CHAPTER 5

Section 5.3

1. (a) From the given end points $(0,0)$, $(2,2)$ it follows that we can represent the curve C in the form $y = x$, $0 \le x \le 2$. Hence, by (5.6) we find

$$\int_{(0,0)}^{(2,2)} y^2\,dx = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

(b) Given the end points $(2,1)$, $(1,2)$ we will parameterise the curve C according to: $x = 2 - t$, $y = 1 + t$, $0 \le t \le 1$. Then by (5.4) we find

$$\int_{(2,1)}^{(1,2)} y\,dx = -\int_0^1 (1+t)\,dt = -\left[t + \frac{t^2}{2}\right]_0^1 = -\frac{3}{2}$$

(c) Given the end points $(1,1)$, $(2,1)$ we will parameterise the curve C according to $x = 1 + t$, $y = 1$, $0 \le t \le 1$. Then by (5.5) we find

$$\int_{(1,1)}^{(2,1)} x\,dy = \int_0^1 (1+t)\,(0)\,dt = 0$$

2. (a) Let us represent the curve $C : x = \sqrt{1 - y^2}$ in the form $x = \cos t$, $y = \sin t$, $-\pi/2 \le t \le \pi/2$. Then by (5.4) and (5.5)

$$\begin{aligned}
\int_{(0,-1)}^{(0,1)} y^2\,dx + x^2\,dy &= \int_{-\pi/2}^{\pi/2} -\sin^3 t\,dt + \cos^3 t\,dt \\
&= \int_{-\pi/2}^{\pi/2} -\left(1 - \cos^2 t\right)\sin t + \left(1 - \sin^2 t\right)\cos t\,dt \\
&= \left[\cos t - \frac{\cos^3 t}{3} + \sin t - \frac{\sin^3 t}{3}\right]_{-\pi/2}^{\pi/2} = \frac{4}{3}
\end{aligned}$$

(b) Let C be the parabola $y = x^2$. Then by (5.6) and (5.7) we find

$$\int_{(0,0)}^{(2,4)} y\,dx + x\,dy = \int_0^2 \left(x^2 + 2x^2\right)\,dx = \left[\frac{x^3}{3} + \frac{2}{3}x^3\right]_0^2 = 8$$

(c) Let C be the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi/2$ and let us use the substitution $u = \tan^3 t$. Then by (5.4) and (5.5) we can rewrite the integral as

$$\begin{aligned}
\int_{(1,0)}^{(0,1)} \frac{y\,dx - x\,dy}{x^2 + y^2} &= -3\int_0^{\pi/2} \frac{\sin^4 t\cos^2 t + \sin^2 t\cos^4 t}{\cos^6 t + \sin^6 t}\,dt = \int_0^{\pi/2} \frac{-3\sin^2 t\cos^2 t}{\cos^6 t + \sin^6 t}\,dt \\
&= -\int_0^{\infty} \frac{\cos^6 t}{\cos^6 t + \sin^6 t}\,du = -\int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} -\int_0^b \frac{du}{1+u^2} \\
&= \lim_{b\to\infty} -\tan^{-1} u\Big|_0^b = \lim_{b\to\infty} -\tan^{-1} b = -\frac{\pi}{2}
\end{aligned}$$

3. (a) Let C be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Then the integral

$$\oint_C y^2\,dx + xy\,dy$$

can be evaluated by computing the sum of the four integrals

$$\underbrace{\int_{(1,1)}^{(-1,1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(-1,1)}^{(-1,-1)} xy\,dy}_{dx=0} \qquad \underbrace{\int_{(-1,-1)}^{(1,-1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(1,-1)}^{(1,1)} xy\,dy}_{dx=0}$$

Hence,

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy &= \int_1^{-1} dx - \int_1^{-1} y\,dy + \int_{-1}^{1} dx + \int_{-1}^{1} y\,dy \\ &= x\Big|_1^{-1} - \frac{y^2}{2}\Big|_1^{-1} + x\Big|_{-1}^{1} + \frac{y^2}{2}\Big|_{-1}^{1} = 0\end{aligned}$$

(b) Let C be the circle $x^2 + y^2 = 1$. Using the parameterization $x = \cos t$, $y = \sin t$ where $0 \le t \le 2\pi$, then by (5.4) and (5.5) the integral

$$\oint_C y\,dx - x\,dy$$

may be written as

$$\begin{aligned}\oint_C y\,dx - x\,dy = \int_0^{2\pi} -\sin^2 t\,dt - \cos^2 t\,dt = -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\,dt &= -\int_0^{2\pi} dt \\ &= -2\pi\end{aligned}$$

(c) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Then the integral

$$\oint_C x^2y^2\,dx - xy^3\,dy$$

can be evaluated by computing the sum of the three integrals

$$\underbrace{\int_{(0,0)}^{(1,0)} x^2y^2\,dx = 0}_{dy=0} \qquad \underbrace{-\int_{(1,0)}^{(1,1)} xy^3\,dy}_{dx=0} \qquad \int_{(1,1)}^{(0,0)} x^2y^2\,dx - xy^3\,dy$$

Hence,

$$\begin{aligned}\oint_C x^2y^2\,dx - xy^3\,dy &= -\int_0^1 y^3\,dy + \int_0^1 x^4\,dx - \int_0^1 y^4\,dy \\ &= -\frac{y^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 - \frac{y^5}{5}\Big|_0^1 = -\frac{1}{4}\end{aligned}$$

4. (a) Let C be the circle $x^2+y^2=4$. Then using the parametrisation $x=4\cos t$, $y=4\sin t$, where $0\le t\le 2\pi$ and (5.12) the integral

$$\oint_C \left(x^2-y^2\right)\,ds$$

may be written as

$$\oint_C \left(x^2-y^2\right)\,ds = 64\int_0^{2\pi}\left(\cos^2 t-\sin^2 t\right)\,dt = 64\int_0^{2\pi}\cos 2t\,dt = 32\sin 2t\Big|_0^{2\pi} = 0$$

(b) Let C be the line $y=x$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) the integral

$$\int_{(0,0)}^{(1,1)} x\,ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} x\,ds = \sqrt{2}\int_0^1 x\,dx = \frac{\sqrt{2}}{2}x^2\Big|_0^1 = \frac{1}{\sqrt{2}}$$

(c) Let C be the parabola $y=x^2$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) and using the substitution $x=(1/2)\tan u$, such that $dx=(1/2)\sec^2 u\,du$ the integral

$$\int_{(0,0)}^{(1,1)} ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1\sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^{\tan^{-1}2}\sec^3 u\,du$$

In order to solve the integral on the right hand side, let us solve the indefinite integral

$$\begin{aligned}\int \sec^3 x\,dx = \int_0 \sec^2 x\sec x\,dx &= \sec x\tan x - \int \sec x\tan^2 x\,dx + C\\ &= \sec x\tan x - \int \sec x\left(\sec^2 x-1\right)\,dx + C\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx + C\end{aligned}$$

Adding the term $\int \sec^3 x\,dx$ to both sides and dividing by two then gives

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx + C\\ &= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x+\tan x| + C\end{aligned}$$

Substituting in the original equation then gives

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3 u\,du$$
$$= \frac{1}{4}\sec u \tan u\Big|_0^{\tan^{-1}2} + \frac{1}{4}\ln|\sec u + \tan u|\Big|_0^{\tan^{-1}2}$$
$$= \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5}+2\right)}{4}$$

5. Let a path $x = \phi(t)$, $y = \psi(t)$, $h \le t \le k$, where x and y are continuous and have continuous derivatives for $h \le t \le k$ like (5.1) be given. Next, let us make a change of parameter by the equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. Then by (5.4) the line integral $\int_C f(x,y)\,dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$, such that $dx = (d/d\tau)\phi(g(\tau))\,d\tau$, is given by

$$\int\limits_C f(x,y)\,dx = \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d}{d\tau}\phi\left(g\left(\tau\right)\right)\,d\tau$$
$$= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d\phi}{dt}\frac{d}{d\tau}g\left(\tau\right)\,d\tau$$
$$= \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right] \frac{d\phi}{dt}\frac{dt}{d\tau}\,d\tau = \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right]\phi'\left(t\right)\,dt$$

6. (a) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFG$ may be approximated as

$$\int\limits_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right]$$
$$+ \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] = 7$$

(b) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFGKH$ may be approximated as

$$\int\limits_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(1+4)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right]$$
$$+ \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+9)\cdot 1\right] + \left[\frac{1}{2}(0+2)\cdot 1 + \frac{1}{2}(9+8)\cdot -1\right]$$
$$= 5$$

(c) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABCDHLSONMIEA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+8)\cdot 1 + \frac{1}{2}(2+3)\cdot 0\right] \\
& + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+4)\cdot 0\right] + \left[\frac{1}{2}(5+2)\cdot 0 + \frac{1}{2}(4+8)\cdot 1\right] \\
& + \left[\frac{1}{2}(2+1)\cdot 0 + \frac{1}{2}(8+2)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 0 + \frac{1}{2}(2+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(4+3)\cdot -1 + \frac{1}{2}(6+2)\cdot 0\right] + \left[\frac{1}{2}(3+7)\cdot -1 + \frac{1}{2}(2+8)\cdot 0\right] \\
& + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] \\
& + \left[\frac{1}{2}(8+3)\cdot 0 + \frac{1}{2}(3+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
= & \ 8
\end{aligned}$$

(d) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFJNMIJFA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(4+1)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 0 + \frac{1}{2}(4+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(5+7)\cdot 0 + \frac{1}{2}(6+8)\cdot 1\right] + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] \\
& + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+6)\cdot 0\right] \\
& + \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+4)\cdot -1\right] + \left[\frac{1}{2}(0+0)\cdot -1 + \frac{1}{2}(4+1)\cdot -1\right] \\
= & \ \frac{11}{2}
\end{aligned}$$

(e) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFEAEFBA$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right] \\ & + \left[\frac{1}{2}(0+3)\cdot -1 + \frac{1}{2}(4+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\ & + \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(1+2)\cdot 1\right] + \left[\frac{1}{2}(3+0)\cdot 1 + \frac{1}{2}(2+4)\cdot 0\right] \\ & + \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(4+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot -1 + \frac{1}{2}(2+1)\cdot 0\right] \\ = & \, 0\end{aligned}$$

7. Let C be a smooth curve in the xy-plane and let $f(x, y) > 0$ be a continuous function defined over a region of the xy-plane containing the curve C. The equation $z = f(x, y)$ then is the equation of a surface that lies above the region of the xy-plane containing the curve C. Next, we imagine moving a straight line along C perpendicular to the xy-plane, effectively tracing out a "wall" standing on C, orthogonal to the xy-plane. This "wall" cuts the surface $z = f(x, y)$, forming a curve on it that lies above the curve C. In fact, the curve C may be interpreted as the projection of the surface curve onto the xy-plane. Using (5.11), the line integral

$$\int\limits_C f(x, y)\, ds = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta_i s$$

then may be interpreted as an infinite sum of the length of each straight line directed from C to the surface curve lying above it in the limit where the distance Δs between each subsequent line becomes infinitely small, effectively tracing out a "wall" with height at each point (x, y) given by $f(x, y)$. This may be interpreted the as the area of the cylindrical surface $0 \le z \le f(x, y)$, (x, y) on C.

Section 5.5

1. Let the vector $v = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$ be given. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_T\, ds$ along the path $C \to y = x$ from $(0, 0)$ to $(1, 1)$ may be evaluated as

   $$\int\limits_C v_T\, ds = \int\limits_C \left(x^2 + y^2\right)\, dx + 2xy\, dy \overset{(5.6),(5.9)}{=} \int_0^1 2x^2\, dx + \int_0^1 2y^2\, dy = \frac{4}{3}$$

   (b) The integral $\int_C v_T\, ds$ along the path $C \to y = x^2$ from $(0, 0)$ to $(1, 1)$ may be evaluated as

   $$\int\limits_C v_T\, ds = \int\limits_C \left(x^2 + y^2\right)\, dx + 2xy\, dy \overset{(5.6),(5.7)}{=} \int_0^1 \left(x^2 + 5x^4\right)\, dx = \frac{4}{3}$$

(c) The integral $\int_C v_T\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_T\,ds &= \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_{(0,0)}^{(1,0)} \left(x^2+y^2\right)\,dx + 2xy\,dy + \int_{(1,0)}^{(1,1)} \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{4}{3}\end{aligned}$$

2. Let $\mathbf{v} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be the same vector as given in Problem 1, and let $\mathbf{n}$ be the unit normal vector 90° behind the tangent vector $\mathbf{T}$ as given by (5.37). Then the normal component of $\mathbf{v}$ is given by $v_n = \mathbf{v}\cdot\mathbf{n} = (P\mathbf{i}+Q\mathbf{j})\cdot(y_s\mathbf{i} - x_s\mathbf{j}) = -Qx_s + Py_s$. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_n\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.9)}{=} \int_0^1 -2x^2\,dx + \int_0^1 2y^2\,dy = 0$$

   (b) The integral $\int_C v_n\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.7)}{=} \int_0^1 2x^5\,dx = \frac{1}{3}$$

   (c) The integral $\int_C v_n\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_n\,ds &= \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_{(0,0)}^{(1,0)} -2xy\,dx + \left(x^2+y^2\right)\,dy + \int_{(1,0)}^{(1,1)} -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_0^1 \left(1+y^2\right)\,dy = \frac{4}{3}\end{aligned}$$

3. Let the gravitational force near a point on the earth's surface be represented approximately by the vector $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = -mg\mathbf{j}$, where the y-axis points upwards. Then by (5.29) and the fact that $P(x,y) = 0$ the work done by the force $\mathbf{F}$ on a body moving in a vertical plane from height h_1 to height h_2 along any path is equal to

$$\int\limits_C F_T\,ds = \int\limits_C \left(P\cos\alpha + Q\sin\alpha\right)\,ds = \int\limits_C Q\,dy = -\int_{h_1}^{h_2} mg\,dy = -mgy\Big|_{h_1}^{h_2} = mg\left(h_1 - h_2\right)$$

4. Let the gravitational force $\mathbf{F}$ be given by $\mathbf{F} = -(kMm/r^2)(\mathbf{r}/r)$. Then in order to compute the work by the gravitational force $\mathbf{F}$ in bringing a particle to its present position r from infinite distance along the ray through the earth's center, we will represents the curve C in terms of parameter t and then use (5.34) to get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\infty^r \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt = \int_\infty^r \left(-\frac{kMm}{t^2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right)dt = \int_\infty^r -\frac{kMm}{t^2}\,dt = \left.\frac{kMm}{t}\right|_\infty^r \\ = kMm\left(\frac{1}{r}-\frac{1}{\infty}\right) = \frac{kMm}{r} \\ = -U\end{aligned}$$

where $(d\mathbf{r}/dr)\cdot(d\mathbf{r}/dr) = 1$ follows from the fact that $d\mathbf{r}/dr$ is a unit vector.

5. (a) By (5.40) the integral $\oint_C ay\,dx + bx\,dy$ may be written as

$$\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\;dx\,dy = (b-a)\,A$$

where A is the area enclosed by the curve C.

(b) By (5.40) the integral $\oint e^x \sin y\,dx + e^x\cos y\,dy$ around the rectangle with vertices $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$ may be written as

$$\oint e^x\sin y\,dx + e^x\cos y\,dy = \int_0^{\pi/2}\int_0^1 \left(e^x\cos y - e^x\cos y\right)\,dx\,dy = 0$$

(c) By (5.40) and (4.61) the integral $\oint(2x^3-y^3)\,dx + (x^3+y^3)\,dy$ around the circle $x^2+y^2=1$ may be written as

$$\oint \left(2x^3-y^3\right)\,dx + \left(x^3+y^3\right)\,dy = 3\int_0^1\int_0^{2\pi} r^3\,d\theta\,dr = 6\pi\int_0^1 r^3\,dr = \frac{3\pi}{2}$$

(d) By (5.43) and (3.31) the integral $\oint_C u_T\,ds$, where $\mathbf{u} = \mathrm{grad}(x^2y)$ and C is the circle $x^2+y^2=1$ may be written as

$$\oint_C u_T\,ds = \iint_R \mathrm{curl}_z\,\mathbf{u}\,dx\,dy = \iint_R \mathrm{curl}_z\;\mathrm{grad}\left(x^2y\right)\,dx\,dy = 0$$

(e) By (5.44) the integral $\oint_C v_n\,ds$, where $\mathbf{v} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j}$ and C is the circle $x^2+y^2=1$ ($\mathbf{n}$ being the outer normal) may be written as

$$\begin{aligned}\oint_C v_n\,ds = \iint_R \mathrm{div}\,\mathbf{v}\,dx\,dy = \iint_R \mathrm{div}\,\left[\left(x^2+y^2\right)\mathbf{i} - 2xy\mathbf{j}\right]\,dx\,dy = \iint_R (2x-2x)\,dx\,dy \\ = 0\end{aligned}$$

(f) Let $F = (x-2)^2 + y^2$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44) and (4.64)

$$\oint_C v_n\, ds = \iint_R \text{div}\ (\nabla F)\ dx\, dy = \iint_R \nabla \cdot \nabla F\, dx\, dy = \iint_R \nabla^2 F\, dx\, dy$$
$$= \iint_R \nabla^2 \left[(x-2)^2 + y^2\right]\ dx\, dy$$
$$= 4 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = 4\pi$$

(g) Let $F = \ln[(x-2)^2 + y^2]^{-1}$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44)

$$\oint_C v_n\, ds = \iint_R \text{div}\ (\nabla F)\ dx\, dy = \iint_R \nabla \cdot \nabla F\, dx\, dy = \iint \nabla^2 F\, dx\, dy$$
$$= \iint_R \nabla \ln \frac{1}{(x-2)^2 + y^2}\, dx\, dy$$
$$= 2 \iint_R \frac{x^2 - 4x + 4 - y^2 - (x-2)^2 + y^2}{\left[(x-2)^2 + y^2\right]^2}\, dx\, dxy = 0$$

(h) By (5.40) the integral $\oint_C f(x)\, dx + g(y)\, dy$ may be written as

$$\oint_C f\ (x)\ dx + g\ (y)\ dy = \iint_R \left[\frac{\partial}{\partial x} g\ (y) - \frac{\partial}{\partial y} f\ (x)\right]\ dx\, dy = 0$$

6. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of an arbitrary point (x, y) and let $\mathbf{n}$ be the outer normal to some arbitrary closed curve C. Then by (5.44)

$$\frac{1}{2}\oint_C r_n\, ds = \frac{1}{2}\oint_C \mathbf{r} \cdot \mathbf{n}\, ds = \frac{1}{2}\iint_R \text{div}\, \mathbf{r}\, dx\, dy = \frac{1}{2}\iint_R \nabla \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy$$
$$= \frac{1}{2}\iint_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy$$
$$= \iint_R dx\, dy = A$$

7. As for Problem 2(a), let the line integral $\int_{(0,-1)}^{(0,1)} y^2\, dx + x^2\, dy$, where C is the semi-circle

$x = \sqrt{1 - y^2}$ be given. Then by (5.40) and (4.64)

$$
\begin{aligned}
\oint\limits_C y^2\, dx + x^2\, dy = \iint\limits_R \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2 \right) dx\, dy &= 2 \iint\limits_R (x - y)\ dx\, dy \\
&= 2 \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} (x - y)\ dx\, dy \\
&= 2 \int_{-\pi/2}^{\pi/2} \int_{0}^{1} (\cos\theta - \sin\theta)\, r^2\, dr\, d\theta \\
&= \frac{2}{3} \int_{\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\ d\theta = \frac{4}{3}
\end{aligned}
$$

As for Problem 3(a), let the line integral $\oint_C y^2\, dx + xy\, dy$, where C is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ be given. Then by (5.40)

$$
\begin{aligned}
\oint\limits_C y^2\, dx + xy\, dy = \iint\limits_R \left[\frac{\partial}{\partial x} (xy) - \frac{\partial}{\partial y} y^2 \right] dx\, dy = - \iint\limits_R y\, dx\, dy &= - \int_{-1}^{1} \int_{-1}^{1} y\, dx\, dy \\
&= - \int_{-1}^{1} xy\Big|_{-1}^{1}\, dy = -2 \int_{-1}^{1} y\, dy \\
&= -y^2\Big|_{-1}^{1} = 0
\end{aligned}
$$

As for Problem 3(b), let the line integral $\oint_C y\, dx - x\, dy$, where C is the circle $x^2 + y^2 = 1$ be given. Then by (5.40) and (4.64)

$$
\begin{aligned}
\oint\limits_C y\, dx - x\, dy = - \iint\limits_R \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y \right) dx\, dy = -2 \iint\limits_R dx\, dy &= -2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta \\
&= - \int_0^{2\pi} d\theta = -2\pi
\end{aligned}
$$

As for Problem 3(c), let the line integral $\oint_C x^2 y^2\, dx - xy^3\, dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ be given. Then by (5.40)

$$
\begin{aligned}
\oint\limits_C x^2 y^2\, dx - xy^3\, dy &= - \iint\limits_R \left[\frac{\partial}{\partial x} \left(xy^3 \right) + \frac{\partial}{\partial y} \left(x^2 y^2 \right) \right] dx\, dy \\
&= - \iint\limits_R \left(y^3 + 2x^2 y \right) dx\, dy = - \int_0^1 \int_0^x \left(y^3 + 2x^2 y \right) dy\, dx \\
&= - \int_0^1 \left[\frac{y^4}{4} + x^2 y^2 \right]_0^x dx = -\frac{5}{4} \int_0^1 x^4\, dx = -\frac{1}{4}
\end{aligned}
$$

As for Problem 4(a), let the line integral $\oint_C (x^2-y^2)\,ds$, where C is the circle $x^2+y^2=4$ be given. Then by (5.44) and the fact that $\mathbf{n}$ may be written as $\mathbf{n}=(x\mathbf{i}+y\mathbf{j})/|x+y|$

$$\oint\limits_C (x^2-y^2)\,ds = a\iint\limits_R \text{div }(x\mathbf{i}-y\mathbf{j})\;dx\,dy = a\iint\limits_R \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}\right)\cdot(x\mathbf{i}-y\mathbf{j})\;dx\,dy = 0$$

Section 5.7

1. (a) Let

$$dF = 2xy\,dx + x^2\,dy \qquad\qquad \int_C{}_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$$

where C is the curve $y=x^{3/2}$. To determine the function $F(x,y)$ we firstly note that

$$dF = 2xy\,dx + x^2\,dy = P\,dx + Q\,dy$$

where the functions $P(x,y)$ and $Q(x,y)$ are defined and continuous in the domain D given by $\{x,y\in\mathbb{R}\mid -\infty<x,y<\infty\}$. From inspection it then follows that

$$F\,(x,y) = x^2y + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = 2xy \qquad\qquad \frac{\partial F}{\partial y} = x^2$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 2xy\,dx + x^2\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$$

is independent of path (and hence curve C given by $y=x^{3/2}$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = F\,(1,1) - F\,(0,0) = 1$$

(b) Let

$$dF = ye^{xy}\,dx + xe^{xy}\,dy \qquad\qquad \int_C \int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

where C is the curve $y = \sin^3 x$. From inspection it follows that

$$F(x, y) = e^{xy} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = ye^{xy} \qquad\qquad \frac{\partial F}{\partial y} = xe^{xy}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = ye^{xy}\,dx + xe^{xy}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

is independent of path (and hence curve C given by $y = \sin^3 x$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy = F(\pi, 0) - F(0, 0) = 0$$

(c) Let

$$dF = \frac{x\,dx + y\,dy}{(x^2 + y^2)^{3/2}} \qquad\qquad \int_C \int_{(1,0)}^{(e^{2\pi},0)} \frac{x\,dx + y\,dy}{(x^2 + y^2)^{3/2}}$$

where C is the curve $x = e^t \cos t\, y = e^t \sin t$. From inspection it follows that

$$F(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = \frac{x}{(x^2 + y^2)^{3/2}} \qquad\qquad \frac{\partial F}{\partial y} = \frac{y}{(x^2 + y^2)^{3/2}}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy$$

is independent of path (and hence curve C given by $x = e^t \cos t\, y = e^t \sin t$) and can easily be evaluated by (5.48) to have the value

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}} = F\left(e^{2\pi},0\right) - F\left(1,0\right) = 1 - e^{-2\pi}$$

2. (a) Let

$$\int_{C\ (1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2} = \int_{C\ (1,-2)}^{(3,4)} P\,dx + Q\,dy$$

where C is the line $y = 3x - 5$ be given. From inspection we can define the function $F(x,y) = -(y/x) + D$, where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{1}{x} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $x = 0$. And so by (5.48) the integral has the value $F(3,4) - F(1,-2) = -10/3$.

(b) Let

$$\int_{C\ (0,2)}^{(1,3)} \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy$$

where C is the parabola $y = 2 + x^2$ be given. From inspection we can define the function $F(x,y) = x^3/y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $y = 0$. And so by (5.48) the integral has the value $F(1,3) - F(0,2) = 1/3$.

(c) Let

$$\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy$$

where C is the circular arc $y=\sqrt{1-x^2}$, $-1 \le x \le 1$ be given. From inspection it follows that we cannot define a function $F(x,y)$ such that (5.46) holds. Hence, the given integral is not independent of path. Instead we use (5.6) and (5.7) to find

$$\begin{aligned}\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy &= \int_1^{-1} \left(2x\sqrt{1-x^2} - 1 - \frac{x^3}{\sqrt{1-x^2}} - 3x\right)\ dx \\ &= 2\end{aligned}$$

where we have made use of the fact the first, third and fourth term in the integral on the right hand side are odd and hence, will be zero when integrated from $x=1$ to $x=-1$.

