CHAPTER 6

Section 6.4

1. (a)
$$\lim_{n\to\infty}\frac{n^2+1}{n^3+1}=\lim_{n\to\infty}\frac{(1/n^2)+1/n^3}{1+1/n^3}=\frac{0}{1}=0$$

   (b)
$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{d(\ln n)/dn}{d(n)/dn}=\lim_{n\to\infty}\frac{1/n}{1}=\frac{0}{1}=0$$

   (c)
$$\lim_{n\to\infty}\frac{n}{2^n}=\lim_{n\to\infty}\frac{1}{2^n\ln 2}=\frac{1}{\infty}=0$$

   (d)
$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+1/n)}{n^{-1}}&=\lim_{n\to\infty}\frac{d[\ln(1+1/n)]/dn}{d(n^{-1})/dn}\\&=\lim_{n\to\infty}\frac{1/(n^2+n)}{1/n^2}\\&=\lim_{n\to\infty}\frac{1}{1+1/n}\\&=\frac{1}{1+0}=1\end{aligned}$$

   (e)
$$\lim_{n\to\infty}s_n=1\text{ for }n=1,2,3,\cdots\implies\lim_{n\to\infty}s_n=1$$

2. (a)
$$\overline{\lim}_{n\to\infty}\cos n\pi=1\qquad\qquad\underline{\lim}_{n\to\infty}\cos n\pi=-1\tag{1}$$

   (b)
$$\overline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx 0.951\qquad\qquad\underline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx -0.951$$

   (c)
$$\overline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=\infty\qquad\qquad\underline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=-\infty$$

3. (a) A sequence
$$s_n=1+\cos n\pi$$
   has limits
$$\overline{\lim}_{n\to\infty}s_n=2\qquad\qquad\underline{\lim}_{n\to\infty}s_n=0$$

(b) A sequence

$$s_n = -n^2 \sin^2\left(\frac{1}{2}n\pi\right)$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = 0 \qquad \underline{\lim}_{n\to\infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left| \left(1 + \frac{1}{n}\right)^n - e \right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

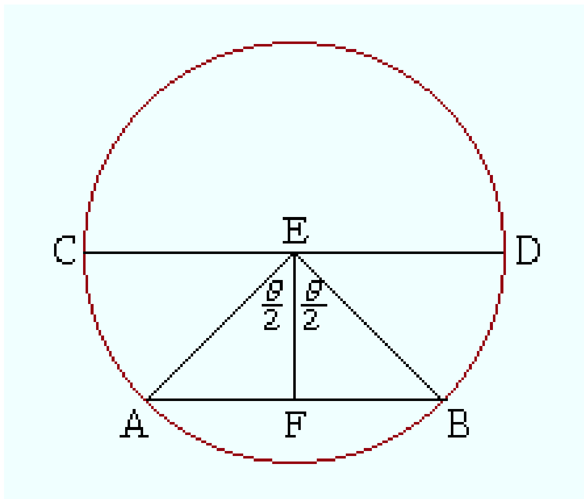
$$e \approx \left(1 + \frac{1}{164}\right)^{164} \approx 2.71$$

6.

$$\overline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } |x| > 1$$
$$\overline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = \pm 1$$
$$\overline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$

$$\underline{\lim}_{n\to\infty} x^n = -\infty \qquad \text{for } x < -1$$
$$\underline{\lim}_{n\to\infty} x^n = -1 \qquad \text{for } x = -1$$
$$\underline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$
$$\underline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = 1$$
$$\underline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } x > 1$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE\sin\frac{\theta}{2} \times AE\cos\frac{\theta}{2} = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \to \infty$:

$$A_{S_1} = s_n = \lim_{n\to\infty}\frac{n}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\pi\frac{n}{2\pi}\sin\frac{2\pi}{n}$$

Now using the fact that $\lim_{x\to 0}\sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n\to\infty}\pi\frac{n}{2\pi}\sin\frac{2\pi}{n} = \lim_{x\to 0}\pi\frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π, it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n\to\infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the nth term test $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$ diverges.

(b) Since

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^{n-1}}{3n} = \lim_{n\to\infty} \frac{(n-1)\,2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the nth term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3-1} = \left|\frac{1}{n^3-1}\right| < \frac{1}{n-1}$$

for $n = 2, 3, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \frac{1/n}{1-(1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3-1)$ is absolutely convergent.

(b) Since $|\sin n| < 1$ for $n \geq 1$ it follows that

$$\left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$$

for $n = 1, 2, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a) Since $n+5 > n$ and $n^2 - 3n - 5 < n^2$ for $n \geq 1$ it follows that

$$\frac{n+5}{n^2-3n-5} > \frac{n}{n^2} = \frac{1}{n}$$

for $n = 1, 2, \ldots$. Now since $\sum_{n=1}^{\infty} 1/n$ is the *harmonic series*, which diverges, it follows by the comparison test for divergence that $\sum_{n=1}^{\infty}(n+5)/(n^2-3n-5)$ diverges as well.

