CHAPTER 2

Section 2.4

1. Some examples are:

$$\begin{aligned} \text{volume of a cylinder}: &\quad \pi r^2 h \\ \text{surface area of a cone}: &\quad \pi r\left(l+r\right) \\ \text{volume of a cuboid}: &\quad lwh \end{aligned}$$

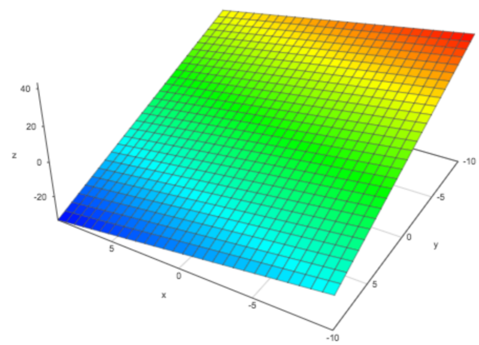

Figure 1: $z = 3 - x - 3y$

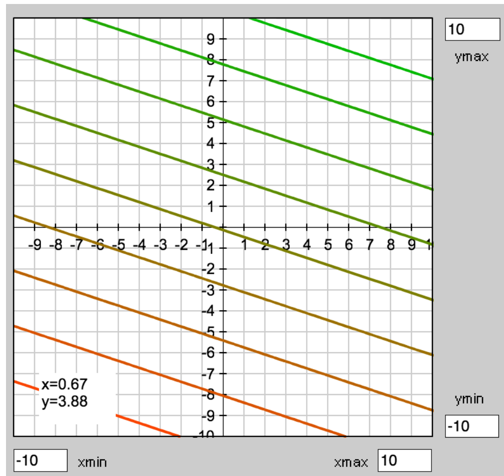

Figure 2: $z = 3 - x - 3y$

2. (a)

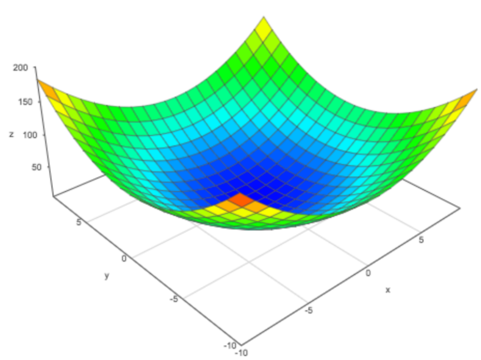

Figure 3: $z = x^2 + y^2 + 1$

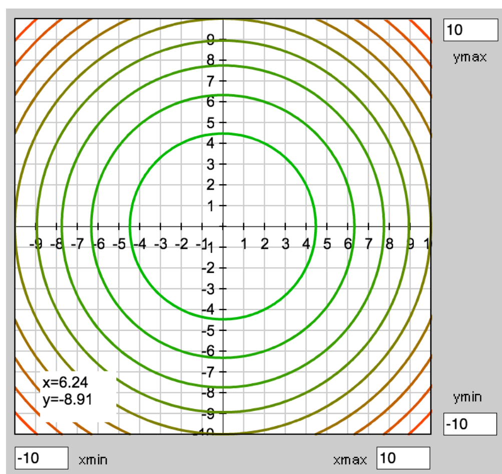

Figure 4: $z = x^2 + y^2 + 1$

(b)

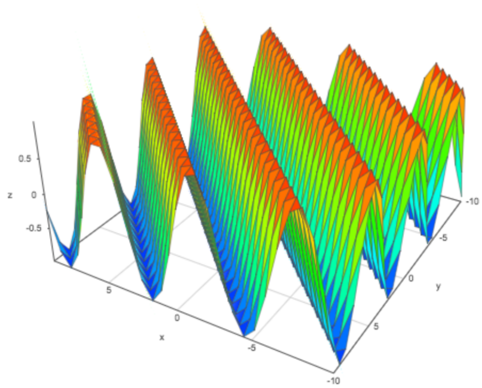

Figure 5: $z = \sin(x + y)$

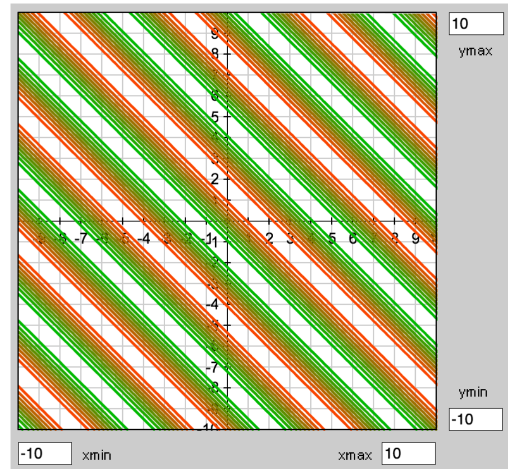

Figure 6: $z = \sin(x + y)$

(c)

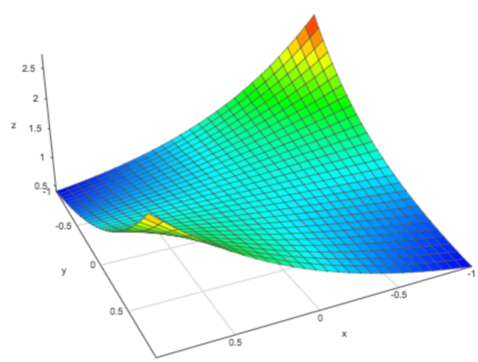

Figure 7: $z = e^{xy}$

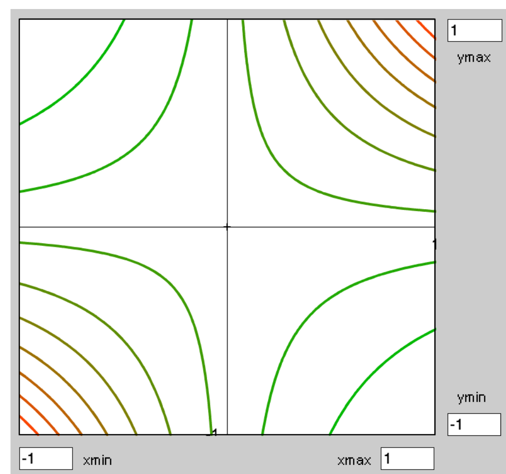

Figure 8: $z = e^{xy}$

(d)

3. (a) The level surfaces of $u = x^2 + y^2 + z^2$ are spheres centered at the point $(x, y, z) = (0, 0, 0)$ and with radius $r = \sqrt{u}$.

   (b) The level surfaces of $u = x + y + z$ are planes, where a particular value for u denotes the point of intersection of the plane with the x, y and z axes.

   (c) The level surfaces of $w = x^2 + y^2 - z$ are hyperbolic paraboloids with the saddle point located at point $(x, y, z) = (0, 0, -w)$.

   (d) The level surface of $w = x^2 + y^2$ is a hyperbolic paraboloid with its saddle point located at at point $(x, y, z) = (0, 0, 0)$.

4. (a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{1+x^2+y^2} = \frac{0+0}{1+0+0} = 0$$

(b) Let $x = y$. Then the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2x} = \infty$$

Next, let $x = 0$. Then the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{0+y^2} = \frac{0}{0+0} = 0$$

Hence, the limit does not exist.

(c)

$$\lim_{(x,y)\to(0,0)} \frac{(1+y^2)\sin x}{x} = \left(\lim_{y\to 0} 1+y^2\right) \lim_{x\to 0} \frac{\sin x}{x} = (1)\,(1) = 1$$

To show that $\lim_{x\to 0} \sin x/x = 1$ we use the sandwich theorem:

$$\sin x \le x \le \tan x \iff 1 \le \frac{x}{\sin x} \le \frac{\tan x}{\sin x} \implies 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Next, note that

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$$

Hence,

$$\lim_{x\to 0} \frac{x}{\sin x} = 1 \implies \lim_{x\to 0} \frac{\sin x}{x} = 1$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{1+x-y}{x^2+y^2} = \frac{1+0-0}{0+0} = \infty$$

5. (a) Let us consider the limit

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \frac{x}{x-y} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x}{x-y}\right) = \lim_{x\to 0} \frac{x}{x-0} = \lim_{x\to 0} 1 = 1$$

However

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \frac{x}{x-y} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x}{x-y}\right) = \lim_{y\to 0} \frac{0}{0-y} = 0$$

Hence, the limit at $(x,y) = (0,0)$ does not exist and so the function z is discontinuous at that point.

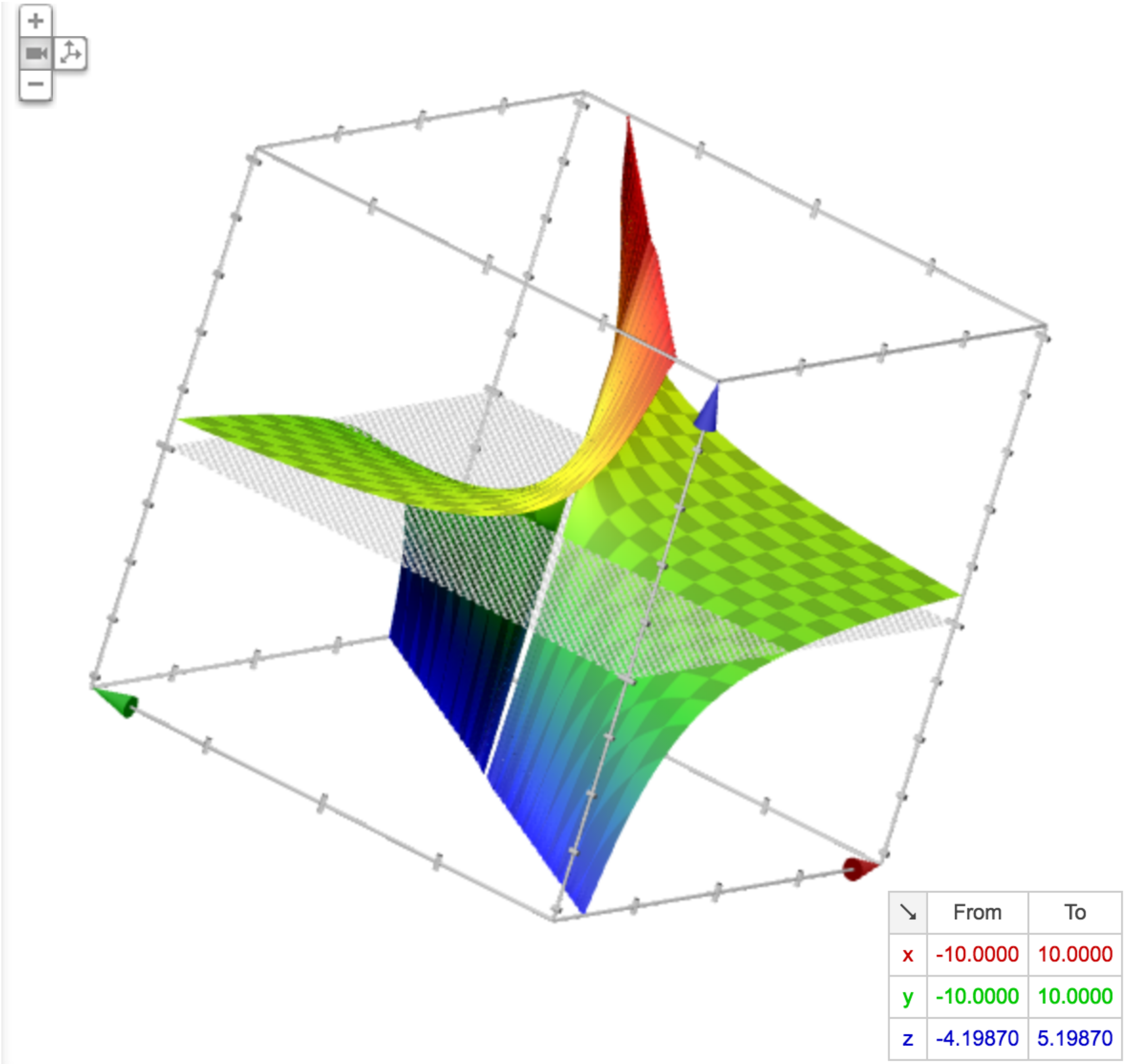

Figure 9: $z = x/(x-y)$

(b) Let us consider the limit

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right) = \lim_{x\to 0}\left(\lim_{y\to 0} \ln\left(x^2+y^2\right)\right) = \lim_{x\to 0} \ln x^2 = -\infty$$

However, the point $(x,y) = (0,0)$ is not in the domain of z, as the function is not defined there. Hence, strictly speaking z is continuous over its domain of definition $x, y \in (0,\infty)$ and has an infinite discontinuity at the point $(x,y) = (0,0)$ as it is not defined there.

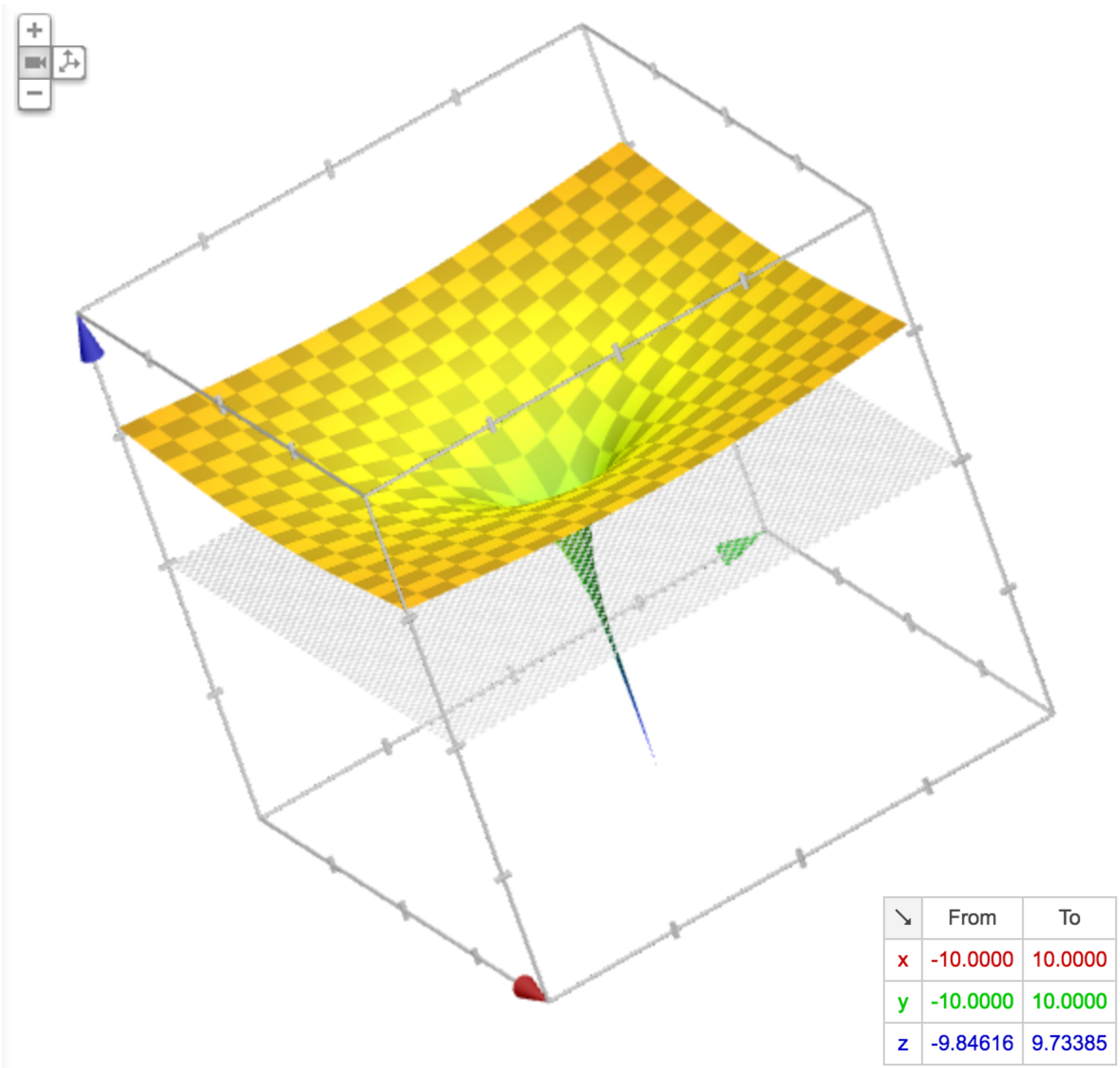

Figure 10: $z = x/(x-y)$

6. (a) The function e^a where a is an arbitrary real valued scalar is defined for any a, positive or negative. Hence, the domain may be formally defined as $\{x, y \in \mathbb{R} \mid \infty < x, y < \infty\}$.

   (b) The domain for the function $z = \ln(x^2+y^2-1)$ is given by $\{x, y \in \mathbb{R} \mid x^2+y^2 > 1\}$.

   (c) The set in which the function $z = \sqrt{1-x^2-y^2}$ is defined is the closed region $\{x, y \in \mathbb{R} \mid x^2+y^2 \leq 1\}$.

   (d) The set in which the function $u = xy/z$ is defined is an open set, excluding the points lying in the xy plane (i.e. $z = 0$). It is not a domain, since not all points in the open set can be joined by a broken line.

7. Let $f(x,y)$ be defined in domain D and continuous at the point (x_1,y_1) of D, so that $\lim_{(x,y)\to(x_1,y_1)} f(x,y) = c = f(x_1,y_1)$. Substituting $\epsilon = (1/2)f(x_1,y_1)$ in (2.3) then gives
$$|f(x,y) - f(x_1,y_1)| < \frac{1}{2} f(x_1,y_1)$$
which is equivalent to stating that there is a neighbourhood of (x_1,y_1) in which $f(x,y) > (1/2)f(x_1,y_1) > 0$. Rewriting the absolute inequality gives
$$\begin{aligned} -\frac{1}{2} f(x_1,y_1) < f(x,y) - f(x_1,y_1) &< \frac{1}{2} f(x_1,y_1) \\ 0 < f(x,y) - \frac{1}{2} f(x_1,y_1) &< f(x_1,y_1) \\ f(x_1,y_1) > f(x,y) - \frac{1}{2} f(x_1,y_1) &> 0 \end{aligned}$$
And so we find
$$\frac{1}{2} f(x_1,y_1) < f(x,y) < \frac{3}{2} f(x_1,y_1)$$
Since the starting assumption was that $f(x_1,y_1) > 0$ clearly $(1/2)f(x_1,y_1) > 0$ as well, and since $f(x,y) > (1/2)f(x_1,y_1)$ we may conclude that $f(x,y) > (1/2)f(x_1,y_1) > 0$.

8. Suppose that the domain D could consist of two open sets E_1 and E_2 with no point in common. Next let us choose point P in E_1 and Q in E_2 and join them by a broken line in D. We will regard this line as a path from point P to Q and let s be the distance from P along the path so that the path is given by continuous functions $x = x(s)$ and $y = y(s)$, where $0 \le s \le L$, with $s = 0$ at point P and $s = L$ at point Q. Now consider a function $f(s)$ and let $f(s) = -1$ if $(x(s), y(s))$ is in E_1 and let $f(s) = 1$ if $(x(s), y(s))$ is in E_2. Furthermore, let this function $f(s)$ be some linear combination of $x(s)$ and $y(s)$, i.e. $f(s) = ax(s) + by(s)$, where a and b are arbitrary scalars. Now since both $x(s)$ and $y(s)$ are continuous for $0 \le s \le L$ then according to (2.7) so will be $f(s)$. Next, we apply the *intermediate value theorem*: If $f(x)$ is continuous for $a \le x \le b$ and $f(a) < 0$, $f(b) > 0$, then $f(x) = 0$ for some x between a and b. Hence, since $f(s)$ is continuous for $0 \le s \le L$ and $f(0) = -1 < 0$ and $f(L) = 1 > 0$, then $f(s) = 0$ for some $s = s_0$ between 0 and L. But $f(s_0) = 0$ does not correspond to a point $(x(s_0), y(s_0))$ lying in either E_1 or E_2. In other words, a section of the path representing the broken line connecting points P and Q and given by continuous functions $x(s)$ and $y(s)$ doesn't belong to either E_1 or E_2. But this contradicts the definition of a domain D, which states that two points P and Q belonging to two different non-overlapping open sets E_1 and E_2 cannot be joined by a broken line.

9. Let the set A consist of all points (x,y) for which the continuous function $f(x,y) > 0$ in domain D. Let (x_1,y_1) be such a point. We can choose $f(x_1,y_1)$ arbitrarily small as long as $f(x_1,y_1) > 0$. Then with the help of the answer to Problem 7 we can verify

that there is a neighborhood of $f(x_1, y_1)$ in which $f(x, y) > (1/2)f(x_1, y_1) > 0$. In other words, no matter how small $f(x_1, y_1)$ is, as long as $f(x_1, y_1) > 0$ and $f(x, y)$ is continuous, there will always exist a neighborhood of (x_1, y_1) of radius δ where $f(x, y) > (1/2)f(x_1, y_1) > 0$. Hence, the set A is an open set. A similar reasoning can be applied to conclude that the set B is an open set. Together, A and B form two non-overlapping open sets. Next, imagine choosing a point $(x, y) = P$ in A and a point $(x, y) = Q$ in B and join them by a continuous (broken) line in D. Let s, $0 \leq s \leq L$ denote the distance along the path in the same way as for Problem 8, i.e. the path is from P to Q and is given by the continuous functions $x = x(s)$ and y = y(s). Now we apply the *intermediate value theorem*; since $f(s)$ is continuous for $0 \leq s \leq L$ and $f(0) > 0$ and $f(L) < 0$, then $f(s) = 0$ for some $s = s_0$ between $s = 0$ and $s = L$. Let us suppose the opposite; that $f(s) \neq 0$ for any s. This would imply that D consists of the two non-overlapping open sets A and B only, which as we concluded in Problem 8 contradicts the definition of a domain D; stating that two points P and Q belonging to two different non-overlapping open sets A and B cannot be joined by a (broken) line.

10. (a) Let $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$ in V^n. If $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2} < \epsilon$ then

$$|\mathbf{x}|^2 = x_1^2 + \cdots + x_n^2 < \epsilon^2 \implies |x_1| < \epsilon, \ldots, |x_n| < \epsilon$$

For $n = 2$, the result may be geometrically interpreted by stating that if the length of a vector $\mathbf{x}$ with origin at point $(x_1, x_2) = (0, 0)$, i.e. $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ is smaller than some $\epsilon > 0$, then there exists a neighborhood of $(0, 0)$ where $|x_1| < \epsilon$ and $|x_2| < \epsilon$.

(b) Suppose that $|x_1| < \delta, \ldots, |x_n| < \delta$ then $x_1^2 < \delta^2, \ldots, x_n^2 < \delta^2$ and so

$$x_1^2 + \cdots + x_n^2 < \delta^2 + \cdots + \delta^2 = n\delta^2 \qquad n \geq 0$$

Taking square roots next gives

$$\sqrt{x_1^2 + \cdots + x_n^2} = |\mathbf{x}| < \sqrt{n}\delta < n\delta$$

where the right most inequality clearly holds, since $\sqrt{n} < n$.

(c) To show continuity of the mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ at the point $\mathbf{x}^0$, where $\mathbf{x} \in V^n$ and $\mathbf{y} \in V^m$, we choose a $\delta > 0$ for a given $\epsilon > 0$ small enough such that

$$|f_1(x_1, \ldots, x_n) - f_1(x_1^0, \ldots, x_n^0)| < \frac{\epsilon}{m}, \ldots, |f_m(x_1, \ldots, x_n) - f_m(x_1^0, \ldots, x_n^0)| < \frac{\epsilon}{m}$$

for $|\mathbf{x} - \mathbf{x}^0| = \sqrt{(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2} < \delta$. Squaring each inequality and then summing gives

$$\left[f_1(x_1, \ldots, x_n) - f_1(x_1^0, \ldots, x_n^0)\right]^2 + \cdots + \left[f_m(x_1, \ldots, x_n) - f_m(x_1^0, \ldots, x_n^0)\right]^2 < \frac{\epsilon^2}{m}$$

Finally, taking square roots of both sides of the inequality gives

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}\left(\mathbf{x}^0\right)| < \frac{\epsilon}{\sqrt{m}} \leq \epsilon$$

In conclusion, since we haven chosen δ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon/\sqrt{m}$, it will certainly satisfy $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon$, since $\epsilon \geq \epsilon/\sqrt{m}$.

(d) Squaring the inequality that signifies continuity for the mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ gives

$$\left[f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)\right]^2 + \cdots + \left[f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2$$

which implies that

$$\left[f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2, \ldots, \left[f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2$$

Taking square roots of both sides then finally results in

$$|f_1(\mathbf{x}) - f_1\left(\mathbf{x}^0\right)| < \epsilon, \ldots, |f_m(\mathbf{x}) - f_m\left(\mathbf{x}^0\right)| < \epsilon$$

from which we may conclude that each of the functions $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ is continuous at $(x_1^0, \ldots, x_n^0)$.

11. Let us for the moment assume that the limit $P_n \to P_0$ is not unique so that there exists a $P_n \to P_0'$, $P_0' \neq P_0$ and lets take $\epsilon = (1/3)d(P_0, P_0') = (1/3)|P_0' - P_0|$. We thus have $|P_n - P_0| < \epsilon$ and $|P_n - P_0'| < \epsilon$. Then

$$|P_0' - P_0| = \underbrace{|P_n - P_0 + P_0' - P_n| \leq |P_n - P_0| + |P_0' - P_n|}_{\text{triangle inequality}} < \frac{2}{3}|P_0' - P_0|$$

which is clearly a contradiction and so we must conclude that in fact $P_0 = P_0'$, i.e. the limit P_0 is unique.

12. To show that a set E in the plane is closed if and only if for every convergent sequence of points P_n in E the limit of the sequence is in E we will try to prove the opposite and see that it produces a contradiction. First, suppose E is closed and $P_n \to P_0$, with P_n in E for all n, but the limit P_0 not in E (i.e. $P_0 \in \mathbb{R} \setminus E$). According to section (2.2), since E is closed, $\mathbb{R} \setminus E$ is open. Now since $P_0 \in \mathbb{R} \setminus E$ and the set is open there will exist a neighborhood of P_0 of radius ϵ such that $\mathrm{d}(P, P_0) < \epsilon$ which is completely contained in $\mathbb{R} \setminus E$ and so implies $P \notin E$. But this would mean there exists an N such that for all $n \geq N$, $P_n \in \mathbb{R} \setminus E$, which contradicts the assumption that the sequence P_n is entirely contained in E.

Next, suppose E is such that whenever $P_n \in E$ and $P_n \to P_0$, then $P_0 \in E$. To show that E is closed, we need to prove that $\mathbb{R} \setminus E$ is open, meaning that a neighborhood of a point $P \in \mathbb{R} \setminus E$ of radius $\epsilon > 0$ is contained entirely in $\mathbb{R} \setminus E$. Let us suppose the opposite however, that P_0 is a point not in E ($P_0 \in \mathbb{R} \setminus E$), but has at least one point of its neighborhood in E. In other words, suppose a neighborhood of P_0 of arbitrary radius $\epsilon > 0$ will contain at least one point that lies in E, in particular consider $\epsilon = 1, \epsilon = 1/2, \ldots, \epsilon = 1/n$. Let $P_n \in E$ be such a point and let its distance to $P_0 \in \mathbb{R} \setminus E$ satisfy the condition $\mathrm{d}(P_n, P_0) < 1/n$. Then if $\epsilon = 1/n$ and $n \to \infty$ we arrive at $P_n \to P_0$ (see Problem 11 for the definition of the limit of a convergent series), which implies $P_n \in \mathbb{R} \setminus E$. But this is contradictory to the original assumption that $P_n \in E$. Hence, a neighborhood of a point P not in E must always be contained entirely in $\mathbb{R} \setminus E$, implying that $\mathbb{R} \setminus E$ is open and consequently, that E is closed. In conclusion, we have proven that a set E is closed if and only if for every convergent sequence of point P_n in E, the limit of the sequence P_0 is in E.

13. (a) A set is called open if we can form a neighborhood of a point in the set of radius ϵ that is contained entirely in the set. In other words, this neighborhood does not contain any elements that are not part of the set. Since by definition the empty set does not contain any elements, the above statement can be applied to it without any problems and so it can be considered to be open.

    (b) To show that a set E in the plane and its boundary is closed is equivalent to showing that the complement to this is an open set $\mathbb{R} \setminus \bar{E}$ where the set $\bar{E}$ denotes the union of E and its boundary. To show that $\mathbb{R} \setminus \bar{E}$ is open is equivalent to showing that a neighborhood of a point $P \in \mathbb{R} \setminus \bar{E}$ of radius $\epsilon > 0$ is contained entirely in $\mathbb{R} \setminus \bar{E}$. We have already proven this as part of the second part of the solution to Problem 12 and so we won't repeat it again. Hence, we may conclude that a set E in the plane and its boundary are indeed closed.

Section 2.6

1. (a)

$$\frac{\partial z}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial z}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

   (b)

$$\frac{\partial z}{\partial x} = y^2 \cos xy \qquad \frac{\partial z}{\partial y} = \sin xy + xy \cos xy$$

   (c)

$$\frac{\partial z}{\partial x} = \frac{3x^2+2xy-2xz}{x^2-3z^2} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{x^2-3z^2}$$

   (d)

$$\frac{\partial z}{\partial x} = \frac{e^{x+2y}}{2\sqrt{e^{x+2y}-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{e^{x+2y}-y}{\sqrt{e^{x+2y}-y^2}}$$

(e)

$$\frac{\partial z}{\partial x} = 3x\sqrt{x^2+y^2} \qquad \frac{\partial z}{\partial y} = 3y\sqrt{x^2+y^2}$$

(f)

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x+2y)^2}} \qquad \frac{\partial z}{\partial y} = \frac{2}{\sqrt{1-(x+2y)^2}}$$

(g)

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^z+1} \qquad \frac{\partial z}{\partial y} = \frac{2e^y}{e^z+1}$$

(h)

$$\frac{\partial z}{\partial x} = -\frac{y+z}{x+2z} \qquad \frac{\partial z}{\partial y} = -\frac{2xy+z^2+xz}{2yz+xy}$$

2. Using a forward difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(2,1)-f(1,1)}{1} = \frac{2-(-1)}{1} = 3$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,2)-f(1,1)}{1} = \frac{-3-(-1)}{1} = -2$$

Using a backward difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(1,1)-f(0,1)}{1} = \frac{-1-(-2)}{1} = 1$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,1)-f(1,0)}{1} = \frac{-1-1}{1} = -2$$

Using a centered difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(2,1)-f(0,1)}{2} = \frac{2-(-2)}{2} = 2$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,2)-f(1,0)}{2} = \frac{-3-(-1)}{2} = -2$$

3. (a)

$$\left(\frac{\partial u}{\partial x}\right)_y = 2x \qquad \left(\frac{\partial v}{\partial y}\right)_x = -2$$

(b)

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial x}{\partial u} = e^u \cos v \qquad \left(\frac{\partial y}{\partial v}\right)_u = e^u \cos v$$

(c)

$$\left(\frac{\partial x}{\partial u}\right)_y = \left[\frac{\partial}{\partial u}(u+2y)\right]_y = 1 \qquad \left(\frac{\partial y}{\partial v}\right)_u = \left[\frac{1}{2}\frac{\partial}{\partial v}(u-v)\right]_u = -\frac{1}{2}$$

(d)

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2+y^2}} \qquad \left(\frac{\partial r}{\partial \theta}\right)_x = \frac{x\sin\theta}{\cos^2\theta} = x\sec\theta\tan\theta$$

4. (a)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{ydx - xdy}{y^2}$$

(b)

$$dz = \frac{xdx + ydy}{x^2+y^2}$$

(c)

$$dz = \frac{(y-y^2)\,dx + (x-x^2)\,dy}{(1-x-y)^2}$$

(d)

$$dz = (x-2y)^4 e^{xy}\left[\left(5+xy-2y^2\right)dx + \left(-10-2xy+x^2\right)dy\right]$$

(e)

$$dz = \frac{-ydx + xdy}{x^2+y^2}$$

(f)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = -\frac{xdx + ydy + zdz}{(x^2+y^2+z^2)^{3/2}}$$

5. (a)

$$\begin{aligned}
\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2 + 2(x+\Delta x)(y+\Delta y) - x^2 - 2xy\big|_{(1,1)} \\
&= 2(x+y)\Delta x + 2x\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2\big|_{(1,1)} \\
&= 2(1+1)\Delta x + 2\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2 \\
&= 4\Delta x + 2\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2 \\
dz &= 2(x+y)\Delta x + 2x\Delta y \\
&= 2(1+1)\Delta x + 2\Delta y \\
&= 4\Delta x + 2\Delta y
\end{aligned}$$

(b)

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = \left.\frac{x+\Delta x}{x+\Delta x+y\Delta y} - \frac{x}{x+y}\right|_{(1,1)}$$
$$= \frac{1+\Delta x}{2+\Delta x+\Delta y} - \frac{1}{2}$$
$$= \frac{\Delta x - \Delta y}{2(2+\Delta x+\Delta y)}$$
$$= \frac{(\Delta x-\Delta y)(2+\Delta x+\Delta y) - (\Delta x-\Delta y)(\Delta x+\Delta y)}{4(2+\Delta x+\Delta y)}$$
$$= \frac{\Delta x-\Delta y}{4} - \frac{(\Delta x-\Delta y)(\Delta x+\Delta y)}{4(2+\Delta x+\Delta y)}$$
$$dz = \frac{\Delta x-\Delta y}{4}$$

6. Given the data for point $(x,y)=(1,2)$ we get $\Delta x = 0.1$ and $\Delta y = -0.2$ for the point $(x,y)=(1.1,1.8)$, and so

$$dz = f_x(1,2)\,\Delta x + f_y(1,2)\,\Delta y = 2(0.1) + 5(-0.2) = -0.8$$

which gives the estimate

$$f(1.1,1.8) = f(1,2) + dz = 3 - 0.8 = 2.2$$

Next, for the point $(x,y)=(1.2,1.8)$ we have $\Delta x = 0.2$ and $\Delta y = -0.2$ and so

$$dz = 2(0.2) + 5(-0.2) = -0.6$$

which gives the estimate

$$f(1.2,1.8) = 3 - 0.6 = 2.4$$

And lastly, for the point $(x,y)=(1.3,1.8)$ we have $\Delta x = 0.3$ and $\Delta y = -0.2$ and so

$$dz = 2(0.3) + 5(-0.2) = -0.4$$

which gives the estimate

$$f(1.3,1.8) = 3 - 0.4 = 2.6$$

7. First off, we will show that the limit at the point $(x,y)=(0,0)$ does not exist for $z=f(x,y)$. Let us consider approaching the point $(x,y)=(0,0)$ along the line $x=y$, such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{x^2}{2x^2} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

Similarly, approaching the point $(x,y)=(0,0)$ along the line $x=-y$ result in

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{-x^2}{2x^2} = \lim_{x\to 0} -\frac{1}{2} = -\frac{1}{2}$$

Combined with the fact that $f(0,0) = 0$, we may conclude that there does not exists a unique limit at the point $(x, y) = (0, 0)$ and so the function $z = f(x, y) = xy/(x^2 + y^2)$ is discontinuous at this point. Taking partial derivatives gives

$$\frac{\partial f}{\partial x} = -\frac{y\,(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \frac{\partial f}{\partial y} = \frac{x\,(x^2 - y^2)}{(x^2 + y^2)^2}$$

Since $f(x, y)$ is discontinuous at the point $(x, y) = (0, 0)$, so will be $\partial f/\partial x$ and $\partial f/\partial y$ (i.e. the partial derivatives do not exist at this point). However, we can show this explicitly as well by once again taking limits. First, we will take the one-sided limit, approaching zero for positive y along the line $x = 0$ of $\partial f/\partial x$, giving

$$\lim_{y\to 0^+} \frac{\partial f}{\partial x} = -\frac{y\,(0 - y^2)}{(0 + y^2)^2} = \lim_{y\to 0^+} \frac{y^3}{y^4} = \lim_{y\to 0^+} \frac{1}{y} = \infty$$

However, approaching zero for negative y along the line $x = 0$ gives

$$\lim_{y\to 0^-} \frac{\partial f}{\partial x} = -\frac{y\,(0 - y^2)}{(0 + y^2)^2} = \lim_{y\to 0^-} \frac{y^3}{y^4} = \lim_{y\to 0^-} \frac{1}{y} = -\infty$$

Hence, the limit does not exist. A similar analysis for $\partial f/\partial y$ reveals that the limit for $\partial f/\partial y$ at the point $(x, y) = (0, 0)$ does not exist either and so we may conclude that $\partial f/\partial x$ and $\partial f/\partial y$ exist for all (x, y) and are continuous except at the point $(x, y) = (0, 0)$.

