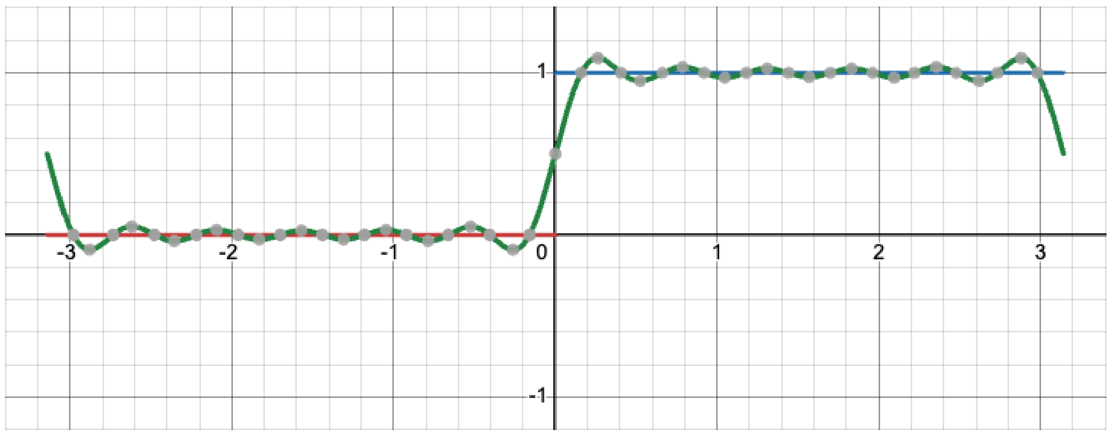
CHAPTER 7

Section 7.4

1. (a) Let the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x \le \pi \end{cases}$$

be given.



Using (7.9), the coefficients for the Fourier series of $f(x)$ are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} \cos nx\, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1, 2, 3, \ldots \end{cases}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx = -\frac{\cos nx}{n\pi}\Bigg|_0^{\pi} = \begin{cases} \frac{2}{n\pi} & \text{if } n = 1, 3, 5, \ldots \\ 0 & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

Hence, by (7.10) the Fourier series of $f(x)$ is given by

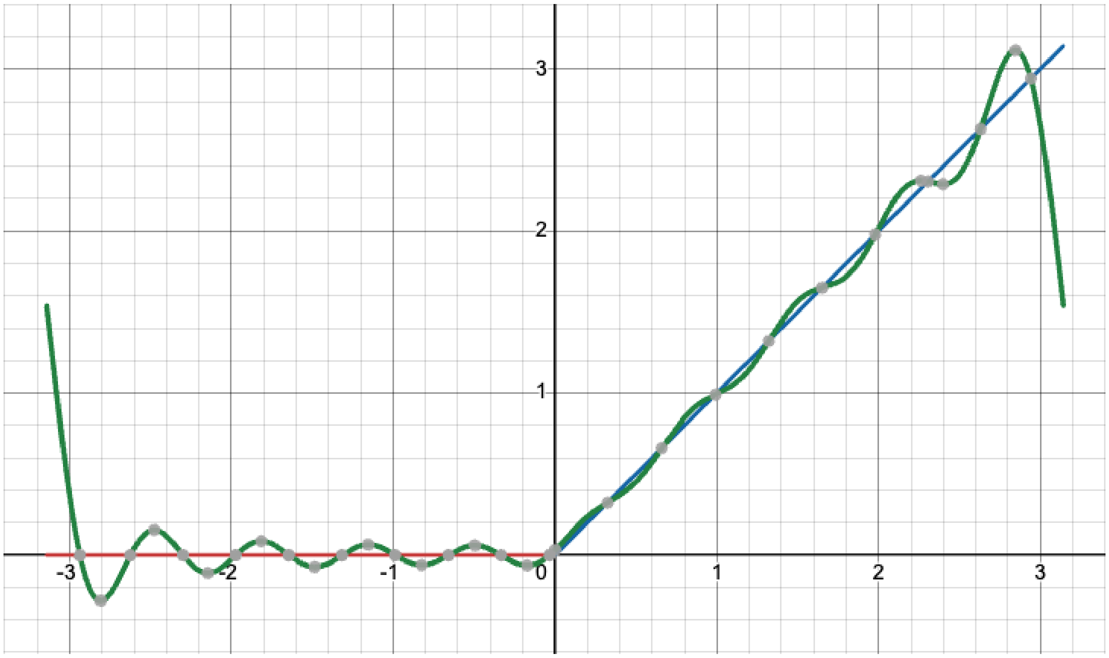
$$\frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots + \frac{2}{(2n-1)\pi}\sin(2n-1)x + \cdots$$

for $n = 1, 2, 3, \ldots$. The figure shows S_{11}, i.e. the sum of the first eleven terms of the Fourier series of $f(x)$.