(d) Let

$$\int_{C\ (0,0)}^{(\pi/4,\pi/4)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$$

where C is the curve $y = 16x^3/\pi^2$. From inspection we can define the function $F(x,y) = \tan x \tan y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \sec^2 x \tan y = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = \sec^2 y \tan x = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\{x, y \mid x, y \neq k\pi/2\}$ for $k = \pm 1, 3, 5, \ldots$. And so by (5.48) the integral has the value $F(\pi/4, \pi/4) - F(0,0) = 1$.

3. (a) Let

$$\oint_C [\sin\left(xy\right) + xy\cos\left(xy\right)]\ dx + x^2\cos\left(xy\right)\ dy = \oint_C P\, dx + Q\, dy$$

where C is the circle $x^2+y^2=1$. Now since $P(x,y)$ and $Q(x,y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x\cos\left(xy\right) - x^2 y \sin\left(xy\right)$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\, dx + Q\, dy$ when integrated on the circle $x^2+y^2=1$ is equal to 0.

(b) Let

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2 + y^2 = 4$. Furthermore, we note that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(x-1)^2 - y^2}{\left[(x-1)^2 + y^2\right]^2}$$

in the doubly connected region D with hole A at point $(1,0)$. Hence, according to the discussion in Section 5.7 and (5.57) the integral is equal to some constant k, which is the same for all curves enclosing the hole A. We will use this fact to simplify the integral $\oint P\,dx + Q\,dy$ by evaluating it on the circle $(x-1)^2 + y^2 = 1$ instead. This is permitted since both the circles $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ enclose the hole A at point $(1,0)$. Thus we can write

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C y\,dx - (x-1)\;dy \underset{(5.40)}{=} \iint_R -2\,dx\,dy = -2\pi$$

where C now denotes the circle $(x-1)^2 + y^2 = 1$ and the last step follows from Example 1 following Section 5.7.

(c) Let

$$\oint_C y^3\,dx - x^3\,dy = \oint_C P\,dx + Q\,dy$$

where C is the square $|x| + |y| = 1$. Using (5.6) and (5.7) with $x = x$ and $y = 1 - x$ for the first quadrant, $y = 1 + x$ for the second quadrant, $y = -1 - x$ for the third quadrant and $y = -1 + x$ for the fourth quadrant, this integral can be written as

$$\oint_C y^3\,dx - x^3\,dy = \int_1^0 \left(1 - 3x + 3x^2\right)\,dx + \int_0^{-1} \left(1 + 3x + 3x^2\right)\,dx$$
$$+ \int_{-1}^0 \left(-1 - 3x - 3x^2\right)\,dx + \int_0^1 \left(-1 + 3x - 3x^2\right)\,dx = -2$$

(d) Let

$$\oint_C xy^6\,dx + \left(3x^2y^5 + 6x\right)\,dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $x^2 + 4y^2 = 4$. Next, let us define the parametrisation $x = 2\cos t$, $y = \sin t$ such that the integral on the ellipse becomes

$$\int_0^{2\pi} \left(-4\sin^7 t\cos t + 12\sin^5 t\cos^3 t + 12\cos^2 t\right)\,dt$$

Using integration by parts on the first term gives

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t\Big|_0^{2\pi} - \int_0^{2\pi} 7\sin^7 t\cos t\,dt\right]$$

from which follows

$$\int_0^{2\pi} 7\sin^7 t\cos t\,dt = \frac{7}{8}\sin^8 t\Big|_0^{2\pi}$$

and so

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t - \frac{7}{8}\sin^8 t\right]_0^{2\pi} = -\frac{\sin^8 t}{2}\Bigg|_0^{2\pi} = 0$$

Evaluating the second term gives

$$\int_0^{2\pi} 12\sin^5 t\cos^3 t = 12\int_0^{2\pi} \sin^4 t\sin t\cos^3 t\,dt = 12\int_0^{2\pi} \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt$$

Now applying the substitution $u = \cos t$, such that $du = -\sin t\,dt$ the integral becomes

$$\begin{aligned}12\int \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt &= -12\int u^3\left(1-u^2\right)^2\,du\\ &= -12\int \left(u^3 - 2u^5 + u^7\right)\,du\\ &= -12\left[\frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8}\right] + C\\ &= -12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right] + C\end{aligned}$$

Evaluating at the endpoints then gives

$$-12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right]_0^{2\pi} = -\frac{1}{2} + \frac{1}{2} = 0$$

Lastly, evaluating the third term gives

$$\int_0^{2\pi} 12\cos^2 t\,dt = 12\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = 6\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 12\pi$$

(e) Let

$$\oint_C (7x - 3y + 2)\,dx + (4y - 3x - 5)\,dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $2x^2+3y^2 = 1$. Now since $P(x, y)$ and $Q(x, y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -3$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\,dx + Q\,dy$ when integrated on the ellipse $2x^2 + 3y^2$ is equal to 0.

(f) Let

$$\oint_C \frac{(e^x \cos y - 1)\ dx + e^x \sin y\, dy}{e^{2x} - 2e^x \cos y + 1} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2 + y^2 = 1$. The denominator may also be written as $e^{2x} - 2e^x \cos y + 1 = (e^x - 1)^2 + 2e^x(1 - \cos y)$, from which follows that the denominator is equal to 0 only for $x = 0$ and $y = 2n\pi$, $n = \pm 0, 1, 2, \ldots$. Hence, according to (5.53), the integral $\int P\,dx + Q\,dy$ is independent of path in any simply connected domain D not containing the points $(0, 2n\pi)$, $n = \pm 0, 1, 2, \ldots$ for

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(e^x - e^{3x}) \sin y}{(e^{2x} - 2e^x \cos y + 1)^2}$$

except at the aforementioned points. However, since the circle $x^2+y^2 = 1$ encloses the point $(0, 0)$ we are dealing with a doubly connected domain and hence, Theorem IV does not hold. As such, let us make the substitution $x = \cos t$, $y = \sin t$ where $-\pi \le t \le \pi$ so that the integral can be written as

$$\int_{-\pi}^{\pi} \frac{[1 - e^{\cos t} \cos(\sin t)] \sin t + e^{\cos t} \sin(\sin t) \cos t}{e^{2\cos t} - 2e^{\cos t} \cos(\sin t) + 1}\, dt = \int_{-\pi}^{\pi} \frac{f(t)}{g(t)}\, dt = \int_{-\pi}^{\pi} h(t)\ dt$$

Now since $f(-t) = -f(t)$ and $g(-t) = g(t)$ it follows that $h(-t) = -h(t)$ and hence, that $h(t)$ is an odd function. Using the fact that integrating an odd function on an interval $[-a, a]$ always gives 0, we may finally conclude that the integral $\int P\,dx + Q\,dy$ evaluated on the circle $x^2 + y^2$ is in fact 0.

4. Let

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_{(1,0)}^{(2,2)} P\,dx + Q\,dy$$

where C is an arbitrary path connecting the points $(1, 0)$ and $(2, 2)$ not passing through the origin. As stated in Example 2 of Section 2, $P\,dx + Q\,dy$ is a familiar differential, namely that of the polar coordinate angle θ:

$$d\theta = d\left(\tan^{-1} \frac{y}{x}\right) = \frac{-y\,dx + x\,dx}{x^2 + y^2}$$

and so we can write

$$\int_A^B \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_A^B d\theta = \theta_B - \theta_A = \text{total increase in } \theta \text{ from } A \text{ to } B$$

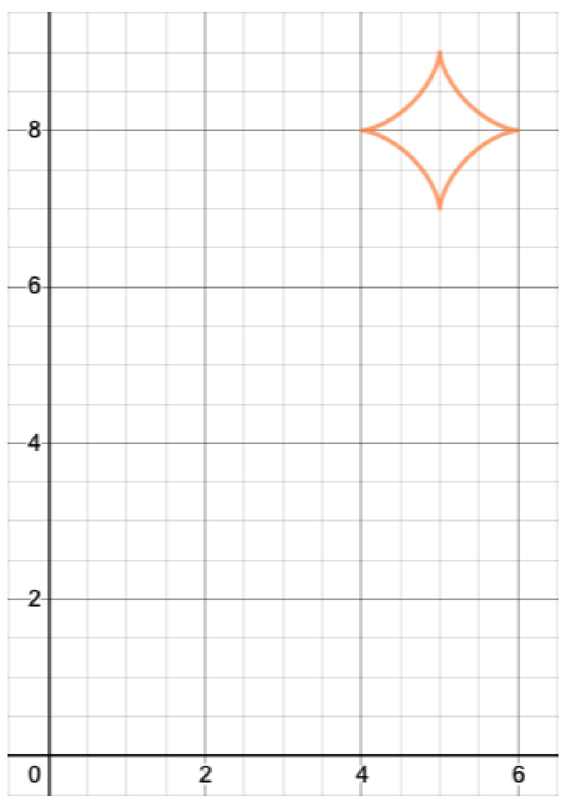
as θ varies continuously on the path C. The integral is thus not independent of path, but depends on the number of times C goes around the origin. As such we find that

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2+y^2} = \tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{0}{1}\right) + 2n\pi = \frac{\pi}{4} + 2n\pi$$

where $n = \pm 0, 1, 2, \ldots$.

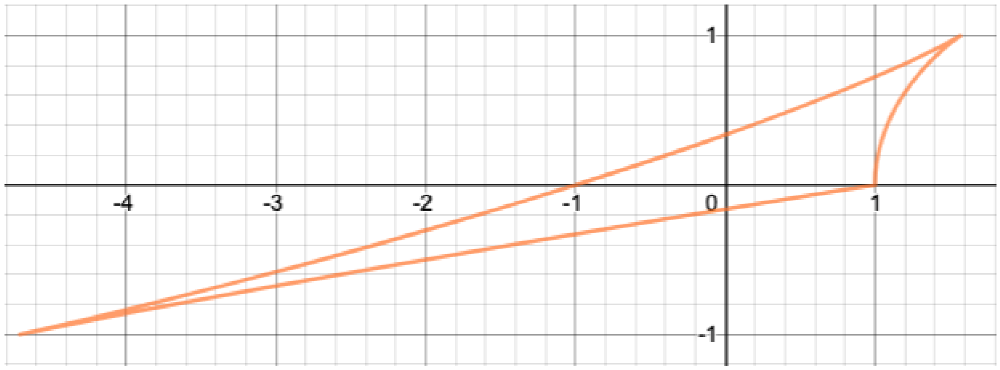
5. Let
$$\int\limits_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int\limits_C P\,dx + Q\,dy$$
where C is a path given by $x = f(t)$, $y = g(t)$, $a \le t \le b$ not passing through the origin, for which $f(a) = f(b)$, $g(a) = g(b)$. The analysis of Section 5.6 shows that $\int_C P\,dx + Q\,dy$ equals $n \cdot 2\pi$, where n is the number of times C encircles the origin. The value of n can be determined from plotting the path.

   (a) From a plot of the path $C:\ x = 5 + \cos^3 t$, $y = 8 + \sin^3 t$, $0 \le t \le 2\pi$



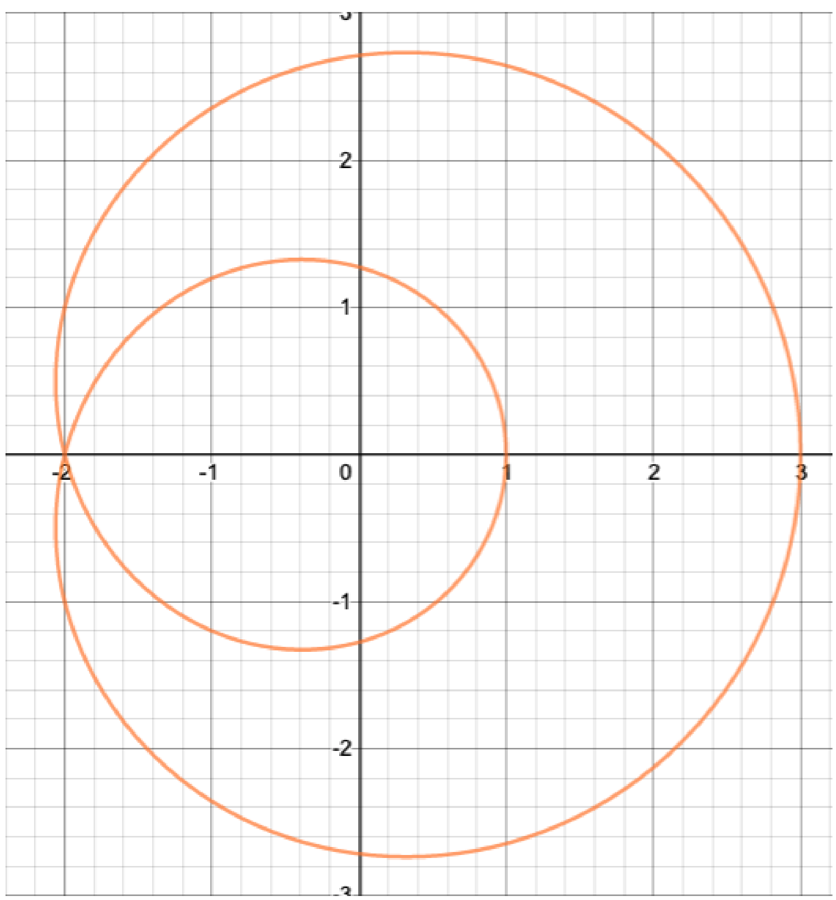
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

(b) From a plot of the path $C:\ x = \cos t + t\sin t,\ y = \sin t,\ 0 \le t \le 2\pi$



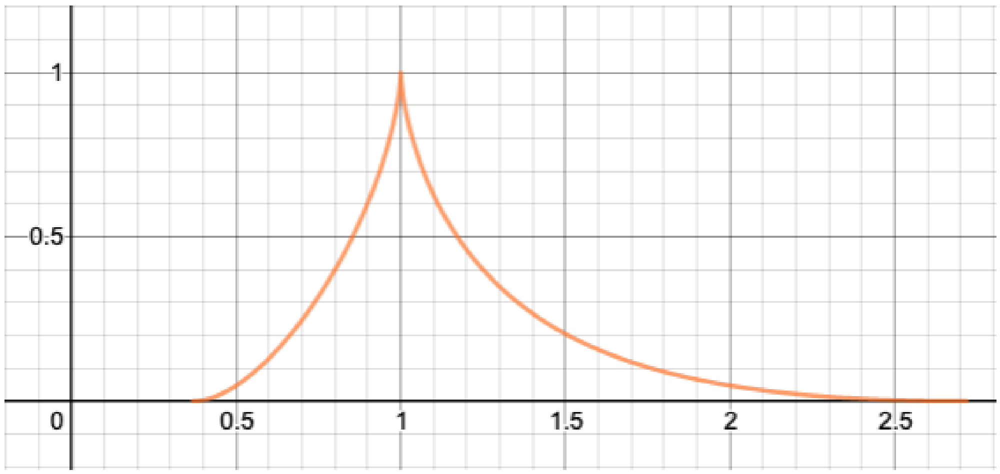
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 2π, since $n = 1$ as the path encircles the origin once.

(c) From a plot of the path $C:\ x = 2\cos 2t - \cos t,\ y = 2\sin 2t - \sin t,\ 0 \le t \le 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 4π, since $n = 2$ as the path encircles the origin twice.

(d) From a plot of the path $C: \, x = e^{\cos^3 t}$, $y = \sin^4 t$, $0 \leq t \leq 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

6. (a) Let

$$\int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 - y^2\right)\,dy = \int_{(1,1)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(1,1)$ to $(x,1)$ and $(x,1)$ to (x,y) such that the integral becomes

$$\begin{aligned} F &= \int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_{(1,1)}^{(x,1)} 2xy\,dx + \left(x^2 + y^2\right)\,dy + \int_{(x,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_1^x 2x\,dx + \int_1^y \left(x^2 + y^2\right)\,dy = x^2\Big|_1^x + \left[x^2 y + \frac{y^3}{3}\right]_1^y = x^2 y - \frac{1}{3}\left(y^3 + 2\right) \end{aligned}$$

(b) Let

$$\int_{(0,0)}^{(x,y)} \sin y\,dx + x\cos y\,dy = \int_{(0,0)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \cos y$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(0,0)$ to $(x,0)$ and $(x,0)$ to (x,y) such that the integral becomes

$$\begin{aligned}
F &= \int_{(0,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_{(0,0)}^{(x,0)} \sin y\,dy + x\cos y\,dy + \int_{(x,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_0^y x\cos y\,dy = x\sin y\big|_0^y = x\sin y
\end{aligned}$$

7. The integral

$$\oint \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} = \oint P\,dx + Q\,dy$$

is independent of path in any simply connected domain D not containing the origin, for

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{x^2\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}$$

except at the origin. Hence, the integral is 0 for any path not enclosing the origin. For the square with vertices $(\pm 1, \pm 1)$ however, the integral has a certain value k. To find k, we thus have to evaluate the integral

$$\begin{aligned}
k &= \int_{(1,1)}^{(-1,1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} + \int_{(-1,1)}^{(-1,-1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} + \int_{(-1,-1)}^{(1,-1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} \\
&\quad + \int_{(1,-1)}^{(1,1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} \\
&= \int_1^{-1} \frac{x^2\,dx}{\left(x^2+1\right)^2} + \int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} - \int_{-1}^1 \frac{x^2\,dx}{\left(x^2+1\right)^2} - \int_{-1}^1 \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\int_1^{-1} \frac{x^2\,dx}{\left(x^2+1\right)^2} + 2\int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\int_1^{-1} \left[\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right] dx + 2\int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\tan^{-1} x\big|_1^{-1} - 2\int_{\pi/4}^{-\pi/4} du + 2\int_{\pi/4}^{-\pi/4} dv = 2\tan^{-1} x\big|_1^{-1} = -\pi
\end{aligned}$$

where we have used the substitution $x = \tan u$, $y = \tan v$ such that $dx = \sec^2 u\,du$, $dy = \sec^2 v\,dv$.

8. Let D be a domain with a finite number of holes at points $A_1, A_2, \ldots, A_k$ so that D is $(k+1)$-tuply connected as in Figure 5.23. Let P and Q be continuous and have continuous derivatives in D and let $\partial P/\partial y = \partial Q/\partial x$ in D. Let C_1 denote a circle around the point A_1 in D, enclosing none of the other A's. Let C_2 be chosen similar for A_2 and so on. Futhermore, let

$$\oint_{C_1} P\,dx + Q\,dy = \alpha_1, \oint_{C_2} P\,dx + Q\,dy = \alpha_2, \ldots, \oint_{C_k} P\,dx + Q\,dy = \alpha_k$$

(a) Let C be an arbitrary simple closed path in D enclosing $A_1, A_2, \ldots, A_k$. Furthermore, we assume that the circles $C_1, C_2, \ldots, C_k$ do not intersect C at any point. Let us also define the closed region R in D whose boundaries are given by the simply closed path C and all of the circles $C_1, C_2, \ldots, C_k$ that are interior to C. Next, let us introduce auxiliary arcs from C to C_1, C to $C_2, \ldots,$ two to each so that we end up with a figure similar to Figure 5.21. These decompose the region R into $k+1$ smaller regions, each of which is simply connected (i.e. does not contain any holes in its interior). If we integrate in a positive direction around the boundary of each sub region and then add the results we find that the integrals along the auxiliary arcs cancel out, leaving just the integral around C in the positive direction plus the integrals around $C_1, C_2, \ldots, C_k$ in the negative direction. On the other hand, the line integral around the boundary of each sub region can be expressed as a double integral

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

over the sub region by Green's theorem. Hence, the sum of the line integrals is equal to the double integral over R:

$$\oint_{C} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

However, since $\partial P/\partial y = \partial Q/\partial x$ in D the integral on the right hand side is equal to zero and hence, we end up with

$$\oint_{C} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

(b) Let

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P\,dx + Q\,dy$$

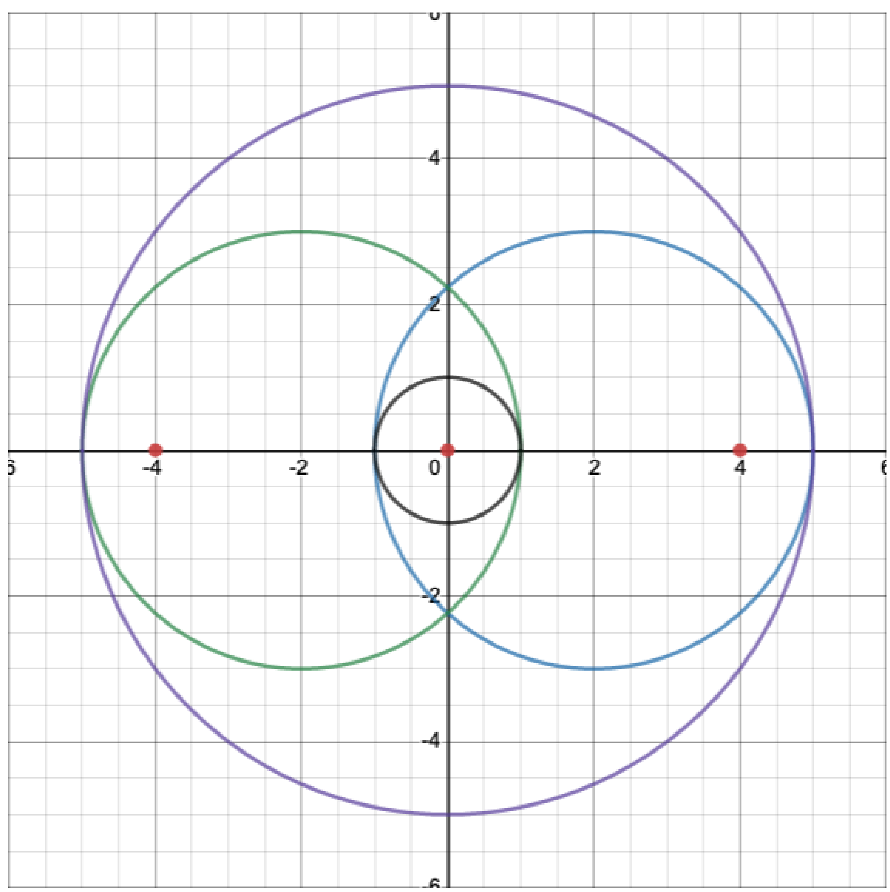
where (x_1, y_1), (x_2, y_2) are two fixed points in D and let this integral have the value K for one particular path. All possible values of the integral are then given by

$$K + n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k$$

where $n_1, \ldots, n_k$ are positive or negative integers or 0. The exact value of $n_1, \ldots, n_k$ depends on how many times a chosen path between (x_1, y_1), (x_2, y_2) encircles one of the holes at $A_1, A_2, \ldots, A_k$ and in which direction (i.e. negative for clockwise or positive for anti-clockwise).

9. Let D be a domain with holes at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$ and let P and Q be continuous and have continuous derivatives in D, with $\partial P/\partial y = \partial Q/\partial x$ except at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$. Let C_1 denote the circle $(x-2)^2 + y^2 = 9$; let C_2 denote the circle $(x+2)^2 + y^2 = 9$; let C_3 denote the circle $x^2 + y^2 = 25$. Furthermore, let it be given that

$$\oint_{C_1} P\,dx + Q\,dy = 11, \qquad \oint_{C_2} P\,dx + Q\,dy = 9, \qquad \oint_{C_3} P\,dx + Q\,dy = 13$$



From inspection of the figure it follows that

$$\oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy$$

and so

$$\oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy - \oint_{C_3} P\,dx + Q\,dy = 11 + 9 - 13 = 7$$

10. Let $F(x, y) = x^2 - y^2$

   (a) By Theorem I of Section 5.6 the integral

   $$\int_{(0,0)}^{(2,8)} \nabla F \cdot d\mathbf{r} = \int_{(0,0)}^{(2,8)} 2x\,dx - 2y\,dy = \int_{(0,0)}^{(2,8)} \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \int_{(0,0)}^{(2,8)} P\,dx + Q\,dy$$

   is independent of path in D, where D is the xy-plane. Hence, since the points $(0, 0)$ and $(2, 8)$ both lie on the curve $y = x^3$ the integral on the curve $y = x^3$ is equal to $F(2, 8) - F(0, 0) = 4 - 60 = -60$.

   (b) By (5.37) and (5.38) the integral

   $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C (Q\mathbf{i} - P\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right) ds = \oint_C \left(Q\frac{dy}{ds} + P\frac{dx}{ds}\right) ds$$
   $$= \oint_C P\,dx + Q\,dy$$

   where C is the circle $x^2 + y^2 = 1$, $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the outer normal to C and $\partial F/\partial n = \nabla F \cdot \mathbf{n}$ is the directional derivative of F in the direction of $\mathbf{n}$ (see Section 2.4). Since

   $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2$$

   then by Theorem III of Section 5.6 the integral is independent of path in D and hence, by Theorem II of Section 5.6 it follows that

   $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C P\,dx + Q\,dy = 0$$

11. Let $F(x, y)$ and $G(x, y)$ be continuous and have continuous derivatives in a domain D and let R be a closed region in D with directed boundary B_R consisting of closed curves $C_1, \ldots, C_n$ as in Figure 5.21. Let $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ be the outer normal of R and let $\partial F/\partial n$ and $\partial G/\partial n$ denote the directional derivatives of F and G in the direction of $\mathbf{n}$: $\partial F/\partial n = \nabla F \cdot \mathbf{n}$, $\partial G/\partial n = \nabla G \cdot \mathbf{n}$.

