbf{v}| = |\mathbf{v} - \mathbf{u} + \mathbf{u}| \le |\mathbf{v} - \mathbf{u}| + |\mathbf{u}| &\implies |\mathbf{u}| - |\mathbf{v}| \ge -|\mathbf{u} - \mathbf{v}|
\end{aligned}$$

And so

$$-|\mathbf{u} - \mathbf{v}| \le |\mathbf{u}| - |\mathbf{v}| \le |\mathbf{u} - \mathbf{v}| \implies \big||\mathbf{u}| - |\mathbf{v}|\big| \le |\mathbf{u} - \mathbf{v}|$$

(b) Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector in E^n, where $0 \le x_1, x_2, \ldots, x_n < \infty$. Then it follows immediately that

$$f(\mathbf{x}) = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

it a continuous mapping of E^n into E^1. In order to determine if this mapping is uniformly continuous, we look for a $\delta > 0$ such that $c = |\mathbf{x}_2 - \mathbf{x}_1| < \delta$ implies $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$ for two different vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in E^n. If $|\mathbf{x}_2 - \mathbf{x}_1| = c$, then, using part (a), $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| = \big||\mathbf{x}_1| - |\mathbf{x}_2|\big| \le |\mathbf{x}_1 - \mathbf{x}_2| = c$. Hence, since $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \le c$, we can always find a $\delta > 0$ such that $c < \delta$ implies $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$ and so we conclude that the mapping $f(\mathbf{x})$ is in fact a uniformly continuous mapping of E^n into E^1 for $0 \le x_1, x_2, \ldots, x_n < \infty$.

2. (a) By Theorem K, the function $y = f(x) = e^x$, $0 \le x \le 1$ is uniformly continuous, since $y = f(x)$ is a continuous mapping of the bounded closed set $G : 0 \le x \le 1$ into E^1.

(b) The function $y = f(x) = \ln x$, $0 < x \le 1$, is a continuous mapping of the set $G : 0 < x \le 1$ into E^1. However, this mapping is *not* uniformly continuous. In order to prove this, let us take $\epsilon = 1$ and ask how close x_1, x_2 must be so that $|f(x_1) - f(x_2)| < 1$. Substituting for $f(x)$ gives

$$|f(x_1) - f(x_2)| = |\ln x_1 - \ln x_2| < 1 \qquad \text{or} \qquad \frac{x_1}{x_2} < e,\ 0 < x \le 1$$

But we can choose x_1, x_2 as close together as we wish while x_1/x_2 is arbitrarily large. Thus there is no $\delta > 0$ such that $c = |x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < 1$. For instance, let us approximate $c = |x_1 - x_2|$ as $c = |x_1 - x_1^2|$, while letting $x_1 \to 0$. Clearly $c \to 0$, so that x_1, x_2 are as close together as possibly desired. However,

$$\lim_{x_1 \to 0} \frac{x_1}{x_2} = \lim_{x_1 \to 0} \frac{x_1}{x_1^2} = \lim_{x_1 \to 0} \frac{1}{x_1} = \infty > e$$

(c) The function $y = f(x) = \ln x$, $1 \le x < \infty$, is a continuous mapping of the set $G : 1 \le x < \infty$ into E^1. This mapping is also uniformly continuous, since for the set $G : 1 \le x < \infty$ we can always find a $\delta > 0$ such that $c = |x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \epsilon$. To this end, let us define $x_2 = x_1 + h$, where $0 < h < \delta$, so that $c = h < \delta$. Again, taking $\epsilon = 1$ and substituting for $f(x)$ gives

$$|f(x_1) - f(x_2)| = |\ln x_1 - \ln(x_1 + h)| < 1 \qquad \text{or} \qquad \frac{x_1}{x_1 + h} < e,\ 1 \le x < \infty$$

Now we let $x_1 \to \infty$ and let $h \to 0$ and take limits to get

$$\lim_{\substack{x_1 \to \infty \\ h \to 0}} \frac{x_1}{x_1 + h} = \lim_{\substack{x_1 \to \infty \\ h \to 0}} \frac{1}{1 + h/x_1} = \frac{1}{1 + (0)(0)} = 1 < e$$

Similarly, we can choose to define $x_1 = x_2 + h$ and let $x_2 \to \infty$, which gives

$$\lim_{\substack{x_2 \to \infty \\ h \to 0}} \frac{x_2 + h}{x_2} = \lim_{\substack{x_2 \to \infty \\ h \to 0}} \left(1 + \frac{h}{x_2}\right) = 1 + (0)\,(0) = 1 < e$$

This concludes the prove for the set G being right-open. Furthermore, G is bounded and closed from the left and hence, by Theorem K and the above analysis we may conclude that $y = f(x)$ is uniformly continuous.