(b) Let the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x \le \pi \end{cases}$$

be given.



Using (7.9) the coefficients for the Fourier series of $f(x)$ are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

For $n = 0$ the integral reduces to

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

For $n = 1, 2, 3, \ldots$ we get

$$\begin{aligned} a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{x\sin nx}{n\pi}\bigg|_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx &= \left[\frac{x\sin nx}{n\pi} + \frac{\cos nx}{n^2\pi}\right]_0^{\pi} \\ &= \frac{\cos n\pi - 1}{n^2\pi} \\ &= \begin{cases} -\frac{2}{n^2\pi} & \text{if } n = 1, 3, 5, \ldots \\ 0 & \text{if } n = 2, 4, 6, \ldots \end{cases} \end{aligned}$$

and

$$\begin{aligned} b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{x\cos nx}{n\pi}\bigg|_0^{\pi} + \frac{1}{n\pi}\int_0^{\pi}\cos nx\,dx &= \left[\frac{\sin nx}{n^2\pi} - \frac{x\cos nx}{n\pi}\right]_0^{\pi} \\ &= -\frac{\cos n\pi}{n} \\ &= -\frac{(-1)^n}{n} \end{aligned}$$

Hence, by (7.10) the Fourier series of $f(x)$ is given by

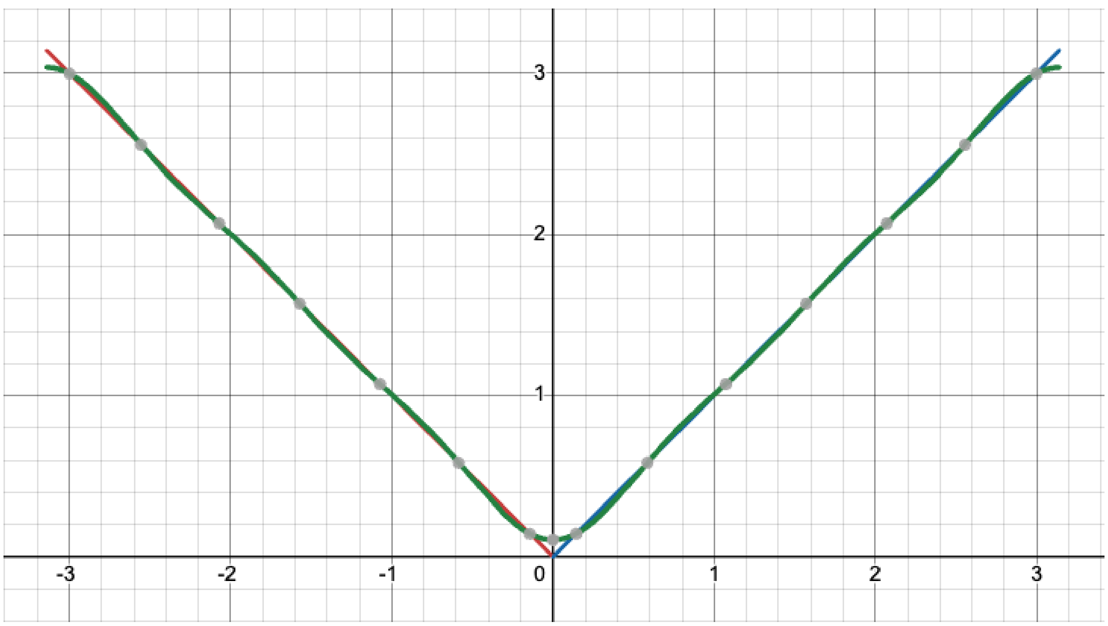
$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$$

The figure shows S_9, i.e. the sum of the first nine terms of the Fourier series of $f(x)$.

