

# CHAPTER 6

## Section 6.4

1. (a)

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{(1/n^2) + 1/n^3}{1 + 1/n^3} = \frac{0}{1} = 0$$

(b)

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{d(\ln n)/dn}{d(n)/dn} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

(c)

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$$

(d)

$$\begin{aligned} \lim_{n \rightarrow \infty} n \ln \left( 1 + \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n)}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{d[\ln(1 + 1/n)]/dn}{d(n^{-1})/dn} \\ &= \lim_{n \rightarrow \infty} \frac{1/(n^2 + n)}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} \\ &= \frac{1}{1 + 0} = 1 \end{aligned}$$

(e)

$$\lim_{n \rightarrow \infty} s_n = 1 \text{ for } n = 1, 2, 3, \dots \implies \lim_{n \rightarrow \infty} s_n = 1$$

2. (a)

$$\overline{\lim}_{n \rightarrow \infty} \cos n\pi = 1 \qquad \underline{\lim}_{n \rightarrow \infty} \cos n\pi = -1 \qquad (1)$$

(b)

$$\overline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx 0.951 \qquad \underline{\lim}_{n \rightarrow \infty} \sin \frac{1}{5} n\pi \approx -0.951$$

(c)

$$\overline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = \infty \qquad \underline{\lim}_{n \rightarrow \infty} n \sin \frac{1}{2} n\pi = -\infty$$

3. (a) A sequence

$$s_n = 1 + \cos n\pi$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 2 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = 0$$

(b) A sequence

$$s_n = -n^2 \sin^2 \left( \frac{1}{2} n \pi \right)$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = 0 \qquad \underline{\lim}_{n \rightarrow \infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n \rightarrow \infty} s_n = \underline{\lim}_{n \rightarrow \infty} s_n = \infty$$

4. Let a sequence  $s_n = 1/n$  be given. Now this sequence converges, since

$$s = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Hence, for every  $\epsilon > 0$  an  $N$  can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all  $n > N$ . Hence, for all  $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define  $e$  to 2 decimal places from its definition

$$e = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n$$

we let  $\epsilon = 0.00828$  in order to find a value  $N$  such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left| \left( 1 + \frac{1}{n} \right)^n - e \right| < 0.00828$$

which is satisfied for  $n = 164$ . Hence,

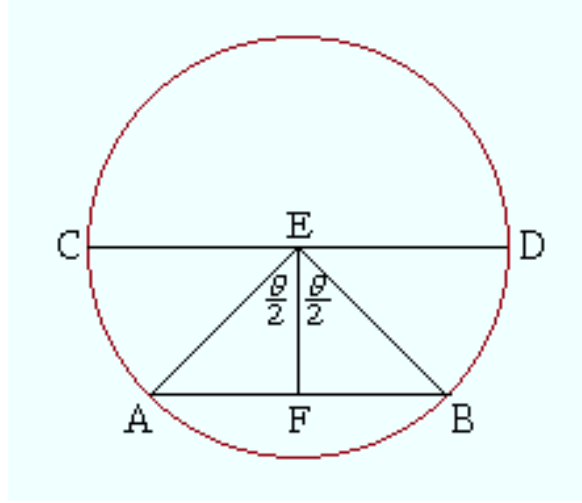
$$e \approx \left( 1 + \frac{1}{164} \right)^{164} \approx 2.71$$

6.

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } |x| > 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = \pm 1 \\ \overline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1\end{aligned}$$

$$\begin{aligned}\underline{\lim}_{n \rightarrow \infty} x^n &= -\infty && \text{for } x < -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= -1 && \text{for } x = -1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 0 && \text{for } |x| < 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= 1 && \text{for } x = 1 \\ \underline{\lim}_{n \rightarrow \infty} x^n &= \infty && \text{for } x > 1\end{aligned}$$

7.



Assuming the figure above represents the unit circle, it follows that  $AE = BE = 1$  and that the area of the polygon  $AEB$  is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin \frac{\theta}{2} \times AE \cos \frac{\theta}{2} = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of  $n$  such polygons in the limit  $n \rightarrow \infty$ :

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n}$$

Now using the fact that  $\lim_{x \rightarrow 0} \sin(x)/x = 1$  and setting  $x = 2\pi/n$  we find

$$A_{S_1} = s_n = \lim_{n \rightarrow \infty} \pi \frac{n}{2\pi} \sin \frac{2\pi}{n} = \lim_{x \rightarrow 0} \pi \frac{\sin x}{x} = \pi$$

Hence, since the sequence  $s_n$  is bounded and has limit  $\pi$ , it is monotone increasing.

## Section 6.7

1. (a) Since

$$\overline{\lim}_{n \rightarrow \infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the  $n$ th term test  $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$  diverges.

- (b) Since

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^3} = \lim_{n \rightarrow \infty} \frac{2^{n-1}}{3n} = \lim_{n \rightarrow \infty} \frac{(n-1)2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the  $n$ th term test  $\sum_{n=1}^{\infty} 2^n/n^3$  diverges.

2. (a) Since  $n^3 > n$  for  $n > 0$  it follows that

$$\frac{1}{n^3 - 1} = \left| \frac{1}{n^3 - 1} \right| < \frac{1}{n - 1}$$

for  $n = 2, 3, \dots$ . Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n - 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 - (1/n)} = 0$$

then  $\sum_{n=2}^{\infty} 1/(n-1)$  converges and hence, by the comparison test for convergence  $\sum_{n=2}^{\infty} 1/(n^3 - 1)$  is absolutely convergent.

- (b) Since  $|\sin n| < 1$  for  $n \geq 1$  it follows that

$$\left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}$$

for  $n = 1, 2, \dots$ . Now since

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

then  $\sum_{n=1}^{\infty} 1/n^2$  converges and hence, by the comparison test for convergence  $\sum_{n=1}^{\infty} \sin(n)/n^2$  is absolutely convergent.

3. (a) Since  $n + 5 > n$  and  $n^2 - 3n - 5 < n^2$  for  $n \geq 1$  it follows that

$$\frac{n + 5}{n^2 - 3n - 5} > \frac{n}{n^2} = \frac{1}{n}$$

for  $n = 1, 2, \dots$ . Now since  $\sum_{n=1}^{\infty} 1/n$  is the *harmonic series*, which diverges, it follows by the comparison test for divergence that  $\sum_{n=1}^{\infty} (n + 5)/(n^2 - 3n - 5)$  diverges as well.

(b) Since  $\sqrt{n} \ln n < n \ln n$  for  $n \geq 2$  it follows that

$$\frac{1}{\sqrt{n} \ln n} > \frac{1}{n \ln n}$$

for  $n = 2, 3, \dots$ . Using the inequality  $\ln(1+x) \leq x$  we may continue to write

$$\frac{1}{n \ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln \left( 1 + \frac{\ln(1+1/n)}{\ln n} \right) \geq \ln \frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n} \ln n} > \ln \frac{\ln(1+n)}{\ln n} = \ln \ln(1+n) - \ln \ln n$$

Now let us consider the series

$$\sum_{n=2}^N \ln \ln(1+n) - \ln \ln n = \ln \ln(1+N) - \ln \ln 2$$

Hence, when  $N \rightarrow \infty$

$$\sum_{n=2}^{\infty} \ln \ln(1+n) - \ln \ln n = \lim_{N \rightarrow \infty} \ln \ln(1+N) - \ln \ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that  $\sum_{n=2}^{\infty} 1/(\sqrt{n} \ln n)$  diverges as well.

4. (a) Let  $y = f(x) = 1/(x^2 + 1)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^{\infty} f(x) dx$  with  $c = 1$  then evaluates to

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^2 + 1} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2 + 1} = \lim_{b \rightarrow \infty} \int_{\pi/4}^{\tan^{-1} b} du = \lim_{b \rightarrow \infty} u \Big|_{\pi/4}^{\tan^{-1} b} = \lim_{b \rightarrow \infty} \tan^{-1} b - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

where we have used the substitution  $x = \tan u$ . Hence, by the integral test, since the improper integral  $\int_1^{\infty} f(x) dx$  converges so will the series  $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2 + 1)$ .

- (b) let  $y = f(x) = 1/(x \ln^2 x)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^{\infty} f(x) dx$  with  $c = 2$  then evaluates to

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln^2 x} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln^2 x} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b \rightarrow \infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \lim_{b \rightarrow \infty} \frac{1}{\ln b} \\ &= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2} \end{aligned}$$

where we have used the substitution  $u = \ln x$ . Hence, by the integral test, since the improper integral  $\int_2^\infty f(x) dx$  converges so will the series  $a_n = f(n) = \sum_{n=2}^\infty 1/(n \ln^2 n)$ .

5. (a) Let  $y = f(x) = x/(x^2 + 1)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $(f(x))$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^\infty f(x) dx$  with  $c = 1$  then evaluates to

$$\begin{aligned} \int_1^\infty \frac{x}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2 + 1} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_2^{b^2+1} \frac{du}{u} = \lim_{b \rightarrow \infty} \left. \frac{\ln u}{2} \right|_2^{b^2+1} \\ &= \lim_{b \rightarrow \infty} \frac{\ln |b^2 + 1| - \ln 2}{2} \\ &= \infty - \frac{\ln 2}{2} = \infty \end{aligned}$$

where we have used the substitution  $u = x^2 + 1$ . Hence, by the integral test, since the improper integral  $\int_1^\infty f(x) dx$  diverges so will the series  $a_n = f(n) = \sum_{n=1}^\infty n/(n^2 + 1)$ .

- (b) Let  $y = f(x) = 1/(x \ln x \ln \ln x)$ . As such,  $f(x)$  is defined and continuous for  $c \leq x < \infty$ ,  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$ . The improper integral  $\int_c^\infty f(x) dx$  with  $c = 10$  then evaluates to

$$\begin{aligned} \int_{10}^\infty \frac{dx}{x \ln x \ln \ln x} &= \lim_{b \rightarrow \infty} \int_{10}^b \frac{dx}{x \ln x \ln \ln x} = \lim_{b \rightarrow \infty} \int_{\ln 10}^{\ln b} \frac{du}{u \ln u} \\ &= \lim_{b \rightarrow \infty} \int_{\ln \ln 10}^{\ln \ln b} \frac{dv}{v} \\ &= \lim_{b \rightarrow \infty} \left. \ln v \right|_{\ln \ln 10}^{\ln \ln b} \\ &= \lim_{b \rightarrow \infty} \ln \ln \ln b - \ln \ln \ln 10 \\ &= \infty - \ln \ln \ln 10 = \infty \end{aligned}$$

where we have used the substitutions  $u = \ln x$  and  $v = \ln u$ . Hence, by the integral test, since the improper integral  $\int_{10}^\infty f(x) dx$  diverges so will the series  $a_n = f(n) = \sum_{n=10}^\infty 1/(n \ln n \ln \ln n)$ .

6. (a) Let  $a_n = (-1)^n/n!$ . As such we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = \lim_{n \rightarrow \infty} \left| \frac{n!}{(-1)^n} \frac{(-1)^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| -\frac{1}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence,  $L < 1$  and so according to the ratio test the series  $\sum_{n=1}^\infty (-1)^n/n!$  is absolutely convergent.

(b) Let  $a_n = 2^n + 1/(3^n + n)$ . As such we find

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= L = \lim_{n \rightarrow \infty} \left| \frac{3^n + n}{2^n + 1} \frac{2^{n+1} + 1}{3^{n+1} + n + 1} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3^n (1 + n/3^n)}{2^n (1 + 1/2^n)} \frac{2^{n+1} (1 + 1/2^{n+1})}{3^{n+1} [1 + (n/3^{n+1}) + (1/3^{n+1})]} \right| \\ &= \frac{2}{3} \lim_{n \rightarrow \infty} \left| \frac{(1 + n/3^n) (1 + 1/2^{n+1})}{(1 + 1/2^n) (1 + n/3^{n+1} + 1/3^{n+1})} \right| \\ &= \frac{2}{3} \frac{(1 + 0) \cdot (1 + 0)}{(1 + 0) \cdot (1 + 0 + 0)} = \frac{2}{3}\end{aligned}$$

where we have used the fact that

$$\lim_{x \rightarrow \infty} \frac{x}{a^x} = \lim_{x \rightarrow \infty} \frac{1}{x a^{x-1}} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. Hence,  $L < 1$  and so according to the ratio test the series  $\sum_{n+1}^{\infty} 2^n + 1/(3^n + n)$  is absolutely convergent.

7. (a) Let  $a_n = 1/\ln n$ . Then for  $2 \leq n < \infty$  we find

$$a_n = \frac{(-1)^n}{\ln n} = \frac{1}{\ln n}$$

Now since  $\ln n$  is monotonically increasing for  $2 \leq n < \infty$  we may conclude that  $a_n = 1/\ln n$  is monotonically decreasing for  $2 \leq n < \infty$  and so  $a_{n+1} \leq a_n$ . Furthermore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0$$

provided  $n \geq 2$  and so by the alternating series test we may conclude that the series  $\sum_{n=2}^{\infty} (-1)^n / \ln n$  converges.

(b) Let  $f(x) = \ln x/x$ . Hence,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{\ln x}{x} = \frac{1 - \ln x}{x^2}$$

Note that the derivative of  $f(x)$  becomes negative when  $x > e \approx 2.71828$  and hence, that  $f(x)$  becomes monotonically decreasing when  $e < x < \infty$ . As such, the terms of the sequence  $a_n = f(n) = \ln n/n$  are decreasing (i.e.  $a_{n+1} \leq a_n$ ) when  $3 \leq n < \infty$ . Furthermore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. As such, by the alternating series test we may conclude that the series  $\sum_{n=3}^{\infty} (-1)^n \ln n/n$  converges.

8. (a) Let  $a_n = 1/n^n$ . Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R = \lim_{n \rightarrow \infty} \sqrt[n]{n^{-n}} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

provided  $n \geq 1$ . Hence, since  $R < 1$  it follows from the root test that the series  $\sum_{n=1}^{\infty} 1/n^n$  is absolutely convergent.

- (b) Let  $a_n = [n/(n+1)]^{n^2}$ . Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

provided  $n \geq 1$ . Hence, since  $R < 1$  it follows from the root test that the series  $\sum_{n=1}^{\infty} [n/(n+1)]^{n^2}$  is absolutely convergent.

9. (a) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left( \frac{n+1}{n+2} - \frac{n}{n+1} \right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{2}{3} - \frac{1}{2} + \frac{3}{4} - \frac{2}{3} + \cdots + \frac{n+1}{n+2} - \frac{n}{n+1} = -\frac{1}{2} + \frac{n+1}{n+2}$$

Taking the limit of  $S_n$  as  $n \rightarrow \infty$  then gives

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} - \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{1+1/n}{1+2/n} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence, the series converges.

- (b) Let the series

$$\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{\infty} \left( \frac{n+1}{2^{n+1}} - \frac{n}{2^n} \right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{1}{2} - \frac{1}{2} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} - \frac{3}{8} + \cdots + \frac{n+1}{2^{n+1}} = -\frac{1}{2} + \frac{n+1}{2^{n+1}}$$

Taking the limit of  $S_n$  as  $n \rightarrow \infty$  then gives

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n+1}{2^{n+1}} - \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)2^n} - \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$$

using *L'Hospital's rule*. Hence, the series converges.

10. Let  $y = f(x)$  satisfy the following conditions:



- (a)  $f(x)$  is defined and continuous for  $c \leq x < \infty$
- (b)  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$
- (c)  $f(n) = a_n$

Let us suppose the improper integral  $\int_c^\infty f(x) dx$  diverges. Assumptions (b) and (c) imply that  $a_n > 0$  for  $n$  sufficiently large. Hence, by Theorem 7 of Section 6.5 the series  $\sum a_n$  is either convergent or properly divergent. Let the integer  $m$  be chosen so that  $m > c$ . Then, since  $f(x)$  is decreasing

$$\int_n^{n+1} f(x) dx \leq f(n) = a_n \quad \text{for } n \geq m$$

Hence,  $a_m + \cdots + a_{m+p} \geq \int_m^{m+p+1} f(x) dx$ . However, since  $\int_c^\infty f(x) dx$  diverges it follows that  $\lim_{p \rightarrow \infty} \int_m^{m+p+1} f(x) dx$  diverges, which thus ultimately implies that the series  $\sum_m^\infty a_n$  must be divergent as well.

11. Let an alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n, \quad a_n > 0$$

be given along with the two conditions

- (a)  $a_{n+1} \leq a_n$  for  $n = 1, 2, \dots$
- (b)  $\lim_{n \rightarrow \infty} a_n = 0$

What remains to be proven is that such a series converges given the aforementioned conditions. Let  $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$  denote the  $n$ th partial sum of an alternating series. Then  $S_1 = a_1$ ,  $S_2 = a_1 - a_2 < S_1$ ,  $S_3 = S_2 + a_3 > S_2$ ,  $S_3 = S_1 - (a_2 - a_3) < S_1$ , so that  $S_2 < S_3 < S_1$ . As such, we may conclude that  $S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$  or  $S_n \leq a_1$  and that each  $S_n \geq 0$  for  $n = 1, 2, \dots$ .

Next, let an  $\epsilon > 0$  be given. By the Cauchy criterion our goal is to find an  $N$  so that whenever  $m > n > N$  then

$$|S_m - S_n| = |a_{n+1} - a_{n+2} + \cdots \pm a_m| < \epsilon$$

Now since each partial sum is non-negative (i.e.  $S_n \geq 0$ ) and acknowledging that all partial sums are  $\leq$  the first term  $a_1$ , but now applied to the alternating series starting at  $a_{n+1}$  instead of  $a_1$  we can write

$$|S_m - S_n| \leq a_{n+1} < \epsilon$$

Now because  $\lim_{n \rightarrow \infty} a_n = 0$  we can find  $N$  such that  $a_{n+1} < \epsilon$  whenever  $n > N$ . Hence,

$$m > n > N \implies |S_m - S_n| \leq a_{n+1} < \epsilon$$

which thus satisfies our initial condition

$$|S_m - S_n| = |a_{n+1} - a_{n+2} + \cdots \pm a_m| < \epsilon$$

We may conclude that the sequence of partial sums  $S_n$  of our original alternating series subject to conditions (a) and (b) satisfies the Cauchy criterion and therefore, is convergent. Hence, the alternating series itself is convergent.

12. (a) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+4}{2n^3-1}$$

be given. In order to determine convergence or divergence we first try the comparison test for convergence. To this end, note that  $n+4 \leq 5n$  and  $2n^3-1 \geq n^3$  for  $n = 1, 2, \dots$ . Hence,

$$|a_n| = \frac{n+4}{2n^3-1} \leq \frac{5n}{n^3} = \frac{5}{n^2} = b_n \quad \text{for } n = 1, 2, \dots$$

As such, if we can prove that  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent. Now let  $y = f(x) = 5/x^2$ , which satisfies the following conditions:

- i.  $f(x)$  is defined and continuous for  $c \leq x < \infty$  for  $c \neq 0$
- ii.  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$
- iii.  $f(n) = b_n$

Then by the integral test the series  $\sum_{n=1}^{\infty} b_n$  converges or diverges according to whether the improper integral  $\int_c^{\infty} f(x) dx$  converges or diverges. As such, we evaluate

$$\int_1^{\infty} \frac{5}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{5}{x^2} dx = \lim_{b \rightarrow \infty} -\frac{5}{x} \Big|_1^b = 5 - \lim_{b \rightarrow \infty} \frac{5}{b} = 5 - \frac{5}{\infty} = 5$$

Hence, since the improper integral  $\int_c^{\infty} f(x) dx$  converges, so do the series  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} a_n$ .

