CHAPTER 6

Section 6.4

1. (a)

$$\lim_{n\to\infty}\frac{n^2+1}{n^3+1}=\lim_{n\to\infty}\frac{(1/n^2)+1/n^3}{1+1/n^3}=\frac{0}{1}=0$$

(b)

$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{d(\ln n)/dn}{d(n)/dn}=\lim_{n\to\infty}\frac{1/n}{1}=\frac{0}{1}=0$$

(c)

$$\lim_{n\to\infty}\frac{n}{2^n}=\lim_{n\to\infty}\frac{1}{2^n\ln 2}=\frac{1}{\infty}=0$$

(d)

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+1/n)}{n^{-1}}&=\lim_{n\to\infty}\frac{d\left[\ln(1+1/n)\right]/dn}{d\left(n^{-1}\right)/dn}\\&=\lim_{n\to\infty}\frac{1/\left(n^2+n\right)}{1/n^2}\\&=\lim_{n\to\infty}\frac{1}{1+1/n}\\&=\frac{1}{1+0}=1\end{aligned}$$

(e)

$$\lim_{n\to\infty}s_n=1\text{ for }n=1,2,3,\cdots\implies\lim_{n\to\infty}s_n=1$$

2. (a)

$$\overline{\lim}_{n\to\infty}\cos n\pi=1\qquad\qquad \underline{\lim}_{n\to\infty}\cos n\pi=-1\tag{1}$$

(b)

$$\overline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx 0.951\qquad\qquad \underline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx -0.951$$

(c)

$$\overline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=\infty\qquad\qquad \underline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=-\infty$$

3. (a) A sequence

$$s_n=1+\cos n\pi$$

has limits

$$\overline{\lim}_{n\to\infty}s_n=2\qquad\qquad \underline{\lim}_{n\to\infty}s_n=0$$

(b) A sequence

$$s_n = -n^2 \sin^2\left(\frac{1}{2}n\pi\right)$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = 0 \qquad\qquad \underline{\lim}_{n\to\infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left|\left(1 + \frac{1}{n}\right)^n - e\right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

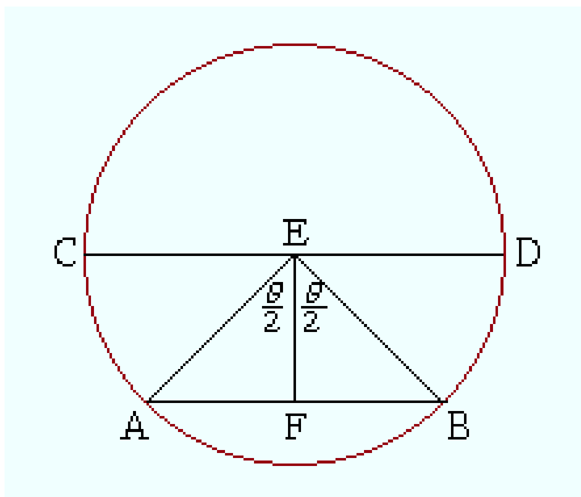
$$e \approx \left(1 + \frac{1}{164}\right)^{164} \approx 2.71$$

6.

$$\overline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } |x| > 1$$
$$\overline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = \pm 1$$
$$\overline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$

$$\underline{\lim}_{n\to\infty} x^n = -\infty \qquad \text{for } x < -1$$
$$\underline{\lim}_{n\to\infty} x^n = -1 \qquad \text{for } x = -1$$
$$\underline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$
$$\underline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = 1$$
$$\underline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } x > 1$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2}AB \times EF = AF \times EF = AE \sin\frac{\theta}{2} \times AE \cos\frac{\theta}{2} = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \to \infty$:

$$A_{S_1} = s_n = \lim_{n\to\infty} \frac{n}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty} \pi\frac{n}{2\pi}\sin\frac{2\pi}{n}$$

Now using the fact that $\lim_{x\to 0}\sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n\to\infty} \pi\frac{n}{2\pi}\sin\frac{2\pi}{n} = \lim_{x\to 0}\pi\frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π, it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n\to\infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the nth term test $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$ diverges.