(b) Since $\sqrt{n}\ln n < n\ln n$ for $n \geq 2$ it follows that

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n\ln n}$$

for $n = 2, 3, \ldots$. Using the inequality $\ln(1+x) \leq x$ we may continue to write

$$\frac{1}{n\ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln\left(1+\frac{\ln(1+1/n)}{\ln n}\right) \geq \ln\frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n}\ln n} > \ln\frac{\ln(1+n)}{\ln n} = \ln\ln(1+n) - \ln\ln n$$

Now let us consider the series

$$\sum_{n=2}^{N} \ln\ln(1+n) - \ln\ln n = \ln\ln(1+N) - \ln\ln 2$$

Hence, when $N \to \infty$

$$\sum_{n=2}^{\infty} \ln\ln(1+n) - \ln\ln n = \lim_{N\to\infty} \ln\ln(1+N) - \ln\ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that $\sum_{n=2}^{\infty} 1/(\sqrt{n}\ln n)$ diverges as well.

4. (a) Let $y = f(x) = 1/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^{\infty} f(x)\,dx$ with $c = 1$ then evaluates to

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_1^{b} \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_{\pi/4}^{\tan^{-1}b} du = \lim_{b\to\infty} u\Big|_{\pi/4}^{\tan^{-1}b} = \lim_{b\to\infty}\tan^{-1}b - \frac{\pi}{4}$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

where we have used the substitution $x = \tan u$. Hence, by the integral test, since the improper integral $\int_1^{\infty} f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2+1)$.

(b) let $y = f(x) = 1/(x\ln^2 x)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^{\infty} f(x)\,dx$ with $c = 2$ then evaluates to

$$\int_2^{\infty} \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_2^{b} \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \lim_{b\to\infty}\frac{1}{\ln b}$$
$$= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}$$

where we have used the substitution $u = \ln x$. Hence, by the integral test, since the improper integral $\int_2^\infty f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=2}^\infty 1/(n \ln^2 n)$.

5. (a) Let $y = f(x) = x/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, (fx) decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 1$ then evaluates to

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\frac{1}{2}\int_2^{b^2+1}\frac{du}{u} &= \lim_{b\to\infty}\left.\frac{\ln u}{2}\right|_2^{b^2+1}\\ &= \lim_{b\to\infty}\frac{\ln|b^2+1| - \ln 2}{2}\\ &= \infty - \frac{\ln 2}{2} = \infty\end{aligned}$$

where we have used the substitution $u = x^2+1$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=1}^\infty n/(n^2+1)$.

(b) Let $y = f(x) = 1/(x \ln x \ln\ln x)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 10$ then evaluates to

$$\begin{aligned}\int_{10}^\infty \frac{dx}{x\ln x\ln\ln x} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{x\ln x\ln\ln x} &= \lim_{b\to\infty}\int_{\ln 10}^{\ln b}\frac{du}{u\ln u}\\ &= \lim_{b\to\infty}\int_{\ln\ln 10}^{\ln\ln b}\frac{dv}{v}\\ &= \lim_{b\to\infty}\ln v\Big|_{\ln\ln 10}^{\ln\ln b}\\ &= \lim_{b\to\infty}\ln\ln\ln b - \ln\ln\ln 10\\ &= \infty - \ln\ln\ln 10 = \infty\end{aligned}$$

where we have used the substitutions $u = \ln x$ and $v = \ln u$. Hence, by the integral test, since the improper integral $\int_{10}^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=10}^\infty 1/(n\ln n\ln\ln n)$.

6. (a) Let $a_n = (-1)^n/n!$. As such we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty}\left|\frac{n!}{(-1)^n}\frac{(-1)^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|-\frac{1}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^\infty (-1)^n/n!$ is absolutely convergent.

(b) Let $a_n = 2^n + 1/(3^n + n)$. As such we find

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L &= \lim_{n\to\infty}\left|\frac{3^n+n}{2^n+1}\frac{2^{n+1}+1}{3^{n+1}+n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{3^n\left(1+n/3^n\right)}{2^n\left(1+1/2^n\right)}\frac{2^{n+1}\left(1+1/2^{n+1}\right)}{3^{n+1}\left[1+\left(n/3^{n+1}\right)+\left(1/3^{n+1}\right)\right]}\right| \\
&= \frac{2}{3}\lim_{n\to\infty}\left|\frac{\left(1+n/3^n\right)\left(1+1/2^{n+1}\right)}{\left(1+1/2^n\right)\left(1+n/3^{n+1}+1/3^{n+1}\right)}\right| \\
&= \frac{2}{3}\frac{(1+0)\cdot(1+0)}{(1+0)\cdot(1+0+0)} = \frac{2}{3}
\end{aligned}$$

where we have used the fact that

$$\lim_{x\to\infty}\frac{x}{a^x} = \lim_{x\to\infty}\frac{1}{xa^{x-1}} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^{\infty} 2^n + 1/(3^n+n)$ is absolutely convergent.

7. (a) Let $a_n = 1/\ln n$. Then for $2 \le n < \infty$ we find

$$a_n = \frac{(-1)^n}{\ln n} = \frac{1}{\ln n}$$

Now since $\ln n$ is monotonically increasing for $2 \le n < \infty$ we may conclude that $a_n = 1/\ln n$ is monotonically decreasing for $2 \le n < \infty$ and so $a_{n+1} \le a_n$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\ln n} = \frac{1}{\infty} = 0$$

provided $n \ge 2$ and so by the alternating series test we may conclude that the series $\sum_{n=2}^{\infty}(-1)^n/\ln n$ converges.