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n-5}{n2^n}$$

be given. Since  $a_n \neq 0$  for  $n = 1, 2, \dots$  we can try the ratio test in order to determine convergence or divergence:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= L = \lim_{n \rightarrow \infty} \left| \frac{3(n+1)-5}{(n+1)2^{n+1}} \frac{n2^n}{3n-5} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{3(n+1)-5}{n+1} \frac{n}{3n-5} \right| \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{3(1+1/n)-5/n}{1+1/n} \frac{1}{3-5/n} \right| \\ &= \frac{1}{2} \frac{3+0-0}{1+0} \frac{1}{3-0} = \frac{1}{2} \end{aligned}$$

Hence, since  $L = 1/2 < 1$  the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n}{n+1}$$

be given. Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{e^n}{n+1} = \lim_{n \rightarrow \infty} e^n = \infty$$

using *L'Hospital's rule*. Hence, it follows from the  $n$ th term test that the series  $\sum_{n=1}^{\infty} a_n$  diverges.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n! + 1}$$

be given. Since

$$|a_n| = \frac{n^2}{n! + 1} < \frac{n^2}{n!} = b_n \quad \text{for } n = 1, 2, \dots$$

the comparison test for convergence tells us that if  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent. Since  $b_n \neq 0$  for  $n = 1, 2, \dots$  we can use the ratio test in order to determine if  $\sum_{n=1}^{\infty} b_n$  converges:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| &= L = \lim_{n \rightarrow \infty} \frac{(n+1)^2 n!}{(n+1)! n^2} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2 (n+1)} = \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = \lim_{n \rightarrow \infty} \frac{1+1/n}{n} \\ &= \frac{1+0}{\infty} = 0 \end{aligned}$$

Hence, since  $L = 0 < 1$  we may conclude that  $\sum_{n=1}^{\infty} b_n$  is absolutely convergent by the ratio test and thus, that  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent by the comparison test for convergence.

(e) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{3 \cdot 5 \cdots (2n+3)}$$

be given. Since  $a_n \neq 0$  for  $n = 1, 2, \dots$  we can use the ratio test to determine convergence:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= L = \lim_{n \rightarrow \infty} \frac{(n+1)!}{3 \cdot 5 \cdots [2(n+1)+3]} \frac{3 \cdot 5 \cdots (2n+3)}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2(n+1)+3} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2(1+1/n)+3/n} = \frac{1}{2(1+0)+0} = \frac{1}{2} \end{aligned}$$

Hence, since  $L = 1/2 < 1$  we may conclude that  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

(f) Let the series

$$\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{2n+3}$$

be given. This is an alternating series. Note that for  $n = 1, 2, 3, 4$  its terms are actually increasing (i.e.  $a_{n+1} > a_n$ ) in absolute value and  $a_{n+1} \leq a_n$  only becomes true when  $n = 5, 6, \dots$ . This is not a problem for the alternating series test to be valid however. Furthermore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{2n+3} = \lim_{n \rightarrow \infty} \frac{1/n}{2} = \frac{0}{2} = 0$$

using *L'Hospital's rule*. Hence, the alternating series converges.

(g) Let the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1 + \ln^2 n}{n \ln^2 n}$$

be given. As such, let us define the function  $y = f(x) = (1 + \ln^2 x)/n \ln^2 x$ . Then

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1 + \ln^2 x}{x \ln^2 x} = \lim_{x \rightarrow \infty} \left( \frac{1}{x \ln^2 x} + \frac{1}{x} \right) = \frac{1}{\infty} + \frac{1}{\infty} = 0$$

Furthermore,  $f(x)$  satisfies the following conditions:

- i.  $f(x)$  is defined and continuous for  $c \leq x < \infty$
- ii.  $f(x)$  decreases as  $x$  increases for  $x \geq 2$  and  $\lim_{x \rightarrow \infty} f(x) = 0$
- iii.  $f(n) = a_n$

Hence, we can use the integral test to determine whether the series  $\sum_{n=2}^{\infty} a_n$  converges or diverges:

$$\begin{aligned} \int_2^{\infty} \frac{1 + \ln^2 x}{x \ln^2 x} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1 + \ln^2 x}{x \ln^2 x} dx = \lim_{b \rightarrow \infty} \int_2^b \left( \frac{1}{x \ln^2 x} + \frac{1}{x} \right) dx \\ &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln^2 x} + \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} + \lim_{b \rightarrow \infty} \ln |x| \Big|_2^b \\ &= \lim_{b \rightarrow \infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln b} + \lim_{b \rightarrow \infty} (\ln b - \ln 2) \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{\ln b} + \frac{1}{\ln 2} + \ln b - \ln 2 \right) = \infty \end{aligned}$$

In conclusion, since the improper integral  $\int_c^{\infty} f(x) dx$  diverges, so will the series  $\sum_{n=2}^{\infty} a_n$ .

(h) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n+2} \equiv \sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$$

be given.  $\sum_{n=1}^{\infty} (-1)^n b_n$  is an alternating series with terms that are decreasing in absolute value:  $b_{n+1} < b_n$  for  $n = 1, 2, \dots$  and  $\lim_{n \rightarrow \infty} b_n = 0$ . Hence, by the alternating series test the series  $\sum_{n=1}^{\infty} (-1)^n b_n$  converges and thus, so will the series  $\sum_{n=1}^{\infty} a_n$ .

(i) Let the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + \ln n}$$

be given. Now since  $a \geq 0$  and  $n + \ln n < 2n$  for  $n = 1, 2, \dots$  we can define  $b_n = \ln n / 2n$  such that  $a_n > b_n \geq 0$ . Then by the comparison test for divergence if  $\sum_{n=1}^{\infty} b_n$  diverges so will  $\sum_{n=1}^{\infty} a_n$ . To this end, let us define the function  $y = f(x) = \ln x / 2x$ . Now since  $\ln x < 2x$  for  $1 \leq x < \infty$  and

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{2x} = \lim_{x \rightarrow \infty} \frac{1/x}{2} = 0$$

using *L'Hospital's rule*, we find that

- i.  $f(x)$  is defined and continuous for  $c \leq x < \infty$ , where  $c = 1$
- ii.  $f(x)$  decreases as  $x$  increases and  $\lim_{x \rightarrow \infty} f(x) = 0$
- iii.  $f(n) = a_n$

Then the series  $\sum_{n=1}^{\infty} b_n$  converges or diverges according to whether the improper integral  $\int_c^{\infty} f(x) dx$  converges or diverges:

$$\int_1^{\infty} \frac{\ln x}{2x} dx = \frac{1}{2} \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx = \frac{1}{2} \lim_{b \rightarrow \infty} \int_0^{\ln b} u du = \lim_{b \rightarrow \infty} \frac{u^2}{4} \Big|_0^{\ln b} = \lim_{b \rightarrow \infty} \frac{\ln^2 b}{4} = \infty$$

where we have used the substitution  $u = \ln x$ . Hence, by the integral test the series  $\sum_{n=1}^{\infty} b_n$  diverges and so by the comparison test for divergence the series  $\sum_{n=1}^{\infty} a_n$  diverges as well.

(j) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left( \frac{n+1}{2n} \right)^n$$

be given. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n+1}{2n} \right)^n} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \lim_{n \rightarrow \infty} \frac{1 + 1/n}{2} = \frac{1}{2}$$

Then by the root test, since  $R = 1/2 < 1$  the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

13. Let  $a_n > 0$  and  $b_n > 0$  for  $n = 1, 2, \dots$  and let the sequence  $a_n/b_n$  have limit  $k$ , possibly infinite.

- (a) Suppose  $0 < k < \infty$ , i.e.  $\lim_{n \rightarrow \infty} a_n/b_n = k$  is some positive number. Then for some  $\epsilon > 0$  we know that there must exist a positive integer  $N$  such that for all  $n > N$  it is true that

$$\left| \frac{a_n}{b_n} - k \right| < \epsilon \iff (k - \epsilon) b_n < a_n < (k + \epsilon) b_n$$

As  $k > 0$  we can choose  $\epsilon$  sufficiently small so that  $k - \epsilon > 0$ . Hence,

$$b_n < \frac{a_n}{k - \epsilon}$$

As such, by the comparison test for convergence, if  $\sum a_n$  converges then so must  $\sum b_n$ . Similarly  $a_n < (k + \epsilon)b_n$ . Hence, if  $\sum a_n$  diverges then by the comparison test for divergence so will  $\sum b_n$ . In conclusion, both series  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge.

- (b) Suppose  $k = 0$ . Then for some  $\epsilon > 0$  there must exist a positive integer  $N$  such that for all  $n > N$  it is true that

$$\frac{a_n}{b_n} < \epsilon \iff a_n < \epsilon b_n$$

Hence, by the comparison test for convergence, if  $\sum b_n$  converges then so must  $\sum a_n$ . Additionally, as long as  $\sum a_n$  converges the inequality can still be satisfied if  $\sum b_n$  diverges by choosing  $\epsilon$  sufficiently small.

- (c) Suppose  $k = \infty$ . Then for some  $\epsilon > 0$  we know that there must exist a positive integer  $N$  such that for all  $n > N$  it is true that

$$(k - \epsilon) b_n < a_n < (k + \epsilon) b_n$$

From the first inequality we see that

$$a_n > (k - \epsilon) b_n$$

from which we may gather that  $\sum a_n$  may diverge while  $\sum b_n$  converges, since  $k = \infty$ . Similarly, since  $a_n < (k + \epsilon)b_n$  then the comparison test for divergence tells us that divergence of  $\sum a_n$  implies divergence of  $\sum b_n$ .

14. (a) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{3n^2+n+1}$$

be given and let  $b_n = 1/n$ . Using Problem 13 we thus find

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{3n^2+n+1} = \lim_{n \rightarrow \infty} \frac{2+1/n}{3+1/n+1/n^2} = \frac{2}{3}$$

and so the series  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge. Since the series  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$  diverges we conclude that the series  $\sum_{n=1}^{\infty} a_n$  must diverge as well.

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^3 - 3n^2 + 5}{n^5 + n + 1}$$

be given and let  $b_n = 1/n^2$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^5 - 3n^4 + 5n^2}{n^5 + n + 1} = \lim_{n \rightarrow \infty} \frac{1 - 3/n + 5/n^3}{1 + 1/n^4 + 1/n^5} = 1$$

and so the series  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge. Since the series  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$  converges we conclude that the series  $\sum_{n=1}^{\infty} a_n$  must converge as well.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin \frac{1}{n}$$

be given and let  $b_n = 1/n$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{\cos(1/n)/n^2}{1/n^2} = \lim_{n \rightarrow \infty} \cos \frac{1}{n} = \cos \frac{1}{\infty} = 1$$

using *L'Hospital's rule* and so the series  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge. Since the series  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$  diverges we conclude that the series  $\sum_{n=1}^{\infty} a_n$  must diverge as well.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$$

be given and let  $b_n = 1/n^2$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \frac{1}{2}$$

using *L'Hospital's rule* and so the series  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge. Since the series  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$  converges we conclude that the series  $\sum_{n=1}^{\infty} a_n$  must converge as well.

## Section 6.9

1. (a) Let the sum  $\sum_{n=1}^{\infty} 1/n^2$  be given and let us define the allowed error as  $\epsilon = 1$ . We know from the previous section that this series converges by the integral test of

Theorem 14. Hence, by Theorem 23 we find

$$|R_n| = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x) dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

and so the condition  $T_n \leq \epsilon$  then translates to the inequality  $n \geq 1$ , which is satisfied for  $n = 1$ . Hence, one term is sufficient to compute the sum with given allowed error  $\epsilon = 1$  and so  $S_1 = 1$ .

- (b) Let the sum  $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$  be given and let us define the allowed error as  $\epsilon = 1/10$ . Now since this series converges by the alternating series test then by Theorem 26

$$|R_n| < a_{n+1} = T_n$$

Hence, we end up with the inequality  $a_{n+1} \leq \epsilon$  or  $1/(n+1)^2 \leq 1/10 \iff (n+1)^2 \geq 10$ , which is satisfied for  $n = 3$ . Hence, three terms is sufficient to compute the sum with the given allowed error  $\epsilon = 1/10$  and so  $S_3 \approx 0.86$ .

- (c) Let the sum  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n/(n^3 + 5)$  be given and let us define the allowed error as  $\epsilon = 1/5$ . It is true that  $n^3 + 5 > n^3$  and so we can define  $b_n = 1/n^2$  such that  $|a_n| < b_n$  for  $n \geq n_1 = 1$ . Now since  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$  converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x) dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find  $T_n \leq \epsilon \implies n \geq 5$ . Hence, five terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/5$  and so  $S_5 \approx 0.51$ .

- (d) Let the sum  $\sum n = 1^\infty 1/(n^2 + 1)$  be given and let us define the allowed error as  $\epsilon = 1/2$ . It is true that  $n^2 + 1 > n^2$  and so we can define  $b_n = 1/n^2$  such that  $|a_n| < b_n$  for  $n \geq n_1 = 1$ . Now since  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$  converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m \leq \sum_{m=n+1}^{\infty} b_m \frac{1}{m^2} < \int_n^{\infty} f(x) dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find  $T_n \leq \epsilon \implies n \geq 2$ . Hence, two terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/2$  and so  $S_2 = 0.7$ .

- (e) Let the sum  $\sum_{n=1}^{\infty} 1/n^n$  be given and let us define the allowed error as  $\epsilon = 1/100$ . Then

$$\sqrt[n]{|a_n|} = \frac{1}{n} \leq r < 1$$

for  $n \geq 2$ , so that the series  $\sum a_n$  converges by the root test. Hence, by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = T_n \implies \frac{1}{(n+1)^{n+1}} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n(n+1)^n} \leq \epsilon$$



for  $n \geq 2$ . In other words, we are looking for the smallest integer  $n \geq 2$  such that  $n(n+1)^n \geq 100$ , which is satisfied for  $n = 3$ . Hence, three terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/100$  and so  $S_3 \approx 1.287$ .

- (f) Let the sum  $\sum_{n+1}^{\infty} 1/n!$  be given and let us define the allowed error as  $\epsilon = 1/100$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq r < 1$$

for  $n \geq 1$ , so that the series  $\sum a_n$  converges by the ratio test. Hence, by Theorem 24

$$|R_n| \leq \frac{|a_{n+1}|}{1-r} = T_n \implies \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+2}} = \frac{1}{(n+1)!} \left( 1 + \frac{1}{n+1} \right) \leq \epsilon$$

for  $n \geq 1$ . In other words, we are looking for the smallest integer  $n \geq 1$  such that  $T_n \leq \epsilon$ , which is satisfied for  $n = 4$ . Hence, four terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/100$  and so  $S_4 \approx 1.708$ .

- (g) Let the sum  $\sum_{n+1}^{\infty} (-1)^{n+1}/(2n-1)!$  be given and let us define the allowed error as  $\epsilon = 1/1000$ . Since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n-1)!}{(2n+1)!} < 1 \quad \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{(2n-1)!} = 0$$

the series  $\sum a_n$  converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(2n+1)!} = T_n \implies \frac{1}{(2n+1)!} \leq \epsilon$$

and so we are looking for the smallest integer such that  $(2n+1)! \geq 1000$ , which is satisfied for  $n = 3$ . Hence, three terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/1000$  and so  $S_3 \approx 0.8417$ .

- (h) Let the sum  $\sum_{n+2}^{\infty} (-1)^n/(n \ln n)$  be given and let us define the allowed error as  $\epsilon = 1/2$ . Since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n \ln n}{(n+1) \ln(n+1)} < 1 \quad \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$$

the series  $\sum a_n$  converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(n+1) \ln(n+1)} = T_n \implies \frac{1}{(n+1) \ln(n+1)} \leq \epsilon$$

and so we are looking for the smallest integer such that  $(n+1) \ln(n+1) \geq 2$ , which is satisfied for  $n = 2$ . Hence, one term is sufficient to compute the sum with the given allowed error  $\epsilon = 1/2$  and so  $S_1 \approx 0.72$ .

- (i) Let the sum  $\sum_{n=2}^{\infty} 1/(n^3 \ln n)$  be given and let us define the allowed error as  $\epsilon = 1/2$ . It is true that  $n^3 \ln n > n^2$  for  $n \geq 2$  and so we can define  $b_n = 1/n^2$  such that  $|a_n| < b_n$  for  $n \geq n_1 = 2$ . Now since  $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} 1/n^2$  converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x) dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find  $T_n \leq \epsilon \implies n \geq 2$ . Hence, one term is sufficient to compute the sum with the given allowed error  $\epsilon = 1/2$  and so  $S_1 \approx 0.18$ .

- (j) Let the sum  $\sum_{n=1}^{\infty} 2^n/(3^n + 1)$  be given and let us define the allowed error as  $\epsilon = 1/10$ . It is true that  $3^n + 1 > 3^n$  for  $n \geq 1$  and so we can define  $b_n = 2^n/3^n$  such that  $|a_n| < b_n$  for  $n \geq n_1 = 1$ . Now since  $\sqrt[n]{|b_n|} = \sqrt[n]{2^n/3^n} = 2/3 \leq r < 1$  for  $n \geq 1$  we may conclude that the series  $\sum b_n$  converges by the root test. Hence, choosing  $r = 2/3$  then by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = \frac{2^{n+1}}{3^n} = T_n \implies \frac{2^{n+1}}{3^n} \leq \epsilon$$

and so we are looking for the smallest integer such that  $3^n/2^{n+1} \geq 10$ , which is satisfied for  $n = 8$ . Hence, eight terms are sufficient to compute the sum with the given allowed error  $\epsilon = 1/10$  and so  $S_8 \approx 1.697$ .

2. Let  $\sum a_n$  be the geometric series  $1 + r + r^2 + \dots = \sum_{n=0}^{\infty} r^n$ . By Theorem 16 this series converges for  $-1 < r < 1$ . Hence, by Theorem 23

$$|R_n| = \sum_{m=n+1}^{\infty} r^m < \int_n^{\infty} r^x dx = T_n$$

Or

$$\begin{aligned} T_n &= \int_n^{\infty} r^x dx = \lim_{b \rightarrow \infty} \int_n^b r^x dx = \lim_{b \rightarrow \infty} \int_n^b e^{x \ln r} dx = \lim_{b \rightarrow \infty} \int_{n \ln r}^{b \ln r} \frac{e^u}{\ln r} du \\ &= \lim_{b \rightarrow \infty} \frac{e^u}{\ln r} \Big|_{n \ln r}^{b \ln r} \\ &= \lim_{b \rightarrow \infty} \frac{e^{b \ln r}}{\ln r} - \frac{e^{n \ln r}}{\ln r} \\ &= -\frac{e^{n \ln r}}{\ln r} = -\frac{r^n}{\ln r} \end{aligned}$$

assuming  $0 < r < 1$ .