(a) From (5.37), (5.38), (5.44) and (5.56) it follows that

$$\int\limits_{B_R} \frac{\partial F}{\partial n}\, ds = \int\limits_{B_R} \nabla F \cdot \mathbf{n}\, ds = \int\limits_{B_R} (Q\mathbf{i} - P\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \right)\, ds$$

$$= \int\limits_{B_R} P\, dx + Q\, dy = \iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dx\, dy$$

$$= \iint\limits_{R} \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) \cdot (Q\mathbf{i} - P\mathbf{j})\; dx\, dy$$

$$= \iint\limits_{R} \nabla \cdot \nabla F\, dx\, dy = \iint\limits_{R} \nabla^2 F\, dx\, dy$$

(b) By (5.56) and (2.124)

$$\int\limits_{B_R} \nabla F \cdot d\mathbf{r} = \int\limits_{B_R} \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = \iint\limits_{B_R} \underbrace{\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x} \right)}_{0}\, dx\, dy = 0$$

(c) By (2.126) if a function $z = f(x, y)$ has continuous second derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be harmonic in D. Hence, it follows automatically from Problem 11(a) that

$$\int\limits_{B_R} \frac{\partial F}{\partial n}\, ds = \iint\limits_{R} \underbrace{\nabla^2 F}_{0}\, dx\, dy = 0$$

(d) Using the identity $\nabla \cdot (f\mathbf{u}) = f\nabla \cdot \mathbf{u} + \nabla f \cdot \mathbf{u}$ and the solution to Problem 11(a) we can write

$$\int\limits_{B_R} F\frac{\partial G}{\partial n}\, ds = \int\limits_{B_R} F\,(\nabla G \cdot \mathbf{n})\; ds = \iint\limits_{R} \nabla \cdot (F\nabla G)\; dx\, dy$$

$$= \iint\limits_{R} (F\nabla \cdot \nabla G + \nabla F \cdot \nabla G)\; dx\, dy$$

$$= \iint\limits_{R} F\nabla^2 G\, dx\, dy + \iint\limits_{R} (\nabla F \cdot \nabla G)\; dx\, dy$$

12. (a) Using the solution to Problem 11(d) we find

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R F\nabla^2 G\, dx\, dy + \iint\limits_R (\nabla F \cdot \nabla G)\ dx\, dy - \iint\limits_R G\nabla^2 F\, dx\, dy$$
$$- \iint\limits_R (\nabla G \cdot \nabla F)\ dx\, dy$$
$$= \iint\limits_R F\nabla^2 G\, dx\, dy - \iint\limits_R G\nabla^2 F\, dx\, dy$$
$$= \iint\limits_R \left(F\nabla^2 G - G\nabla^2 F \right)\ dx\, dy$$

where we have utilised the fact that $\nabla F \cdot \nabla G = \nabla G \cdot \nabla F$.

(b) If F and G are harmonic in R, i.e. when $\nabla^2 F = 0$, $\nabla^2 G = 0$ in R, then

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R \left(F\underbrace{\nabla^2 G}_{0} - G\underbrace{\nabla^2 F}_{0} \right)\ dx\, dy = 0$$

Section 5.10

1. (a)

$$\int\limits_C{}_{(1,0,0)}^{(1,0,2\pi)} z\, dx + x\, dy + y\, dz = \int_0^{2\pi} \left(\omega\frac{d\phi}{dt} + \phi\frac{d\psi}{dt} + \psi\frac{d\omega}{dt} \right)\ dt$$
$$= \int_0^{2\pi} \left(-t\sin t + \cos^2 t + \sin t \right)\ dt$$
$$= -\int_0^{2\pi} t\sin t\, dt + \int_0^{2\pi} \cos^2 t\, dt + \int_0^{2\pi} \sin t\, dt$$
$$= t\cos t\Big|_0^{2\pi} + \int_0^{2\pi} \cos t\, dt + \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt - \cos t\Big|_0^{2\pi}$$
$$= 2\pi + \pi = 3\pi$$

where C is the curve $x = \phi(t) = \cos t,\ y = \psi(t) = \sin t,\ z = \omega(t) = t,\ 0 \le t \le 2\pi$.

(b) Let

$$\int_{(1,0,1)}^{(2,3,2)} x^2\, dx - xz\, dy + y^2\, dz$$

To evaluate this integral on the straight line joining the two points $(1,0,1),\ (2,3,2)$ we use the parametrisation $x = \phi(t) = z = \omega(t) = 1+t,\ y = \psi(t) = 3t,\ 0 \le t \le 1$, so that

$$\int_{(1,0,1)}^{(2,3,2)} x^2\,dx - xz\,dy + y^2\,dz = \int_0^1 \left(\phi^2 \frac{d\phi}{dt} - \phi\omega \frac{d\psi}{dt} + \psi^2 \frac{d\omega}{dt}\right) dt$$

$$= \int_0^1 \left(-2 - 4t + 7t^2\right) dt = \left[-2t - 2t^2 + \frac{7t^3}{3}\right]_0^1 = -\frac{5}{3}$$

(c)

$$\int_{C\ (1,1,0)}^{\left(0,0,\sqrt{2}\right)} x^2 yz\,ds = \int_0^{\pi/2} \phi^2\psi\omega \sqrt{\left(\frac{d\phi}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\omega}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} 2\sin t\cos^3 t\,dt = \int_{-1}^{0} -2\theta^3\,d\theta = -\frac{\theta^4}{2}\bigg|_{-1}^{0} = \frac{1}{2}$$

where C is the curve $x = \phi(t) = \cos t$, $y = \psi(t) = \cos t$, $z = \omega(t) = \sqrt{2}\sin t$, $0 \le t \le \pi/2$ and we have used the substitution $\theta = -\cos t$.

(d) Using (5.65) and (5.34)

$$\int_C u_T\,ds = \int_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} \left(2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + x^2y^2\mathbf{k}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) dt$$

$$= \int_0^{2\pi} \left(4\sin^2 t\cos t\mathbf{i} + 4\sin t\cos^2 t\mathbf{j} + \sin^2 t\cos^2 t\mathbf{k}\right)\cdot\left(-\sin t\mathbf{i} + \cos t\mathbf{j}\right) dt$$

$$= 4\int_0^{2\pi} \left(-\sin^3 t\cos t + \sin t\cos^3 t\right) dt$$

$$= 4\int_0^{2\pi} -\sin^3 t\cos t\,dt + 4\int_0^{2\pi} \sin t\cos^3 t\,dt = -4\int_{u_0}^{u_1} u^3\,du - 4\int_{v_0}^{v_1} v^3\,dv$$

$$= -\sin^4 t\Big|_0^{2\pi} - \cos^4 t\Big|_0^{2\pi} = 0$$

where C is the circle $x = \cos t$, $y = \sin t$, $z = 2$, $0 \le t \le 2\pi$ and we have used the substitution $u = \sin t$, $v = -\cos t$.

(e) Using (5.65), (5.34) and (3.23)

$$\int_C u_T\,ds = \int_C \mathbf{u}\cdot d\mathbf{r} = \int_0^1 \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right)\,dt = \int_0^1 \left(\nabla\times\mathbf{v}\cdot\frac{d\mathbf{r}}{dt}\right)\,dt$$
$$= \int_0^1 -2\left(z\mathbf{i}+x\mathbf{j}+y\mathbf{k}\right)\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right)\,dt$$
$$= -2\int_0^1 \left[\left(1+t^3\right)\mathbf{i}+(2t+1)\,\mathbf{j}+t^2\mathbf{k}\right]\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right)\,dt$$
$$= -2\int_0^1 \left(2+2t+4t^2+2t^3+3t^4\right)\,dt = -2\left[2t+t^2+\frac{4t^3}{3}+\frac{t^4}{2}+\frac{3t^5}{5}\right]_0^1$$
$$= -\frac{163}{15}$$

2. Let $\mathbf{u}=\nabla F$ in a domain D.

(a) By (5.65) and the chain rule (see Section 2.8) we find

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} u_T\,ds = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \mathbf{u}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \nabla F\cdot d\mathbf{r}$$
$$= \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}\right)\,dt$$
$$= \int_{t_1}^{t_2}\frac{dF}{dt}\,dt = F\big|_{t=t_2} - F\big|_{t=t_1}$$
$$= F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right)$$

where we have used the parametrisation $x=\phi(t)$, $y=\psi(t)$, $z=\omega(t)$, such that $(x_1,y_1,z_1)=(\phi_1,\psi_1,\omega_1)$, $(x_2,y_2,z_2)=(\phi_2,\psi_2,\omega_2)$ for $t_1\le t\le t_2$ and the integral is along any path in D joining the two points.

(b) On any closed path in D it follows that $(x_1,y_1,z_1)=(x_2,y_2,z_2)$ and so

$$\int_C u_T\,ds = F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right) = 0$$

3. Let a curve C in space represent a wire and let its density (mass per unit length) be given by $\delta=\delta(x,y,z)$, where (x,y,z) is a variable point on C.

(a) For a smooth or piecewise smooth path C arc length s in well defined, i.e. as the distance traversed from some initial point $t=h$ up to a general t:

$$s = \int_h^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

If the curve is directed with increasing t, then s also increases in the direction of motion, going from 0 up to the length L of C. We can subdivide C so that $\Delta_i s$ denotes the increment in s from t_{i-1} to t_i, that is, the distance moved in this interval, which leads to the definition

$$\lim_{\substack{n\to\infty \\ \max \Delta_i s\to 0}} \sum_{i=1}^{n} \Delta_i s = \int\limits_C ds = L = \text{length of wire}$$

(b) Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the total mass of the wire can be computed by summing over all of the products of the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max \Delta_i s \to 0$, or put more succinctly:

$$\lim_{\substack{n\to\infty \\ \max \Delta_i s\to 0}} \sum_{i=1}^{n} \delta\left(x_i^*, y_i^*, z_i^*\right) \Delta_i s = \int\limits_C \delta\left(x, y, z\right)\, ds = M = \text{total mass of wire}$$

(c) Let the center of mass of the wire be given by point $(\bar{x}, \bar{y}, \bar{z})$. Furthermore, it is a given that the first moment (i.e. expected value) in any direction about the center of mass is equal to zero. As such, this implies

$$\int\limits_C (x - \bar{x})\, \delta\, ds = 0 \qquad \int\limits_C (y - \bar{y})\, \delta\, ds = 0 \qquad \int\limits_C (z - \bar{z})\, \delta\, ds = 0$$

Focusing on the x-coordinate for now, we thus find

$$\int\limits_C (x - \bar{x})\, \delta\, ds = 0 \iff \bar{x} \underbrace{\int\limits_C \delta\, ds}_{M} = \int\limits_C x\delta\, ds$$

and so

$$M\bar{x} = \int\limits_C x\delta\, ds \qquad M\bar{y} = \int\limits_C y\delta\, ds \qquad M\bar{z} = \int\limits_C z\delta\, ds$$

(d) The moment of inertia can be defined as a measure of the resistance of an object to a change in its rotational motion. Because it has to do with rotational motion, the moment of inertia is always measured about a reference line, i.e. the axis of rotation. For a point mass m the moment of inertia about the z-axis is defined as $I_z = md^2$, where d is the distance of the mass m to the z-axis. Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the moment of inertia about the z-axis of the wire can be computed by summing over all of the products of the distance to the z-axis at a given point (x_i^*, y_i^*, z_i^*), the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max \Delta_i s \to 0$, or put more succinctly:

$$I_z = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} \left[(x_i^*)^2 + (y_i^*)^2 \right] \delta\left(x_i^*, y_i^*, z_i^*\right) \Delta_i s = \int\limits_C \left(x^2 + y^2\right) \delta \, ds$$

4. Let a smooth surface S in space represent a thin curved sheet of metal and let its density (mass per unit area) be given by $\mu = \mu(x, y, z)$, where (x, y, z) is a variable point on S. The surface may be represented in any of the forms given by (5.66), (5.67) or (5.68).

   (a) We assume the surface S is cut into n pieces as in Fig. 5.25. The quantity $\Delta_i \sigma$ then denotes the area of the ith piece and it is assumed that the ith piece shrinks to a point as $n \to \infty$ in an appropriate manner. The total surface area of S then can be obtained by summing over all $\Delta_i \sigma$ in the limit that $n \to \infty$ and $\max \Delta_i \sigma \to 0$, which leads to the definition

   $$\lim_{\substack{n\to\infty \\ \max \Delta_i \sigma \to 0}} \sum_{i=1}^{n} \Delta_i \sigma = \iint\limits_S d\sigma = \text{surface area of sheet}$$

   (b) Given that the density (mass per unit area) of the thin metal sheet is $\mu = \mu(x, y, z)$ for some point (x, y, z) on C, the total mass of the sheet can be computed by summing over the product of all $\Delta_i \sigma$'s and the density at the corresponding point (x_i^*, y_i^*, z_i^*) in the limit that the number of area segments $n \to \infty$ and the area of the ith piece $\Delta_i \sigma \to 0$, or put more succinctly:

   $$\lim_{\substack{n\to\infty \\ \max \Delta_i \sigma \to 0}} \sum_{i=1}^{n} \mu\left(x_i^*, y_i^*, z_i^*\right) \Delta_i \sigma = \iint\limits_S \mu\,(x, y, z)\; d\sigma = M = \text{total mass of sheet}$$

   (c) Let the center of mass of the sheet be given by point $(\bar{x}, \bar{y}, \bar{z})$. Furthermore, it is a given that the first moment (i.e. expected value) in any direction about the center of mass is equal to zero. As such, this implies

   $$\iint\limits_S (x - \bar{x})\, \mu \, d\sigma = 0 \qquad \iint\limits_S (y - \bar{y})\, \mu \, d\sigma = 0 \qquad \iint\limits_S (z - \bar{z})\, \mu \, d\sigma = 0$$

   Focusing on the x-coordinate for now, we thus find

   $$\iint\limits_S (x - \bar{x})\, \mu \, d\sigma = 0 \iff \bar{x} \underbrace{\iint\limits_S \mu \, d\sigma}_{M} = \iint\limits_S x\mu \, d\sigma$$

and so

$$M\bar{x} = \iint\limits_S x\mu \, d\sigma \qquad M\bar{y} = \iint\limits_S y\mu \, d\sigma \qquad M\bar{z} = \iint\limits_S z\mu \, d\sigma$$

(d) Given that the density (mass per unit area) of the sheet is $\mu = \mu(x, y, z)$ for some point (x, y, z) on the surface S, the moment of inertia about the z-axis of the sheet can be computed by summing over the product of all $\Delta_i\sigma$'s, the density at the corresponding point (x_i^*, y_i^*, z_i^*) and the distance to the z-axis at (x_i^*, y_i^*, z_i^*) in the limit that the number of area segments $n \to \infty$ and the area of the ith piece $\Delta_i\sigma \to 0$, or put more succinctly:

$$I_z = \lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \left[(x_i^*)^2 + (y_i^*)^2\right] \mu\left(x_i^*, y_i^*, z_i^*\right) \Delta_i\sigma = \iint\limits_S \left(x^2 + y^2\right) \mu \, d\sigma$$

5. (a) Let

$$\iint\limits_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy$$

where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and the normal is pointing away from $(0, 0, 0)$. From geometry it follows that the normal $\mathbf{n}$ to our triangle S pointing away from the origin is given by $\sqrt{3}\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Using (1.23) and (1.24) we then can formulate the equation for the plane in space that coincides with S: $x + y + z = 1$, and so $z = f(x, y) = 1 - x - y$. The region R_{xy} may be expressed as $y = 1 - x$, where $0 \leq x \leq 1$. Finally, we can use (5.80) to evaluate the surface integral to get

$$\begin{aligned}
\iint\limits_S L \, dy \, dz + M \, dz \, dx + N \, dx \, dy &= \iint\limits_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy \\
&= \iint\limits_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N\right) dx \, dy \\
&= \int_0^1 \int_0^{1-x} (x + y + z) \; dy \, dx = \int_0^1 \int_0^{1-x} dy \, dx \\
&= \int_0^1 (1 - x) \; dx = \frac{1}{2}
\end{aligned}$$

(b) Let

$$\iint\limits_S dy \, dz + dz \, dx + dx \, dy$$

where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \leq 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Next, let us introduce the parametrisation $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{1 - r^2}$, $r^2 \leq 1$, where $(r, \theta) \in R_{r\theta}$. Then by (5.81)

$$\begin{aligned}
\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy &= \iint\limits_S dy\,dz + dz\,dx + dx\,dy \\
&= \iint\limits_{R_{r\theta}} \left[L\frac{\partial(y,z)}{\partial(r,\theta)} + M\frac{\partial(z,x)}{\partial(r,\theta)} + N\frac{\partial(x,y)}{\partial(r,\theta)} \right] dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[\frac{r^2}{\sqrt{1-r^2}}(\sin\theta + \cos\theta) + r \right] dr\,d\theta \\
&= \underbrace{\int_0^{2\pi} (\sin\theta + \cos\theta)\,d\theta \int_0^1 \frac{r^2\,dr}{\sqrt{1-r^2}}}_{0} + \int_0^{2\pi}\int_0^1 r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2}\bigg|_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{\theta}{2}\bigg|_0^{2\pi} = \pi
\end{aligned}$$

(c) Let

$$\iint\limits_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma$$

where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \leq 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Again, we introduce the parametrisation $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{1 - r^2}$, $r^2 \leq 1$, where $(r, \theta) \in R_{r\theta}$. Then by (5.78) and (5.81)

$$\begin{aligned}
\iint\limits_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma &= \iint\limits_S (L\cos\alpha + M\cos\beta + N\cos\gamma)\,d\sigma \\
&= \iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{R_{r\theta}} \left[L\frac{\partial(y,z)}{\partial(r,\theta)} + M\frac{\partial(z,x)}{\partial(r,\theta)} + N\frac{\partial(x,y)}{\partial(r,\theta)} \right] dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2\sqrt{u}}\, du\, d\theta \\
&= \int_0^{2\pi} \sqrt{u}\Big|_0^1\, d\theta = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi
\end{aligned}
$$

where we have used the substitution $u = 1 - r^2$ such that $du = -2r\, dr$.

(d) Let

$$
\iint\limits_S x^2 z\, d\sigma
$$

where S is the cylindrical surface $x^2 + y^2 = 1$, $0 \leq z \leq 1$. Using the parametrisation $x = \cos\theta$, $y \sin\theta$, $z = z$, the surface area element $d\sigma$ can be represented by the product $d\theta\, dz$ and so the integral can be evaluated as

$$
\begin{aligned}
\iint\limits_S x^2 z\, d\sigma = \int_0^{2\pi}\int_0^1 z\cos^2\theta\, dz\, d\theta = \frac{1}{2}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta \\
= \frac{1}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{\pi}{2}
\end{aligned}
$$

6. (a) Let

$$
\iint\limits_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy
$$

where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and the normal is pointing away from $(0,0,0)$. Furthermore, let the parametrisation $x = u + v$, $y = u - v$, $z = 1 - 2u$, where $0 \leq u \leq 1/2$, $0 \leq v \leq 1/2$ be a given. From Problem 5(a) we know that $\sqrt{3}\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence, utilising (5.82) we find that

$$
\mathbf{n} = -\frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}
$$

where

$$
\mathbf{P}_1 = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad\qquad \mathbf{P}_2 = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}
$$

and so by (5.81)

$$\begin{aligned}
\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy &= \iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \\
&= -\iint\limits_{R_{uv}} \left[L\frac{\partial\,(y,z)}{\partial\,(u,v)} + M\frac{\partial\,(z,x)}{\partial\,(u,v)} + N\frac{\partial\,(x,y)}{\partial\,(u,v)} \right] du\,dv \\
&= 2\int_0^{1/2}\int_0^{1/2} du\,dv = \frac{1}{2}
\end{aligned}$$

(b) Let

$$\iint\limits_S dy\,dz + dz\,dx + dx\,dy$$

where S is the hemisphere $z = \sqrt{1-x^2-y^2}$, $x^2+y^2 \leq 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

Furthermore, let the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$, where $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.81)

$$\begin{aligned}
\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy &= \iint\limits_S dy\,dz + dz\,dx + dx\,dy \\
&= \iint\limits_{R_{uv}} \left[L\frac{\partial\,(y,z)}{\partial\,(u,v)} + M\frac{\partial\,(z,x)}{\partial\,(u,v)} + N\frac{\partial\,(x,y)}{\partial\,(u,v)} \right] du\,dv \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left[\sin^2 u\,(\sin v + \cos v) + \sin u \cos u\right] du\,dv \\
&= \underbrace{\int_0^{2\pi} (\sin v + \cos v)\;dv}_{0} \int_0^{\pi/2} \sin^2 u\,du \\
&\quad + \int_0^{2\pi}\int_0^{\pi/2} \sin u \cos u\,du\,dv \\
&= \int_0^{2\pi}\int_0^{1} w\,dw\,dv = \frac{1}{2}\int_0^{2\pi} dv = \pi
\end{aligned}$$

(c) Let

$$\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\ d\sigma$$

where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \le 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Furthermore, let the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$, where $0 \le u \le \pi/2$, $0 \le v \le 2\pi$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.78) and (5.81)

$$\begin{aligned}
\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\ d\sigma &= \iint_S (L\cos\alpha + M\cos\beta + N\cos\gamma)\ d\sigma \\
&= \iint_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy \\
&= \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right] du\,dv \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left[\sin u + \sin^2 u\cos u\,(\sin v + \cos v)\right] du\,dv \\
&= \int_0^{2\pi}\int_0^{\pi/2} \sin u\,du\,dv \\
&\quad + \underbrace{\int_0^{2\pi} (\sin v + \cos v)\ dv}_{0} \int_0^{\pi/2} \sin^2 u\cos u\,du \\
&= \int_0^{2\pi} dv = 2\pi
\end{aligned}$$

(d) Let

$$\iint_S x^2 z\,d\sigma$$

where S is the cylindrical surface $x^2 + y^2 = 1$, $0 \le z \le 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$$

Furthermore, let the parametrisation $x = \cos u$, $y = \sin u$, $z = v$, where $0 \le u \le 2\pi$, $0 \le v \le 1$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.78) and (5.81)

$$\begin{aligned}
\iint\limits_S x^2 z\, d\sigma = \iint\limits_S L \cos\alpha\, d\sigma = \iint\limits_S L\, dy\, dz &= \iint\limits_{R_{uv}} L \frac{\partial (y,z)}{\partial (u,v)}\, du\, dv \\
&= \int_0^1 \int_0^{2\pi} v \cos^2 u\, du\, dv \\
&= \int_0^1 v\, dv \int_0^{2\pi} \frac{1 + \cos 2u}{2}\, du \\
&= \pi \int_0^1 v\, dv = \frac{\pi}{2}
\end{aligned}$$

7. (a) Let

$$\iint\limits_S \mathbf{w} \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{w} = xy^2 z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$ and $\mathbf{n}$ is the upper / outer normal. Furthermore, let us introduce the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos^2 u$, where $0 \le u \le \pi/2$, $0 \le v \le 2\pi$. Then by (5.79) and (5.81)

$$\begin{aligned}
\iint\limits_S \mathbf{w} \cdot \mathbf{n}\, d\sigma &= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S xy^2 z\, dy\, dz - 2x^3\, dz\, dx + yz^2\, dx\, dy \\
&= \iint\limits_{R_{uv}} \left[L \frac{\partial (y,z)}{\partial (u,v)} + M \frac{\partial (z,x)}{\partial (u,v)} + N \frac{\partial (x,y)}{\partial (u,v)} \right] du\, dv \\
&= 2 \int_0^{2\pi} \int_0^{\pi/2} \cos^3 u \sin^5 u \sin^2 v \cos^2 v\, du\, dv \\
&\quad - 4 \int_0^{2\pi} \int_0^{\pi/2} \cos u \sin^5 u \cos^3 v \sin v\, du\, dv + \int_0^{2\pi} \int_0^{\pi/2} \cos^5 u \sin^2 u \sin v\, du\, dv
\end{aligned}$$

$$= 2\int_0^{2\pi} \sin^2 v\cos^2 v\,dv \int_0^{\pi/2} \cos^3 u \sin^5 u\,du - \underbrace{4\int_0^{2\pi} \cos^3 v \sin v\,dv}_{0}$$

$$\times \int_0^{\pi/2} \cos u \sin^5 u\,du + \underbrace{\int_0^{2\pi} \sin v\,dv}_{0} \int_0^{\pi/2} \cos^5 u \sin^2 u\,du$$

$$= 2\int_0^{2\pi} \sin^2 v\cos^2 v\,dv \int_0^{\pi/2} \cos^3 u \sin^5 u\,du$$

$$= \frac{1}{4}\int_0^{2\pi} (1-\cos 4v)\,dv \int_0^{\pi/2} \cos^3 u \left(1-\cos^2 u\right)^2 \sin u\,du$$

$$= \frac{1}{4}\left[v - \frac{\sin 4v}{4}\right]_0^{2\pi} \times \int_0^1 w^3\left(1-w^2\right)^2\,dw = \frac{\pi}{2}\int_0^1 \left(w^3 - 2w^5 + w^7\right)\,dw$$

$$= \frac{\pi}{2}\left[\frac{w^4}{4} - \frac{w^6}{3} + \frac{w^8}{8}\right]_0^1 = \frac{\pi}{48}$$

where we have used the substitution $w = \cos u$ so that $dw = -\sin u\,du$.