(b) Let $f(x) = \ln x/x$. Hence,

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2}$$

Note that the derivative of $f(x)$ becomes negative when $x > e \approxeq 2.71828$ and hence, that $f(x)$ becomes monotonically decreasing when $e < x < \infty$. As such, the terms of the sequence $a_n = f(n) = \ln n/n$ are decreasing (i.e. $a_{n+1} \le a_n$) when $3 \le n < \infty$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. As such, by the alternating series test we may conclude that the series $\sum_{n=3}^{\infty}(-1)^n \ln n/n$ converges.

8. (a) Let $a_n = 1/n^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{n^{-n}} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} 1/n^n$ is absolutely convergent.

(b) Let $a_n = [n/(n+1)]^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \frac{1}{e}$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} [n/(n+1)]^{n^2}$ is absolutely convergent.

9. (a) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{2}{3} - \frac{1}{2} + \frac{3}{4} - \frac{2}{3} + \cdots + \frac{n+1}{n+2} - \frac{n}{n+1} = -\frac{1}{2} + \frac{n+1}{n+2}$$

Taking the limit of S_n as $n \to \infty$ then gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{n+2} - \frac{1}{2} = \lim_{n\to\infty} \frac{1+1/n}{1+2/n} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence, the series converges.

(b) Let the series

$$\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{n+1}{2^{n+1}} - \frac{n}{2^n}\right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{1}{2} - \frac{1}{2} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} - \frac{3}{8} + \cdots + \frac{n+1}{2^{n+1}} = -\frac{1}{2} + \frac{n+1}{2^{n+1}}$$

Taking the limit of S_n as $n \to \infty$ then gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{(n+1)\,2^n} - \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$$

using *L'Hospital's rule*. Hence, the series converges.

10. Let $y = f(x)$ satisfy the following conditions:

(a) $f(x)$ is defined and continuous for $c \leq x < \infty$

(b) $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$

(c) $f(n) = a_n$

Let us suppose the improper integral $\int_c^\infty f(x)\,dx$ diverges. Assumptions (b) and (c) imply that $a_n > 0$ for n sufficiently large. Hence, by Theorem 7 of Section 6.5 the series $\sum a_n$ is either convergent or properly divergent. Let the integer m be chosen so that $m > c$. Then, since $f(x)$ is decreasing

$$\int_n^{n+1} f\left(x\right)\,dx \leq f\left(n\right) = a_n \quad \text{for } n \geq m$$

Hence, $a_m + \cdots + a_{m+p} \geq \int_m^{m+p+1} f\left(x\right)\,dx$. However, since $\int_c^\infty f(x)\,dx$ diverges it follows that $\lim_{p\to\infty} \int_m^{m+p+1} f(x)\,dx$ diverges, which thus ultimately implies that the series $\sum_m^\infty a_n$ must be divergent as well.

11. Let an alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} \left(-1\right)^{n+1} a_n, \quad a_n > 0$$

be given along with the two conditions

(a) $a_{n+1} \leq a_n$ for $n = 1, 2, \ldots$

(b) $\lim_{n\to\infty} a_n = 0$

What remains to be proven is that such a series converges given the aforementioned conditions. Let $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$ denote the nth partial sum of an alternating series. Then $S_1 = a_1$, $S_2 = a_1 - a_2 < S_1$, $S_3 = S_2 + a_3 > S_2$, $S_3 = S_1 - (a_2 - a_3) < S_1$, so that $S_2 < S_3 < S_1$. As such, we may conclude that $S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$ or $S_n \leq a_1$ and that each $S_n \geq 0$ for $n = 1, 2, \ldots$.

Next, let an $\epsilon > 0$ be given. By the Cauchy criterion our goal is to find an N so that whenever $m > n > N$ then

$$\left|S_m - S_n\right| = \left|a_{n+1} - a_{n+2} + \cdots \pm a_m\right| < \epsilon$$

Now since each partial sum is non-negative (i.e. $S_n \geq 0$) and acknowledging that all partial sums are $\leq$ the first term a_1, but now applied to the alternating series starting at a_{n+1} instead of a_1 we can write

$$\left|S_m - S_n\right| \leq a_{n+1} < \epsilon$$

Now because $\lim_{n\to\infty} a_n = 0$ we can find N such that $a_{n+1} < \epsilon$ whenever $n > N$. Hence,

$$m > n > N \implies \left|S_m - S_n\right| \leq a_{n+1} < \epsilon$$

which thus satisfies our initial condition

$$\left|S_m - S_n\right| = |a_{n+1} - a_{n+2} + \cdots \pm a_m| < \epsilon$$

We may conclude that the sequence of partial sums S_n of our original alternating series subject to conditions (a) and (b) satisfies the Cauchy criterion and therefore, is convergent. Hence, the alternating series itself is convergent.

12. (a) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+4}{2n^3-1}$$

be given. In order to determine convergence or divergence we first try the comparison test for convergence. To this end, note that $n+4 \leq 5n$ and $2n^3 - 1 \geq n^3$ for $n = 1, 2, \ldots$. Hence,

$$|a_n| = \frac{n+4}{2n^3-1} \leq \frac{5n}{n^3} = \frac{5}{n^2} = b_n \quad \text{for } n = 1, 2, \ldots$$

As such, if we can prove that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Now let $y = f(x) = 5/x^2$, which satisfies the following conditions:

i. $f(x)$ is defined and continuous for $c \leq x < \infty$ for $c \neq 0$

ii. $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$

iii. $f(n) = b_n$

Then by the integral test the series $\sum_{n=1}^{\infty} b_n$ converges or diverges according to whether the improper integral $\int_c^{\infty} f(x)\,dx$ converges or diverges. As such, we evaluate

$$\int_1^{\infty} \frac{5}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{5}{x^2}\,dx = \lim_{b\to\infty} -\frac{5}{x}\bigg|_1^b = 5 - \lim_{b\to\infty} \frac{5}{b} = 5 - \frac{5}{\infty} = 5$$

Hence, since the improper integral $\int_c^{\infty} f(x)\,dx$ converges, so do the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$.