(c) Let the function

$$f(x) = \begin{cases} -x & \text{if } -\pi \leq x \leq 0 \\ x & \text{if } 0 \leq x \leq \pi \end{cases}$$

be given.



Using (7.9) the coefficients for the Fourier series of $f(x)$ are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right)$$

For $n = 0$ the integral reduces to

$$a_0 = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} x\,dx\right) = \pi$$

For $n = 1, 2, 3, \ldots$ we get

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right) \\
&= \frac{x\sin nx}{n\pi}\Big|_{0}^{-\pi} - \frac{1}{n\pi}\int_{0}^{-\pi}\sin nx\,dx + \frac{x\sin nx}{n\pi}\Big|_{0}^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi}\sin nx\,dx \\
&= \left[\frac{x\sin nx}{n\pi} + \frac{\cos nx}{n^2\pi}\right]_{0}^{-\pi} + \left[\frac{x\sin nx}{n\pi} + \frac{\cos nx}{n^2\pi}\right]_{0}^{\pi} = \begin{cases} -\frac{4}{n^2\pi} & \text{if } n = 1, 3, 5, \ldots \\ 0 & \text{if } n = 2, 4, 6, \ldots \end{cases}
\end{aligned}$$

and

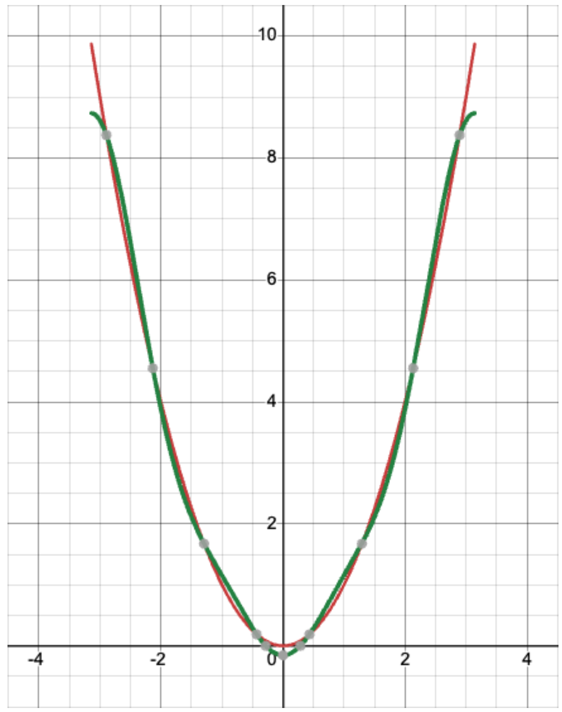
$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right) \\
&= \frac{x\cos nx}{n\pi}\Big|_{-\pi}^{0} - \frac{1}{n\pi}\int_{-\pi}^{0}\cos nx\,dx - \frac{x\cos nx}{n\pi}\Big|_{0}^{\pi} + \frac{1}{n\pi}\int_{0}^{\pi}\cos nx\,dx \\
&= \left[\frac{x\cos nx}{n\pi} - \frac{\sin nx}{n^2\pi}\right]_{-\pi}^{0} - \left[\frac{x\cos nx}{n\pi} - \frac{\sin nx}{n^2\pi}\right]_{0}^{\pi} = 0
\end{aligned}$$

Hence, by (7.10) the Fourier series of $f(x)$ is given by

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

The figure shows S_5, i.e. the sum of the first five terms of the Fourier series of $f(x)$.

(d) Let the function $f(x) = x^2$, $-\pi \le x \le \pi$ be given.