(b) Let

$$\iint_S \mathbf{w}\cdot\mathbf{n}\,d\sigma$$

where $\mathbf{w} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, S is the surface $x = e^u\cos v,\ y = e^u \sin v,\ z = \cos v \sin v,\ 0 \le u \le 1,\ 0 \le v \le \pi/2$ and $\mathbf{n}$ is given by (5.82) with the + sign. Then by (5.79) and (5.81)

$$\iint_S \mathbf{w}\cdot\mathbf{n}\,d\sigma = \iint_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy$$

$$= \iint_S dy\,dz + 2\,dz\,dx + 3\,dx\,dy$$

$$= \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right]\,du\,dv$$

$$= \int_0^{\pi/2}\int_0^1 e^u \sin v\left(\cos^2 v - \sin^2 v\right)\,du\,dv$$

$$- \int_0^{\pi/2}\int_0^1 2e^u\cos v\left(\cos^2 v - \sin^2 v\right)\,du\,dv + \int_0^{\pi/2}\int_0^1 3e^{2u}\,du\,dv$$

$$= \int_0^{\pi/2}\int_0^1 e^u \sin v\left(2\cos^2 v - 1\right)\,du\,dv$$

$$- \int_0^{\pi/2}\int_0^1 2e^u\cos v\left(1 - 2\sin^2 v\right)\,du\,dv + \int_0^{\pi/2}\int_0^1 3e^{2u}\,du\,dv$$

$$= \int_0^{\pi/2} \sin v \left(2\cos^2 v - 1\right)\, dv \int_0^1 e^u\, du - \int_0^{\pi/2} \cos v \left(1 - 2\sin^2 v\right)\, dv \int_0^1 2e^u\, du$$
$$+ \frac{3\left(e^2-1\right)}{2} \int_0^{\pi/2} dv$$
$$= (e-1) \int_0^1 \left(2p^2 - 1\right)\, dp - 2(e-1) \int_0^1 \left(1 - 2q^2\right)\, dq + \frac{3\pi\left(e^2-1\right)}{4}$$
$$= (e-1) \left[\frac{2p^3}{3} - p\right]_0^1 - 2(e-1) \left[q - \frac{2q^3}{3}\right]_0^1 + \frac{3\pi\left(e^2-1\right)}{4}$$
$$= -(e-1) + \frac{3\pi\left(e^2-1\right)}{4}$$

(c) Let
$$\iint_S \frac{\partial w}{\partial n}\, d\sigma$$
where $w = x^2y^2z$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$ and $\mathbf{n}$ is the upper / outer normal. Furthermore, let us introduce the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos^2 u$, where $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$. Then by (5.79), (5.81) and (2.113)
$$\iint_S \frac{\partial w}{\partial n}\, d\sigma = \iint_S \nabla w \cdot \mathbf{n}\, d\sigma$$
$$= \iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy$$
$$= \iint_S 2xy^2z\, dy\, dz + 2x^2yz\, dz\, dx + x^2y^2\, dx\, dy$$
$$= \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right] du\, dv$$
$$= \int_0^{2\pi} \int_0^{\pi/2} \left(8\cos^3 u \sin^5 u + \cos u \sin^5 u\right) \cos^2 v \sin^2 v\, du\, dv$$
$$= \int_0^{2\pi} \cos^2 v \sin^2 v\, dv \int_0^{\pi/2} \left(8\cos^3 u \sin^5 u + \cos u \sin^5 u\right)\, du$$
$$= \int_0^{2\pi} \frac{1 - \cos 4v}{8}\, dv \left[8\int_0^1 p^3\left(1-p^2\right)^2\, dp + \int_0^1 q^5\, dq\right]$$
$$= \frac{1}{8}\left[v - \frac{\sin 4v}{4}\right]_0^{2\pi} \left\{8\left[\frac{p^4}{4} - \frac{p^6}{3} + \frac{p^8}{8}\right]_0^1 + \left.\frac{q^6}{6}\right|_0^1\right\} = \frac{\pi}{4}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{\pi}{8}$$
where we have used the substitutions $p = \cos u$ so that $dp = -\sin u\, du$ and $q = \sin u$ so that $dq = \cos u\, du$.

(d) Let

$$\iint\limits_S \frac{\partial w}{\partial n}\, d\sigma$$

where $w = x^2 - y^2 + z^2$, S is the surface $x = e^u \cos v,\ y = e^u \sin v,\ z = \cos v \sin v,\ 0 \leq u \leq 1,\ 0 \leq v \leq \pi/2$ and $\mathbf{n}$ is given by (5.82) with the + sign. Then by (5.79), (5.81) and (2.113)

$$\begin{aligned}
\iint\limits_S \frac{\partial w}{\partial n}\, d\sigma &= \iint\limits_S \nabla w \cdot \mathbf{n}\, d\sigma \\
&= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S 2x\, dy\, dz - 2y\, dz\, dx + 2z\, dx\, dy \\
&= \iint\limits_{R_{uv}} \left[L\frac{\partial (y,z)}{\partial (u,v)} + M\frac{\partial (z,x)}{\partial (u,v)} + N\frac{\partial (x,y)}{\partial (u,v)} \right] du\, dv \\
&= \int_0^{\pi/2} \int_0^1 \left[4e^u \cos v \sin v \left(\cos^2 v - \sin^2 v \right) + 2e^{2u} \cos v \sin v \right] du\, dv \\
&= \int_0^{\pi/2} \cos v \sin v \left(\cos^2 v - \sin^2 v \right)\, dv \int_0^1 4e^u\, du + \int_0^{\pi/2} \cos v \sin v\, dv \int_0^1 2e^{2u}\, du \\
&= 2\left(e - 1 \right) \int_0^{\pi/2} \sin 2v \cos 2v\, dv + \frac{e^2 - 1}{2} \int_0^{\pi/2} \sin 2v\, dv \\
&= \left(e - 1 \right) \int_0^{\pi/2} \sin 4v\, dv - \frac{e^2 - 1}{4} \cos 2v \Big|_0^{\pi/2} = \underbrace{-\frac{e-1}{4} \cos 4v \Big|_0^{\pi/2}}_{0} + \frac{e^2 - 1}{2} \\
&= \frac{e^2 - 1}{2}
\end{aligned}$$

(e) Let

$$\iint\limits_S \left(\nabla \times \mathbf{u} \right) \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{u} = yz\mathbf{i} - xz\mathbf{j} + xz\mathbf{k}$, S is the triangle with vertices $P_1 : (1, 2, 8)$, $P_2 : (3, 1, 9)$, $P_3 : (2, 1, 7)$ and $\mathbf{n}$ is the upper / outer normal. To find $\mathbf{n}$ we utilise the fact that a vector $\mathbf{v} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is by definition perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. As such, we find

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

and note that $\mathbf{n} = a\mathbf{v}$, where a is an arbitrary positive scalar. Furthermore, in Section 1.3 it was stated that if $\mathbf{n}$ is a nonzero normal vector and $P_1 : (x_1, y_1, z_1)$ is a point of a plane, then $P : (x, y, z)$ is in the plane precisely when $\mathbf{v} \cdot \overrightarrow{P_1P} = 0$. Hence, we find

$$2(x-1) + 3(y-2) - (z-8) = 0 \iff z = 2x + 3y$$

Then by (5.79), (5.80) and (3.23)

$$\begin{aligned}
\iint\limits_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, d\sigma &= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S x\, dy\, dz + (y-z)\; dz\, dx - 2z\, dx\, dy \\
&= \iint\limits_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N \right) dx\, dy \\
&= \iint\limits_{R_{xy}} \underbrace{[-2x - 3(y-z) - 2z]}_{0}\, dx\, dy = 0
\end{aligned}$$

8. (a) Let a surface $S : z = f(x, y)$ be defined by an implicit equation $F(x, y, z) = 0$. Assuming F is differentiable we may write

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy + \frac{\partial F}{\partial z}\, dz \iff dz = -\frac{F_x}{F_z}\, dx - \frac{F_y}{F_z}\, dy$$

and hence,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

From (5.70) and (5.75) it then follows that

$$\begin{aligned}
\iint\limits_S H\, d\sigma = \iint\limits_{R_{xy}} H\,[x, y, f(x, y)] \sec\gamma\, dx\, dy &= \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, H\, dx\, dy \\
= \iint\limits_{R_{xy}} \sqrt{1 + \left(-\frac{F_x}{F_z}\right)^2 + \left(-\frac{F_y}{F_z}\right)^2}\, H\, dx\, dy \\
= \iint\limits_{R_{xy}} \sqrt{F_x^2 + F_y^2 + F_z^2}\, \frac{H}{|F_z|}\, dx\, dy \\
= \iint\limits_{R_{xy}} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, \frac{H}{\left|\dfrac{\partial F}{\partial z}\right|}\, dx\, dy
\end{aligned}$$

provided that $\partial F/\partial z \neq 0$.

(b) Let $\mathbf{n} = \nabla F / |\nabla F|$. Then using the result to part (a), (1.4) and (3.8) we find

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma = \iint_S H\, d\sigma = \iint_{R_{xy}} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, \frac{H}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy$$

$$= \iint_{R_{xy}} |\nabla F|\,(\mathbf{v}\cdot\mathbf{n})\, \frac{1}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy = \iint_{R_{xy}} |\nabla F| \left(\mathbf{v}\cdot\frac{\nabla F}{|\nabla F|}\right) \frac{1}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy$$

$$= \iint_{R_{xy}} (\mathbf{v}\cdot\nabla F)\, \frac{1}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy$$

(c) Using (5.82), (5.72), (5.73) and (5.79) we find

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma = \iint_{R_{uv}} \left(\mathbf{v}\cdot \pm\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right) d\sigma = \pm\iint_S \left(\mathbf{v}\cdot\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right)\sqrt{EG-F^2}\, du\, dv$$

$$= \pm\iint_{R_{uv}} \left(\mathbf{v}\cdot\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right) |\mathbf{P}_1\times\mathbf{P}_2|\, du\, dv$$

$$= \pm\iint_{R_{uv}} (\mathbf{v}\cdot\mathbf{P}_1\times\mathbf{P}_2)\, du\, dv$$

$$= \pm\iint_{R_{uv}} (L\mathbf{i}+M\mathbf{j}+N\mathbf{k})\cdot\left[\frac{\partial(y,z)}{\partial(u,v)}\mathbf{i} + \frac{\partial(z,x)}{\partial(u,v)}\mathbf{j} + \frac{\partial(x,y)}{(u,v)}\mathbf{k}\right] du\, dv$$

$$= \pm\iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right] du\, dv$$

(d) When $x = u$, $y = v$, $z = f(u,v)$

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial u} = -\frac{\partial z}{\partial x}\frac{dx}{du} = -\frac{\partial z}{\partial x}$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial y}\frac{dy}{dv} = -\frac{\partial z}{\partial y}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = 1$$

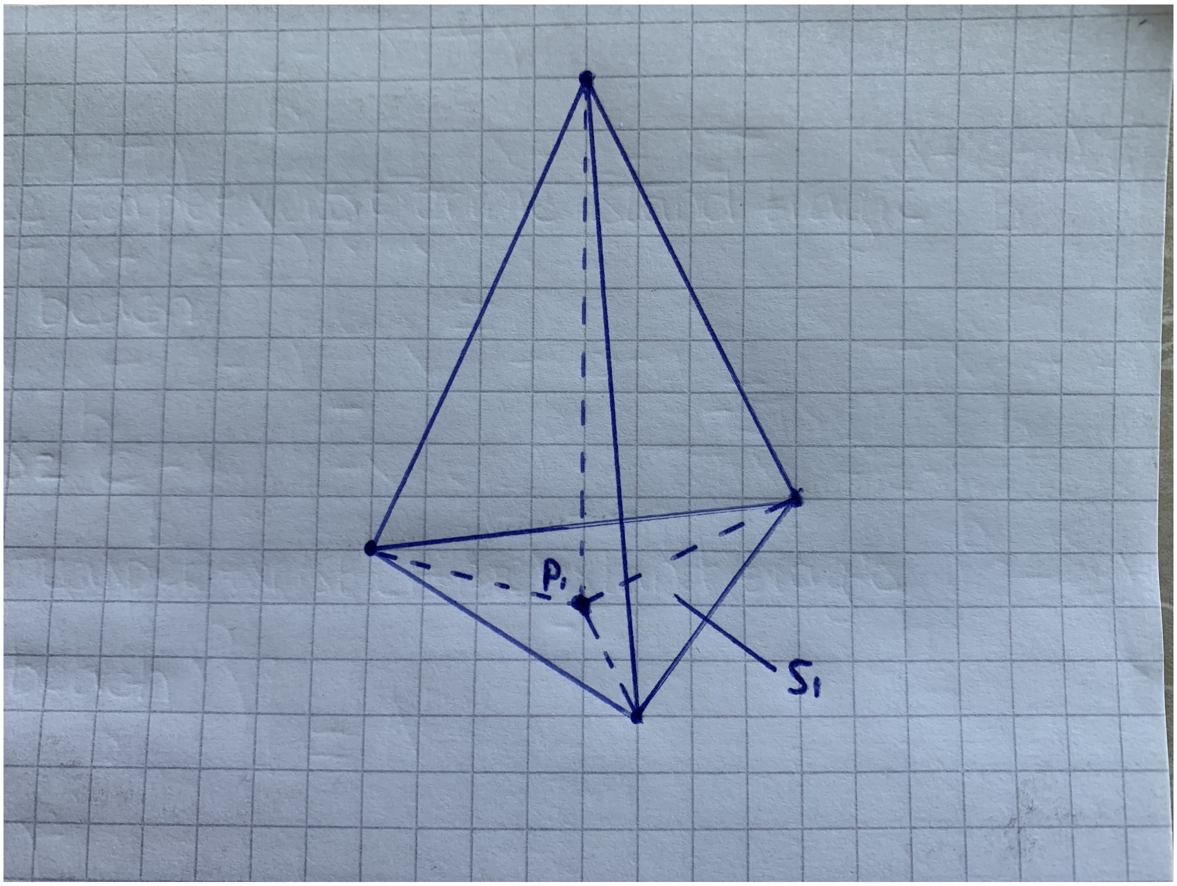
and $dx = du$, $dy = dv$. Hence, (5.81) reduces to

$$\pm\iint_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N\right) dx\, dy = \iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma$$

$$= \iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy$$

which is none other than (5.80)

9. Let S be an oriented surface in space that is planar. With S one can associate the vector $\mathbf{S}$, which has the direction of the normal chosen on S and has a length equal to the area of S.



(a) Let S_1, S_2, S_3, S_4 be the faces of a tetrahedron, oriented so that the normal is the exterior normal. With each $S_1, \ldots, S_4$ we can then associate the vector $\mathbf{S}_i = A_i \mathbf{n}_i \, (A_i > 0)$ for $i = 1, \ldots, 4$, where A_i is the area of face S_i and $\mathbf{n}_i$ is the exterior normal to S_i. Next, let the point p_1 be the foot of the altitude on face S_1. Then we can join p_1 to the vertices of S_1 to form three triangles of areas A_{12}, A_{13}, A_{14}, such that $A_{12} + A_{13} + A_{14} = A_2$ (see the above figure). Analogous to (4.79), it follows readily from geometry that $A_{1j} = \pm A_j \mathbf{n}_j \cdot \mathbf{n}_1$, with $+$ when $\mathbf{n}_j \cdot \mathbf{n}_1 > 0$ or $-$ when $\mathbf{n}_j \cdot \mathbf{n} < 0$ and $A_{1j} = 0$ if $\mathbf{n}_j \cdot \mathbf{n}_1 = 0 \, (j = 2, 3, 4)$, i.e. $A_{1j} = 0$ when $\mathbf{n}_j$ and $\mathbf{n}_1$ are perpendicular. In other words, the area of each triangle A_{12}, A_{13}, A_{14} is equal to the projection of S_2, S_3, S_4 onto S_1. Let us now introduce the vector $\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{b}$ and consider the dot product $\mathbf{b} \cdot \mathbf{n}_j$

for $j = 1,2,3,4$. Fixing $j = 2$ for example, we find

$$\begin{aligned}
\mathbf{b}\cdot\mathbf{n}_2 &= \mathbf{S}_1\cdot\mathbf{n}_2 + \mathbf{S}_2\cdot\mathbf{n}_2 + \mathbf{S}_3\cdot\mathbf{n}_2 + \mathbf{S}_4\cdot\mathbf{n}_2 \\
&= A_1\mathbf{n}_1\cdot\mathbf{n}_2 + A_2\mathbf{n}_2\cdot\mathbf{n}_2 + A_3\mathbf{n}_3\cdot\mathbf{n}_2 + A_4\mathbf{n}_4\cdot\mathbf{n}_2 \\
&= A_1\mathbf{n}_1\cdot\mathbf{n}_2 + A_2 + A_3\mathbf{n}_3\cdot\mathbf{n}_2 + A_4\mathbf{n}_4\cdot\mathbf{n}_2 \\
&= -A_{21} + A_2 - A_{23} - A_{24} = 0
\end{aligned}$$

where the minus sign for the first, third and fourth term on the right hand side follows from the fact that the angle between the normal $\mathbf{n}_2$ to face S_2 and any of the other normal vectors $\mathbf{n}_1$, $\mathbf{n}_3$, $\mathbf{n}_4$ associated with faces S_1, S_3, S_4 is always at least equal to or larger than $\pi/2$ and hence, $\mathbf{n}_i\cdot\mathbf{n}_2 \leq 0\,(i = 1,3,4)$. The same reasoning applies to the other faces and hence, we may conclude that $\mathbf{b}\cdot\mathbf{n}_j = 0$ and so clearly $\mathbf{b}\cdot A_j\mathbf{n}_j = 0$ as well. Summing over j then gives

$$\begin{aligned}
\mathbf{b}\cdot A_1\mathbf{n}_1 + \mathbf{b}\cdot A_2\mathbf{n}_2 + \mathbf{b}\cdot A_3\mathbf{n}_3 + \mathbf{b}\cdot A_4\mathbf{n}_4 &= \mathbf{b}\cdot(A_1\mathbf{n}_1 + A_2\mathbf{n}_2 + A_3\mathbf{n}_3 + A_4\mathbf{n}_4) \\
&= \mathbf{b}\cdot(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4) \\
&= \mathbf{b}\cdot\mathbf{b} \\
&= 0
\end{aligned}$$

By (1.102) $\mathbf{b}\cdot\mathbf{b} = 0$ if only if $\mathbf{b} = \mathbf{0}$ and hence, this proves that

$$\mathbf{b} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}$$

(b) Let an arbitrary convex polyhedron with faces $S_1,\ldots,S_n$, oriented so that the normal is the exterior normal, be given. With each $S_1,\ldots,S_n$ we can again associate the vector $\mathbf{S}_i = A_i\mathbf{n}_i\,(A_i > 0)$ for $i = 1,2,\ldots,n$, where A_i is the area of face S_i and $\mathbf{n}_i$ is the exterior normal to S_i. Just like an arbitrary polygon can be subdivided into a finite number of triangles, an arbitrary (convex) polyhedron can be subdivided into a finite number of tetrahedra. Let $\mathbf{S}_1^{'}$ and $\mathbf{S}_1^{''}$ be the vectors associated with sides $S_1^{'}$ and $S_1^{''}$ of two such tetrahedra that are glued together. By definition $\mathbf{S}_1^{'}$ and $\mathbf{S}_1^{''}$ will have the same magnitude, but point in opposite directions and so $\mathbf{S}_1^{'} + \mathbf{S}_1^{''} = \mathbf{0}$. From part (a) we know that

$$\mathbf{S}_1^{'} = -\left(\mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}\right) \qquad \mathbf{S}_1^{''} = -\left(\mathbf{S}_2^{''} + \mathbf{S}_3^{''} + \mathbf{S}_4^{''}\right)$$

Hence,

$$\mathbf{S}_1^{'} + \mathbf{S}_1^{''} = -\left(\mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}\right) - \left(\mathbf{S}_2^{''} + \mathbf{S}_3^{''} + \mathbf{S}_4^{''}\right) = \mathbf{0}$$

The thing to take away from this exercise is that the vector associated with the face that is hidden when gluing two tetrahedra together on one tetrahedron is equal to the sum of the remaining vectors on the other tetrahedron, i.e.

$$\mathbf{S}_1^{''} = \mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}$$

Hence, starting with the fact that for a single tetrahedron we proved that $\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}$, we find that as we build up our arbitrary convex polyhedron by gluing tetrahedra together we can always express the vector associated with one of its connecting faces in terms of the outward pointing vectors associated with the remaining faces on the tetrahedron that is glued on next. As we continue this operation, the vectors associated with the faces that connect to another tetrahedron will cancel each other out, leaving only the n vectors associated with the faces that do not connect to the face of another tetrahedron, and so we may conclude that

$$\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n = \mathbf{0}$$

(c) Using the definition

$$\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma\to 0}} \sum_{i=1}^{n} \Delta_i\sigma = \iint\limits_S d\sigma$$

then, provided that $\mathbf{v}$ is a constant vector and so will have the same magnitude and direction at any point (x, y, z) in 3D space, we can write

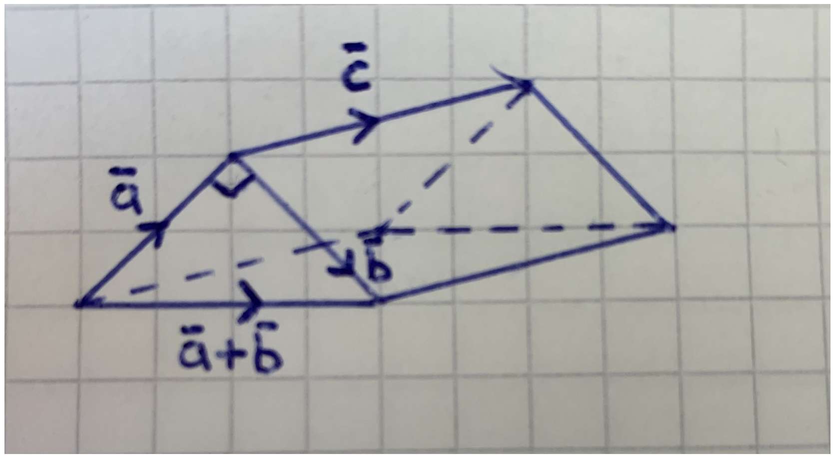
$$\iint\limits_S \mathbf{v}\cdot d\boldsymbol{\sigma} = \iint\limits_S \mathbf{v}\cdot\mathbf{n}\, d\sigma = \lim_{\substack{n\to\infty \\ \max \Delta_i\sigma\to 0}} \sum_{i=1}^{n} \mathbf{v}\cdot\mathbf{n}_i\, \Delta_i\sigma = \mathbf{v}\cdot\left(\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma\to 0}} \sum_{i=1}^{n} \mathbf{n}_i\Delta_i\sigma\right)$$

Now from part (a) and (b) we know that any arbitrary convex polyhedron can be build up from simple tetrahedra and that the sum of their face vectors, i.e. $\mathbf{S}_i = A_i\mathbf{n}_i\,(A_i > 0)$ for $i = 1, 2, \ldots, n$, adds up to zero. An arbitrary convex closed surface S (such as the surface of a sphere or ellipsoid) can be thought of as a convex polyhedron in the limit that the number of faces $n \to \infty$, such that $\max A_i \to 0$. Identifying that $A_i = \Delta_i\sigma$ we thus may conclude that

$$\iint\limits_S \mathbf{v}\cdot d\boldsymbol{\sigma} = \mathbf{v}\cdot\left(\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma\to 0}} \sum_{i=1}^{n} \mathbf{n}_i\Delta_i\sigma\right) = \mathbf{v}\cdot\mathbf{0} = 0$$

provided, as stated before, that $\mathbf{v}$ is a constant vector and S is any convex closed surface.