- (a) let the given allowed error  $\epsilon = 1/100$ . In order to determine how many terms are needed to compute the sum with error less than  $\epsilon$  we require  $T_n < \epsilon$ . For  $r = 1/2$  this results in

$$-\frac{1}{2^n \ln 2^{-1}} < \frac{1}{100} \iff n > \frac{\ln(100/\ln 2)}{\ln 2}$$

which is satisfied for  $n = 8$ . Hence, when  $r = 1/2$ , 8 terms are sufficient to compute the sum with error less than  $\epsilon = 1/100$ . For  $r = 0.9 = 9/10$  we get

$$-\frac{1}{\ln(9/10)} \left(\frac{9}{10}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 9/10)}{\ln 10/9}$$

which is satisfied for  $n = 66$ . Hence, when  $r = 0.9$ , 66 terms are sufficient to compute the sum with error less than  $\epsilon = 1/100$ . For  $r = 0.99 = 99/100$  we get

$$-\frac{1}{\ln(99/100)} \left(\frac{99}{100}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 99/100)}{\ln 100/99}$$

which is satisfied for  $n = 916$ . Hence, when  $r = 0.99$ , 916 terms are sufficient to compute the sum with error less than  $\epsilon = 1/100$ .

- (b) The closed form formula (6.17) for a geometric series  $1 + ar + ar^2 + \dots$ , with  $a = 1$  and  $-1 < r < 1$  is given by  $S = 1/(1 - r)$ . Likewise, the closed form formula for the partial sum of the same geometric series is given by  $S_n = (1 - r^{n+1})/(1 - r)$ . The remainder  $R_n$  after  $n$  terms thus can be defined as

$$|R_n| = |S_n - S| = \left| \frac{1 - r^{n+1}}{1 - r} - \frac{1}{1 - r} \right| = \left| \frac{-r^{n+1}}{1 - r} \right| < \epsilon \iff -\epsilon < -\frac{r^{n+1}}{1 - r} < \epsilon$$

The inequality on the right hand side can be further manipulated to finally get

$$\begin{aligned} -\frac{r^{n+1}}{1 - r} &< \epsilon \\ r^{n+1} &> -\epsilon(1 - r) \\ \ln |r|^{n+1} &> \ln |-\epsilon(1 - r)| \\ n &> \frac{\ln \epsilon(1 - r)}{\ln |r|} \end{aligned}$$

where  $-1 < r < 1$ .

- (c) When  $r$  approaches 1 from the left we note that

$$\lim_{r \rightarrow 1^-} \frac{\ln \epsilon(1 - r)}{\ln |r|} = \lim_{r \rightarrow 1^-} \frac{\ln \epsilon(1 - r)}{\ln r} = \lim_{r \rightarrow 1^-} \ln \epsilon(1 - r) \cdot \lim_{r \rightarrow 1^-} \frac{1}{\ln r} = -\infty \cdot -\infty = \infty$$

Hence, it follows from (b) that  $n \rightarrow \infty$  when  $r \rightarrow 1^-$ , or in other words; that the number of terms needed to compute the sum with error less than a fixed  $\epsilon$  becomes infinite.

3. Let the series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where  $p > 0$  be given. As such,  $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$  and so  $S_1 = a_1 = 1$ ,  $S_2 = a_1 - a_2 = 1 - 2^{-p}$  so that  $0 < S_2 < S_1$ ,  $S_3 = S_1 - (a_2 - a_3) = 1 - 2^{-p} + 3^{-p}$  so that  $0 < S_3 < S_1$  and  $S_2 < S_3 < S_1$ . Reasoning in this way, we conclude that

$$S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$$

Hence, the smallest partial sum is  $S_2$ , but we just established that  $S_2 = 1 - 2^{-p} > 0$ . Hence, it follows that the sum  $S = \lim_{n \rightarrow \infty} S_n$  must be positive whenever  $p > 0$ .

## Section 6.10

1. Let the following relations be given:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

Then by (6.15)

(a)

$$\sum_{n=1}^{\infty} \frac{6}{n^2} = 6 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90} =$$

(c)

$$\sum_{n=1}^{\infty} \frac{2n^2 - 3}{n^4} = \sum_{n=1}^{\infty} \frac{2}{n^2} - \sum_{n=1}^{\infty} \frac{3}{n^4} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 3 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{3} - \frac{\pi^4}{30}$$

(d)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{9 + 3n^2 + 5n^4}{n^6} &= \sum_{n=1}^{\infty} \frac{9}{n^6} + \sum_{n=1}^{\infty} \frac{3}{n^4} + \sum_{n=1}^{\infty} \frac{5}{n^2} = 9 \sum_{n=1}^{\infty} \frac{1}{n^6} + 3 \sum_{n=1}^{\infty} \frac{1}{n^4} + 5 \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{5\pi^2}{6} + \frac{\pi^4}{30} + \frac{\pi^6}{105} \end{aligned}$$

(e)

$$\sum_{n=3}^{\infty} \frac{n^4 - 1}{n^6} = \sum_{n=3}^{\infty} \frac{1}{n^2} - \sum_{n=3}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{5}{4} - \sum_{n=1}^{\infty} \frac{1}{n^6} + \frac{65}{64} = \frac{\pi^2}{6} - \frac{\pi^6}{945} - \frac{15}{64}$$

(f)

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n^2+1}{(n^2-1)^2} &= \sum_{n=2}^{\infty} \left[ \frac{1}{2(n+1)^2} + \frac{1}{2(n-1)^2} \right] = \sum_{n=2}^{\infty} \frac{1}{2(n+1)^2} + \sum_{n=2}^{\infty} \frac{1}{2(n-1)^2} \\
&= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{(n+1)^2} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \\
&= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{8} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^2} + \frac{1}{2} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{8} - \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{2} - \frac{1}{2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{5}{8} = \frac{\pi^2}{6} - \frac{5}{8}
\end{aligned}$$

2. (a)

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} + 1 - 1 = \sum_{n=2}^{\infty} \frac{1}{n^3} + 1 = \sum_{n=2}^{\infty} \frac{1}{(n-1)^3}$$

(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \sum_{n=1}^{\infty} f(n+1) - \sum_{n=1}^{\infty} f(n) \\
&= \sum_{n=1}^{\infty} f(n) + \lim_{n \rightarrow \infty} f(n) - f(1) - \sum_{n=1}^{\infty} f(n) \\
&= \lim_{n \rightarrow \infty} f(n) - f(1)
\end{aligned}$$

if the limit exists.

(c)

$$\begin{aligned}
\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] &= \sum_{n=2}^{\infty} f(n+1) - \sum_{n=2}^{\infty} f(n-1) \\
&= \sum_{n=1}^{\infty} f(n+1) + \lim_{n \rightarrow \infty} f(n+1) - f(2) - \sum_{n=1}^{\infty} f(n) \\
&= \sum_{n=1}^{\infty} f(n) - f(1) + \lim_{n \rightarrow \infty} f(n) + \lim_{n \rightarrow \infty} f(n+1) - f(2) \\
&\quad - \sum_{n=1}^{\infty} f(n) \\
&= \lim_{n \rightarrow \infty} [f(n) + f(n+1)] - f(1) - f(2)
\end{aligned}$$

if the limit exists.

3. (a) Let  $f(n) = 1/n^2$ . Then using 2(b)

$$\begin{aligned}\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n \rightarrow \infty} f(n) - f(1) \\ \sum_{n=1}^{\infty} \left[ \frac{1}{(n+1)^2} - \frac{1}{n^2} \right] &= \lim_{n \rightarrow \infty} \frac{1}{n^2} - 1 \\ \sum_{n=1}^{\infty} -\frac{2n+1}{n^2(n+1)^2} &= 0 - 1 \\ \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} &= 1\end{aligned}$$

- (b) Let  $f(n) = 1/n$ . Then using 2(b)

$$\begin{aligned}\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n \rightarrow \infty} f(n) - f(1) \\ \sum_{n=1}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n} \right] &= \lim_{n \rightarrow \infty} \frac{1}{n} - 1 \\ \sum_{n=1}^{\infty} -\frac{1}{n(n+1)} &= 0 - 1 \\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= 1\end{aligned}$$

- (c) Let  $f(n) = 1/n$ . Then using 2(c)

$$\begin{aligned}\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] &= \lim_{n \rightarrow \infty} [f(n) + f(n+1)] - f(1) - f(2) \\ \sum_{n=2}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n-1} \right] &= \lim_{n \rightarrow \infty} \left( \frac{1}{n} + \frac{1}{n+1} \right) - 1 - \frac{1}{2} \\ \sum_{n=2}^{\infty} -\frac{2}{n^2-1} &= 0 + 0 - 1 - \frac{1}{2} \\ -2 \sum_{n=2}^{\infty} \frac{1}{n^2-1} &= -\frac{3}{2} \\ \sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \frac{3}{4}\end{aligned}$$

(d) Let  $f(n) = 1/n^2$ . Then using 2(c)

$$\begin{aligned}\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] &= \lim_{n \rightarrow \infty} [f(n) + f(n+1)] - f(1) - f(2) \\ \sum_{n=2}^{\infty} \left[ \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} \right] &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right] - 1 - \frac{1}{4} \\ \sum_{n=2}^{\infty} -\frac{4n}{(n^2-1)^2} &= 0 + 0 - 1 - \frac{1}{4} \\ \sum_{n=2}^{\infty} \frac{4n}{(n^2-1)^2} &= \frac{5}{4}\end{aligned}$$

4. Let the relation

$$\frac{1}{1-r} = 1 + r + \cdots + r^n + \cdots = \sum_{n=0}^{\infty} r^n \quad -1 < r < 1$$

be given.

(a) Using the *Cauchy product* as illustrated in Fig. 6.6 we can thus write

$$\begin{aligned}\frac{1}{(1-r)^2} &= \frac{1}{1-r} \cdot \frac{1}{1-r} \\ &= (1 + r + \cdots + r^n + \cdots) \cdot (1 + r + \cdots + r^n + \cdots) \\ &= 1 + (1 \cdot r + r \cdot 1) + (1 \cdot r^2 + r \cdot r + r^2 \cdot 1) + \cdots \\ &\quad + (1 \cdot r^n + r \cdot r^{n-1} + \cdots + r^n \cdot 1) + \cdots \\ &= 1 + 2r + 3r^2 + \cdots + (n+1)r^n + \cdots\end{aligned}$$

(b) Firstly, we will derive the formula for a sum of an arithmetic sequence  $a_m = a_1 + (m-1)d$ , where  $d$  denotes the common difference between successive terms. We will start by expressing the arithmetic series in two different ways:

$$\begin{aligned}S_m &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (m-2)d] + [a_1 + (m-1)d] \\ S_m &= [a_m - (m-1)d] + [a_m - (m-2)d] + \cdots + (a_m - 2d) + (a_m - d) + a_m\end{aligned}$$

Adding both equations, we find that all terms involving  $d$  cancel and so we are left with

$$2S_m = m(a_1 + a_m) \iff S_m = \frac{m(a_1 + a_m)}{2}$$

Now, again using the *Cauchy product*

$$\begin{aligned}
\frac{1}{(1-r)^3} &= \frac{1}{(1-r)^2} \cdot \frac{1}{1-r} \\
&= [1 + 2r + 3r^2 + \dots + (n+1)r^n + \dots] \cdot (1 + r + \dots + r^n + \dots) \\
&= 1 + (1 \cdot r + 2r \cdot 1) + \dots + [1 \cdot r^n + 2r \cdot r^{n-1} + \dots + (n+1) \cdot r^n] + \dots \\
&= 1 + 3r + \dots + [1 + 2 + \dots + (n+1)]r^n + \dots \\
&= 1 + 3r + \dots + \frac{(n+2)(n+1)}{2}r^n + \dots
\end{aligned}$$

where we have used the fact that the arithmetic sequence  $1 + 2 + \dots + (n+1)$  can be written as  $(n+2)(n+1)/2$  using the derived formula above.

5. We want to prove that

$$(1-r)^{-k} = 1 + kr + \frac{k(k+1)}{1 \cdot 2}r^2 + \dots + \frac{k(k+1) \dots (k+n-1)}{1 \cdot 2 \dots n}r^n + \dots$$

for  $-1 < r < 1$ ,  $k = 1, 2, \dots$ . Using the solutions to 4(a) and 4(b) we can confirm the above equation is true for  $k = 1, 2, 3$ . It remains to be proven that the equation is true for  $k = 1, 2, \dots$ . In order to simplify the discussion we will write the coefficients appearing in the equation above as binomial coefficients and also make use of *Pascal's identity*:

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} \qquad \binom{k}{n} = \binom{k-1}{n-1} + \binom{k-1}{n}$$

Next, we assume the equation is true for some positive integer  $k \geq 2$  and consider the expansion

$$\begin{aligned}
(1-r)^{-k-1} &= (1-r)^{-1} \left[ 1 + kr + \frac{k(k+1)}{1 \cdot 2}r^2 + \dots + \frac{k(k+1) \dots (k+n-1)}{1 \cdot 2 \dots n}r^n + \dots \right] \\
&= (1-r)^{-1} \left[ 1 + \binom{k}{1}r + \binom{k+1}{2}r^2 + \dots + \binom{k+n-1}{n}r^n + \dots \right] \\
&= (1+r+r^2+\dots+r^n) \left[ 1 + \binom{k}{1}r + \binom{k+1}{2}r^2 + \dots + \binom{k+n-1}{n}r^n + \dots \right] \\
&= 1 + \left[ 1 + \binom{k}{1} \right] r + \left[ 1 + \binom{k}{1} + \binom{k+1}{2} \right] r^2 + \dots \\
&\quad + \left[ 1 + \binom{k}{1} + \binom{k+1}{2} + \dots + \binom{k+n-2}{n-1} + \binom{k+n-1}{n} \right] r^n + \dots \\
&= 1 + \binom{k+1}{1}r + \binom{k+2}{2}r^2 + \dots + \binom{k+n}{n}r^n + \dots \\
&= 1 + (k+1)r + \frac{(k+1)(k+2)}{1 \cdot 2}r^2 + \dots + \frac{(k+1)(k+2) \dots (k+n)}{1 \cdot 2 \dots n}r^n + \dots
\end{aligned}$$



Hence, the equation is true for  $k + 1$  and so by induction the equation must be true for any positive integer  $k \geq 1$ .

6. Let  $\sin x$  and  $\cos x$  be represented for all  $x$  by the absolutely convergent series

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} a_n \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} b_n\end{aligned}$$

Then by (6.24) it follows that

$$\begin{aligned}\sin x \cos x &= \sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n \\ &= \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \right] \\ &\quad \times \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right] \\ &= x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots + x^{2n+1} \sum_{k=0}^m \frac{(-1)^k (-1)^{m-k}}{(2k+1)! (2m-2k)!} + \cdots\end{aligned}$$

and

$$\frac{1}{2} \sin 2x = x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \cdots + (-1)^n \frac{2^{2n} x^{2n+1}}{(2n+1)!} + \cdots$$

Hence, we need to prove that

$$\sum_{k=0}^m \frac{1}{(2m+1)! (2m-2k)!} = \frac{2^{2m}}{(2k+1)!} \iff \sum_{k=0}^m \frac{(2m+1)!}{(2k+1)! (2m-2k)!} = 2^{2m}$$

To this end (making use of *Pascal's identity*)

$$\begin{aligned}\sum_{k=0}^m \frac{(2m+1)!}{(2k+1)! (2m-2k)!} &= \sum_{k=0}^m \binom{2m+1}{2k+1} \\ &= \sum_{k=0}^m \left[ \binom{2m}{2k} + \binom{2m}{2k+1} \right] \\ &= \sum_{k=0}^m \left[ \frac{(2m)!}{(2k)! (2m-2k)!} + \frac{(2m)!}{(2k+1)! (2m-2k-1)!} \right] \\ &= \sum_{k=0}^{2m} \frac{(2m)!}{k! (2m-k)!} \\ &= \sum_{k=0}^{2m} \binom{2m}{k}\end{aligned}$$

Now from the definition of the binomial formula

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

it then finally follows that

$$\sum_{k=0}^{2m} \binom{2m}{k} = (1 + 1)^{2m} = 2^{2m}$$

which thus completes the proof.

7. Let the sequence  $a_n$  be close to the sequence  $b_n$  and let  $\sum_{n=1}^{\infty} b_n$  be known. We can write  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n)$ .

- (a) Let  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2^n + 1)^{-1}$  and let us choose  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2^{-n} = 1$ . Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n) \\ \sum_{n=1}^{\infty} (2^n + 1)^{-1} &= \sum_{n=1}^{\infty} 2^{-n} - \sum_{n=1}^{\infty} (4^n + 2^n)^{-1} \\ &= 1 - \sum_{n=1}^{\infty} (4^n + 2^n)^{-1} \end{aligned}$$

As such, we find that for  $n \geq 7$  the expression  $\sum_{n=1}^{\infty} a_n = 1 - \sum_{n=1}^{\infty} (4^n + 2^n)^{-1} \cong 0.7645$ , whereas we require  $n \geq 15$  for  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2^n + 1)^{-1} \cong 0.7645$ .

- (b) Let  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2^n + 9^{-1})^{-1}$  and let us choose  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2^{-n} = 1$ . Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n) \\ \sum_{n=1}^{\infty} (2^n + 9^{-1})^{-1} &= \sum_{n=1}^{\infty} 2^{-n} - \sum_{n=1}^{\infty} (4^n 9 + 2^n)^{-1} \\ &= 1 - \sum_{n=1}^{\infty} (4^n 9 + 2^n)^{-1} \end{aligned}$$

As such, we find that for  $n \geq 6$  the expression  $\sum_{n=1}^{\infty} a_n = 1 - \sum_{n=1}^{\infty} (4^n 9 + 2^n)^{-1} \cong 0.9646$ , whereas we require  $n \geq 14$  for  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2^n + 9^{-1})^{-1} \cong 0.9646$ .

- (c) Let  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (n^2 + 1)^{-1}$  and let us choose  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ . Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n) \\ \sum_{n=1}^{\infty} (n^2 + 1)^{-1} &= \sum_{n=1}^{\infty} n^{-2} - \sum_{n=1}^{\infty} (n^4 + n^2)^{-1} \\ &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} (n^4 + n^2)^{-1}\end{aligned}$$

For  $n \geq 16$  we then find  $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - \sum_{n=1}^{\infty} (n^4 + n^2)^{-1} \cong 1.0767$ . Next, we use  $b_n = n^{-4}$ . Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} (n^4 + n^2)^{-1} \\ &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} n^{-4} + \sum_{n=1}^{\infty} (n^6 + n^4)^{-1} \\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty} (n^6 + n^4)^{-1}\end{aligned}$$

For  $n \geq 6$  we then find  $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - (\pi^4/90) + \sum_{n=1}^{\infty} (n^6 + n^4)^{-1} \cong 1.0767$ . Lastly, we use  $b_n = n^{-6}$ . Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty} (n^6 + n^4)^{-1} \\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty} n^{-6} - \sum_{n=1}^{\infty} (n^8 + n^6)^{-1} \\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \frac{\pi^6}{945} - \sum_{n=1}^{\infty} (n^8 + n^6)^{-1}\end{aligned}$$

For  $n \geq 3$  we then find  $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - (\pi^4/90) + (\pi^6/945) - \sum_{n=1}^{\infty} (n^8 + n^6)^{-1} \cong 1.0767$ .