(b) Since

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^{n-1}}{3n} = \lim_{n\to\infty} \frac{(n-1)\,2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the nth term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3-1} = \left|\frac{1}{n^3-1}\right| < \frac{1}{n-1}$$

for $n = 2, 3, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \frac{1/n}{1-(1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3-1)$ is absolutely convergent.

(b) Since $|\sin n| < 1$ for $n \geq 1$ it follows that

$$\left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$$

for $n = 1, 2, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a) Since $n+5 > n$ and $n^2 - 3n - 5 < n^2$ for $n \geq 1$ it follows that

$$\frac{n+5}{n^2-3n-5} > \frac{n}{n^2} = \frac{1}{n}$$

for $n = 1, 2, \ldots$. Now since $\sum_{n=1}^{\infty} 1/n$ is the *harmonic series*, which diverges, it follows by the comparison test for divergence that $\sum_{n=1}^{\infty}(n+5)/(n^2-3n-5)$ diverges as well.

(b) Since $\sqrt{n}\ln n < n\ln n$ for $n \geq 2$ it follows that

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n\ln n}$$

for $n = 2, 3, \ldots$. Using the inequality $\ln(1+x) \leq x$ we may continue to write

$$\frac{1}{n\ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln\left(1 + \frac{\ln(1+1/n)}{\ln n}\right) \geq \ln\frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n}\ln n} > \ln\frac{\ln(1+n)}{\ln n} = \ln\ln(1+n) - \ln\ln n$$

Now let us consider the series

$$\sum_{n=2}^{N} \ln\ln(1+n) - \ln\ln n = \ln\ln(1+N) - \ln\ln 2$$

Hence, when $N \to \infty$

$$\sum_{n=2}^{\infty} \ln\ln(1+n) - \ln\ln n = \lim_{N\to\infty} \ln\ln(1+N) - \ln\ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that $\sum_{n=2}^{\infty} 1/(\sqrt{n}\ln n)$ diverges as well.

4. (a) Let $y = f(x) = 1/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 1$ then evaluates to

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_{\pi/4}^{\tan^{-1}b} du = \lim_{b\to\infty} u\Big|_{\pi/4}^{\tan^{-1}b} &= \lim_{b\to\infty}\tan^{-1}b - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

where we have used the substitution $x = \tan u$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2+1)$.

(b) let $y = f(x) = 1/(x\ln^2 x)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 2$ then evaluates to

$$\begin{aligned}\int_2^\infty \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln b} &= \frac{1}{\ln 2} - \lim_{b\to\infty}\frac{1}{\ln b} \\ &= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}\end{aligned}$$

where we have used the substitution $u = \ln x$. Hence, by the integral test, since the improper integral $\int_2^\infty f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=2}^\infty 1/(n \ln^2 n)$.

5. (a) Let $y = f(x) = x/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, (fx) decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 1$ then evaluates to

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\frac{1}{2}\int_2^{b^2+1}\frac{du}{u} &= \lim_{b\to\infty}\left.\frac{\ln u}{2}\right|^{b^2+1} \\ &= \lim_{b\to\infty}\frac{\ln|b^2+1| - \ln 2}{2} \\ &= \infty - \frac{\ln 2}{2} = \infty\end{aligned}$$

where we have used the substitution $u = x^2+1$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=1}^\infty n/(n^2+1)$.

(b) Let $y = f(x) = 1/(x \ln x \ln\ln x)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 10$ then evaluates to

$$\begin{aligned}\int_{10}^\infty \frac{dx}{x\ln x\ln\ln x} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{x\ln x\ln\ln x} &= \lim_{b\to\infty}\int_{\ln 10}^{\ln b}\frac{du}{u\ln u} \\ &= \lim_{b\to\infty}\int_{\ln\ln 10}^{\ln\ln b}\frac{dv}{v} \\ &= \lim_{b\to\infty}\ln v\Big|_{\ln\ln 10}^{\ln\ln b} \\ &= \lim_{b\to\infty}\ln\ln\ln b - \ln\ln\ln 10 \\ &= \infty - \ln\ln\ln 10 = \infty\end{aligned}$$

where we have used the substitutions $u = \ln x$ and $v = \ln u$. Hence, by the integral test, since the improper integral $\int_{10}^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=10}^\infty 1/(n\ln n\ln\ln n)$.