(d)



Let the vectors **a**, **b**, **a** + **b** and **c** represent the edges of a triangular prism (see figure above). We can then introduce the vector $\mathbf{c} \times \mathbf{a}$, which is perpendicular to and pointing away from the face formed by vectors **a** and **c**. Similarly, we can define the vector $\mathbf{c} \times \mathbf{b}$, which is perpendicular to and pointing away from the face formed by vectors **b** and **c**. Lastly, we can define the vector $\mathbf{c} \times (\mathbf{a} + \mathbf{b})$ which is pointing towards the face formed by vectors **a** + **b** and **c**. Moreover, from inspection of the figure above and visualising these 3 newly defined vectors in space it follows readily that

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}$$

Section 5.11

1. (a) Let S be the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{n}$ be the outer normal. Then by (5.85), (2.84) and (4.68)

$$\begin{aligned}
\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy &= \iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \\
&= \iiint\limits_R \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) dx\,dy\,dz \\
&= 3 \iiint\limits_R dx\,dy\,dz = 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \sin\phi\,d\phi\,d\theta = 2 \int_0^{2\pi} d\theta = 4\pi
\end{aligned}$$

(b) Let S be the surface of the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, $\mathbf{v} =$

$x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{n}$ be the outer normal. Then by (5.84)

$$\iint\limits_S \mathbf{v}\cdot\mathbf{n}\,d\sigma = \iint\limits_S v_n\,d\sigma = \iiint\limits_R \nabla\cdot\mathbf{v}\,dx\,dy\,dz = 2\int_0^1\int_0^1\int_0^1 (x+y+z)\;dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2}+y+z\right)\,dy\,dz$$

$$= 2\int_0^1 (1+z)\;dz = 3$$

(c) Let S be the sphere $x^2+y^2+z^2=1$ and $\mathbf{n}$ be the outer normal. Then by (5.85), (2.84) and (4.68)

$$\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy = \iint\limits_S e^y\cos z\,dy\,dz + e^x\sin z\,dz\,dx + e^x\cos y\,dx\,dy$$

$$= \iiint\limits_R \left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right)\,dx\,dy\,dz$$

$$= \iiint\limits_R 0\,dx\,dy\,dz = 0$$

(d) Let S bound a solid region R, $F = x^2+y^2+z^2$ and $\mathbf{n}$ be the outer normal. Then by (5.84), (4.53) and (2.117)

$$\iint\limits_S \nabla F\cdot\mathbf{n}\,d\sigma = \iint\limits_S \nabla_n F\,d\sigma = \iiint\limits_R \nabla\cdot\nabla F\,dx\,dy\,dz = 6\iiint\limits_R dx\,dy\,dz = 6V$$

where V is the volume of R.

(e) Let S bound a solid region R, $F = 2x^2-y^2-z^2$ and $\mathbf{n}$ be the outer normal. Then by (5.84) and (2.117)

$$\iint\limits_S \nabla F\cdot\mathbf{n}\,d\sigma = \iint\limits_S \nabla_n F\,d\sigma = \iiint\limits_R \nabla\cdot\nabla F\,dx\,dy\,dz = \iiint\limits_R 0\,dx\,dy\,dz = 0$$

(f) Let S be the sphere $x^2+y^2+z^2=1$, $F = [(x-2)^2+y^2+z^2]^{-1/2}$ and $\mathbf{n}$ be the outer normal. Then by (5.84) and (2.117)

$$\iint\limits_S \nabla F\cdot\mathbf{n}\,d\sigma = \iint\limits_S \nabla_n F\,d\sigma = \iiint\limits_R \nabla\cdot\nabla F\,dx\,dy\,dz$$

$$= \iiint\limits_R \frac{0}{\left[(x-2)^2+y^2+z^2\right]^{5/2}}\,dx\,dy\,dz$$

$$= \iiint\limits_R 0\,dx\,dy\,dz = 0$$

2. Let S be the boundary surface of a region R in space and let $\mathbf{n}$ be its outer normal.

   (a) By definition (4.53) the volume V of a region R in space is given by

$$V = \iiint\limits_R dx\,dy\,dz$$

   As such, we find that by (5.84) and (5.85)

$$\begin{aligned}
\frac{1}{3}\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy &= \frac{1}{3}\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy \\
&= \frac{1}{3}\iiint\limits_R \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) dx\,dy\,dz \\
&= \iiint\limits_R dx\,dy\,dz = V
\end{aligned}$$

   Furthermore, from (5.86) it follows that

$$\iint\limits_S z\,dx\,dy = \iint\limits_S N\,dx\,dy = \iiint\limits_R \frac{\partial N}{\partial z}\,dx\,dy\,dz = \iiint\limits_R dx\,dy\,dz = V$$

   Applying the same logic to (5.87) and (5.88) gives

$$\iint\limits_S x\,dy\,dz = V \qquad\qquad \iint\limits_S y\,dz\,dx = V$$

   (b) By definition; the centroid $(\bar{x}, \bar{y}, \bar{z})$ is the average position of all the points of an object occupying a region R in space. For the $\bar{x}$ coordinate this can be expressed symbolically as

$$\bar{x} = \frac{\iiint_R x\,dx\,dy\,dz}{\iiint_R dx\,dy\,dz} = \frac{1}{V}\iiint\limits_R x\,dx\,dy\,dz$$

   Hence, we find

$$\begin{aligned}
\iint\limits_S x^2\,dy\,dz + 2xy\,dz\,dx + 2xz\,dx\,dy &= \iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy \\
&= \iiint\limits_R \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) dx\,dy\,dz \\
&= 6\iiint\limits_R x\,dx\,dy\,dz = 6V\bar{x}
\end{aligned}$$

(c) Let $\mathbf{v}$ be an arbitrary vector field. Then by (5.84) and (3.33)

$$\iint\limits_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, d\sigma = \iint\limits_S (\nabla \times \mathbf{v})_n \; d\sigma = \iiint\limits_R \nabla \cdot (\nabla \times \mathbf{v}) \; dx\, dy\, dz = 0$$

3. Let the vector field $\mathbf{v}$ denote the constant velocity of the flow of some fluid. Now if this flow is incompressible, the outgoing volume precisely equals the incoming volume and hence, by (5.91) this implies that $\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = 0$. As such, we find that by (5.84)

$$\iiint\limits_R \operatorname{div} \mathbf{v}\, dx\, dy\, dz = \iint\limits_S v_n\, d\sigma = \iint\limits_S \mathbf{v} \cdot d\boldsymbol{\sigma} = \iint\limits_S \mathbf{v} \cdot \mathbf{n}\, d\sigma = 0$$

Using the definition

$$\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \Delta_i \sigma = \iint\limits_S d\sigma$$

and recognising that $\mathbf{v}$ is constant we find

$$\begin{aligned} \iiint\limits_R \operatorname{div} \mathbf{v}\, dx\, dy\, dz = \iint\limits_S \mathbf{v} \cdot \mathbf{n}\, d\sigma &= \lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \mathbf{v} \cdot \mathbf{n}_i \Delta_i \sigma \\ &= \mathbf{v} \cdot \left(\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \mathbf{n}_i \Delta_i \sigma \right) = 0 \end{aligned}$$

which in turn implies that

$$\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \mathbf{n}_i \Delta_i \sigma = \mathbf{0}$$

since $\mathbf{v} \neq \mathbf{0}$ necessarily (but could be). This is exactly the same result we arrived at when solving Problem 9(c) and so recognising once more that $\Delta_i \sigma = A_i$, such that $\mathbf{S}_i = A_i \mathbf{n}_i$ may be thought of as representing the i'th face vector of an arbitrarily convex polyhedron in the case that $n \not\to \infty$ and the fact that any arbitrary convex polyhedron can be build out of an arbitrary number of tetrahydra; we may again conclude that for an arbitrary convex polyhedron with faces $S_1, \ldots, S_n$ it holds that

$$\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n = \mathbf{0}$$

and reduces to

$$\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}$$

for a single tetrahedron.

4. Let S be the boundary surface of a region R, with outer normal $\mathbf{n}$. Let $f(x, y, z)$ and $g(x, y, z)$ be functions defined and continuous, with continuous first and second derivatives, in a domain D containing R.

(a) Using the identity $\nabla \cdot (f\mathbf{u}) = \nabla f \cdot \mathbf{u} + f(\nabla \cdot \mathbf{u})$ and (2.117) we find

$$\begin{aligned}
\iint\limits_S f\frac{\partial g}{\partial n}\,d\sigma = \iint\limits_S f\,(\nabla g \cdot \mathbf{n})\,d\sigma = \iint\limits_S f\nabla_n g\,d\sigma &= \iiint\limits_R \nabla \cdot (f\nabla g)\,dx\,dy\,dz \\
&= \iiint\limits_R f\nabla \cdot \nabla g\,dx\,dy\,dz + \iiint\limits_R (\nabla f \cdot \nabla g)\,dx\,dy\,dz \\
&= \iiint\limits_R f\nabla^2 g\,dx\,dy\,dz + \iiint\limits_R (\nabla f \cdot \nabla g)\,dx\,dy\,dz
\end{aligned}$$

(b) When g is harmonic in D then $\nabla^2 g = 0$ in R. Setting $f = 1$ in (a) we thus find

$$\iint\limits_S \frac{\partial g}{\partial n}\,d\sigma = \iiint\limits_R \underbrace{\nabla^2 g}_{0}\,dx\,dy\,dz + \iiint\limits_R \left(\underbrace{\nabla f}_{\mathbf{0}} \cdot \nabla g\right)\,dx\,dy\,dz = 0$$

(c) When f is harmonic in D then $\nabla^2 f = 0$ in R. As such, we find

$$\begin{aligned}
\iint\limits_S f\frac{\partial f}{\partial n}\,d\sigma = \iint\limits_S f\,(\nabla f \cdot \mathbf{n})\,d\sigma = \iint\limits_S f\nabla_n f\,d\sigma &= \iiint\limits_R \nabla \cdot (f\nabla f)\,dx\,dy\,dz \\
&= \iiint\limits_R f\,\underbrace{\nabla^2 f}_{0}\,dx\,dy\,dz + \iiint\limits_R (\nabla f \cdot \nabla f)\,dx\,dy\,dz \\
&= \iiint\limits_R |\nabla f|^2\,dx\,dy\,dz
\end{aligned}$$

(d) Let f be harmonic in D, such that $\nabla^2 f = 0$ in R and let $f \equiv 0$ on S. It then follows from (c) that

$$\begin{aligned}
\iiint\limits_R f\,\underbrace{\nabla^2 f}_{0}\,dx\,dy\,dz + \iiint\limits_R |\nabla f|^2\,dx\,dy\,dz &= \iint\limits_S \underbrace{f}_{0}\,\frac{\partial f}{\partial n}\,d\sigma \\
\iiint\limits_R |\nabla f|^2\,dx\,dy\,dz &= 0
\end{aligned}$$

The last equality implies that $\nabla f = 0$ in R, so that $f = c$ in R, where c is an arbitrary real constant. However, we know that $f \equiv 0$ on S and since S is bounded and closed and contained in R it follows by continuity that $f = c \equiv 0$ in R.

(e) Let f and g be harmonic in D, such that $\nabla^2 f = \nabla^2 g = 0$ in R and let $f \equiv g$ on

S. Using (d) we thus find

$$\iiint\limits_R \left(f\nabla^2 f + |\nabla f|^2 - g\nabla^2 g - |\nabla g|^2\right)\, dx\, dy\, dz = \iint\limits_S \left(f\frac{\partial f}{\partial n} - g\frac{\partial g}{\partial n}\right)\, d\sigma$$

$$\iiint\limits_R \left(|\nabla f|^2 - |\nabla g|^2\right)\, dx\, dy\, dz = 0$$

The last equality implies that $\nabla f = \nabla g$ in R, so that $f = g + c$, where c is an arbitrary real constant. However, we know that $f \equiv g$ on S and since S is bounded and closed and contained in R it follows by continuity that $c = 0$ and so $f \equiv g$ in R.

(f) Let f be harmonic in D, such that $\nabla^2 f = 0$ in R and let $\partial f/\partial n = 0$ on S. Using (c) we find

$$\iiint\limits_R |\nabla f|^2\, dx\, dy\, dz = \iint\limits_S f\, \underbrace{\frac{\partial f}{\partial n}}_{0}\, d\sigma = 0$$

which implies that $\nabla f = 0$ in R, so that $f = c$ in R, where c is an arbitrary real constant.

(g) Let f and g be harmonic in D, such that $\nabla^2 f = \nabla^2 g = 0$ in R and let $\partial f/\partial n = \partial g/\partial n$ so that $f = g + c$, where c is an arbitrary real constant, on S. Using (e) we find

$$\iint\limits_S \left(f\frac{\partial f}{\partial n} - g\frac{\partial g}{\partial n}\right)\, d\sigma = \iiint\limits_R \left(|\nabla f|^2 - |\nabla g|^2\right)\, dx\, dy\, dz$$

$$\bar{c}\iint\limits_S \frac{\partial f}{\partial n}\, d\sigma = \iiint\limits_R \left(|\nabla f|^2 - |\nabla g|^2\right)\, dx\, dy\, dz$$

$$\bar{c}\iint\limits_S \nabla_n f\, d\sigma = \iiint\limits_R \operatorname{div}(\nabla f - \nabla g)\, dx\, dy\, dz$$

Now from (5.84) and the fact that f is harmonic it follow that

$$\bar{c}\iint\limits_S \nabla_n f\, d\sigma = \bar{c}\iiint\limits_R \nabla^2 f\, dx\, dy\, dz = 0$$

and so it must be true that

$$\bar{c}\iint\limits_S \nabla_n f\, d\sigma = \iiint\limits_R \operatorname{div}(\nabla f - \nabla g)\, dx\, dy\, dz = \iiint\limits_R \nabla\cdot(\nabla f - \nabla g)\, dx\, dy\, dz = 0$$

which implies that $\nabla f = \nabla g$ in R, so that $f = g + \text{const}$ in R.

(h) Let f and g be harmonic in R, such that $\nabla^2 f = \nabla^2 g = 0$ in R and let

$$\frac{\partial f}{\partial n} = -f + h \qquad \frac{\partial g}{\partial n} = -g + h$$

on S, where $h = h(x, y, z)$. Using (a) we find

$$\iint\limits_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma = \iint\limits_S (f - g)\, h\, d\sigma = 0$$

which implies that $f \equiv g$ on S. Knowing this, it follows from (e) that $f \equiv g$ in R as well.

(i) Let f and g both satisfy the same Poisson equation in R:

$$\nabla^2 f = -4\pi h \qquad \nabla^2 g = -4\pi h \qquad h = h\,(x, y, z)$$

and let $f = g$ on S. Using (a) it then follows that

$$\begin{aligned} &\iiint\limits_R f\nabla^2 g\, dx\, dy\, dz - \iiint\limits_R g\nabla^2 f\, dx\, dy\, dz = \iint\limits_S f\frac{\partial g}{\partial n}\, d\sigma - \iint\limits_S g\frac{\partial f}{\partial n}\, d\sigma \\ &-4\pi \iiint\limits_R fh\, dx\, dy\, dz + 4\pi \iiint\limits_R gh\, dx\, dy\, dz = 0 \end{aligned}$$

The last equality implies that $f \equiv g$ in R.

(j) Using (a) we find

$$\begin{aligned} \iint\limits_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma &= \iint\limits_S f\frac{\partial g}{\partial n}\, d\sigma - \iint\limits_S g\frac{\partial f}{\partial n}\, d\sigma \\ &= \iiint\limits_R \nabla \cdot (f\nabla g)\ dx\, dy\, dz - \iiint\limits_R \nabla \cdot (g\nabla f)\ dx\, dy\, dz \\ &= \iiint\limits_R \left(f\nabla^2 g + \nabla f \cdot \nabla g\right)\ dx\, dy\, dz - \iiint\limits_R \left(g\nabla^2 f + \nabla g \cdot \nabla f\right)\ dx\, dy\, dz \\ &= \iiint\limits_R \left(f\nabla^2 g - g\nabla^2 f\right)\ dx\, dy\, dz \end{aligned}$$

(k) Let f and g be harmonic in R, such that $\nabla^2 f = \nabla^2 g = 0$ in R. It then follows from (j) that

$$\iint\limits_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma = \iiint\limits_R \Bigg(f\,\underbrace{\nabla^2 g}_{0} - g\,\underbrace{\nabla^2 f}_{0} \Bigg)\ dx\, dy\, dz = 0$$

(l) Let f and g satisfy the equations

$$\nabla^2 f = hf \qquad \nabla^2 g = hg \qquad h = h(z, y, z)$$

in R. Using (j) it then follows that

$$\iint\limits_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma = \iiint\limits_R \left(f\nabla^2 g - g\nabla^2 f \right) dx\,dy\,dz = \iiint\limits_R (fgh - fgh)\; dV = 0$$

5. Let S and R be as in Problem 4

(a) Using (5.84) it follows that

$$\begin{aligned}\iint\limits_S f\mathbf{n}\cdot\mathbf{i}\,d\sigma = \iint\limits_S (f\mathbf{i})_n\; d\sigma = \iiint\limits_R \nabla\cdot(f\mathbf{i})\; dx\,dy\,dz &= \iiint\limits_R (\nabla f\cdot\mathbf{i} + f\nabla\cdot\mathbf{i})\; dx\,dy\,dz \\ = \iiint\limits_R \nabla f\cdot\mathbf{i}\,dx\,dy\,dz &= \iiint\limits_R \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}\,dV \\ = \iiint\limits_R \frac{\partial f}{\partial x}\,dV &= \iiint\limits_R \frac{\partial f}{\partial x}\,dV\end{aligned}$$

(b) From (a) we know that

$$\iint\limits_S f\mathbf{n}\cdot\mathbf{i}\,d\sigma = \iiint\limits_R \frac{\partial f}{\partial x}\,dV$$

In the same way it can be shown that

$$\iint\limits_S f\mathbf{n}\cdot\mathbf{j}\,d\sigma = \iiint\limits_R \frac{\partial f}{\partial y}\,dV \qquad \iint\limits_S f\mathbf{n}\cdot\mathbf{k}\,d\sigma = \iiint\limits_R \frac{\partial f}{\partial z}\,dV$$

These are integrals of vectors as in Section 4.5 and hence, analogous to (4.58) we find

$$\iint\limits_S f\mathbf{n}\,d\sigma = \iiint\limits_R \frac{\partial f}{\partial x}\,dV\mathbf{i} + \iiint\limits_R \frac{\partial f}{\partial y}\,dV\mathbf{j} + \iiint\limits_R \frac{\partial f}{\partial z}\,dV\mathbf{k} = \iiint\limits_R \nabla f\,dV$$

(c) By (5.84) and (3.35) we find

$$\begin{aligned}\iint\limits_S \mathbf{v}\times\mathbf{i}\cdot\mathbf{n}\,d\sigma = \iint\limits_S (\mathbf{v}\times\mathbf{i})_n\; d\sigma &= \iiint\limits_R \operatorname{div}(\mathbf{v}\times\mathbf{i})\; dx\,dy\,dz \\ &= \iiint\limits_R (\mathbf{i}\cdot\operatorname{curl}\mathbf{v} - \mathbf{v}\cdot\operatorname{curl}\mathbf{i})\; dV \\ &= \iiint\limits_R \Big(\mathbf{i}\cdot\operatorname{curl}\mathbf{v} - \mathbf{v}\cdot\underbrace{\operatorname{curl}\mathbf{i}}_{\mathbf{0}}\Big)\; dV = \iiint\limits_R \operatorname{curl}\mathbf{v}\cdot\mathbf{i}\,dV\end{aligned}$$

(d) From (c) we know that

$$\iint\limits_S \mathbf{v}\times\mathbf{i}\cdot\mathbf{n}\,d\sigma \underset{(1.34)}{=} \iint\limits_S \mathbf{n}\times\mathbf{v}\cdot\mathbf{i}\,d\sigma = \iiint\limits_R \operatorname{curl}\mathbf{v}\cdot\mathbf{i}\,dV$$

In the same way it can be shown that

$$\iint\limits_S \mathbf{n}\times\mathbf{v}\cdot\mathbf{j}\,d\sigma = \iiint\limits_R \operatorname{curl}\mathbf{v}\cdot\mathbf{j}\,dV \qquad \iint\limits_S \mathbf{n}\times\mathbf{v}\cdot\mathbf{k}\,d\sigma = \iiint\limits_R \operatorname{curl}\mathbf{v}\cdot\mathbf{k}\,dV$$

These are integrals of vectors as in Section 4.5 and hence, analogous to (4.58) we find

$$\iint\limits_S \mathbf{n}\times\mathbf{v}\,d\sigma = \iiint\limits_R \operatorname{curl}\mathbf{v}\,dV$$

Section 5.13

1. (a) Let C be the circle $x^2+y^2=1$, $z=2$ directed so that y increases for positive x, and $\mathbf{u}=-3y\mathbf{i}+3x\mathbf{j}+\mathbf{k}$. Then by (5.96) and (4.64) we find

$$\begin{aligned}\int\limits_C u_T\,ds &= \int\limits_C L\,dx+M\,dy+N\,dz = \iint\limits_S (\operatorname{curl}\mathbf{u}\cdot\mathbf{n})\,d\sigma\\ &= \iint\limits_S \left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right)dy\,dz+\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)dz\,dx+\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dx\,dy\\ &= 6\iint\limits_S dx\,dy \underset{(4.64)}{=} 6\int_0^1\int_0^{2\pi} r\,d\theta\,dr = 12\pi\int_0^1 r\,dr = 6\pi\end{aligned}$$

(b) Let C be the curve $x=\cos t$, $y=\sin t$, $z=\sin t$, $0\le t\le 2\pi$, directed with increasing t. Then by (5.96(we find

$$\begin{aligned}\int\limits_C u_T\,ds &= \int\limits_C 2xy^2z\,dx+2x^2yz\,dy+\left(x^2y^2-2z\right)dz = \int\limits_C L\,dx+M\,dy+N\,dz\\ &= \iint\limits_S (\operatorname{curl}\mathbf{u}\cdot\mathbf{n})\,d\sigma\\ &= \iint\limits_S \left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right)dy\,dz+\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)dz\,dx+\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dx\,dy\\ &= \iint\limits_S \underbrace{\left[\frac{\partial}{\partial y}\left(x^2y^2-2z\right)-\frac{\partial}{\partial z}2x^2yz\right]}_{0}dy\,dz+\underbrace{\left[\frac{\partial}{\partial z}2xy^2z-\frac{\partial}{\partial x}\left(x^2y^2-2z\right)\right]}_{0}dz\,dx\\ &\quad+\underbrace{\left(\frac{\partial}{\partial x}2x^2yz-\frac{\partial}{\partial y}2xy^2z\right)}_{0}dx\,dy = 0\end{aligned}$$

2. (a) Let the integral

$$\int u_T\,ds = \int X\,dx + Y\,dy + Z\,dz = \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$$

be given. This integral is independent of path, since we can define the function $F(x,y,z) = xyz$ such that

$$\frac{\partial F}{\partial x} = yz = X \qquad \frac{\partial F}{\partial y} = xz = Y \qquad \frac{\partial F}{\partial z} = xy = Z$$

Or in other words, the integral is independent of path, since $\mathbf{u} = \nabla F$. Hence, by Theorem I of Section 5.13

$$\begin{aligned}\int_{(1,1,2)}^{(3,5,0)} X\,dx + Y\,dy + Z\,dz &= \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz \\ &= \int_{(1,1,2)}^{(3,5,0)} dF = F\,(3,5,0) - F\,(1,1,2) = -2\end{aligned}$$

(b) Let the integral

$$\int u_T\,ds = \int X\,dx + Y\,dy + Z\,dz = \int_{(1,0,0)}^{(1,0,2\pi)} \sin yz\,dx + xz\,\cos yz\,dy + xy\,\cos yz\,dz$$

on the helix $x = \cos t$, $y = \sin t$, $z = t$ be given. This integral is independent of path, since we can define the function $F(x,y,z) = x\sin yz$ such that

$$\frac{\partial F}{\partial x} = \sin yz = X \qquad \frac{\partial F}{\partial y} = xz\cos yz = Y \qquad \frac{\partial F}{\partial z} = xy\cos yz = Z$$

Or in other words, the integral is independent of path, since $\mathbf{u} = \nabla F$. Hence, by Theorem I of Section 5.13

$$\begin{aligned}\int_{(1,0,0)}^{(1,0,2\pi)} X\,dx + Y\,dy + Z\,dz &= \int_{(1,0,0)}^{(1,0,2\pi)} \sin yz\,dx + xz\,\cos yz\,dy + xy\,\cos yz\,dz \\ &= \int_{(1,0,0)}^{(1,0,2\pi)} dF = F\,(1,0,2\pi) - F\,(1,0,0) = 0\end{aligned}$$

3. Let C be a simple closed plane curve in space. Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a unit vector normal to the plane of C and let the direction on C match that of $\mathbf{n}$. Also, let $\mathbf{u} = (bz - cy)\mathbf{i} + (cx - az)\mathbf{j} + (ay - bx)\mathbf{k}$ be given. Then by (5.33), (5.93), (5.96), (5.78)

and (5.79) we find

$$\begin{aligned}
\frac{1}{2}\int\limits_C u_T\,ds &= \frac{1}{2}\int\limits_C \mathbf{u}\cdot d\mathbf{r} = \frac{1}{2}\int\limits_C (bz-cy)\;dx + (cx-az)\;dy + (ay-bx)\;dz \\
&= \frac{1}{2}\iint\limits_S \operatorname{curl}_n \mathbf{u}\,d\sigma \\
&= \frac{1}{2}\iint\limits_S \left[\frac{\partial}{\partial y}(ay-bx) - \frac{\partial}{\partial z}(cx-az)\right] dy\,dz + \left[\frac{\partial}{\partial z}(bz-cy) - \frac{\partial}{\partial x}(ay-bx)\right] dz\,dx \\
&\quad + \left[\frac{\partial}{\partial x}(cx-az) - \frac{\partial}{\partial y}(bz-cy)\right] dx\,dy \\
&= \iint\limits_S a\,dy\,dz + b\,dz\,dx + c\,dx\,dy = \iint\limits_S (a\cos\alpha + b\cos\beta + c\cos\gamma)\;d\sigma \\
&= \iint\limits_S (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(\cos\alpha\mathbf{i}+\cos\beta\mathbf{j}+\cos\gamma\mathbf{k})\;d\sigma = \iint\limits_S (\mathbf{n}\cdot\mathbf{n})\;d\sigma = \iint\limits_S d\sigma
\end{aligned}$$

which proves (see Problem 4 following Section 5.10) that $(1/2)\int_C u_T\,ds$ equals the plane area enclosed by C.

