CHAPTER 6

Section 6.4

1. (a)

$$\lim_{n\to\infty} \frac{n^2+1}{n^3+1} = \lim_{n\to\infty} \frac{(1/n^2)+1/n^3}{1+1/n^3} = \frac{0}{1} = 0$$

(b)

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{d\,(\ln n)\,/dn}{d\,(n)\,/dn} = \lim_{n\to\infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

(c)

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$$

(d)

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+1/n)}{n^{-1}} &= \lim_{n\to\infty} \frac{d\left[\ln(1+1/n)\right]/dn}{d\left(n^{-1}\right)/dn} \\ &= \lim_{n\to\infty} \frac{1/\left(n^2+n\right)}{1/n^2} \\ &= \lim_{n\to\infty} \frac{1}{1+1/n} \\ &= \frac{1}{1+0} = 1\end{aligned}$$

(e)

$$\lim_{n\to\infty} s_n = 1 \text{ for } n = 1,2,3,\cdots \implies \lim_{n\to\infty} s_n = 1$$

2. (a)

$$\overline{\lim}_{n\to\infty} \cos n\pi = 1 \qquad \underline{\lim}_{n\to\infty} \cos n\pi = -1 \tag{1}$$

(b)

$$\overline{\lim}_{n\to\infty} \sin\frac{1}{5}n\pi \approx 0.951 \qquad \underline{\lim}_{n\to\infty} \sin\frac{1}{5}n\pi \approx -0.951$$

(c)

$$\overline{\lim}_{n\to\infty} n \sin\frac{1}{2}n\pi = \infty \qquad \underline{\lim}_{n\to\infty} n \sin\frac{1}{2}n\pi = -\infty$$

3. (a) A sequence

$$s_n = 1 + \cos n\pi$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = 2 \qquad \underline{\lim}_{n\to\infty} s_n = 0$$

(b) A sequence

$$s_n = -n^2 \sin^2\left(\frac{1}{2}n\pi\right)$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = 0 \qquad\qquad \underline{\lim}_{n\to\infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left|\left(1 + \frac{1}{n}\right)^n - e\right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

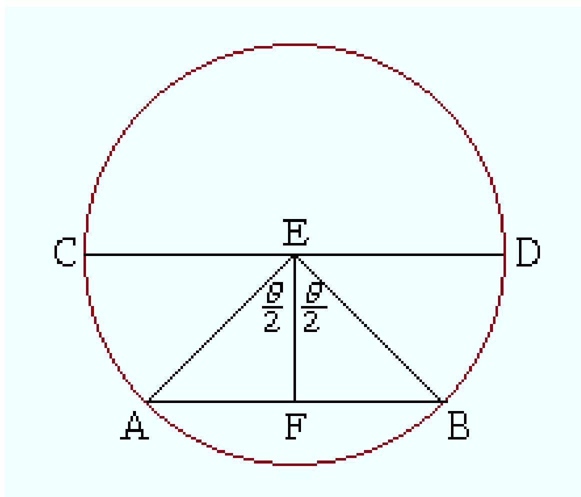
$$e \approx \left(1 + \frac{1}{164}\right)^{164} \approx 2.71$$

6.

$$\overline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } |x| > 1$$
$$\overline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = \pm 1$$
$$\overline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$

$$\underline{\lim}_{n\to\infty} x^n = -\infty \qquad \text{for } x < -1$$
$$\underline{\lim}_{n\to\infty} x^n = -1 \qquad \text{for } x = -1$$
$$\underline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$
$$\underline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = 1$$
$$\underline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } x > 1$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin\frac{\theta}{2} \times AE \cos\frac{\theta}{2} = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \to \infty$:

$$A_{S_1} = s_n = \lim_{n\to\infty} \frac{n}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty} \pi\frac{n}{2\pi}\sin\frac{2\pi}{n}$$

Now using the fact that $\lim_{x\to 0}\sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n\to\infty} \pi\frac{n}{2\pi}\sin\frac{2\pi}{n} = \lim_{x\to 0} \pi\frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π, it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n\to\infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the nth term test $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$ diverges.

(b) Since

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^{n-1}}{3n} = \lim_{n\to\infty} \frac{(n-1)\,2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the nth term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3-1} = \left|\frac{1}{n^3-1}\right| < \frac{1}{n-1}$$

for $n = 2, 3, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \frac{1/n}{1-(1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3-1)$ is absolutely convergent.

(b) Since $|\sin n| < 1$ for $n \geq 1$ it follows that

$$\left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$$

for $n = 1, 2, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a) Since $n+5 > n$ and $n^2-3n-5 < n^2$ for $n \geq 1$ it follows that

$$\frac{n+5}{n^2-3n-5} > \frac{n}{n^2} = \frac{1}{n}$$

for $n = 1, 2, \ldots$. Now since $\sum_{n=1}^{\infty} 1/n$ is the *harmonic series*, which diverges, it follows by the comparison test for divergence that $\sum_{n=1}^{\infty}(n+5)/(n^2-3n-5)$ diverges as well.

(b) Since $\sqrt{n}\ln n < n\ln n$ for $n \geq 2$ it follows that

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n\ln n}$$

for $n = 2, 3, \ldots$. Using the inequality $\ln(1+x) \leq x$ we may continue to write

$$\frac{1}{n\ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln\left(1 + \frac{\ln(1+1/n)}{\ln n}\right) \geq \ln\frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n}\ln n} > \ln\frac{\ln(1+n)}{\ln n} = \ln\ln(1+n) - \ln\ln n$$

Now let us consider the series

$$\sum_{n=2}^{N} \ln\ln(1+n) - \ln\ln n = \ln\ln(1+N) - \ln\ln 2$$

Hence, when $N \to \infty$

$$\sum_{n=2}^{\infty} \ln\ln(1+n) - \ln\ln n = \lim_{N\to\infty} \ln\ln(1+N) - \ln\ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that $\sum_{n=2}^{\infty} 1/(\sqrt{n}\ln n)$ diverges as well.

4. (a)