## Section 6.13

1. (a) Let the series  $\sum_{n=1}^{\infty} x^n/(2n^2 - n)$  be given. By the ratio test we find

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{2(n+1)^2 - (n+1)} \frac{2n^2 - n}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{2n^2 - n}{2n^2 + 3n + 1} \\ &= |x| \lim_{n \rightarrow \infty} \frac{2 - 1/n}{2 + 3/n + 1/n^2} = |x|\end{aligned}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = |x| < 1$  or  $-1 < x < 1$ . To test for convergence when  $x = \pm 1$  we employ the integral test:

$$\begin{aligned} \int_1^\infty \frac{(\pm 1)}{2y^2 - y} dy &= (\pm 1) \lim_{b \rightarrow \infty} \int_1^b \left( \frac{2}{2y - 1} - \frac{1}{y} \right) dy = (\pm 1) \lim_{b \rightarrow \infty} \left( \int_1^{2b-1} \frac{du}{u} - \int_1^b \frac{dy}{y} \right) \\ &= (\pm 1) \lim_{b \rightarrow \infty} (\ln |2b - 1| - \ln |b|) \\ &= (\pm 1) \lim_{b \rightarrow \infty} \ln \frac{2b - 1}{b} \\ &= (\pm 1) \lim_{b \rightarrow \infty} \ln \left( 2 - \frac{1}{b} \right) = (\pm 1) \ln 2 \end{aligned}$$

Since the improper integral  $\int_c^\infty f(y) dy$  converges, so will the series  $\sum_{n=1}^\infty (\pm 1)/(2n^2 - n)$ . Hence, the series  $\sum_{n=1}^\infty x^n/(2n^2 - n)$  converges for  $-1 \leq x \leq 1$ .

(b) Let the series  $\sum_{n=1}^\infty nx^n/2^n$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{2^{n+1}} \frac{2^n}{nx^n} \right| = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{|x|}{2} \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = \frac{|x|}{2}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = |x|/2 < 1$  or  $-2 < x < 2$ .

(c) Let the series  $\sum_{n=1}^\infty 1/nx^{2n}$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{nx^{2n}}{(n+1)x^{2(n+1)}} \right| = \frac{1}{x^2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{1}{x^2} \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} = \frac{1}{x^2}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = 1/x^2 < 1$  or  $|x| > 1 \iff x > 1, x < -1$ .

(d) Let the series  $\sum_{n=0}^\infty 1/2^{nx}$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{nx}}{2^{(n+1)x}} \right| = \frac{1}{|2^x|}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = 1/2^x < 1$  or  $2^x > 2^0 \implies x > 0$ .

(e) Let the series  $\sum_{n=1}^\infty x^n/(1-x)^n$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(1-x)^{n+1}} \frac{(1-x)^n}{x^n} \right| = \left| \frac{x}{1-x} \right|$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = |x/(1-x)| < 1$  or  $x-1 < x < 1-x \implies x < 1/2$ .

(f) Let the series  $\sum_{n=1}^{\infty} 2^n \sin^n x / n^2$  be given. By the ratio test we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} \sin^{n+1} x}{(n+1)^2} \frac{n^2}{2^n \sin^n x} \right| = 2 |\sin x| \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} \\ &= 2 |\sin x| \lim_{n \rightarrow \infty} \frac{1}{1 + 2/n + 1/n^2} = 2 |\sin x| \end{aligned}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = 2 |\sin x| < 1$  or  $-1/2 < \sin x < 1/2 \iff \sin^{-1}(-1/2) < x < \sin^{-1}(1/2)$ , which is satisfied when  $(-\pi/6) + n\pi < x < (\pi/6) + n\pi$  for  $n = 0, \pm 1, \pm 2, \dots$ .

(g) Let the series  $\sum_{n=1}^{\infty} (x-1)^n / n^2$  be given. By the ratio test we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(n+1)^2} \frac{n^2}{(x-1)^n} \right| = |x-1| \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} \\ &= |x-1| \lim_{n \rightarrow \infty} \frac{1}{1 + 2/n + 1/n^2} = |x-1| \end{aligned}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = |x-1| < 1$  or  $-1 < x-1 < 1 \iff 0 < x < 2$ . For  $x = 0$  and  $x = 2$  the series converges by comparison with the harmonic series of order 2:

$$\left| \frac{(\pm 1)^n}{n^2} \right| \leq \frac{1}{n^2}$$

Hence, the series converges for  $0 \leq x \leq 2$ .

(h) Let the series  $\sum_{n=1}^{\infty} 1/x^n \ln(n+1)$  be given. By the ratio test we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^n \ln(n+1)}{x^{n+1} \ln(n+2)} \right| = \frac{1}{|x|} \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)} = \frac{1}{|x|} \lim_{n \rightarrow \infty} \frac{1/(n+1)}{1/(n+2)} \\ &= \frac{1}{|x|} \lim_{n \rightarrow \infty} \frac{1 + 2/n}{1 + 1/n} = \frac{1}{|x|} \end{aligned}$$

Hence, by the ratio test the series converges when  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L = 1/|x| < 1$  or  $|x| > 1 \iff x > 1, x \leq -1$ .

(i) Let the series  $\sum_{n=1}^{\infty} (x-2)^{3n} / n!$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{3(n+1)}}{(n+1)!} \frac{n!}{(x-2)^{3n}} \right| = |(x-2)^3| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

Hence, the series converges for all  $x$ .

(j) Let the series  $\sum_{n=2}^{\infty} x^n / \ln^n n$  be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\ln^{n+1} n} \frac{\ln^n n}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Hence, the series converges for all  $x$ .

2. (a) Let the series  $\sum_{n=1}^{\infty} x^n/n^3$ , where  $-1 \leq x \leq 1$  be given. The ratio test gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^3} \frac{n^3}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + 3n^2 + 3n + 1} \\ &= |x| \lim_{n \rightarrow \infty} \frac{1}{1 + 3/n + 3/n^2 + 1/n^3} = |x| \end{aligned}$$

Hence, the series converges for  $|x| < 1 \iff -1 < x < 1$ . For  $x = \pm 1$  the series converges by comparison with the harmonic series of order 2:

$$\left| \frac{(\pm 1)^n}{n^3} \right| \leq \frac{1}{n^3} \leq \frac{1}{n^2}$$

Hence, the series converges for  $-1 \leq x \leq 1$ . The convergence is uniform for this range, since the comparison

$$\left| \frac{x^n}{n^3} \right| \leq \frac{1}{n^2} = M_n$$

holds for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} 1/n^2$  converges.

- (b) Let the series  $\sum_{n=1}^{\infty} \tanh^n x/n!$ , where  $x$  is any real number be given. This series converges uniformly for all  $x$ , since

$$\left| \frac{\tanh^n x}{n!} \right| \leq \frac{1}{n!} = M_n$$

holds for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} 1/n!$  converges.

- (c) Let the series  $\sum_{n=1}^{\infty} \sin nx/(n^2 + 1)$ , where  $x$  is any real number be given. This series converges uniformly for all  $x$ , since

$$\left| \frac{\sin nx}{n^2 + 1} \right| \leq \frac{1}{n^2 + 1} < \frac{1}{n^2} = M_n$$

holds for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} 1/n^2$  converges.

- (d) Let the series  $\sum_{n=1}^{\infty} e^{nx}/2^n$ , where  $x \leq \ln(3/2)$  be given. The ratio test gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{e^{(n+1)x}}{2^{n+1}} \frac{2^n}{e^{nx}} = \lim_{n \rightarrow \infty} \frac{e^x}{2} = \frac{e^x}{2}$$

Hence, the series converges for  $e^x/2 < 1 \iff x < \ln 2$ . Because  $\ln 3/2 < \ln 2$  the series converges uniformly, since

$$\frac{e^{nx}}{2^n} \leq \frac{e^{n \ln 3/2}}{2^n} = \frac{3^n}{4^n} = M_n$$

holds for all  $x < \ln 3/2$  and the series  $\sum M_n = \sum_{n=1}^{\infty} (3/4)^n$  converges according to the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{3^{n+1}}{4^{n+1}} \frac{4^n}{3^n} = \frac{3}{4} = L < 1$$

- (e) Let the series  $\sum_{n=0}^{\infty} x^n/n! = \sum_{n=1}^{\infty} x^n/n! + 1$ , where  $-1 \leq x \leq 1$  be given. The ratio test gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1}$$

Hence, the series  $\sum_{n=1}^{\infty} x^n/n!$  converges for  $|x| < 1 \iff -1 < x < 1$ . For  $x = \pm 1$  the series converges by comparison with:

$$\left| \frac{(\pm 1)^n}{n!} \right| \leq \frac{1}{n!}$$

Hence, the series converges for  $-1 \leq x \leq 1$ . The convergence is uniform for this range, since the comparison

$$\left| \frac{x^n}{n!} \right| \leq \frac{1}{n!} = M_n$$

holds for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} 1/n!$  converges.

- (f) Let the series  $\sum_{n=1}^{\infty} nx^n$ , where  $-1/2 \leq x \leq 1/2$  be given. The ratio test gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = |x| \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = |x|$$

Hence, the series converges for  $|x| < 1 \iff -1 < x < 1$ . This series converges uniformly for  $-1/2 \leq x \leq 1/2$ , since

$$|nx^n| \leq \frac{n}{2^n} = M_n$$

for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} n/2^n$  converges according to the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{1}{2} \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = \frac{1}{2} = L < 1$$

- (g) Let the series  $\sum_{n=1}^{\infty} nx^n$ , where  $-0.9 \leq x \leq 0.9$  be given. From (f) it follows that the series converges for  $|x| < 1 \iff -1 < x < 1$ . The series converges uniformly for  $-0.9 \leq x \leq 0.9$ , since

$$|nx^n| \leq n \left( \frac{9}{10} \right)^n = M_n$$

for all  $x$  of the range and the series  $\sum M_n = \sum_{n=1}^{\infty} n(9/10)^n$  converges according to the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} \left( \frac{9}{10} \right)^{n+1} \left( \frac{10}{9} \right)^n = \frac{9}{10} \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = 0.9 = L < 1$$

- (h) Let the series  $\sum_{n=1}^{\infty} nx^n$ , where  $-a \leq x \leq a$ ,  $a < 1$  be given. From (f) it follows that the series converges for  $|x| < 1 \iff -1 < x < 1$ . The series converges uniformly for  $-a \leq x \leq a$ , since

$$|nx^n| \leq na^n = M_n < n^n$$

and the series  $\sum M_n = \sum_{n=1}^{\infty} na^n$  converges according to the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)a^{n+1}}{na^n} = a \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = a = L < 1$$

3. Let  $\sum_{n=1}^{\infty} u_n(x)$  be uniformly convergent for the interval  $a \leq x \leq b$ . In other words, some convergent series of constants  $\sum_{n=1}^{\infty} M_n$  exists such that

$$|u_n(x)| \leq M_n \quad a \leq x \leq b$$

Note that each constant  $M_n$  is the *same* for all  $x \in [a, b]$ . Hence, it must be the same for any smaller interval contained in  $a \leq x \leq b$ , since this smaller interval is just some subset  $E_1$  that is part of the set  $E$  of values of  $x$  that represents the interval  $a \leq x \leq b$ . As such, the series must be uniformly convergent in each smaller interval contained in  $a \leq x \leq b$  as well.

4. Let  $\sum_{n=1}^{\infty} v_n(x)$  be uniformly convergent for a set  $E$  of values of  $x$ . Hence, some convergent series of constants  $\sum_{n=1}^{\infty} M_n$  exists such that

$$|v_n(x)| \leq M_n \quad \text{for all } x \text{ in } E$$

Furthermore, let  $|u_n(x)| \leq v_n(x)$  for  $x \in E$ . In other words, for each fixed  $x$ , each term of the series  $\sum_{n=1}^{\infty} |u_n(x)|$  is less than or equal to the  $n$ th term  $v_n(x)$  of the uniformly convergent series  $\sum_{n=1}^{\infty} v_n(x)$ . Hence, by the comparison test (Section 6.6, Theorem 12) the series  $\sum_{n=1}^{\infty} u_n(x)$  is absolutely convergent for  $x \in E$  and since

$$|u_n(x)| \leq |v_n(x)| \leq M_n \quad \text{for all } x \text{ in } E$$

it follows that  $\sum_{n=1}^{\infty} u_n(x)$  is uniformly convergent for  $x \in E$ .

5. Let  $0 < u_n(x) < 1/n$  (which implies that  $\lim_{n \rightarrow \infty} u_n(x) = 0$ ) and  $u_{n+1}(x) \leq u_n(x)$  for  $a \leq x \leq b$ . Hence, by the alternating series test (Section 6.6, Theorem 18) the series  $\sum_{n=1}^{\infty} (-1)^n u_n(x)$  converges. Furthermore,  $|u_n(x)| < 1/n = M_n$  for all  $x$  of the range considered and hence, the alternating series converges uniformly for  $a \leq x \leq b$ .
6. Let a convergent series  $\sum_{n=1}^{\infty} M_n$  of constants  $M_n > 0$  be given. Hence, for some  $\epsilon > 0$  and  $N$  can be found such that  $|M_{n+1} + M_{n+2} + \cdots + M_m| \leq \epsilon$  for  $m > n > N$  (Section 6.5, Theorem 9). Next, let a sequence  $f_n(x)$  be given such that  $|f_{n+1}(x) - f_n(x)| \leq M_n$  for all  $x \in E$ . Since  $M_{n+1} \leq \epsilon$  for  $n > N$  it is true (after relabelling) that  $|f_{n+1}(x) - f_n(x)| \leq \epsilon$ . In other words, there exists some  $n > N$  such that the difference between  $f_{n+1}(x)$  and  $f_n(x)$  is not greater than  $\epsilon > 0$  (which can be chosen arbitrarily small) for each  $x \in E$ . Hence, the sequence  $f_n(x)$  is uniformly convergent for all  $x \in E$ .



7. (a) Let the sequence  $(n+x)/x$ , where  $0 \leq x \leq 1$  be given. This sequence converges uniformly for the range of  $x$  given, since

$$|f_{n+1}(x) - f_n(x)| = \left| \frac{n+1+x}{n+1} - \frac{n+x}{n} \right| = \left| \frac{-x}{n(n+1)} \right| \leq \frac{1}{n^2} = M_n$$

and the series of constants  $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n^2$  converges.

- (b) Let the sequence  $x^n/n!$ , where  $-1 \leq x \leq 1$  be given. This sequence converges uniformly for the range of  $x$  given, since

$$|f_{n+1}(x) - f_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} - \frac{x^n}{n!} \right| = \frac{|x^n|}{n!} \left| \frac{x}{n+1} - 1 \right| \leq \frac{3}{2} \frac{1}{n!} = M_n$$

and the series of constants  $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n!$  converges. The constant  $3/2$  is justified by noting that

$$\max \left| \frac{x}{n+1} - 1 \right| = \max(a)$$

in the interval  $-1 \leq x \leq 1$  occurs when  $x = -1$ ,  $n = 1$ . Furthermore, as  $n \rightarrow \infty$  we see that  $a \rightarrow 1$ .

- (c) Let the sequence  $f_n(x) = \ln(1+nx)/n$ , where  $1 \leq x \leq 2$  be given. Firstly, we note that

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\ln(1+nx)}{n} = \lim_{n \rightarrow \infty} \frac{x}{1+nx} = 0$$

and since  $f_n(x) > 0$  for  $1 \leq x \leq 2$  this implies  $f_{n+1}(x) < f_n(x)$ . As such

$$\frac{\ln(1+nx)}{n+1} < \frac{\ln[1+(n+1)x]}{n+1} < \frac{\ln(1+nx)}{n}$$

or equivalently

$$\frac{\ln(1+nx)}{n+1} - \frac{\ln(1+nx)}{n} < \frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n} < 0$$

And so we learn that

$$\left| \frac{\ln(1+nx)}{n+1} - \frac{\ln(1+nx)}{n} \right| > \left| \frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n} \right|$$

Hence, for  $1 \leq x \leq 2$  we find

$$\begin{aligned} |f_{n+1}(x) - f_n(x)| &= \left| \frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n} \right| < \ln(1+nx) \left| \frac{1}{n+1} - \frac{1}{n} \right| \\ &= \frac{\ln(1+nx)}{n(n+1)} \\ &< \frac{\ln(1+nx)}{n^2} \\ &\leq \frac{\ln(1+2n)}{n^2} = M_n \end{aligned}$$

It remains to be shown that the series of constants  $\sum_{n=1}^{\infty} M_n$  converges. To this end we employ the *integral test*:

$$\begin{aligned}
\int_1^{\infty} \frac{\ln(1+2x)}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln(1+2x)}{x^2} dx \\
&= \lim_{b \rightarrow \infty} \left. -\frac{\ln(1+2x)}{x} \right|_1^b + \lim_{b \rightarrow \infty} \int_1^b \frac{2}{x(1+2x)} dx \\
&= \ln 3 - \lim_{b \rightarrow \infty} \frac{\ln(1+2b)}{b} + 2 \lim_{b \rightarrow \infty} \int_1^b \left( \frac{1}{x} - \frac{2}{1+2x} \right) dx \\
&= \ln 3 + 2 \lim_{b \rightarrow \infty} [\ln|x| - \ln|1+2x|]_1^b \\
&= 3 \ln 3 + 2 \lim_{b \rightarrow \infty} [\ln b - \ln(1+2b)] \\
&= 3 \ln 3 + 2 \lim_{b \rightarrow \infty} \ln \frac{b}{1+2b} = 3 \ln 3 + 2 \ln \left( \lim_{b \rightarrow \infty} \frac{b}{1+2b} \right) \\
&= 3 \ln 3 - 2 \ln 2
\end{aligned}$$

Since this integral converges and all the conditions of the integral test are satisfied, we may conclude that  $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \ln(1+2n)/n^2$  converges. Hence, the original sequence  $f_n(x) = \ln(1+nx)/n$  converges uniformly for  $1 \leq x \leq 2$ .