4. Let

$$\mathbf{u} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + z\mathbf{k}$$

and let D be the interior of the torus obtained by rotating the circle $(x-2)^2+z^2 = 1$, $y=0$ about the z-axis. Using (3.23) we find

$$\begin{aligned}
\operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u} &= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k} \\
&= \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\mathbf{k} \\
&= \left[\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right]\mathbf{k} = \mathbf{0}
\end{aligned}$$

Now in general the integral

$$\int u_T\,ds = \int \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + z\,dz$$

will not equal zero for some arbitrary path in D, since D is not simply connected (i.e. it is doubly connected). Let C be the circle $x^2+y^2=4$, $z=0$. We can now simplify (see Example 2 following Section 5.6) the line integral above to

$$\int\limits_C u_T\,ds = \frac{1}{4}\int\limits_C -y\,dx + x\,dy = \pm\frac{1}{2}\iint\limits_R dx\,dy = \pm\frac{1}{2}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = \pm 2\pi$$

where the $\pm$ accounts for the direction of integration along C. Hence, the integral is not independent of path, but depends on the number of times C goes around the origin. Taking a hint from Problem 4 following Section 5.7 it then follows that

$$\int_{(2,0,0)}^{(0,2,0)} u_T\,ds = \int_{(2,0,0)}^{(0,2,0)} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + z\,dz = \int_{(2,0)}^{(0,2)} \frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{\pi}{2} + 2n\pi$$

where $n = \pm 0, 1, 2, \ldots$.

5. (a) Let $\mathbf{u}$ be a vector in a simply connected domain D and let $\mathbf{v}$ be one solution of the equation $\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \mathbf{u}$. It is then trivial to see that all solutions to the same equation are given by $\mathbf{v} + \operatorname{grad} f = \mathbf{v} + \nabla f$, where f is an arbitrary differentiable scalar in D, since from (3.27) and (3.31) it follows that

$$\operatorname{curl}(\mathbf{v} + \operatorname{grad} f) = \operatorname{curl}\mathbf{v} + \operatorname{curl}\operatorname{grad} f = \operatorname{curl}\mathbf{v}$$

(b) Let $\mathbf{u} = (2xyz^2 + xy^3)\mathbf{i} + (x^2y^2 - y^2z^2)\mathbf{j} - (y^3z + 2x^2yz)\mathbf{k}$. Recognising that

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

then in order to find all vectors $\mathbf{v}$ such that $\operatorname{curl}\mathbf{v} = \mathbf{u}$ we need to solve the vector equation

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 2xyz^2 + xy^3 \quad \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = x^2y^2 - y^2z^2 \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -y^3z - 2x^2yz$$

which is satisfied for $\mathbf{v} = x^2y^2z\mathbf{i} - xy^3z\mathbf{j} + xy^2z^2\mathbf{k}$. Hence, analogous to part (a) all solutions to the equation $\operatorname{curl}\mathbf{v} = \mathbf{u}$ are given by $\mathbf{v} + \operatorname{grad} f$.

6. Let f and g be scalars having continuous second partial derivatives in a domain D. It then follows from (3.35) and (3.31) that

$$\operatorname{div}\mathbf{u} = \operatorname{div}(\nabla f \times \nabla g) = \operatorname{grad} g \cdot \underbrace{\operatorname{curl}\operatorname{grad} f}_{\mathbf{0}} - \operatorname{grad} f \cdot \underbrace{\operatorname{curl}\operatorname{grad} g}_{\mathbf{0}} = 0$$

and hence, the vector $\mathbf{u} = \nabla f \times \nabla g$ is solenoidal in D.

7. Let S be an oriented spherical surface in a domain D that forms the complete boundary of a bounded closed region R in D. Let $\mathbf{u}$ be a vector field in D whose components have continuous derivatives in D and let $\mathbf{n}$ be the outer normal of S with respect to R. Then by Gauss's Theorem (5.84)

$$\iint_S u_n\,d\sigma = \iiint_R \operatorname{div}\mathbf{u}\,dx\,dy\,dz$$

tells us that if $\iint_S u_n \, d\sigma = 0$ then

$$\iint\limits_S u_n \, d\sigma = \iiint\limits_R \operatorname{div} \mathbf{u} \, dx \, dy \, dz = 0 \implies \operatorname{div} \mathbf{u} \equiv 0$$

and hence, $\mathbf{u}$ is solenoidal in D. Now the converse is not necessarily true. That is, div$\mathbf{u} = 0$ does not always imply $\iint_S u_n \, d\sigma = 0$. Let us suppose that $\mathbf{u} = \operatorname{curl}_n \mathbf{v}$ for some vector field $\mathbf{v}$ in D and let C be a piecewise smooth simple closed curve in D directed in accordance with the given orientation in S. Then by Stokes's Theorem (5.95)

$$\iint\limits_S u_n \, d\sigma = \iint\limits_S \operatorname{curl}_n \mathbf{v} \, d\sigma = \int\limits_C v_T \, ds$$

However, we know from Problem 4 that for some chosen $\mathbf{v}$ it is possible that $\operatorname{curl} \mathbf{v} = \mathbf{0}$ and hence, $\operatorname{div} \mathbf{u} = \operatorname{div} \operatorname{curl} \mathbf{v} = 0$, but in general $\int_C v_T \, ds = \iint_S u_n \, d\sigma \neq 0$ if D is not simply connected.

8. Let C and S be as in Stokes's theorem

   (a) Using (5.95), (3.28) and (1.34) we find

$$\begin{aligned}\int\limits_C f\mathbf{T} \cdot \mathbf{i} \, ds = \iint\limits_S \operatorname{curl}_n (f\mathbf{i}) \, d\sigma &= \iint\limits_S \operatorname{curl} (f\mathbf{i}) \cdot \mathbf{n} \, d\sigma \\ &= \iint\limits_S (f \operatorname{curl} \mathbf{i} + \operatorname{grad} f \times \mathbf{i}) \cdot \mathbf{n} \, d\sigma \\ &= \iint\limits_S \nabla f \times \mathbf{i} \cdot \mathbf{n} \, d\sigma = \iint\limits_S \mathbf{n} \times \nabla f \cdot \mathbf{i} \, d\sigma\end{aligned}$$

   (b) From (a) we know that

$$\int\limits_C f\mathbf{T} \cdot \mathbf{i} \, ds = \iint\limits_S \mathbf{n} \times \nabla f \cdot \mathbf{i} \, d\sigma$$

   In the same way it can be shown that

$$\int\limits_C f\mathbf{T} \cdot \mathbf{j} \, ds = \iint\limits_S \mathbf{n} \times \nabla f \cdot \mathbf{j} \, d\sigma \qquad \int\limits_C f\mathbf{T} \cdot \mathbf{k} \, ds = \iint\limits_S \mathbf{n} \times \nabla f \cdot \mathbf{k} \, d\sigma$$

   These are integrals of vectors as in Section 4.5 and hence, analogous to (4.58) we find

$$\int\limits_C f\mathbf{T} \, ds = \iint\limits_S \mathbf{n} \times \nabla f \, d\sigma$$

9. Using the formal definition $\mathbf{v} \times \nabla = (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \times (\nabla_x\mathbf{i} + \nabla_y\mathbf{j} + \nabla_z\mathbf{k}) = (v_y\nabla_z - v_z\nabla_y)\mathbf{i} + (v_z\nabla_x - v_x\nabla_z)\mathbf{j} + (v_x\nabla_y - v_y\nabla_x)\mathbf{k}$ we can show that:

   (a)

$$\begin{aligned}
(\mathbf{v} \times \nabla) \cdot \mathbf{u} &= [(v_y\nabla_z - v_z\nabla_y)\mathbf{i} + (v_z\nabla_x - v_x\nabla_z)\mathbf{j} + (v_x\nabla_y - v_y\nabla_x)\mathbf{k}] \cdot (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \\
&= (v_y\nabla_z - v_z\nabla_y)u_x + (v_z\nabla_x - v_x\nabla_z)u_y + (v_x\nabla_y - v_y\nabla_x)u_z \\
&= v_y\nabla_z u_x - v_z\nabla_y u_x + v_z\nabla_x u_y - v_x\nabla_z u_y + v_x\nabla_y u_z - v_y\nabla_x u_z \\
&= v_x\nabla_y u_z - v_x\nabla_z u_y + v_y\nabla_z u_x - v_y\nabla_x u_z + v_z\nabla_x u_y - v_z\nabla_y u_x \\
&= v_x\left(\nabla_y u_z - \nabla_z u_y\right) + v_y\left(\nabla_z u_x - \nabla_x u_z\right) + v_z\left(\nabla_x u_y - \nabla_y u_x\right) \\
&= (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \cdot \left[\left(\nabla_y u_z - \nabla_z u_y\right)\mathbf{i} + \left(\nabla_z u_x - \nabla_x u_z\right)\mathbf{j} + \left(\nabla_x u_y - \nabla_y u_x\right)\mathbf{k}\right] \\
&= \mathbf{v} \cdot \nabla \times \mathbf{u} = \mathbf{v} \cdot \operatorname{curl}\mathbf{u}
\end{aligned}$$

   (b)

$$\begin{aligned}
(\mathbf{v} \times \nabla) \times \mathbf{u} &= [(v_y\nabla_z - v_z\nabla_y)\mathbf{i} + (v_z\nabla_x - v_x\nabla_z)\mathbf{j} + (v_x\nabla_y - v_y\nabla_x)\mathbf{k}] \times (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \\
&= \left[\left(v_z\nabla_x - v_x\nabla_z\right)u_z - \left(v_x\nabla_y - v_y\nabla_x\right)u_y\right]\mathbf{i} \\
&\quad + \left[\left(v_x\nabla_y - v_y\nabla_x\right)u_x - \left(v_y\nabla_z - v_z\nabla_y\right)u_z\right]\mathbf{j} \\
&\quad + \left[\left(v_y\nabla_z - v_z\nabla_y\right)u_y - \left(v_z\nabla_x - v_x\nabla_z\right)u_x\right]\mathbf{k} \\
&= \left(v_z\nabla_x u_z - v_x\nabla_z u_z - v_x\nabla_y u_y + v_y\nabla_x u_y\right)\mathbf{i} \\
&\quad + \left(v_x\nabla_y u_x - v_y\nabla_x u_x - v_y\nabla_z u_z + v_z\nabla_y u_z\right)\mathbf{j} \\
&\quad + \left(v_y\nabla_z u_y - v_z\nabla_y u_y - v_z\nabla_x u_x + v_x\nabla_z u_x\right)\mathbf{k} \\
&= \left(v_z\nabla_x u_z - v_x\nabla_z u_z - v_x\nabla_y u_y + v_y\nabla_x u_y + v_x\nabla_x u_x - v_x\nabla_x u_x\right)\mathbf{i} \\
&\quad + \left(v_x\nabla_y u_x - v_y\nabla_x u_x - v_y\nabla_z u_z + v_z\nabla_y u_z + v_y\nabla_y u_y - v_y\nabla_y u_y\right)\mathbf{j} \\
&\quad + \left(v_y\nabla_z u_y - v_z\nabla_y u_y - v_z\nabla_x u_x + v_x\nabla_z u_x + v_z\nabla_z u_z - v_z\nabla_z u_z\right)\mathbf{k} \\
&= \left(v_x\nabla_x u_x\mathbf{i} + v_x\nabla_y u_x\mathbf{j} + v_x\nabla_z u_x\mathbf{k}\right) + \left(v_y\nabla_x u_y\mathbf{i} + v_y\nabla_y u_y\mathbf{j} + v_y\nabla_z u_y\mathbf{k}\right) \\
&\quad + \left(v_z\nabla_x u_z\mathbf{i} + v_z\nabla_y u_z\mathbf{j} + v_z\nabla_z u_z\mathbf{k}\right) - \left(\nabla_x u_x + \nabla_y u_y + \nabla_z u_z\right)v_x\mathbf{i} \\
&\quad - \left(\nabla_x u_x + \nabla_y u_y + \nabla_z u_z\right)v_y\mathbf{j} - \left(\nabla_x u_x + \nabla_y u_y + \nabla_z u_z\right)v_z\mathbf{k} \\
&= v_x\nabla u_x + v_y\nabla u_y + v_z\nabla u_z - \left(\nabla_x u_x + \nabla_y u_y + \nabla_z u_z\right)\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) \\
&= \nabla_u\left(\mathbf{v} \cdot \mathbf{u}\right) - \left(\nabla \cdot \mathbf{u}\right)\mathbf{v}
\end{aligned}$$

10. Let C and S be as in Stokes's theorem

(a) Using (5.95) and Problem 9(a) we find

$$\int\limits_C \mathbf{T}\times\mathbf{u}\cdot\mathbf{i}\,ds \underset{9(a)}{=} \int\limits_C \mathbf{u}\times\mathbf{i}\cdot\mathbf{T}\,ds = \int\limits_C (\mathbf{u}\times\mathbf{i})_T\ ds = \iint\limits_S \operatorname{curl}(\mathbf{u}\times\mathbf{i})\cdot\mathbf{n}\,d\sigma$$

$$= \iint\limits_S \nabla\times(\mathbf{u}\times\mathbf{i})\cdot\mathbf{n}\,d\sigma$$

$$\underset{9(a)}{=} \iint\limits_S (\mathbf{n}\times\nabla)\cdot(\mathbf{u}\times\mathbf{i})\ d\sigma$$

$$\underset{9(a)}{=} \iint\limits_S (\mathbf{n}\times\nabla)\times\mathbf{u}\cdot\mathbf{i}\,d\sigma$$

(b) From (a) we know that

$$\int\limits_C \mathbf{T}\times\mathbf{u}\cdot\mathbf{i}\,ds = \iint\limits_S (\mathbf{n}\times\nabla)\times\mathbf{u}\cdot\mathbf{i}\,d\sigma$$

In the same way it can be shown that

$$\int\limits_C \mathbf{T}\times\mathbf{u}\cdot\mathbf{j}\,ds = \iint\limits_S (\mathbf{n}\times\nabla)\times\mathbf{u}\cdot\mathbf{j}\,d\sigma \quad \int\limits_C \mathbf{T}\times\mathbf{u}\cdot\mathbf{k}\,ds = \iint\limits_S (\mathbf{n}\times\nabla)\times\mathbf{u}\cdot\mathbf{k}\,d\sigma$$

These are integrals of vectors as in Section 4.5 and hence, analogous to (4.58) we find

$$\int\limits_C \mathbf{T}\times\mathbf{u}\,ds = \iint\limits_S (\mathbf{n}\times\nabla)\times\mathbf{u}\,d\sigma$$

Section 5.14

1. (a) Let $u = y$, $v = x$. Then by (5.104)

$$\iint\limits_{R_{xy}} F(x\,y)\ dx\,dy = \int_0^1\int_0^y \left(x^2+y^2\right)\ dx\,dy = \iint\limits_{R_{uv}} F\left[f(u,v),g(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^1\int_0^u \left(u^2+v^2\right)\ dv\,du$$

(b) Let R_{xy} be the region $x^2+y^2\le 1$, where $x = u+(1-u^2-v^2)$, $y = v+(1-u^2-v^2)$. The region R_{xy} thus is bounded by the circle $C_{xy} : x^2+y^2 = 1$. Next, let R_{uv} be the region $u^2+v^2\le 1$, so that R_{uv} is bounded by the circle $C_{uv} : u^2+v^2 = 1$ and so $x = u$, $y = v$ when (u,v) is on C_{uv}. Hence, the corresponding point (x,y), $x = f(u,v) = u$, $y = g(u,v) = v$ is on C_{xy}, and as (u,v) traces C_{uv} once in the positive direction, (x, y) traces C_{xy} once in the positive direction. Then by (5.106)

$$\iint\limits_{R_{xy}} F(x\,y)\;dx\,dy = \iint\limits_{R_{xy}} (x-y)\;dx\,dy = \pm\iint\limits_{R_{uv}} F\left[f(u,v), g(u,v)\right]\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= \iint\limits_{R_{uv}} (u-v)(1-2u-2v)\;du\,dv$$

where in the last step the $+$ sign has been chosen, since, again, when a point (u, v) on C_{uv} is traced in the positive direction, a point (x, y) on C_{xy} will be traced in the positive direction as well.

(c) Let R_{xy} be the region $x^2 + y^2 \leq 1$, where $x = u^2 - v^2$, $y = 2uv$. The region R_{xy} thus is bounded by the circle $C_{xy} : x^2 + y^2 = 1$. Next, let us choose R_{uv} as the region $u^2 + v^2 \leq 1$, so that R_{uv} is bounded by the circle $C_{uv} : u^2 + v^2 = 1$. Now let us make the substitution $u = \cos\phi$, $v = \sin\phi$ for a point (u, v) on C_{uv}. Hence, for a point (u, v) on C_{uv} the corresponding point (x, y), $x = \cos^2\phi - \sin^2\phi = \cos 2\phi$, $y = 2\sin\phi\cos\phi = \sin 2\phi$ is on C_{xy}. Now if we increment ϕ from zero to 2ϕ, the point (u, v) goes around C_{uv} exactly once in the positive (counter clockwise) direction and the point (x, y) goes around C_{xy} exactly twice in the positive direction. Hence, we may conclude that the degree of the mapping of C_{uv} into C_{xy} is given by $\delta = 2$. Note that δ is positive, as both C_{uv} and C_{xy} are traced in the same direction. Then by (5.105)

$$\delta\iint\limits_{R_{xy}} F(x\,y)\;dx\,dy = 2\iint\limits_{R_{xy}} xy\,dx\,dy = \iint\limits_{R_{uv}} F\left[f(u,v), g(u,v)\right]\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= 8\iint\limits_{R_{uv}} uv\left(u^4 - v^4\right)\;du\,dv$$

and so

$$\iint\limits_{R_{xy}} xy\,dx\,dy = 4\iint\limits_{R_{uv}} uv\left(u^4 - v^4\right)\;du\,dv$$

2. Let $x = f(u, v)$, $y = g(u, v)$ be given as in Theorem II and let the curve C_{xy} enclose the origin O. We know from Example 2 of Section 5.6 that

$$\oint\limits_{C_{xy}} \frac{-y\,dx + x\,dy}{x^2 + y^2} = 2\pi$$

for any closed path C_{xy} enclosing the origin O. Next, let us define

$$P(x,y) = \frac{-y}{x^2+y^2} \qquad\qquad Q(x,y) = \frac{x}{x^2+y^2}$$

Then using Theorem II we find

$$\begin{aligned}
2\pi\delta = \delta \oint_{C_{xy}} \frac{-y\,dx + x\,dy}{x^2+y^2} &= \delta \oint_{C_{xy}} P(x,y)\ dx + Q(x,y)\ dy \\
&= \delta \oint_{C_{xy}} P(x,y)\ dx + \delta \oint_{C_{xy}} Q(x,y)\ dy \\
&= \oint_{C_{uv}} P\left(\frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv\right) + \oint_{C_{uv}} Q\left(\frac{\partial g}{\partial u}\,du + \frac{\partial g}{\partial v}\,dv\right) \\
&= \oint_{C_{uv}} P\left(\frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv\right) + Q\left(\frac{\partial g}{\partial u}\,du + \frac{\partial g}{\partial v}\,dv\right) \\
&= \oint_{C_{uv}} \frac{-y\,dx + x\,dy}{x^2+y^2}
\end{aligned}$$

where in the last step x, y, dx, dy are expressed in terms of u, v:

$$x = f(u,v) \qquad dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv \qquad y = g(u,v) \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv$$

3. Using the formula of Problem 2 we can evaluate the degree for the following mappings of the circle $u^2+v^2=1$ into the circle $x^2+y^2=1$:

   (a) When $x = (3u+4v)/5$, $y = (4u-3v)/5$ we find

   $$\begin{aligned}
   2\pi\delta = \oint_{C_{uv}} \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{C_{uv}} v\,du - u\,dv &\underset{(5.40)}{=} -2 \iint_{R_{uv}} du\,dv \\
   &\underset{(4.64)}{=} -2 \int_0^1 \int_0^{2\pi} r\,d\theta\,dr = -2\pi
   \end{aligned}$$

   and so $\delta = -1$.

   (b) When $x = u^2 - v^2$, $y = 2uv$ we find

   $$\begin{aligned}
   2\pi\delta = \oint_{C_{uv}} \frac{-y\,dx + x\,dy}{x^2+y^2} = -2 \oint_{C_{uv}} v\,du - u\,dv &\underset{(5.40)}{=} 4 \iint_{R_{uv}} du\,dv \\
   &\underset{(4.64)}{=} 4 \int_0^1 \int_0^{2\pi} r\,d\theta\,dr = 4\pi
   \end{aligned}$$

   and so $\delta = 2$.

   (c) ***Note****: There is actually an error in the problem statement. $x = u^3 - uv^2$ most likely is incorrect, since otherwise the relation $x^2+y^2=1$ does not hold. Instead $x = u^3 - 3uv^2$ is probably meant here.*

When $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$ we find

$$\begin{aligned}
2\pi\delta &= \oint\limits_{C_{uv}} \frac{-y\,dx + x\,dy}{x^2+y^2} \\
&= -3 \oint\limits_{C_{uv}} \left(v^5 + 2u^2v^3 + u^4v\right)\,du + \left(-u^5 - 2u^3v^2 - uv^4\right)\,dv \\
&= -3 \iint\limits_{R_{uv}} \left[\frac{\partial}{\partial u}\left(-u^5 - 2u^3v^2 - uv^4\right) - \frac{\partial}{\partial v}\left(v^5 + 2u^2v^3 + u^4v\right)\right] du\,dv \\
&= 18 \iint\limits_{R_{uv}} du\,dv = 18 \int_0^1 \int_0^{2\pi} r\,d\theta\,dr = 18\pi
\end{aligned}$$

and so $\delta = 9$.

4. Let R be the circular region $x^2+y^2 \le 1$, where $x = r\cos\theta$, $y = r\sin\theta$. Furthermore, let $R = R_1 + R_2$, where R_1 is the semicircular region $x^2+y^2 \le 1$, $y \ge 0$ and R_2 is the semicircular region $x^2+y^2 \le 1$, $y \le 0$. With R_1 we can then associate the corresponding rectangle $0 \le \theta \le \pi$, $0 \le r \le 1$ in the $r\theta$ plane. Both R_1 and the rectangle in the $r\theta$ plane are bounded closed regions that are bounded by piecewise smooth simple closed curves C_{xy} and $C_{r\theta}$ respectively. Furthermore, we assume (since this is not explicitly stated in the problem statement) that the function $F(x,y)$ is defined and has continuous first derivatives in R_1. Now from the definition $x = r\cos\theta$, $y = r\sin\theta$ it follows that the functions $x = f(r,\theta)$, $y = g(r,\theta)$ have continuous second derivatives in the $r\theta$ plane. Hence, when (r,θ) is in the rectangle $0 \le \theta \le \pi$, $0 \le r \le 1$, the point (x,y) is in R_1. As such, conditions a), b) and c) of Theorem I are all satisfied. Furthermore, when (r,θ) is on $C_{r\theta}$, the corresponding point (x,y) is on C_{xy} and when $(r, theta)$ traces $C_{r\theta}$ once on the positive (counter clockwise) direction, (x,y) traces C_{xy} once in the positive direction, which thus satisfies condition d) of Theorem I as well. Or alternatively, this satisfies condition d') of Theorem II with the degree of the mapping δ of $C_{r\theta}$ onto C_{xy} defined as $\delta = 1$. Note that the correspondence between $C_{r\theta}$ and C_{xy} is *not* one-to-one, since the line $r = 0, 0 \le \theta \le \pi$ on $C_{r\theta}$ maps to the singular point $(0,0)$ on C_{xy}. Hence, this proves the validity of the transformation formula

$$\begin{aligned}
\iint\limits_{R_1} F(x,y)\,dx\,dy &= \iint\limits_{r\theta_1} F\left[f(r,\theta), g(r,\theta)\right] \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta \\
&= \int_0^{\pi} \int_0^1 F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta
\end{aligned}$$

Following the same reasoning a similar result may be obtained for R_2:

$$\iint\limits_{R_2} F(x,y)\; dx\, dy = \iint\limits_{r\theta_2} F\left[f(r,\theta), g(r,\theta)\right] \frac{\partial(x,y)}{\partial(r,\theta)}\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi}\int_0^1 F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Adding the results for R_1 and R_2 then finally gives

$$\iint\limits_{R_1} F(x,y)\; dx\, dy + \iint\limits_{R_2} F(x,y)\; dx\, dy = \iint\limits_{R} F(x,y)\; dx\, dy$$

$$= \int_0^{\pi}\int_0^1 F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta + \int_{\pi}^{2\pi}\int_0^1 F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

5. Let S be a plane surface, oriented in accordance with a unit normal $\mathbf{n}$. The solid angle Ω of S with respect to a point O not in S is defined as

$$\Omega(O,S) = \pm \text{area of projection of } S \text{ on } S_1$$

where S_1 is the sphere of radius 1 about O and the $+$ or $-$ sign is chosen according to whether $\mathbf{n}$ points away from or toward the side of S on which O lies.