Using (7.9) the coefficients for the Fourier series of $f(x)$ are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(x\right)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

For $n = 0$ the integral reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \left.\frac{x^3}{3\pi}\right|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

For $n = 1, 2, 3, \ldots$ we get

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx &= \left.\frac{x^2\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \left.\frac{x^2\sin nx}{n\pi}\right|_{-\pi}^{\pi} + \left.\frac{2x\cos nx}{n^2\pi}\right|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos nx\,dx \\ &= \left[\frac{x^2\sin nx}{n\pi} + \frac{2x\cos nx}{n^2\pi} - \frac{2\sin nx}{n^3\pi}\right]_{-\pi}^{\pi} = (-1)^n\,\frac{4}{n^2} \end{aligned}$$

and

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx &= -\frac{x^2 \cos nx}{n\pi}\Bigg|_{-\pi}^{\pi} + \frac{2}{n^2\pi}\int_{-\pi}^{\pi} x \cos nx \, dx \\
&= -\frac{x^2 \cos nx}{n\pi}\Bigg|_{-\pi}^{\pi} + \frac{2x \sin nx}{n^2\pi}\Bigg|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi} \sin nx \, dx \\
&= \left[-\frac{x^2 \cos nx}{n\pi} + \frac{2x \sin nx}{n^2\pi} + \frac{2\cos nx}{n^3\pi}\right]_{-\pi}^{\pi} = 0
\end{aligned}
$$

Hence, by (7.10) the Fourier series of $f(x)$ is given by

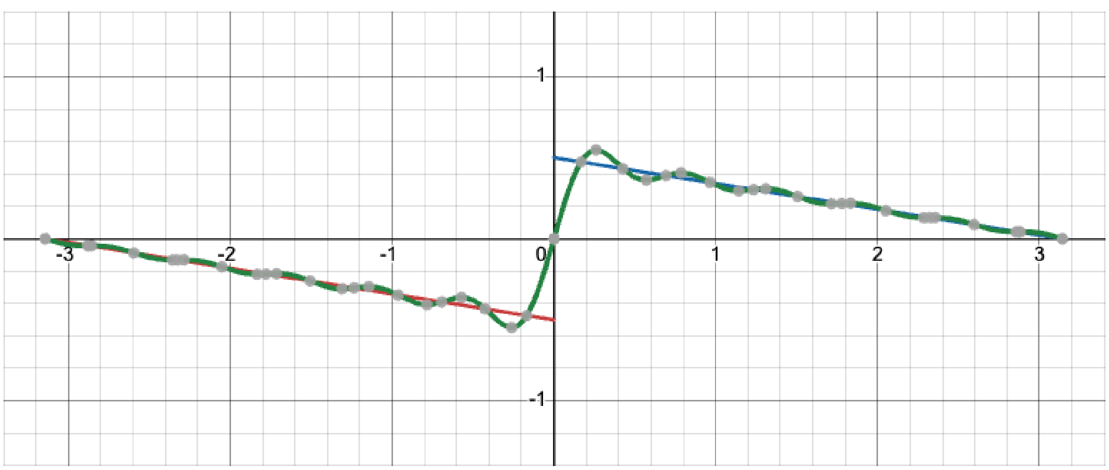
$$
\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}
$$

The figure shows S_3, i.e. the sum of the first third terms of the Fourier series of $f(x)$.

(e) Let the function

$$
F(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & \text{if } -\pi \le x < 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{2} - \frac{x}{2\pi} & \text{if } 0 < x \le \pi \end{cases}
$$

be given.



Using (7.9) the coefficients for the Fourier series of $F(x)$ are given by

$$
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos nx \, dx = -\frac{1}{2\pi}\int_{-\pi}^{0}\left(1 + \frac{x}{\pi}\right) \cos nx \, dx + \frac{1}{2\pi}\int_{0}^{\pi}\left(1 - \frac{x}{\pi}\right) \cos nx \, dx
$$

For $n = 0$ the integral reduces to

$$
a_0 = \frac{1}{2\pi}\left[\int_0^{-\pi}\left(1 + \frac{x}{\pi}\right) dx + \int_0^{\pi}\left(1 - \frac{x}{\pi}\right) dx\right] = \frac{1}{2\pi}\left[x + \frac{x^2}{2\pi}\right]_0^{-\pi} + \frac{1}{2\pi}\left[x - \frac{x^2}{2\pi}\right]_0^{\pi} = 0
$$