- (d) Let the sequence  $f_n(x) = n/e^{nx^2}$ , where  $1/2 \leq x \leq 1$  be given. Firstly, we note that

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{n}{e^{nx^2}} = \lim_{n \rightarrow \infty} \frac{1}{x^2 e^{nx^2}} = 0$$

and since  $f_n(x) > 0$  for  $1/2 \leq x \leq 1$  this implies  $f_{n+1}(x) < f_n(x)$ . However, if we plot  $f_n(x)$  for various values of  $n$  (keeping  $x$  fixed) we see that  $f_{n+1} < f_n(x)$  only is true for  $n \geq 4$ . As stated at the start of Section 6.6, the convergence or divergence of a series is unaffected if a finite number of terms of the series are discarded. Hence, in testing for convergence of  $\sum M_n$  we can simply ignore the first four terms and aim to prove  $\sum_{n=4}^{\infty} M_n$  does converge for a certain  $M_n$  yet to be determined. Continuing with our sequence, we conclude (for  $n \geq 4$ )

$$\frac{n}{e^{(n+1)x^2}} < \frac{n+1}{e^{(n+1)x^2}} < \frac{n}{e^{nx^2}}$$

or equivalently

$$\frac{n}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} < \frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} < 0$$

And so we learn that

$$\left| \frac{n}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} \right| > \left| \frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} \right|$$

Hence, for  $1/2 \leq x \leq 1$ ,  $n \geq 4$  we find

$$\begin{aligned}
|f_{n+1}(x) - f_n(x)| &= \left| \frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} \right| < \frac{n}{e^{nx^2}} \left| \frac{1}{e^{x^2}} - 1 \right| \\
&= \frac{n}{e^{nx^2}} \left( 1 - \frac{1}{e^{x^2}} \right) \\
&\leq \max_{1/2 \leq x \leq 1} \left[ \frac{n}{e^{nx^2}} \left( 1 - \frac{1}{e^{x^2}} \right) \right] \\
&= \frac{n}{e^{n/4}} \left( 1 - \frac{1}{e^{1/4}} \right) = M_n
\end{aligned}$$

It remains to be shown that the series of constants  $\sum_{n=4}^{\infty} M_n$  converges. To this end we employ the *integral test*:

$$\begin{aligned}
\int_4^{\infty} \frac{x}{e^{x/4}} dx &= \lim_{b \rightarrow \infty} \int_4^b \frac{x}{e^{x/4}} dx = \lim_{b \rightarrow \infty} \int_4^b x e^{-x/4} dx \\
&= \lim_{b \rightarrow \infty} -4x e^{-x/4} \Big|_4^b + \lim_{b \rightarrow \infty} \int_4^b 4e^{-x/4} dx \\
&= \lim_{b \rightarrow \infty} [-4x e^{-x/4} - 16e^{-x/4}]_4^b \\
&= \lim_{b \rightarrow \infty} (-4b e^{-b/4} - 16e^{-b/4}) + 32e^{-1} = \frac{32}{e}
\end{aligned}$$

Since this integral converges and all the conditions of the integral test are satisfied, we may conclude that  $\sum_{n=4}^{\infty} M_n = \sum_{n=4}^{\infty} n e^{-n/4} (1 - e^{-1/4})$  converges. Hence, the original sequence  $f_n(x) = n/e^{nx^2}$  converges uniformly for  $1/2 \leq x \leq 1$ .

## Section 6.16

1. (a) Let the relation

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

where  $-1 < x < 1$  be given. Integrating both sides gives

$$\begin{aligned}
\int \frac{dx}{1-x} &= \sum_{n=0}^{\infty} \int x^n dx \\
\ln \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n} \\
&= x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots
\end{aligned}$$

To verify (6.43) we note that

$$f(a) = f(0) = \ln \frac{1}{1-0} = 0$$

This checks out, since  $c_0 = 0$  for  $\sum_{n=1}^{\infty} x^n/n$ . Also

$$f'(a) = f'(0) = \left( \frac{d}{dx} \ln \frac{1}{1-x} \right)_{x=0} = \frac{1}{1-x} \Big|_{x=0} = 1$$

Again, this checks out, since  $c_1 = 1$  for  $\sum_{n=1}^{\infty} x^n/n$ . And in general

$$f^{(n)}(a) = f^{(n)}(0) = \left( \frac{d^{(n)}}{dx^{(n)}} \ln \frac{1}{1-x} \right)_{x=0} = \frac{(n-1)!}{(1-x)^n} \Big|_{x=0} = (n-1)!$$

which checks out, since  $c_n = f^{(n)}(a)/n! = (n-1)!/n! = 1/n$  for  $\sum_{n=1}^{\infty} x^n/n$ .

(b) For  $x = -1$  the series  $\sum_{n=1}^{\infty} x^n/n$  reduces to the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n a_n$$

Since  $a_{n+1} < a_n$  for  $n = 1, 2, \dots$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 1/n = 0$  it follows from Theorem 18 that the alternating series converges. Hence,

$$\begin{aligned} \ln \frac{1}{1-(-1)} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\ &= -\ln 2 \\ \ln 2 &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \end{aligned}$$

2. From the relation

$$\frac{1}{1-x} = 1 + x + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1$$

we obtain by successive differentiation the relations

$$\begin{aligned} \frac{1}{(1-x)^2} &= 1 + 2x + \dots + nx^{n-1} + \dots &= \sum_{n=0}^{\infty} (n+1)x^n & -1 < x < 1 \\ \frac{2}{(1-x)^3} &= 2 + 6x + \dots + n(n-1)x^{n-2} + \dots &= \sum_{n=0}^{\infty} (n+2)(n+1)x^n & -1 < x < 1 \end{aligned}$$

Hence, in general

$$\begin{aligned}\frac{1}{(1-x)^k} &= (1-x)^{-k} \\ &= 1 + \frac{kx}{1} + \frac{k(k+1)}{1 \cdot 2}x^2 + \cdots + \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n}x^n + \cdots\end{aligned}$$

where  $-1 < x < 1$  for some known  $k = 1, 2, 3, \dots$ . Differentiating both sides of this equation gives

$$\begin{aligned}\frac{d}{dx}(1-x)^{-k} &= \frac{d}{dx} \left[ 1 + \frac{kx}{1} + \frac{k(k+1)}{1 \cdot 2}x^2 + \cdots + \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n}x^n + \cdots \right] \\ k(1-x)^{-k-1} &= k + \frac{2k(k+1)}{1 \cdot 2}x + \frac{3k(k+1)(k+2)}{1 \cdot 2 \cdot 3}x^2 + \cdots + \frac{nk(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n}x^{n-1} \\ &\quad + \cdots \\ (1-x)^{-k-1} &= 1 + \frac{k(k+1)}{1}x + \frac{k(k+1)(k+2)}{1 \cdot 2}x^2 + \cdots + \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n-1}x^{n-1} + \cdots \\ &= 1 + \frac{k(k+1)}{1}x + \frac{k(k+1)(k+2)}{1 \cdot 2}x^2 + \cdots + \frac{k(k+1) \cdots (k+n)}{1 \cdot 2 \cdots n}x^n + \cdots\end{aligned}$$

We see that this is none other than (6.41), i.e. the generalised relation we started with, but for  $k+1$  instead of  $k$ . In other words, if we know that (6.41) is true for some known  $k$  we have established that it will be true for  $k+1$  also and hence, by induction for any  $k = 1, 2, 3, \dots$ .

3. From (6.41) we know that

$$\frac{1}{(1-r)^k} = 1 + \frac{kr}{1} + \frac{k(k+1)}{2}r^2 + \cdots + \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n}r^n + \cdots$$

for  $-1 < r < 1$ ,  $k = 1, 2, 3, \dots$ .

(a) Let the function  $f(x) = 1/x$  be given. To expand this function in a Taylor series about  $x = 1$  we note that

$$\frac{1}{x} = \frac{1}{1 - (1-x)} = \frac{1}{1-r}$$

using the substitution  $r = 1 - x$ . Hence, by (6.41) setting  $k = 1$  we find

$$\begin{aligned}\frac{1}{x} &= \frac{1}{1-r} = 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^n + \cdots \\ &= 1 - (x-1) + (x-1)^2 + \cdots + (-1)^n(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n\end{aligned}$$

for  $-1 < r < 1 \implies 0 < x < 2$ .

- (b) Let the function  $f(x) = 1/(x+2)$  be given. To expand this function in a Maclaurin series (i.e. expand it around  $x = 0$ ) we note that

$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1 - (-x/2)} = \frac{1}{2} \cdot \frac{1}{1-r}$$

using the substitution  $r = -x/2$ . Hence, by (6.41) setting  $k = 1$  we find

$$\begin{aligned} \frac{1}{x+2} &= \frac{1}{2} \cdot \frac{1}{1-r} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} + \cdots + (-1)^n \frac{x^n}{2^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}} \end{aligned}$$

for  $-1 < r < 1 \implies -2 < x < 2$ .

- (c) Let the function  $f(x) = 1/(3x+5)$  be given. To expand this function in a Maclaurin series we employ (6.44) to get

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ \frac{1}{3x+5} &= \frac{1}{5} - \frac{3}{5^2}x + \frac{9}{5^3}x^2 + \cdots + (-1)^n \frac{3^n}{5^{n+1}}x^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^{n+1}}x^n \end{aligned}$$

Since we require  $3x+5 > 0$  in order for the function  $f(x)$  to be differentiable the convergence interval is given by  $-5/3 < x < 5/3$ .

- (d) Let the function  $f(x) = 1/(3x+5)$  be given. To expand this function in a Taylor series about  $x = 1$  we note that

$$\frac{1}{3x+5} = \frac{1}{3(x-1)+8} = \frac{1}{8} \cdot \frac{1}{1+3(x-1)/8} = \frac{1}{8} \cdot \frac{1}{1-r}$$

using the substitution  $r = 3(1-x)/8$ . Hence, by (6.41) setting  $k = 1$  we find

$$\begin{aligned} \frac{1}{3x+5} &= \frac{1}{8} \cdot \frac{1}{1-r} = \frac{1}{8} + \frac{3}{8^2}(1-x) + \frac{9}{8^3}(1-x)^2 + \cdots + \frac{3^n}{8^{n+1}}(1-x)^n + \cdots \\ &= \frac{1}{8} - \frac{3}{8^2}(x-1) + \frac{9}{8^3}(x-1)^2 + \cdots + (-1)^n \frac{3^n}{8^{n+1}}(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{8^{n+1}}(x-1)^n \end{aligned}$$

for  $-1 < r < 1 \implies -5/3 < x < 11/3$ .

- (e) Let the function  $f(x) = 1/(ax + b)$  be given. To expand this function in a Taylor series about  $x = c$  we employ (6.43) to get

$$\begin{aligned} f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\ \frac{1}{ax+b} &= \frac{1}{ac+b} - \frac{a}{(ac+b)^2}(x-c) + \cdots + (-1)^n \frac{a^n}{(ac+b)^{n+1}}(x-c)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{(ac+b)^{n+1}}(x-c)^n \end{aligned}$$

for

$$c - \left| \frac{ac+b}{a} \right| < x < c + \left| \frac{ac+b}{a} \right|$$

which follows from (6.38) and (6.42).

- (f) Let the function  $f(x) = 1/(1-x^2)$  be given. To expand this function in a Maclaurin series we employ (6.44) to get

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ \frac{1}{1-x^2} &= 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots \\ &= \sum_{n=0}^{\infty} x^{2n} \end{aligned}$$

for  $-1 < x < 1$ .

- (g) Let the function  $f(x) = 1/[(x-2)(x-3)]$  be given. To expand this function in a Maclaurin series we note that

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3} = \frac{1}{2} \cdot \frac{1}{1-x/2} - \frac{1}{3} \cdot \frac{1}{1-x/3}$$

Hence, by (6.41) setting  $k = 1$  we find

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{1-x/2} &= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots + \frac{x^n}{2^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \end{aligned}$$

for  $-2 < x < 2$  and

$$\begin{aligned} \frac{1}{3} \cdot \frac{1}{1-x/3} &= \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \cdots + \frac{x^n}{3^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \end{aligned}$$

for  $-3 < x < 3$ . And so

$$\frac{1}{(x-2)(x-3)} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) x^n$$

for  $-2 < x < 2$ .

- (h) Let the function  $f(x) = 1/x^2$  be given. To expand this function in a Taylor series about  $x = 1$  we note that

$$\frac{1}{x^2} = \frac{1}{[1 - (1-x)]^2} = \frac{1}{(1-r)^2}$$

using the substitution  $r = 1 - x$ . Hence, by (6.41) setting  $k = 2$  we find

$$\begin{aligned} \frac{1}{x^2} &= \frac{1}{(1-r)^2} = 1 + 2(1-x) + 3(1-x)^2 + \cdots + (n+1)(1-x)^n + \cdots \\ &= 1 - 2(x-1) + 3(x-1)^2 + \cdots + (-1)^n (n+1)(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n \end{aligned}$$

for  $-1 < r < 1 \implies 0 < x < 2$ .

- (i) Let the function  $f(x) = 1/(3x+5)^2$  be given. To expand this function in a Taylor series about  $x = 1$  we note that

$$\frac{1}{(3x+5)^2} = \frac{1}{[3(x-1)+8]^2} = \frac{1}{8^2} \cdot \frac{1}{[1+3(x-1)/8]^2} = \frac{1}{8^2} \cdot \frac{1}{(1-r)^2}$$

using the substitution  $r = 3(1-x)/8$ . Hence, by (6.41) setting  $k = 2$  we find

$$\begin{aligned} \frac{1}{(3x+5)^2} &= \frac{1}{8^2} \cdot \frac{1}{(1-r)^2} \\ &= \frac{1}{8^2} + \frac{2 \cdot 3}{8^3} (1-x) + \frac{2 \cdot 3 \cdot 3^2}{8^4 \cdot 1 \cdot 2} (1-x)^2 + \cdots + \frac{3^n (n+1)}{8^{n+2}} (1-x)^n + \cdots \\ &= \frac{1}{8^2} - \frac{2 \cdot 3}{8^3} (x-1) + \frac{2 \cdot 3 \cdot 3^2}{8^4 \cdot 1 \cdot 2} (x-1)^2 + \cdots + (-1)^n \frac{3^n (n+1)}{8^{n+2}} (x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n (n+1)}{8^{n+2}} (x-1)^n \end{aligned}$$

for  $-1 < r < 1 \implies -5/3 < x < 11/3$ .

- (j) Let the function  $f(x) = 1/(ax+b)^k$  be given. To expand this function in a Taylor



series about  $x = c$  we employ (6.43) to get

$$\begin{aligned}
f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\
\frac{1}{(ax+b)^k} &= \frac{1}{(ac+b)^k} - \frac{k}{1} \frac{a}{(ax+b)^{k+1}}(x-c) + \frac{k(k+1)}{1 \cdot 2} \frac{a^2}{(ax+b)^{k+2}}(x-c)^2 + \cdots \\
&\quad + (-1)^n \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \frac{a^n}{(ax+b)^{k+n}}(x-c)^n + \cdots \\
&= \frac{1}{(ac+b)^k} + \sum_{n=1}^{\infty} (-1)^n \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \frac{a^n}{(ax+b)^{k+n}}(x-c)^n
\end{aligned}$$

for

$$c - \left| \frac{ac+b}{a} \right| < x < c + \left| \frac{ac+b}{a} \right|$$

which follows from (6.38) and (6.42).

4. Let  $f(x) = \sum_{n=1}^{\infty} x^n/n^n$ .

- (a) From the formal definition of the limit:  $\lim_{x \rightarrow x_1} f(x) = c$  means that given any  $\epsilon > 0$ , a  $\delta > 0$  can be found such that for every  $x$  in a domain  $D$  where  $|x - x_1| < \delta$  an  $\epsilon$  can be found such that  $|f(x) - c| < \epsilon$ , follows that  $\lim_{x \rightarrow x_1} k = k$  for some constant function  $f(x) = k$  and  $\lim_{x \rightarrow x_1} x = x_1$  for some linear function  $f(x) = x$ . Hence, both are continuous for any  $x \in D : (-\infty, \infty)$ . Now if both  $f(x)$  and  $g(x)$  are continuous in  $D$  then so will be  $f(x)g(x)$ . And thus we may conclude at once that  $kx^n = kxx^{n-1} = kxx^{n-2} = \cdots = kx^{n-1}x$  is continuous in  $D$ . Furthermore, if  $f(x)$  and  $g(x)$  are continuous in  $d$  so will be  $f(x) + g(x)$ . Hence  $a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=1}^{\infty} a_nx^n$  is continuous in  $D$ . Lastly, choosing  $a_n = 1/n^n$  for  $n = 1, 2, \dots$  then proves that  $f(x) = \sum_{n=1}^{\infty} x^n/n^n$  is continuous (and hence, defined) for all  $x \in D : (-\infty, \infty)$ .

(b)

$$f(0) = \sum_{n=1}^{\infty} \frac{0^n}{n^n} = \frac{0^1}{1^1} + \frac{0^2}{2^2} + \cdots + \frac{0^n}{n^n} + \cdots = 0$$

$$\begin{aligned}
f(1) &= \sum_{n=1}^{\infty} \frac{1^n}{n^n} = \frac{1^1}{1^1} + \frac{1^2}{2^2} + \cdots + \frac{1^n}{n^n} + \cdots \\
&= 1 + \frac{1}{4} + \cdots + \frac{1}{n^n} + \cdots \cong 1.29
\end{aligned}$$

$$f'(0) = \sum_{n=1}^{\infty} \frac{0^{n-1}}{n^{n-1}} = \frac{0^0}{1^0} + \frac{0^1}{2^1} + \cdots + \frac{0^{n-1}}{n^{n-1}} + \cdots = 1$$

$$\begin{aligned}
f'(1) &= \sum_{n=1}^{\infty} \frac{1^{n-1}}{n^{n-1}} = \frac{1^0}{1^0} + \frac{1^1}{2^1} + \cdots + \frac{1^{n-1}}{n^{n-1}} + \cdots \\
&= 1 + \frac{1}{2} + \cdots + \frac{1^{n-1}}{n^{n-1}} + \cdots \approx 1.63
\end{aligned}$$

$$f''(0) = \sum_{n=1}^{\infty} \frac{(n-1)0^{n-2}}{n^{n-1}} = \frac{0 \cdot 0^{-1}}{1^0} + \frac{1 \cdot 0^0}{2^1} + \cdots + \frac{(n-1)0^{n-2}}{n^{n-1}} + \cdots = \frac{1}{2}$$

(c) Using (6.44) we find

$$\begin{aligned}
f'(x) &= f'(0) + \frac{f''(0)}{1!}x + \frac{f'''(0)}{2!}x^2 + \cdots + \frac{f^{(n+1)}(0)}{n!}x^n + \cdots \\
&= 1 + \frac{x}{2^1} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{(n+1)^n} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^n}
\end{aligned}$$

and

$$\begin{aligned}
f''(x) &= f''(0) + \frac{f'''(0)}{1!}x + \frac{f^{(4)}(0)}{2!}x^2 + \cdots + \frac{f^{(n+2)}(0)}{n!}x^n + \cdots \\
&= \frac{1}{2^1} + \frac{2x}{3^2} + \frac{3x^2}{4^3} + \cdots + \frac{(n+1)x^n}{(n+2)^{n+1}} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)^{n+1}}
\end{aligned}$$

5. Let the function

$$y = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

be given. From Problem 4(a) it follows that this function is defined for all  $x \in D : (-\infty, \infty)$ . Also, we may conclude at once that

$$f(0) = 1 + 0 + \frac{0^2}{2!} + \cdots + \frac{0^n}{n!} + \cdots = 1$$

Furthermore, it is easy to show that

$$\begin{aligned}
\frac{dy}{dx} = f'(x) &= \frac{d}{dx} \left( 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \right) \\
&= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = y = f(x)
\end{aligned}$$

and that generally

$$\frac{d^{(n)}y}{dx^{(n)}} = f^{(n)}(x) = \frac{d}{dx} \left( \frac{d^{(n-1)}y}{dx^{(n-1)}} \right) = y = f(x) \implies f(0) = f'(0) = \cdots = f^{(n)}(0) = 1$$

Hence,

$$\begin{aligned} y = f(x) &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \end{aligned}$$

is a Maclaurin series valid for all  $x$ .

## Section 6.18

1. (a) Since  $\sinh x = (1/2)(e^x - e^{-x})$  then by (6.46) we find

$$\begin{aligned} \sinh x &= \frac{1}{2} (e^x - e^{-x}) \\ &= \frac{1}{2} \left[ \left( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \right) - \left( 1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots \right) \right] \\ &= \frac{x}{1!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \end{aligned}$$

for  $-\infty < x < \infty$ .

- (b) Since  $\cos^2 x = (1 + \cos 2x)/2$  then by (6.48) we find

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ &= \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} + \cdots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} + \cdots \right) \\ &= 1 - \frac{2^1 x^2}{2!} + \frac{2^3 x^4}{4!} + \cdots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} \end{aligned}$$

for  $-\infty < x < \infty$ .