(a) Let us define $\mathbf{r}/r$ as the unit vector corresponding to the position vector $\mathbf{r}$ directed from point O to the origin of S and let θ represent the angle between $\mathbf{r}$ and $\mathbf{n}$. Hence, by (1.9) we have

$$\frac{\mathbf{r}\cdot\mathbf{n}}{r} = \cos\theta$$

Let us consider the case where the point O lies in the plane of S, but not actually in S so that $\theta = \pi/2$. Since the solid angle Ω of S with respect to O is equal to $\pm$ the area of projection of S on S_1, or in other words:

$$\Omega(O,S) \propto \pm\, \mathbf{r}\cdot\mathbf{n}$$

we may conclude that if O lies in the plane of S the solid angle $\Omega(O,S) = 0$, since then $\mathbf{r} \perp \mathbf{n}$ and so $\cos\pi/2 = 0$.

(b) When S is a complete (that is, infinite) plane, then the area of projection of S on the unit sphere S_1 will be equal to $\pm$ half the surface area of S_1, since only half of S_1 will always be in the field of view of S at any given moment. As the surface area of S_1 is 4π, the solid angle Ω will be given by $\Omega(O,S) = \pm 2\pi$, with the $+$ or $-$ sign depending on if the normal $\mathbf{n}$ is pointing away or toward the side of S on which O lies.

(c) Let S be an arbitrary oriented surface in space. This surface can be thought of as made up of infinitesimal elements of surface area $d\sigma$, each of which is approximately planar and having a normal $\mathbf{n}$. Furthermore, with each element we can associate the unit vector $\mathbf{r}/r$, where $\mathbf{r}$ is the position vector with respect to the point O. The *element of solid angle* for the surface element $d\sigma$ may then be defined as

$$d\Omega = \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, d\sigma$$

where the extra factor of $1/r^2$ is due to the fact that the projection of $d\sigma$ on S_1 (i.e. $d\Omega$) is inversely proportional to the square of the distance from $d\sigma$.

(d) On the basis of the formula of (c) we obtain the following equation for the solid angle for a general oriented surface S:

$$\Omega\,(O, S) = \iint\limits_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, d\sigma$$

Let us now consider the parametric equations

$$x = x\,(u, v) \qquad\qquad y = y\,(u, v) \qquad\qquad z = z\,(u, v)$$

where $(u, v) \in R_{uv}$. These equations can be regarded as a mapping of R_{uv} onto a curved region in space as discussed in Section 4.7. Using (4.81), (4.82), (5.73), (1.33), (1.34) and taking a hint from (4.83) we thus find

$$\begin{aligned}
\Omega\,(O, S) &= \iint\limits_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, d\sigma \\
&= \iint\limits_{R_{uv}} \frac{\mathbf{r} \cdot \mathbf{P}_1 \times \mathbf{P}_2}{r^3 |\mathbf{P}_1 \times \mathbf{P}_2|} |\mathbf{P}_1 \times \mathbf{P}_2|\, du\, dv \\
&= \iint\limits_{R_{uv}} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}} \cdot \left[\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\right] du\, dv \\
&= \iint\limits_{R_{uv}} \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \frac{1}{\left(x^2 + y^2 + z^2\right)^{3/2}}\, du\, dv
\end{aligned}$$

(e) Let the normal $\mathbf{n}$ on S_1 be the outer one and let $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$ for $0 \le u \le \pi$, $0 \le v \le 2\pi$. Using the formula for $\Omega(O, S)$ from (d) we thus find

$$\Omega\,(O, S) = \iint\limits_{R_{uv}} \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \frac{1}{\left(x^2 + y^2 + z^2\right)^{3/2}}\, du\, dv = \int_0^{2\pi} \int_0^{\pi} \sin u\, du\, dv = 4\pi$$

(f) Let S from the boundary of a bounded, closed, simply connected region R. Any such region R could in turn be imagined to be a sub set of some spherical region. Now consider a point O inside S that is the origin of the spherical region containing S. Then we can again apply the formula for $\Omega(O,S)$ from (d), using the parameterisation

$$x = a\sin u\cos v \qquad y = a\sin u\sin v \qquad z = a\cos u \ (u,v)$$

where a is the radius of the sphere containing S and $(u,v) \in R_{uv}$. Hence, we find

$$\Omega(O,S) = \pm \iint\limits_{R_{uv}} \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \frac{1}{(x^2+y^2+z^2)^{3/2}}\,du\,dv = \pm \int_0^{2\pi}\int_0^{\pi} \sin u\,du\,dv = \pm 4\pi$$

where the + sign applies when $\mathbf{n}$ is the outer normal and $-$ for the inward normal. Next, imagine that the point O is located outside of S. In this case S always has a far side and a near side facing O. For instance, let S be a sphere. Each hemisphere contributes to the solid angle in equal, but opposite parts due to the orientation of the normal $\mathbf{n}$ associated with each hemisphere. Hence, the net sum of the contributions of each hemisphere will sum up to zero, i.e. $\Omega(O,S) = 0$.

(g) Let S be a fixed circular disk and let the point O be variable. Let $\mathbf{r}/r$ again denote the unit vector corresponding to the position vector $\mathbf{r}$ with respect to O and let $\mathbf{n}$ be the outer normal of S. Furthermore, let the angle between $\mathbf{r}$ and $\mathbf{n}$ be given by θ such that $\mathbf{r}\cdot\mathbf{n}/r = \cos\theta$ and let $r = a$ be fixed. Next, let $0 \le \theta \le \pi$ and let S be parameterised by the equations $x = \cos u$, $y = \sin u$. Utilising the formula for S from (d) we thus find

$$\Omega(O,S) = \int\limits_{R_u} \frac{\cos\theta}{(x^2+y^2)^{3/2}}\,du = \int_0^{2\pi} \cos\theta\,du$$

Now when θ varies from 0 to π, the expression inside the integral will vary between 1 and -1 and so $-2\pi \le \Omega(O,S) \le 2\pi$.

6. Let S_{uvw} be the sphere $u = \sin s\cos t$, $v = \sin s\sin t$, $w = \cos s$ where $0 \le s \le \pi$, $0 \le t \le 2\pi$ and let S_{xyz} be the sphere $x^2+y^2+z^2 = 1$.

   (a) Let S_{xyz} be parameterised by the equations $x = v$, $y = -w$, $z = u$. The degree of the mapping of S_{uvw} into S_{xyz} is then given by

$$\delta = \frac{1}{4\pi}\iint\limits_{R_{st}} \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \frac{1}{(x^2+y^2+z^2)^{3/2}}\,ds\,dt$$

$$= \frac{1}{4\pi}\iint\limits_{R_{st}} \begin{vmatrix} \sin s\sin t & -\cos s & \sin s\cos t \\ \cos s\sin t & \sin s & \cos s\cos t \\ \sin s\cos t & 0 & -\sin s\sin t \end{vmatrix} ds\,dt = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin s\,ds\,dt = -1$$

(b) Let S_{xyz} be parameterised by the equations $x = u^2 - v^2$, $y = 2uv$, $z = w\sqrt{2 - w^2}$. The degree of the mapping of S_{uvw} into S_{xyz} is then given by

$$\begin{aligned}
\delta &= \frac{1}{4\pi} \iint\limits_{R_{st}} \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\, ds\, dt \\
&= \frac{1}{4\pi} \iint\limits_{R_{st}} \begin{vmatrix} \sin^2 s \cos 2t & \sin^2 s \sin 2t & \cos s\sqrt{2 - \cos^2 s} \\ \sin 2s \cos 2t & \sin 2s \sin 2t & -\sin s\sqrt{2 - \cos^2 s} + \dfrac{\sin 2s \cos s}{2\sqrt{2 - \cos^2 s}} \\ -2\sin^2 s \sin 2t & 2\sin^2 s \cos 2t & 0 \end{vmatrix} ds\, dt \\
&= \frac{1}{\pi} \iint\limits_{R_{st}} \frac{\sin^3 s}{\sqrt{2 - \cos^2 s}}\, ds\, dt = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\pi} \frac{1 - \cos^2 s}{\sqrt{2 - \cos^2 s}} \sin s\, ds\, dt \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_{-1}^{1} \frac{1 - p^2}{\sqrt{2 - p^2}}\, dp\, dt = \frac{1}{\pi} \int_0^{2\pi} \int_{-1}^{1} \left(\frac{1}{\sqrt{2 - p^2}} - \frac{p^2}{\sqrt{2 - p^2}} \right) dp\, dt \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_{-\pi/4}^{\pi/4} \left(\frac{\cos q}{\sqrt{1 - \sin^2 q}} - \frac{2\sin^2 q \cos q}{\sqrt{1 - \sin^2 q}} \right) dq\, dt = \frac{1}{\pi} \int_0^{2\pi} \int_{-\pi/4}^{\pi/4} \left(1 - 2\sin^2 q\right) dq\, dt \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_{-\pi/4}^{\pi/4} \cos 2q\, dq\, dt = \frac{1}{\pi} \int_0^{2\pi} dt = 2
\end{aligned}$$

where we have used the substitutions $p = -\cos s \implies dp = \sin s\, ds$ and $p = \sqrt{2}\sin q \implies dp = \sqrt{2}\cos q\, dq$.

Section 5.15

1. (a) Let a particle of mass m move on a straight line, the x-axis, subject to a force $F = -k^2 x$. Using (5.112), the potential energy U then is given by

$$U = -\int_{x_1}^{x} F_T\, ds + \text{const} = k^2 \int_{x_1}^{x} x'\, dx' + \text{const} = \frac{k^2 x^2}{2} - \frac{k^2 x_1^2}{2} + \text{const}$$

where const is an arbitrary scalar constant that we are free to choose. Hence, by setting const $= k^2 x_1^2/2$ the potential energy U becomes

$$U = \frac{k^2 x^2}{2}$$

In order for the total energy of the particle to be conserved (i.e. E doesn't change with time) we require

$$E = \frac{mv^2}{2} + U = \frac{mv^2}{2} + \frac{k^2 x^2}{2} = \text{const}$$

Furthermore, the total energy E would *not* be conserved if a resistance $-cx_t$ is added, since then

$$E = K + U = \frac{mv^2}{2} + \frac{k^2x^2}{2} = c\frac{dx}{dt} \neq \text{const}$$

(b) Let a particle of mass m move in the xy-plane subject to a force $\mathbf{F} = -a^2x\mathbf{i} - b^2y\mathbf{j}$. Using (5.112), the potential energy then is given by

$$\begin{aligned} U = -\int_{(x_1,y_1)}^{(x,y)} F_T\, ds + \text{const} &= \int_{(x_1,y_1)}^{(x,y)} a^2x'\, dx' + b^2y'\, dy' + \text{const} \\ &= \frac{a^2x^2 + b^2y^2}{2} - \frac{a^2x_1^2 + b^2y_1^2}{2} + \text{const} \end{aligned}$$

where const is an arbitrary scalar constant that we are free to choose. Hence, by setting const $= (a^2x_1^2 + b^2y_1^2)/2$ the potential energy U becomes

$$U = \frac{a^2x^2 + b^2y^2}{2}$$

In order for the total energy of the particle to be conserved (i.e. E doesn't change with time) we require

$$E = \frac{mv^2}{2} + U = \frac{mv^2}{2} + \frac{a^2x^2 + b^2y^2}{2} = \text{const}$$

2. Let D be a simply connected domain in the xy-plane and let $\mathbf{w} = u\mathbf{i} - v\mathbf{j}$ be the velocity vector of an irrotational, incompressible flow in D.

(a) Since the flow is incompressible we find by (5.117)

$$\operatorname{div}\mathbf{w} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ in } D$$

and hence, $\mathbf{w}$ is solenoidal. Furthermore, since the flow is irrotational we find by (5.118) and (3.23)

$$\operatorname{curl}\mathbf{w} = \mathbf{0} \iff \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)\mathbf{k} = \mathbf{0} \iff \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ in } D$$

(b) From (a), (3.15) and (3.16) follows

$$\begin{aligned} \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} &= \frac{\partial v}{\partial y}\mathbf{i} - \frac{\partial v}{\partial x}\mathbf{j} \\ \operatorname{div}\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right) &= \operatorname{div}\left(\frac{\partial v}{\partial y}\mathbf{i} - \frac{\partial v}{\partial x}\mathbf{j}\right) \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} \\ \nabla^2 u &= 0 \end{aligned}$$

Hence, by (2.126) and (2.127) u is harmonic in D. Similarly, we find

$$\begin{aligned}
\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} &= -\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j} \\
\operatorname{div}\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right) &= \operatorname{div}\left(-\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j}\right) \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= -\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial x \partial y} \\
\nabla^2 v &= 0
\end{aligned}$$

and so v is harmonic in D also.

(c) Let $P(x, y) = u$, $Q(x, y) = -v$ and let C be a piecewise smooth simple closed curve in D whose interior is also in D. Then by Green's Theorem (5.40) we find

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

on every closed path C in D. Hence, by Theorem II of Section 5.6, since the integral

$$\oint_C P\,dx + Q\,dy = \oint_C u\,dx - v\,dy = 0$$

the integral $\int u\,dx - v\,dy$ is independent of path in D. Next, let $\bar{P}(x, y) = v$, $\bar{Q}(x, y) = u$ and let C again be a piecewise smooth simple closed curve in D whose interior is also in D. Green's Theorem gives

$$\oint_C \bar{P}\,dx + \bar{Q}\,dy = \iint_R \left(\frac{\partial \bar{Q}}{\partial x} - \frac{\partial \bar{P}}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

on every closed path C in D. Hence, by Theorem II of Section 5.6, since the integral

$$\oint_C \bar{P}\,dx + \bar{Q}\,dy = \oint_C v\,dx + u\,dy = 0$$

the integral $\int v\,dx + u\,dy$ is also independent of path in D.

(d) Let us assume that the vector $\mathbf{w}$ can be defined as the gradient of some scalar function $\phi(x, y)$. In addition to the vector $\mathbf{w} = \operatorname{grad}\phi = u\mathbf{i} - v\mathbf{j}$ let us define the vector $\mathbf{w}' = \operatorname{grad}\psi = v\mathbf{i} + u\mathbf{j}$ for some scalar ψ. Or written out more explicitly:

$$\mathbf{w} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} = u\mathbf{i} - v\mathbf{j} \qquad\qquad \mathbf{w}' = \frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j} = v\mathbf{i} + u\mathbf{j}$$

From which follows

$$\frac{\partial \phi}{\partial x} = u = \frac{\partial \psi}{\partial y} \qquad\qquad \frac{\partial \phi}{\partial y} = -v = -\frac{\partial \psi}{\partial x}$$

The functions $\phi(x, y)$ and $\psi(x, y)$ in turn can be thought of as the x and y components of a vector $\mathbf{F}$, that is $\mathbf{F} = \phi\mathbf{i} - \psi\mathbf{j}$, such that $F_x = \phi$, $F_y = -\psi$ and so $\operatorname{grad} F_x = \operatorname{grad}\phi = \mathbf{w}$, $\operatorname{grad} F_y = -\operatorname{grad}\psi = -\mathbf{w}'$.

(e) Using the definition for $\mathbf{F}$ from (d) we find

$$\operatorname{div}\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot (\phi\mathbf{i} - \psi\mathbf{j}) = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} = 0$$

and using (3.23)

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} = \left(-\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

(f) Since D is simply connected and the flow is irrotational we can write $\mathbf{w} = \operatorname{grad}\phi$ for some scalar ϕ, termed the *velocity potential.* Since the flow is both irrotational and incompressible then by (5.117) ϕ must satisfy the equation $\operatorname{div}\operatorname{grad}\phi = 0$, or writing this out explicitly

$$\operatorname{div}\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}\right) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

and so ϕ is harmonic in D. Similarly, we can write $\mathbf{w}' = \operatorname{grad}\psi$ for some scalar ψ, termed the *stream function*, where ψ satisfies the equation $\operatorname{div}\operatorname{grad}\psi = 0$, or writing this out explicitly

$$\operatorname{div}\left(\frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j}\right) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot \left(\frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and so ψ is harmonic in D also.

(g) By definition, streamlines are defined as the lines along which the stream function ψ is constant. Since $\mathbf{w} = \operatorname{grad}\phi = u\mathbf{i} - v\mathbf{j} \perp v\mathbf{i} + u\mathbf{j} = \operatorname{grad}\psi = \mathbf{w}'$ a tangent drawn at any point on a streamline gives the direction of velocity. The slope of a streamline is given by $dy/dx = v/u$. Similarly, the lines along which the velocity potential is constant are called equipotential lines. The slope of an equipotential line is given by $dy/dx = -u/v$. Note that the product of the slope of a streamline and an equipotential line is -1, again illustrating that the lines are perpendicular to each other.

3. let a wire occupying the line segment from $(0, -c)$ to $(0, c)$ in the xy-plane have a constant charge density equal to ρ. Using (5.137) the electrostatic potential due to

this wire at a point (x_1, y_1) of the xy-plane is given by

$$\begin{aligned}
\Phi\left(x_1, y_1\right) &= \int\limits_C \frac{\rho\, ds}{r_1} + \text{const} = \int_{-c}^{c} \frac{\rho\, dy}{\sqrt{x_1^2 + \left(y - y_1\right)^2}} + k = \int_{-c-y_1}^{c-y_1} \frac{\rho\, du}{\sqrt{x_1^2 + u^2}} + k \\
&= \rho \int_{\tan^{-1} \frac{-c-y_1}{x_1}}^{\tan^{-1} \frac{c-y_1}{x_1}} \sec v\, dv + k = \rho \ln \left|\sec v + \tan v\right|_{\tan^{-1} \frac{-c-y_1}{x_1}}^{\tan^{-1} \frac{c-y_1}{x_1}} + k \\
&= \rho \ln \left|\frac{\sqrt{x_1^2 + \left(c - y_1\right)^2} + c - y_1}{x_1}\right| - \rho \ln \left|\frac{\sqrt{x_1^2 + \left(c + y_1\right)^2} - c - y_1}{x_1}\right| + k \\
&= \rho \ln \frac{\sqrt{x_1^2 + \left(c - y_1\right)^2} + c - y_1}{\sqrt{x_1^2 + \left(c + y_1\right)^2} - c - y_1} + k
\end{aligned}$$

where we have used the substitutions $u = y - y_1$, $du = dy$ and $u = x_1 \tan v$, $du = x_1 \sec^2 v\, dv$ and the relations $\int \sec x\, dx = \ln\left|\sec x + \tan x\right|$, $\sec \tan^1 x = \sqrt{1 + x^2}$ and k is an arbitrary constant. Let us choose k such that $\Phi(1, 0) = 0$, or

$$\Phi\left(1, 0\right) = \rho \ln \frac{\sqrt{1 + c^2} + c}{\sqrt{1 + c^2} - c} + k = 0 \implies k = -\rho \ln \frac{\sqrt{1 + c^2} + c}{\sqrt{1 + c^2} - c}$$

Taking the limit when $c \to \infty$ then gives

$$\begin{aligned}
\lim_{c \to \infty} \Phi &= \lim_{c \to \infty} \rho \ln \frac{\sqrt{x_1^2 + \left(c - y_1\right)^2} + c - y_1}{\sqrt{x_1^2 + \left(c + y_1\right)^2} - c - y_1} - \lim_{c \to \infty} \rho \ln \frac{\sqrt{1 + c^2} + c}{\sqrt{1 + c^2} - c} \\
&= \lim_{c \to \infty} \rho \ln \left(\frac{\sqrt{x_1^2 + \left(c - y_1\right)^2} + c - y_1}{\sqrt{x_1^2 + \left(c + y_1\right)^2} - c - y_1} \frac{\sqrt{1 + c^2} - c}{\sqrt{1 + c^2} + c} \right) \\
&\simeq \lim_{c \to \infty} \rho \ln \left(\frac{\sqrt{x_1^2 + c^2} + c}{\sqrt{x_1^2 + c^2} - c} \frac{\sqrt{1 + c^2} - c}{\sqrt{1 + c^2} + c} \right) \\
&= \lim_{c \to \infty} \rho \ln \frac{\left(\sqrt{x_1^2 + c^2} + c\right)^2 \left(\sqrt{1 + c^2} - c\right)^2}{x_1^2} \\
&= \lim_{c \to \infty} \rho \ln \left[\left(\sqrt{x_1^2 + c^2} + c\right)^2 \left(\sqrt{1 + c^2} - c\right)^2 \right] - \rho \ln x_1^2 \\
&\simeq \lim_{c \to \infty} \rho \ln \left[\left(\sqrt{1 + c^2} + c\right)^2 \left(\sqrt{1 + c^2} - c\right)^2 \right] - \rho \ln x_1^2 \\
&= \lim_{c \to \infty} 2\rho \ln \left[\left(\sqrt{1 + c^2} + c\right) \left(\sqrt{1 + c^2} - c\right) \right] - 2\rho \ln \left|x_1\right| \\
&= 2\rho \ln 1 - 2\rho \ln \left|x_1\right| = -2\rho \ln \left|x_1\right|
\end{aligned}$$

4. (a) An electromagnetic field is described by two vector fields **E** and **H**, where **E** is the electric force and **H** is the magnetic field strength. Both **E** and **H** in general vary with time t. In the absence of conductors, **E** and **H** satisfy the *Maxwell equations*:

$$\operatorname{div}\mathbf{E} = 4\pi\rho \qquad \operatorname{div}\mathbf{H} = 0 \qquad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where ρ is the charge density and c is a universal constant. Furthermore, we assume that all derivatives appearing here are continuous, so that the order of differentiation can be changed and that the discussion is confined to a simply connected domain such that Theorem IV of Section 5.13 is applicable, i.e. each irrotational field is a gradient and each solenoidal field is a curl. The latter thus implies that we can choose a vector field **A** (the *vector potential*) such that

$$\operatorname{div}\mathbf{H} = \operatorname{div}\operatorname{curl}\mathbf{A} = 0$$

Just as **H**, the vector field **A** depends on x, y, z and in general also on time t. Furthermore, **A** is not unique (see Problem 5 following Section 5.13); to any choice of **A** we can add $\operatorname{grad}\phi$, where ϕ is an arbitrary function of x, y, z and t. Using the fact that $\mathbf{H} = \operatorname{curl}\mathbf{A}$, the third Maxwell equation becomes

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{A} \iff \operatorname{curl}\left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}$$

Since we assume the domain D is simply connected we can choose a function Φ (the *scalar potential*) such that

$$-\operatorname{curl}\operatorname{grad}\Phi = \mathbf{0} \iff \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\operatorname{grad}\Phi = -\nabla\Phi$$

Taking the divergence of both sides and using the first Maxwell equation then gives

$$\nabla\cdot\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\nabla^2\Phi \iff \nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

By substituting $\mathbf{H} = \operatorname{curl}\mathbf{A}$ in the fourth Maxwell equation and applying (3.40) we get

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \iff \operatorname{curl}\operatorname{curl}\mathbf{A} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0} \iff \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$$

Differentiation leads to

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\Phi\right) = \mathbf{0} \iff \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c}\nabla\frac{\partial \Phi}{\partial t} = \mathbf{0}$$

Adding the last two equations then gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = \mathbf{0}$$

We now utilise the freedom to choose $\mathbf{A}$ to achieve the *Lorentz condition*:

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} \equiv 0$$

Substitution then finally leads to the two partial differential equations

$$\nabla^2 \Phi + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}\right) = -4\pi\rho \iff \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = \mathbf{0} \iff \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0}$$

(b) Taking the curl of both sides of the third Maxwell equation gives

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{curl}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{H}$$

We now apply identity (3.40) and use the fourth Maxwell equation to get

$$\nabla^2 \mathbf{E} - \nabla\left(\nabla \cdot \mathbf{E}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) \iff \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0}$$

where we have also utilised the fact that the first Maxwell equation reduces to $\operatorname{div}\mathbf{E} = \nabla \cdot \mathbf{E} = 4\pi\rho = 0$ in a domain D free of charge (i.e. since the charge density $\rho = 0$ then).

5. Let a solid whose boundaries are two parallel planes, d units apart, kept at temperatures T_1, T_2 respectively, be given. We can take the boundaries to be the planes $x = 0$, $x = d$ and furthermore, we note that, by symmetry, the temperature distribution T in the solid must be independent of y and z. Now since T_1 and T_2 are fixed, the solid is in temperature equilibrium (i.e. $\partial T/\partial t = 0$) and we may conclude that T is harmonic:

$$\operatorname{div}\operatorname{grad}T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} = 0$$

where the last equality holds, since as mentioned before, T is independent of both y and z. From the fact that $\partial^2 T/\partial x^2 = 0$ it follows that $\partial T/\partial x = a$, where a is some constant, and so $T = T(x) = ax + b$, where b is some other constant, changes linearly with distance. We can then determine b by noting that $T(0) = T_1 \implies b = T_1$ and a by noting that $T(d) = T_2 \implies (T_2 - T_1)/d$. Hence,

$$T = T\left(x\right) = \frac{T_2 - T_1}{d}x + T_1$$

6. As stated in Section 5.15, it is an experimental law that the integral

$$\int \frac{1}{T}\,dU + \frac{p}{T}\,dV$$

is independent of path in the UV plane. We can accordingly introduce a scalar S (termed the *entropy*) whose differential is the expression being integrated:

$$dS = \frac{1}{T}\,dU + \frac{p}{T}\,dV$$

Solving for dU we get

$$dU = T\,dS - p\,dV = d\,(TS) - S\,dT - p\,dV \iff d\,(U - TS) = dF = -S\,dT - p\,dV$$

where $F = U - TS$ is called the *Helmholtz free energy.* Hence, it follows that the integral

$$\int S\,dT + p\,dV = -\int dF$$

is independent of path in the TV plane.