(d) The function $y = f(x) = \sin x$, $-\infty < x < \infty$, is a continuous mapping of the set $G : -\infty < x < \infty$ onto E^1. Now since the function $f(x) = \sin x$ is periodic with period 2π, it suffices to show that $f(x)$ is uniformly continuous for the bounded closed (compact) set $H \subset G : 0 \leq x \leq 2\pi$. But then by Theorem K it follows immediately that $y = f(x)$ is uniformly continuous.

3. Let $f(x)$ be defined for $a < x < b$, $f'(x)$ be continuous for $a < x < b$ and let $f'(x)$ be bounded. Then since $f'(x)$ is bounded, there exists a scalar $K > 0$ such that $|f'(x)| \leq K$. Hence, utilising the mean value theorem: $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$, where $x_1 < \xi < x_2$ we find $|f(x_1) - f(x_2)| \leq K|x_2 - x_1| = Kc$. As such, choosing $c < \epsilon/K$ we can always find a $\delta > 0$, such that $c < \delta$ implies $|f(x_1) - f(x_2)| < \epsilon$, i.e. δ depends only on ϵ and the bound $K > 0$ and *not* on the point x.

4. Let $f(x, y)$ have continuous partial derivatives in the domain D in the xy-plane and let $|\nabla f| \leq K$, where K is a constant.

(a) Let us first consider the case when D is the domain $x^2 + y^2 < 1$. Next, let s be the distance along an arbitrary line segment from (x_1, y_1) to (x_2, y_2), where (x_1, y_1), $(x_2, y_2) \in D$, and let the unit vector

$$\mathbf{u} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

be in the direction of increasing s along this line segment. Then by (2.120) f has directional derivative $df/ds = \nabla f \cdot \mathbf{u}$ along the line segment. Furthermore, note that

$$\left|\frac{df}{ds}\right| = |\nabla f \cdot \mathbf{u}| \leq |\nabla f||\mathbf{u}| = |\nabla f|$$

where $|\nabla f \cdot \mathbf{u}| \leq |\nabla f||\mathbf{u}|$ follows from (1.107). Next, let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and let s range from 0 to 1, such that $P_1 + s(P_2 - P_1)$ traces out the line segment from (x_1, y_1) to (x_2, y_2). Hence, along the line segment the function f may be written as $f(x, y) = f[P_1 + s(P_2 - P_1)] = f(s)$, $0 \leq s \leq 1$. As for the previous problem, utilising the mean value theorem: $f(s_2) - f(s_1) = f'(s^*)(s_2 - s_1)$, where $s_1 < s^* < s_2$, we find

$$|f(s_1) - f(s_2)| = |f'(s^*)||s_2 - s_1| \leq |\nabla f||s_2 - s_1| \leq K|s_2 - s_1| = Kc$$

As such, choosing $c < \epsilon/K$ we can always find a $\delta > 0$, such that $c < \delta$ implies $|f(s_1) - f(s_2)| < \epsilon$, i.e. δ depends only on ϵ and the bound $K > 0$ and *not* on the point (x, y). Note that uniform continuity of f holds by the fact that the domain D is simply connected. In other words, it is always possible to choose two point (x_1, y_1), (x_2, y_2) in D, such that all the points on the line segment from (x_1, y_1) to (x_2, y_2) are in D as well, and hence, that the mean-value theorem in terms of distance along the line segment is applicable.

(b) When D is the domain $|x| < 1$, $|y| < 1$, excluding the points $(x, 0)$ for $0 \leq x \leq 1$ the function f need not be uniformly continuous, as a line segment from (x_1, y_1) to (x_2, y_2) may intersect the line traced out by the points $(x, 0)$, $0 \leq x \leq 1$, which is not in D. If this indeed is the case, then there will exist an s^*, $s_1 < s^* < s_2$ such that the corresponding point $P^* \notin D$. Hence, at the point P^*, $|\nabla f|$ need not be equal or less than K necessarily and as such, f might not be uniformly continuous.

5. Let $f(x)$ and $f'(x)$ be continuous for the interval $a \leq x \leq b$. Furthermore, let $|f'(x)| \leq K = \text{const}$ for $a \leq x \leq b$, such that $f'(x)$ is bounded. Utilising the fact that $f'(x)$ is bounded in conjunction with the mean value theorem (see Problem 3), we find that $|f(x') - f(x'')| \leq K|x'' - x'|$, assuming x', x'' are on the interval $a \leq x \leq b$. Next, let us require (as is done in the text) that $|f(x') - f(x'')| < \epsilon/[2(b-a)]$. Hence, we want to choose $|x'' - x'| = c$ such that $c < \delta = \epsilon/[2K(b-a)]$. Then we can proceed in the same way as done in the text (last part of the proof for Theorem L) to show that each sum $\sum f(x_i^*)\Delta_i x$ differs from $\int_a^b f(x)\,dx$ by less than ϵ.