(c) Since  $\sin^2 x = (1 - \cos 2x)/2$  then by (6.47) we find

$$\begin{aligned}
 \sin^2 x &= \frac{1 - \cos 2x}{2} \\
 &= \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \cdots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} + \cdots \right) \\
 &= \frac{2^1 x^2}{2!} - \frac{2^3 x^4}{4!} + \cdots + \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}
 \end{aligned}$$

for  $-\infty < x < \infty$ .

(d) Since  $\ln x = \int_1^x du/u$  and using the substitution  $x' = u - 1$  so that

$$\frac{1}{u} = \frac{1}{1 + x'}$$

then by (6.49) (recognising that  $m = -1$ ) we find

$$\begin{aligned}
 \ln x &= \int_1^x \frac{du}{u} = \int_0^{x-1} \frac{dx'}{1 + x'} \\
 &= \int_0^{x-1} \left[ 1 - x' + (x')^2 - \cdots + (-1)^n (x')^n + \cdots \right] dx' \\
 &= \left[ x' - \frac{(x')^2}{2} + \frac{(x')^3}{3} - \cdots + \frac{(-1)^{n+1} (x')^n}{n} + \cdots \right]_0^{x-1} \\
 &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n+1} (x-1)^n}{n} + \cdots \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}
 \end{aligned}$$

(e) Since  $\sqrt{1-x} = (1+x')^m$  for  $|x| < 1$ , using the substitutions  $x = -x'$  and  $m = 1/2$  then by (6.49) we find

$$\begin{aligned}
 \sqrt{1-x} &= (1+x')^m \\
 &= 1 + \frac{m}{1!} (x') + \frac{m(m-1)}{2!} (x')^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!} (x')^n + \cdots \\
 &= 1 - \frac{1}{2}x - \frac{1}{2^2 2!}x^2 - \frac{1 \cdot 3}{2^3 3!}x^3 - \cdots
 \end{aligned}$$

(f) Since  $(1-x^2)^{-1/2} = (1+x')^m$  for  $|x| < 1$ , using the substitutions  $x' = -x^2$  and

$m = -1/2$  then by (6.49) we find

$$\begin{aligned}
(1 - x^2)^{-1/2} &= (1 + x')^m \\
&= 1 + \frac{m}{1!} (x') + \frac{m(m-1)}{2!} (x')^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!} (x')^n + \cdots \\
&= 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2^2 2!} x^4 + \frac{1 \cdot 3 \cdot 5}{2^3 2!} x^6 + \cdots
\end{aligned}$$

(g) Since  $d(\sin^{-1} x)/dx = (1 - x^2)^{-1/2}$  for  $|x| < 1$  we can find the Taylor series expansion for  $\sin^{-1} x$  by integrating the series of Problem 1(f):

$$\begin{aligned}
\sin^{-1} x &= \int_0^x \frac{du}{\sqrt{1-u^2}} = \int_0^x \left( 1 + \frac{1}{2} u^2 + \frac{1 \cdot 3}{2^2 2!} u^4 + \frac{1 \cdot 3 \cdot 5}{2^3 2!} u^6 + \cdots \right) du \\
&= \left[ u + \frac{1}{2} \frac{u^3}{3} + \frac{1 \cdot 3}{2^2 2!} \frac{u^5}{5} + \frac{1 \cdot 3 \cdot 5}{2^3 2!} \frac{u^7}{7} + \cdots \right]_0^x \\
&= x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2^2 2!} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2^3 2!} \frac{x^7}{7} + \cdots
\end{aligned}$$

2. (a) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
f(x) &= e^x \sin x \\
&= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots \\
&= e^0 \sin(0) + [e^0 \sin(0) + e^0 \cos(0)] x + e^0 \cos(0) x^2 + \frac{e^0 \cos(0) - e^0 \sin(0)}{3} x^3 + \cdots \\
&= x + x^2 + \frac{x^3}{3}
\end{aligned}$$

- (b) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
f(x) &= \tan x \\
&= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots \\
&= \tan(0) + \sec^2(0) x + \sec^2(0) \tan(0) x^2 + \left[ \frac{-2}{3} \sec^2(0) + \sec^4(0) \right] x^3 \\
&\quad + \frac{\sec^2(0) \tan(0) [3 \sec^2(0) - 1]}{3} x^4 \\
&\quad + \frac{12 \sec^4(0) \tan^2(0) - 2 \sec^2(0) \tan^2(0) + 3 \sec^6(0) - \sec^4(0)}{15} x^5 + \cdots \\
&= x + \frac{1}{3} x^3 + \frac{2}{15} x^5
\end{aligned}$$

- (c) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= \ln^2(1+x) \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= \ln^2(1) + 2\ln(1)x + [1 - \ln(1)]x^2 + \left[\frac{2}{3}\ln(1) - 1\right]x^3 + \left[\frac{11}{12} - \frac{\ln(1)}{2}\right]x^4 + \cdots \\
 &= x^2 - x^3 + \frac{11}{12}x^4
 \end{aligned}$$

- (d) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= \ln(1-x^2) \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= \ln(1) - x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 + \cdots \\
 &= -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6
 \end{aligned}$$

- (e) Using (6.43) we find that the first three non-zero terms of the Taylor series about  $x = 2$  of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= x^3 + 3x + 1 \\
 &= f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\
 &= 15 + 15(x-2) + 6(x-2)^2
 \end{aligned}$$

- (f) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= e^{\tan x} \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= 1 + e^{\tan 0} \sec^2(0)x + \left[ \frac{e^{\tan 0}}{2} \sec^4(0) + e^{\tan 0} \sec^2(0) \tan(0) \right] x^2 + \cdots \\
 &= 1 + x + \frac{x^2}{2}
 \end{aligned}$$

- (g) Using (6.44) we find that the first three non-zero terms of the Taylor series about

$x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= \sinh^{-1} x \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= \sinh^{-1}(0) + x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots \\
 &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5
 \end{aligned}$$

(h) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= \tanh x \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= \tanh(0) + \operatorname{sech}^2(0)x - \operatorname{sech}^2(0)\tanh(0)x^2 + \frac{2\operatorname{sech}^2(0)\tanh^2(0) - \operatorname{sech}^4(0)}{3}x^3 \\
 &\quad + \frac{\operatorname{sech}^2(0)\tanh(0)[2\operatorname{sech}^2(0) - \tanh^2(0)]}{3}x^4 \\
 &\quad + \frac{2\tanh^4(0)\operatorname{sech}^2(0) - 11\operatorname{sech}^4(0)\tanh^2(0) + 2\operatorname{sech}^6(0)}{15}x^5 + \cdots \\
 &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5
 \end{aligned}$$

(i) Using (6.44) we find that the first three non-zero terms of the Taylor series about  $x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
 f(x) &= \tanh^{-1} x \\
 &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
 &= \tanh^{-1}(0) + x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \\
 &= x + \frac{x^3}{3} + \frac{x^5}{5}
 \end{aligned}$$

(j) Using (6.44) we find that the first three non-zero terms of the Taylor series about

$x = 0$  (Maclaurin series) of  $f(x)$  are given by

$$\begin{aligned}
f(x) &= \ln \sec x \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\
&= \ln \sec(0) + \tan(0)x + \frac{\sec^2(0)}{2}x^2 + \frac{\sec^2(0)\tan(0)}{3}x^3 + \frac{3\sec^4(0) - 2\sec^2(0)}{12}x^4 \\
&\quad \frac{\sec^2(0)\tan(0)[3\sec^2(0) - 1]}{15}x^5 + \\
&\quad \frac{12\sec^4(0)\tan^2(0) - 2\sec^2(0)\tan^2(0) + 3\sec^6(0) - \sec^4(0)}{90}x^6 + \dots \\
&= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}
\end{aligned}$$

3. Let  $n \geq 1$  be a positive integer and  $x_2$  be a fixed number in the interval  $a - r_0 < x < a + r_0$ ,  $x_2 \neq a$ . Furthermore, let

$$\bar{G}(x) = G(x) - \left( \frac{x_2 - x}{x_2 - a} \right)^n G(a)$$

where

$$G(x) = f(x_2) - f(x) - (x_2 - x)f'(x) - \dots - \frac{(x_2 - x)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$

We assume that  $f(x)$  is defined and continuous and has continuous derivatives up to the  $(n+1)^{st}$  order in the given interval, which implies  $\bar{G}(x)$  is defined and continuous for  $x$  in the same interval. Also, note that

$$\bar{G}(a) = G(a) - \left( \frac{x_2 - a}{x_2 - a} \right)^n G(a) = 0$$

and

$$\begin{aligned}
\bar{G}(x_2) &= G(x_2) - \left( \frac{x_2 - x_2}{x_2 - a} \right)^n G(a) \\
&= G(x_2) \\
&= f(x_2) - f(x_2) - (x_2 - x_2)f'(x_2) - \dots - \frac{(x_2 - x_2)^{n-1}}{(n-1)!}f^{(n-1)}(x_2) \\
&= 0
\end{aligned}$$



Hence, by the Mean Value theorem,  $\bar{G}'(x) = 0$  for some  $x_1$  between  $a$  and  $x_2$ . Now

$$\begin{aligned}\bar{G}'(x) &= G'(x) + \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} G(a) \\ &= -\frac{(x_2 - x)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} G(a) \\ &= \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} \left[ G(a) - \frac{1}{n!} f^{(n)}(x) (x_2 - a)^n \right]\end{aligned}$$

and so the equation  $\bar{G}'(x_1) = 0$  thus becomes the equation

$$\begin{aligned}\bar{G}'(x_1) &= 0 \\ \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} \left[ G(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) \right] &= \\ G(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= \\ f(x_2) - f(a) - (x_2 - a) f'(a) - \dots - \frac{(x_2 - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= \\ f(a) + (x_2 - a) f'(a) + \dots + \frac{(x_2 - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= f(x_2)\end{aligned}$$

If  $x_2$  is now replaced by a variable  $x$ , we get the desired result:

$$f(x) = f(a) + (x - a) f'(a) + \dots + \frac{(x - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x - a)^n}{n!} f^{(n)}(x_1)$$

4. Starting from the premise that

$$R_n(x) = f(x) - f(a) - \left[ \frac{f'(a)}{1!} (x - a) + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n \right]$$

we see that for  $n = 0$  the term  $R_0$ , using the *fundamental theorem of calculus*, can be written as

$$R_0 = f(x) - f(a) = \int_a^x f'(t) dt$$

For  $n = 1$  we find (using integration by parts)

$$\begin{aligned}R_1 &= f(x) - f(a) - \frac{f'(a)}{1!} (x - a) = R_0 - (x - a) f'(a) \\ &= -(x - a) f'(a) + \int_a^x f'(t) dt \\ &= \underbrace{(x - t) f'(t)}_{uv} \Big|_a^x - \int_a^x \underbrace{-f'(t)}_{v du} dt \\ &= \int_a^x (x - t) f''(t) dt\end{aligned}$$

Likewise, for  $n = 2$  we find

$$\begin{aligned}
R_2 &= f(x) - f(a) - \frac{f'(a)}{1!}(x-a) - \frac{f''(a)}{2!}(x-a)^2 \\
&= R_1 - (x-a)^2 \frac{f''(a)}{2} \\
&= -(x-a)^2 \frac{f''(a)}{2} + \int_a^x (x-t) f''(t) dt \\
&= \underbrace{\frac{(x-t)^2}{2} f''(t)}_{uv} \Big|_a^x - \int_a^x \underbrace{-f''(t)(x-t)}_{v du} dt = \int_a^x \frac{(x-t)^2}{2} f^{(3)}(t) dt
\end{aligned}$$

and  $n = 3$ :

$$\begin{aligned}
R_3 &= f(x) - f(a) - \frac{f'(a)}{1!}(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \frac{f'''(a)}{3!}(x-a)^3 \\
&= R_2 - \frac{f'''(a)}{3!}(x-a)^3 \\
&= -(x-a)^3 \frac{f^{(3)}(a)}{3!} + \int_a^x \frac{(x-t)^2}{2!} f^{(3)}(t) dt \\
&= \underbrace{\frac{(x-t)^3}{3!} f^{(3)}(t)}_{uv} \Big|_a^x - \int_a^x \underbrace{-f^{(3)}(t) \frac{(x-t)^2}{2!}}_{v du} dt = \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t) dt
\end{aligned}$$

As such, we consider the formula

$$R_m = \int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t) dt$$

to be true for some known, positive integer  $m \geq 0$ . Next, using integration by parts with  $u = f^{(m+1)}(t)$  and  $dv = (x-t)^m dt/m!$  we can write

$$\begin{aligned}
R_m &= \int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t) dt \\
&= \int_a^x u dv \\
&= uv \Big|_a^x - \int_a^x v du \\
&= -\frac{(x-t)^{m+1}}{(m+1)!} f^{(m+1)}(t) \Big|_a^x + \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t) dt \\
&= \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) + \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t) dt \\
R_m - \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) &= \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t) dt
\end{aligned}$$

From (6.45) it follows that the last expression can be written as

$$R_m - \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) = R_{m+1} = \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t) dt$$

Thus by induction, since the integral formula for  $R_n$  is true for an some fixed positive integer  $m$  and  $m+1$  it must be true for any arbitrary, positive integer  $n \geq 0$ .

5. (a) To evaluate the integral  $\int_0^1 e^{-x^2} dx$  to three decimal places we will expand the integrand in a Maclaurin series using (6.44) and integrate term by term:

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \int_0^1 \left[ 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \right] dx \\ &= \left[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \right]_0^1 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} \Big|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \end{aligned}$$

Using (6.45) we learn that in order to evaluate the integral to three decimal places we need to add the first five terms of the Maclaurin series since:

$$R_4 = \frac{1}{(2 \cdot 5 + 1) \cdot 5!} \cong 0.00076$$

Hence,

$$\int_0^1 e^{-x^2} dx \cong \sum_{n=0}^4 \frac{(-1)^n}{(2n+1)n!} \cong 0.747$$

- (b) To evaluate the integral  $\int_0^{1/2} (1+x^4)^{-1/2} dx$  to three decimal places we will use (6.41) to expand the integrand in a power series:

$$\begin{aligned} \frac{1}{\sqrt{1+x^4}} &= (1+x^4)^{-1/2} \\ &= 1 + \frac{1/2}{1} (-x^4) + \frac{(1/2)(3/2)}{1 \cdot 2} (-x^4)^2 + \frac{(1/2)(3/2)(5/2)}{1 \cdot 2 \cdot 3} (-x^4)^3 \\ &\quad + \cdots + \frac{(1/2)(3/2) \cdots (1/2+n-1)}{1 \cdot 2 \cdots n} (-x^4)^n + \cdots \end{aligned}$$

Focusing on the  $n^{\text{th}}$  for now, this can be written as:

$$\begin{aligned} \frac{(1/2)(3/2) \cdots (1/2+n-1)}{1 \cdot 2 \cdots n} (-x^4)^n &= (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (1 \cdot 2 \cdot 3 \cdots n)} x^{4n} \\ &= (-1)^n \frac{1 \cdot 2 \cdot 3 \cdots 2n}{2^{2n} (1 \cdot 2 \cdot 3 \cdots n)^2} x^{4n} = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} x^{4n} \end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{1/2} \frac{dx}{\sqrt{1+x^4}} &= \int_0^{1/2} (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} x^{4n} dx = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} \int_0^{1/2} x^{4n} dx \\ &= (-1)^n \frac{(2n)!}{2^{2n} (4n+1) (n!)^2} x^{4n+1} \Big|_0^{1/2} \\ &= (-1)^n \frac{(2n)!}{2^{6n+1} (4n+1) (n!)^2}\end{aligned}$$

Using (6.45) we learn that in order to evaluate the integral to three decimal places we need to add the first two terms of the power series since:

$$|R_1| = \frac{3!}{2^{13} \cdot 9} \cong 0.00008$$

And so

$$\int_0^{1/2} \frac{dx}{\sqrt{1+x^4}} \cong \sum_{n=0}^1 (-1)^n \frac{(2n)!}{2^{6n+1} (4n+1) (n!)^2} \cong 0.497$$

6. Let  $f(x) = e^{-1/x^2}$  for  $x \neq 0$  and  $f(0) = 0$ .

(a) A differentiable function is continuous by definition. Hence, since the derivative of  $f(x)$ :

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} \left( e^{-1/x^2} \right) = \frac{d}{dx} \left( -\frac{1}{x^2} \right) e^{-1/x^2} = \frac{2}{x^3} e^{-1/x^2} = P_1(x) f(x)$$

is defined for all  $x \neq 0$  it follows that  $f(x)$  is continuous for all  $x \neq 0$ . Furthermore, since we have explicitly defined the value of  $f(x)$  at  $x = 0$  as  $f(0) = 0$  and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^{-1/x^2} = \lim_{x \rightarrow -\infty} e^{x^2} = 0 = \lim_{x \rightarrow 0^+} f(x)$$

we may conclude that  $f(x)$  is continuous at  $x = 0$  and hence, that  $f(x)$  is continuous for all  $x$ .

(b) Since  $f'(x) = P_1(x)f(x)$ , i.e. the product of a polynomial and the original function  $f(x)$  and the product of two continuous functions is itself continuous it follows at once that  $f'(x)$  is continuous for all  $x \neq 0$ . Next, using the definition of the derivative, we find

$$\begin{aligned}\lim_{x \rightarrow 0} f'(x) &= f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^{-1}}{e^{1/h^2}} \\ &= \lim_{h \rightarrow 0} \frac{-h^{-2}}{-2h^{-3}e^{1/h^2}} \\ &= \lim_{h \rightarrow 0} \frac{h}{2e^{1/h^2}} = 0\end{aligned}$$

Hence,  $f'(x)$  is continuous for all  $x$ .

- (c) To prove that  $f^{(n)}(x)$  is continuous for all  $x$  and  $f^{(n)}(0) = 0$  we will use induction. First, let us assume that  $f^{(k)}(x) = P_k(x)f(x)$  for some fixed, positive integer  $k \geq 0$  and some polynomial  $P_k(x)$ ,  $P_0(x) = 1$ . For  $k + 1$  we thus have

$$\begin{aligned} f^{(k+1)}(x) &= (P_k(x)f(x))' = P_k(x)f'(x) + f(x)P_k'(x) = \frac{2}{x^3}P_k(x)f(x) + P_k'(x)f(x) \\ &= (P_1(x)P_k(x) + P_k'(x))f(x) \\ &= P_{k+1}(x)f(x) \end{aligned}$$

Thus by induction, since the formula for  $f^{(n)}(x)$  is true for fixed, positive integers  $k$  and  $k + 1$  it must be true for any arbitrary positive integer  $n \geq 0$ . Hence, since  $f^{(n)}(x)$  exists for all  $x \neq 0$  it follows that  $f^{(n)}(x)$  is continuous for all  $x \neq 0$ .