7. Let a particle occupying position (x_0, y_0, z_0) at time $t = 0$ occupy position (x, y, z) at time t represent a fluid motion in space. Thus x, y, z become functions of x_0, y_0, z_0, t:

$$x = \phi\,(x_0, y_0, z_0, t) \qquad y = \psi\,(x_0, y_0, z_0, t) \qquad z = \chi\,(x_0, y_0, z_0, t)$$

Let the ∇ symbol be used as follows:

$$\nabla = \frac{\partial}{\partial x_0}\mathbf{i} + \frac{\partial}{\partial y_0}\mathbf{j} + \frac{\partial}{\partial z_0}\mathbf{k}$$

and let J denote the Jacobian

$$J = \frac{\partial\,(x, y, z)}{\partial\,(x_0, y_0, z_0)}$$

Let $\mathbf{v}$ denote the velocity vector:

$$\mathbf{v} = \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k} = \frac{\partial \phi}{\partial t}\mathbf{i} + \frac{\partial \psi}{\partial t}\mathbf{j} + \frac{\partial \chi}{\partial t}\mathbf{k}$$

(a) Writing out J explicitly and using (2.32) and (1.21) gives

$$\begin{aligned}
J &= \frac{\partial (x,y,z)}{\partial (x_0,y_0,z_0)} = \begin{vmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{vmatrix} \\
&= \frac{\partial x}{\partial x_0}\left(\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0} - \frac{\partial z}{\partial y_0}\frac{\partial y}{\partial z_0}\right) + \frac{\partial x}{\partial y_0}\left(\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial z_0} - \frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}\right) + \frac{\partial x}{\partial z_0}\left(\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0} - \frac{\partial z}{\partial x_0}\frac{\partial y}{\partial y_0}\right) \\
&= \left(\frac{\partial x}{\partial x_0}\mathbf{i} + \frac{\partial x}{\partial y_0}\mathbf{j} + \frac{\partial x}{\partial z_0}\mathbf{k}\right) \\
&\quad \cdot \left[\left(\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0} - \frac{\partial z}{\partial y_0}\frac{\partial y}{\partial z_0}\right)\mathbf{i} + \left(\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial z_0} - \frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0} - \frac{\partial z}{\partial x_0}\frac{\partial y}{\partial y_0}\right)\mathbf{k}\right] \\
&= \nabla x \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{vmatrix} = \nabla x \cdot \nabla y \times \nabla z
\end{aligned}$$

(b) From

$$\begin{aligned}
J\frac{\partial x_0}{\partial x} = (\nabla x \cdot \nabla y \times \nabla z)\frac{\partial x_0}{\partial x} &= \frac{\partial x_0}{\partial x}\left(\frac{\partial x}{\partial x_0}\mathbf{i} + \frac{\partial x}{\partial y_0}\mathbf{j} + \frac{\partial x}{\partial z_0}\mathbf{k}\right) \cdot \nabla y \times \nabla z \\
&= \left(\frac{\partial x_0}{\partial x}\frac{\partial x}{\partial x_0}\mathbf{i} + \underbrace{\frac{\partial x_0}{\partial x}\frac{\partial x}{\partial y_0}}_{0}\mathbf{j} + \underbrace{\frac{\partial x_0}{\partial x}\frac{\partial x}{\partial z_0}}_{0}\mathbf{k}\right) \cdot \nabla y \times \nabla z \\
&= \mathbf{i} \cdot \nabla y \times \nabla z
\end{aligned}$$

follows

$$\frac{\partial x_0}{\partial x} = \mathbf{i} \cdot \frac{\nabla y \times \nabla z}{J}$$

Similarly, from

$$\begin{aligned}
J\frac{\partial y_0}{\partial x} = (\nabla x \cdot \nabla y \times \nabla z)\frac{\partial y_0}{\partial x} &= \frac{\partial y_0}{\partial x}\left(\frac{\partial x}{\partial x_0}\mathbf{i} + \frac{\partial x}{\partial y_0}\mathbf{j} + \frac{\partial x}{\partial z_0}\mathbf{k}\right) \cdot \nabla y \times \nabla z \\
&= \left(\underbrace{\frac{\partial y_0}{\partial x}\frac{\partial x}{\partial x_0}}_{0}\mathbf{i} + \frac{\partial y_0}{\partial x}\frac{\partial x}{\partial y_0}\mathbf{j} + \underbrace{\frac{\partial y_0}{\partial x}\frac{\partial x}{\partial z_0}}_{0}\mathbf{k}\right) \cdot \nabla y \times \nabla z \\
&= \mathbf{j} \cdot \nabla y \times \nabla z
\end{aligned}$$

follows

$$\frac{\partial y_0}{\partial x} = \mathbf{j} \cdot \frac{\nabla y \times \nabla z}{J}$$

and from

$$J\frac{\partial z_0}{\partial x} = (\nabla x \cdot \nabla y \times \nabla z)\frac{\partial z_0}{\partial x} = \frac{\partial z_0}{\partial x}\left(\frac{\partial x}{\partial x_0}\mathbf{i} + \frac{\partial x}{\partial y_0}\mathbf{j} + \frac{\partial x}{\partial z_0}\mathbf{k}\right) \cdot \nabla y \times \nabla z$$

$$= \left(\underbrace{\frac{\partial z_0}{\partial x}\frac{\partial x}{\partial x_0}}_{0}\mathbf{i} + \underbrace{\frac{\partial z_0}{\partial x}\frac{\partial x}{\partial y_0}}_{0}\mathbf{j} + \frac{\partial z_0}{\partial x}\frac{\partial x}{\partial z_0}\mathbf{k}\right) \cdot \nabla y \times \nabla z$$

$$= \mathbf{k} \cdot \nabla y \times \nabla z$$

follows

$$\frac{\partial z_0}{\partial x} = \mathbf{k} \cdot \frac{\nabla y \times \nabla z}{J}$$

From identity (1.34) it follows that

$$\nabla x \cdot \nabla y \times \nabla z = \nabla z \cdot \nabla x \times \nabla y = \nabla y \cdot \nabla z \times \nabla x$$

Using these other two *scalar triple product* expressions and going through the same steps as before then leads to the additional equations

$$\frac{\partial x_0}{\partial y} = \mathbf{i} \cdot \frac{\nabla z \times \nabla x}{J} \qquad \frac{\partial y_0}{\partial y} = \mathbf{j} \cdot \frac{\nabla z \times \nabla x}{J} \qquad \frac{\partial z_0}{\partial y} = \mathbf{k} \cdot \frac{\nabla z \times \nabla x}{J}$$

$$\frac{\partial x_0}{\partial z} = \mathbf{i} \cdot \frac{\nabla x \times \nabla y}{J} \qquad \frac{\partial y_0}{\partial z} = \mathbf{j} \cdot \frac{\nabla x \times \nabla y}{J} \qquad \frac{\partial z_0}{\partial z} = \mathbf{k} \cdot \frac{\nabla x \times \nabla y}{J}$$

(c) From Problem 3(c) and 3(d) following Section 2.13 and using identity (1.34) it follows that

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial (x, y, z)}{\partial (x_0, y_0, z_0)}\right] = \frac{\partial}{\partial t}(\nabla x \cdot \nabla y \times \nabla z)$$

$$= \frac{\partial}{\partial t}(\nabla x) \cdot \nabla y \times \nabla z + \nabla x \cdot \frac{\partial}{\partial t}(\nabla y \times \nabla z)$$

$$= \frac{\partial}{\partial t}(\nabla x) \cdot \nabla y \times \nabla z + \nabla x \cdot \left[\nabla y \times \frac{\partial}{\partial t}(\nabla z) + \frac{\partial}{\partial t}(\nabla y) \times \nabla z\right]$$

$$= \nabla\left(\frac{\partial x}{\partial t}\right) \cdot \nabla y \times \nabla z + \nabla x \cdot \left[\nabla y \times \nabla\left(\frac{\partial z}{\partial t}\right) + \nabla\left(\frac{\partial y}{\partial t}\right) \times \nabla z\right]$$

$$= \nabla v_x \cdot \nabla y \times \nabla z + \nabla x \cdot (\nabla y \times \nabla v_z + \nabla v_y \times \nabla z)$$

$$= \nabla v_x \cdot \nabla y \times \nabla z + \nabla x \cdot \nabla y \times \nabla v_z + \nabla x \cdot \nabla v_y \times \nabla z$$

$$= \nabla v_x \cdot \nabla y \times \nabla z + \nabla v_y \cdot \nabla z \times \nabla x + \nabla v_z \cdot \nabla x \times \nabla y$$

(d) By the chain rule (2.40) we find

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial v_x}{\partial y_0}\frac{\partial y_0}{\partial x} + \frac{\partial v_x}{\partial z_0}\frac{\partial z_0}{\partial x} = \left(\frac{\partial v_x}{\partial x_0}\mathbf{i} + \frac{\partial v_x}{\partial y_0}\mathbf{j} + \frac{\partial v_x}{\partial z_0}\mathbf{k}\right) \cdot \left(\frac{\partial x_0}{\partial x}\mathbf{i} + \frac{\partial y_0}{\partial x}\mathbf{j} + \frac{\partial z_0}{\partial x}\mathbf{k}\right)$$

$$= \nabla v_x \cdot \left(\frac{\partial x_0}{\partial x}\mathbf{i} + \frac{\partial y_0}{\partial x}\mathbf{j} + \frac{\partial z_0}{\partial x}\mathbf{k}\right)$$

Next, From (b) follows

$$\frac{\partial x_0}{\partial x}\mathbf{i} + \frac{\partial y_0}{\partial x}\mathbf{j} + \frac{\partial z_0}{\partial x}\mathbf{k} = \frac{\nabla y \times \nabla z}{J}$$

Hence,

$$\frac{\partial v_x}{\partial x} = \nabla v_x \cdot \left(\frac{\partial x_0}{\partial x}\mathbf{i} + \frac{\partial y_0}{\partial x}\mathbf{j} + \frac{\partial z_0}{\partial x}\mathbf{k}\right) = \frac{\nabla v_x \cdot \nabla y \times \nabla z}{J}$$

Similarly, we find

$$\begin{aligned}
\frac{\partial v_y}{\partial y} &= \frac{\partial v_y}{\partial x_0}\frac{\partial x_0}{\partial y} + \frac{\partial v_y}{\partial y_0}\frac{\partial y_0}{\partial y} + \frac{\partial v_y}{\partial z_0}\frac{\partial z_0}{\partial y} = \left(\frac{\partial v_y}{\partial x_0}\mathbf{i} + \frac{\partial v_y}{\partial y_0}\mathbf{j} + \frac{\partial v_y}{\partial z_0}\mathbf{k}\right) \cdot \left(\frac{\partial x_0}{\partial y}\mathbf{i} + \frac{\partial y_0}{\partial y}\mathbf{j} + \frac{\partial z_0}{\partial y}\mathbf{k}\right) \\
&= \frac{\nabla v_y \cdot \nabla z \times \nabla x}{J} \\
\frac{\partial v_z}{\partial z} &= \frac{\partial v_z}{\partial x_0}\frac{\partial x_0}{\partial z} + \frac{\partial v_z}{\partial y_0}\frac{\partial y_0}{\partial z} + \frac{\partial v_z}{\partial z_0}\frac{\partial z_0}{\partial z} = \left(\frac{\partial v_z}{\partial x_0}\mathbf{i} + \frac{\partial v_z}{\partial y_0}\mathbf{j} + \frac{\partial v_z}{\partial z_0}\mathbf{k}\right) \cdot \left(\frac{\partial x_0}{\partial z}\mathbf{i} + \frac{\partial y_0}{\partial z}\mathbf{j} + \frac{\partial z_0}{\partial z}\mathbf{k}\right) \\
&= \frac{\nabla v_z \cdot \nabla x \times \nabla y}{J}
\end{aligned}$$

Adding the results and utilising (c) then finally gives

$$\begin{aligned}
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} &= \frac{\nabla v_x \cdot \nabla y \times \nabla z + \nabla v_y \cdot \nabla z \times \nabla x + \nabla v_z \cdot \nabla x \times \nabla y}{J} \\
\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \mathbf{v} &= \frac{\nabla v_x \cdot \nabla y \times \nabla z + \nabla v_y \cdot \nabla z \times \nabla x + \nabla v_z \cdot \nabla x \times \nabla y}{J} \\
\operatorname{div} \mathbf{v} &= \frac{1}{J}\frac{\partial J}{\partial t}
\end{aligned}$$

8. Let a piece of the fluid of Problem 7 occupy a region $R = R(t)$ at time t and a region $R_0 = R(0)$ when $t = 0$. Let $F(x, y, z, t)$ be a function differentiable throughout the part of space concerned.

   (a) Reiterating that $x = \phi(x_0, y_0, z_0, t)$, $y = \psi(x_0, y_0, z_0, t)$, $z = \chi(x_0, y_0, z_0, t)$ and $J = \partial(x, y, z)/\partial(x_0, y_0, z_0)$ we can use (5.109) to form the relation

$$\iiint\limits_{R(t)} F(x, y, z, t)\; dx\, dy\, dz = \iiint\limits_{R_0} F\left[\phi(x_0, y_0, z_0, t), \ldots\right] J\, dx_0\, dy_0\, dz_0$$

   Note that $\delta = 1$, since the mapping from $R(t)$ to R_0 by definition is one-to-one (i.e. two or more particles representing the fluid flow cannot occupy the exact same position in space at any time t), as can also be readily deduced from looking at Figure 5.39.

(b) Using (a), (4.94) and the result of part (d) of Problem 7 we find

$$
\begin{aligned}
\frac{d}{dt}\iiint\limits_{R(t)} F\left(x,y,z,t\right)\,dx\,dy\,dz &= \frac{d}{dt}\iiint\limits_{R_0} F\left[\phi\left(x_0,y_0,z_0,t\right),\ldots\right] J\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R_0} \frac{\partial}{\partial t}\left\{F\left[\phi\left(x_0,y_0,z_0,t\right),\ldots\right] J\right\}\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R_0} \left(\frac{\partial F}{\partial t}J + F\frac{\partial J}{\partial t}\right)\,dx_0\,dy_0,\,dz_0 \\
&= \iiint\limits_{R_0} \left(\frac{\partial F}{\partial t} + F\,\mathrm{div}\,\mathbf{v}\right) J\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R(t)} \left[\frac{\partial F}{\partial t} + \mathrm{div}\left(F\mathbf{v}\right)\right]\,dx\,dy\,dz
\end{aligned}
$$

9. Let $\rho = \rho(x,y,z,t)$ be the density of the fluid motion of Problems 7 and 8. The integral $\iiint_{R(t)} \rho\,dx\,dy\,dz$ represents the mass of the fluid filling $R(t)$. The conservation of mass implies that this integral is constant:

$$
\frac{d}{dt}\iiint\limits_{R(t)} \rho\,dx\,dy\,dz = 0
$$

Using the result of part (b) of Problem 8 this can be written as

$$
\begin{aligned}
\frac{d}{dt}\iiint\limits_{R(t)} \rho\,dx\,dy\,dz = \frac{d}{dt}\iiint\limits_{R_0} \rho J\,dx_0\,dy_0\,dz_0 &= \iiint\limits_{R_0} \frac{\partial}{\partial t}\left(\rho J\right)\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R_0} \left(\frac{\partial \rho}{\partial t}J + \rho\frac{\partial J}{\partial t}\right)\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R_0} \left(\frac{\partial \rho}{\partial t} + \rho\,\mathrm{div}\,\mathbf{v}\right) J\,dx_0\,dy_0\,dz_0 \\
&= \iiint\limits_{R(t)} \left[\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho\mathbf{v}\right)\right]\,dx\,dy\,dz = 0
\end{aligned}
$$

Since the last integral needs to be equal to zero for any $R(t)$ we may conclude that

$$
\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho\mathbf{v}\right) = 0
$$

Section 5.18

1. From the result of Problem 6 following Section 4.9 we know that

$$\int_0^{2\pi} \ln\left(1 - c\cos\theta\right)\, d\theta = 2\pi \ln \frac{1+\sqrt{1-c^2}}{2} \qquad 0 \le c < 1$$

As such, we may use this to evaluate the logarithmic potential U of the example in Section 5.16:

$$\begin{aligned}
U &= b\int_0^a \int_0^{2\pi} \ln \frac{1}{\sqrt{R^2+\rho^2-2R\rho\cos\left(\theta-\alpha\right)}}\rho\, d\theta\, d\rho \\
&= b\int_0^a \int_0^{2\pi} \ln \frac{1}{\sqrt{R^2+\rho^2}\sqrt{1-\frac{2R\rho}{R^2+\rho^2}\cos\left(\theta-\alpha\right)}}\rho\, d\theta\, d\rho \\
&= -\frac{b}{2}\left[\int_0^a \rho\ln\left(R^2+\rho^2\right)\int_0^{2\pi} d\theta\, d\rho + \int_0^a \rho \int_0^{2\pi} \ln\left(1-\frac{2R\rho}{R^2+\rho^2}\cos\left(\theta-\alpha\right)\right) d\theta\, d\rho\right] \\
&= -\pi b\left[\int_0^a \rho\ln\left(R^2+\rho^2\right) d\rho + \int_0^a \rho\ln\frac{1+\sqrt{1-\frac{4R^2\rho^2}{(R^2+\rho^2)^2}}}{2}\, d\rho\right] \\
&= -\pi b\left[\int_0^a \rho\ln\left(R^2+\rho^2\right) d\rho + \int_0^a \rho\ln\left(1+\sqrt{1-\frac{4R^2\rho^2}{\left(R^2+\rho^2\right)^2}}\right) d\rho - \int_0^a \rho\ln 2\, d\rho\right] \\
&= -\pi b\left[\int_0^a \rho\ln\left(R^2+\rho^2\right) d\rho + \int_0^a \rho\ln\left(\frac{R^2+\rho^2+|R^2-\rho^2|}{R^2+\rho^2}\right) d\rho - \int_0^a \rho\ln 2\, d\rho\right] \\
&= -\pi b\left[\int_0^a \rho\ln\left(R^2+\rho^2+|R^2-\rho^2|\right) d\rho - \int_0^a \rho\ln 2\, d\rho\right]
\end{aligned}$$

First, let us consider the case when $R \le a$. Then the integral can be written and evaluated as

$$\begin{aligned}
U &= -\pi b\left[\int_0^R \rho\ln\left(2R^2\right) d\rho + \int_R^a \rho\ln\left(2\rho^2\right) d\rho - \int_0^a \rho\ln 2\, d\rho\right] \\
&= -\pi b\left(\int_0^R \rho\ln 2\, d\rho + 2\int_0^R \rho\ln R\, d\rho + \int_R^a \rho\ln 2\, d\rho + 2\int_R^a \rho\ln\rho\, d\rho - \int_0^a \rho\ln 2\, d\rho\right) \\
&= -2\pi b\left(\int_0^R \rho\ln R\, d\rho + \int_R^a \rho\ln\rho\, d\rho\right) \\
&= -2\pi b\left(\frac{R^2\ln R}{2} + \left.\frac{\rho^2\ln\rho}{2}\right|_R^a - \frac{1}{2}\int_R^a \rho\, d\rho\right) \\
&= \frac{\pi b}{2}\left(a^2 - R^2 - 2a^2\ln a\right)
\end{aligned}$$

Finally, when $R > a$ the integral may be evaluated as

$$U = -\pi b\left[\int_0^a \rho\ln\left(2R^2\right) d\rho - \int_0^a \rho\ln 2\, d\rho\right] = -2\pi b\int_0^a \rho\ln R\, d\rho = \pi a^2 b\ln\frac{1}{R}$$

2. Alternatively, we can solve for the potential U of the example in Section 5.16 using the following information:

   (a) $\nabla^2 U = -2\pi b$ for $R < a$,
   (b) $\nabla^2 U = 0$ for $R > a$,
   (c) $U = M \ln \frac{1}{R} + \frac{p(x,y)}{R}$ for large R, as in (5.172),
   (d) U and $\operatorname{grad} U$ are continuous for all (x, y),
   (e) U depends on R alone, by symmetry.

   Since $\nabla^2 U = \partial^2 U/\partial R^2 + (1/R)\partial U/\partial R$ in polar coordinates, for a function depending only on R (see Section 2.17), (a) and (b) give the following system of second order ordinary differential equations for U:

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} = -2\pi b, \quad R < a$$
$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} = 0, \quad R > a$$

   Let us first solve the second equation, which is a second order homogeneous ordinary differential equation. As a first step, we make the substitution $v = \partial U/\partial R$, such that the second order differential equation reduces to the first order differential equation

$$\frac{\partial v}{\partial R} + \frac{1}{R}v = 0 \iff \frac{dv}{v} = -\frac{dR}{R}$$

   Integrating both sides gives

$$\ln v = \ln \frac{1}{R} + \bar{C}_1 \iff v = \frac{C_1}{R} \iff \frac{\partial U}{\partial R} = \frac{C_1}{R} \iff dU = \frac{C_1}{R}\,dR$$

   where $C_1 = e^{\bar{C}_1}$ and $\bar{C}_1$ is an arbitrary constant. The general solution for U may then be obtained by integrating once more to get

$$U = -C_1 \ln \frac{1}{R} + C_2$$

   where again, C_1 and C_2 are arbitrary constants. Next, we use (c) to determine the constants C_1, C_2 in the case $R > a$. For large R (i.e. $R \gg a$) we know that the potential U behaves as if the mass M were concentrated at the origin, with a small error $p(x, y)/R$, which approaches 0 as $R \to \infty$. For our circular disk $M = \pi a^2 b$ and so

$$U = M \ln \frac{1}{R} + \frac{p\,(x, y)}{R} = \pi a^2 b \ln \frac{1}{R} + \frac{p\,(x, y)}{R}$$

   Hence, we conclude that $C_1 = -\pi a^2 b$. Furthermore, since the second term approaches 0 when $R \to \infty$ this implies $C_2 = 0$. As such, we find

$$U = \pi a^2 b \ln \frac{1}{R}, \qquad R > a$$

The first differential equation for the case $R < a$ is a non-homogeneous ordinary differential equation, the solution of which is given by $U = U_h + U_p$, where U_h is the *complementary equation* and U_p is the *particular solution*. The complementary equation is just the associated homogeneous equation and so its solution is given by $U_h = -C_1 \ln \frac{1}{R} + C_2$, where the constants C_1, C_2 are still unknown. In order to find the particular solution we will employ *the method of integrating factor*. We again make the substitution $v = \partial U_p/\partial R$ such that we end up with the first order differential equation

$$\frac{\partial v}{\partial R} + \frac{1}{R} v = -2\pi b$$

which is of the form $v' + P(R)v = Q(R)$, where $P(R) = 1/R$, $Q(R) = -2\pi b$. After multiplying by $w(R) = \exp[\int P(R)\,dR]$ it can be written as $[w(R)v]' = Q(R)w(R)$ so that the solution is given by $w(R)v = \int Q(R)w(R)\,dR + C$. Hence, the particular solution to our first order non-homogeneous ordinary differential equation becomes

$$\begin{aligned} v = -2\pi b e^{-\int dR/R}\left(\int e^{\int dR/R}\,dR + C\right) &= -\frac{2\pi b}{R}\left(\int e^{\int dR/R}\,dR + C\right) \\ &= -\frac{2\pi b}{R}\int R\,dR + C \\ &= -\pi b R \end{aligned}$$

Note that we have set all constants due to the integration operations equal to 0. Since $\partial U_p/\partial R = v$ we integrate once more to finally arrive at

$$U_p = \int v\,dR = -\pi b \int R\,dR = -\frac{\pi b R^2}{2}$$

where again, we have set the constant due to the integration operation equal to 0. The general solution for the case $R < a$ thus is given by

$$U = U_h + U_p = C_1 \ln R + C_2 - \frac{\pi b R^2}{2}$$

In order to determine the constants C_1 and C_2 we note that in order for U to be continuous at the point $R = a$ we require

$$U(a) = -\pi a^2 b \ln a$$

which is satisfied by choosing $C_1 = -\pi a^2 b$ and $C_2 = \pi a^2 b/2$. Hence,

$$U = \frac{\pi b}{2}\left(a^2 - R^2 - 2a^2 \ln R\right), \qquad R \le a$$

However, if we also require $\operatorname{grad} U$ to be continuous at the point $R = a$, i.e. that $\partial U/\partial R$ is continuous at $R = a$ we define U as

$$U = \frac{\pi b}{2}\left(a^2 - R^2 - 2a^2 \ln a\right), \qquad R \le a$$

This still satisfies the condition $\nabla^2 U = -2\pi b$ for $R < a$ and ensures both U and $\operatorname{grad} U$ are continuous at $R = a$.

3. Let U be the logarithmic potential of a distribution of mass with constant density b on the circle $x^2 + y^2 = a^2$.

   (a)