The fundamental lemma states that if a function $z = f(x, y)$ has continuous partial derivatives in D, then z has a differential

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

at every point (x, y) of D. Since we have just verified that the function $z = f(x, y) = xy/(x^2 + y^2)$ has continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ except at $(x, y) = (0, 0)$ (as $f(x, y)$ is discontinuous there), we may conclude that $z = f(x, y)$ has a differential for $(x, y) \neq (0, 0)$, which is of the form

$$dz = -\frac{y\,(x^2 - y^2)}{(x^2 + y^2)^2}\Delta x + \frac{x\,(x^2 - y^2)}{(x^2 + y^2)^2}\Delta y = \frac{x^2 - y^2}{(x^2 + y^2)^2}\,(-y\Delta x + x\Delta y)$$

Section 2.7

1. (a)

$$\begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 & 2x_2 \\ 3x_2 & 3x_1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ 2x_1x_3 & 0 & x_1^2 \end{bmatrix}$$

(d)

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \\ 2x & 0 \end{bmatrix}$$

(e)

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xyz & x^2z & x^2y \end{bmatrix}$$

(f)

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$$

(g)

$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} 2t \\ 3t^2 \\ 4t^3 \end{bmatrix}$$

2. (a)

$$\begin{aligned} d\mathbf{y} = \mathbf{f_x}|_{\mathbf{x}=(2,1)}d\mathbf{x} &= \begin{bmatrix} \left.\frac{\partial y_1}{\partial x_1}\right|_{x_1=2,x_2=1} & \left.\frac{\partial y_1}{\partial x_2}\right|_{x_1=2,x_2=1} \\ \left.\frac{\partial y_2}{\partial x_1}\right|_{x_1=2,x_2=1} & \left.\frac{\partial y_2}{\partial x_2}\right|_{x_1=2,x_2=1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 2x_1|_{x_1=2} & 2x_2|_{x_2=1} \\ x_2|_{x_2=1} & x_1|_{x_1=2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.06 \end{bmatrix} \\ \mathbf{f}(2.04, 1.01) = \mathbf{f}(2,1) + d\mathbf{y} &= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.18 \\ 0.06 \end{bmatrix} = \begin{bmatrix} 5.18 \\ 2.06 \end{bmatrix} \end{aligned}$$

(b)

$$d\mathbf{y} = \mathbf{f_x}|_{\mathbf{x}=(3,2,1)} d\mathbf{x} = \begin{bmatrix} x_2|_{x_2=2} & x_1|_{x_1=3} & -2x_3|_{x_3=1} \\ [x_2+x_3]_{x_2=2,x_3=1} & x_1|_{x_1=3} & x_1|_{x_1=3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0.01 \\ -0.01 \\ 0.03 \end{bmatrix} = \begin{bmatrix} -0.07 \\ 0.09 \end{bmatrix}$$

$$\mathbf{f}(3.01, 1.99, 1.03) = \mathbf{f}(3,2,1) + d\mathbf{y} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} + \begin{bmatrix} -0.07 \\ 0.09 \end{bmatrix} = \begin{bmatrix} 4.93 \\ 9.09 \end{bmatrix}$$

(c)

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} \left.\frac{\partial u}{\partial x}\right|_{x=0,y=\pi/2} & \left.\frac{\partial u}{\partial y}\right|_{x=0,y=\pi/2} \\ \left.\frac{\partial v}{\partial x}\right|_{x=0,y=\pi/2} & \left.\frac{\partial v}{\partial y}\right|_{x=0,y=\pi/2} \\ \left.\frac{\partial w}{\partial x}\right|_{x=0,y=\pi/2} & \left.\frac{\partial w}{\partial y}\right|_{x=0,y=\pi/2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$= \begin{bmatrix} e^x \cos y|_{x=0,y=\pi/2} & -e^x \sin y|_{x=0,y=\pi/2} \\ e^x \sin y|_{x=0,y=\pi/2} & e^x \cos y|_{x=0,y=\pi/2} \\ 2e^x|_{x=0} & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1.6 - (\pi/2) \end{bmatrix} \approxeq \begin{bmatrix} -0.03 \\ 0.1 \\ 0.2 \end{bmatrix}$$

$$\begin{bmatrix} u(0.1, 1.6) \\ v(0.1, 1.6) \\ w(0.1, 1.6) \end{bmatrix} = \begin{bmatrix} u(0, \pi/2) \\ v(0, \pi/2) \\ w(0, \pi/2) \end{bmatrix} + \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \approxeq \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -0.03 \\ 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -0.03 \\ 1.1 \\ 2.2 \end{bmatrix}$$

(d)

$$d\mathbf{y} = \mathbf{f_x}|_{\mathbf{x}=(1,0,\ldots,0)} d\mathbf{x} = \begin{bmatrix} \left.\dfrac{\partial y_1}{\partial x_1}\right|_{x_1=1,x_2=0,\ldots,x_n=0} & \ldots & \left.\dfrac{\partial y_1}{\partial x_n}\right|_{x_1=1,x_2=0,\ldots,x_n=0} \\ \vdots & & \vdots \\ \left.\dfrac{\partial y_n}{\partial x_1}\right|_{x_1=1,x_2=0,\ldots,x_n=0} & \ldots & \left.\dfrac{\partial y_n}{\partial x_n}\right|_{x_1=1,x_2=0,\ldots,x_n=0} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2x_2|_{x_2=0} & \ldots & \ldots & 2x_n|_{x_2=0} \\ 2x_1|_{x_1=1} & 0 & 2x_3|_{x_3=0} & \ldots & 2x_n|_{x_n=0} \\ \vdots & & \ddots & & \vdots \\ 2x_1|_{x_1=1} & \ldots & 2x_{n-2}|_{x_{n-2}=0} & 0 & 2x_n|_{x_n=0} \\ 2x_1|_{x_1=1} & \ldots & \ldots & 2x_{n-1}|_{x_{n-1}=0} & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 2 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ \vdots \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{f}(1, 0.1, \ldots, 0.1) = \mathbf{f}(1, 0, \ldots, 0) + d\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

3. (a)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3(x^2-y^2) & -6xy \\ 6xy & 3(x^2-y^2) \end{vmatrix} = 9\left(x^2-y^2\right)^2 + 36x^2y^2$$

$$= 9\left(x^2+y^2\right)^2$$

(b)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} e^y\cos z & xe^y\cos z & -xe^y\sin z \\ e^y\sin z & xe^y\sin z & xe^y\cos z \\ e^y & xe^y & 0 \end{vmatrix} = 0$$

(c)

$$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{vmatrix} = \begin{vmatrix} 2uvw & u^2w \\ 2uv^2 & 2u^2v \end{vmatrix} = 2u^3v^2w$$

(d)

$$\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & 2 & 2z \\ yz & xz & xy \\ 0 & 0 & 2z \end{vmatrix} = 4z^2\left(x^2 - y\right)$$

4. (a) The Jacobian determinant is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x}\left(\cos^2 y + \sin^2 y\right) = e^{2x}$$

Evaluating the Jacobian determinant at the point $(x,y) = (1,0)$ then gives $\partial(u,v)/\partial(x,y) = e^2 \approxeq 7.39$.

(b) Squaring and adding the equations $u = e^x \cos y$ and $v = e^x \sin y$ gives

$$u^2 + v^2 = e^{2x}\left(\cos^2 y + \sin^2 y\right) = e^{2x} \qquad 0.9 \le x \le 1.1$$

Dividing the second equation by the first gives

$$\frac{v}{u} = \frac{\sin y}{\cos y} = \tan y \iff v = (\tan y)\,u \qquad -0.1 \le y \le 0.1$$

These two equations describe a region R_{uv} which is bounded by arcs of the circles $u^2 + v^2 = e^{1.8}$, $u^2 + v^2 = e^{2.2}$ and the rays $v = (\tan -0.1)u$, $v = (\tan 0.1)u \implies v = \pm(\tan 0.1)u$ (see the right (or left) half of Figure 11).

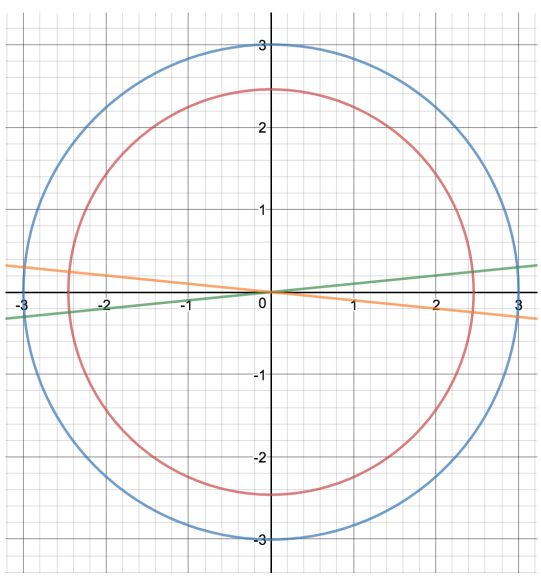

Figure 11: $u^2 + v^2 = e^{1.8}$, $u^2 + v^2 = e^{2.2}$, $v = \pm(\tan 0.1)\,u$

To find the area A_{uv} of this region we make use of the formula $A = (\theta/2)r^2$ and so

$$A_{uv} = 2\frac{0.1}{2}\left(e^{2.2} - e^{1.8}\right)$$

which gives for the ratio of the are of R_{uv} to that of R_{xy}

$$\frac{A_{uv}}{A_{xy}} = \frac{0.1}{0.04}\left(e^{2.2} - e^{1.8}\right) \approxeq 7.44$$

This answer is slightly higher than the value of the Jacobian determinant from part (a).

(c) The approximating linear mapping is given by

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

The region R'_{uv} corresponding to the square R_{xy} of part (b) under this linear mapping is a tilted square in the uv plane. For $dy = 0$ we have $du = e^x \cos y dx$ and $dv = e^x \sin$, so that (du, dv) follows a line of slope $\tan y$. For $dx = 0$ we have $du = -e^x \sin y dy$ and $dv = e^x \cos y dy$, so that (du, dv) follows a line of slope $-\cot y$. At the point $(x, y) = (1, 0)$ we have $du = edx$ and $dv = edy$ and so the area of the square region R'_{uv} is given by $A'_{uv} = dudv = e^2 dxdy$. The ratio of the area of R'_{uv} to that of R_{xy} then is

$$\frac{A'_{uv}}{A_{xy}} = e^2 \approxeq 7.39$$

This is the same answer as was found for part (a) and slightly smaller than the answer to part (b).

5. (a) Any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V^2 that are not parallel (i.e. such that $\mathbf{u} \neq a\mathbf{v}$ for some arbitrary scalar a) are linearly independent. As Figure 12 shows, the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ forms the edges of a parallelogram, since $\mathbf{a} = \mathbf{v}$ and $\mathbf{u} = \mathbf{b}$. Now consider keeping the vector $\mathbf{v}$ fixed while scaling the vector $\mathbf{u}$ by some scalar $0 \leq a \leq 1$, such that $\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + \mathbf{v}$. As should be clear from looking at the figure, the point P will then lie somewhere on the line segment formed by the vector $\mathbf{b} = \mathbf{u}$, which is the rightmost edge of the parallelogram. Similarly, keeping the vector $\mathbf{u}$ fixed while scaling the vector $\mathbf{v}$ by some scalar $0 \leq b \leq 1$, such that $\mathbf{x} = \overrightarrow{OP} = \mathbf{u} + b\mathbf{v}$ will result in the point P lying somewhere on the line segment formed by the vector $\mathbf{v}$, which is the top edge of the parallelogram. Hence, it should not be hard to see that any combination $\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v}$, $0 \leq a \leq 1$, $0 \leq b \leq 1$ will result in a point P that is located somewhere inside or

on an edge of the parallelogram formed by the two linearly independent vectors $\mathbf{u} = \mathbf{b}$ and $\mathbf{v} = \mathbf{a}$.

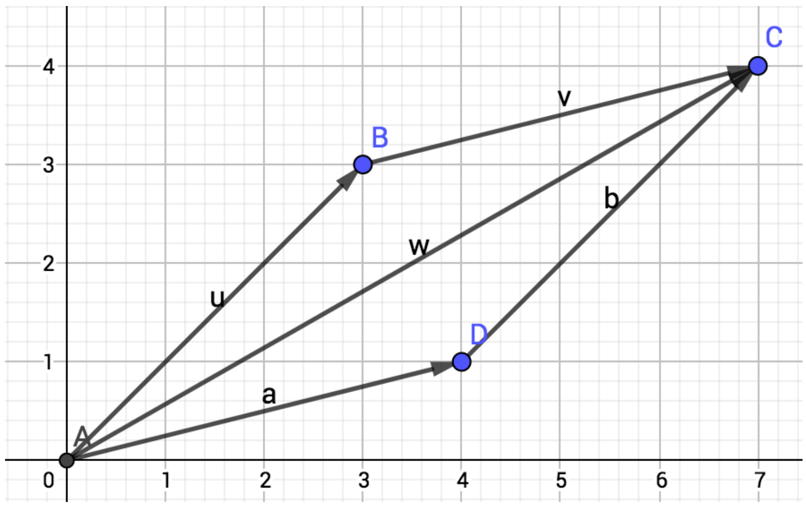

Figure 12: $\mathbf{w} = \mathbf{u} + \mathbf{v}$

(b) Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

be the matrix associated with the parallelogram of part (a) and the vector $\mathbf{x}$, such that

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1$$

Then according to Section 1.4 the determinant of $\mathbf{B}$ may be interpreted as the area of the parallelogram: $A_{\mathbf{x}} = \det \mathbf{B}$. Similarly, let

$$\mathbf{C} = \begin{bmatrix} \mathbf{Au} & \mathbf{Av} \end{bmatrix} = \mathbf{AB}$$

be the matrix associated with the parallelogram obtained by the linear mapping $\mathbf{y} = \mathbf{Ax}$, such that

$$\mathbf{y} = \overrightarrow{OQ} = \mathbf{A}\left(a\mathbf{u} + b\mathbf{v}\right) = a\mathbf{Au} + b\mathbf{Av} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1$$

The area of this parallelogram then is given by

$$A_{\mathbf{y}} = \det \mathbf{C} = \det\left(\mathbf{AB}\right) = \det \mathbf{A}\left(\det \mathbf{B}\right)$$

where the last equality holds because the determinant of a product of matrices is equal to the product of the determinant of each individual matrix. Hence, we observe that indeed as claimed $A_{\mathbf{y}} = \det \mathbf{A}\left(A_{\mathbf{x}}\right)$.

6. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in V^3. A point P for which

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \qquad 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1$$

will then fill a parallelepiped in 3-dimensional space whose edges, properly directed, represent $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

be the matrix associated with the parallelepiped and the vector $\mathbf{x}$, such that

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1$$

Then according to Section 1.4 the determinant of $\mathbf{B}$ may be interpreted as the volume of the parallelepiped: $V_{\mathbf{x}} = \det \mathbf{B}$. Similarly, let

$$\mathbf{C} = \begin{bmatrix} \mathbf{Au} & \mathbf{Av} \end{bmatrix} = \mathbf{AB}$$

be the matrix associated with the parallelepiped obtained by the linear mapping $\mathbf{y} = \mathbf{Ax}$, such that

$$\begin{aligned} \mathbf{y} = \overrightarrow{OQ} = \mathbf{A}\left(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}\right) &= a\mathbf{Au} + b\mathbf{Av} + c\mathbf{Aw} \qquad 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1 \\ &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned}$$

The volume of this parallelepiped then is given by

$$V_{\mathbf{y}} = \det \mathbf{C} = \det\left(\mathbf{AB}\right) = \det \mathbf{A}\left(\mathbf{B}\right)$$

where the last equality holds because the determinant of a product of matrices is equal to the product of the determinant of each individual matrix. Hence, we observe that indeed as claimed $V_{\mathbf{y}} = \det \mathbf{A}\left(V_{\mathbf{x}}\right)$.

Section 2.8

1. (a)

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

(b)

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

(c)

$$\frac{dy}{dx} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{1}{v}\left(\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right)$$

2.

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$$

3.

$$\begin{aligned}\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} = \left(\frac{\partial}{\partial u}\frac{1}{\log_v u}\right)\frac{du}{dx} + \frac{1}{v\ln u}\frac{dv}{dx} &= -\frac{1}{\log_v^2 u}\frac{1}{u\ln v}\frac{du}{dx} + \frac{1}{v\ln u}\frac{dv}{dx}\\ &= -\frac{\ln^2 v}{\ln^2 u}\frac{1}{u\ln v}\frac{du}{dx} + \frac{1}{v\ln u}\frac{dv}{dx}\\ &= -\frac{\ln v}{u\ln^2 u}\frac{du}{dx} + \frac{1}{v\ln u}\frac{dv}{dx}\end{aligned}$$

4. Let us start by finding dx/dt and dy/dt:

$$\frac{d}{dt}\left(x^3 + e^x - t^2 - t\right) = \frac{d}{dt}1 \iff 3x^2\frac{dx}{dt} + e^x\frac{dx}{dt} - 2t - 1 = 0 \iff \frac{dx}{dt} = \frac{2t+1}{3x^2+e^x}$$

and

$$\frac{d}{dt}\left(yt^2 + y^2t - t + y\right) = \frac{d}{dt}0 \iff t^2\frac{dy}{dt} + 2yt + 2yt\frac{dy}{dt} + y^2 - 1 + \frac{dy}{dt} = 0 \iff \frac{dy}{dt} = \frac{1-2yt-y^2}{1+2yt+t^2}$$

Hence,

$$\left.\frac{dy}{dt}\right|_{t=0} = \left[\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right]_{t=0} = \left[e^x\cos y\frac{2t+1}{3x^2+e^x} - e^x\sin y\frac{1-2yt-y^2}{1+2yt+t^2}\right]_{t=0} = 1$$

5. There is an error in the problem statement. It should read: *Find dz/dt for $t = 5$.*

$$\left.\frac{dz}{dt}\right|_{t=5} = \left[3x^2 - 6xy\right]_{x=7,y=2}\left.\frac{dx}{dt}\right|_{t=5} - \left.3x^2\right|_{x=7}\left.\frac{dy}{dt}\right|_{t=5} = (63)(3) - (147)(-1) = 336$$

6. We first compute dx/dt and dy/dt:

$$\frac{dx}{dt} = 6e^{3t} + 2t - 1 \qquad\qquad \frac{dy}{dt} = 15e^{3t} + 3$$

Hence,

$$\left.\frac{dz}{dt}\right|_{t=0} = \left.\frac{\partial z}{\partial x}\right|_{x=4,y=4}\left.\frac{dx}{dt}\right|_{t=0} + \left.\frac{\partial z}{\partial y}\right|_{x=4,y=4}\left.\frac{dy}{dt}\right|_{t=0} = (7)(5) + (9)(18) = 197$$

7. We start by finding $\partial u/\partial r$ and $\partial u/\partial\theta$:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

Squaring both sides then gives

$$\left(\frac{\partial u}{\partial r}\right)^2 = \cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2 + 2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$
$$\left(\frac{\partial u}{\partial \theta}\right)^2 = r^2\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + r^2\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2 - 2r^2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

Finally, multiplying the second equation by $1/r^2$ and adding the result to the first gives

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

8. We start by finding $\partial w/\partial u$ and $\partial w/\partial v$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = \cosh v\frac{\partial w}{\partial x} + \sinh v\frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = u\sinh v\frac{\partial w}{\partial x} + u\cosh v\frac{\partial w}{\partial y}$$

Squaring both sides then gives

$$\left(\frac{\partial w}{\partial u}\right)^2 = \cosh^2 v\left(\frac{\partial w}{\partial x}\right)^2 + \sinh^2 v\left(\frac{\partial w}{\partial y}\right)^2 + 2\sinh v\cosh v\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$
$$\left(\frac{\partial w}{\partial v}\right)^2 = u^2\sinh^2 v\left(\frac{\partial w}{\partial x}\right)^2 + u^2\cosh^2 v\left(\frac{\partial w}{\partial y}\right)^2 + 2u^2\sinh v\cosh v\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Finally, multiplying the second equation by $1/u^2$ and subtracting the result from the first gives

$$\left(\frac{\partial w}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial w}{\partial v}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2$$

where we have made use of the identity $\cosh^2 v - \sinh^2 v = 1$.

9. Let us define $u = ax + by$. Then

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = a\frac{dz}{du} \qquad\qquad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = b\frac{dz}{du}$$

Multiplying $\partial z/\partial x$ by b, $\partial z/\partial y$ by a and subtracting finally gives

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = ab\frac{dz}{du} - ab\frac{dz}{du} = 0$$

10. (a)
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 2\cot\left(x^2y^2 - 1\right)\left(xy^2dx + x^2ydy\right)$$
and so
$$\frac{\partial z}{\partial x} = 2\cot\left(x^2y^2 - 1\right)xy^2 \qquad \frac{\partial z}{\partial y} = 2\cot\left(x^2y^2 - 1\right)x^2y$$

(b)
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{2xy^2 - 3x^3y^2 - 2xy^4}{\sqrt{1 - x^2 - y^2}}dx + \frac{2x^2y - 3x^2y^3 - 2x^4y}{\sqrt{1 - x^2 - y^2}}dy$$
and so
$$\frac{\partial z}{\partial x} = \frac{2xy^2 - 3x^3y^2 - 2xy^4}{\sqrt{1 - x^2 - y^2}} \qquad \frac{\partial z}{\partial y} = \frac{2x^2y - 3x^2y^3 - 2x^4y}{\sqrt{1 - x^2 - y^2}}$$

(c)
$$2xdx + 4ydy - 2zdz = 0 \implies dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{xdx + 2ydy}{z}$$
and so
$$\frac{\partial z}{\partial x} = \frac{x}{z} \qquad \frac{\partial z}{\partial y} = \frac{2y}{z}$$

11. Let $x' = xt$ and $y' = yt$. Then
$$\begin{aligned}
\frac{\partial}{\partial t}f\left(x', y'\right) &= \frac{\partial}{\partial t}\left(t^n f\left(x, y\right)\right) \\
\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t} &= nt^{n-1}f\left(x, y\right) \\
x\frac{\partial f}{\partial\left(xt\right)} + y\frac{\partial f}{\partial\left(yt\right)} &= nt^{n-1}f\left(x, y\right)
\end{aligned}$$
Setting $t = 1$ then gives
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf\left(x, y\right)$$

12. If $w = F(x, y, z, t)$ and $x = f(t)$, $y = g(t)$ and $z = h(t)$ then
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}\frac{dt}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}$$

Section 2.9

1. (a)

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \left(\frac{\partial y_i}{\partial u_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} u_2 - 3 & u_1 \\ 2u_2 + 2 & 2u_2 + 2u_1 - 1 \end{bmatrix} \begin{bmatrix} \cos 3x_2 & -3x_1 \sin 3x_2 \\ \sin 3x_2 & 3x_1 \cos 3x_2 \end{bmatrix}$$

$$\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x_1=0, x_2=0} = \begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 2 & 0 \end{bmatrix}$$

(b)

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix} = \left(\frac{\partial y_i}{\partial u_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)$$

$$= \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \frac{\partial y_1}{\partial u_3} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \frac{\partial y_2}{\partial u_3} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 - 3 & 2u_2 & 1 \\ 2u_1 + 2 & -2u_2 & -3 \end{bmatrix} \begin{bmatrix} x_2 x_3^2 & x_1 x_3^2 & 2x_1 x_2 x_3 \\ x_2^2 x_3 & 2x_1 x_2 x_3 & x_1 x_2^2 \\ 2x_1 x_2 x_3 & x_1^2 x_3 & x_1^2 x_2 \end{bmatrix}$$

$$\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x_1=1, x_2=1, x_3=1} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 1 \\ -4 & -3 & 3 \end{bmatrix}$$

(c)

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \end{bmatrix} = \left(\frac{\partial y_i}{\partial u_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \\ \frac{\partial y_3}{\partial u_1} & \frac{\partial y_3}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{u_2} & u_1 e^{u_2} \\ e^{-u_2} & -u_1 e^{-u_2} \\ 2u_1 & 0 \end{bmatrix} \begin{bmatrix} 2x_1 & 1 \\ 4x_1 & -1 \end{bmatrix}$$

$$\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x_1=1, x_2=0} = \begin{bmatrix} e^2 & e^2 \\ e^{-2} & -e^{-2} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6e^2 & 0 \\ -2e^{-2} & 2e^{-2} \\ 4 & 2 \end{bmatrix}$$

(d)

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \vdots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} \end{bmatrix} = \left(\frac{\partial y_i}{\partial u_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \dots & \frac{\partial y_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial u_1} & \dots & \frac{\partial y_n}{\partial u_n} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2u_2 & 2u_3 & \dots & 2u_n \\ 2u_1 & 0 & 2u_3 & \dots & 2u_n \\ \vdots & & \ddots & & \vdots \\ 2u_1 & \dots & 2u_{n-2} & 0 & 2u_n \\ 2u_1 & \dots & 2u_{n-2} & 2u_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 2x_1 + x_2 & x_1 \\ 2x_1 + 2x_2 & 2x_1 \\ \vdots & \vdots \\ 2x_1 + nx_2 & nx_1 \end{bmatrix}$$

$$\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x_1=1, x_2=0} = \begin{bmatrix} 0 & 2 & 2 & \dots & 2 \\ 2 & 0 & 2 & \dots & 2 \\ \vdots & & \ddots & & \vdots \\ 2 & \dots & 2 & 0 & 2 \\ 2 & \dots & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ 2 & n \end{bmatrix}$$

$$= \begin{bmatrix} 4(n-1) & n^2 + n - 2 \\ 4(n-1) & n^2 + n - 4 \\ \vdots & \vdots \\ 4(n-1) & n^2 + n - 2n \end{bmatrix}$$

2. (a)

$$
\begin{aligned}
\frac{\partial(z,w)}{\partial(x,y)} &= \begin{vmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial w}{\partial u} & \dfrac{\partial w}{\partial v} \end{vmatrix} \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} 3u^2+6uv+2u & 3u^2-3v^2-2v \\ 3u^2-4u & 3v^2 \end{vmatrix} \\
&\quad \times \begin{vmatrix} \cos xy - xy\sin xy & -x^2\sin xy \\ \sin xy + xy\cos xy + 2x & x^2\cos xy - 2y \end{vmatrix} \\
\left.\frac{\partial(z,w)}{\partial(x,y)}\right|_{x=1,y=0} &= \begin{vmatrix} 11 & -2 \\ -1 & 3 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\
&= 31
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(s,t)} &= \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} \begin{vmatrix} \dfrac{\partial z}{\partial s} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial w}{\partial s} & \dfrac{\partial w}{\partial t} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{z}{\sqrt{z^2+w^2}} & \dfrac{w}{\sqrt{z^2+w^2}} \\ -\dfrac{wz}{(z^2+w^2)^{3/2}} & \dfrac{z^2}{(z^2+w^2)^{3/2}} \end{vmatrix} \begin{vmatrix} -\dfrac{1}{(s+t+1)^2} & -\dfrac{1}{(s+t+1)^2} \\ -\dfrac{2}{(2s-t+1)^2} & \dfrac{1}{(2s-t+1)^2} \end{vmatrix} \\
\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{s=0,t=0} &= \begin{vmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{2^{3/2}} & \dfrac{1}{2^{3/2}} \end{vmatrix} \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} \\
&= -\frac{3}{2}
\end{aligned}
$$

3. (a) Let us consider component $\partial y_1/\partial x_1$ of $\mathbf{y_x} = (\partial y_i/\partial x_j)$:

$$
\begin{aligned}
\frac{\partial y_1}{\partial x_1} &= \frac{\partial y_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial y_1}{\partial u_m}\frac{\partial u_m}{\partial x_1} \\
&= \frac{\partial y_1}{\partial u_1}\left(\frac{\partial u_1}{\partial v_1}\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial u_1}{\partial v_n}\frac{\partial v_n}{\partial x_1}\right) + \cdots + \frac{\partial y_1}{\partial u_m}\left(\frac{\partial u_m}{\partial v_1}\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial u_m}{\partial v_n}\frac{\partial v_n}{\partial x_1}\right) \\
&= \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \cdots & \frac{\partial y_1}{\partial u_m} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial v_1}\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial u_1}{\partial v_n}\frac{\partial v_n}{\partial x_1} \\ \vdots \\ \frac{\partial u_m}{\partial v_1}\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial u_m}{\partial v_n}\frac{\partial v_n}{\partial x_1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \cdots & \frac{\partial y_1}{\partial u_m} \end{pmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial v_1} & \cdots & \frac{\partial u_1}{\partial v_n} \\ \vdots & & \vdots \\ \frac{\partial u_m}{\partial v_1} & \cdots & \frac{\partial u_m}{\partial v_n} \end{bmatrix} \begin{pmatrix} \frac{\partial v_1}{\partial x_1} \\ \vdots \\ \frac{\partial v_n}{\partial x_1} \end{pmatrix} \\
&= \left(\frac{\partial y_1}{\partial u_m}\right)\left(\frac{\partial u_m}{\partial v_n}\right)\left(\frac{\partial v_n}{\partial x_1}\right)
\end{aligned}
$$

The other components of $\mathbf{y_x}$ can be derived in a similar way and so we find that

$$
\begin{aligned}
\mathbf{y_x} = \left(\frac{\partial y_i}{\partial x_j}\right) &= \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \cdots & \frac{\partial y_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial y_i}{\partial u_1} & \cdots & \frac{\partial y_i}{\partial u_m} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial v_1} & \cdots & \frac{\partial u_1}{\partial v_n} \\ \vdots & & \vdots \\ \frac{\partial u_m}{\partial v_1} & \cdots & \frac{\partial u_m}{\partial v_n} \end{bmatrix} \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_j} \\ \vdots & & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_j} \end{bmatrix} \\
&= \mathbf{y_u u_v v_x}
\end{aligned}
$$

(b) We start by finding all components of $\partial(z,w)/\partial(x,y)$:

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right) \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right) \\
\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\right) + \frac{\partial w}{\partial v}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right) \\
\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right) + \frac{\partial w}{\partial v}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right)
\end{aligned}
$$

which subsequently can all be arranged as a matrix multiplication:

$$\frac{\partial z}{\partial x} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \\ \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{pmatrix} \frac{\partial s}{\partial x} \\ \frac{\partial t}{\partial x} \end{pmatrix}$$

$$\frac{\partial z}{\partial y} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \\ \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{pmatrix} \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial y} \end{pmatrix}$$

$$\frac{\partial w}{\partial x} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \\ \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{pmatrix} \frac{\partial s}{\partial x} \\ \frac{\partial t}{\partial x} \end{pmatrix}$$

$$\frac{\partial w}{\partial y} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \\ \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{pmatrix} \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial y} \end{pmatrix}$$

And so we find that

$$\begin{aligned} \frac{\partial(z,w)}{\partial(x,y)} &= \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \\ &= \frac{\partial(z,w)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(s,t)} \frac{\partial(s,t)}{\partial(x,y)} \end{aligned}$$

4.

$$\begin{aligned} \frac{\partial(z,w)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \frac{\partial(z,w)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} &= \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 7 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 26 \\ -8 & 1 \end{bmatrix} \end{aligned}$$

5. Let us start by finding $\mathbf{u_x}$:

$$\mathbf{u_x} = \left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1+2x_2 & -3+2x_1 \\ 2-3x_2 & 5-3x_1 \end{bmatrix}$$

which at $\mathbf{x} = (2, 1)$ reduces to

$$\mathbf{u_x}\big|_{x_1=2, x_2=1} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$$

And so $\mathbf{w_x}$ at $\mathbf{u} = (3, 3)$ and $\mathbf{x} = (2, 1)$ is given by

$$\mathbf{w_x} = \mathbf{w_u u_x} = \begin{bmatrix} 2 & 11 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -9 \\ 16 & 2 \end{bmatrix}$$

6. (a)

$$\begin{aligned} d(\mathbf{u} + \mathbf{v}) = d\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} d(u_1 + v_1) \\ d(u_2 + v_2) \\ d(u_3 + v_3) \end{bmatrix} = \begin{bmatrix} du_1 + dv_1 \\ du_2 + dv_2 \\ du_3 + dv_3 \end{bmatrix} &= \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} + \begin{bmatrix} dv_1 \\ dv_2 \\ dv_3 \end{bmatrix} \\ &= d\mathbf{u} + d\mathbf{v} \end{aligned}$$

(b)

$$\begin{aligned} d(a\mathbf{u} + b\mathbf{v}) = d\left(\begin{bmatrix} au_1 + bv_1 \\ au_2 + bv_2 \\ au_3 + bv_3 \end{bmatrix}\right) &= \begin{bmatrix} d(au_1 + bv_1) \\ d(au_2 + bv_2) \\ d(au_3 + bv_3) \end{bmatrix} = \begin{bmatrix} adu_1 + bdv_1 \\ adu_2 + bdv_2 \\ adu_3 + bdv_3 \end{bmatrix} \\ &= \begin{bmatrix} adu_1 \\ adu_2 \\ adu_3 \end{bmatrix} + \begin{bmatrix} bdv_1 \\ bdv_2 \\ bdv_3 \end{bmatrix} \\ &= ad\mathbf{u} + bd\mathbf{v} \end{aligned}$$

(c)

$$\begin{aligned} d(\mathbf{A}\mathbf{u}) &= d\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = d\left(\begin{bmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_{11}du_1 + a_{12}du_2 + a_{13}du_3 \\ a_{21}du_1 + a_{22}du_2 + a_{23}du_3 \\ a_{31}du_1 + a_{32}du_2 + a_{33}du_3 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} \\ &= \mathbf{A}d\mathbf{u} \end{aligned}$$

(d)

$$\begin{aligned}
d\left(\mathbf{u}\cdot\mathbf{v}\right) &= d\left(u_1v_1+u_2v_2+u_3v_3\right)\\
&= d\left(u_1v_1\right)+d\left(u_2v_2\right)+d\left(u_3v_3\right)\\
&= u_1dv_1+\left(du_1\right)v_1+u_2dv_2+\left(du_2\right)v_2+u_3dv_3+\left(du_3\right)v_3\\
&= \left(u_1dv_1+u_2dv_2+u3dv_3\right)+\left(v_1du_1+v_2du_2+v_3du_3\right)\\
&= \mathbf{u}\cdot d\mathbf{v}+\mathbf{v}\cdot d\mathbf{u}
\end{aligned}$$