To show  $f^{(n)}(0) = 0$  we will again use induction. Let us assume that  $f^{(k)}(0) = 0$  for some fixed, positive integer  $k \geq 0$ . For  $k + 1$  we thus have

$$\begin{aligned} \lim_{x \rightarrow 0} f^{(k+1)}(x) &= f^{(k+1)}(0) = \lim_{h \rightarrow 0} \frac{f^{(k)}(0+h) - f^{(k)}(0)}{h} = \lim_{h \rightarrow 0} \frac{P_k(h)f(h) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{P_k(h)e^{-1/h^2}}{h} \end{aligned}$$

Before we continue with attempting to evaluate the limit let us first rewrite our polynomial  $P_k(x)$  as

$$P_k(x) = \sum_{n=1}^{3k} \frac{a_n}{x^n}$$

Here  $a_n$  denotes the  $n^{\text{th}}$  polynomial coefficient, which will be equal to zero in most cases and  $x^n$  is simply  $x$  raised to the  $n^{\text{th}}$  power. For example: when  $k = 2$  then

$$P_2(x) = \sum_{n=1}^6 \frac{a_n}{x^n} = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_6}{x^6} = -\frac{6}{x^4} + \frac{4}{x^6}$$

Thus, we see that in this case  $a_4 = -6$ ,  $a_6 = 4$  and  $a_1 = a_2 = a_3 = a_5 = 0$ . Furthermore, it should be noted that  $P_k(x)$  is finite. Hence,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{P_k(h)e^{-1/h^2}}{h} &= \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} \sum_{n=1}^{3k} \frac{a_n}{h^n} \\ &= \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} \left( \frac{a_1}{h} + \frac{a_2}{h^2} + \cdots + \frac{a_{3k}}{h^{3k}} \right) \\ &= a_1 \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^2} + a_2 \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^3} + \cdots + a_{3k} \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^{3k+1}} \end{aligned}$$

Let us consider the arbitrary  $j^{\text{th}}$  term of the last expression, since, if we can prove that

$$\lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^j} = 0$$

then by the additive law of limits it will follow that

$$\lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} \sum_{n=1}^{3k} \frac{a_n}{h^n} = 0$$

also. In order to prove that the limit above is equal to zero we need to consider the following one sided limits:

$$\lim_{h \rightarrow 0^-} \frac{e^{-1/h^2}}{h^j} \qquad \lim_{h \rightarrow 0^+} \frac{e^{-1/h^2}}{h^j}$$

Now instead of attempting to evaluate the above two limits it will be more convenient to evaluate the following two equivalent limits:

$$\lim_{h \rightarrow \infty} \frac{h^j}{e^{h^2}} \qquad \lim_{h \rightarrow -\infty} \frac{h^j}{e^{h^2}}$$

Because  $f(x) = e^x$  is a strictly increasing function for  $x > 1$  it follows that  $e^x < e^{x^2}$ . Hence, if we can prove that

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty \implies \lim_{x \rightarrow \infty} \frac{e^{x^2}}{x^n} = \infty$$

Expanding  $e^x$  in a Maclaurin series we note that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots > \frac{x^{n+1}}{(n+1)!}$$

and so

$$\frac{e^x}{x^n} > \frac{x}{(n+1)!}$$

Now since it is trivial to see that since

$$\lim_{x \rightarrow \infty} \frac{x}{(n+1)!} = \infty \implies \lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty \implies \lim_{x \rightarrow \infty} \frac{e^{x^2}}{x^n} = \infty$$

Of course the inverse of the last expression must be equal to zero then, i.e.

$$\lim_{x \rightarrow \infty} \frac{e^{x^2}}{x^n} = \infty \implies \lim_{x \rightarrow \infty} \frac{x^n}{e^{x^2}} = 0$$

Hence, we may conclude that

$$\lim_{h \rightarrow \infty} \frac{h^j}{e^{h^2}} = 0$$

Now if  $j$  is even we find that

$$\lim_{h \rightarrow -\infty} \frac{h^j}{e^{h^2}} = \lim_{h \rightarrow \infty} \frac{h^j}{e^{h^2}} = 0$$

If  $j$  is odd then

$$\lim_{h \rightarrow -\infty} \frac{h^j}{e^{h^2}} = - \lim_{h \rightarrow \infty} \frac{h^j}{e^{h^2}} = -1 \times 0 = 0$$

It then follows that

$$\lim_{h \rightarrow \infty} \frac{h^j}{e^{h^2}} = 0 \implies \lim_{h \rightarrow 0} \frac{1/h^j}{e^{1/h^2}} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^j} = 0$$

And so

$$\lim_{h \rightarrow 0} \frac{P_k(h) e^{-1/h^2}}{h} = \lim_{x \rightarrow 0} f^{(k+1)}(x) = f^{(k+1)}(0) = 0$$

Concluding, since we have shown that  $f^{(n)}(x) = P_k(x)f(x)$  is continuous for all  $x \neq 0$  and  $f^{(n)}(0) = 0$  and  $f(0) = 0$ , it follows that  $f^{(n)}(x)$  is continuous for all  $x$ .

(d)



7. (a) Let

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

be given. Furthermore, we assume that it is known that  $e < 3$ . Then by (6.45) and (6.54), setting  $a = 0$ ,  $x = 1$  and acknowledging that  $|f^{(6)}(1)| = e \leq M_6 \implies M_6 = 3$ , an estimate for the error is given by

$$|R_5| < \frac{M_6 |x - a|^6}{6!} = \frac{3}{6!} \approx 0.0042$$

(b) Let

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!}$$

be given. Furthermore, we know that  $\cos(1) < 1$ . Then by (6.45) and (6.54), setting  $a = 0$ ,  $x = 1$  and acknowledging that  $|f^{(7)}(x)| = |-\cos(x)| = \cos(x) \leq M_7 \implies M_7 = 1$ , an estimate for the error is given by

$$|R_6| < \frac{M_7|x-a|^7}{7!} = \frac{1}{7!} \approx 0.000198$$

(c) Let

$$\ln \frac{3}{2} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3}$$

be given. The expression above is equal to the Maclaurin series of  $\ln(1+x)$  evaluated at  $x = 1/2$ :

$$\begin{aligned} \ln(1+x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 0 + \frac{x}{1} - \frac{x^2}{2} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots \end{aligned}$$

Acknowledging that  $|f^{(4)}(x)| = |-6/(1+x)^4| \leq M_4 \implies M_4 = 6$ , an estimate for the error is given by

$$|R_3| < \frac{M_4|x-a|^4}{4!} = \frac{6}{2^4 \cdot 4!} \approx 0.0156$$

8. The problem statement is a little screwy, as Theorem 41 clearly states the function  $f(x)$  only has continuous derivatives up to the  $(n+1)^{\text{st}}$  order and nothing is being said about any derivatives beyond the  $(n+1)^{\text{st}}$  order. In any case, we will instead provide a proof of the rule deduced in Section 2.19: *Let  $f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0$ , but  $f^{(n+1)}(a) \neq 0$ . Then  $f(x)$  has a relative maximum at  $x = a$  if  $n$  is odd and  $f^{(n+1)}(a) < 0$ ;  $f(x)$  has a relative minimum at  $x = a$  if  $n$  is odd and  $f^{(n+1)}(a) > 0$ ;  $f(x)$  has neither relative maximum nor relative minimum at  $x = a$ , but a horizontal inflection point at  $x = a$  if  $n$  is even* in order to avoid any further confusion.

Let a function  $f(x)$  be defined and continuous and have continuous derivatives up to the  $(n+1)^{\text{st}}$  order for  $a - r_0 < x < a + r_0$ . Then for each  $x$  of this interval except  $x = a$ :

$$f(x) = f(a) + \frac{f'(a)}{1} (x-a) + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \frac{f^{(n+1)}(x_1)}{(n+1)!} (x-a)^{n+1}$$

for some  $x_1$  such that  $a < x_1 < x$  or, if  $x < a$ ,  $x < x_1 < a$ . Furthermore, let  $f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0$ , but  $f^{(n+1)}(a) \neq 0$ . Hence, the expression for  $f(x)$  reduces to

$$f(x) = f(a) + \frac{f^{(n+1)}(x_1)}{(n+1)!} (x-a)^{n+1} \iff f(x) - f(a) = \frac{f^{(n+1)}(x_1)}{(n+1)!} (x-a)^{n+1}$$



Let us first consider the case where  $n$  is odd, such that  $(x - a)^{n+1} > 0$  always. Then  $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$  for  $x_1 > a$  and  $x_1$  sufficiently close to  $a$ . Hence,  $f(x) - f(a) > 0$  for  $x > a$  and  $x$  sufficiently close to  $a$ . Similarly,  $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$  for  $x_1 < a$  and  $x_1$  sufficiently close to  $a$ . Again,  $f(x) - f(a) > 0$ . However, now for  $x < a$  and  $x$  sufficiently close to  $a$ . Combined with the knowledge that  $f'(a) = 0$  we thus may conclude that  $f(x)$  has a relative minimum at the point  $x = a$ . Still assuming  $n$  is odd, then when  $f^{(n+1)}(a) < 0 \implies f^{(n+1)}(x_1) < 0$  for  $x_1 > a$  and  $x_1$  sufficiently close to  $a$ . Hence,  $f(x) - f(a) < 0$  for  $x > a$  and  $x$  sufficiently close to  $a$ . Similarly,  $f^{(n+1)}(a) < 0 \implies f^{(n+1)}(x_1) < 0$  for  $x_1 < a$  and  $x_1$  sufficiently close to  $a$ . Again,  $f(x) - f(a) < 0$ . However, now for  $x < a$  and  $x$  sufficiently close to  $a$ . Combined with the knowledge that  $f'(a) = 0$  we thus may conclude that  $f(x)$  has a relative maximum at the point  $x = a$ .

Lastly, we consider the case where  $n$  is even. Now the sign of  $(x - a)^{n+1}$  will depend on whether  $x > a$  or  $x < a$ . Assuming that  $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$  for  $x_1 > a$  and  $x_1$  sufficiently close to  $a$ . Hence,  $f(x) - f(a) > 0$  for  $x > a$  and  $x$  sufficiently close to  $a$  because  $(x - a)^{n+1} > 0$ . So far there is no difference with the previously analysed case when  $n$  is odd and  $x > a$ . However, when  $x_1 < a$  for  $x_1$  sufficiently close to  $a$  we still have  $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$ , but now  $f(x) - f(a) < 0$  for  $x < a$  and  $x$  sufficiently close to  $a$  because  $(x - a)^{n+1} < 0$ . Thus the function  $f(x)$  has a horizontal inflection point at  $a$ .

9. (a) Let

$$g_1(x, h) = \frac{f(x+h) - f(x)}{h}$$

be given, where it is assumed  $h$  is a small, positive number. Taylor's formula with remainder gives the series expansion

$$f(x+h) = f(x) + h \frac{f'(x)}{1!} + h^2 \frac{f''(x_1)}{2!}$$

for some  $x_1$  such that  $x < x_1 < x+h$ . Rearranging gives

$$\frac{f(x+h) - f(x)}{h} - f'(x) = g_1(x, h) - f'(x) = h \frac{f''(x_1)}{2!}$$

And so

$$\lim_{h \rightarrow 0} h \frac{f''(x_1)}{2!} = 0 \implies \lim_{h \rightarrow 0} g_1(x, h) - f'(x) = 0$$

In a similar way we find that

$$\frac{g_1(x, h) - f'(x)}{h} = \frac{f''(x_1)}{2!}$$

Hence,

$$\lim_{h \rightarrow 0} \frac{f''(x_1)}{2!} = \frac{f''(x_1)}{2} \implies \lim_{h \rightarrow 0} \frac{g_1(x, h) - f'(x)}{h} = \frac{f''(x_1)}{2}$$

(b) Let

$$g_2(x, h) = \frac{f(x+h) - f(x-h)}{2h}$$

be given. Taylor's formula with remainder gives the series expansions

$$\begin{aligned} f(x+h) &= f(x) + h \frac{f'(x)}{1!} + h^2 \frac{f''(x)}{2!} + h^3 \frac{f'''(x_1)}{3!} \\ f(x-h) &= f(x) - h \frac{f'(x)}{1!} + h^2 \frac{f''(x)}{2!} - h^3 \frac{f'''(x_2)}{3!} \end{aligned}$$

for some  $x_1$  such that  $x < x_1 < x+h$  and some  $x_2$  such that  $x-h < x_2 < x$ . Rearranging, subtracting both equations and finally dividing by a factor of  $2h$  gives

$$\frac{f(x+h) - f(x-h)}{2h} - f'(x) = g_2(x, h) - f'(x) = h^2 \left( \frac{f'''(x_1)}{12} + \frac{f'''(x_2)}{12} \right)$$

And so

$$\lim_{h \rightarrow 0} h^2 \left( \frac{f'''(x_1)}{12} + \frac{f'''(x_2)}{12} \right) = 0 \implies \lim_{h \rightarrow 0} g_2(x, h) - f'(x) = 0$$

Similarly, we find that

$$\frac{g_2(x, h) - f'(x)}{h^2} = \frac{f'''(x_1)}{12} + \frac{f'''(x_2)}{12}$$

Hence,

$$\lim_{h \rightarrow 0} \frac{f'''(x_1)}{12} + \frac{f'''(x_2)}{12} \implies \lim_{h \rightarrow 0} \frac{g_2(x, h) - f'(x)}{h^2} = \frac{f'''(x_1)}{12} + \frac{f'''(x_2)}{12}$$

(c) Let

$$g_3(x, h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

be given. Taylor's formula with remainder gives the series expansions

$$\begin{aligned} f(x+h) &= f(x) + h \frac{f'(x)}{1!} + h^2 \frac{f''(x)}{2!} + h^3 \frac{f'''(x_1)}{3!} + h^4 \frac{f''''(x_1)}{4!} \\ f(x-h) &= f(x) - h \frac{f'(x)}{1!} + h^2 \frac{f''(x)}{2!} - h^3 \frac{f'''(x_1)}{3!} + h^4 \frac{f''''(x_2)}{4!} \end{aligned}$$

for some  $x_1$  such that  $x < x_1 < x+h$  and some  $x_2$  such that  $x-h < x_2 < x$ . Rearranging, adding both equations and finally dividing by a factor of  $h^2$  gives

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) = g_3(x, h) - f''(x) = h^2 \left( \frac{f''''(x_1)}{24} + \frac{f''''(x_2)}{24} \right)$$

And so

$$\lim_{h \rightarrow 0} h^2 \left( \frac{f'''(x_1)}{24} + \frac{f'''(x_2)}{24} \right) = 0 \implies \lim_{h \rightarrow 0} g_3(x, h) - f''(x) = 0$$

Note that the book divides by a factor of  $h$  instead of  $h^2$ . This would *not* result in a finite limit when  $h \rightarrow 0$  for the expression  $[g_3(x, h) - f''(x)]/h$ . This is most likely a mistake and we will instead divide by a factor of  $h^2$ . And so we find that

$$\frac{g_3(x, h) - f''(x)}{h^2} = \frac{f'''(x_1)}{24} + \frac{f'''(x_2)}{24}$$

Hence,

$$\lim_{h \rightarrow 0} \frac{f'''(x_1)}{24} + \frac{f'''(x_2)}{24} \implies \lim_{h \rightarrow 0} \frac{g_3(x, h) - f''(x)}{h^2} = \frac{f'''(x_1)}{24} + \frac{f'''(x_2)}{24}$$

Thus, the error in approximating  $f''(x)$  by  $g_3(x, h)$  is *not* of order  $h$ , but of order  $h^2$ .

## Section 6.19

1. (a) Let  $z = i$  be given. Then

$$\lim_{n \rightarrow \infty} \frac{z^n}{n} = \lim_{n \rightarrow \infty} \frac{i^n}{n}$$

Now

$$\lim_{n \rightarrow \infty} i^n$$

is undefined, since  $i^n$  keeps cycling between the values  $i, -1, -i$  and  $1$ . However, it is true that

$$\lim_{n \rightarrow \infty} \left| \frac{z^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{|z^n|}{n} = \lim_{n \rightarrow \infty} \frac{|z|^n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

Can we thus conclude that  $\lim_{n \rightarrow \infty} z^n/n = \lim_{n \rightarrow \infty} i^n/n = 0$ ? The answer is yes, since Theorem 46 is satisfied when taking  $z_n = i^n/n$  and  $z_0 = 0$  for then there will exist an  $N$  such that when  $n > N$  for some  $\epsilon > 0$  then  $|z_n - z_0| = |z_n| = 1/n < \epsilon \implies n > 1/\epsilon$ .

- (b)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(1+i)n^3 - 2in + 3}{in^3 - 1} &= \lim_{n \rightarrow \infty} \frac{1+i - 2i/n^2 + 3/n^3}{i - 1/n^3} = \frac{1+i}{i} = \frac{1}{i} + 1 = \frac{i}{i^2} + 1 \\ &= 1 - i \end{aligned}$$

2. (a) Let the series

$$\frac{1+i}{2} + \left(\frac{1+i}{2}\right)^2 + \cdots + \left(\frac{1+i}{2}\right)^n + \cdots = \sum_{n=1}^{\infty} \left(\frac{1+i}{2}\right)^n$$

be given. Employing the root test gives

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{1+i}{2}\right)^n\right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{1+i}{2}\right|^n} = \left|\frac{1+i}{2}\right| = \frac{|1+i|}{2} = \frac{1}{\sqrt{2}} < 1$$

Hence, the series is absolutely convergent.

(b) Let the series

$$\sum_{n=1}^{\infty} ni^n$$

be given. Employing the  $n^{\text{th}}$  term test gives

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} ni^n = \infty \cdot i^\infty \neq 0$$

Hence, the series diverges.

(c) Let the series

$$\sum_{n=1}^{\infty} \frac{ni^n}{n^2 + 1}$$

be given. Now

$$|a_n| = \left| \frac{ni^n}{n^2 + 1} \right| = \frac{n}{n^2 + 1} |i^n| = \frac{n}{n^2 + 1} |i|^n = \frac{n}{n^2 + 1} < \frac{1}{n} = b_n$$

for  $n = 1, 2, \dots$ . The series  $\sum_{n=1}^{\infty} 1/n$  diverges and hence, the series  $\sum_{n=1}^{\infty} a_n$  is not absolutely convergent. However, the series of real parts is

$$0 - \frac{1}{2} + 0 + \frac{1}{4} + 0 - \frac{1}{6} + \cdots$$

and the series of imaginary parts is

$$1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \cdots$$

If the zeros are disregarded, these are convergent alternating series. Hence,  $\sum_{n=1}^{\infty} i^n/n$  converges and since

$$\frac{ni^n}{n^2 + 1} < \frac{i^n}{n}$$

the series  $\sum_{n=1}^{\infty} a_n$  converges.

(d) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)^2}$$

be given. Since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{1}{(n+i)^2} \right| = \sum_{n=1}^{\infty} \frac{1}{|(n+i)^2|} = \sum_{n=1}^{\infty} \frac{1}{|n+i|^2} = \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the series  $\sum_{n=1}^{\infty} 1/n^2$  converges implies the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

3. Let the series

$$e^z = 1 + z + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

be given. Employing the ratio test gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{z}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{|z|}{n+1} = |z| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$$

Hence, the series converges for all  $z$ .

Let the series

$$\sin z = z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$$

be given. Employing the ratio test gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{z^{2n+1}}{(2n+1)!} \frac{(2n-1)!}{z^{2n-1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^2}{2n(2n+1)} \right| \\ &= |z|^2 \lim_{n \rightarrow \infty} \frac{1}{2n(2n+1)} \\ &= 0 < 1 \end{aligned}$$

Hence, the series converges for all  $z$ .