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n-5}{n2^n}$$

be given. Since $a_n \neq 0$ for $n = 1, 2, \ldots$ we can try the ratio test in order to determine convergence or divergence:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty} \left|\frac{3(n+1)-5}{(n+1)\,2^{n+1}} \frac{n2^n}{3n-5}\right| &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{3(n+1)-5}{n+1} \frac{n}{3n-5}\right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left|\frac{3(1+1/n)-5/n}{1+1/n} \frac{1}{3-5/n}\right| \\
&= \frac{1}{2} \frac{3+0-0}{1+0} \frac{1}{3-0} = \frac{1}{2}
\end{aligned}$$

Hence, since $L = 1/2 < 1$ the series $\sum_{n+1}^{\infty} a_n$ is absolutely convergent.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n}{n+1}$$

be given. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n+1} = \lim_{n\to\infty} e^n = \infty$$

using *L'Hospital's rule*. Hence, it follows from the nth term test that the series $\sum_{n=1}^{\infty} a_n$ diverges.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n!+1}$$

be given. Since

$$|a_n| = \frac{n^2}{n!+1} < \frac{n^2}{n!} = b_n \quad \text{for } n = 1, 2, \ldots$$

the comparison test for convergence tells us that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Since $b_n \neq 0$ for $n = 1, 2, \ldots$ we can use the ratio test in order to determine if $\sum_{n=1}^{\infty} b_n$ converges:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = L = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2(n+1)} = \lim_{n\to\infty} \frac{n+1}{n^2} &= \lim_{n\to\infty} \frac{1+1/n}{n} \\ &= \frac{1+0}{\infty} = 0\end{aligned}$$

Hence, since $L = 0 < 1$ we may conclude that $\sum_{n=1}^{\infty} b_n$ is absolutely convergent by the ratio test and thus, that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent by the comparison test for convergence.

(e) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{3 \cdot 5 \cdots (2n+3)}$$

be given. Since $a_n \neq 0$ for $n = 1, 2, \ldots$ we can use the ratio test to determine convergence:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L &= \lim_{n\to\infty} \frac{(n+1)!}{3 \cdot 5 \cdots [2(n+1)+3]} \frac{3 \cdot 5 \cdots (2n+3)}{n!} \\ &= \lim_{n\to\infty} \frac{n}{2(n+1)+3} \\ &= \lim_{n\to\infty} \frac{1}{2(1+1/n)+3/n} = \frac{1}{2(1+0)+0} = \frac{1}{2}\end{aligned}$$

Hence, since $L = 1/2 < 1$ we may conclude that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(f) Let the series

$$\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{2n+3}$$

be given. This is an alternating series. Note that for $n = 1, 2, 3, 4$ its terms are actually increasing (i.e. $a_{n+1} > a_n$) in absolute value and $a_{n+1} \leq a_n$ only becomes true when $n = 5, 6, \ldots$. This is not a problem for the alternating series test to be valid however. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{2n+3} = \lim_{n\to\infty} \frac{1/n}{2} = \frac{0}{2} = 0$$

using *L'Hospital's rule.* Hence, the alternating series converges.

(g) Let the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1+\ln^2 n}{n \ln^2 n}$$

be given. As such, let us define the function $y = f(x) = (1+\ln^2 x)/n \ln^2 x$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+\ln^2 x}{x \ln^2 x} = \lim_{x\to\infty} \left(\frac{1}{x \ln^2 x} + \frac{1}{x} \right) = \frac{1}{\infty} + \frac{1}{\infty} = 0$$

Furthermore, $f(x)$ satisfies the following conditions:

i. $f(x)$ is defined and continuous for $c \leq x < \infty$

ii. $f(x)$ decreases as x increases for $x \geq 2$ and $\lim_{x\to\infty} f(x) = 0$

iii. $f(n) = a_n$

Hence, we can use the integral test to determine whether the series $\sum_{n=2}^{\infty} a_n$ converges or diverges:

$$\begin{aligned}
\int_2^{\infty} \frac{1+\ln^2 x}{x \ln^2 x}\, dx = \lim_{b\to\infty} \int_2^b \frac{1+\ln^2 x}{x\ln^2 x}\, dx &= \lim_{b\to\infty} \int_2^b \left(\frac{1}{x \ln^2 x} + \frac{1}{x} \right) dx \\
&= \lim_{b\to\infty} \int_2^b \frac{dx}{x \ln^2 x} + \lim_{b\to\infty} \int_2^b \frac{dx}{x} \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} + \lim_{b\to\infty} \ln |x| \Big|_2^b \\
&= \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln b} + \lim_{b\to\infty} (\ln b - \ln 2) \\
&= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2} + \ln b - \ln 2 \right) = \infty
\end{aligned}$$

In conclusion, since the improper integral $\int_c^{\infty} f(x)\, dx$ diverges, so will the series $\sum_{n=2}^{\infty} a_n$.