For $n = 1, 2, 3, \ldots$ we get

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_0^{-\pi}\left(1+\frac{x}{\pi}\right)\cos nx\,dx + \frac{1}{2\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\cos nx\,dx \\ &= \left[\frac{\sin nx}{2n\pi} + \frac{x\sin nx}{2n\pi^2} + \frac{\cos nx}{2n^2\pi^2}\right]_0^{-\pi} + \left[\frac{\sin nx}{2n\pi} - \frac{x\sin nx}{2n\pi^2} - \frac{\cos nx}{2n^2\pi^2}\right]_0^{\pi} = 0 \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{-\pi}\left(1+\frac{x}{\pi}\right)\sin nx\,dx + \frac{1}{2\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\sin nx\,dx \\ &= \left[-\frac{\cos nx}{2n\pi} - \frac{x\cos nx}{2n\pi^2} + \frac{\sin nx}{2n^2\pi^2}\right]_0^{-\pi} + \left[-\frac{\cos nx}{2n\pi} + \frac{x\cos nx}{2n\pi^2} - \frac{\sin nx}{2n^2\pi^2}\right]_0^{\pi} = \frac{1}{n\pi} \end{aligned}$$

Hence, by (7.10) the Fourier series of $F(x)$ is given by

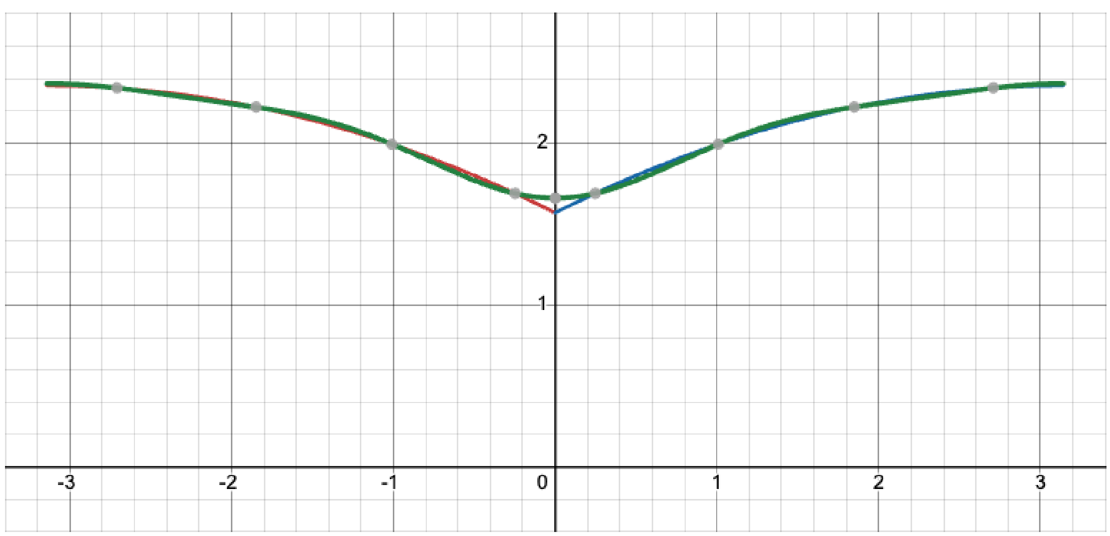
$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

The figure shows S_{11}, i.e. the sum of the first eleven terms of the Fourier series of $F(x)$.

(f) Let the function

$$G(x) = \begin{cases} \frac{\pi}{2} - \frac{x}{2} - \frac{x^2}{4\pi} & \text{if } -\pi \le x \le 0 \\ \frac{\pi}{2} + \frac{x}{2} - \frac{x^2}{4\pi} & \text{if } 0 \le x \le \pi \end{cases}$$

be given.



Using (7.9) the coefficients for the Fourier series of $G(x)$ are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} G\left(x\right)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2}-\frac{x}{2}-\frac{x^2}{4\pi}\right)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)\cos nx\,dx \end{aligned}$$

For $n=0$ the integral reduces to

$$\begin{aligned} a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(\frac{\pi}{2}-\frac{x}{2}-\frac{x^2}{4\pi}\right)\,dx+\int_{0}^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)\,dx\right] \\ &= \left[\frac{x}{2}-\frac{x^2}{4\pi}-\frac{x^3}{12\pi^2}\right]_{-\pi}^{0}+\left[\frac{x}{2}+\frac{x^2}{4\pi}-\frac{x^3}{12\pi^2}\right]_{0}^{\pi}=\frac{4\pi}{3} \end{aligned}$$