6. (a) Let $a_n = (-1)^n/n!$. As such we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty}\left|\frac{n!}{(-1)^n}\frac{(-1)^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|-\frac{1}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^\infty (-1)^n/n!$ is absolutely convergent.

(b) Let $a_n = 2^n + 1/(3^n + n)$. As such we find

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L &= \lim_{n\to\infty}\left|\frac{3^n+n}{2^n+1}\frac{2^{n+1}+1}{3^{n+1}+n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{3^n\left(1+n/3^n\right)}{2^n\left(1+1/2^n\right)}\frac{2^{n+1}\left(1+1/2^{n+1}\right)}{3^{n+1}\left[1+\left(n/3^{n+1}\right)+\left(1/3^{n+1}\right)\right]}\right| \\
&= \frac{2}{3}\lim_{n\to\infty}\left|\frac{\left(1+n/3^n\right)\left(1+1/2^{n+1}\right)}{\left(1+1/2^n\right)\left(1+n/3^{n+1}+1/3^{n+1}\right)}\right| \\
&= \frac{2}{3}\frac{(1+0)\cdot(1+0)}{(1+0)\cdot(1+0+0)} = \frac{2}{3}
\end{aligned}$$

where we have used the fact that

$$\lim_{x\to\infty}\frac{x}{a^x} = \lim_{x\to\infty}\frac{1}{xa^{x-1}} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^{\infty} 2^n + 1/(3^n + n)$ is absolutely convergent.

7. (a) Let $a_n = 1/\ln n$. Then for $2 \le n < \infty$ we find

$$a_n = \frac{(-1)^n}{\ln n} = \frac{1}{\ln n}$$

Now since $\ln n$ is monotonically increasing for $2 \le n < \infty$ we may conclude that $a_n = 1/\ln n$ is monotonically decreasing for $2 \le n < \infty$ and so $a_{n+1} \le a_n$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\ln n} = \frac{1}{\infty} = 0$$

provided $n \ge 2$ and so by the alternating series test we may conclude that the series $\sum_{n=2}^{\infty}(-1)^n/\ln n$ converges.

(b) Let $f(x) = \ln x/x$. Hence,

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2}$$

Note that the derivative of $f(x)$ becomes negative when $x > e \approxeq 2.71828$ and hence, that $f(x)$ becomes monotonically decreasing when $e < x < \infty$. As such, the terms of the sequence $a_n = f(n) = \ln n/n$ are decreasing (i.e. $a_{n+1} \le a_n$) when $3 \le n < \infty$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. As such, by the alternating series test we may conclude that the series $\sum_{n=3}^{\infty}(-1)^n \ln n/n$ converges.

8. (a) Let $a_n = 1/n^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{n^{-n}} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} 1/n^n$ is absolutely convergent.

(b) Let $a_n = [n/(n+1)]^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

provided $n \geq 1$. Hence, since $R < 1$ it follows from the root test that the series $\sum_{n=1}^{\infty} [n/(n+1)]^{n^2}$ is absolutely convergent.

9. (a) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{2}{3} - \frac{1}{2} + \frac{3}{4} - \frac{2}{3} + \cdots + \frac{n+1}{n+2} - \frac{n}{n+1} = -\frac{1}{2} + \frac{n+1}{n+2}$$

Taking the limit of S_n as $n \to \infty$ then gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{n+2} - \frac{1}{2} = \lim_{n\to\infty} \frac{1+1/n}{1+2/n} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence, the series converges.