(e)

$$\begin{aligned}
d\left(\mathbf{u}\times\mathbf{v}\right) = d\left(\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\right) &= \begin{bmatrix}d\left(u_2v_3-u_3v_2\right)\\d\left(u_3v_1-u_1v_3\right)\\d\left(u_1v_2-u_2v_1\right)\end{bmatrix}\\
&= \begin{bmatrix}u_2dv_3+v_3du_2-u_3dv_2-v_2du_3\\u_3dv_1+v_1du_3-u_1dv_3-v_3du_1\\u_1dv_2+v_2du_1-u_2dv_1-v_1du_2\end{bmatrix}\\
&= \begin{bmatrix}u_2dv_3-u_3dv_2\\u_3dv_1-u_1dv_3\\u_1dv_2-u_2dv_1\end{bmatrix}+\begin{bmatrix}v_3du_2-v_2du_3\\v_1du_3-v_3du_1\\v_2du_1-v_1du_2\end{bmatrix}\\
&= \mathbf{u}\times d\mathbf{v}+d\mathbf{u}\times\mathbf{v}
\end{aligned}$$

Section 2.11

1. (a) We start by defining $F(x,y,z)=2x^2+y^2-z^2-3=0$. Next,

$$F_xdx+F_ydy+F_zdz=4xdx+2ydy-2zdz=0 \implies dz=\frac{2x}{z}dx+\frac{y}{z}dy$$

And so

$$\frac{\partial z}{\partial x}=\frac{2x}{z} \qquad\qquad \frac{\partial z}{\partial y}=\frac{y}{z}$$

(b) We start by defining $F(x,y,z)=xyz+2x^2z+3xz^2-1=0$. Next,

$$F_xdx+F_ydy+F_zdz=\left(yz+4xz+3z^2\right)dx+xzdy+\left(xy+2x^2+6xz\right)dz=0$$

which gives

$$dz=-\frac{yz+4xz+3z^2}{xy+2x^2+6xz}dx-\frac{z}{y+2x+6z}dy$$

And so

$$\frac{\partial z}{\partial x}=-\frac{yz+4xz+3z^2}{xy+2x^2+6xz} \qquad\qquad \frac{\partial z}{\partial y}=-\frac{z}{y+2x+6z}$$

(c) We start by defining $F(x, y, z) = z^3 + xz + 2yz - 1 = 0$. Next,

$$F_x dx + F_y dy + F_z dz = z dx + 2z dy + \left(3z^2 + x + 2y\right) dz = 0$$

which gives

$$dz = -\frac{z dx + 2z dy}{3z^2 + x + 2y}$$

And so

$$\frac{\partial z}{\partial x} = -\frac{z}{3z^2 + x + 2y} \qquad \frac{\partial z}{\partial y} = -\frac{2z}{3z^2 + x + 2y}$$

(d) We start by defining $F(x, y, z) = e^{xz} + e^{yz} + z - 1 = 0$. Next,

$$F_x dx + F_y dy + F_z dz = ze^{xz} dx + ze^{yz} dy + (xe^{xz} + ye^{yz} + 1) dz = 0$$

which gives

$$dz = -\frac{ze^{xz} dx + ze^{yz} dy}{xe^{xz} + ye^{yz} + 1}$$

And so

$$\frac{\partial z}{\partial x} = -\frac{ze^{xz}}{xe^{xz} + ye^{yz} + 1} \qquad -\frac{ze^{yz}}{xe^{xz} + ye^{yz} + 1}$$

2. Let us define $F(x, y, z, u) = 2x + y - 3z - 2u = 0$ and $G(x, y, z, u) = 2x + 2y + z + u = 0$ and the associated system of equations

$$\begin{aligned} F_z dz + F_u du &= -F_x dx - F_y dy \\ G_z dz + G_u du &= -G_x dx - G_y dy \end{aligned}$$

With the above system we associate the determinants

$$D = \begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}, \qquad D_1 = \begin{vmatrix} -F_x dx - F_y dy & F_u \\ -G_x dx - G_y dy & G_u \end{vmatrix}, \qquad D_2 = \begin{vmatrix} F_z & -F_x dx - F_y dy \\ G_z & -F_x dx - F_y dy \end{vmatrix}$$

Hence,

$$dz = -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dy \qquad du = -\frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dx - \frac{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dy$$

Re-arranging then gives

$$dx = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} dz - \frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dy = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} dz - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} dy$$

from which follows

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix}} = -\frac{5}{4}$$

From the second equation we find similarly

$$dy = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} du - \frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} \frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dx = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} du - \frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} dx$$

and so

$$\left(\frac{\partial y}{\partial x}\right)_u = -\frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}} = -\frac{5}{7}$$

Rewriting dz as

$$\begin{aligned}
dz &= -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} \left(-\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} du - \frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} dx \right) \\
&= -\left(\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} \frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} \right) dx + \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} \frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} du \\
&= -\left(\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} \frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} \right) dx + \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} du
\end{aligned}$$

gives

$$\left(\frac{\partial z}{\partial u}\right)_x = \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}} = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}} = -\frac{5}{7}$$

Lastly

$$dy = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}} \frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}} dx - \frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}} dz = -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}} dx - \frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}} dz$$

Hence

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\begin{vmatrix} F_z & F_u \\ G_z & G_u \end{vmatrix}}{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}} = \frac{1}{5}$$

3.

$$du = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dy$$

(a)

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{3x}{u}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -2y & 4v \\ 2y & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{y}{u}$$

(b)

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} u & -y \\ -v & e^v - x \end{vmatrix}}{\begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix}} = \frac{xu + yv - ue^v}{e^{u+v} - xe^u + xe^v - x^2 + y^2}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -v & -y \\ u & e^v - x \end{vmatrix}}{\begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix}} = \frac{ve^v - xv - yu}{e^{u+v} - xe^u + xe^v - x^2 + y^2}$$

(c)

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2x + u & -2yv + u \\ u & -2y \end{vmatrix}}{\begin{vmatrix} x + v & -2yv + u \\ x & -2y \end{vmatrix}} = -\frac{4xy + 2yu - 2yuv + u^2}{2xy + 2yv - 2xyv + xu}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -v^2 & -2yv + u \\ -2v & -2y \end{vmatrix}}{\begin{vmatrix} x + v & -2yv + u \\ x & -2y \end{vmatrix}} = -\frac{2yv^2 - 2uv}{2xy + 2yv - 2xyv + xu}$$

4. Let us define $F(x,y,z,u,v) = x^2+y^2+z^2-u^2+v^2-1=0$ and $G(x,y,z,u,v) = x^2-y^2+z^2+u^2+2v^2-21=0$ and the associated system of equations

$$F_u du + F_v dv = -F_x dx - F_y dy - F_z dz$$
$$G_u du + G_v dv = -G_x dx - G_y dy - G_z dz$$

With the above system we associate the determinants

$$D = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}, \qquad D_1 = \begin{vmatrix} -F_x dx - F_y dy - F_z dz & F_v \\ -G_x dx - G_y dy - G_z dz & G_v \end{vmatrix},$$
$$D_2 = \begin{vmatrix} F_u & -F_x dx - F_y dy - F_z dz \\ G_u & -G_x dx - G_y dy - G_z dz \end{vmatrix}$$

Hence,

$$du = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dy - \frac{\begin{vmatrix} F_z & F_v \\ G_z & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dz = \frac{xv}{3uv} dx + \frac{yv}{uv} dy + \frac{zv}{3uv} dz$$

$$dv = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dx - \frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dy - \frac{\begin{vmatrix} F_u & F_z \\ G_u & G_z \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} dz = -\frac{2xu}{3uv} dx - \frac{2zu}{3uv} dz$$

(a)

$$du\,(1,1,2) = \frac{1}{9}\,(dx + 3dy + 2dz) \qquad dv\,(1,1,2) = -\frac{1}{3}\,(dx + 2dz)$$

(b)

$$\left.\frac{\partial u}{\partial x}\right|_{x=1,y=1,z=2} = \frac{1}{9} \qquad \left.\frac{\partial v}{\partial y}\right|_{x=1,y=1,z=2} = 0$$

(c) Let us start by finding dx, dy and dz:

$$dx = 0.1 \qquad dy = 0.2 \qquad dz = -0.2$$

and so

$$du\,(1,1,2) = \frac{1}{9}\frac{3}{10} = \frac{1}{30} \qquad dv\,(1,1,2) = -\frac{1}{3}\left(-\frac{3}{10}\right) = \frac{1}{10}$$

which gives

$$u\,(1.1, 1.2, 1.8) \approxeq u\,(1,1,2) + du\,(1,1,2) = 3 + \frac{1}{30} \approxeq 3.033$$
$$v\,(1.1, 1.2, 1.8) \approxeq v\,(1,1,2) + dv\,(1,1,2) = 2 + \frac{1}{10} = 2.1$$

5.

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix} & \begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix} \\ \begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix} & \begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{3yu - 4xu + 4x^2 + 9xy + 8xv} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$a = 8uv + 4u^2 - 8xy - 4xu \qquad b = -3u^2 - 8x^2 - 9xu$$
$$c = 4xy - 3y^2 - 6yu - 4xv - 9yv - 4uv - 8v^2 \qquad d = 4x^2 - 3xy + 6xu + 3uv$$

6. (a)

$$\left(\frac{\partial x_1}{\partial x_3}\right)_{x_4} = -\frac{\dfrac{\partial (F_1, F_2)}{\partial (x_3, x_2)}}{\dfrac{\partial (F_1, F_2)}{\partial (x_1, x_2)}} = -\frac{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}} = \frac{1}{2}, \quad \left(\frac{\partial x_1}{\partial x_4}\right)_{x_3} = -\frac{\dfrac{\partial (F_1, F_2)}{\partial (x_4, x_2)}}{\dfrac{\partial (F_1, F_2)}{\partial (x_1, x_2)}} = -\frac{\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}} = -1$$

(b)

$$\left(\frac{\partial x_1}{\partial x_3}\right)_{x_2} = -\frac{\dfrac{\partial (F_1, F_2)}{\partial (x_3, x_4)}}{\dfrac{\partial (F_1, F_2)}{\partial (x_1, x_4)}} = -\frac{\begin{vmatrix} 0 & 2 \\ -1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}} = -1, \quad \left(\frac{\partial x_4}{\partial x_3}\right)_{x_2} = -\frac{\dfrac{\partial (F_1, F_2)}{\partial (x_3, x_1)}}{\dfrac{\partial (F_1, F_2)}{\partial (x_4, x_1)}} = -\frac{\begin{vmatrix} 0 & 3 \\ -1 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}} = \frac{3}{2}$$

(c)

$$\frac{\partial (x_1, x_2)}{\partial (x_3, x_4)} = \begin{vmatrix} \left(\dfrac{\partial x_1}{\partial x_3}\right)_{x_4} & \left(\dfrac{\partial x_1}{\partial x_4}\right)_{x_3} \\ \left(\dfrac{\partial x_2}{\partial x_3}\right)_{x_4} & \left(\dfrac{\partial x_2}{\partial x_4}\right)_{x_3} \end{vmatrix} = \begin{vmatrix} 1/2 & -1 \\ -3/2 & 1 \end{vmatrix} = -1$$

$$\frac{\partial (x_3, x_4)}{\partial (x_1, x_2)} = \begin{vmatrix} \left(\dfrac{\partial x_3}{\partial x_1}\right)_{x_2} & \left(\dfrac{\partial x_3}{\partial x_2}\right)_{x_1} \\ \left(\dfrac{\partial x_4}{\partial x_1}\right)_{x_2} & \left(\dfrac{\partial x_4}{\partial x_2}\right)_{x_1} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3/2 & -1/2 \end{vmatrix} = -1$$

7. If $F(x, y, z) = 0$ then $F_x dx + F_y dy + F_z dz = 0$ and so provided that $F_x \neq 0$, $F_y \neq 0$ and $F_z \neq 0$ at the points considered

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z} \qquad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x} \qquad \left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$$

from which follows

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right) = -1$$

8. (a) The relation

$$\frac{\partial U}{\partial V} - T\frac{\partial p}{\partial T} + p = 0$$

implies that if $dU = adV + bdT$ and $dp = cdV + edT$ are the expressions for dU and dp in terms of dV and dT then

$$\frac{\partial U}{\partial V} = a, \qquad \frac{\partial p}{\partial T} = e$$

Substituting for this in the relation (a) gives $a - Te + p = 0$.

(b) Treating U and V as the independent variables we assume the relations $dT = \alpha dV + \beta dU$ and $dp = \gamma dV + \delta dU$. Solving these two equation for dU and dp in terms of dV and dT gives

$$dU = -\frac{\alpha}{\beta}dV + \frac{1}{\beta}dT \qquad dp = \left(\gamma + \frac{\alpha\delta}{\beta}\right)dV + \frac{\delta}{\beta}dT$$

from which follows $a = -\alpha/\beta$ and $e = \delta/\beta$. Substituting for a and e in the equation $a - Te + p = 0$ from part (a) gives $\alpha + T\delta - p\beta = 0$. Since

$$\alpha = \frac{\partial T}{\partial V} \qquad \delta = \frac{\partial p}{\partial U} \qquad \beta = \frac{\partial T}{\partial U}$$

the equation $\alpha + T\delta - p\beta = 0$ can be written as

$$\frac{\partial T}{\partial V} + T\frac{\partial p}{\partial U} - p\frac{\partial T}{\partial U} = 0$$

(c) Treating V and p as the independent variables we assume the relations $dT = \alpha dp + \beta dV$ and $dU = \gamma dp + \delta dV$. Solving these two equations for dU and dp in terms of dV and dT gives

$$dU = \frac{\gamma}{\alpha}dT + \left(\delta - \frac{\beta\gamma}{\alpha}\right)dV \qquad dp = \frac{1}{\alpha}dT - \frac{\beta}{\alpha}dV$$

from which follows $a = \delta - (\beta\gamma)/\alpha$ and $e = 1/\alpha$. Substituting for a and e in the equation $a - Te + p = 0$ from part (a) gives $\alpha(\delta + p) - \beta\gamma - T = 0$. Since

$$\alpha = \frac{\partial T}{\partial p} \qquad \beta = \frac{\partial T}{\partial V} \qquad \gamma = \frac{\partial U}{\partial p} \qquad \delta = \frac{\partial U}{\partial V}$$

the equation $\alpha(\delta + p) - \beta\gamma - T = 0$ can be written as

$$T - p\frac{\partial T}{\partial p} + \left(\frac{\partial T}{\partial V}\frac{\partial U}{\partial p} - \frac{\partial T}{\partial p}\frac{\partial U}{\partial V}\right) = T - p\frac{\partial T}{\partial p} + \frac{\partial\,(T, U)}{\partial\,(V, p)} = 0$$

(d) Treating p and T as the independent variables we assume the relations $dV = \alpha dp + \beta dT$ and $dU = \gamma dp + \delta dT$. Solving these two equations for dU and dp in terms of dV and dT gives

$$dU = \frac{\gamma}{\alpha}dV + \left(\delta - \frac{\beta\gamma}{\alpha}\right)dT \qquad\qquad dp = \frac{1}{\alpha}dV - \frac{\beta}{\alpha}dT$$

from which follows $a = \gamma/\alpha$ and $e = -\beta/\alpha$. Substituting for a and e in the equation $a - Te + p = 0$ from part (a) gives $\gamma + T\beta + p\alpha = 0$. Since

$$\alpha = \frac{\partial V}{\partial p} \qquad\qquad \beta = \frac{\partial V}{\partial T} \qquad\qquad \gamma = \frac{\partial U}{\partial p}$$

the equation $\gamma + T\beta + p\alpha = 0$ can be written as

$$\frac{\partial U}{\partial p} + T\frac{\partial V}{\partial T} + p\frac{\partial V}{\partial p} = 0$$

(e) Treating U and p as the independent variables we assume the relations $dT = \alpha dp + \beta dU$ and $dV = \gamma dp + \delta dU$. Solving these two equations for dU and dp in terms of dV and dT gives

$$dU = \frac{1}{\beta\gamma - \alpha\delta}\left(\gamma dT - \alpha dV\right) \qquad\qquad dp = \frac{1}{\beta\gamma - \alpha\delta}\left(-\delta dT + \beta dV\right)$$

from which follows $a = \alpha/(\alpha\delta - \beta\gamma)$ and $e = \delta/(\alpha\delta - \beta\gamma)$. Substituting for a and e in the equation $a - Te + p = 0$ from part (a) gives $\alpha - T\delta + p(\alpha\delta - \beta\gamma) = 0$. Since

$$\alpha = \frac{\partial T}{\partial p} \qquad\qquad \beta = \frac{\partial T}{\partial U} \qquad\qquad \gamma = \frac{\partial V}{\partial p} \qquad\qquad \delta = \frac{\partial V}{\partial U}$$

the equation $\alpha - T\delta + p(\alpha\delta - \beta\gamma) = 0$ can be written as

$$\frac{\partial T}{\partial p} - T\frac{\partial V}{\partial U} + p\left(\frac{\partial T}{\partial p}\frac{\partial V}{\partial U} - \frac{\partial T}{\partial U}\frac{\partial V}{\partial p}\right) = \frac{\partial T}{\partial p} - T\frac{\partial V}{\partial U} + p\frac{\partial\left(V,T\right)}{\partial\left(U,p\right)} = 0$$

(f) Treating T and U as the independent variables we assume the relations $dV = \alpha dT + \beta dU$ and $dp = \gamma dT + \delta dU$. Solving these two equations for dU and dp in terms of dV and dT gives

$$dU = \frac{1}{\beta}dV - \frac{\alpha}{\beta}dT \qquad\qquad dp = \frac{\delta}{\beta}dV + \left(\gamma - \frac{\alpha\delta}{\beta}\right)dT$$

from which follows $a = 1/\beta$ and $e = \gamma - (\alpha\delta)/\beta$. Substituting for a and e in the equation $a - Te + p = 0$ from part (a) gives $T(\beta\gamma - \alpha\delta) - p\beta - 1 = 0$. Since

$$\alpha = \frac{\partial V}{\partial T} \qquad\qquad \beta = \frac{\partial V}{\partial U} \qquad\qquad \gamma = \frac{\partial p}{\partial T} \qquad\qquad \delta = \frac{\partial p}{\partial U}$$

the equation $T(\beta\gamma - \alpha\delta) - p\beta - 1 = 0$ can be written as

$$T\left(\frac{\partial V}{\partial U}\frac{\partial p}{\partial T} - \frac{\partial V}{\partial T}\frac{\partial p}{\partial U}\right) - p\frac{\partial V}{\partial U} - 1 = T\frac{\partial\left(p,V\right)}{\partial\left(T,U\right)} - p\frac{\partial V}{\partial U} - 1 = 0$$

Section 2.12

1. Using the relations $x = r\cos\theta$ and $y = r\sin\theta$ we find $F(x,y,r) = x^2 + y^2 - r^2 = 0$ and $G(x,y,\theta) = y - x\tan\theta = 0$.

   (a)

$$dx = -\frac{\begin{vmatrix} F_r & F_y \\ G_r & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} dr - \frac{\begin{vmatrix} F_\theta & F_y \\ G_\theta & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} d\theta = \frac{1+0}{\cos\theta + \sin\theta\tan\theta} dr - \frac{0 + r\tan\theta}{\cos\theta + \sin\theta\tan\theta} d\theta$$
$$= \cos\theta dr - r\sin\theta d\theta$$
$$dy = -\frac{\begin{vmatrix} F_x & F_r \\ G_x & G_r \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} dr - \frac{\begin{vmatrix} F_x & F_\theta \\ G_x & G_\theta \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} d\theta = \frac{0+\tan\theta}{\cos\theta + \sin\theta\tan\theta} dr + \frac{r}{\cos\theta + \sin\theta\tan\theta} d\theta$$
$$= \sin\theta dr + r\cos\theta d\theta$$

   (b)

$$dr = -\frac{\begin{vmatrix} F_x & F_\theta \\ G_x & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_\theta \\ G_y & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} dy = \frac{1+0}{\sec\theta - 0} dx + \frac{\tan\theta + 0}{\sec\theta - 0} dy$$
$$= \cos\theta dx + \sin\theta dy$$
$$d\theta = -\frac{\begin{vmatrix} F_r & F_x \\ G_r & G_x \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} dx - \frac{\begin{vmatrix} F_r & F_y \\ G_r & G_y \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} dy = -\frac{\tan\theta - 0}{r\sec\theta - 0} dx + \frac{1+0}{r\sec\theta - 0} dy$$
$$= -\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy$$

   (c)

$$dx = \cos\theta dr - r\sin\theta d\theta \implies \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$
$$dr = \cos\theta dx + \sin\theta dy \iff dx = \sec\theta dr - \tan\theta dy \implies \left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$$
$$dr = \cos\theta dx + \sin\theta dy \implies \left(\frac{\partial r}{\partial x}\right)_y = \cos\theta$$

$$dr = \cos\theta dx + \sin\theta\left(\sin\theta dr + r\cos\theta d\theta\right) \iff$$

$$dr = \sec\theta dx + r\tan\theta d\theta \implies \left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta$$

(d)

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \left(\frac{\partial x}{\partial r}\right)_\theta & \left(\frac{\partial x}{\partial \theta}\right)_r \\ \left(\frac{\partial y}{\partial r}\right)_\theta & \left(\frac{\partial y}{\partial \theta}\right)_r \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \left(\frac{\partial r}{\partial x}\right)_y & \left(\frac{\partial r}{\partial y}\right)_x \\ \left(\frac{\partial \theta}{\partial x}\right)_y & \left(\frac{\partial \theta}{\partial y}\right)_x \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -(\sin\theta)/r & (\cos\theta)/r \end{vmatrix} = \frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r} = \frac{1}{r}$$

Note that

$$\frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}\begin{vmatrix} \cos\theta & \sin\theta \\ -(\sin\theta)/r & (\cos\theta)/r \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence, the Jacobian of the inverse mapping is the reciprocal of the Jacobian of the mapping.

2. (a)

$$u = \frac{1}{5}x + \frac{2}{5}y \qquad v = -\frac{2}{5}x + \frac{1}{5}y$$

(b)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \left(\frac{\partial x}{\partial u}\right)_v & \left(\frac{\partial x}{\partial v}\right)_u \\ \left(\frac{\partial y}{\partial u}\right)_v & \left(\frac{\partial y}{\partial v}\right)_u \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial v}{\partial x}\right)_y & \left(\frac{\partial v}{\partial y}\right)_x \end{vmatrix} = \begin{vmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{vmatrix} = \frac{1}{5}$$

3. (a)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \left(\frac{\partial x}{\partial u}\right)_v & \left(\frac{\partial x}{\partial v}\right)_u \\ \left(\frac{\partial y}{\partial u}\right)_v & \left(\frac{\partial y}{\partial v}\right)_u \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right)$$

(b) We start by computing

$$dx = 2udu - 2vdv \qquad\qquad dy = 2udv + 2vdu$$

Solving the second equation for du and inserting for du in the first equation results in

$$dx = 2u\left(\frac{dy - 2udv}{2v}\right) - 2vdu = \frac{u}{v}dy - 2\left(\frac{u^2}{v} + v\right)dv$$

Rearranging gives

$$dv = \frac{v}{2\left(u^2+v^2\right)}\left(-dx + \frac{u}{v}dy\right)$$

From which we can read of

$$\left(\frac{dv}{dx}\right)_y = -\frac{v}{2\left(u^2+v^2\right)}$$

Similarly, solving the second equation for dv and inserting for dv in the first equation gives

$$du = \frac{u}{2\left(u^2+v^2\right)}\left(dx + \frac{v}{u}dy\right)$$

And so

$$\left(\frac{du}{dx}\right)_y = \frac{u}{2\left(u^2+v^2\right)}$$

4. If $J = \partial(x,y)/\partial(u,v)$ then

$$\frac{1}{J} = \frac{\partial\left(u,v\right)}{\partial\left(x,y\right)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Hence,

$$\begin{aligned}
\frac{1}{J}\frac{\partial y}{\partial v} &= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - 0\cdot\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} \\
-\frac{1}{J}\frac{\partial x}{\partial v} &= \left(-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial v} = \left(-\frac{\partial u}{\partial x}\frac{\partial x}{\partial v}\right)\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} = 0\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \\
-\frac{1}{J}\frac{\partial y}{\partial u} &= \left(-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\frac{\partial y}{\partial u} = \left(-\frac{\partial u}{\partial x}\frac{\partial y}{\partial u}\right)\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = 0\cdot\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \\
\frac{1}{J}\frac{\partial x}{\partial u} &= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial u}\right) = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot 0 = \frac{\partial v}{\partial y}
\end{aligned}$$

Using the results from Problem 1 we find

$$\begin{aligned}
\frac{1}{J}\frac{\partial y}{\partial \theta} &= \frac{1}{r}\left(r\cos\theta\right) = \cos\theta = \frac{\partial r}{\partial x} & -\frac{1}{J}\frac{\partial x}{\partial \theta} &= -\frac{1}{r}\left(-r\sin\theta\right) = \sin\theta = \frac{\partial r}{\partial y} \\
-\frac{1}{J}\frac{\partial y}{\partial r} &= -\frac{1}{r}\sin\theta = \frac{\partial \theta}{\partial x} & \frac{1}{J}\frac{\partial x}{\partial r} &= \frac{1}{r}\cos\theta = \frac{\partial \theta}{\partial y}
\end{aligned}$$

5. Let us start by defining $1/J$ explicitly by expanding the determinant along the first row, which gives

$$\begin{aligned}\frac{1}{J} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} &= (v_y w_z - v_z w_y)\, u_x - (v_x w_z - v_z w_x)\, u_y + (v_x w_y - v_y w_x)\, u_z \\ &= \frac{\partial (v,w)}{\partial (y,z)} u_x - \frac{\partial (v,w)}{\partial (x,z)} u_y + \frac{\partial (v,w)}{\partial (x,y)} u_z\end{aligned}$$

And so

$$\begin{aligned}\frac{1}{J}\frac{\partial (y,z)}{\partial (v,w)} &= \frac{\partial (v,w)}{\partial (y,z)}\frac{\partial (y,z)}{\partial (v,w)} u_x - \frac{\partial (v,w)}{\partial (x,z)}\frac{\partial (y,z)}{\partial (v,w)} u_y + \frac{\partial (v,w)}{\partial (x,y)}\frac{\partial (y,z)}{\partial (v,w)} u_z \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{\partial u}{\partial x} - (0)\,\frac{\partial u}{\partial y} + (0)\,\frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial x} \\ \frac{1}{J}\frac{\partial (z,x)}{\partial (v,w)} &= \frac{\partial (v,w)}{\partial (y,z)}\frac{\partial (z,x)}{\partial (v,w)} u_x - \frac{\partial (v,w)}{\partial (x,z)}\frac{\partial (z,x)}{\partial (v,w)} u_y + \frac{\partial (v,w)}{\partial (x,y)}\frac{\partial (z,x)}{\partial (v,w)} u_z \\ &= (0)\,\frac{\partial u}{\partial x} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{\partial u}{\partial y} + (0)\,\frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial y} \\ \frac{1}{J}\frac{\partial (x,y)}{\partial (v,w)} &= \frac{\partial (v,w)}{\partial (y,z)}\frac{\partial (x,y)}{\partial (v,w)} u_x - \frac{\partial (v,w)}{\partial (x,z)}\frac{\partial (x,y)}{\partial (v,w)} u_y + \frac{\partial (v,w)}{\partial (x,y)}\frac{\partial (x,y)}{\partial (v,w)} u_z \\ &= (0)\,\frac{\partial u}{\partial x} - (0)\,\frac{\partial u}{\partial y} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial z}\end{aligned}$$

Similarly, expanding $1/J$ across the second row gives

$$\begin{aligned}\frac{1}{J} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} &= -(u_y w_z - u_z w_y)\, v_x + (u_x w_z - u_z w_x)\, v_y - (u_x w_y - u_y w_x)\, v_z \\ &= -\frac{\partial (u,w)}{\partial (y,z)} v_x + \frac{\partial (u,w)}{\partial (x,z)} v_y - \frac{\partial (u,w)}{\partial (x,y)} v_z\end{aligned}$$

And so

$$
\begin{aligned}
\frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)} &= -\frac{\partial(u,w)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(w,u)}v_x + \frac{\partial(u,w)}{\partial(x,z)}\frac{\partial(y,z)}{\partial(w,u)}v_y - \frac{\partial(u,w)}{\partial(x,y)}\frac{\partial(y,z)}{\partial(w,u)}v_z \\
&= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial v}{\partial x} + (0)\frac{\partial v}{\partial y} - (0)\frac{\partial v}{\partial z} \\
&= \frac{\partial v}{\partial x} \\
\frac{1}{J}\frac{\partial(z,x)}{\partial(w,u)} &= -\frac{\partial(u,w)}{\partial(y,z)}\frac{\partial(z,x)}{\partial(w,u)}v_x + \frac{\partial(u,w)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(w,u)}v_y - \frac{\partial(u,w)}{\partial(x,y)}\frac{\partial(z,x)}{\partial(w,u)}v_z \\
&= (0)\frac{\partial v}{\partial x} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial v}{\partial y} - (0)\frac{\partial v}{\partial z} \\
&= \frac{\partial v}{\partial y} \\
\frac{1}{J}\frac{\partial(x,y)}{\partial(w,u)} &= -\frac{\partial(u,w)}{\partial(y,z)}\frac{\partial(x,y)}{\partial(w,u)}v_x + \frac{\partial(u,w)}{\partial(x,z)}\frac{\partial(x,y)}{\partial(w,u)}v_y - \frac{\partial(u,w)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(w,u)}v_z \\
&= (0)\frac{\partial v}{\partial x} + (0)\frac{\partial v}{\partial y} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial v}{\partial z} \\
&= \frac{\partial v}{\partial z}
\end{aligned}
$$

Lastly, expanding $1/J$ across the third row gives

$$
\begin{aligned}
\frac{1}{J} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} &= (u_y v_z - u_z v_y)\,w_x - (u_x v_z - u_z v_x)\,w_y + (u_x v_y - u_y v_x)\,w_z \\
&= \frac{\partial(u,v)}{\partial(y,z)}w_x - \frac{\partial(u,v)}{\partial(x,z)}w_y + \frac{\partial(u,v)}{\partial(x,y)}w_z
\end{aligned}
$$

And so

$$
\begin{aligned}
\frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)} &= \frac{\partial(u,v)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(u,v)}w_x - \frac{\partial(u,v)}{\partial(x,z)}\frac{\partial(y,z)}{\partial(u,v)}w_y + \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(y,z)}{\partial(u,v)}w_z \\
&= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial w}{\partial x} - (0)\,\frac{\partial w}{\partial y} + (0)\,\frac{\partial w}{\partial z} \\
&= \frac{\partial w}{\partial x} \\
\frac{1}{J}\frac{\partial(z,x)}{\partial(u,v)} &= \frac{\partial(u,v)}{\partial(y,z)}\frac{\partial(z,x)}{\partial(u,v)}w_x - \frac{\partial(u,v)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(u,v)}w_y + \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(z,x)}{\partial(u,v)}w_z \\
&= (0)\,\frac{\partial w}{\partial x} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial w}{\partial y} + (0)\,\frac{\partial w}{\partial z} \\
&= \frac{\partial w}{\partial y} \\
\frac{1}{J}\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial(u,v)}{\partial(y,z)}\frac{\partial(x,y)}{\partial(u,v)}w_x - \frac{\partial(u,v)}{\partial(x,z)}\frac{\partial(x,y)}{\partial(u,v)}w_y + \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}w_z \\
&= (0)\,\frac{\partial w}{\partial x} - (0)\,\frac{\partial w}{\partial y} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial w}{\partial z} \\
&= \frac{\partial w}{\partial z}
\end{aligned}
$$

6. (a) The Jacobian J is given by

$$
\begin{aligned}
J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \rho^2\sin\phi
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial\rho}{\partial y} &= \frac{1}{J}\frac{\partial(z,x)}{\partial(\phi,\theta)} = \frac{1}{J}\left(z_\phi x_\theta - z_\theta x_\phi\right) = \frac{1}{\rho^2\sin\phi}\left(\rho^2\sin^2\phi\sin\theta\right) = \sin\phi\sin\theta \\
\frac{\partial\phi}{\partial z} &= \frac{1}{J}\frac{\partial(x,y)}{\partial(\theta,\rho)} = \frac{1}{J}\left(x_\theta y_\rho - x_\rho y_\theta\right) = \frac{1}{\rho^2\sin\phi}\left(-\rho\sin^2\phi\right) = -\frac{\sin\phi}{\rho} \\
\frac{\partial\theta}{\partial x} &= \frac{1}{J}\frac{\partial(y,z)}{\partial(\rho,\phi)} = \frac{1}{J}\left(y_\rho z_\phi - y_\phi z_\rho\right) = \frac{1}{\rho^2\sin\phi}\left(-\rho\sin\theta\right) = -\frac{\sin\theta}{\rho\sin\phi}
\end{aligned}
$$

7. Let $F(x,y,u,v) = f(u,v) - x = 0$ and $G(x,y,u,v) = g(u,v) - y = 0$, so that (see

section 2.10)

$$\left(\frac{\partial x}{\partial u}\right)_v = -\frac{\dfrac{\partial (F,G)}{\partial (u,y)}}{\dfrac{\partial (F,G)}{\partial (x,y)}} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}} = f_u$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\dfrac{\partial (F,G)}{\partial (x,v)}}{\dfrac{\partial (F,G)}{\partial (u,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{g_v}{f_u g_v - f_v g_u}$$

and

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{\dfrac{\partial (F,G)}{\partial (v,x)}}{\dfrac{\partial (F,G)}{\partial (y,x)}} = -\frac{\begin{vmatrix} f_v & -1 \\ g_v & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}} = g_v$$

$$\left(\frac{\partial v}{\partial y}\right)_x = -\frac{\dfrac{\partial (F,G)}{\partial (y,u)}}{\dfrac{\partial (F,G)}{\partial (v,u)}} = -\frac{\begin{vmatrix} 0 & f_u \\ -1 & g_u \end{vmatrix}}{\begin{vmatrix} f_v & f_u \\ g_v & g_u \end{vmatrix}} = -\frac{f_u}{f_v g_u - f_u g_v}$$

Hence,

$$\left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y = \frac{f_u g_v}{f_u g_v - f_v g_u} = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

Similarly

$$\left(\frac{\partial x}{\partial v}\right)_u = -\frac{\dfrac{\partial (F,G)}{\partial (v,y)}}{\dfrac{\partial (F,G)}{\partial (x,y)}} = -\frac{\begin{vmatrix} f_v & 0 \\ g_v & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}} = f_v$$

$$\left(\frac{\partial v}{\partial x}\right)_y = -\frac{\dfrac{\partial (F,G)}{\partial (x,u)}}{\dfrac{\partial (F,G)}{\partial (v,u)}} = -\frac{\begin{vmatrix} -1 & f_u \\ 0 & g_u \end{vmatrix}}{\begin{vmatrix} f_v & f_u \\ g_v & g_u \end{vmatrix}} = \frac{g_u}{f_v g_u - f_u g_v}$$

and

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = -\frac{f_v}{f_u g_v - f_v g_u}$$

$$\left(\frac{\partial y}{\partial u}\right)_v = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(y,x)}} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}} = g_u$$

So that again

$$\left(\frac{\partial x}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y = \frac{f_v g_u}{f_v g_u - f_u g_v} = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

Lastly

$$\left(\frac{\partial x}{\partial y}\right)_u = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(x,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}} = \frac{f_v}{g_v} \quad \left(\frac{\partial y}{\partial x}\right)_u = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(y,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}} = \frac{g_v}{f_v}$$

which implies

$$\left(\frac{\partial x}{\partial y}\right)_u \left(\frac{\partial y}{\partial x}\right)_u = \frac{f_v}{g_v}\frac{g_v}{f_v} = 1$$

8. (a) If unique polar coordinates $(r, \phi_1, \ldots, \phi_n)$ are to be assigned to $\mathbf{x} = (x_1, \ldots, x_n)$, $n \geq 3$ then it is is trivial to see that $\mathbf{x} \neq \mathbf{0}$, since this would imply $\sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{|x|^2} = r = 0$ and hence $\phi_1, \ldots, \phi_{n-1}$ can take on any arbitrary value. Next, consider $\phi_1 = \cos^{-1}(x_1/r)$, $0 \leq \phi_1 \leq \pi$ and choose $\phi_1 = 0$ or $\phi_1 = \pi$. This would imply $|x_1| = r$ and hence, $x_2 = x_3 = \cdots = 0$, which contradicts the assumption that unique polar coordinates are assigned to $\mathbf{x}$ whenever (x_{n-1}, x_n) is not $(0, 0)$, since then $\phi_2, \ldots, \phi_{n-1}$ can be chosen arbitrarily. Thus we conclude that if unique polar coordinates are to be assigned to $\mathbf{x}$, $n \geq 3$ then $0 < \phi_1 < \pi$.