Let the series

$$\cos z = 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

be given. Employing the ratio test gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{z^{2n+2}}{(2n+2)!} \frac{(2n)!}{z^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^2}{(2n+2)(2n+1)} \right| \\ &= |z|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} \\ &= 0 < 1 \end{aligned}$$

Hence, the series converges for all  $z$ .

4. (a) Setting  $z = iy$  in (6.57) we find

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots + \frac{(iy)^n}{n!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \cdots + (-1)^n \frac{y^{2n}}{(2n)!} + \cdots\right) + i \left(y - \frac{y^3}{3!} + \cdots + (-1)^{n-1} \frac{y^{2n-1}}{(2n-1)!} + \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

where the last equality follows from (6.59) and (6.58).

(b)

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

where the last expression follows from (a) directly.

Setting  $z = x + iy$  in (6.57) we find

$$e^{x+iy} = 1 + (x + iy) + \frac{(x + iy)^2}{2!} + \frac{(x + iy)^3}{3!} + \cdots + \frac{(x + iy)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(x + iy)^n}{n!}$$

The term  $(x + iy)^n$  can be expanded using the Binomial formula:

$$(x + iy)^n = \sum_{k=0}^n \binom{n}{k} x^k (iy)^{n-k}$$

And so

$$\begin{aligned} e^{x+iy} &= \sum_{n=0}^{\infty} \frac{(x + iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k (iy)^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k! (n-k)!} x^k (iy)^{n-k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=k}^{\infty} \frac{(iy)^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\ &= e^x e^{iy} = e^x (\cos y + i \sin y) \end{aligned}$$

5. (a) By (6.57)

$$\begin{aligned}
\frac{e^{iz} + e^{-iz}}{2} &= \frac{1}{2} \left( 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots + \frac{(iz)^n}{n!} + \cdots \right) \\
&\quad + \frac{1}{2} \left( 1 - iz + \frac{(-iz)^2}{2!} + \frac{(-iz)^3}{3!} + \cdots + \frac{(-iz)^n}{n!} + \cdots \right) \\
&= 1 + \frac{(iz)^2}{2 \cdot 2!} + \frac{(-iz)^2}{2 \cdot 2!} + \cdots + \frac{(iz)^n}{2 \cdot n!} + \frac{(-iz)^n}{2 \cdot n!} + \cdots \\
&= 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \\
&= \cos z
\end{aligned}$$

(b) By (6.57)

$$\begin{aligned}
\frac{e^{iz} - e^{-iz}}{2i} &= \frac{1}{2i} \left( 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots + \frac{(iz)^n}{n!} + \cdots \right) \\
&\quad - \frac{1}{2i} \left( 1 - iz + \frac{(-iz)^2}{2!} + \frac{(-iz)^3}{3!} + \cdots + \frac{(-iz)^n}{n!} + \cdots \right) \\
&= z + \frac{(iz)^2}{2 \cdot 2!} + \frac{(-iz)^2}{2 \cdot 2!} + \cdots + \frac{(iz)^n}{2 \cdot n!} + \frac{(-iz)^n}{2 \cdot n!} + \cdots \\
&= z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \\
&= \sin z
\end{aligned}$$

(c) By (6.57)

$$\begin{aligned}
e^{z_1+z_2} &= 1 + (z_1 + z_2) + \cdots + \frac{(z_1 + z_2)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{n=k}^{\infty} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{n=0}^{\infty} \frac{z_2^n}{n!} = e^{z_1} e^{z_2}
\end{aligned}$$

(d) By (6.58)

$$\begin{aligned}
\sin(-z) &= (-z) - \frac{(-z)^3}{3!} + \cdots + (-1)^{n-1} \frac{(-z)^{2n-1}}{(2n-1)!} + \cdots \\
&= -z + \frac{z^3}{3!} - \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots \\
&= - \left( z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right) = -\sin z
\end{aligned}$$

(e) By (6.59)

$$\begin{aligned}
\cos(-z) &= 1 - \frac{(-z)^2}{2!} + \cdots + (-1)^n \frac{(-z)^{2n}}{(2n)!} + \cdots \\
&= 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots = \cos z
\end{aligned}$$

(f) By (6.58) and (6.59) and using the Cauchy product

$$\left( \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \right)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^k}{k!} (-1)^{n-k} \frac{z^{n-k}}{(n-k)!}$$



$$\begin{aligned}
\sin^2 z + \cos^2 z &= \left( z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right)^2 \\
&\quad + \left( 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \right)^2 \\
&= \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \right)^2 + \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 \\
&= \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right)^2 + \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{z^{2(n-k)+1}}{[2(n-k)+1]!} \\
&\quad + \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^{2k}}{(2k)!} (-1)^{n-k} \frac{z^{2(n-k)}}{[2(n-k)]!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n+2}{2k+1} \frac{z^{2n+2}}{(2n+2)!} + \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-1} \binom{2n}{2k+1} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \left[ \sum_{k=0}^n \binom{2n}{2k} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \right] \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n (1-1)^{2n} \frac{z^{2n}}{(2n)!} = 1
\end{aligned}$$

where in the last step we have made use of the Binomial formula

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

with  $x = 1$  and  $y = -1$ .

(g) By (6.58) and (6.59)

$$\begin{aligned}
\cos^2 z - \sin^2 z &= \left( 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \right)^2 \\
&\quad - \left( z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right)^2 \\
&= \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 - \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \right)^2 \\
&= \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 - \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right)^2 \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^{2k}}{(2k)!} (-1)^{n-k} \frac{z^{2(n-k)}}{[2(n-k)]!} \\
&\quad - \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{z^{2(n-k)+1}}{[2(n-k)+1]!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n+2}{2k+1} \frac{z^{2n+2}}{(2n+2)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-1} \binom{2n}{2k+1} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \left[ \sum_{k=0}^n \binom{2n}{2k} + \sum_{k=0}^{n-1} \binom{2n}{2k+1} \right] \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{2n} \binom{2n}{k} \frac{z^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^n (1+1)^{2n} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!} \\
&= 1 - \frac{(2z)^2}{2!} + \cdots + (-1)^n \frac{(2z)^{2n}}{(2n)!} + \cdots = \cos 2z
\end{aligned}$$

(h) By (6.58) and (6.59)

$$\begin{aligned}
2 \sin z \cos z &= 2 \left( z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right) \\
&\quad \times \left( 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \right) \\
&= 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\
&= 2 \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \frac{z^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{z^{2(n-k)}}{[2(n-k)]!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \binom{2n+1}{2k+1} \frac{z^{2n+1}}{(2n+1)!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^n \left[ \binom{2n}{2k} + \binom{2n}{2k+1} \right] \frac{z^{2n+1}}{(2n+1)!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{k=0}^{2n} (-1)^n \binom{2n}{k} \frac{z^{2n+1}}{(2n+1)!} = 2 \sum_{n=0}^{\infty} (-1)^n (1+1)^{2n} \frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2z)^{2n-1}}{(2n-1)!} \\
&= 2z - \frac{(2z)^3}{3!} + \cdots + (-1)^{n-1} \frac{(2z)^{2n-1}}{(2n-1)!} + \cdots = \sin 2z
\end{aligned}$$

where we have made use of the relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

6. (a) Let the series

$$z + \frac{z^2}{2} + \cdots + \frac{z^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

be given. By the ratio test we find

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^{n+1}}{n+1} \frac{n}{z^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{nz}{n+1} \right| = |z| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |z| \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = |z|$$

Hence, the series converges for  $|z| < 1$  and diverges for  $|z| > 1$ .

(b) Let the series

$$1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n$$

where  $z = x + iy$  be given. By the extended ratio test we find

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z^{n+1}}{z^n} \right| = |z| = \sqrt{(x + iy)(x - iy)} = \sqrt{x^2 + y^2}$$

Hence, the series converges for  $|z| = \sqrt{x^2 + y^2} < 1$  and diverges for  $|z| > 1$ .

## Section 6.21

1. (a) Let  $F(x, y) = e^{x^2 - y^2}$ . Now let us make the substitution  $t = x^2 - y^2 \implies F(t) = e^t$ . Expanding  $F(t)$  in a Taylor series about the point  $(0, 0)$ , using (6.46), and finally substituting back for  $t$  in terms of  $x, y$  gives

$$\begin{aligned} F(t) &= 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots \\ &= 1 + (x^2 - y^2) + \frac{1}{2!} (x^2 - y^2)^2 + \cdots + \frac{1}{n!} (x^2 - y^2)^n + \cdots \end{aligned}$$

Since (6.46) converges for all  $x$  it follows that  $F(x, y) = F(t)$  converges for all  $t$  and hence,  $(x, y) \in \mathbb{R}$ .

- (b) Let  $F(x, y) = \sin(xy)$ . Again, let us make a substitution, this time of the form  $t = xy$ . Expanding  $F(t)$  in a Taylor series about the point  $(0, 0)$ , using (6.47), and finally substituting back for  $t$  in terms of  $x, y$  gives

$$\begin{aligned} F(t) &= t - \frac{t^3}{3!} + \cdots + (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} + \cdots \\ &= xy - \frac{(xy)^3}{3!} + \cdots + (-1)^{n-1} \frac{(xy)^{2n-1}}{(2n-1)!} + \cdots \end{aligned}$$

Since (6.47) converges for all  $x$  it follows that  $F(x, y) = F(t)$  converges for all  $t$  and hence,  $(x, y) \in \mathbb{R}$ .

- (c) Let  $F(x, y) = 1/(1 - x - y)$ . Let us make the substitution  $t = x + y$ . Using (6.17), setting  $a = 1$  and  $r = t$ , and finally substituting back for  $t$  in terms of  $x, y$  we thus find

$$\begin{aligned} F(t) &= 1 + t + t^2 + \cdots + t^n + \cdots \\ &= 1 + (x + y) + (x + y)^2 + \cdots + (x + y)^n + \cdots \end{aligned}$$

Again, from (6.17) it follows that this series converges for  $-1 < t < 1 \implies -1 < x + y < 1$ .

- (d) Let  $F(x, y) = 1/(1 - x - y - z)$ . Let us make the substitution  $t = x + y + z$ . Using (6.17), setting  $a = 1$  and  $r = t$ , and finally substituting back for  $t$  in terms of  $x, y, z$  we thus find

$$\begin{aligned} F(t) &= 1 + t + t^2 + \cdots + t^n + \cdots \\ &= 1 + (x + y + z) + (x + y + z)^2 + \cdots + (x + y + z)^n + \cdots \end{aligned}$$

Again, from (6.17) it follows that this series converges for  $-1 < t < 1 \implies -1 < x + y + z < 1$ .

2. Let

$$\phi(t) = F[x_1 + t(x - x_1), y_1 + t(y - y_1)]$$

where  $0 \leq t \leq 1$ . Hence, as stated in Section 6.21, it follows that

$$\begin{aligned} \phi'(t) &= (x - x_1) F_x[x_1 + t(x - x_1), y_1 + t(y - y_1)] + (y - y_1) F_y[x_1 + t(x - x_1), y_1 + t(y - y_1)] \\ &= dF[x_1 + t(x - x_1), y_1 + t(y - y_1)] \end{aligned}$$

As such, we assume that in general

$$\begin{aligned} \phi^{(k)}(t) &= (x - x_1)^k \frac{\partial^k F}{\partial x^k} [x_1 + t(x - x_1), y_1 + t(y - y_1)] + \cdots \\ &= \sum_{r=0}^k \binom{k}{r} (x - x_1)^r (y - y_1)^{k-r} \frac{\partial^k F}{\partial x^r \partial y^{k-r}} [x_1 + t(x - x_1), y_1 + t(y - y_1)] \\ &= d^k F[x_1 + t(x - x_1), y_1 + t(y - y_1)] \end{aligned}$$

for some arbitrary, fixed integer  $k \geq 0$ . For  $k + 1$  we find

$$\begin{aligned} \phi^{(k+1)}(t) &= [\phi^{(k)}(t)]' \\ &= \left[ (x - x_1)^k \frac{\partial^k F}{\partial x^k} [x_1 + t(x - x_1), y_1 + t(y - y_1)] + \cdots \right]' \\ &= (x - x_1)^{k+1} \frac{\partial^{k+1} F}{\partial x^{k+1}} [x_1 + t(x - x_1), y_1 + t(y - y_1)] \\ &\quad + (x - x_1)^k (y - y_1) \frac{\partial^{k+1} F}{\partial x^k \partial y} [x_1 + t(x - x_1), y_1 + t(y - y_1)] + \cdots \\ &= \sum_{r=0}^{k+1} \binom{k+1}{r} (x - x_1)^r (y - y_1)^{k+1-r} \frac{\partial^{k+1} F}{\partial x^r \partial y^{k+1-r}} [x_1 + t(x - x_1), y_1 + t(y - y_1)] \\ &= d^{k+1} F[x_1 + t(x - x_1), y_1 + t(y - y_1)] \end{aligned}$$

And so by induction the equation

$$\phi^{(n)}(t) = d^n F[x_1 + t(x - x_1), y_1 + t(y - y_1)]$$

must be true for any positive integer  $n$ .

3. Let it be a given that a power series

$$c_{0,0} + (c_{1,0}x + c_{1,1}y) + (c_{2,0}x^2 + c_{2,1}xy + c_{2,2}y^2) + \cdots$$

converges absolutely at a point  $(x_0, y_0)$ .

*Note: the problem statement only speaks of convergence of a power series at the point  $(x_0, y_0)$ . However, in itself this is not a sufficient enough condition to prove convergence at every point  $(\lambda x_0, \lambda y_0)$ , for  $|\lambda| < 1$ . We require absolute convergence, since absolute convergence implies convergence, but the inverse is not necessarily true.*

The  $n^{\text{th}}$  term of this power series may be written as

$$u_n(x, y) = \sum_{r=0}^n \binom{n}{r} c_{n,r} x^{n-r} y^r$$

Then it follows that if  $|\lambda| < 1$  it must be true that

$$\begin{aligned} |u_n(\lambda x, \lambda y)| &= \left| \sum_{r=0}^n \binom{n}{r} c_{n,r} (\lambda x)^{n-r} (\lambda y)^r \right| = \left| \sum_{r=0}^n \binom{n}{r} c_{n,r} \lambda^n x^{n-r} y^r \right| \\ &= \sum_{r=0}^n \binom{n}{r} |\lambda|^n |c_{n,r} x^{n-r} y^r| \\ &< \sum_{r=0}^n \binom{n}{r} |c_{n,r} x^{n-r} y^r| \\ &= \left| \sum_{r=0}^n \binom{n}{r} c_{n,r} x^{n-r} y^r \right| = |u_n(x, y)| \end{aligned}$$

for all  $x$ . Hence, since  $u_n(x, y)$  is absolutely convergent at the point  $(x_0, y_0)$  and we have shown that  $|u_n(\lambda x_0, \lambda y_0)| < |u_n(x_0, y_0)|$ , for  $|\lambda| < 1$ , it follows at once that it is also absolutely convergent (and hence, convergent) at every point  $(\lambda x_0, \lambda y_0)$ , for  $|\lambda| < 1$ .

4. From Problem 1(b) it follows that we can write the integral  $\int_0^1 \int_0^1 \sin(xy) \, dx dy$  as

$$\begin{aligned} \int_0^1 \int_0^1 \sin(xy) \, dx dy &= \int_0^1 \int_0^1 \left[ xy - \frac{(xy)^3}{3!} + \cdots + (-1)^{n-1} \frac{(xy)^{2n-1}}{(2n-1)!} + \cdots \right] dx dy \\ &= \int_0^1 \left[ \frac{y}{2} - \frac{y^3}{4 \cdot 3!} + \cdots + (-1)^{n-1} \frac{y^{2n-1}}{2n(2n-1)} + \cdots \right] dy \\ &= \frac{1}{4} - \frac{1}{16 \cdot 3!} + \cdots + (-1)^{n-1} \frac{1}{4n^2(2n-1)} + \cdots \cong 0.240 \end{aligned}$$

5. (a) Taylor's formula with remainder for functions of several variables (6.67) gives the series expansions

$$\begin{aligned}
f(x+h, y+h) &= f(x, y) + h \left( \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y) \right) \\
&\quad + \frac{h^2}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_1^*, y_1^*) + 2 \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_1^*) \right) \\
f(x+h, y-h) &= f(x, y) + h \left( \frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right) \\
&\quad + \frac{h^2}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_1^*, y_2^*) - 2 \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_2^*) \right) \\
f(x-h, y+h) &= f(x, y) - h \left( \frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right) \\
&\quad + \frac{h^2}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_2^*, y_1^*) - 2 \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_1^*) \right) \\
f(x-h, y-h) &= f(x, y) - h \left( \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y) \right) \\
&\quad + \frac{h^2}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_2^*, y_2^*) + 2 \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_2^*) \right)
\end{aligned}$$

where  $x_1^* = x + t^*h$ ,  $y_1^* = y + t^*h$  and  $x_2^* = x - t^*h$ ,  $y_2^* = y - t^*h$  respectively for  $0 < t^* < 1$ . Adding these 4 equations, rearranging and dividing by a factor of  $h^2$  gives

$$\begin{aligned}
&\frac{f(x+h, y+h) + f(x+h, y-h) + f(x-h, y+h) + f(x-h, y-h) - 4f(x, y)}{h^2} = \\
g_1(x, y, h) &= \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_1^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_2^*) \right) + \\
&\quad \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_2^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_2^*) \right)
\end{aligned}$$

Taking the limit when  $h \rightarrow 0$  of the right-hand side results in

$$\begin{aligned}
&\frac{1}{2} \lim_{h \rightarrow 0} \left( \frac{\partial^2 f}{\partial x^2}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_1^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*, y_2^*) \right) + \\
&\quad \frac{1}{2} \lim_{h \rightarrow 0} \left( \frac{\partial^2 f}{\partial x^2}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_2^*, y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*, y_2^*) \right) = \\
&\quad 2 \left( \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) \right)
\end{aligned}$$

since  $\lim_{h \rightarrow 0} x_1^* = \lim_{h \rightarrow 0} x_2^* = x$  and  $\lim_{h \rightarrow 0} y_1^* = \lim_{h \rightarrow 0} y_2^* = y$ . Hence, we find that

$$\frac{1}{2} \lim_{h \rightarrow 0} g_1(x, y, h) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = \nabla^2 f(x, y)$$

- (b) Adding the first and last equations and subtracting the second and third equations from (a), rearranging and dividing by a factor of  $4h^2$  gives

$$\frac{f(x+h, y+h) - f(x+h, y-h) - f(x-h, y+h) + f(x-h, y-h)}{4h^2} = g_2(x, y, h) = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_2^*) + \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_2^*) \right)$$

Taking the limit when  $h \rightarrow 0$  of the right-hand side results in

$$\frac{1}{4} \lim_{h \rightarrow 0} \left( \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_1^*) + \frac{\partial^2 f}{\partial x \partial y}(x_1^*, y_2^*) + \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_1^*) + \frac{\partial^2 f}{\partial x \partial y}(x_2^*, y_2^*) \right) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

since  $\lim_{h \rightarrow 0} x_1^* = \lim_{h \rightarrow 0} x_2^* = x$  and  $\lim_{h \rightarrow 0} y_1^* = \lim_{h \rightarrow 0} y_2^* = y$ . Hence, we find that

$$\lim_{h \rightarrow 0} g_2(x, y, h) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

6.