(h) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n+2} \equiv \sum_{n=1}^{\infty} (-1)^n\, b_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$$

be given. $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series with terms that are decreasing in absolute value: $b_{n+1} < b_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} b_n = 0$. Hence, by the alternating series test the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges and thus, so will the series $\sum_{n=1}^{\infty} a_n$.

(i) Let the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + \ln n}$$

be given. Now since $a \geq 0$ and $n + \ln n < 2n$ for $n = 1, 2, \ldots$ we can define $b_n = \ln n/2n$ such that $a_n > b_n \geq 0$. Then by the comparison test for divergence if $\sum_{n=1}^{\infty} b_n$ diverges so will $\sum_{n=1}^{\infty} a_n$. To this end, let us define the function $y = f(x) = \ln x/2x$. Now since $\ln x < 2x$ for $1 \leq x < \infty$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{2x} = \lim_{x\to\infty} \frac{1/x}{2} = 0$$

using *L'Hospital's rule*, we find that

i. $f(x)$ is defined and continuous for $c \leq x < \infty$, where $c = 1$
ii. $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$
iii. $f(n) = a_n$

Then the series $\sum_{n=1}^{\infty} b_n$ converges or diverges according to whether the improper integral $\int_c^{\infty} f(x)\, dx$ converges or diverges:

$$\int_1^{\infty} \frac{\ln x}{2x}\, dx = \frac{1}{2} \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x}\, dx = \frac{1}{2} \lim_{b\to\infty} \int_0^{\ln b} u\, du = \lim_{b\to\infty} \left.\frac{u^2}{4}\right|_0^{\ln b} = \lim_{b\to\infty} \frac{\ln^2 b}{4} = \infty$$

where we have used the substitution $u = \ln x$. Hence, by the integral test the series $\sum_{n=1}^{\infty} b_n$ diverges and so by the comparison test for divergence the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

(j) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{n+1}{2n}\right)^n$$

be given. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{2n}\right)^n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}$$

Then by the root test, since $R = 1/2 < 1$ the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

13. Let $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \ldots$ and let the sequence a_n/b_n have limit k, possibly infinite.

   (a) Suppose $0 < k < \infty$, i.e. $\lim_{n\to\infty} a_n/b_n = k$ is some positive number. Then for some $\epsilon > 0$ we know that there must exist a positive integer N such that for all $n > N$ it is true that

$$\left|\frac{a_n}{b_n} - k\right| < \epsilon \iff (k-\epsilon)\, b_n < a_n < (k+\epsilon)\, b_n$$

   As $k > 0$ we can choose ϵ sufficiently small so that $k - \epsilon > 0$. Hence,

$$b_n < \frac{a_n}{k-\epsilon}$$

   As such, by the comparison test for convergence, if $\sum a_n$ converges then so must $\sum b_n$. Similarly $a_n < (k+\epsilon)b_n$. Hence, if $\sum a_n$ diverges then by the comparison test for divergence so will $\sum b_n$. In conclusion, both series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

   (b) Suppose $k = 0$. Then for some $\epsilon > 0$ there must exist a positive integer N such that for all $n > N$ it is true that

$$\frac{a_n}{b_n} < \epsilon \iff a_n < \epsilon b_n$$

   Hence, by the comparison test for convergence, if $\sum b_n$ converges then so must $\sum a_n$. Additionally, as long as $\sum a_n$ converges the inequality can still be satisfied if $\sum b_n$ diverges by choosing ϵ sufficiently small.

   (c) Suppose $k = \infty$. Then for some $\epsilon > 0$ we know that there must exist a positive integer N such that for all $n > N$ it is true that

$$(k-\epsilon)\, b_n < a_n < (k+\epsilon)\, b_n$$

   From the first inequality we see that

$$a_n > (k-\epsilon)\, b_n$$

   from which we may gather that $\sum a_n$ may diverge while $\sum b_n$ converges, since $k = \infty$. Similarly, since $a_n < (k+\epsilon)b_n$ then the comparison test for divergence tells us that divergence of $\sum a_n$ implies divergence of $\sum b_n$.

14. (a) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{3n^2+n+1}$$

   be given and let $b_n = 1/n$. Using Problem 13 we thus find

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{3n^2+n+1} = \lim_{n\to\infty} \frac{2+1/n}{3+1/n+1/n^2} = \frac{2}{3}$$

and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ diverges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must diverge as well.

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^3 - 3n^2 + 5}{n^5 + n + 1}$$

be given and let $b_n = 1/n^2$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^5 - 3n^4 + 5n^2}{n^5 + n + 1} = \lim_{n\to\infty} \frac{1 - 3/n + 5/n^3}{1 + 1/n^4 + 1/n^5} = 1$$

and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must converge as well.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin \frac{1}{n}$$

be given and let $b_n = 1/n$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(1/n)/n^2}{1/n^2} = \lim_{n\to\infty} \cos\frac{1}{n} = \cos\frac{1}{\infty} = 1$$

using *L'Hospital's rule* and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ diverges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must diverge as well.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right)$$

be given and let $b_n = 1/n^2$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 - \cos(1/n)}{1/n^2} = \frac{1}{2} \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \frac{1}{2}$$

using *L'Hospital's rule* and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must converge as well.