   Similarly, consider $\phi_2 = \cos^{-1}(x_2/(r \sin\phi_1))$, $0 \leq \phi_2 \leq \pi$ and choose $\phi_2 = 0$ or $\phi_2 = \pi$. This would imply $|x_2| = r\sin\phi_1$, which for $\phi_1 = \pi/2$ gives $x_1 = x_3 = x_4 = \cdots = 0$. Again, this contradicts the assumption that unique polar coordinates are assigned to $\mathbf{x}$ whenever (x_{n-1}, x_n) is not $(0, 0)$, since $\phi_3, \ldots, \phi_{n-1}$ can be chosen arbitrarily. In conclusion, if unique polar coordinates are to be assigned to $\mathbf{x}$, $n \geq 3$ then $0 < \phi_2 < \pi$.

   By induction, we can prove that a similar reasoning holds for all ϕ up to ϕ_{n-2}. Consider $\phi_{n-2} = \cos^{-1}(x_{n-2}/(r \sin\phi_1 \ldots \sin\phi_{n-3}))$, $0 \leq \phi_{n-2} \leq \pi$ and choose $\phi_{n-2} = 0$ or $\phi_{n-2} = \pi$. This would imply $|x_{n-2}| = r\sin\phi_1 \ldots \sin\phi_{n-3}$, which for $\phi_1 = \cdots = \phi_{n-3} = \pi/2$ gives $x_1 = x_2 = \cdots = x_{n-3} = x_{n-1} = x_n = 0$.

This contradicts the assumption that unique polar coordinates are assigned to $\mathbf{x}$ whenever (x_{n-1}, x_n) is not $(0,0)$, since ϕ_{n-1} could be chosen arbitrarily. Hence, if unique polar coordinates are to be assigned to $\mathbf{x}$, $n \geq 3$ then $0 < \phi_{n-2} < \pi$ and more generally, we conclude $0 < \phi_1, \ldots, \phi_{n-2} < \pi$.

Lastly, consider $\phi_{n-1} = \cos^{-1}(x_{n-1}/(r \sin\phi_1 \ldots \sin\phi_{n-2}))$, $0 \leq \phi_{n-1} \leq 2\pi$ and choose $\phi_{n-1} = 0$ or $\phi_{n-1} = 2\pi$. This would imply $x_{n-1} = r\sin\phi_1 \ldots \sin\phi_{n-2}$, which for $\phi_1 = \cdots = \phi_{n-2} = \pi/2$ gives $x_{n-1} = r$ and so $x_1 = x_2 = \cdots = x_{n-2} = x_n = 0$. This contradicts the assumption that unique polar coordinates are assigned to $\mathbf{x}$ whenever (x_{n-1}, x_n) is not $(c, 0)$ with $c = r > 0$ for $\phi_1 = \cdots = \phi_{n-2} = \pi/2$, since $\phi_{n-1} = 0$ and $\phi_{n-1} = 2\pi$ results in the same value for x_{n-1}. Thus we conclude that if unique polar coordinates are to be assigned to $\mathbf{x}$, $n \geq 3$ then $0 < \phi_{n-1} < 2\pi$.

(b) The Jacobian J_n can be written as

$$J_n = \frac{\partial(x_1, \ldots, x_n)}{\partial(r, \phi_1, \ldots, \phi_{n-1})} = \frac{\partial(x_1, \rho, \phi_2, \ldots, \phi_{n-1})}{\partial(r, \phi_1, \ldots, \phi_{n-1})} \frac{\partial(x_1, \ldots, x_n)}{\partial(x_1, \rho, \phi_2, \ldots, \phi_{n-1})}$$

where $x_1 = r\cos\phi_1$ and $\rho = r\sin\phi_1$. The first term on the right hand side is of the form

$$\begin{aligned}
\frac{\partial(x_1, \rho, \phi_2, \ldots, \phi_{n-1})}{\partial(r, \phi_1, \ldots, \phi_{n-1})} &= \begin{vmatrix} \cos\phi_1 & -r\sin\phi_1 & 0 & 0 & \cdots & 0 \\ \sin\phi_1 & r\cos\phi_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ & & & & & \\ & & & & & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} r\cos^2\phi_1 + \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} r\sin^2\phi_1 + 0 + \cdots + 0 \\
&= r
\end{aligned}$$

The second term on the right hand side is of the form

$$J_{n-1} = \begin{vmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cos\phi_2 & -\rho\sin\phi_2 & 0 & \cdots & \cdots & 0 \\ 0 & \sin\phi_2\cos\phi_3 & \rho\cos\phi_2\cos\phi_3 & -\rho\sin\phi_2\sin\phi_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_4 & 0 \\ 0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \cdots & \beta_4 \\ 0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & \gamma_4 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi_2 & -\sin\phi_2 & 0 & \cdots & \cdots & 0 \\ \sin\phi_2\cos\phi_3 & \cos\phi_2\cos\phi_3 & -\sin\phi_2\sin\phi_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_4 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \cdots & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & \gamma_4 \end{vmatrix} \rho^{n-2}$$

where

$\alpha_1 = \sin\phi_2 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$ $\qquad$ $\alpha_2 = \cos\phi_2\sin\phi_3 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$

$\alpha_3 = \sin\phi_2\cos\phi_3\sin\phi_4 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$ $\qquad$ $\alpha_4 = -\sin\phi_2 \cdots \sin\phi_{n-3}\sin\phi_{n-2}$

$\beta_1 = \sin\phi_2 \cdots \sin\phi_{n-2}\cos_{n-1}$ $\qquad$ $\beta_2 = \cos\phi_2\sin\phi_3 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$

$\beta_3 = \sin\phi_2\cos\phi_3\sin\phi_4 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$ $\qquad$ $\beta_4 = -\sin\phi_2 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$

$\gamma_1 = \sin\phi_2 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$ $\qquad$ $\gamma_2 = \cos\phi_2\sin\phi_3 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$

$\gamma_3 = \sin\phi_2\cos\phi_3\sin\phi_4 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$ $\qquad$ $\gamma_4 = \sin\phi_2 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$

Expanding J_{n-1} along the first row gives

$$J_{n-1} = \begin{vmatrix} \cos\phi_3 & -\sin\phi_3 & 0 & \cdots & \cdots & 0 \\ \sin\phi_3\cos\phi_4 & \cos\phi_3\cos\phi_4 & -\sin\phi_3\sin\phi_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \alpha_1^* & \alpha_2^* & \alpha_3^* & \cdots & \alpha_4^* & 0 \\ \beta_1^* & \beta_2^* & \beta_3^* & \cdots & \cdots & \beta_4^* \\ \gamma_1^* & \gamma_2^* & \gamma_3^* & \cdots & \cdots & \gamma_4^* \end{vmatrix} \rho^{n-2}\sin^{n-3}\phi_2$$

where we have made use of the identity $\sin^2\theta + \cos^2\theta = 1$ and repeated application of rule III for determinants from section 1.4 and where

$\alpha_1^* = \sin\phi_3 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$ $\qquad$ $\alpha_2^* = \cos\phi_3\sin\phi_4 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$

$\alpha_3^* = \sin\phi_3\cos\phi_4\sin\phi_5 \cdots \sin\phi_{n-3}\cos\phi_{n-2}$ $\qquad$ $\alpha_4^* = -\sin\phi_3 \cdots \sin\phi_{n-3}\sin\phi_{n-2}$

$$\beta_1^* = \sin\phi_3 \cdots \sin\phi_{n-2}\cos\phi_{n-1} \qquad \beta_2^* = \cos\phi_3 \sin\phi_4 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$$
$$\beta_3^* = \sin\phi_3 \cos\phi_4 \sin\phi_5 \cdots \sin\phi_{n-2}\cos\phi_{n-1} \quad \beta_4^* = -\sin\phi_3 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$$

$$\gamma_1^* = \sin\phi_3 \cdots \sin\phi_{n-2}\sin\phi_{n-1} \qquad \gamma_2^* = \cos\phi_3 \sin\phi_4 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$$
$$\gamma_3^* = \sin\phi_3 \cos\phi_4 \sin\phi_5 \cdots \sin\phi_{n-2}\sin\phi_{n-1} \quad \gamma_4^* = \sin\phi_3 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$$

Repeating the same procedure of expanding the remaining determinant and making use of the identity $\sin^2\theta + \cos^2\theta$ and rule III from section 1.4 will then eventually result in

$$\begin{aligned} \frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\rho,\phi_2,\ldots,\phi_{n-1})} = J_{n-1} &= \rho^{n-2}\sin^{n-3}\phi_2 \cdots \sin\phi_{n-2} \begin{vmatrix} \cos\phi_{n-1} & -\sin\phi_{n-1} \\ \sin\phi_{n-1} & \cos\phi_{n-1} \end{vmatrix} \\ &= \rho^{n-2}\sin^{n-3}\phi_2 \cdots \sin\phi_{n-2} \end{aligned}$$

And so finally we find

$$\begin{aligned} J_n = \frac{\partial(x_1,\ldots,x_n)}{\partial(r,\phi_1,\ldots,\phi_{n-1})} &= \frac{\partial(x_1,\rho,\phi_2,\ldots,\phi_{n-1})}{\partial(r,\phi_1,\ldots,\phi_{n-1})}\frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\rho,\phi_2,\ldots,\phi_{n-1})} \\ &= r J_{n-1} \\ &= r^{n-1}\sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2} \end{aligned}$$

Section 2.13

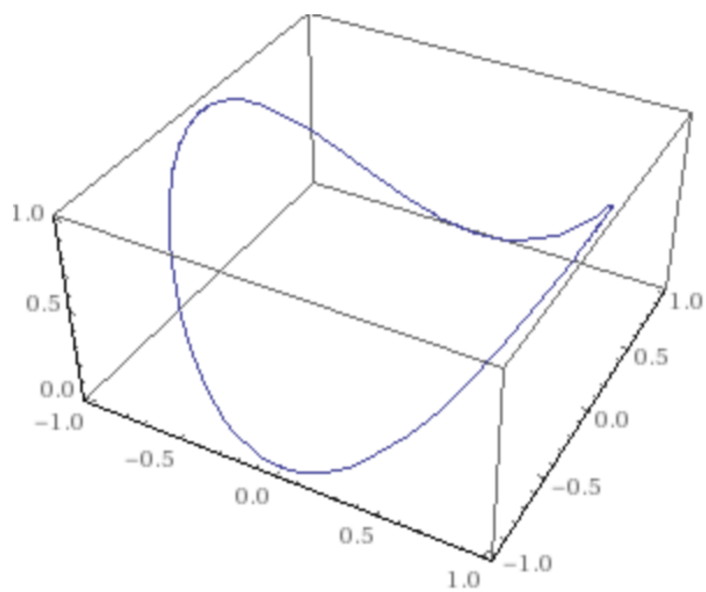

Figure 13: $x = \sin t,\ y = \cos t,\ z = \sin^2 t$

1. (a)

(b) Using the equation $\overrightarrow{P_1P} = (t - t_1)\mathbf{r}'(t_1)$ with $t_1 = \pi/3$ we find

$$x = \frac{1}{2}\left(t - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} \qquad y = -\frac{\sqrt{3}}{2}\left(t - \frac{\pi}{3}\right) + \frac{1}{2} \qquad z = \frac{\sqrt{3}}{2}\left(t - \frac{\pi}{3}\right) + \frac{3}{4}$$

(c) Firstly, note that the velocity vector $\mathbf{v} = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}$ will be tangent to the curve. If we are looking for a plane cutting the curve at right angles at the point P we can use the equation $\mathbf{n} \cdot \overrightarrow{P_1P} = 0$ where we let $\mathbf{n} = \mathbf{v}$ and $P_1 = (\sqrt{3}/2, 1/2, 3/4)$, i.e. the point P for which $t = \pi/3$ from the previous question. Hence, we find

$$\begin{aligned}
\mathbf{v} \cdot \overrightarrow{P_1P} &= \left(\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}\right) \cdot \left[\left(x - \frac{\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{1}{2}\right)\mathbf{j} + \left(z - \frac{3}{4}\right)\mathbf{k}\right] \\
&= \frac{1}{2}\left(x - \sqrt{3}y + \sqrt{3}z - \frac{3\sqrt{3}}{4}\right)
\end{aligned}$$

And so an equation for a plane cutting the curve at right angles at P is given by

$$x - \sqrt{3}y + \sqrt{3}z - \frac{3\sqrt{3}}{4} = 0$$

2. (a) If the curve from problem 1(b) lies in the surface $x^2 + 2y^2 + z = 2$ it should hold that

$$F(x, y, z) = F[f(t), g(t), h(t)] = 0$$

For the curve from problem 1(b) this implies

$$F[f(t), g(t), h(t)] = \sin^2 t + 2\cos^2 t + \sin^2 t - 2 = 0$$

which indeed is the case.

(b) The equation for a plane containing the tangent line to a surface and a curve that lies in this surface is given by

$$\left.\frac{\partial F}{\partial x}\right|_{(x_1,y_1,z_1)}(x - x_1) + \left.\frac{\partial F}{\partial y}\right|_{(x_1,y_1,z_1)}(y - y_1) + \left.\frac{\partial F}{\partial z}\right|_{(x_1,y_1,z_1)}(z - z_1) = 0$$

For the surface $x^2 + 2y^2 + z = 2$ and the curve from problem 1(b) and the point $P = (\sqrt{3}/2, 1/2, 3/4)$ for which $t = \pi/3$ which is both on the surface and curve this gives

$$\sqrt{3}x + 2y + z = \frac{13}{4}$$

(c) If the tangent line to the curve at the point P lies in the tangent plane to the surface then the dot product between a vector parallel to the tangent line and the normal vector to the tangent plane at the point P will be 0. Let the vector $\mathbf{v} = 2(x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}) = \sqrt{3}\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ be a vector parallel to the tangent line to the curve at the point P and let the normal to the tangent plane at the point P be given by $\mathbf{n} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. Then it is easy to verify that

$$\mathbf{v} \cdot \mathbf{n} = \left(\sqrt{3}\mathbf{i} + 2\mathbf{j} + \mathbf{k}\right) \cdot \left(\mathbf{i} - \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}\right) = 0$$

which thus concludes the prove.

3. (a)

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}+\mathbf{v}) &= \frac{d}{dt}\left[(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) + (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})\right] \\
&= \left(\frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k}\right) + \left(\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}\right) \\
&= \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d}{dt}\left[g(t)\,\mathbf{u}\right] &= \frac{d}{dt}\left[g(t)\,(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\right] \\
&= \frac{d}{dt}\left[g(t)\,u_x\mathbf{i} + g(t)\,u_y\mathbf{j} + g(t)\,u_z\mathbf{k}\right] \\
&= \left[g(t)\frac{du_x}{dt} + g'(t)\,u_x\right]\mathbf{i} + \left[g(t)\frac{du_y}{dt} + g'(t)\,u_y\right]\mathbf{j} + \left[g(t)\frac{du_z}{dt} + g'(t)\,u_z\right]\mathbf{k} \\
&= g(t)\left(\frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k}\right) + g'(t)\,(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \\
&= g(t)\frac{d\mathbf{u}}{dt} + g'(t)\,\mathbf{u}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \frac{d}{dt}\left[(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \cdot (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})\right] \\
&= \frac{d}{dt}(u_xv_x + u_yv_y + u_zv_z) \\
&= u_x\frac{dv_x}{dt} + \frac{du_x}{dt}v_x + u_y\frac{dv_y}{dt} + \frac{du_y}{dt}v_y + u_z\frac{dv_z}{dt} + \frac{du_z}{dt}v_z \\
&= \left(u_x\frac{dv_x}{dt} + u_y\frac{dv_y}{dt} + u_z\frac{dv_z}{dt}\right) + \left(\frac{du_x}{dt}v_x + \frac{du_y}{dt}v_y + \frac{du_z}{dt}v_z\right) \\
&= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \cdot \left(\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}\right) \\
&\quad + \left(\frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k}\right) \cdot (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \\
&= \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{v}
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \frac{d}{dt}\left[(u_y v_z - u_z v_y)\,\mathbf{i} + (u_z v_x - u_x v_z)\,\mathbf{j} + (u_z v_x - u_x v_z)\,\mathbf{k}\right] \\
&= \left(u_y\frac{dv_z}{dt} + \frac{du_y}{dt}v_z - u_z\frac{dv_y}{dt} - \frac{du_z}{dt}v_y\right)\mathbf{i} + \left(u_z\frac{dv_x}{dt} + \frac{du_z}{dt}v_x - u_x\frac{dv_z}{dt} - \frac{du_x}{dt}v_z\right)\mathbf{j} \\
&\quad + \left(u_z\frac{dv_x}{dt} + \frac{du_z}{dt}v_x - u_x\frac{dv_z}{dt} - \frac{du_x}{dt}v_z\right)\mathbf{k} \\
&= \left(u_y\frac{dv_z}{dt} - u_z\frac{dv_y}{dt} + \frac{du_y}{dt}v_z - \frac{du_z}{dt}v_y\right)\mathbf{i} + \left(u_z\frac{dv_x}{dt} - u_x\frac{dv_z}{dt} + \frac{du_z}{dt}v_x - \frac{du_x}{dt}v_z\right)\mathbf{j} \\
&\quad + \left(u_z\frac{dv_x}{dt} - u_x\frac{dv_z}{dt} + \frac{du_z}{dt}v_x - \frac{du_x}{dt}v_z\right)\mathbf{k} \\
&= \left(u_y\frac{dv_z}{dt} - u_z\frac{dv_y}{dt}\right)\mathbf{i} + \left(u_z\frac{dv_x}{dt} - u_x\frac{dv_z}{dt}\right)\mathbf{j} + \left(u_z\frac{dv_x}{dt} - u_x\frac{dv_z}{dt}\right)\mathbf{k} \\
&\quad + \left(\frac{du_y}{dt}v_z - \frac{du_z}{dt}v_y\right)\mathbf{i} + \left(\frac{du_z}{dt}v_x - \frac{du_x}{dt}v_z\right)\mathbf{j} + \left(\frac{du_z}{dt}v_x - \frac{du_x}{dt}v_z\right)\mathbf{k} \\
&= \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v}
\end{aligned}$$

4. If $|\mathbf{u}(t)| \equiv a$ then $|\mathbf{u}'(t)| \equiv 0$, since the derivative of a constant is zero. Hence,

$$|\mathbf{u}'(t)| = \frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{u} = 2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0 \iff \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0$$

By definition, two non-zero vectors are said to be perpendicular when their dot product is zero, and so we may conclude that indeed $\mathbf{u}$ is perpendicular to $d\mathbf{u}/dt$. The locus of P such that $\overrightarrow{OP} = \mathbf{u}$ will be a constant as well.

5. (a) If $v = |\mathbf{v}(t)| \equiv 1$ then $v_t = |\mathbf{v}'(t)| \equiv 0$, since the derivative of a constant is zero. Hence,

$$|\mathbf{v}'(t)| = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \iff \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 0$$

By definition, two non-zero vectors are said to be perpendicular when their dot product is zero, and so we may conclude that indeed $\mathbf{v}$ is perpendicular to $d\mathbf{v}/dt = d\mathbf{v}/ds = \mathbf{a}$. Since $\mathbf{a} = d^2\mathbf{r}/dt^2$ and $\mathbf{v} = d\mathbf{r}/dt$ the equation for the osculating place may be written as

$$\overrightarrow{PQ}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2} = 0$$

where t is an arbitrary parameter along the curve, provided that $(d\mathbf{r}/dt)\times(d^2\mathbf{r}/dt^2) \neq \mathbf{0}$.

(b) The vectors $\mathbf{v}$ and $\mathbf{a}$ at the point $t = \pi/3$ for the curve of problem 1 are given by

$$\mathbf{v}(\pi/3) = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k} \qquad \mathbf{a}(\pi/3) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

The cross product of $\mathbf{v}$ and $\mathbf{a}$ then becomes

$$\mathbf{v} \times \mathbf{a} = (v_y a_z - v_z a_y)\,\mathbf{i} + (v_z a_x - v_x a_z)\,\mathbf{j} + (v_x a_y - v_y a_x)\,\mathbf{k} = \frac{3\sqrt{3}}{4}\mathbf{i} - \frac{1}{4}\mathbf{j} - \mathbf{k}$$

The point P for $t = \pi/3$ is given by

$$P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{3}{4}\right) \implies \overrightarrow{PQ} = \left(x - \frac{\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{1}{2}\right)\mathbf{j} + \left(z - \frac{3}{4}\right)\mathbf{k}$$

And so we find

$$\overrightarrow{PQ} \cdot \mathbf{v} \times \mathbf{a} = 0 \iff 3\sqrt{3}x - y - 4z = 1$$

6. (a) Assuming $v = |\mathbf{v}| = 1$ we can write

$$|\mathbf{r}' \times \mathbf{r}''| = \left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = |\mathbf{v} \times \mathbf{a}| = |\mathbf{T} \times \frac{d\mathbf{T}}{ds}| = |\mathbf{T} \times \kappa\mathbf{N}| = \kappa|\mathbf{T} \times \mathbf{N}| = \kappa$$

where $|\mathbf{T} \times \mathbf{N}| = 1$ follows from the fact that $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors. The binormal vector $\mathbf{B}$ may then be written as

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{T} \times \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{\mathbf{r}' \times \mathbf{r}''}{\kappa} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$$

To show that $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a positive triple of unit vectors we compute

$$\mathbf{B} \cdot \mathbf{B} = \mathbf{T} \times \mathbf{N} \cdot \mathbf{B} = 1$$

which is positive and hence, concludes the proof.

(b) From $|\mathbf{T} \times \mathbf{N}| = |\mathbf{B}| = 1$ it follows that

$$\frac{d|\mathbf{B}|}{ds} = \frac{d}{ds}(\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0 \iff \frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0$$

Next, rewriting $d\mathbf{B}/ds$ as

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \mathbf{T} \times \frac{d\mathbf{N}}{ds} + \frac{d\mathbf{T}}{ds} \times \mathbf{N}$$

it follows that

$$\begin{aligned}\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T} \times \frac{d\mathbf{N}}{ds} + \mathbf{T} \cdot \frac{d\mathbf{T}}{ds} \times \mathbf{N} &= \frac{d\mathbf{N}}{ds} \cdot \mathbf{T} \times \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{ds} \times \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\ &= \frac{d\mathbf{N}}{ds} \cdot \mathbf{0} + \frac{1}{\kappa}\mathbf{T} \cdot \mathbf{0} \\ &= 0\end{aligned}$$

(c) Since $\mathbf{N} \cdot \mathbf{N} = 1$ it follows that

$$\frac{d}{ds}(\mathbf{N} \cdot \mathbf{N}) = 2\mathbf{N} \cdot \frac{d\mathbf{N}}{ds} = 0 \iff \mathbf{N} \cdot \frac{d\mathbf{N}}{ds} = 0$$

Hence, $\mathbf{N}$ and $d\mathbf{N}/ds$ are perpendicular and $d\mathbf{N}/ds$ lies in the plane spanned by the vectors $\mathbf{T}$ and $\mathbf{B}$ and can be expressed as a linear combination of $\mathbf{T}$ and $\mathbf{B}$: $d\mathbf{N}/ds = \mu\mathbf{T} + \tau\mathbf{B}$. Next, substituting for $d\mathbf{N}/ds$ in the equation

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \mathbf{T} \times \frac{d\mathbf{N}}{ds} + \frac{d\mathbf{T}}{ds} \times \mathbf{N} = \mathbf{T} \times \frac{d\mathbf{N}}{ds} + \frac{d\mathbf{T}}{ds} \times \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

results in

$$\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times (\mu\mathbf{T} + \tau\mathbf{B}) = \mathbf{T} \times \tau\mathbf{B} = -\tau\mathbf{B} \times \mathbf{T} = -\tau\mathbf{N}$$

The constant τ may also be written as

$$\begin{aligned}
\tau = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds} = -\frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \cdot \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right) = -\frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \cdot \left(\mathbf{T} \times \frac{1}{\kappa}\frac{d\mathbf{T}}{ds^2}\right) &= -\frac{1}{\kappa^2}\mathbf{r}'' \cdot (\mathbf{r}' \times \mathbf{r}''') \\
&= -\frac{1}{\kappa^2}(\mathbf{r}'' \times \mathbf{r}') \cdot \mathbf{r}''' \\
&= \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} \\
&= |\mathbf{v} \times \mathbf{a}|^{-2}\mathbf{v} \times \mathbf{a} \cdot \mathbf{w}
\end{aligned}$$

where $\mathbf{w} = \mathbf{r}''' = d\mathbf{a}/ds = d\mathbf{a}/dt$, since $ds/dt = v = 1$.

(d) Starting from $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and using the fact that

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \mathbf{T} = \mathbf{N} \times \mathbf{B} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

we find

$$\frac{d\mathbf{N}}{ds} = \mathbf{B} \times \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{B}}{ds} \times \mathbf{T} = \mathbf{B} \times \kappa\mathbf{N} - \tau\mathbf{N} \times \mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

7. (a) First, note that

$$\mathbf{v} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\mathbf{T}$$

Hence,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{dv}{dt}\mathbf{T} + v^2\kappa\mathbf{N} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}$$

(b)

$$
\begin{aligned}
v^{-3}|\mathbf{v}\times\mathbf{a}| = v^{-3}\left|v\mathbf{T}\times\left(\frac{dv}{dt}\mathbf{T}+v^2\kappa\mathbf{N}\right)\right| &= v^{-3}\left|\mathbf{T}\times v\frac{dv}{dt}\mathbf{T}+\mathbf{T}\times v^3\kappa\mathbf{N}\right| \\
&= v^{-3}\left|\mathbf{T}\times v^3\kappa\mathbf{N}\right| \\
&= \kappa|\mathbf{T}\times\mathbf{N}| \\
&= \kappa
\end{aligned}
$$

where $|\mathbf{T}\times\mathbf{N}| = 1$ because $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors.

(c) We have already proved this as part of problem 6(c).

(d) If $\kappa \equiv 0$ then $d\mathbf{T}/ds \equiv \mathbf{0}$, which implies $\mathbf{T} = \mathbf{b}$ is a constant vector. Hence, note that

$$\frac{d}{ds}(\mathbf{r}\times\mathbf{T}) = \mathbf{r}\times\frac{d\mathbf{T}}{ds}+\mathbf{T}\times\mathbf{T} = \mathbf{r}\times\mathbf{0} = \mathbf{0}$$

which implies the vector $\mathbf{r}\times\mathbf{T} = \mathbf{r}\times\mathbf{b} = \mathbf{c}$ is a constant vector also. This is a vector equation for a straight line, since

$$\mathbf{r}\times\mathbf{b} = \mathbf{c} \iff (b_3 y - b_2 z)\,\mathbf{i} + (b_1 z - b_3 x)\,\mathbf{j} + (b_2 x - b_1 y)\,\mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

And so

$$z = \bar{b}_1 y + \bar{c}_1 \qquad z = \bar{b}_2 x + \bar{c}_2 \qquad y = \bar{3}x + \bar{c}_3$$

where $\bar{b}_1 = b_3/b_2$, $\bar{c}_1 = -c_1/b_2$, $\bar{b}_2 = b_3/b_1$, $\bar{c}_2 = c_2/b_1$, $\bar{b}_3 = b_2/b_1$ and $\bar{c}_3 = -c_3/b_1$.

(e) From the equation $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ it follows that $\mathbf{T}\perp\mathbf{B}$ ($\mathbf{T}$ is perpendicular to $\mathbf{B}$). Hence

$$\mathbf{T}\cdot\mathbf{B} = \left[\frac{d}{ds}(\mathbf{r}-\mathbf{r}_0)\right]\cdot\mathbf{B} = 0$$

where $\mathbf{r}_0$ is some arbitrary constant vector. Next, since $\tau = 0$ we find that $d\mathbf{b}/ds = \mathbf{0}$ and so $\mathbf{B} = \mathbf{b}$ where $\mathbf{b}$ is some constant vector. As such, we can write

$$\left[\frac{d}{ds}(\mathbf{r}-\mathbf{r}_0)\right]\cdot\mathbf{B} = \frac{d}{ds}\left[\mathbf{b}\cdot(\mathbf{r}-\mathbf{r}_0)\right] = 0 \implies \mathbf{b}\cdot(\mathbf{r}-\mathbf{r}_0) = a$$

where a is some arbitrary scalar. Lastly, we can rewrite $\mathbf{b}\cdot(\mathbf{r}-\mathbf{r}_0) = a$ as

$$
\begin{aligned}
\mathbf{b}\cdot(\mathbf{r}-\mathbf{r}_0) &= a \\
b_1(x-x_0) + b_2(y-y_0) + b_3(z-z_0) &= a \\
b_1 x + b_2 y + b_3 z - (b_1 x_0 + b_2 y_0 + b_3 z_0 + a) &= 0
\end{aligned}
$$

This last form may be identified with equation (1.25) with $A = b_1$, $B = b_2$, $C = b_3$ and $D = -b_1 x_0 - b_2 y_0 - b_3 z_0 - a$. In other words, we have proven that when $\tau \equiv 0$ the binormal vector $\mathbf{B}$ becomes the constant vector $\mathbf{b}$, which is the normal vector to the plane in which lies the vector $\mathbf{r}$, representing the path of the curve traced out by the point P moving in space at speed $v = ds/dt \neq 0$.

8. In order to find an equation for the tangent plane at the point $P = (x_1, y_1, z_1)$ for the given surface we make use of equation (2.100). In order to find a vector equation for the normal line to the tangent plane at the point $P = (x_1, y_1, z_1)$ we will make use of the fact that $\nabla F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ is parallel to this line and the point P lies on the line so that $\overrightarrow{P_1P} = t\nabla F|_{(x_1,y_1,z_1)}$ where t is an arbitrary parameter describes this line (this essentially is equation (1.27) with $\mathbf{v} = \nabla F|_{(x_1,y_1,z_1)}$).

   (a) Let us first define $F(x, y, z)$ as

   $$F\left(x, y, z\right) = x^2 + y^2 + z^2 - 9 = 0$$

   The equation for the tangent plane at the point $P = (2, 2, 1)$ is then given by

   $$2x + 2y + z = 9$$

   The equations for the normal line at the point $P = (2, 2, 1)$ are given by

   $$x = 2 + 4t \qquad y = 2 + 4t \qquad z = 1 + 2t$$

   or alternatively

   $$\frac{x-2}{2} = \frac{y-2}{2} = z - 1$$

   (b) $F(x, y, z)$ is of the form

   $$F\left(x, y, z\right) = e^{x^2+y^2} - z^2 = 0$$

   The equation for the tangent plane at the point $P = (0, 0, 1)$ is given by

   $$z = 1$$

   The equations for the normal line at the point $P = (0, 0, 1)$ are given by

   $$x = 0 \qquad y = 0 \qquad z = 1 - 2t$$

   (c) $F(x, y, z)$ is of the form

   $$F\left(x, y, z\right) = x^3 - xy^2 + yz^2 - z^3 = 0$$

   The equation for the tangent plane at the point $P = (1, 1, 1)$ is given by

   $$2x - y - z = 0$$

   The equations for the normal line at the point $P = (1, 1, 1)$ are given by

   $$x = 1 + 2t \qquad y = 1 - t \qquad z = 1 - t$$

   or alternatively

   $$\frac{x-1}{2} = 1 - y = 1 - z$$

(d) The procedure breaks down for the surface given by $F(x,y,z) = x^2 + y^2 - z^2 = 0$ at the point $P = (0,0,0)$, since $\nabla F|_{(0,0,0)} = \mathbf{0}$ and so equation (2.100) fails to determine a plane.

(e) $F(x,y,z)$ is of the form

$$F(x,y,z) = xy - z = 0$$

The equation for the tangent plane at the point $P = (x_1, y_1, z_1)$ where $x_1 y_1 = z_1$ is given by

$$xy_1 + x_1 y - z = x_1 y_1 = z_1$$

The equations for the normal line at the point $P = (x_1, y_1, z_1)$ where $x_1 y_1 = z_1$ are given by

$$x = x_1 + y_1 t \qquad y = y_1 + x_1 t \qquad z = z_1 - t$$

or alternatively

$$\frac{x - x_1}{y_1} = \frac{y - y_1}{x_1} = z_1 - z$$

(f) $F(x,y,z)$ is of the form

$$F(x,y,z) = xy + yz + xz - 1 = 0$$

The equation for the tangent plane at the point $P = (x_1, y_1, z_1)$ where $x_1 y_1 + y_1 z_1 + x_1 z_1 = 1$ is given by

$$(y_1 + z_1)\, x + (x_1 + z_1)\, y + (y_1 + x_1)\, z = 2\,(x_1 y_1 + x_1 z_1 + y_1 z_1) = 2$$

The equations for the normal line at the point $P = (x_1, y_1, z_1)$ where $x_1 y_1 + y_1 z_1 + x_1 z_1 = 1$ are given by

$$x = x_1 + (y_1 + z_1)\, t \qquad y = y_1 + (x_1 + z_1)\, t \qquad z = z_1 + (y_1 + x_1)\, t$$

or alternatively

$$\frac{x - x_1}{y_1 + z_1} = \frac{y - y_1}{x_1 + z_1} = \frac{z - z_1}{y_1 + x_1}$$

9. Let us define $F(x,y,z)$ as $F(x,y,z) = f(x,y) - z = 0$. Then according to equation (2.100)

$$\left.\frac{\partial F}{\partial x}\right|_{(x_1,y_1,z_1)} (x - x_1) + \left.\frac{\partial F}{\partial y}\right|_{(x_1,y_1,z_1)} (y - y_1) + \left.\frac{\partial F}{\partial z}\right|_{(x_1,y_1,z_1)} (z - z_1) = 0$$

we end up with

$$z - z_1 = \left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1,z_1)} (x - x_1) + \left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1,z_1)} (y - y_1)$$

The equations of the normal line may be determined from the vector equation $\overrightarrow{P_1P} = t\nabla F|_{(x_1,y_1,z_1)}$ where $P_1 = (x_1, y_1, z_1)$, t is some arbitrary parameter and $\nabla F|_{(x_1,y_1,z_1)} = F_x(x_1)\mathbf{i} + F_y(y_1)\mathbf{j} + F_z(z_1)\mathbf{k}$. Hence,

$$x = x_1 + \left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1,z_1)} t \qquad y = y_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1,z_1)} t \qquad z = z_1 - t$$

or alternatively

$$\frac{x - x_1}{f_x\,(x_1, y_1, z_1)} = \frac{y - y_1}{f_y\,(x_1, y_1, z_1)} = z_1 - z$$

10. (a) The tangent plane at the point $P = (1, 1, 2)$ is given by

$$z - 2 = 2\,(x - 1) + 2\,(y - 1)$$

The equations for the normal line at the point $P = (1, 1, 2)$ are given by

$$\frac{x - 1}{2} = \frac{y - 1}{2} = 2 - z$$

(b) The tangent plane at the point $P = (2/3, 2/3, 1/3)$ is given by

$$z - \frac{1}{3} = -2\left(x - \frac{2}{3}\right) - 2\left(y - \frac{2}{3}\right)$$

The equations for the normal line at the point $P = (2/3, 2/3, 1/3)$ are given by

$$\frac{x - 2/3}{2} = \frac{y - 2/3}{2} = z - \frac{1}{3}$$

(c) The tangent plane at the point $P = (2, 1, 2)$ is given by

$$z = x - 2y + 2$$

The equations for the normal line at the point $P = (2, 1, 2)$ are given by

$$x - 2 = \frac{1 - y}{2} = 2 - z$$

The tangent plane at the point $P = (3/5, 4/5, 0)$ is given by

$$5z = 6x + 8y - 10$$

The equations for the normal line at the point $P = (3/5, 4/5, 0)$ are given by

$$\frac{x - 3/5}{6/5} = \frac{y - 4/5}{8/5} = -z$$

11. Let two surfaces in space be given by $F(x, y, z$ and $G(x, y, z)$. Then the vector $\nabla F \times \nabla G$ is of the form

$$\nabla F \times \nabla G = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \mathbf{i} + \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix} \mathbf{j} + \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \mathbf{k}$$

and so the vector equation $d\mathbf{r} \times (\nabla F \times \nabla G)$ representing the tangent line (i.e. the intersection of the two tangent planes of each surface at the point P) at the point $P = (x_1, y_1, z_1)$ is of the form

$$\begin{aligned} d\mathbf{r} \times (\nabla F \times \nabla G) = & \left(\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} (y - y_1) - \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix} (z - z_1) \right) \mathbf{i} \\ & + \left(\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} (z - z_1) - \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} (x - x_1) \right) \mathbf{j} \\ & + \left(\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix} (x - x_1) - \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} (y - y_1) \right) \mathbf{k} = \mathbf{0} \end{aligned}$$

where it is assumed all partial derivatives are evaluated at the point $P = (x_1, y_1, z_1)$. Rearrangement and substitution then allows for the equation above to be written as equation (2.108). Alternatively, we can use equation (1.27) with $\mathbf{v} = \nabla F|_{(x_1,y_1,z_1)} \times \nabla G|_{(x_1,y_1,z_1)}$ and $\overrightarrow{P_1P} = d\mathbf{r}$.