For $n=1,2,3,\ldots$ we get

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2}-\frac{x}{2}-\frac{x^2}{4\pi}\right)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)\cos nx\,dx \\ &= \left[\frac{\sin nx}{2n}-\frac{x\sin nx}{2n\pi}-\frac{\cos nx}{2n^2\pi}\right]_{-\pi}^{0}-\frac{1}{4\pi^2}\int_{-\pi}^{0}x^2\cos nx\,dx \\ &\quad+\left[\frac{\sin nx}{2n}+\frac{x\sin nx}{2n\pi}+\frac{\cos nx}{2n^2\pi}\right]_{0}^{\pi}-\frac{1}{4\pi^2}\int_{0}^{\pi}x^2\cos nx\,dx \\ &= \frac{(-1)^n-1}{n^2\pi}+\frac{1}{4\pi^2}\left(\int_{0}^{-\pi}x^2\cos nx\,dx-\int_{0}^{\pi}x^2\cos nx\,dx\right) \\ &= \cdots+\left.\frac{x^2\sin nx}{4n\pi^2}\right|_{0}^{-\pi}-\frac{1}{2n\pi^2}\int_{0}^{-\pi}x\sin nx\,dx-\left.\frac{x^2\sin nx}{4n\pi^2}\right|_{0}^{\pi}+\frac{1}{2n\pi^2}\int_{0}^{\pi}x\sin nx\,dx \\ &= \cdots+\left.\frac{x\cos nx}{2n^2\pi^2}\right|_{0}^{-\pi}-\frac{1}{2n^2\pi^2}\int_{0}^{-\pi}\cos nx\,dx-\left.\frac{x\cos nx}{2n^2\pi^2}\right|_{0}^{\pi}+\frac{1}{2n^2\pi^2}\int_{0}^{\pi}\cos nx\,dx \\ &= -\frac{1}{n^2\pi}-\left.\frac{\sin nx}{2n^3\pi^2}\right|_{0}^{-\pi}+\left.\frac{\sin nx}{2n^3\pi^2}\right|_{0}^{\pi}=-\frac{1}{n^2\pi} \end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2}-\frac{x}{2}-\frac{x^2}{4\pi}\right)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)\sin nx\,dx \\
&= \left[\frac{\cos nx}{2n}-\frac{x\cos nx}{2n\pi}+\frac{\sin nx}{2n^2\pi}\right]_0^{-\pi} - \frac{1}{4\pi^2}\int_{-\pi}^{0} x^2\sin nx\,dx \\
&\quad + \left[-\frac{\cos nx}{2n}-\frac{x\cos nx}{2n\pi}+\frac{\sin nx}{2n^2\pi}\right]_0^{\pi} - \frac{1}{4\pi^2}\int_{0}^{\pi} x^2\sin nx\,dx \\
&= \frac{1}{4\pi^2}\left(\int_0^{-\pi} x^2\sin nx - \int_0^{\pi} x^2\sin nx\,dx\right) \\
&= -\frac{x^2\cos nx}{4n\pi^2}\Bigg|_0^{-\pi} + \frac{1}{2n\pi^2}\int_0^{-\pi} x\cos nx\,dx + \frac{x^2\cos nx}{4n\pi^2}\Bigg|_0^{\pi} - \frac{1}{2n\pi^2}\int_0^{\pi} x\cos nx\,dx \\
&= \frac{x\sin nx}{2n^2\pi^2}\Bigg|_0^{-\pi} - \frac{1}{2n^2\pi^2}\int_0^{-\pi}\sin nx\,dx - \frac{x\sin nx}{2n^2\pi^2}\Bigg|_0^{\pi} + \frac{1}{2n^2\pi^2}\int_0^{\pi}\sin nx\,dx \\
&= \frac{\cos nx}{2n^3\pi^2}\Bigg|_0^{-\pi} - \frac{\cos nx}{2n^3\pi^2}\Bigg|_0^{\pi} = 0
\end{aligned}
$$

Hence, by (7.10) the Fourier series of $F(x)$ is given by

$$\frac{2\pi}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

The figure shows S_3, i.e. the sum of the first eleven terms of the Fourier series of $G(x)$.

2. (a)