Section 6.9

1. (a) Let the sum $\sum_{n=1}^{\infty} 1/n^2$ be given and let us define the allowed error as $\epsilon = 1$. We know from the previous section that this series converges by the integral test of

Theorem 14. Hence, by Theorem 23 we find

$$|R_n| = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

and so the condition $T_n \leq \epsilon$ then translates to the inequality $n \geq 1$, which is satisfied for $n = 1$. Hence, one term is sufficient to compute the sum with given allowed error $\epsilon = 1$ and so $S_1 = 1$.

(b) Let the sum $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ be given and let us define the allowed error as $\epsilon = 1/10$. Now since this series converges by the alternating series test then by Theorem 26

$$|R_n| < a_{n+1} = T_n$$

Hence, we end up with the inequality $a_{n+1} \leq \epsilon$ or $1/(n+1)^2 \leq 1/10 \iff (n+1)^2 \geq 10$, which is satisfied for $n = 3$. Hence, three terms is sufficient to compute the sum with the given allowed error $\epsilon = 1/10$ and so $S_3 \approxeq 0.86$.

(c) Let the sum $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n/(n^3 + 5)$ be given and let us define the allowed error as $\epsilon = 1/5$. It is true that $n^3 + 5 > n^3$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 5$. Hence, five terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/5$ and so $S_5 \approxeq 0.51$.

(d) Let the sum $\sum n = 1^{\infty} 1/(n^2 + 1)$ be given and let us define the allowed error as $\epsilon = 1/2$. It is true that $n^2 + 1 > n^2$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m =\leq \sum_{m=n+1}^{\infty} b_m \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 2$. Hence, two terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_2 = 0.7$.

(e) Let the sum $\sum_{n=1}^{\infty} 1/n^n$ be given and let us define the allowed error as $\epsilon = 1/100$. Then

$$\sqrt[n]{|a_n|} = \frac{1}{n} \leq r < 1$$

for $n \geq 2$, so that the series $\sum a_n$ converges by the root test. Hence, by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = T_n \implies \frac{1}{(n+1)^{n+1}} \cdot \frac{1}{1 - \dfrac{1}{n+1}} = \frac{1}{n\,(n+1)^n} \leq \epsilon$$

for $n \geq 2$. In other words, we are looking for the smallest integer $n \geq 2$ such that $n(n+1)^n \geq 100$, which is satisfied for $n = 3$. Hence, three terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/100$ and so $S_3 \approxeq 1.287$.

(f) Let the sum $\sum_{n+1}^{\infty} 1/n!$ be given and let us define the allowed error as $\epsilon = 1/100$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq r < 1$$

for $n \geq 1$, so that the series $\sum a_n$ converges by the ratio test. Hence, by Theorem 24

$$|R_n| \leq \frac{|a_{n+1}|}{1-r} = T_n \implies \frac{1}{(n+1)!} \cdot \frac{1}{1 - \dfrac{1}{n+2}} = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1}\right) \leq \epsilon$$

for $n \geq 1$. In other words, we are looking for the smallest integer $n \geq 1$ such that $T_n \leq \epsilon$, which is satisfied for $n = 4$. Hence, four terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/100$ and so $S_4 \approxeq 1.708$.

(g) Let the sum $\sum_{n+1}^{\infty}(-1)^{n+1}/(2n-1)!$ be given and let us define the allowed error as $\epsilon = 1/1000$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n-1)!}{(2n+1)!} < 1 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$$

the series $\sum a_n$ converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(2n+1)!} = T_n \implies \frac{1}{(2n+1)!} \leq \epsilon$$

and so we are looking for the smallest integer such that $(2n+1)! \geq 1000$, which is satisfied for $n = 3$. Hence, three terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/1000$ and so $S_3 \approxeq 0.8417$.

(h) Let the sum $\sum_{n+2}^{\infty}(-1)^n/(n \ln n)$ be given and let us define the allowed error as $\epsilon = 1/2$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n \ln n}{(n+1)\ln(n+1)} < 1 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

the series $\sum a_n$ converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(n+1)\ln(n+1)} = T_n \implies \frac{1}{(n+1)\ln(n+1)} \leq \epsilon$$

and so we are looking for the smallest integer such that $(n+1)\ln(n+1) \geq 2$, which is satisfied for $n = 2$. Hence, one term is sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_1 \approxeq 0.72$.

(i) Let the sum $\sum_{n=2}^{\infty} 1/(n^3 \ln n)$ be given and let us define the allowed error as $\epsilon = 1/2$. It is true that $n^3 \ln n > n^2$ for $n \geq 2$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 2$. Now since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 2$. Hence, one term is sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_1 \approxeq 0.18$.

(j) Let the sum $\sum_{n=1}^{\infty} 2^n/(3^n + 1)$ be given and let us define the allowed error as $\epsilon = 1/10$. It is true that $3^n + 1 > 3^n$ for $n \geq 1$ and so we can define $b_n = 2^n/3^n$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sqrt[n]{|b_n|} = \sqrt[n]{2^n/3^n} = 2/3 \leq r < 1$ for $n \geq 1$ we may conclude that the series $\sum b_n$ converges by the root test. Hence, choosing $r = 2/3$ then by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = \frac{2^{n+1}}{3^n} = T_n \implies \frac{2^{n+1}}{3^n} \leq \epsilon$$

and so we are looking for the smallest integer such that $3^n/2^{n+1} \geq 10$, which is satisfied for $n = 8$. Hence, eight terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/10$ and so $S_8 \approxeq 1.697$.