(a) Defining $F(x, y, z)$ and $G(x, y, z)$ as

$$F(x, y, z) = 2x + y - z - 6 = 0 \qquad G(x, y, z) = x + 2y + 2z - 7 = 0$$

gives at the point $P = (3, 1, 1)$

$$\nabla F|_{(3,1,1)} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \qquad \nabla G|_{(3,1,1)} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

And so the equations for the tangent line are given by

$$\frac{x-3}{4} = \frac{y-1}{-5} = \frac{z-1}{3}$$

or in terms of an arbitrary parameter t

$$x = 3 + 4t \qquad y = 1 - 5t \qquad z = 1 + 3t$$

(b) Defining $F(x, y, z)$ and $G(x, y, z)$ as

$$F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0 \qquad G(x, y, z) = x^2 + y^2 - 8z^2 = 0$$

gives at the point $P = (2, 2, 1)$

$$\nabla F|_{(2,2,1)} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \qquad \nabla G|_{(2,2,1)} = 4\mathbf{i} + 4\mathbf{j} - 16\mathbf{k}$$

And so

$$\nabla F|_{(2,2,1)} \times \nabla G|_{(2,2,1)} = -72\mathbf{i} + 72\mathbf{j} = -72(\mathbf{i} - \mathbf{j})$$

Hence, the equations for the tangent line become

$$x = 2 + t \qquad y = 2 - t \qquad z = 1$$

(c) Defining $F(x, y, z)$ and $G(x, y, z)$ as

$$F(x, y, z) = x^2 + y^2 - 1 = 0 \qquad G(x, y, z) = x + y + z = 0$$

gives at the point $P = (1, 0, -1)$

$$\nabla F|_{(1,0,-1)} = 2\mathbf{i} \qquad \nabla G|_{(1,0,-1)} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

And so

$$\nabla F|_{(1,0,-1)} \times \nabla G|_{(1,0,-1)} = -2\mathbf{j} + 2\mathbf{k}$$

Hence, the equations for the tangent line become

$$x = 1 \qquad y = -t \qquad z = -1 + t$$

(d) Defining $F(x, y, z)$ and $G(x, y, z)$ as

$$F(x, y, z) = x^2 + y^2 + x^2 - 9 = 0 \qquad G(x, y, z) = x^2 + 2y^2 + 3z^2 - 9 = 0$$

gives at the point $P = (3, 0, 0)$

$$\nabla F|_{(3,0,0)} = 6\mathbf{i} \qquad \nabla G|_{(3,0,0)} = 6\mathbf{i}$$

The procedure breaks down, since

$$\nabla F|_{(3,0,0)} \times \nabla G|_{(3,0,0)} = \mathbf{0}$$

In other words, the tangent planes at the point $P = (3, 0, 0)$ are parallel (i.e. they do not intersect each other) and so the tangent line at the given point does not exist.

12. To show that the curve given by the cross-section of the two surfaces

$$F(x, y, z) = x^2 - y^2 + z^2 - 1 = 0 \qquad G(x, y, z) = xy + xz - 2 = 0$$

is tangent to the surface $H(x, y, z) = xyz - x^2 - 6y + 6 = 0$ at the point $P_1 = (1, 1, 1)$ is equivalent to showing that the dot product of the normal to the surface $H(x, y, z)$ (i.e. the gradient) at the point P_1 is perpendicular to the tangent vector of the curve at the point P_1. Let us first find the tangent vector to the curve. To this end, we first will use the vector equation $d\mathbf{r} \times \left(\nabla F|_{(1,1,1)} \times \nabla G|_{(1,1,1)}\right) = \mathbf{0}$ to find the equations for the tangent line. Hence, at the point $P_1 = (1, 1, 1)$ we find

$$\nabla F_{(1,1,1)} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \qquad \nabla G_{(1,1,1)} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

And so

$$\nabla F_{(1,1,1)} \times \nabla G_{(1,1,1)} = -4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

Hence, the equations for the tangent line become

$$x = 1 - 4t \qquad y = 1 + 2t \qquad z = 1 + 6t$$

Choosing $t = 1$ we find $P_2 = (-3, 3, 7)$. Then $\overrightarrow{P_2P_1} = 2\left(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}\right) = 2\mathbf{v}$, where $\mathbf{v}$ is a tangent vector to the curve at the point $P_1 = (1, 1, 1)$. Lastly, we evaluate

$$\mathbf{v} \cdot \nabla H|_{(1,1,1)} = (-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 0$$

which thus confirms that the given curve is tangent to the given surface at the point $(1, 1, 1)$.

13. Firstly, note that the vector $\nabla F \times \nabla G$ is perpendicular to both ∇F and ∇G and is tangent to the curve $F(x, y, z)$, $G(x, y, z$ at the point (x_1, y_1, z_1). As such, a plane normal to the curve at the point (x_1, y_1, z_1) will be perpendicular to the vector $\nabla F \times \nabla G$. Next, let the vector $d\mathbf{r} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$ be some vector lying wholly in the normal plane. It then follows that $d\mathbf{r} \cdot \nabla F \times \nabla G = 0$, which is the equation for a plane normal to the curve $F(x, y, z)$, $G(x, y, z)$ at the point (x_1, y_1, z_1). In rectangular coordinates the equation for the plane is given by

$$Ax + By + Cz + D = 0$$

where $A = f_y g_z - f_z g_y$, $B = f_z g_x - f_x g_z$, $C = f_x g_y - f_y g_x$ and $D = -Ax_1 - By_1 - Cz_1$. The normal plane to the curve of Problem 11(b) at the point (2, 2, 1) is given by

$$x - y = 0$$

The normal plane to the curve of Problem 11(c) at the point (1, 0, -1) is given by

$$y - z + 1 = 0$$

14. Let the vector $\mathbf{v} = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}$ be a vector tangent to the curve $x = t^2$, $y = t$, $z = 2t$ at the point $(0, 0, 0)$. Hence,

$$\mathbf{v}\left(0\right) = 2\left(0\right)\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{j} + 2\mathbf{k}$$

Let $\mathbf{v}$ be the normal vector to some plane that contains the point $(1, 0, 0)$. Hence, the vector $d\mathbf{r} = (x - 1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ will be a vector lying wholly in this plane whenever

$$d\mathbf{r} \cdot \mathbf{v} = (x - 1)\left(0\right) + y\left(1\right) + z\left(2\right) = y + 2z = 0$$

15. Let some arbitrary level surface be given by the function $F(x, y, z) = c$, where c is a real valued constant and let $P = (x_0, y_0, z_0)$ be a point on this level surface, i.e. $F(x_0, y_0, z_0) = c$. Furthermore, let $\mathbf{r}(t)$ be a parametric representation for a curve on this surface with $\mathbf{r}(t_0) = (x_0, y_0, z_0)$. Next, consider the function $G(t) =$

$F(x(t), y(t), z(t))$. Since the curve is on the level surface we have $G(t) = F(x(t), y(t), z(t)) = c$. Differentiation with respect to t and evaluating at the point t_0 then gives

$$\frac{dG}{dt} = \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)} \left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)} \left.\frac{dy}{dt}\right|_{t_0} + \left.\frac{\partial F}{\partial z}\right|_{(x_0,y_0,z_0)} \left.\frac{dz}{dt}\right|_{t_0} = 0$$

which also may be written in vector form as the dot product

$$\nabla F|_{(x_0,y_0,z_0)} \cdot \left.\frac{d\mathbf{r}}{dt}\right|_{t_0} = 0$$

Since this dot product is zero it shows that the gradient to the surface $F(x, y, z) = c$ is perpendicular to the tangent to any curve that lies on this surface.

(a) The gradient vector of the function $F = x^2 + y^2 + z^2$ is given by

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

(b) The gradient vector of the function $G = 2x^2 + y^2$ is given by

$$\nabla G = 4x\mathbf{i} + 2y\mathbf{j}$$

16. (a) Let us define the parametric equation $G(u, v) = F(f(u, v), g(u, v), h(u, v)) = c$ corresponding to the surface $F(x, y, z) = c$. Partial differentiation then gives

$$\frac{\partial G}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial g}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial h}{\partial u} = \nabla F \cdot \left(\frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}\right) = 0$$
$$\frac{\partial G}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial f}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial g}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial h}{\partial v} = \nabla F \cdot \left(\frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}\right) = 0$$

Since it follows from this that ∇F is perpendicular to both $f_u\mathbf{i} + g_u\mathbf{j} + h_u\mathbf{k}$ and $f_v\mathbf{i} + g_v\mathbf{j} + h_v\mathbf{k}$ we may conclude that

$$\nabla F \times \left[\left(\frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}\right)\right] = \mathbf{0}$$

which in turn implies that ∇F and the vector $(f_u\mathbf{i} + g_u\mathbf{j} + h_u\mathbf{k}) \times (f_v\mathbf{i} + g_v\mathbf{j} + h_v\mathbf{k})$ are parallel. Hence, since ∇F is normal to the surface at the point (x_1, y_1, z_1), it follows that the vector

$$\left(\frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}\right) \equiv \frac{\partial\,(g,h)}{\partial\,(u,v)}\mathbf{i} + \frac{\partial\,(h,f)}{\partial\,(u,v)}\mathbf{j} + \frac{\partial\,(f,g)}{\partial\,(u,v)}\mathbf{k}$$

is normal to the surface at the point (x_1, y_1, z_1).

(b) The equation for the tangent plane in terms of the parametric variables u and v is given by

$$\left(\frac{\partial\,(g,h)}{\partial\,(u,v)}\mathbf{i} + \frac{\partial\,(h,f)}{\partial\,(u,v)}\mathbf{j} + \frac{\partial\,(f,g)}{\partial\,(u,v)}\mathbf{k}\right) \cdot d\mathbf{r} = 0$$

(c) Firstly, note that $x = f(u,v) = \cos u \cos v$, $y = g(u,v) = \cos u \sin v$ and $z = h(u,v) = \sin u$. Evaluating the relevant partial derivatives at the point for which $u = \pi/4$, $v = \pi/4$ gives

$$\begin{aligned}
\left.\frac{\partial f}{\partial u}\right|_{(\pi/4,\pi/4)} &= -\frac{1}{2} & \left.\frac{\partial g}{\partial u}\right|_{(\pi/4,\pi/4)} &= -\frac{1}{2} & \left.\frac{\partial h}{\partial u}\right|_{(\pi/4,\pi/4)} &= \frac{\sqrt{2}}{2} \\
\left.\frac{\partial f}{\partial v}\right|_{(\pi/4,\pi/4)} &= -\frac{1}{2} & \left.\frac{\partial g}{\partial v}\right|_{(\pi/4,\pi/4)} &= \frac{1}{2} & \left.\frac{\partial h}{\partial v}\right|_{(\pi/4,\pi/4)} &= 0
\end{aligned}$$

The normal to the tangent plane to the surface at the point for which $u = \pi/4$ and $v = \pi/4$ then is given by

$$\left[\left(\frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}\right)\right]_{(\pi/4,\pi/4)} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1/2 & -1/2 & \sqrt{2}/2 \\ -1/2 & 1/2 & 0 \end{vmatrix}$$

And so the tangent plane itself is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1/2 & -1/2 & \sqrt{2}/2 \\ -1/2 & 1/2 & 0 \end{vmatrix} \cdot \left[\left(x - \frac{1}{2}\right)\mathbf{i} + \left(y - \frac{1}{2}\right)\mathbf{j} + \left(z - \frac{\sqrt{2}}{2}\right)\mathbf{k}\right] = x + y + \sqrt{2}z = 2$$

(d) Firstly, note that $\sqrt{x^2 + y^2 + z^2} = \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u} = 1$, which is the radius of the unit sphere. Next, the azimuth of the sphere is given by $\tan^{-1}(y/x) = \tan^{-1}(\cos u \sin v/(\cos u \cos v)) = \tan^{-1}\tan v = v$. Lastly, the inclination angle of the sphere is given by $\cos^{-1} z = \cos^{-1}\sin u = \cos^{-1}\cos(u - \pi/2) = u - \pi/2$.

17. Let us define the two surfaces $F(x,y,z) = f(x,y) - z = 0$ and $G(x,y,z) = g(x,y) - z$. Then using (2.108) the equation for the tangent line is given by

$$\frac{x - x_1}{\begin{vmatrix} \partial f/\partial y & -1 \\ \partial g/\partial y & -1 \end{vmatrix}} = \frac{y - y_1}{\begin{vmatrix} -1 & \partial f/\partial x \\ -1 & \partial g/\partial x \end{vmatrix}} = \frac{z - z_1}{\begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}}$$

or more succinctly

$$\frac{x - x_1}{\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}} = \frac{y - y_1}{\dfrac{\partial f}{\partial x} - \dfrac{\partial g}{\partial x}} = \frac{z - z_1}{\dfrac{\partial(f,g)}{\partial(x,y)}}$$

18. The differential relations corresponding to the equations $F(x,y,z,t)$, $G(x,y,z,t)$ and $H(x,y,z,t)$ can be treated like a system of linear equations:

$$\begin{aligned}
F_x dx + F_y dy + F_z dz &= -F_t dt \\
G_x dx + G_y dy + G_z dz &= -G_t dt \\
H_x dx + H_y dy + H_z dz &= -H_t dt
\end{aligned}$$

Cramer's rule (section 1.5) then tells us that

$$dx = -\frac{\begin{vmatrix} F_t & F_y & F_z \\ G_t & G_y & G_z \\ H_t & H_y & H_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}} dt \qquad dy = -\frac{\begin{vmatrix} F_x & F_t & F_z \\ G_x & G_t & G_z \\ H_x & H_t & H_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}} dt \qquad dz = -\frac{\begin{vmatrix} F_x & F_y & F_t \\ G_x & G_y & G_t \\ H_x & H_y & H_t \end{vmatrix}}{\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}} dt$$

Similarly to (2.102) these differential relations represents the intersection of three tangent planes at the point (x_1, y_1, z_1) at the point considered; the intersection of these planes is the tangent line to the curve. Recognizing that $dx = x - x_1$, $dy = y - y_1$ and $dz = z - z_1$ the equations for the tangent line may, similar to (2.108), be written in the symmetric form

$$\frac{x - x_1}{\begin{vmatrix} F_t & F_y & F_z \\ G_t & G_y & G_z \\ H_t & H_y & H_z \end{vmatrix}} = \frac{y - y_1}{\begin{vmatrix} F_x & F_t & F_z \\ G_x & G_t & G_z \\ H_x & H_t & H_z \end{vmatrix}} = \frac{z - z_1}{\begin{vmatrix} F_x & F_y & F_t \\ G_x & G_y & G_t \\ H_x & H_y & H_t \end{vmatrix}}$$

or

$$\frac{x - x_1}{\dfrac{\partial (F, G, H)}{\partial (t, y, z)}} = \frac{y - y_1}{\dfrac{\partial (F, G, H)}{(x, t, z)}} = \frac{z - z_1}{\dfrac{\partial (F, G, H)}{\partial (x, y, t)}}$$

19. (a) If $d\mathbf{r}/dt \equiv 0$ then $\mathbf{r}(t) = \mathbf{c} = \overrightarrow{OP}$, where $\mathbf{c}$ is some constant vector for any value $t_1 \leq t \leq t_2$. Hence, since the point O is fixed this implies that the point P has to be fixed also.

(b) Since both $\mathbf{r}(t)$ and $\mathbf{r}(t_0)$ correspond to a point P and P_0 on the curve, the vector $\mathbf{r}(t) - \mathbf{r}(t_0)/(t - t_0)^{n+1}$ will intersect the curve at least twice (i.e. once at the point P and once at the point P_0) and hence, is a secant to the curve. To obtain the tangent vector to the curve at point $P_0 = \mathbf{r}(t_0)$, we let $t \to t_0$ and take the limit

$$\begin{aligned} \lim_{t \to t_0} \frac{\mathbf{r}(r) - \mathbf{r}(t_0)}{(t - t_0)^{n+1}} = \lim_{t \to t_0} \frac{\mathbf{r}'}{(n+1)(t - t_0)^n} = \ldots &= \lim_{t \to t_0} \frac{1}{(n+1)!} \frac{d^{n+1}\mathbf{r}}{dt^{n+1}} \\ &= \frac{1}{(n+1)!} \frac{d^{n+1}\mathbf{r}}{dt^{n+1}} \bigg|_{t = t_0} \end{aligned}$$

were we have used the fact that

$$\frac{d\mathbf{r}}{dt} = \mathbf{0}, \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0}, \ldots, \frac{d^n\mathbf{r}}{dt^n} = \mathbf{0}, \frac{d^{n+1}\mathbf{r}}{dt^{n+1}} \neq \mathbf{0}$$

for $t = t_0$ together with l'Hôspital rule.

(c) Since the arc length s represents the distance along the path we have

$$\left|\frac{d\mathbf{r}}{dt}\right|^2 = \left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

And so

$$\left|\frac{d\mathbf{r}}{ds}\right|^2 = \left(\frac{df}{ds}\right)^2 + \left(\frac{dg}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2 = \left[\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2\right]\left(\frac{dt}{ds}\right)^2 = 1$$

In conclusion, $d\mathbf{r}/ds$ is a unit vector and hence, always a tangent vector to the curve.

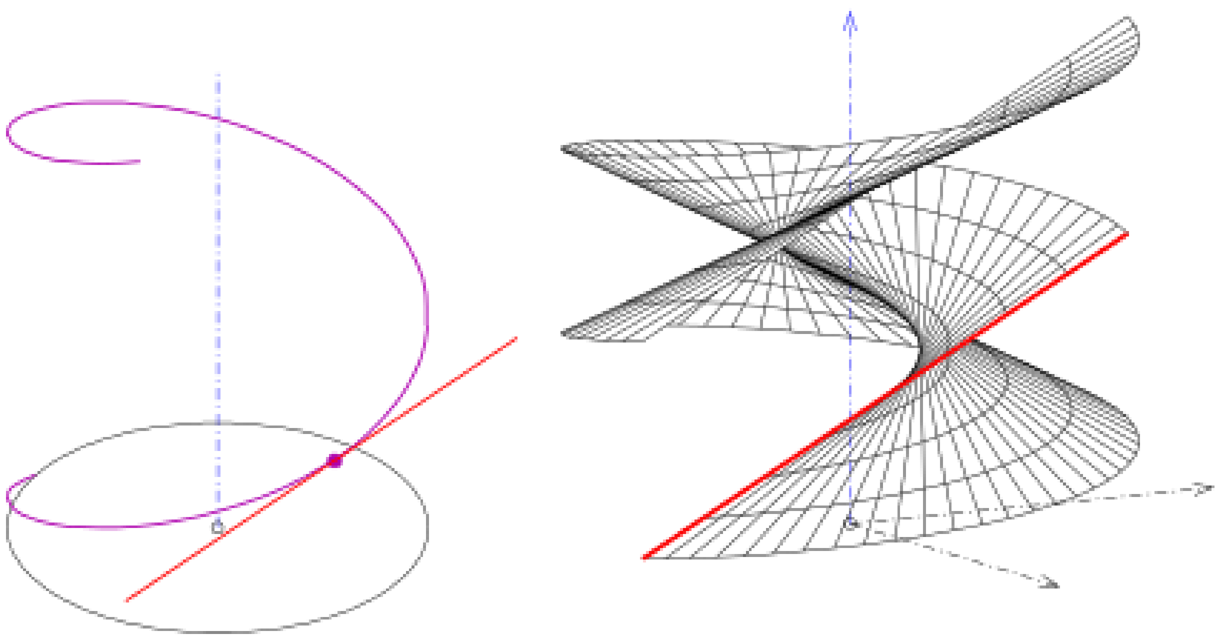

Figure 14: circular helix and its tangential developable

20. (a)

(b) To demonstrate that along a chosen ruling of a tangential developable all points of the surface have the same tangent plane, we compute

$$\frac{d\mathbf{r}}{du} \cdot \mathbf{v}\,(u) \times \frac{d\mathbf{v}}{du} = \frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du} \times \frac{d\mathbf{v}}{du} = \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{du} = 0$$

where $d\mathbf{r}/du$ represents the tangent vector to the curve that is parallel to the chosen ruling and $\mathbf{v}(u) \times d\mathbf{v}/du$ is a vector that is both perpendicular to $\mathbf{v}(u)$ and $d\mathbf{v}/du$.

Section 2.14

1. (a)

$$\nabla F|_{(1,2,3)} = 4\mathbf{i} - 4\mathbf{j} + 6\mathbf{k} \qquad\qquad \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{22}}(2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$$

And so

$$\nabla_v F|_{(1,2,3)} = \nabla F|_{(1,2,3)} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = -\sqrt{22}$$

(b)

$$\nabla F|_{(0,0,0)} = \mathbf{0} \qquad\qquad \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

And so

$$\nabla_u F|_{(0,0,0)} = \nabla F|_{(0,0,0)} \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = 0$$

(c)

$$\nabla F|_{(0,0)} = \mathbf{i} \qquad\qquad \mathbf{u} = \cos\frac{\pi}{3}\mathbf{i} + \sin\frac{\pi}{3}\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

And so

$$\nabla_u F|_{(0,0)} = \nabla F|_{(0,0)} \cdot \mathbf{u} = \frac{1}{2}$$

(d)

$$\nabla F|_{(0,0)} = 2\mathbf{i} - 3\mathbf{j} \qquad\qquad \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

And so

$$\nabla_v F|_{(1,1)} = \nabla F|_{(1,1)} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{4}{\sqrt{5}}$$

(e)

$$\nabla F|_{(2,2,1)} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$$

Next, let $G(x,y,z) = x^2 + y^2 + z^2 = 9$. The outer normal to the surface is then given by

$$\nabla G|_{(2,2,1)} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad\qquad \mathbf{u} = \frac{\nabla G|_{(2,2,1)}}{|\nabla G|_{(2,2,1)}|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

And so

$$\nabla_u F|_{(2,2,1)} = \nabla F|_{(2,2,1)} \cdot \mathbf{u} = -\frac{2}{3}$$

(f) The tangent line to the curve at the point $(3, 4, 5)$ is given by the intersection of the two tangent planes

$$\nabla F|_{(3,4,5)} \cdot d\mathbf{r} = 0 \qquad\qquad \nabla G|_{(3,4,5)} \cdot d\mathbf{r} = 0$$

As such, the tangent line is perpendicular to both ∇F and ∇G at the point given. Hence, the directional derivative $\nabla_v F$ where $\mathbf{v}$ is a vector parallel to the tangent line to the curve at the point $(3, 4, 5)$ is zero.

2. (a) Let $G(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$, so that

$$\mathbf{n} = \frac{\nabla G}{|\nabla G|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{4}} = \frac{1}{2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)$$

Hence,

$$\frac{\partial F}{\partial n} = \nabla_n F = \nabla F \cdot \mathbf{n} = (2x - 2y) \cdot \mathbf{n} = x^2 - y^2$$

(b) Let $G(x, y, z) = x^2 + 2y^2 + 4z^2 - 8 = 0$, so that

$$\mathbf{n} = \frac{\nabla G}{|\nabla G|} = \frac{x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}}{\sqrt{x^2 + 4y^2 + 16z^2}}$$

Hence,

$$\frac{\partial F}{\partial n} = \nabla_n F = \nabla F \cdot \mathbf{n} = (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot \mathbf{n} = \frac{7xyz}{\sqrt{x^2 + 4y^2 + 16z^2}}$$

3. Let the angle α denote the angle of a unit vector $\mathbf{v}$ with respect to the positive x-axis, so that

$$\begin{aligned}
\nabla_v u = \nabla_\alpha u = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha &= \frac{\partial v}{\partial y}\cos\alpha - \frac{\partial v}{\partial x}\sin\alpha \\
&= -\frac{\partial v}{\partial x}\sin\alpha + \frac{\partial v}{\partial y}\cos\alpha \\
&= \frac{\partial v}{\partial x}\cos\left(\alpha + \frac{\pi}{2}\right) + \frac{\partial v}{\partial y}\sin\left(\alpha + \frac{\pi}{2}\right) \\
&= \nabla_{\alpha+\pi/2} v
\end{aligned}$$

4. In polar coordinates we have the relation $x = r\cos\theta$ and $y = r\sin\theta$ where it is assumed that θ is the angle with respect to the positive x-axis. Then

$$\nabla_\theta u = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial r}$$

and

$$\nabla_{\theta+\pi/2} u = \frac{\partial u}{\partial x}\cos\left(\theta+\frac{\pi}{2}\right)+\frac{\partial u}{\partial y}\sin\left(\theta+\frac{\pi}{2}\right) = -\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta$$
$$= \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)$$
$$= \frac{1}{r}\frac{\partial u}{\partial \theta}$$

5. Assuming that $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ and noting that $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta-\frac{\partial v}{\partial x}\sin\theta$$
$$= \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right)$$
$$= \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Likewise

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) = -\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta = -\frac{\partial v}{\partial y}\sin\theta-\frac{\partial v}{\partial x}\cos\theta$$
$$= -\left(\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}\right)$$
$$= -\frac{\partial v}{\partial r}$$

6. Let the vector $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$ denote the position of a point on the circle $x^2+y^2 = 4$. In order to obtain a vector $\mathbf{v}$ that is tangent to the circle we require $\mathbf{r}\cdot\mathbf{v}$, which is satisfied for the vector $\mathbf{v} = -y\mathbf{i}+x\mathbf{j}$. Hence we can compute

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j} = \frac{1}{2}\left(-y\mathbf{i}+x\mathbf{j}\right)$$

And so
$$\frac{du}{ds} = \nabla_v u = \nabla u\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds} = -2xy$$

In order to determine at what point u attains its smallest value we are looking for the point such that $du/ds = 0$. Given the fact that $du/ds = -2xy$ the points that are to be considered are $(\pm 2, 0)$ and $(0, \pm 2)$, since these points both satisfy the condition $du/ds = 0$ as well as $x^2+y^2 = 4$. However, since we are looking for the smallest value of u the only candidate point is $(0, \pm 2)$, which gives $u\,(0, \pm 2) = -4$.

7. Let the vector $\mathbf{n}$ be a unit vector that is perpendicular to the tangent vector $\mathbf{u} = (dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}$ at every point (x, y) lying on a curve C of the domain in which $u(s, n)$ and $v(s, n)$ are given. In other words, let $\mathbf{n}$ satisfy the condition

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \right) = 0 \implies \mathbf{n} = -\frac{dy}{ds}\mathbf{i} + \frac{dx}{ds}\mathbf{j} = \frac{dx}{dn}\mathbf{i} + \frac{dy}{dn}\mathbf{j}$$

Hence, under the hypotheses of Problem 3 we can compute

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = -\frac{\partial v}{\partial x}\frac{dy}{ds} + \frac{\partial v}{\partial y}\frac{dx}{ds} = \frac{\partial v}{\partial x}\frac{dx}{dn} + \frac{\partial v}{\partial y}\frac{dy}{dn} = \frac{\partial v}{\partial n}$$

8. The directional derivative attains its maximum value in the direction of ∇u and so its value will be

$$|\nabla u| = \sqrt{36x^2 + 4y^2} = 2\sqrt{9x^2 + y^2}$$

Under the condition that $x^2 + y^2 = 1$ it should be trivial to see that the largest value of the directional derivative of $u = 3x^2 + y^2$ is 6 in the direction $\mathbf{i}$ at the point $(1, 0)$ or $-\mathbf{i}$ at the point $(-1, 0)$.

9.

$$\nabla_i F = \frac{F(2,1,1) - F(1,1,1)}{\sqrt{1^2}} = 3 \qquad \nabla_{i+j} F = \frac{F(2,2,1) - F(1,1,1)}{\sqrt{1^2 + 1^2}} = \frac{7}{\sqrt{2}}$$

$$\nabla_{i+j+k} F = \frac{F(2,2,2) - F(1,1,1)}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{15}{\sqrt{3}}$$

Section 2.18

1. (a) If $w = 1/\sqrt{x^2 + y^2}$ then

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2}{(x^2 + y^2)^{5/2}} \qquad \frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2}{(x^2 + y^2)^{5/2}}$$

(b) If $w = \tan^{-1} y/x$ then

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = -\frac{2xy}{(x^2 + y^2)^2}$$

(c) If $w = e^{x^2 - y^2}$ then

$$\frac{\partial^3 w}{\partial x \partial y^2} = 4xe^{x^2 - y^2}\left(2y^2 - 1\right) \qquad \frac{\partial^3 w}{\partial x^2 \partial y} = -4ye^{x^2 - y^2}\left(2x^2 + 1\right)$$

(d) If $w = x^m y^n$ then

$$\frac{\partial^{m+n} w}{\partial x^m \partial y^n} = \frac{\partial^{m+n}}{\partial x^m \partial y^n} x^m y^n = \frac{\partial^{m-1+n}}{\partial x^m \partial y^n} m x^{m-1} y^n = \frac{\partial^{m-1+n-1}}{\partial x^m \partial y^n} mn x^{m-1} y^{n-1} = \cdots = m!n!$$

2. (a) If $z = x/\left(x^2+y^2\right)$ then

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(-\frac{2xy}{\left(x^2+y^2\right)^2}\right) = \frac{2y\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3} \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(-\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right) = \frac{2y\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3}
\end{aligned}$$

and so $\partial^2 z/(\partial x \partial y) = \partial^2 z/(\partial y \partial x)$.

