

CHAPTER 6

Section 6.4

1. (a)

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{(1/n^2) + 1/n^3}{1 + 1/n^3} = \frac{0}{1} = 0$$

(b)

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{d(\ln n)/dn}{d(n)/dn} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

(c)

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$$

(d)

$$\begin{aligned} \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n)}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{d[\ln(1 + 1/n)]/dn}{d(n^{-1})/dn} \\ &= \lim_{n \rightarrow \infty} \frac{1/(n^2 + n)}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} \\ &= \frac{1}{1 + 0} = 1 \end{aligned}$$

(e)

$$\lim_{n \rightarrow \infty} s_n = 1 \text{ for } n = 1, 2, 3, \dots \implies \lim_{n \rightarrow \infty} s_n = 1$$

2. (a)

$$\overline{\lim}_{n \rightarrow \infty} \cos n\pi = 1 \qquad \underline{\lim}_{n \rightarrow \infty} \cos n\pi = -1 \qquad (1)$$

(b)

$$\overline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx 0.951 \qquad \underline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx -0.951$$

(c)

$$\overline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = \infty \qquad \underline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = -\infty$$

3. (a) A sequence

$$s_n = 1 + \cos n\pi$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 2 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = 0$$

(b) A sequence

$$s_n = -n^2 \sin^2 \left(\frac{1}{2} n \pi \right)$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 0 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = \underline{\lim}_{n \rightarrow \infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left| \left(1 + \frac{1}{n} \right)^n - e \right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

$$e \approx \left(1 + \frac{1}{164} \right)^{164} \approx 2.71$$

6.

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } |x| > 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = \pm 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1\end{aligned}$$

$$\begin{aligned}\underline{\lim}_{n \rightarrow \infty} x^n &= -\infty && \text{for } x < -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= -1 && \text{for } x = -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } x > 1\end{aligned}$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin \frac{\theta}{2} \times AE \cos \frac{\theta}{2} = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \rightarrow \infty$:

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n}$$

Now using the fact that $\lim_{x \rightarrow 0} \sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n} = \lim_{x \rightarrow 0} \pi \frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π , it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n \rightarrow \infty} \sin \left(\frac{n^2 \pi}{2} \right) = 1 \neq 0$$

then by the n th term test $\sum_{n=1}^{\infty} \sin(n^2 \pi/2)$ diverges.

- (b) Since

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^3} = \lim_{n \rightarrow \infty} \frac{2^{n-1}}{3n} = \lim_{n \rightarrow \infty} \frac{(n-1)2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the n th term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3 - 1} = \left| \frac{1}{n^3 - 1} \right| < \frac{1}{n - 1}$$

for $n = 2, 3, \dots$. Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n - 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 - (1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3 - 1)$ is absolutely convergent.

- (b) Since $|\sin n| < 1$ for $n > 1$ it follows that

$$\left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}$$

for $n = 1, 2, \dots$. Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a)