2. Let $\sum a_n$ be the geometric series $1 + r + r^2 + \cdots = \sum_{n=0}^{\infty} r^n$. By Theorem 16 this series converges for $-1 < r < 1$. Hence, by Theorem 23

$$|R_n| = \sum_{m=n+1}^{\infty} r^m < \int_n^{\infty} r^x\, dx = T_n$$

Or

$$\begin{aligned} T_n = \int_n^{\infty} r^x\, dx = \lim_{b\to\infty} \int_n^b r^x\, dx = \lim_{b\to\infty} \int_n^b e^{x \ln r}\, dx &= \lim_{b\to\infty} \int_{n \ln r}^{b \ln r} \frac{e^u}{\ln r}\, du \\ &= \lim_{b\to\infty} \left.\frac{e^u}{\ln r}\right|_{n \ln r}^{b \ln r} \\ &= \lim_{b\to\infty} \frac{e^{b \ln r}}{\ln r} - \frac{e^{n \ln r}}{\ln r} \\ &= -\frac{e^{n \ln r}}{\ln r} = -\frac{r^n}{\ln r} \end{aligned}$$

assuming $0 < r < 1$.

(a) let the given allowed error $\epsilon = 1/100$. In order to determine how many terms are needed to compute the sum with error less than ϵ we require $T_n < \epsilon$. For $r = 1/2$ this results in

$$-\frac{1}{2^n \ln 2^{-1}} < \frac{1}{100} \iff n > \frac{\ln(100/\ln 2)}{\ln 2}$$

which is satisfied for $n = 8$. Hence, when $r = 1/2$, 8 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$. For $r = 0.9 = 9/10$ we get

$$-\frac{1}{\ln(9/10)}\left(\frac{9}{10}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 9/10)}{\ln 10/9}$$

which is satisfied for $n = 66$. Hence, when $r = 0.9$, 66 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$. For $r = 0.99 = 99/100$ we get

$$-\frac{1}{\ln(99/100)}\left(\frac{99}{100}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 99/100)}{\ln 100/99}$$

which is satisfied for $n = 916$. Hence, when $r = 0.99$, 916 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$.

(b) The closed form formula (6.17) for a geometric series $1 + ar + ar^2 + \cdots$, with $a = 1$ and $-1 < r < 1$ is given by $S = 1/(1-r)$. Likewise, the closed form formula for the partial sum of the same geometric series is given by $S_n = (1-r^n)/(1-r)$. The remainder R_n after n terms thus can be defined as

$$|R_n| = |S_n - S| = \left|\frac{1-r^n}{1-r} - \frac{1}{1-r}\right| = \left|\frac{-r^n}{1-r}\right| < \epsilon \iff -\epsilon < -\frac{r^2}{1-r} < \epsilon$$

The inequality on the right hand side can be further manipulated to finally get

$$\begin{aligned} -\frac{r^n}{1-r} &< \epsilon \\ r^n &> -\epsilon(1-r) \\ \ln|r|^n &> \ln|-\epsilon(1-r)| \\ n &> \frac{\ln \epsilon(1-r)}{\ln|r|} \end{aligned}$$

where $-1 < r < 1$.

(c) When r approaches 1 from the left we note that

$$\lim_{r\to 1^-} \frac{\ln \epsilon(1-r)}{\ln|r|} = \lim_{r\to 1^-} \frac{\ln \epsilon(1-r)}{\ln r} = \lim_{r\to 1^-} \ln \epsilon(1-r) \cdot \lim_{r\to 1^-} \frac{1}{\ln r} = -\infty \cdot -\infty = \infty$$

Hence, it follows from (b) that $n \to \infty$ when $r \to 1^-$, or in other words; that the number of terms needed to compute the sum with error less than a fixed ϵ becomes infinite.

3. Let the series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $p > 0$ be given. As such, $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$ and so $S_1 = a_1 = 1$, $S_2 = a_1 - a_2 = 1 - 2^{-p}$ so that $0 < S_2 < S_1$, $S_3 = S_1 - (a_2 - a_3) = 1 - 2^{-p} + 3^{-p}$ so that $0 < S_3 < S_1$ and $S_2 < S_3 < S_1$. Reasoning in this way, we conclude that

$$S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$$

Hence, the smallest partial sum is S_2, but we just established that $S_2 = 1 - 2^{-p} > 0$. Hence, it follows that the sum $S = \lim_{n\to\infty} S_n$ must be positive whenever $p > 0$.

Section 6.10

1. Let the following relations be given:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

Then by (6.15)

(a)

$$\sum_{n=1}^{\infty} \frac{6}{n^2} = 6\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90} =$$

(c)

$$\sum_{n=1}^{\infty} \frac{2n^2-3}{n^4} = \sum_{n=1}^{\infty} \frac{2}{n^2} - \sum_{n=1}^{\infty} \frac{3}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 3\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{3} - \frac{\pi^4}{30}$$

(d)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{9+3n^2+5n^4}{n^6} &= \sum_{n=1}^{\infty} \frac{9}{n^6} + \sum_{n=1}^{\infty} \frac{3}{n^4} + \sum_{n=1}^{\infty} \frac{5}{n^2} = 9\sum_{n=1}^{\infty} \frac{1}{n^6} + 3\sum_{n=1}^{\infty} \frac{1}{n^4} + 5\sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{5\pi^2}{6} + \frac{\pi^4}{30} + \frac{\pi^6}{105}\end{aligned}$$