(b) If $w = \sqrt{x^2+y^2+z^2}$ then

$$\begin{aligned}
\frac{\partial^3 w}{\partial x \partial y \partial z} &= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}\right) = z\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-1/2}\right) \\
&= -yz\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2} \\
&= \frac{3xyz}{\left(x^2+y^2+z^2\right)^{5/2}} \\
\frac{\partial^3 w}{\partial z \partial y \partial x} &= \frac{\partial^2}{\partial z \partial y}\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\right) = x\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-1/2}\right) \\
&= -xy\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-3/2} \\
&= \frac{3xyz}{\left(x^2+y^2+z^2\right)^{5/2}} \\
\frac{\partial^3 w}{\partial y \partial z \partial x} &= \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-1/2}\right) \\
&= -xz\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2} \\
&= \frac{3xyz}{\left(x^2+y^2+z^2\right)^{5/2}}
\end{aligned}$$

3. A harmonic function $F(x,y)$ is defined by the relation $\nabla^2 F = 0$. Hence, we check that the given functions satisfy this condition next:

(a) Let $F(x,y) = e^x \cos y$. Then

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = e^x\left(\cos y - \cos y\right) = 0$$

(b) Let $F(x, y) = x^3 - 3xy^2$. Then

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 6x - 6x = 0$$

(c) Let $F(x, y) = \ln \sqrt{x^2 + y^2}$. Then

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{-x^2 + y^2}{\left(x^2 + y^2\right)^2} + \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} = 0$$

4. (a) Let a function $F(x, y)$ be a harmonic function, in other words let $\nabla^2 F(x, y) = 0$. It then follows that

$$\nabla^4 F = \nabla^2 \left(\nabla^2 F\right) = \nabla^2 (0) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(0) = \frac{\partial^2}{\partial x^2}0 + \frac{\partial^2}{\partial y^2}0 + \frac{\partial^2}{\partial z^2}0 = 0$$

(b) Let $F(x, y) = xe^x \cos y$. Then

$$\nabla^4 F = \nabla^2 \left(2e^x \cos y\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) 2e^x \cos y = 2e^x \cos y - 2e^x \cos y = 0$$

Next, let $G(x, y) = x^4 - 3x^2 y^2$. Then

$$\nabla^4 G = \nabla^2 \left(6x^2 - 6y^2\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(6x^2 - 6y^2\right) = 12 - 12 = 0$$

(c) Let $F(x, y) = ax^2 + bxy + cy^2$. Then in order to make $F(x, y)$ harmonic we need to solve the equation

$$\nabla^2 F = 2a + 2c = 0 \implies a = -c$$

(d) Let $F(x, y) = ax^3 + bx^2 y + cxy^2 + dy^3$. Then in order to make $F(x, y)$ harmonic we need to solve the equation

$$\nabla^2 F = (3a + c)\, x + (3d + b)\, y = 0 \implies c = -3a,\ b = -3d$$

5. (a) Let $u = u(x, y)$ and $v = v(x, y)$. Then

$$
\begin{aligned}
\nabla^2 (uv) &= \nabla \cdot \nabla (uv) \\
&= \nabla \cdot (u \nabla v + v \nabla u) \\
&= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot \left[u \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right) + v \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \right] \\
&= u \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right) + \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right) \\
&\quad + v \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) + \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \\
&= u \left[\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) \right] + v \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \right] \\
&\quad + 2 \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \cdot \left(\frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} \right) \\
&= u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 2 \nabla u \cdot \nabla v \\
&= u \nabla^2 v + v \nabla^2 u + 2 \nabla u \cdot \nabla v
\end{aligned}
$$

(b) Let $u = u(x, y, z)$ and $v = v(x, y, z)$. Then

$$\begin{aligned}
\nabla^2 (uv) &= \nabla \cdot \nabla (uv) \\
&= \nabla \cdot (u\nabla v + v\nabla u) \\
&= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left[u\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) + v\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right)\right] \\
&= u\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) \\
&\quad + \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) \\
&\quad + v\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) \\
&\quad + \left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) \\
&= u\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)\right] \\
&\quad + v\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right] \\
&\quad + 2\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) \\
&= u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + 2\nabla u \cdot \nabla v \\
&= u\nabla^2 v + v\nabla^2 u + 2\nabla u \cdot \nabla v
\end{aligned}$$

(c) If $u = u(x, y)$ and $v = v(x, y)$ are harmonic then $\nabla^2 u = 0$ and $\nabla^2 v = 0$, and so

$$\begin{aligned}
\nabla^4 w = \nabla^2\left(\nabla^2 w\right) = \nabla^2\left[\nabla^2 (xu + v)\right] &= \nabla^2\left(x\nabla^2 u + u\nabla^2 x + 2\nabla x \cdot \nabla u + \nabla^2 v\right) \\
&= \nabla^2 (2\nabla x \cdot \nabla u) \\
&= \nabla^2\left(2\frac{\partial u}{\partial x}\right) \\
&= 2\frac{\partial}{\partial x}\left(\nabla^2 u\right) \\
&= 0
\end{aligned}$$

(d) As before, if $u = u(x, y)$ and $v = v(x, y)$ are harmonic then $\nabla^2 u = 0$ and $\nabla^2 v = 0$,

and so

$$\begin{aligned}
\nabla^4 w = \nabla^2\left(\nabla^2 w\right) = \nabla^2\left[\nabla^2\left(r^2 u + v\right)\right] &= \nabla^2\left(r^2\nabla^2 u + u\nabla^2 r^2 + 2\nabla r^2\cdot\nabla u + \nabla^2 v\right)\\
&= \nabla^2\left(u\nabla^2 r^2 + 2\nabla r^2\cdot\nabla u\right)\\
&= \nabla^2\left(6u + 4\mathbf{r}\cdot\nabla u\right)\\
&= 6\nabla^2 u + \nabla^2\left(4\mathbf{r}\cdot\nabla u\right)\\
&= 4\nabla^2\left(\mathbf{r}\cdot\nabla u\right)\\
&= 4\left(\mathbf{r}\cdot\nabla\right)\nabla^2 u\\
&= 0
\end{aligned}$$

6.

$$\begin{aligned}
\frac{\partial^2 z}{\partial u\partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\\
&= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)\\
&= \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u\partial v} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial v}\\
&= \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u\partial v} + \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial u\partial v} + \left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v}\\
&= \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u\partial v} + \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} + \frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial u\partial v}
\end{aligned}$$

7. Using (2.133) we find

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial w}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 w}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 w}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial^2 w}{\partial\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\partial w}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}\\
\frac{\partial^2 w}{\partial y^2} &= \frac{\partial w}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 w}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + 2\frac{\partial^2 w}{\partial r\partial\theta}\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial^2 w}{\partial\theta^2}\left(\frac{\partial\theta}{\partial y}\right)^2 + \frac{\partial w}{\partial\theta}\frac{\partial^2\theta}{\partial y^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial r}\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right) + \frac{\partial^2 w}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + 2\frac{\partial^2 w}{\partial r\partial\theta}\left(\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}\right)\\
+ \frac{\partial^2 w}{\partial\theta^2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2\right] + \frac{\partial w}{\partial\theta}\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right)
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\frac{\partial r}{\partial x} = \cos\theta &&\frac{\partial r}{\partial y} = \sin\theta &&\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} &&\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}\\
&\frac{\partial^2 r}{\partial x^2} = \frac{\sin^2\theta}{r} &&\frac{\partial^2 r}{\partial y^2} = \frac{\cos^2\theta}{r} &&\frac{\partial^2\theta}{\partial x^2} = 2\cos\theta\sin\theta &&\frac{\partial^2\theta}{\partial y^2} = -2\cos\theta\sin\theta
\end{aligned}$$

this reduces to

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}$$

8. We will use the 3-D Laplacian in cylindrical coordinates to deduce the 3-D Laplacian in spherical coordinates. Before we start we note that if the equations of transformation from (z,r) to (ρ,ϕ) are the same as those from (x,y) to (r,θ) then

$$\rho=\sqrt{z^2+r^2} \qquad\qquad \phi=\tan^{-1}\frac{r}{z}$$

and

$$dz=\cos\phi d\rho-\rho\sin\phi d\phi \qquad\qquad dr=\sin\phi d\rho+\rho\cos\phi d\phi$$

$$d\rho=\cos\phi dz+\sin\phi dr \qquad\qquad d\phi=-\frac{\sin\phi}{\rho}dz+\frac{\cos\phi}{\rho}dr$$

The 3-D Laplacian in cylindrical coordinates is given by

$$\nabla^2 w=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}$$

Then we can obtain the 3-D Laplacian in spherical coordinates by substituting for r and z. To this end, we first calculate $\partial w/\partial r$ and $\partial^2 w/\partial r^2$:

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial \rho}\frac{\partial \rho}{\partial r}+\frac{\partial w}{\partial \phi}\frac{\partial \phi}{\partial r}=\sin\phi\frac{\partial w}{\partial \rho}+\frac{\cos\phi}{\rho}\frac{\partial w}{\partial \phi}$$

and so

$$\begin{aligned}\frac{\partial^2 w}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)&=\sin\phi\frac{\partial}{\partial \rho}\left(\frac{\partial w}{\partial r}\right)+\frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\left(\frac{\partial w}{\partial r}\right)\\&=\sin\phi\frac{\partial}{\partial \rho}\left(\sin\phi\frac{\partial w}{\partial \rho}+\frac{\cos\phi}{\rho}\frac{\partial w}{\partial \phi}\right)+\frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial w}{\partial \rho}+\frac{\cos\phi}{\rho}\frac{\partial w}{\partial \phi}\right)\\&=\sin^2\phi\frac{\partial^2 w}{\partial \rho^2}-\frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial w}{\partial \phi}+\frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 w}{\partial \rho\partial \phi}+\frac{\cos^2\phi}{\rho}\frac{\partial w}{\partial \rho}\\&\quad+\frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 w}{\partial \rho\partial \phi}-\frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial w}{\partial \phi}+\frac{\cos^2\phi}{\rho^2}\frac{\partial^2 w}{\partial \phi^2}\end{aligned}$$

Similarly, we calculate $\partial w/\partial z$ and $\partial^2 w/\partial z^2$:

$$\frac{\partial w}{\partial z}=\frac{\partial w}{\partial \rho}\frac{\partial \rho}{\partial z}+\frac{\partial w}{\partial \phi}\frac{\partial \phi}{\partial z}=\cos\phi\frac{\partial w}{\partial \rho}-\frac{\sin\phi}{\rho}\frac{\partial w}{\partial \phi}$$

and so

$$\begin{aligned}
\frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) &= \cos\phi\frac{\partial}{\partial \rho}\left(\frac{\partial w}{\partial z}\right) - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\left(\frac{\partial w}{\partial z}\right) \\
&= \cos\phi\frac{\partial}{\partial \rho}\left(\cos\phi\frac{\partial w}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial w}{\partial \phi}\right) - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial w}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial w}{\partial \phi}\right) \\
&= \cos^2\phi\frac{\partial^2 w}{\partial \rho^2} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial w}{\partial \phi} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 w}{\partial \rho\partial \phi} + \frac{\sin^2\phi}{\rho}\frac{\partial w}{\partial \rho} \\
&\quad - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 w}{\partial \rho\partial \phi} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial w}{\partial \phi} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 w}{\partial \phi^2}
\end{aligned}$$

It then follows that

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2}$$

Hence,

$$\begin{aligned}
\nabla^2 w &= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \\
&= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho\sin\phi}\left(\sin\phi\frac{\partial w}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial w}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial w}{\partial \rho} \\
&= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial \theta^2} + \frac{2}{\rho}\frac{\partial w}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial w}{\partial \phi}
\end{aligned}$$

9. Using (2.138) we can prove that the biharmonic equation in x and y becomes

$$\begin{aligned}
\nabla^4 w &= \nabla^2\left(\nabla^2 w\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\nabla^2 w\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\nabla^2 w\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\nabla^2 w\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \\
&= \frac{\partial^4 w}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \theta^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial \theta^2} - \frac{1}{r^2}\frac{\partial w}{\partial r^2} + \frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial w}{\partial r} \\
&= 0
\end{aligned}$$

10. We start by finding $\partial u/\partial x$ using (2.69) and defining $F(x, y, u, v) = xy + uv - 1 = 0$ and $G(x, y, u, v) = xu + yv - 1 = 0$:

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F, G)}{\partial(x, v)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} y & u \\ u & y \end{vmatrix}}{\begin{vmatrix} v & u \\ x & y \end{vmatrix}} = \frac{u^2 - y^2}{1 - 2xu}$$

It then follows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{2\left(u^2 - y^2\right)}{\left(1 - 2xu\right)^2}\left(x\frac{\partial u}{\partial x} + u\right) + \frac{2u}{1 - 2xu}\frac{\partial u}{\partial x} = \frac{2\left(u^2 - y^2\right)}{\left(1 - 2xu\right)^3}\left(2u - 3xu^2 - xy^2\right)$$

11. If u is harmonic then

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, we start by finding $\partial u/\partial x$ and $\partial u/\partial y$ using (2.69) and defining $F(x, y, u, v) = x - u^2 + v^2 = 0$ and $G(x, y, u, v) = y - 2uv = 0$:

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial\left(F, G\right)}{\partial\left(x, v\right)}}{\dfrac{\partial\left(F, G\right)}{\partial\left(u, v\right)}} = -\frac{\begin{vmatrix} 1 & 2v \\ 0 & -2u \end{vmatrix}}{\begin{vmatrix} -2u & 2v \\ -2v & -2u \end{vmatrix}} = \frac{u}{2\left(u^2 + v^2\right)} = \frac{u}{2\left(2u^2 - x\right)}$$

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial\left(F, G\right)}{\partial\left(y, v\right)}}{\dfrac{\partial\left(F, G\right)}{\partial\left(u, v\right)}} = -\frac{\begin{vmatrix} 0 & 2v \\ 1 & -2u \end{vmatrix}}{\begin{vmatrix} -2u & 2v \\ -2v & -2u \end{vmatrix}} = \frac{v}{2\left(u^2 + v^2\right)} = \frac{y}{4u\left(2u^2 - x\right)}$$

from which follows

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{u\left(2u^2 - 3x\right)}{4\left(2u^2 - x\right)^3} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{y^2\left(2u^2 - x\right) - 4y^2u^2 + 4u^2\left(2u^2 - x\right)^2}{16u^3\left(2u^2 - x\right)^3} \\
&= -\frac{u^2\left(u^2 - x\right)\left(2u^2 - x\right) + 4u^4\left(u^2 - x\right) - u^2\left(2u^2 - x\right)^2}{4u^3\left(2u^2 - x\right)^3} \\
&= -\frac{u\left(2u^2 - 3x\right)}{4\left(2u^2 - x\right)^3}
\end{aligned}$$

And so we can verify that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u\left(2u^2 - 3x\right)}{4\left(2u^2 - x\right)^3} - \frac{u\left(2u^2 - 3x\right)}{4\left(2u^2 - x\right)^3} = 0$$

which proves u is harmonic.

12. (a) If $u = x^2$ then $du/dx = 2x$, and so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{2x}{y + x^2} \implies \frac{dy}{du} = \frac{1}{y + u}$$

(b) If $v = y - 3$ and $u = x - 1$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(v+3)\frac{du}{dx} = \frac{2(u+1)-(v+3)+1}{(u+1)+(v+3)-4}$$

$$\frac{dv}{du} = \frac{2u-v}{u+v}$$

(c) If $t = \ln x$, then $dt/dx = 1/x$, and so

$$\begin{aligned}
x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} &= 0 \\
x^2\frac{d^2}{dx^2}\left(\frac{dy}{dt}\frac{dt}{dx}\right) + 3x\frac{d}{dx}\left(\frac{dy}{dt}\frac{dt}{dx}\right) + \frac{dy}{dt}\frac{dt}{dx} &= \\
x^2\frac{d^2}{dx^2}\left(\frac{1}{x}\frac{dy}{dt}\right) + 3x\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) + \frac{1}{x}\frac{dy}{dt} &= \\
x^2\frac{d^2}{dx^2}\left(\frac{1}{x}\frac{dy}{dt}\right) + 3x\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right) + \frac{1}{x}\frac{dy}{dt} &= \\
x^2\frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right) + 3x\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right) + \frac{1}{x}\frac{dy}{dt} &= \\
x^2\left(\frac{2}{x^3}\frac{dy}{dt} - \frac{3}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^3}\frac{d^3y}{dt^3}\right) + 3x\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right) + \frac{1}{x}\frac{dy}{dt} &= \\
\frac{d^3y}{dt^3} &=
\end{aligned}$$

(d) The solution to the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

is given by $x = (1/2)y^2$, and so

$$\frac{d^2x}{dy^2} - 1 = 0$$

(e) If $v = e^{-x^2}y$ then substituting for y gives

$$\begin{aligned}
\frac{d^2}{dx^2}\left(ve^{x^2}\right) - 4x\frac{d}{dx}\left(ve^{x^2}\right) + ve^{x^2}\left(3x^2-2\right) &= 0 \\
\frac{d}{dx}\left(e^{x^2}\frac{dv}{dx} + 3xve^{x^2}\right) - 4x\left(e^{x^2}\frac{dv}{dx} + 3xve^{x^2}\right) + ve^{x^2}\left(3x^2-2\right) &= \\
\frac{d^2v}{dx^2} + 4x\frac{dv}{dx} + 4x^2v + 2v - 4x\frac{dv}{dx} - 8x^2v + 3x^2v - 2v &= \\
\frac{d^2v}{dx^2} - x^2v &=
\end{aligned}$$

(f) If $z = bx - ay$ and $w = ax + by$ and

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} \qquad\qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}$$

then

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} &= 0 \\ a\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}\right) + b\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}\right) &= \\ a\left(b\frac{\partial u}{\partial z} + a\frac{\partial u}{\partial w}\right) + b\left(-a\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial w}\right) &= \\ \frac{\partial u}{\partial w} &= \end{aligned}$$

(g) If $z = x + y$ and $w = x - y$ then

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}\right) \\ &= \frac{\partial u}{\partial z}\frac{\partial^2 z}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial x} \\ &= \frac{\partial u}{\partial z}\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial z \partial w}\frac{\partial w}{\partial x}\right)\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial x^2} \\ &\quad + \left(\frac{\partial^2 u}{\partial z \partial w}\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial w^2}\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial x} \\ &= \frac{\partial u}{\partial z}\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\left(\frac{\partial z}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial z \partial w}\frac{\partial z}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial^2 u}{\partial w^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial x^2} \\ &= \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial z \partial w} + \frac{\partial^2 u}{\partial w^2} \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial u}{\partial z}\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial z \partial w}\frac{\partial z}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial^2 u}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial y^2} \\ &= \frac{\partial^2 u}{\partial z^2} - 2\frac{\partial^2 u}{\partial z \partial w} + \frac{\partial^2 u}{\partial w^2} \end{aligned}$$

and so

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z \partial w} = 0 \implies \frac{\partial^2 u}{\partial z \partial w} = 0$$

13. (a) If $u = z(x)v$ then

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}\left(z(x)\,v\right) = z(x)\frac{dv}{dx} + z'(x)\,v \\ \frac{d^2 u}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dx}\left(z(x)\frac{dv}{dx} + z'(x)\,v\right) = z(x)\frac{d^2 v}{dx^2} + 2z'(x)\frac{dv}{dx} + z''(x)\,v \end{aligned}$$

where

$$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = h'(x)\frac{dv}{dy}$$
$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(h'(x)\frac{dv}{dy}\right) = h'^2(x)\frac{d^2v}{dy^2} + h''(x)\frac{dv}{dy}$$

Substituting for du/dx and d^2u/dx^2 in the definition of the Sturm-Liouville differential equation then gives

$$p(x)\frac{d^2u}{dx^2} + p'(x)\frac{du}{dx} + [\lambda k(x) - q(x)]u = 0$$
$$pz\frac{d^2v}{dx^2} + (2pz' + p'z)\frac{dv}{dx} + [pz'' + p'z' + z(\lambda k - q)]v =$$
$$pzh'^2\frac{d^2v}{dy^2} + (pzh'' + 2pz'h' + p'zh')\frac{dv}{dy} + [pz'' + p'z' + z(\lambda k - q)]v =$$

(b) To show that for a proper choice of $h(x)$ and $z(x)$ the equation from part (a) can be written as

$$\frac{d^2v}{dy^2} + (\lambda - Q(y))v = 0$$

we require that the coefficient of λv equals the coefficient of d^2v/dy^2, in other words $zk = pzh'^2$ so that $h' = \sqrt{k/p}$ and hence, a suitable function $h(x)$ is given by

$$h(x) = \int_{x_0}^{x}\sqrt{\frac{k(t)}{p(t)}}\,dt$$

where x_0 is on the interval $a \le x \le b$. Furthermore, we require that the coefficient of dv/dy is 0:

$$pzh'' + 2pz'h' + p'zh = 0$$

The left hand side can also be written as $(d/dx)\ln(h'z^2p)$, since

$$\frac{d}{dx}\ln\left(h'z^2p\right) = \frac{1}{h'z^2p}\left(pz^2h'' + 2pzz'h' + p'z^2h'\right)$$

Choosing $h'z^2p = 1$ and $z = (kp)^{-1/4}$ then gives

$$\begin{aligned}\frac{d}{dx}\ln\left(h'z^2p\right) &= p(kp)^{-1/2}h'' - \frac{p}{2}(kp)^{-3/2}(k'p + kp')h' + p'(kp)^{-1/2}h' \\ &= \frac{p}{2k}\left(\frac{k'}{p} - \frac{kp'}{p^2}\right) - \frac{1}{2kp}(k'p + kp') + \frac{p'}{p} \\ &= \frac{k'}{2k} - \frac{p'}{2p} - \frac{k'}{2k} - \frac{p'}{2p} + \frac{p'}{p} \\ &= 0\end{aligned}$$

Taking these conditions into account the equation from part (a) can thus be rewritten as

$$\begin{aligned}\frac{d^2v}{dy^2}+\frac{1}{pzh'^2}\left[pz''+p'z'+z\left(\lambda k-q\right)\right]v=\frac{d^2v}{dy^2}+\left(\frac{z''}{zh'^2}+\frac{p'z'}{pzh'^2}+\lambda-\frac{q}{ph'^2}\right)v=0\\ \frac{d^2v}{dy^2}+\left[\lambda-\left(-\frac{z''}{zh'^2}-\frac{p'z'}{pzh'^2}+\frac{q}{ph'^2}\right)\right]v=\\ \frac{d^2v}{dy^2}+\left[\lambda-\left(-\frac{z''}{zh'^2}-\frac{p'z'}{pzh'^2}+\frac{q}{k}\right)\right]v=\end{aligned}$$

Next, let us consider a function $f=1/z(y)$. Differentiating with respect to y then gives

$$\begin{aligned}\frac{df}{dy}&=\frac{df}{dx}\frac{dx}{dy}=\frac{d}{dx}\left(\frac{1}{z}\right)\frac{d\phi}{dy}=-\frac{z'}{z^2}\frac{d\phi}{dy}\\ \frac{d^2f}{dy^2}&=\frac{d}{dy}\left(\frac{df}{dy}\right)=\frac{d}{dy}\left(-\frac{z'}{z^2}\frac{d\phi}{dy}\right)=\left(\frac{2z'^2}{z^3}-\frac{z''}{z^2}\right)\left(\frac{d\phi}{dy}\right)^2-\frac{z'}{z^2}\frac{d^2\phi}{dy^2}\end{aligned}$$

Now since $(dy/dx)(dx/dy)=1$ and $x=\phi(y)=\phi(h(x))$ we may conclude that

$$\frac{dy}{dx}\frac{dx}{dy}=\frac{dh}{dx}\frac{d\phi}{dy}=1\implies\frac{d\phi}{dy}=\frac{1}{h'}$$

In addition

$$\begin{aligned}\frac{d}{dx}\left(\frac{dy}{dx}\frac{dx}{dy}\right)&=0\\ \frac{d^2y}{dx^2}\frac{dx}{dy}+\frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dy}{dx}&=\\ \frac{d^2y}{dx^2}\frac{dx}{dy}+\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^2&=\\ \frac{d^2y}{dx^2}&=-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3\end{aligned}$$

from which follows $d^2\phi/dy^2=-h''/h'^3$. Substitution then gives

$$\frac{1}{f}\frac{d^2f}{dy^2}=-\frac{z''}{zh'^2}+\frac{2z'^2}{z^2h'^2}+\frac{z'h''}{zh'^3}=-\frac{z''}{zh'^2}-\frac{z'}{zh'^2}\left(-\frac{2z'}{z}-\frac{h''}{h'}\right)=-\frac{z''}{zh'^2}-\frac{p'z'}{pzh'^2}$$

where we have utilized the relation $h'z^2p=1$ to compute

$$\frac{d}{dx}\left(h'z^2p\right)=0\implies\frac{p'}{p}=-\frac{2z'}{z}-\frac{h''}{h'}$$

As such, we can rewrite the transformed Sturm-Liouville differential equation as

$$\frac{d^2v}{dy^2} + (\lambda - Q(y))\, v = 0$$

where

$$Q(y) = -\frac{z''}{zh'^2} - \frac{p'z'}{pzh'^2} + \frac{q}{k} = \frac{1}{f}\frac{d^2 f}{dy^2} + \frac{q}{k} \qquad f = \frac{1}{z(y)}$$

14. (a) Let $g(x, y)$ be continuous on a square region $R : x_0 \leq x \leq x_0 + h,\ y_0 \leq y \leq y_0 + h$ and $g(x, y) > 0$ in R. Furthermore, let M be the absolute minimum of $g(x, y)$ for R. Clearly, $M > 0$, since $g(x, y) > 0$ for R. Hence,

$$\begin{aligned} g(x,y) &\geq M \\ \iint_R g(x,y)\ dxdy &\geq \iint_R M\, dxdy \\ \iint_R g(x,y)\ dxdy &\geq Mh^2 \end{aligned}$$

Now since $M > 0$ so will $Mh^2 > 0$ and hence, we find

$$\iint_R g(x,y)\ dxdy > 0$$

(b) In order to prove that $g(x, y) \equiv 0$ whenever $g(x, y)$ is continuous in domain D and

$$\iint_R g(x,y)\ dxydy = 0$$

for every region R contained in D let us consider the opposite and show that we obtain a contradiction. Suppose $g(x_0, y_0) > 0$ for some point (x_0, y_0) in D. Then by continuity (see Problem 7 of Section 2.4) $g(x, y) > 0$ in some region R contained in D. But by (a) we have just shown that the double integral over R cannot be 0 when $g(x, y) > 0$ and hence, we obtain a contradiction. As such, $g(x, y) \equiv 0$. Similarly, suppose $g(x_0, y_0) < 0$. Then again, by continuity $-g(x, y) > 0$ in some region R contained in D which again contradicts the result of part (a) and implies $g(x, y) \equiv 0$.

Section 2.21

1. (a) If $y = x^3 - 3x$ then

$$\frac{dy}{dx} = 3x^2 - 3 \qquad\qquad \frac{d^2y}{dx^2} = 6x$$

To locate the critical points we solve the equation $x^2 - 1 = 0$ which gives $x = \pm 1$. To classify the critical points we evaluate

$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -6 \qquad\qquad \left.\frac{d^2y}{dx^2}\right|_{x=1} = 6$$

Hence, the critical point at $x = -1$ is a relative maximum and the critical point at $x = 1$ is a relative minimum.

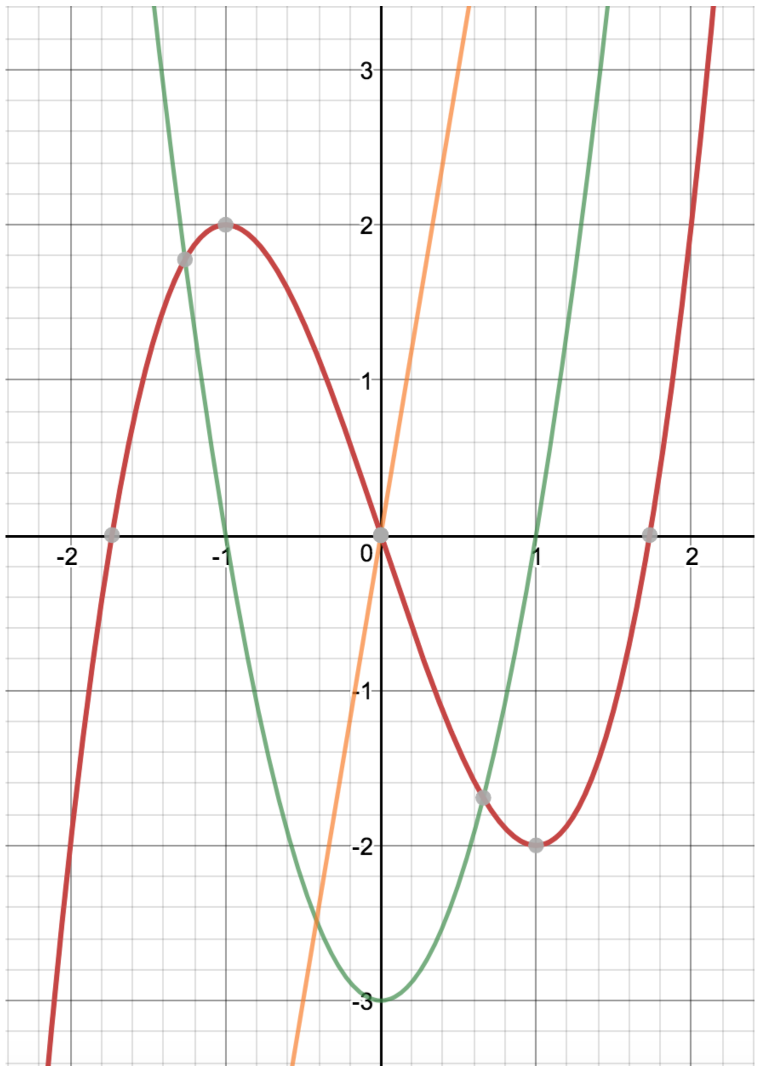

Figure 15: $y = x^3 - 3x$

(b) If $y = 2\sin x + \sin 2x$ then

$$\frac{dy}{dx} = 2\cos x + 2\cos 2x \qquad\qquad \frac{d^2y}{dx^2} = -2\sin x - 4\sin 2x$$

To locate the critical points we need to solve the equation $2\cos x + 2\cos 2x = 0$. This is satisfied for $x = \pi + 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, resulting in a horizontal inflection point, since

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pi+2n\pi} = 0$$

Another solution is given by $x = (\pi/3) + 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, resulting in a relative maximum, since

$$\left.\frac{d^2y}{dx^2}\right|_{x=(\pi/3)+2n\pi} = -3\sqrt{3}$$

and lastly, there is the solution $x = (-\pi/3) + 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, resulting in a relative minimum, since

$$\left.\frac{d^2y}{dx^2}\right|_{x=(-\pi/3)+2n\pi} = 3\sqrt{3}$$

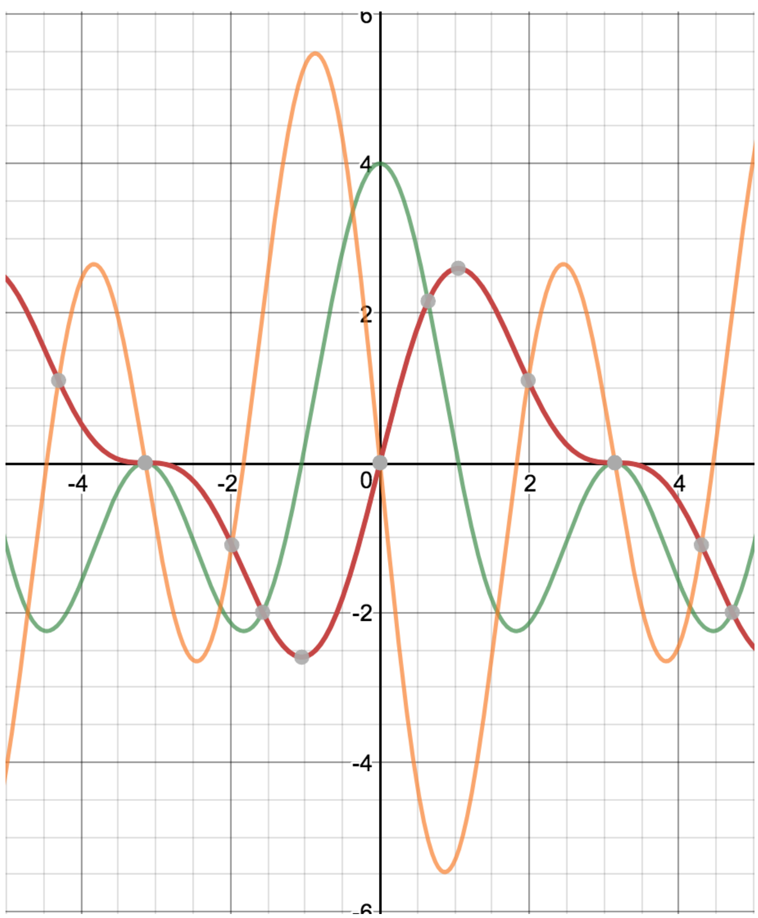

Figure 16: $y = 2\sin x + \sin 2x$

(c) If $y = e^{-x} - e^{-2x}$ then

$$\frac{dy}{dx} = -e^{-x} + 2e^{-2x} \qquad \frac{d^2y}{dx^2} = e^{-x} - 4e^{-2x}$$

To locate the critical points we need to solve the equation $-e^{-x} - e^{-2x} = 0$, which is satisfied for $x = \ln 2$. Since

$$\left.\frac{d^2y}{dx^2}\right|_{x=\ln 2} = -\frac{1}{2}$$

the point $x = \ln 2$ is a relative maximum.

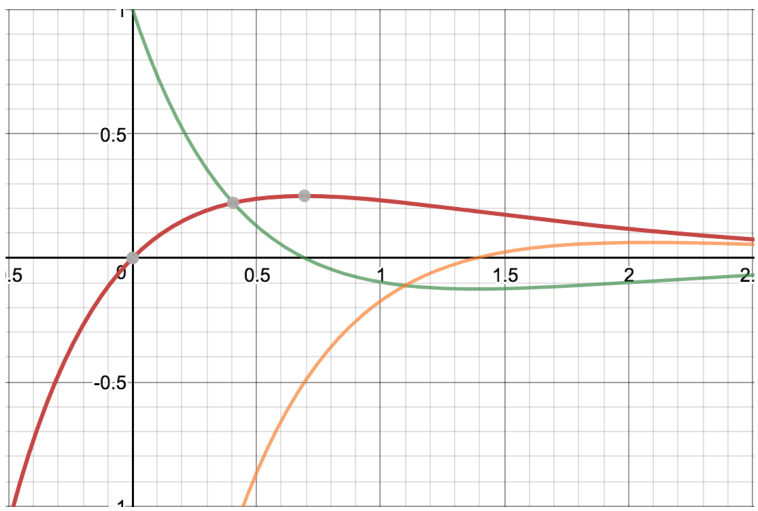

Figure 17: $y = 2\sin x + \sin 2x$

2. Let us first consider the case where n is odd (i.e. $n = 1, 3, 5, \ldots$), so that at the critical point $x_0 = 0$ we have $y'(x_0) = 0,\ y''(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0$, but $y^{(n)} = n! > 0$. Since $n-1$ is even we may conclude that the critical point is a horizontal inflection point. Next, we consider the case where n is even (i.e. $n = 2, 4, 6, \ldots$), so that at the critical point $x_0 = 0$ we have like before $y'(x_0) = 0,\ y''(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0$, but $y^{(n)}(x_0) = n! > 0$. Now $n-1$ will be odd and since $y^{(n)}(x_0) > 0$ and $y = x^n \geq 0$ for $-\infty \leq x \leq \infty$ we may conclude that the critical point is the absolute minimum of y.

3. (a) If $y = \cos x$, where $-\pi/2 \leq x \leq \pi/2$ then $y' = -\sin x$ and $y' = -\sin x = 0$ is satisfied for $x = \pm\pi/2$, which gives the absolute maximum as $y(\pi/2) = 1$ and the absolute minimum as $y(-\pi/2) = 0$.

   (b) If $y = \ln x$, where $0 < x \leq 1$ then the absolute maximum is located at $x = 1$ and is given by $y(1) = 0$.

   (c) The function $y = \tanh x$, for all x has no absolute maximum or minimum.

(d) If $y = x/(1+x^2)$, for all x then $y' = (1-x^2)/(1+x^2)^2$ and $y' = (1-x^2)/(1+x^2)^2 = 0$ is satisfied for $x = \pm 1$, which gives the absolute maximum as $y(1) = 1/2$ and the absolute minimum as $y(-1) = -1/2$.

4. (a) If $z = \sqrt{1 - x^2 - y^2}$ then

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - y^2}} \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

Hence, the critical point is $(0, 0)$. To determine the nature of the critical point we utilize (2.146) to (2.149) and find that the critical point $(0, 0)$ is a relative maximum.

(b) If $z = 1 + x^2 + y^2$ then

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

Hence, the critical point is $(0, 0)$, which is a relative minimum.

(c) If $z = 2x^2 - xy - 3y^2 - 3x + 7y$ then

$$\frac{\partial z}{\partial x} = 4x - y - 3 \qquad \frac{\partial z}{\partial y} = -x - 6y + 7$$

Hence, the critical point is $(1, 1)$, which is a saddle point.

(d) If $z = x^2 - 5xy - y^2$ then

$$\frac{\partial z}{\partial x} = 2x - 5y \qquad \frac{\partial z}{\partial y} = -5x - 2y$$

Hence, the critical point is $(0, 0)$, which is a saddle point.

(e) If $z = x^2 - 2xy + y^2$ then

$$\frac{\partial z}{\partial x} = 2x - 2y \qquad \frac{\partial z}{\partial y} = -2x + 2y$$

Hence, any point on the line $y = x$ is a critical point. Evaluating $B^2 - AC$ give 0 and hence, according to (2.149) the nature of the critical point is undetermined. However, plotting reveals that along the line $x = y$ the function $z = x^2 - 2xy + y^2$ attains a relative minimum.

(f) If $z = x^3 - 3xy^2 + y^3$ then

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial z}{\partial y} = -6xy + 3y^2$$

Hence, there is a triple critical point at $(0, 0)$, which is a saddle point as may be deduced from plotting the function.

(g) If $z = x^2 - 2x\,(\sin y + \cos y) + 1$ then

$$\frac{\partial z}{\partial x} = 2x - 2\,(\sin y + \cos y) \qquad\qquad \frac{\partial z}{\partial y} = -2x\,(\cos y - \sin y)$$

Hence, the critical points are given by $(\sqrt{2}, \pi/4 + 2n\pi)$, $(-\sqrt{2}, 5\pi/4 + 2n\pi)$, which are relative minima and $(0, n\pi - \pi/4)$, which is a saddle point, where $n = 0, \pm1, \pm2, \ldots$.

(h) If $z = xy^2 + x^2y - xy$ then

$$\frac{\partial z}{\partial x} = y^2 + 2xy - y \qquad\qquad \frac{\partial z}{\partial y} = 2xy + x^2 - x$$

Hence the critical points are given by $(1/3, 1/3)$, which is a relative minimum and $(0, 0)$, $(1, 0)$, $(0, 1)$, which are all saddle points.

(i) If $z = x^3 + y^3$ then

$$\frac{\partial z}{\partial x} = 3x^2 \qquad\qquad \frac{\partial z}{\partial y} = 3y^2$$

Hence, the critical point is given by $(0, 0)$, which is undetermined, since it satisfies (2.149): $B^2 - AC = 0$.

(j) If $z = x^4 + 3x^2y^2 + y^4$ then

$$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2 \qquad\qquad \frac{\partial z}{\partial y} = 6x^2y + 4y^3$$

Hence, the critical point is given by $(0, 0)$, which according to (2.149) is undetermined. However, plotting shows that this is the absolute minimum.

(k) If $z = [x^2 + (y+1)^2][x^2 + (y-1)^2]$ then

$$\frac{\partial z}{\partial x} = 4x\left(x^2 + y^2 + 1\right) \qquad\qquad \frac{\partial z}{\partial y} = 4y\left(x^2 + y^2 - 1\right)$$

Hence, the critical points are given by $(0, 0)$, which is a saddle point and $(0, \pm1)$, which are absolute minima.

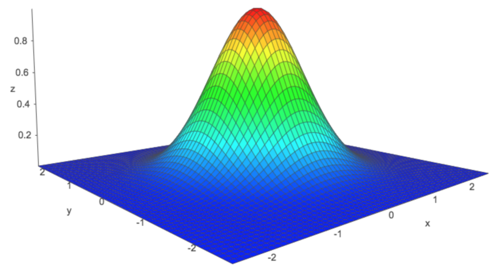

Figure 18: $z = e^{-x^2-y^2}$

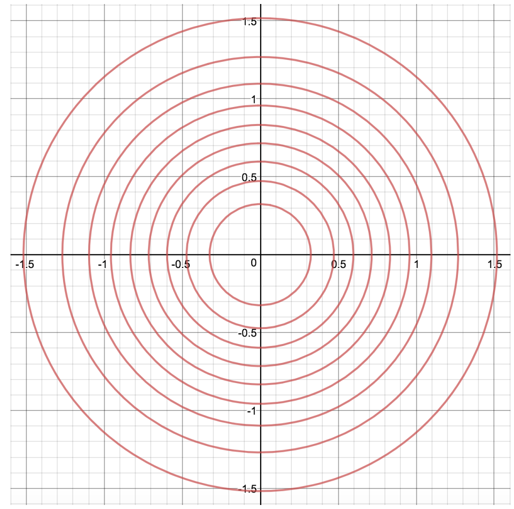

Figure 19: $z = e^{-x^2-y^2}$

5. (a) If $z = e^{-x^2-y^2}$ then

$$\frac{\partial z}{\partial x} = -2xe^{-x^2-y^2} \qquad\qquad \frac{\partial z}{\partial y} = -2ye^{-x^2-y^2}$$

Hence, the critical point is given by $(0, 0)$, which is the absolute maximum.

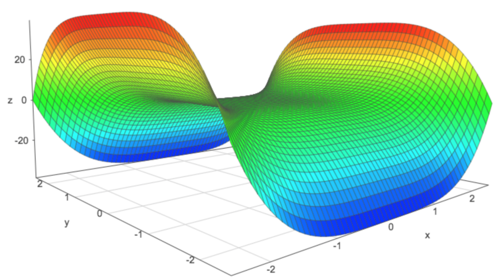

Figure 20: $z = x^4 - y^4$

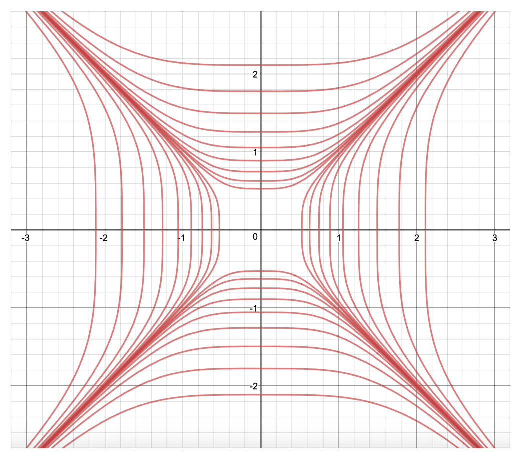

Figure 21: $z = x^4 - y^4$

(b) If $z = x^4 - y^4$ then

$$\frac{\partial z}{\partial x} = 4x^3 \qquad\qquad \frac{\partial z}{\partial y} = -4y^3$$

Hence, the critical point is given by $(0, 0)$, which is a saddle point.

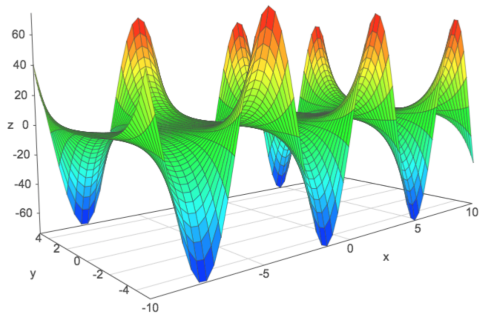

Figure 22: $z = \sin x \cosh y$

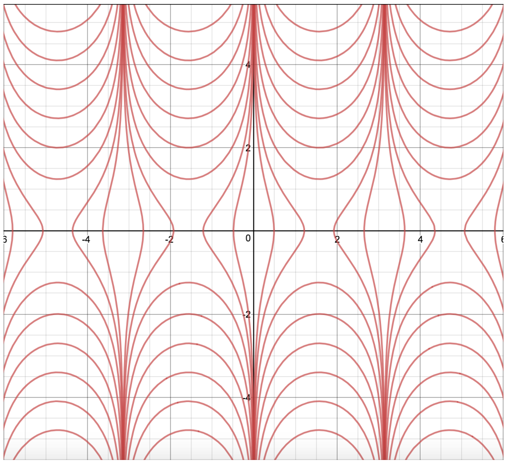

Figure 23: $z = \sin x \cosh y$

(c) If $z = \sin x \cosh y$ then

$$\frac{\partial z}{\partial x} = \cos x \cosh y \qquad \frac{\partial z}{\partial y} = \sin x \sinh y$$

Hence, the critical point is given by $(\pi/2 + 2n\pi, 0)$ where $n = 0, \pm 1, \pm 2, \ldots$, which is a saddle point.

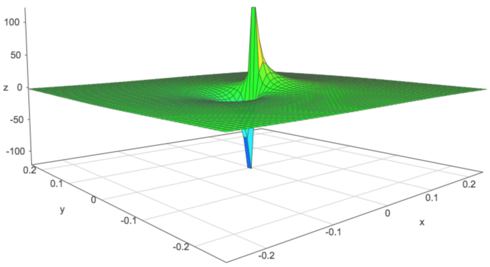

Figure 24: $z = x/\left(x^2 + y^2\right)$

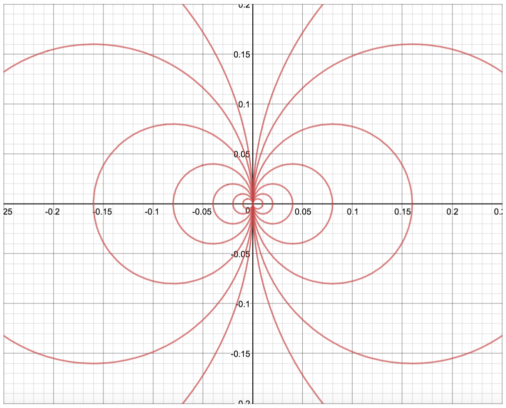

Figure 25: $z = x/\left(x^2 + y^2\right)$

(d) If $z = x/(x^2 + y^2)$ then

$$\frac{\partial z}{\partial x} = \frac{-x^2 + y^2}{\left(x^2 + y^2\right)^2} \qquad \frac{\partial z}{\partial y} = -\frac{2xy}{\left(x^2 + y^2\right)^2}$$

Hence, there are no critical points, but there is a discontinuity at $(0, 0)$.

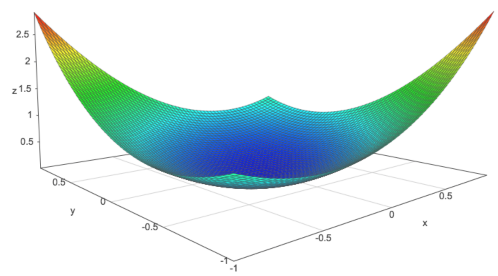

Figure 26: $z = x^2 - xy + y^2$

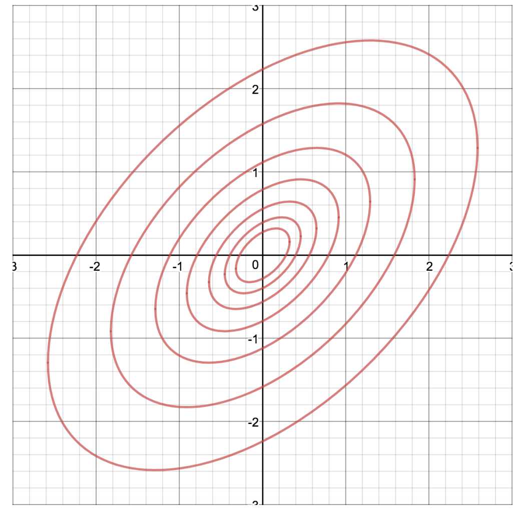

Figure 27: $z = x^2 - xy + y^2$

(e) If $z = x^2 - xy + y^2$ then

$$\frac{\partial z}{\partial x} = 2x - y \qquad\qquad \frac{\partial z}{\partial y} = -x + 2y$$

Hence, there is a critical point at $(0, 0)$, which is the absolute minimum.

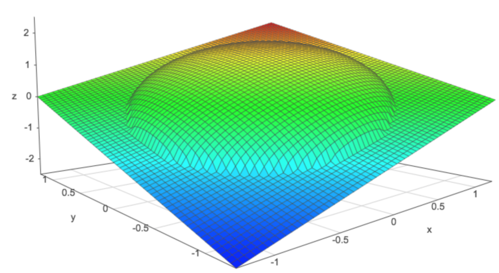

Figure 28: $z = x + y + \sqrt{1 - x^2 - y^2}$

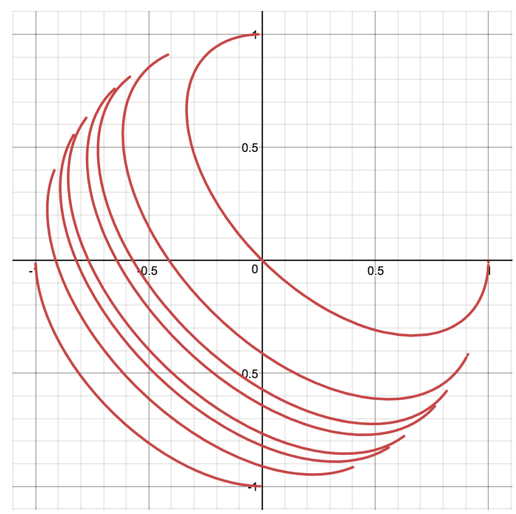

Figure 29: $z = x + y + \sqrt{1 - x^2 - y^2}$

(f) If $z = x + y + \sqrt{1 - x^2 - y^2}$ then

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - y^2}} + 1 \qquad\qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}} + 1$$

Hence, there is a critical point at $(\sqrt{3}/3, \sqrt{3}/3)$, which is the absolute maximum.

6. (a) If $z = 3x + 4y$, where $x^2 + y^2 = 1$ then we define $f(x, y) = z = 3x + 4y$ and $g(x, y) = x^2 + y^2 - 1 = 0$. Next, using relation (2.151): $\nabla f + \lambda \nabla g = \mathbf{0}$ we find the two equations

$$3 + 2\lambda x = 0 \qquad\qquad 4 + 2\lambda y = 0$$

Multiplying the first equation by x, the second by y, adding and using the fact that $x^2+y^2 = 1$ we find $\lambda = -(3x+4y)/2$. Substituting for λ in the two equations above then results in the critical points $(\pm 3/5, \pm 4/5)$. Hence, the absolute maximum is located at the point $(3/5, 4/5)$ and the absolute minimum is located at the point $(-3/5, -4/5)$.

(b) If $z = x^2 + y^2$, where $x^4 + y^4 = 1$ then we define $f(x, y) = z = x^2 + y^2$ and $g(x, y) = x^4 + y^4 - 1 = 0$. Next using (2.151) we find

$$2 + 4\lambda x^2 = 0 \qquad\qquad 2 + 4\lambda y^2 = 0$$

Multiplying the first equation by x^2, the second by y^2, adding and using the fact that $x^4 + y^4 = 1$ we find $\lambda = -(x^2 + y^2)/2$. Substituting for λ in the two equations above then results in the four critical points $(\pm 2^{-1/4}, \pm 2^{-1/4})$, which are the absolute maxima and the four critical points $(\pm 1, 0)$ and $(0, \pm 1)$, which are the absolute minima.

(c) If $z = x^2 + 24xy + 8y^2$, where $x^2 + y^2 = 25$ then we define $f(x, y) = z = x^2 + 24xy + 8y^2$ and $g(x, y) = x^2 + y^2 - 25 = 0$. Next, using (2.151) we find

$$x\,(1 + \lambda) + 12y = 0 \qquad\qquad 12x + y\,(8 + \lambda) = 0$$

Multiplying the first equation by x, the second by y, adding and using the fact that $x^2 + y^2 = 25$ we find $\lambda = -(7y^2 + 24xy + 25)/25$. Substituting for λ in the two equations above then results in the four critical points $(\pm 3, \pm 4)$, which are maxima and the four critical points $(\pm 4, \mp 3)$, which are minima.

(d) If $w = x + z$, where $x^2 + y^2 + z^2 = 1$ then we define $f(x, y, z) = x + z$ and $g(x, y) = x^2 + y^2 + z^2 - 1 = 0$. Next, using (2.151) we find

$$1 + 2\lambda x = 0 \qquad\qquad 2\lambda y = 0 \qquad\qquad 1 + 2\lambda z = 0$$

Multiplying the first equation by x, the second by y, the third by z, adding and using the fact that $x^2 + y^2 + z^2 = 1$ we find $\lambda = -(x + z)/2$. Substituting for λ in the three equations above then results in the two critical points $(1/\sqrt{2}, 0, 1/\sqrt{2})$, which is a maximum and $(-1/\sqrt{2}, 0, -1/\sqrt{2})$, which is a minimum.

(e) If $w = xyz$, where $x^2 + y^2 = 1$ and $x - z = 0$ then we define $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 - 1 = 0$ and $h(x, y, z) = x - z = 0$. Next, using (2.151) we find

$$yz + 2\lambda_1 x + \lambda_2 = 0 \qquad\qquad xz + 2\lambda_1 y = 0 \qquad\qquad xy - \lambda_2 = 0$$

Multiplying the first equation by x, the third by z, adding and using the fact that $x - z = 0$ we find $\lambda_1 = -yz/x$. Similarly, multiplying the first equation by x, the second by y, adding and using the fact that $x^2 + y^2 = 1$ we find $\lambda_2 = 2yz(1/x^2 - 1)$. Substituting for λ_1 and λ_2 in the first equation then results in the two critical points $(\pm\sqrt{2/3}, 1/\sqrt{3}, \pm\sqrt{2/3})$, which is a maximum and $(\pm\sqrt{2/3}, -1/\sqrt{3}, \pm\sqrt{2/3})$, which is a minimum.

(f) If $w = x^2 + y^2 + z^2$, where $x + y + z = 1$ and $x^2 + y^2 - z^2 = 0$ then we define $f(x, y, z) = x^2+y^2+z^2$, $g(x, y, z) = x+y+z-1 = 0$ and $h(x, y, z) = x^2+y^2-z^2 = 0$. Next, using (2.151) we find

$$2x + \lambda_1 + 2\lambda_2 x = 0 \qquad 2y + \lambda_1 + 2\lambda_2 y = 0 \qquad 2z + \lambda_1 - 2\lambda_2 z = 0$$

Multiplying the first equation by x, the second by y, the third by z, adding and using the fact that $x^2+y^2-z^2 = 0$ we find $\lambda_1 = -2(x^2+y^2+z^2) = -4z^2$. Adding the three equations above and substituting for λ_1 gives $\lambda_2 = (6z^2 - 1)/(1 - 2z)$. Substituting for λ_1 and λ_2 in the equations above results in the critical points $(1-\sqrt{2}/2, 1-\sqrt{2}/2, \sqrt{2}-1)$, which is the absolute minimum and $(1+\sqrt{2}/2, 1+\sqrt{2}/2, -\sqrt{2}-1)$, which is a relative minimum.

7. Let the function $f(x, y, z) = x^2 + y^2 + z^2$ denote the square of the distance of a point on the curve $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$ to the origin $(0, 0, 0)$. Finding the point on this curve nearest to the origin then implies minimizing the function $f(x, y, z)$ under the conditions $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$. Hence, let $g(x, y, z) = x^2 - xy + y^2 - z^2 - 1 = 0$ and $h(x, y, z) = x^2 + y^2 - 1 = 0$ and so using (2.151) we find

$$2x + \lambda_1 (2x - y) + 2\lambda_2 x = 0 \qquad 2y + \lambda_1 (2y - x) + 2\lambda_2 y = 0 \qquad 2z (1 - \lambda_1) = 0$$

Let us focus on the last of these three equations. We then have two options; either $z = 0$ or $\lambda_1 = 1$. Let us first consider $\lambda_1 = 1$ and show why this is not a suitable choice. Substituting for λ_1 in the first two equations above and adding gives $\lambda_2 = -3/2$. Substituting for λ_1 and λ_2 in the first two equations and adding then results in the equation $x = y$. Substituting for y in the two side condition $x^2 - xy + y^2 - z^2 = 1$ and $x^2 + y^2 = 1$ then leads to the equation $z^2 = -1/2$, which clearly doesn't have a real valued solution. Hence, we must choose $z = 0$, which reduces the two side conditions to $x^2 - xy + y^2 = 1$ and $x^2 + y^2 = 1$. Using the second side condition, the first one can be written as $xy = 0$. Choosing $x = 0$ then gives $y = \pm 1$ and choosing $y = 0$ gives $x = \pm 1$. Hence, the points nearest to the origin are given by $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$.

8. (a) If $z = 1/(1 + x^2 + y^2)$ for all (x, y) then

$$\frac{\partial z}{\partial x} = -\frac{2x}{(1 + x^2 + y^2)^2} \qquad \frac{\partial z}{\partial y} = -\frac{2y}{(1 + x^2 + y^2)^2}$$

Hence, there is a critical point at $(0, 0)$, which according to (2.146) is a relative maximum. Given the nature of z the point $(0, 0)$ is also the absolute maximum given by $z(0, 0) = 1$.

(b) If $z = xy$, where $x^2 + y^2 \leq 1$ then

$$\frac{\partial z}{\partial x} = y \qquad \frac{\partial z}{\partial y} = x$$

Hence, there is a critical point at $(0, 0)$, which according to (2.148) is a saddle point. In order to determine the absolute maximum and minimum of z we need to consider the boundary $x^2 + y^2 = 1$. Substituting for this in z then gives

$$\frac{\partial z}{\partial x} = \pm\frac{2x^2 - 1}{\sqrt{1 - x^2}} \qquad \frac{\partial z}{\partial y} = \pm\frac{2y^2 - 1}{\sqrt{1 - y^2}}$$

which leads to the critical points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. Hence, the absolute maximum is given by $z(\pm, 1/\sqrt{2}, \pm 1/\sqrt{2}) = 1/2$ and the absolute minimum by $z(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = -1/2$.

(c) If $w = x + y + z$, where $x^2 + y^2 + z^2 \leq 1$ then

$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 1 \qquad \frac{\partial w}{\partial z} = 1$$

Hence, w has no critical points that can give rise to relative minima and maxima. Next, we consider the boundary $x^2 + y^2 + z^2 = 1$. Employing (2.151) with $f(x, y, z) = w$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ then leads to

$$1 + 2\lambda x = 0 \qquad 1 + 2\lambda y = 0 \qquad 1 + 2\lambda z = 0$$

Multiplying the first equation by x, the second by y, the third by z, adding and using the fact that $x^2 + y^2 + z^2$ we find $\lambda = -(x + y + z)/2 = -w/2$. Substituting for λ in the three equations above and adding the results then gives $3 - w^2 = 0$. Hence, the absolute maximum is given by $\sqrt{3}$ and the absolute minimum by $-\sqrt{3}$.

(d) If $w = e^{-x^2-y^2-z^2}$ for all (x, y, z) then

$$\frac{\partial w}{\partial x} = -2xe^{-x^2-y^2-z^2} \qquad \frac{\partial w}{\partial y} = -2ye^{-x^2-y^2-z^2} \qquad \frac{\partial w}{\partial z} = 2ze^{-x^2-y^2-z^2}$$

Hence, there is a critical point at $(0, 0, 0)$, which is a relative maximum given by $w(0, 0, 0) = 1$. Given the nature of the function it is also the absolute maximum of w.

9. As stated at the end of Section 2.21, a quadratic form f is positive definite if and only if all eigenvalues of the $n \times n$ coefficient matrix A associated with the quadratic form f are positive.

   (a) Let $f = 3x^2 + 2xy + y^2 = 3x^2 + xy + yx + y^2$ such that

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues of A are the solutions of (Section 1.11)

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0 \implies \lambda = 2 \pm \sqrt{2}$$

Both eigenvalues are real and hence, the quadratic form is positive definite.

(b) Let $f = x^2 - xy - 2y^2 = x^2 - (1/2)xy - (1/2)yx - 2y^2$ such that

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1/2 \\ -1/2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues of A are the solutions of

$$\begin{vmatrix} 1-\lambda & -1/2 \\ -1/2 & -2-\lambda \end{vmatrix} = 4\lambda^2 + 4\lambda - 9 = 0 \implies \lambda = \frac{-1 \pm \sqrt{10}}{2}$$

Hence, the quadratic form is *not* positive definite nor negative definite.

(c) Let $f = (5/4)x_1^2 + (4/3)x_1x_2 + 2x_2^2 + (4/3)x_2x_3 + (7/3)x_3^2 = (5/4)x_1^2 + (2/3)x_1x_2 + (2/3)x_2x_1 + 2x_2^2 + (2/3)x_2x_3 + (2/3)x_3x_2 + (7/3)x_3^2$ such that

$$f(\mathbf{x}) = \mathbf{x}' \underbrace{\begin{bmatrix} 5/4 & 2/3 & 0 \\ 2/3 & 2 & 2/3 \\ 0 & 2/3 & 7/3 \end{bmatrix}}_{A} \mathbf{x}$$

The eigenvalues of A are the solutions of

$$\begin{vmatrix} 5/4-\lambda & 2/3 & 0 \\ 2/3 & 2-\lambda & 2/3 \\ 0 & 2/3 & 7/3-\lambda \end{vmatrix} = 108\lambda^3 - 603\lambda^2 + 993\lambda - 458 = 0$$

Solving this equation numerically, we may conclude that the quadratic form is positive definite.

10. To prove the validity of (2.143) we start by noting that the function $\nabla_\alpha \nabla_\alpha f(x_0, y_0)$ is continuous in α for $0 \le \alpha \le 2\pi$ (since any combination of $\sin\alpha$ and $\cos\alpha$ will be continuous) and has a minimum M_1 in this interval. If (2.143) is to be upheld then $M_1 > 0$. The fact that $\partial z/\partial x$ and $\partial z/\partial y$ have differentials at (x_0, y_0) implies that

$$\nabla_\alpha \nabla_\alpha f(x_0, y_0) + \epsilon = \frac{\nabla_\alpha f(x_0 + s\cos\alpha, y_0 + s\sin\alpha) - \nabla_\alpha f(x_0, y_0)}{s}$$

In other words, the directional derivative of $\nabla_\alpha f(x, y)$ at the point (x_0, y_0) is equal to the slope of the line passing through the points $(\nabla_\alpha f(x_0, y_0), a)$ and $(\nabla_\alpha f(x_0 + s\cos\alpha, y_0 + s\sin\alpha), b)$, where $s = |a - b|$, plus some deviation ϵ. The closer these points are (i.e. the smaller $s > 0$ will be), the smaller will be ϵ, as from the definition of the derivative we know that

$$\lim_{s\to 0} \frac{\nabla_\alpha f\left(x_0 + s\cos\alpha, y_0 + s\sin\alpha\right) - \nabla_\alpha f\left(x_0, y_0\right)}{s} = \nabla_\alpha \nabla_\alpha f\left(x_0, y_0\right) \implies \epsilon \to 0$$

Defining $x = x_0 + s\cos\alpha$ and $y = y_0 + s\sin\alpha$ and rearranging then leads to

$$\nabla_\alpha f\left(x, y\right) = \nabla_\alpha f\left(x_0, y_0\right) + s\nabla_\alpha \nabla_\alpha f\left(x_0, y_0\right) + \epsilon s = s\nabla_\alpha \nabla_\alpha f\left(x_0, y_0\right) + \epsilon s$$

where the last equality holds, since

$$\nabla_\alpha f\left(x_0, y_0\right) = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} \cos\alpha + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} \sin\alpha = 0$$

Next, we choose δ such that $|\epsilon| < (1/2)M_1 \implies -(1/2)M_1 < \epsilon < (1/2)M_1$ for $0 < s < \delta$. Since $\min_{0\le\alpha\le 2\pi} \nabla_\alpha \nabla_\alpha f(x_0, y_0) = M_1 > 0$ we find

$$\min_{0\le\alpha\le 2\pi} \nabla_\alpha \nabla_\alpha f\left(x_0, y_0\right) + \min_{0<s<\delta} \epsilon = M_1 + \min_{0<s<\delta} \epsilon > \frac{1}{2}M_1 > 0$$

from which we may conclude that $\nabla_\alpha \nabla_\alpha f(x_0, y_0) + \epsilon > 0$ and since $s > 0$ this finally leads to

$$\nabla_\alpha f\left(x, y\right) = s\left(\nabla_\alpha \nabla_\alpha f\left(x_0, y_0\right) + \epsilon\right) > 0 \qquad \text{for } 0 < s < \delta$$

showing that for $0 < s < \delta$ the directional derivative of f is positive and so f increases steadily, as one recedes from the point (x_0, y_0) on a straight line in the neighborhood of radius δ of (x_0, y_0), thus confirming the fact that (2.143) under the conditions stated (i.e. $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$ at (x_0, y_0)) indicates that f has a relative minimum at (x_0, y_0).

11. Making the total square error as small as possible is equivalent to minimizing the function $E(a, b, c)$, or in other words we want to find the values for a, b and c such that $\partial E/\partial a = 0$, $\partial E/\partial b = 0$ and $\partial E/\partial c = 0$. As a fist step we compute

$$\frac{\partial E}{\partial a} = 34a + 10c - 4e_1 - e_2 - e_4 - 4e_5 \qquad \frac{\partial E}{\partial b} = 10b + 2e_1 + e_2 - e_4 - 2e_5$$
$$\frac{\partial E}{\partial c} = 10a + 5c - e_1 - e_2 - e_3 - e_4 - e_5$$

Next, we form the linear system

$$\begin{aligned} 34a \qquad + 10c &= \quad 4e_1 + e_2 + e_4 + 4e_5 \\ 10b \qquad\quad &= -2e_1 - e_2 + e_4 + 2e_5 \\ 10a \qquad + \ 5c &= e_1 + e_2 + e_3 + e_4 + e_5 \end{aligned}$$

which may be solved to give

$$a = \frac{1}{14}\left(2e_1 - e_2 - 2e_3 - e_4 + 2e_5\right) \qquad b = \frac{1}{10}\left(-2e_1 - e_2 + e_4 + 2e_5\right)$$
$$c = \frac{1}{35}\left(-3e_1 + 12e_2 + 17e_3 + 12e_4 - 3e_5\right)$$

12. Let two skew lines be given by the equations

$$x = 1 + 2t \qquad y = 2 - 3t \qquad z = 5t \qquad \text{and} \qquad x = 2 + \tau \qquad y = 3 + 2\tau \qquad z = 1 - \tau$$

We can then define the square of the distance d^2 between the two lines as $d^2 = (-1 + 2t - \tau)^2 + (-1 - 3t - 2\tau)^2 + (-1 + 5t + \tau)^2$. In order to find the shortest distance d between the two lines we will minimize d^2. As a first step we will compute

$$\frac{\partial d^2}{\partial t} = -8 + 76t + 18\tau \qquad\qquad \frac{\partial d^2}{\partial \tau} = 4 + 18t + 12\tau$$

Next, we equate these two partial derivatives to zero and form the linear system

$$\begin{aligned} 38t + 9\tau &= 4 \\ 9t + 6\tau &= -2 \end{aligned}$$

which when solved gives $t = 2/7$ and $\tau = -16/21$. Substituting for t and τ in d^2 the found values then finally gives $d = \sqrt{3}/3$.

13. Firstly, we will show that each level set $E: \ f(\mathbf{x}) = c$ $(c > 0)$ is bounded by noting that

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^\top \mathbf{A} \mathbf{x} = \lambda |\mathbf{x}|^2 = c$$

where $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ. Now since $\lambda_0 > 0$ is considered to be the smallest eigenvalue of $\mathbf{A}$ it holds that $\lambda_0 \le \lambda$ and so

$$\lambda_0 |\mathbf{x}|^2 \le \lambda |\mathbf{x}|^2 = c \implies |\mathbf{x}|^2 \le \frac{c}{\lambda_0}$$

which shows that E is indeed bounded. To show that E is closed we consider the point $\mathbf{x}_0$ that is not in E (i.e. $\mathbf{x} \in \mathbb{R} \setminus E$), so that $|f(\mathbf{x}_0) - c| > 0$. If E is closed, $\mathbb{R} \setminus E$ will be open and there will exists a neighborhood of $\mathbf{x}_0$ of radius ϵ which is completely contained in $\mathbb{R} \setminus E$. Let us choose $\epsilon = 1/n$. Then according to (2.3) we can define

$$|f(\mathbf{x}_0) - c| < \epsilon = \frac{1}{n} \implies -\frac{1}{n} < f(\mathbf{x}_0) - c < \frac{1}{n}$$

Now if we let $n \to \infty$ we can make ϵ arbitrarily small (i.e. choose $\mathbf{x}_0$ such that $f(\mathbf{x}_0)$ is sufficiently close, but not equal, to c) and as such, always find a neighborhood of $\mathbf{x}_0$ that is completely contained in $\mathbb{R} \setminus E$. Hence, this proves that $\mathbb{R} \setminus E$ is open and consequently, that E is closed.

14. (a) Let us use the condition that $g(\mathbf{x}) = 1$ to define the function $h(\mathbf{x}) = 1 - g(\mathbf{x}) = 0$ and so according to Section 2.20 in order to find the critical points of f on the set

$g(\mathbf{x}) = 1$ we can write $\nabla f + \lambda \nabla h = \nabla f - \lambda \nabla g = \mathbf{0}$. Now since $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x}$, $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{B}\mathbf{x}$ and $\mathbf{A}$ and $\mathbf{B}$ are symmetric $n \times n$ matrices we may write

$$\begin{aligned} f(\mathbf{x}) &= a_{11}x_1^2 + a_{12}x_1x_2 + a_{12}x_2x_1 + a_{13}x_1x_3 + a_{13}x_3x_1 + \dots \\ &= a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \dots \end{aligned}$$

where for succinctness only terms in x_1 are suggested and a similar reasoning applies to $g(\mathbf{x})$. As such

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2a_{11}x_1 + 2a_{12}x_2 + 2a_{13}x_3 + \dots \\ \frac{\partial g}{\partial x_1} &= 2b_{11}x_1 + 2b_{12}x_2 + 2b_{13}x_3 + \dots \end{aligned}$$

and hence, the explicit form of equation $\nabla f - \lambda \nabla g = \mathbf{0}$ is given by

$$\begin{aligned} 2a_{11}x_1 + 2a_{12}x_2 + \dots + 2a_{1n}x_n - 2\lambda\left(2b_{11}x_1 + 2b_{12}x_2 + \dots + 2b_{1n}x_n\right) &= 0 \\ &\vdots \\ 2a_{n1}x_1 + 2a_{n2}x_2 + \dots + 2a_{nn}x_n - 2\lambda\left(2b_{n1}x_1 + 2b_{n2}x_2 + \dots + 2b_{nn}x_n\right) &= 0 \end{aligned}$$

which is equivalent to the matrix equation $\mathbf{A}\mathbf{x} - \lambda\mathbf{B}\mathbf{x} = \mathbf{0}$. Next, since we are looking for critical points $\mathbf{x}$ of f on the set $g(\mathbf{x}) = 1$ we furthermore find that $g(\mathbf{x}) = \mathbf{x}^\top\mathbf{B}\mathbf{x} = 1$. Multiplying the aforementioned matrix equation by $\mathbf{x}^\top$ from the left then gives $\mathbf{x}^\top\mathbf{A}\mathbf{x} - \lambda\mathbf{x}^\top\mathbf{B}\mathbf{x} = \mathbf{x}^\top\mathbf{A}\mathbf{x} - \lambda = f(\mathbf{x}) - \lambda = 0$ or $f(\mathbf{x}) = \lambda$.

(b) Let λ_1 and λ_2 be distinct eigenvalues (i.e. roots of the equation $\det(\mathbf{A} - \lambda\mathbf{B})$), such that $\lambda_1 \neq \lambda_2$, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be their corresponding eigenvectors. Next, let us define the two equations

$$\mathbf{x}_1^\top\mathbf{A}\mathbf{x}_2 - \lambda_2\mathbf{x}_1^\top\mathbf{B}\mathbf{x}_2 = 0 \qquad\qquad \mathbf{x}_2^\top\mathbf{A}\mathbf{x}_1 - \lambda_1\mathbf{x}_2^\top\mathbf{B}\mathbf{x}_1 = 0$$

obtained from choosing a specific λ and $\mathbf{x}$ for the equation from part (a) and left multiplying in turn the result by $\mathbf{x}_1^\top$ and $\mathbf{x}_2^\top$ from the left. Now since both $\mathbf{A}$ and $\mathbf{B}$ are real, symmetric $n \times n$ matrices we note that (see Problem 15 following Section 1.13) $\mathbf{x}_1^\top\mathbf{A}\mathbf{x}_2 = \mathbf{x}_2^\top\mathbf{A}\mathbf{x}_1$ and $\mathbf{x}_1^\top\mathbf{B}\mathbf{x}_2 = \mathbf{x}_2^\top\mathbf{B}\mathbf{x}_1$. Making use of this fact, the two equations above may be subtracted to give

$$\mathbf{x}_1^\top\mathbf{A}\mathbf{x}_2 - \lambda_2\mathbf{x}_1^\top\mathbf{B}\mathbf{x}_2 - \left(\mathbf{x}_2^\top\mathbf{A}\mathbf{x}_1 - \lambda_1\mathbf{x}_2^\top\mathbf{B}\mathbf{x}_1\right) = (\lambda_1 - \lambda_2)\,\mathbf{x}_1^\top\mathbf{B}\mathbf{x}_2 = 0$$

However, since $\lambda_1 \neq \lambda_2$ the equation above is satisfied only when $\mathbf{x}_1^\top\mathbf{B}\mathbf{x}_2 = 0$.