(e)

$$\sum_{n=3}^{\infty} \frac{n^4-1}{n^6} = \sum_{n=3}^{\infty} \frac{1}{n^2} - \sum_{n=3}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{5}{4} - \sum_{n=1}^{\infty} \frac{1}{n^6} + \frac{65}{64} = \frac{\pi^2}{6} - \frac{\pi^6}{945} - \frac{15}{64}$$

(f)

$$\sum_{n=2}^{\infty}\frac{n^2+1}{\left(n^2-1\right)^2}=\sum_{n=2}^{\infty}\left[\frac{1}{2\left(n+1\right)^2}+\frac{1}{2\left(n-1\right)^2}\right]=\sum_{n=2}^{\infty}\frac{1}{2\left(n+1\right)^2}+\sum_{n=2}^{\infty}\frac{1}{2\left(n-1\right)^2}$$
$$=\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{\left(n+1\right)^2}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{\left(n-1\right)^2}$$
$$=\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2}-\frac{1}{8}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2}+\frac{1}{2}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{8}-\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{2}-\frac{1}{2}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{5}{8}=\frac{\pi^2}{6}-\frac{5}{8}$$

2. (a)

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}+1-1=\sum_{n=2}^{\infty}\frac{1}{n^3}+1=\sum_{n=2}^{\infty}\frac{1}{\left(n-1\right)^3}$$

(b)

$$\sum_{n=1}^{\infty}\left[f\left(n+1\right)-f\left(n\right)\right]=\sum_{n=1}^{\infty}f\left(n+1\right)-\sum_{n=1}^{\infty}f\left(n\right)$$
$$=\sum_{n=1}^{\infty}f\left(n\right)+\lim_{n\to\infty}f\left(n\right)-f\left(1\right)-\sum_{n=1}^{\infty}f\left(n\right)$$
$$=\lim_{n\to\infty}f\left(n\right)-f\left(1\right)$$

if the limit exists.

(c)

$$\sum_{n=2}^{\infty}\left[f\left(n+1\right)-f\left(n-1\right)\right]=\sum_{n=2}^{\infty}f\left(n+1\right)-\sum_{n=2}^{\infty}f\left(n-1\right)$$
$$=\sum_{n=1}^{\infty}f\left(n+1\right)+\lim_{n\to\infty}f\left(n+1\right)-f\left(2\right)-\sum_{n=1}^{\infty}f\left(n\right)$$
$$=\sum_{n=1}^{\infty}f\left(n\right)-f\left(1\right)+\lim_{n\to\infty}f\left(n\right)+\lim_{n\to\infty}f\left(n+1\right)-f\left(2\right)$$
$$-\sum_{n=1}^{\infty}f\left(n\right)$$
$$=\lim_{n\to\infty}\left[f\left(n\right)+f\left(n+1\right)\right]-f\left(1\right)-f\left(2\right)$$

if the limit exists.

3. (a) Let $f(n) = 1/n^2$. Then using 2(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n\to\infty} f(n) - f(1) \\
\sum_{n=1}^{\infty} \left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] &= \lim_{n\to\infty} \frac{1}{n^2} - 1 \\
\sum_{n=1}^{\infty} -\frac{2n+1}{n^2(n+1)^2} &= 0 - 1 \\
\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} &= 1
\end{aligned}$$

(b) Let $f(n) = 1/n$. Then using 2(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n\to\infty} f(n) - f(1) \\
\sum_{n=1}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n}\right] &= \lim_{n\to\infty} \frac{1}{n} - 1 \\
\sum_{n=1}^{\infty} -\frac{1}{n(n+1)} &= 0 - 1 \\
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= 1
\end{aligned}$$

(c) Let $f(n) = 1/n$. Then using 2(c)

$$\begin{aligned}
\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] &= \lim_{n\to\infty} [f(n) + f(n+1)] - f(1) - f(2) \\
\sum_{n=2}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n-1}\right] &= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1}\right) - 1 - \frac{1}{2} \\
\sum_{n=2}^{\infty} -\frac{2}{n^2-1} &= 0 + 0 - 1 - \frac{1}{2} \\
-2\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= -\frac{3}{2} \\
\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \frac{3}{4}
\end{aligned}$$

(d) Let $f(n) = 1/n^2$. Then using 2(c)

$$
\begin{aligned}
\sum_{n=2}^{\infty} \left[f\left(n+1\right) - f\left(n-1\right)\right] &= \lim_{n\to\infty} \left[f\left(n\right) + f\left(n+1\right)\right] - f\left(1\right) - f\left(2\right) \\
\sum_{n=2}^{\infty} \left[\frac{1}{\left(n+1\right)^2} - \frac{1}{\left(n-1\right)^2}\right] &= \lim_{n\to\infty} \left[\frac{1}{n^2} + \frac{1}{\left(n+1\right)^2}\right] - 1 - \frac{1}{4} \\
\sum_{n=2}^{\infty} -\frac{4n}{\left(n^2-1\right)^2} &= 0 + 0 - 1 - \frac{1}{4} \\
\sum_{n=2}^{\infty} \frac{4n}{\left(n^2-1\right)^2} &= \frac{5}{4}
\end{aligned}
$$

4. (a)