15. Let us define $\mathbf{x} = \mathbf{C}\mathbf{y}$, where $\mathbf{C}$ is an orthogonal matrix such that $\mathbf{C}^\top\mathbf{B}\mathbf{C} = \mathbf{B}_1 = \operatorname{diag}(\mu_1, \dots, \mu_n)$ (see Section 1.13), with all $\mu_j > 0$. Then

$$g(\mathbf{C}\mathbf{y}) = (\mathbf{C}\mathbf{y})^\top\mathbf{B}\mathbf{C}\mathbf{y} = \mathbf{y}^\top\mathbf{C}^\top\mathbf{B}\mathbf{C}\mathbf{y} = \mathbf{y}^\top\mathbf{B}_1\mathbf{y} = \mu_1 y_1^2 + \dots + \mu_n y_n^2 = g_1(\mathbf{y})$$

Next, we set $\mathbf{y} = \mathbf{Dz}$ where $\mathbf{D} = \text{diag}(1/\mu_1^{1/2}, \ldots, 1/\mu_n^{1/2})$ so that

$$g_1(\mathbf{Dz}) = (\mathbf{Dz})^\top \mathbf{B}_1\mathbf{Dz} = \mathbf{z}^\top\mathbf{D}^\top\mathbf{B}_1\mathbf{Dz} = \mathbf{z}^\top\mathbf{Iz} = z_1^2 + \cdots + z_n^2 = g_2(\mathbf{z})$$

Writing the second critical point condition $\mathbf{x}^\top\mathbf{Bx} = 1$ in terms of $\mathbf{z}$ then gives

$$\mathbf{x}^\top\mathbf{Bx} = g(\mathbf{x}) = g(\mathbf{Cy}) = g_1(\mathbf{y}) = g_1(\mathbf{Dz}) = g_2(\mathbf{z}) = \mathbf{z}^\top\mathbf{z} = 1$$

Multiplying the first condition $\mathbf{Ax} - \lambda\mathbf{Bx} = \mathbf{0}$ by $\mathbf{x}^\top$ from the left and substituting for $\mathbf{x}$ gives

$$\begin{aligned}\mathbf{x}^\top\mathbf{Ax} - \lambda\mathbf{x}^\top\mathbf{Bx} = f(\mathbf{x}) - \lambda g(\mathbf{x}) = f(\mathbf{Cy}) - \lambda g(\mathbf{Cy}) &= f_1(\mathbf{y}) - \lambda g_1(\mathbf{y}) \\ &= f_1(\mathbf{Dz}) - \lambda g_1(\mathbf{Dz}) \\ &= f_2(\mathbf{z}) - \lambda g_2(\mathbf{z}) \\ &= \mathbf{z}^\top\mathbf{A}_2\mathbf{z} - \lambda\mathbf{z}^\top\mathbf{z} \\ &= 0\end{aligned}$$

hence, leading to the condition $\mathbf{A}_2\mathbf{z} - \lambda\mathbf{z} = \mathbf{0}$, where $\mathbf{A}_2 = \mathbf{D}^\top\mathbf{A}_1\mathbf{D}$, $\mathbf{A}_1 = \mathbf{C}^\top\mathbf{AC}$. Finally, we may thus conclude that $\mathbf{x} = \mathbf{Ez}$, where $\mathbf{E} = \mathbf{CD}$. Since the matrix $\mathbf{E}$ is a product of two orthogonal matrices, $\mathbf{E}$ will be orthogonal and hence, non-singular according to (1.86).

Section 2.22

1. If $\partial f/\partial y = 0$ then clearly $f(x, y)$ is not dependent on y at all and so if $f(x, 0) = \sin x$ then

$$f(\pi/2, 2) = \sin\frac{\pi}{2} = 1 \qquad f(\pi, 3) = \sin\pi = 0 \qquad f(x, 1) = \sin x$$

2. Let us assume that for two function $f(x, y)$ and $g(x, y)$ it holds that $f \equiv g + c$ for some constant c. Taking the gradient of both sides of the equation then gives

$$\begin{aligned}\nabla f &\equiv \nabla(g + c) \\ &\equiv \nabla g + \nabla c \\ &\equiv \nabla g + 0 \\ &\equiv \nabla g\end{aligned}$$

where $\nabla c = 0$ follows from the fact that the partial derivative (and thus the gradient) of a constant is identically zero.

3. A function of the form $f(x, y) = ax + by + c$, where a, b and c are arbitrary constants satisfies the condition that its second partial derivatives are identically zero.

4. If a function defined for all (x, y), is such that $\partial f/\partial y \equiv 0$ then $y = c$, where c is some arbitrary constant. Hence, we can write $f(x, y) = f(x, c) \equiv g(x)$, since there no longer is a dependence on the variable y (given that it is a constant).

5. Taking a hint from problem 4, let $f(x, y) \equiv g(x) + h(y)$. Hence,

$$\frac{\partial f}{\partial x} \equiv \frac{\partial g}{\partial x} + \frac{\partial h}{\partial x} = \frac{\partial g}{\partial x} \implies \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial y}\frac{\partial g}{\partial x} = 0$$

6. According to Section 2.22, functional dependence of a set of functions is equivalent to the condition that the Jacobian of the functions is equal to zero. Hence, we compute the Jacobian of each set of functions and check that it is indeed equal to zero.

   (a) If $f = y/x$ and $g = (x - y)/(x + y)$ then

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} -y/x^2 & 1/x \\ 2y/(x+y)^2 & -2x/(x+y)^2 \end{vmatrix} = 0$$

   (b) If $f = x^2 + 2xy + y^2 + 2x + 2y$ and $g = e^x e^y$ then

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} 2x + 2y + 2 & 2x + 2y + 2 \\ e^x e^y & e^x e^y \end{vmatrix} = 0$$

   (c) If $f = x^2 y - xy^2 + xyz$, $g = xy + x - y + z$ and $h = x^2 + y^2 + z^2 - 2yz + 2xz$ then

$$\frac{\partial(f, g, h)}{\partial(x, y, z)} = \begin{vmatrix} 2xy - y^2 + yz & x^2 - 2xy + xz & xy \\ y + 1 & x - 1 & 1 \\ 2x + 2z & 2y - 2z & 2z - 2y + 2x \end{vmatrix} = 0$$

   (d) If $f = u + v - x$, $g = x - y + u$ and $h = u - 2v + 5x - 3y$ then we can look for a linear dependency of the form $z_1 f + z_2 g + z_3 h \equiv 0$, where z_1, z_2 and z_3 are constants not all zero. The corresponding matrix equation then becomes

$$\mathbf{z}^\top \mathbf{A} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \mathbf{0}$$

   Since we want this equation to hold for any x, y, u and v this reduces to

$$\mathbf{z}^\top \mathbf{A} = \mathbf{0} \iff \mathbf{A}^\top \mathbf{z} = \mathbf{0}$$

Using Gaussian elimination, we then continue to compute a basis for the subspace W of dimension $n - r$, where n is the number of rows of $\mathbf{A}$ and r is the rank of $\mathbf{A}$, which gives

$$\begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, we find $z_1 = 2z_3$, $z_2 = -3z_3$ and $z_3 = a$, where a is some arbitrary constant and so the solution is given by $\mathbf{z} = a(2, -3, 1)^{\top}$.

7. The functions $f(x,y)$ and $g(x,y)$ from part (a) of Problem 6 are related through the identity

$$g(x,y) = \frac{x-y}{x+y} = \frac{1-y/x}{1+y/x} \equiv \frac{1-f(x,y)}{1+f(x,y)}$$

Hence,

$$F[f(x,y), g(x,y)] = \frac{1-f(x,y)}{1+f(x,y)} - g(x,y) \equiv 0$$

The function $f(x,y)$ from part (b) of Problem 6 can be rewritten as

$$f(x,y) = x^2 + 2xy + y^2 + 2x + 2y = (x+y)^2 + 2(x+y) = (x+y)(x+y+2)$$

and so the function $g(x,y)$ from part (b) of Problem 6 can be rewritten as

$$g(x,y) = e^x e^y = e^{x+y} \equiv e^{f(x,y)/(x+y+2)} \equiv e^{f(x,y)/(2+\ln g(x,y))}$$

Hence,

$$F[f(x,y), g(x,y)] = e^{f(x,y)/(2+\ln g(x,y))} - g(x,y) = \frac{f(x,y)}{2+\ln g(x,y)} - \ln g(x,y) \equiv 0$$

The functions $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ from part (c) of Problem 6 can be rewritten as

$$\begin{aligned} f(x,y,z) &= x^2 y - xy^2 + xyz = xy(x-y+z) \\ g(x,y,z) &= xy + x - y + z \equiv \frac{f(x,y,z)}{xy} + xy \\ h(x,y,z) &= x^2 + y^2 + z^2 - 2yz + 2xz = (x-y+z)^2 + 2xy \equiv \left(\frac{f(x,y,z)}{xy}\right)^2 + 2xy \end{aligned}$$

From this we may deduce the two identities

$$g(x,y,z) - \frac{f(x,y,z)}{xy} - xy \equiv 0 \qquad h(x,y,z) - \left(\frac{f(x,y,z)}{xy}\right)^2 - 2xy \equiv 0$$

which may be added to produce the single identity

$$F(f,g,h) = g(x,y,z) + h(x,y,z) - \frac{f(x,y,z)}{xy}\left(\frac{f(x,y,z)}{xy} + 1\right) - 3xy \equiv 0$$

The identity relating $f(x,y,u,v)$, $g(x,y,u,v)$ and $h(x,y,u,v)$ from part (d) of Problem 6 has already been found as part of solving the aforementioned comment and is given by

$$F(f,g,h) = 2f(x,y,u,v) - 3g(x,y,u,v) + h(x,y,u,v) \equiv 0$$

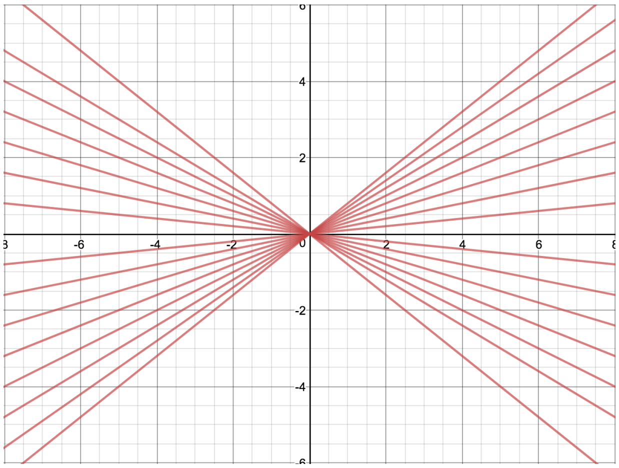

Figure 30: $f = y/x$

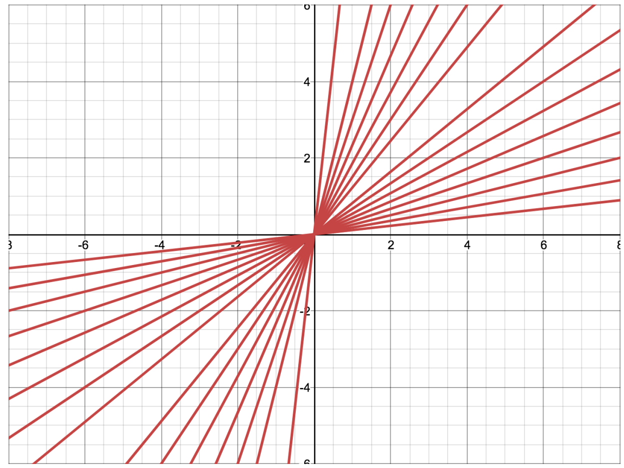

Figure 31: $g = (x - y)/(x + y)$

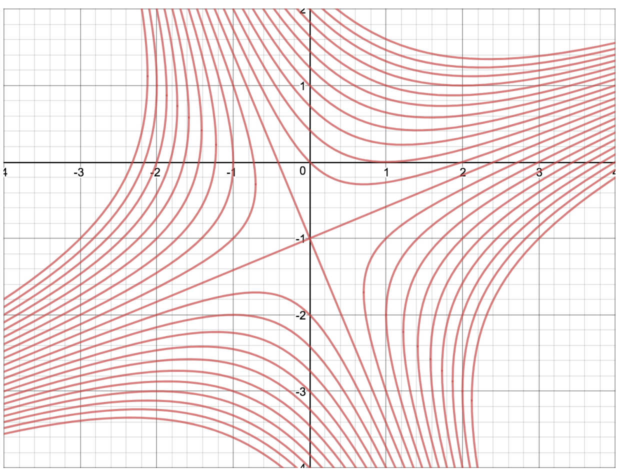

Figure 32: $f = x^2 + 2xy + y^2 + 2x + 2y$

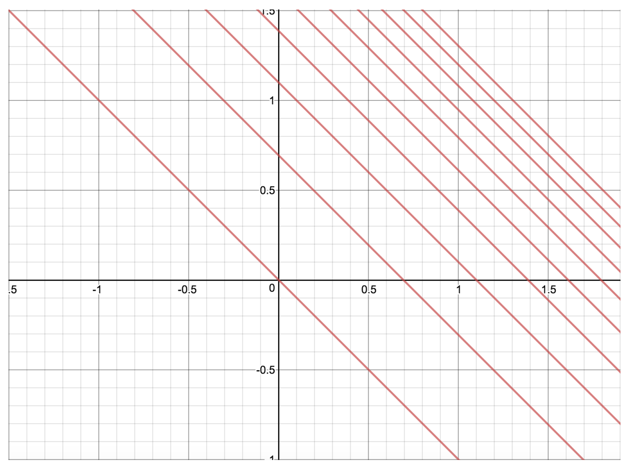

Figure 33: $g = e^x e^y$

8.

9. When $u = f(x, y)$ then $g(x, y, u) = g[x, y, f(x, y)] \equiv g(x, y)$. In other words, if f is a function of the two independent variables x and y then $g[x, y, f(x, y)]$ can just be written as a function g of the two independent variables x and y. If furthermore

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = \frac{\partial\,(f, g)}{\partial\,(x, y)} = 0$$

the level curves of f and g coincide and the equations $u = f(x, y)$ and $v = g(x, y)$ define a mapping from the xy-plane to the uv-plane. Since

$$\frac{\partial\,(f, g)}{\partial\,(x, y)} = \begin{vmatrix} \partial f/\partial x & \partial g/\partial x \\ \partial f/\partial y & \partial g/\partial y \end{vmatrix} = 0$$

we can form the identities

$$\frac{\partial F}{\partial u}\frac{\partial f}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial g}{\partial x}\equiv 0 \qquad\qquad \frac{\partial F}{\partial u}\frac{\partial f}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial g}{\partial y}\equiv 0$$

for the two quantities $\partial F/\partial u$ and $\partial F/\partial v$, not both zero (Section 1.5), the left side of which are just the partial derivatives of the left side of the equation

$$F\left[f\left(x,y\right),g\left(x,y\right)\right]\equiv F\left[f\left(x,y\right),g\left(x,y,f\left(x,y\right)\right)\right]=0$$

Hence, $f(x,y)$ and $g(x,y,u)$ are functionally dependent when $u=f(x,y)$.

10. Let $u(x,y)$ and $v(x,y)$ be harmonic (i.e. $\nabla^2 u=0$ and $\nabla^2 v=0$) in a domain D and have no critical points in D. In other words $\nabla u\neq\mathbf{0}$ and $\nabla v\neq\mathbf{0}$. Next, let u and v be functionally dependent through the relation $u=f(v)$. Taking the Laplacian of both sides of this equation then gives

$$\begin{aligned}\nabla^2 u&=\nabla^2 f\left(v\right)\\ 0&=\nabla^2 f\left(v\right)\end{aligned}$$

since u is harmonic in D. The right hand side expands to

$$\begin{aligned}\nabla^2 f\left(v\right)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}&=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\\ &=\frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial y^2}\\ &=\frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\frac{\partial f}{\partial v}\underbrace{\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}_{\nabla^2 v=0}\\ &=\frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\end{aligned}$$

Now since $v(x,y)$ has no critical points in D (i.e. $\nabla v\neq\mathbf{0}$) the sum $(\partial v/\partial x)^2+(\partial v/\partial y)^2$ cannot be zero. Hence, for the equation $\nabla^2 f(v)=0$ to make sense, we require that $\partial^2 f/\partial v^2=0$, implying that $\partial f/\partial v=a$, where a is some arbitrary constant and finally that $f(v)=av+b$, where b is some arbitrary constant. In conclusion, if u and v are functionally dependent, then they are linearly dependent: $u=av+b$.

11. (a) Let the linear functions $u(x,y)=ax+by$ and $v(x,y)=cx+dy$ be functionally dependent: $F(u,v)\equiv 0$. Differentiating this relation with respect to x and y results in

$$\begin{aligned}\frac{\partial F}{\partial x}&=\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=a\frac{\partial F}{\partial u}+c\frac{\partial F}{\partial v}\equiv 0\\ \frac{\partial F}{\partial y}&=\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=b\frac{\partial F}{\partial u}+d\frac{\partial F}{\partial v}\equiv 0\end{aligned}$$

which can also be written as the single matrix equation

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \partial F/\partial u \\ \partial F/\partial v \end{bmatrix} \equiv \mathbf{0}$$

Since $\partial F/\partial u$ and $\partial F/\partial v$ are not zero the determinant of the coefficients needs to be zero and hence, we require that $ad - bc \equiv 0$. The equations above then are satisfied by choosing $\partial F/\partial u = d$ and $\partial F/\partial v = -b$. Hence, $\partial F/\partial u$ and $\partial F/\partial v$ are constants, which implies that $F(u, v) = hu(x, y) + kv(x, y)$ with $h = d$ and $k = -b$.

(b) Let $f_i(x_1, \ldots, x_m) = a_{i1}x_1 + \cdots + a_{im}x_m$ for $i = 1, \ldots, n$ be n linear functions of m variables which are linearly dependent: $F(f_1, \ldots, f_n) \equiv 0$. Differentiating $F(f_1, \ldots, f_n)$ with respect to x_j for $j = 1, \ldots, m$ then gives the m relations

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= \frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial F}{\partial f_n}\frac{\partial f_n}{\partial x_1} \equiv 0 \\ &\vdots \\ \frac{\partial F}{\partial x_m} &= \frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial x_m} + \cdots + \frac{\partial F}{\partial f_n}\frac{\partial f_n}{\partial x_m} \equiv 0 \end{aligned}$$

This also may be written as the single matrix equation

$$\underbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \vdots & & \vdots \\ \partial f_1/\partial x_m & \ldots & \partial f_n/\partial x_m \end{bmatrix}}_{\mathbf{A}^\top} \underbrace{\begin{bmatrix} \partial F/\partial f_1 \\ \vdots \\ \partial F/\partial f_n \end{bmatrix}}_{\mathbf{z}} = \begin{bmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \ldots & a_{nm} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

which corresponds to (2.166). If $\mathbf{A}$ has rank r then the solutions $\mathbf{z}$ form a subspace W of dimension $n - r$ of V^n (Section 1.17). By employing Gaussian elimination on $\mathbf{A}^\top$ we can find a basis $\mathbf{z}_1, \ldots, \mathbf{z}_{n-r}$ for W and hence, obtain $n-r$ independent linear dependencies of the form

$$\begin{bmatrix} f_1 & \ldots & f_n \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 & \ldots & \mathbf{z}_{n-r} \end{bmatrix} = z_{11}f_1 + \cdots + z_{1n}f_n \equiv 0$$

where $\mathbf{z}_1, \ldots, \mathbf{z}_{n-r} = \operatorname{col}(z_{11}, \ldots, z_{1n}), \ldots, \operatorname{col}(z_{n-r,1}, \ldots, z_{n-r,n})$. If $m < n$ (i.e. the number of independent coordinates is smaller than the number of functions), then $r \leq m < n$ and the remaining $n - (n - r)$ linear dependencies are linear combinations of the $n - r$ independent linear dependencies just found. If instead $n \leq m$ and $\mathbf{A}$ has maximal rank $r = n$ then no such linear dependency relation exists and consequently, the n functions f_i are functionally independent.

12. Note: there is a typo in the book. The start of the problem statement should read as "*Let* $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = \sin(x_1 + x_2 + x_3)$...". If $f_1(x_1, x_2, x_3) =$

$f_2(x_1, x_2, x_3) = \sin(x_1 + x_2 + x_3)$ and $f_3(x_1, x_2, x_3) = \cos(x_1 + x_2 + x_3)$, then the matrix $\mathbf{A} = (\partial f_k/\partial x_l)$ is given by

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix} \\ &= \begin{bmatrix} \cos(x_1+x_2+x_3) & \cos(x_1+x_2+x_3) & \cos(x_1+x_2+x_3) \\ \cos(x_1+x_2+x_3) & \cos(x_1+x_2+x_3) & \cos(x_1+x_2+x_3) \\ -\sin(x_1+x_2+x_3) & -\sin(x_1+x_2+x_3) & -\sin(x_1+x_2+x_3) \end{bmatrix}\end{aligned}$$

Let $t = x_1 + x_2 + x_3$. Employing Gaussian elimination on $\mathbf{A}^\top$ gives

$$\begin{bmatrix} \cos t & \cos t & -\sin t \\ \cos t & \cos t & -\sin t \\ \cos t & \cos t & -\sin t \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -\tan t \\ \cos t & \cos t & -\sin t \\ \cos t & \cos t & -\sin t \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -\tan t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\mathbf{A}^\top$ (and as such, also $\mathbf{A}$) has rank $r = 1$ in E^3 and we find that $z_1 = -z_2 + (\tan t)z_3$, $z_2 = k_1$ and $z_3 = k_2$, where k_1 and k_2 are arbitrary scalars. The solutions are thus given by $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ and $\mathbf{z}_2$ are two linearly independent column vectors:

$$\mathbf{z} = k_1 \mathrm{col}\,(-1, 1, 0) + k_2 \mathrm{col}\,(\tan t, 0, 1)$$

and so the matrix

$$\left(\frac{\partial \bar{F}_j}{\partial u_k}\right) = \begin{bmatrix} k_1 \mathbf{z}_1^\top \\ k_2 \mathbf{z}_2^\top \end{bmatrix} = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_2 \tan t & 0 & k_2 \end{bmatrix}$$

has rank $h = 2$ in the domain D_u. Note that the bar indicates that the rows of $(\partial F_j/\partial u_k)$ are a basis (i.e. are linearly independent row vectors). To show that the F_j can be chosen as stated, we compute

$$\frac{\partial F_1}{\partial u_1} = 1 \qquad \frac{\partial F_1}{\partial u_2} = -1 \qquad \frac{\partial F_1}{\partial u_3} = 0$$

which are just the components of $k_1 \mathbf{z}_1 = -\mathbf{z}_1$ (i.e. $k_1 = -1$). Similarly, computing

$$\frac{\partial F_2}{\partial u_1} = 2u_1 \qquad \frac{\partial F_2}{\partial u_2} = 2u_2 \qquad \frac{\partial F_2}{\partial u_3} = 4u_3$$

shows that these are the components of the vector which is the linear combination of $k_1 \mathbf{z}_1 + k_2 \mathbf{z}_2 = 2 \sin t \mathbf{z}_1 + 4 \cos t \mathbf{z}_2$.

To show that the range of $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is an ellipse, firstly note that

$$\mathbf{u} = \mathbf{f}\,(\mathbf{x}) = \mathrm{col}\,(\sin t, \sin t, \cos t) = \sin t \mathbf{v}_1 + \cos t \mathbf{v}_2$$

where $\mathbf{v}_1 = \mathrm{col}\,(1, 1, 0)$ and $\mathbf{v}_2 = \mathrm{col}\,(0, 0, 1)$. Furthermore, note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ (i.e. $\mathbf{v}_1 \perp \mathbf{v}_2$) and that the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$ are different: $\mathbf{v}_1 = \sqrt{2}$ and $\mathbf{v}_2 = 1$. Hence,

$\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ (where $\hat{\mathbf{v}}_1 = \mathbf{v}_1/\|\mathbf{v}_1\|$ and $\hat{\mathbf{v}}_2 = \mathbf{v}_2/\|\mathbf{v}_2\|$ are unit vectors) can be interpreted as two perpendicular axes of a plane and so $\mathbf{u} = \mathbf{f}(\mathbf{x})$ can be re-written as

$$\mathbf{u} = \mathbf{f}\left(\mathbf{x}\right) = \frac{\sin t}{\sqrt{2}}\hat{\mathbf{v}}_1 + \cos t\hat{\mathbf{v}}_2$$

Hence, the coordinates $\hat{x} = \cos t$ and $\hat{y} = (\sin t/\sqrt{2})$ trace out an ellipse in the plane spanned by the two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$.

Section 2.23

1. Let E be a non-empty set of real numbers and bounded below. Then E has a greatest lower bound A_0; that is, there is a number A_0 such that A_0 is a lower bound of E and $A_0 \geq A$ for every lower bound of E.

   In order to prove this we note that there are integers that are not lower bounds of E, for we can choose an integer greater than some x in E. There is a smallest such integer, s_1. For E has a lower bound A, and every integer that is not a lower bound must be greater than A. Similarly, there is a largest integer k_1, where $0 \leq k_1 \leq 9$, such that $s_2 = s_1 - (k_1/10)$ is not a lower bound of E. Continuing, we obtain a monotone decreasing sequence s_n such that

   $$s_n = s_1 - \frac{k_1}{10} - \cdots - \frac{k_{n-1}}{10^{n-1}}, \qquad 0 \leq k_i \leq 9$$

   and s_n is not a lower bound of E, but $s_n - (1/10^{n-1})$ is a lower bound. Hence, it follows that for every x in E, for all n

   $$s_n - \frac{1}{10^{n-1}} \leq x$$

   Now the sequence s_n is bounded below by A. Therefore, by Theorem A, s_n converges to some A_0. Taking the limit then gives

   $$\lim_{n\to\infty}\left(s_n - \frac{1}{10^{n-1}}\right) = A_0 \leq x$$

   so that A_0 is a lower bound of E. Now $s_n > A$ for all n implies that $A_0 \geq A$. Therefore, A_0 is the greatest lower bound (and is clearly unique).

2. Let s_n be a convergent sequence of real numbers with limit c. In other words, for each $\epsilon > 0$ there is an integer N such that $|s_n - c| < \epsilon$ for $n > N$. Next, let t_n be a sub sequence of s_n: $t_1 = s_{n_1}$, $t_2 = s_{n_2}, \ldots, t_k = s_{n_k}, \ldots$, where $n_1 < n_2 < \cdots < n_k < n_{k+1} < \ldots$. Now since $k \leq n_k$ for all n_k, we have that some $k > N$ corresponds to some $n_k > N$, such that $|s_{n_k} - c| = |t_k - c| < \epsilon$. Hence, $\lim_{k\to\infty} t_k = c$ and so the sub sequence t_n converges to the same limit c as the sequence s_n.

3. To say that the point $P_n = (x_1^n, x_2^n, \ldots, x_N^n)$ converges to the point $P_0 = (x_1^0, x_2^0, \ldots, x_N^0)$ is equivalent to choosing an $\epsilon > 0$, such that the difference

$$|P_N - P_0| = \sqrt{(x_1^n - x_1^0)^2 + \cdots + (x_N^n - x_N^0)^2} < \epsilon$$

becomes arbitrarily small. Squaring the inequality above gives

$$\left(x_1^n - x_1^0\right)^2 + \cdots + \left(x_N^n - x_N^0\right)^2 < \epsilon^2$$

which implies that

$$\left(x_1^n - x_1^0\right)^2 < \epsilon^2, \ldots, \left(x_N^n - x_N^0\right)^2 < \epsilon^2$$

Taking square roots of both sides of all inequalities then gives

$$|x_1^n - x_1^0| < \epsilon, \ldots, |x_N^n - x_N^0| < \epsilon$$

Hence, the point P_n converges to P_0 if and only if

$$x_1^n \to x_1^0, \quad x_2^n \to x_2^0, \ldots, x_N^n \to x_N^0$$

4. (a) Let a closed interval $a \leq x \leq b$ be given and let G be the set of all real numbers x in this interval. Clearly, since $x \leq b$ the set G will be bounded from above. Similarly, since $x \geq a$ the set G will be bounded from below. Since G is both bounded from above *and* below we can simply identify G as a bounded set.

   Now in order to prove that G is closed, we will show that a point x not in G is part of the open set $\mathbb{R} \setminus G$. To prove that $\mathbb{R} \setminus G$ is open means to prove that the neighborhood of a point $x \in \mathbb{R} \setminus G$ of radius $\epsilon > 0$ is contained entirely in $\mathbb{R} \setminus G$. Let us suppose the opposite however, i.e. that x is a point not in G ($x \in \mathbb{R} \setminus G$), but has at least one point of its neighborhood in G. Let $x_n \in G$ be such a point and let its distance to x satisfy the condition $\mathrm{d}\,(x_n, x) < \epsilon = 1/n$. Now let $n \to \infty$, such that $x_n \to x$, which implies that $x_n \in \mathbb{R} \setminus G$. But this is contradictory to the original assumption that $x_n \in G$. Hence, a neighborhood of a point x not in G must always be contained entirely in $\mathbb{R} \setminus G$, implying that $\mathbb{R} \setminus G$ is open and consequently, that G is closed.

   (b) For part(a) we have already proven that the closed interval $a \leq x \leq b$ is a bounded closed set G in E. What remains to be proven is that $y_1(x) < y \leq y_2(x)$, where $y_1(x) \leq y_2(x)$ for $a \leq x \leq b$ is also bounded and closed. For this we can make use of Theorem H by noting that $y_1(x)$ and $y_2(x)$ can be thought of as two different, but nonetheless continuous mappings of the closed bounded set $a \leq x \leq b$ from E^1 into E^1. Hence, according to Theorem H the range of $y_1(x)$ and $y_2(x)$ is also bounded and closed. Suppose that the range of $y_1(x)$ is not bounded, then we can find a sequence x_n in G such that $\mathrm{d}\,(O, y_1(x_n)) > n$ for $n = 1, 2, \ldots$; here O is the origin of E. By Theorem E, x_n has a convergent sub sequence x_{n_k}, $x_{n_k} \to x_0$. By Theorem F, x_0 is in G. By Theorem G, $y_1(x_{n_k}) \to y_1(x_0)$ as $k \to \infty$. But

$n_k < \mathrm{d}\,(O, y_1(x_{n_k})) \leq \mathrm{d}\,(O, y_1(x_0)) + \mathrm{d}\,(y_1(x_0), y_1(x_{n_k}))$. As $k \to \infty$, the last term has limit zero, but $n_k \to \infty$. This would imply $\infty \leq \mathrm{d}\,(O, y_1(x_0))$, which is impossible. Hence, the range of $y_1(x)$ is bounded. The same reasoning applies to $y_2(x)$.

By Theorem F we show that the range of $y_1(x)$ and $y_2(x)$ is closed by showing that if $y_n = y_1(x_n)$ is a convergent sequence in the range, then its limit is also in the range. As before, we choose a sub sequence y_{n_k} converging to y_0 in G and then $y_1(x_{n_k}) \to y_1(x_0)$ by continuity. But the sequence $y_1(x_n)$ and $y_1(x_{n_k})$ have the same limit. Therefore, $y_n \to y_1(x_0)$, which is in the range and so the range of $y_1(x)$ is closed. Again, the same reasoning applies to $y_2(x)$.

Finally, G is not a closed region, since it does not contain all boundary points.

5. Let E be the set of x on the interval $a \leq x \leq b$ for which $f(x) < 0$. The greatest lower bound then is a, since $f(a) < 0$ and $x \geq 0$. Furthermore, let x_0, such that $f(x_0) = 0$, be the least upper bound of E. Then Theorem J tells us that $f(x)$ takes on an absolute maximum in E, given that $f(x)$ is continuous and E is bounded and closed. Since E contains all x for which $f(x) < 0$ and x_0, such that $f(x_0) = 0$, as the least upper bound, this maximum is none other than $f(x_0) = 0$ for some x_0, $a < x_0 < b$.

6. Let f be a real-valued function defined on E^N and as such, be a mapping of a set G in E^N into E^1. Furthermore, let $P_n \to P_0$ and let P_0 be such that $a < f(P_0) < b$, with $P_1, P_2, \ldots$ such that $a < f(P_1) < b, a < f(P_2) < b, \ldots$. In other words, $P_0, P_1, P_2, \cdots \in G$. Then f is continuous at the point P_0 if for all $\epsilon > 0$, a $\delta > 0$ exists such that $\mathrm{d}\,(f(P), f(P_0)) < \epsilon$ whenever $\mathrm{d}\,(P, P_0) < \delta$. Let us choose n_ϵ so that $\mathrm{d}\,(P_n, P_0) < \delta$ for $n > n_\epsilon$. Then $\mathrm{d}\,(f(P_n), f(P_0)) < \epsilon$ for $n > n_\epsilon$ so that $f(P_n) \to f(P_0)$ and clearly, $a < f(P_n) < b$. The fact that the set G of all P for which $a < f(P) < b$ is open then follows from the fact that $\mathrm{d}\,(f(P), f(P_0)) < \epsilon$ in our definition of continuity. In other words, the distance of an arbitrary point P to its limiting value P_0 is given by the open ball of radius ϵ, centered at P.

7. Let G be a closed set in E^N. Then if Q is a point of E^N not in G (i.e. $Q \in \mathbb{R} \setminus G$) we can define the distance from an arbitrary point P in G to Q as $f(P) = \mathrm{d}\,(P, Q)$.

   (a) To show that f is continuous on G, we firstly note that Theorem F tells us that since G is closed, the limit of each convergent sequence P_n in G also is in G. Then by Theorem G, f is continuous at the point P_0 in G if and only if $f(P_n) \to f(P_0)$ for every sequence P_n converging to P_0. To this end, note that by means of the triangle inequality it holds that

$$\mathrm{d}\ (P_n, Q) + \mathrm{d}\ (P_n, P_0) \geq \mathrm{d}\ (P_0, Q) \qquad \mathrm{d}\ (P_0, Q) + \mathrm{d}\ (P_n, P_0) \geq \mathrm{d}\ (P_n, Q)$$

   Combining these two inequalities then gives $|\mathrm{d}\,(P_n, Q) - \mathrm{d}\,(P_0, Q)| \leq \mathrm{d}\,(P_n, P_0)$. Setting $\epsilon = \delta$ and choosing δ such that $\mathrm{d}\,(P_n, P_0) < \delta$ for $n > n_\epsilon$ this becomes $|f(P_n) - f(P_0)| < \epsilon$. Hence, we may conclude that when $P_n \to P_0$, so will

$f(P_n) \to f(P_0)$ and hence, f is continuous at the point P_0. Since P_0 was chosen arbitrarily, the same applies to any point P in G and so f is continuous on G.

(b) It is stated that G is a closed set in E^N. Furthermore, for part (a) of the problem we have proven that f is continuous on G and since f denotes the real-valued distance between two points it is a mapping of G into E^1. We also know from part (a) that the limit of each convergent sequence P_n in G also is in G. Now suppose G is *not* bounded, so that the range of f necessarily also is not bounded. This would imply that we can find a sequence P_n in G such that $\mathrm{d}\,(P_n, Q) = f(P_n) > n$ for $n = 1, 2, \ldots$. Since f is continuous on G there must be a convergent sub sequence P_{n_k} so that when $P_{n_k} \to P_0$ so will $\mathrm{d}\,(P_{n_k}, Q) = f(P_{n_k}) \to f(P_0) = \mathrm{d}\,(P_0, Q)$ as $k \to \infty$. But then $n_k < \mathrm{d}\,(P_{n_k}, Q) \leq \mathrm{d}\,(P_0, Q) + \mathrm{d}\,(P_0, P_{n_k})$. As $k \to \infty$, the last term has limit zero, but $n_k \to \infty$. This would imply $\infty \leq \mathrm{d}\,(P_0, Q) = f(P_0)$, which is impossible. Hence, the range of f is bounded and consequently, G will be bounded too. Then by Theorem J f takes on an absolute maximum and minimum on G. Since $f(P) > 0$ this minimum is positive. Also, since P_0 is the limit of a convergent sequence P_n it will be unique.

8. Let G and H be two closed sets in E^N without a point in common. For example; let G be the set containing points P_n that are of the form $P = (x_1, x_2, \ldots, x_N)$, where $x_i \in \mathbb{N}$ (i.e. x_i is an arbitrary integer) and let H be the set containing points Q_n that are of the form $Q = (x_1 + 2^{-m}, x_2 + 2^{-m}, \ldots, x_N + 2^{-m})$, where $m \in \mathbb{N}$. Both are closed sub sets of $\mathbb{R}$. Then for any ϵ, an m exists such that $2^{-m} < \epsilon$ which implies $\mathrm{d}\,(P, Q) = 0$. Next, suppose G is bounded. Then by Theorem E, if P_n is a sequence of points in G, it will have a convergent sub sequence with limit P_0, which according to Theorem F must be in G. But then $\mathrm{d}\,(P_0, Q) > 0$, since $Q \notin G$.