(b) Let the series

$$\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{n+1}{2^{n+1}} - \frac{n}{2^n}\right)$$

be given. To show that this series converges we consider the partial sum

$$S_n = \frac{1}{2} - \frac{1}{2} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} - \frac{3}{8} + \cdots + \frac{n+1}{2^{n+1}} = -\frac{1}{2} + \frac{n+1}{2^{n+1}}$$

Taking the limit of S_n as $n \to \infty$ then gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{(n+1)\,2^n} - \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$$

using *L'Hospital's rule*. Hence, the series converges.

10. Let $y = f(x)$ satisfy the following conditions:

(a) $f(x)$ is defined and continuous for $c \leq x < \infty$

(b) $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$

(c) $f(n) = a_n$

Let us suppose the improper integral $\int_c^\infty f(x)\,dx$ diverges. Assumptions (b) and (c) imply that $a_n > 0$ for n sufficiently large. Hence, by Theorem 7 of Section 6.5 the series $\sum a_n$ is either convergent or properly divergent. Let the integer m be chosen so that $m > c$. Then, since $f(x)$ is decreasing

$$\int_n^{n+1} f\left(x\right)\,dx \leq f\left(n\right) = a_n \quad \text{for } n \geq m$$

Hence, $a_m + \cdots + a_{m+p} \geq \int_m^{m+p+1} f\left(x\right)\,dx$. However, since $\int_c^\infty f(x)\,dx$ diverges it follows that $\lim_{p\to\infty} \int_m^{m+p+1} f(x)\,dx$ diverges, which thus ultimately implies that the series $\sum_m^\infty a_n$ must be divergent as well.

11. Let an alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} \left(-1\right)^{n+1} a_n, \quad a_n > 0$$

be given along with the two conditions

(a) $a_{n+1} \leq a_n$ for $n = 1, 2, \ldots$

(b) $\lim_{n\to\infty} a_n = 0$

What remains to be proven is that such a series converges given the aforementioned conditions. Let $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$ denote the nth partial sum of an alternating series. Then $S_1 = a_1$, $S_2 = a_1 - a_2 < S_1$, $S_3 = S_2 + a_3 > S_2$, $S_3 = S_1 - (a_2 - a_3) < S_1$, so that $S_2 < S_3 < S_1$. As such, we may conclude that $S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$ or $S_n \leq a_1$ and that each $S_n \geq 0$ for $n = 1, 2, \ldots$.

Next, let an $\epsilon > 0$ be given. By the Cauchy criterion our goal is to find an N so that whenever $m > n > N$ then

$$\left|S_m - S_n\right| = |a_{n+1} - a_{n+2} + \cdots \pm a_m| < \epsilon$$

Now since each partial sum is non-negative (i.e. $S_n \geq 0$) and acknowledging that all partial sums are $\leq$ the first term a_1, but now applied to the alternating series starting at a_{n+1} instead of a_1 we can write

$$\left|S_m - S_n\right| \leq a_{n+1} < \epsilon$$

Now because $\lim_{n\to\infty} a_n = 0$ we can find N such that $a_{n+1} < \epsilon$ whenever $n > N$. Hence,

$$m > n > N \implies \left|S_m - S_n\right| \leq a_{n+1} < \epsilon$$

which thus satisfies our initial condition

$$\left|S_m - S_n\right| = |a_{n+1} - a_{n+2} + \cdots \pm a_m| < \epsilon$$

We may conclude that the sequence of partial sums S_n of our original alternating series subject to conditions (a) and (b) satisfies the Cauchy criterion and therefore, is convergent. Hence, the alternating series itself is convergent.

12. (a) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+4}{2n^3-1}$$

be given. In order to determine convergence or divergence we first try the comparison test for convergence. To this end, note that $n+4 \leq 5n$ and $2n^3 - 1 \geq n^3$ for $n = 1, 2, \ldots$. Hence,

$$|a_n| = \frac{n+4}{2n^3-1} \leq \frac{5n}{n^3} = \frac{5}{n^2} = b_n \quad \text{for } n = 1, 2, \ldots$$

As such, if we can prove that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Now let $y = f(x) = 5/x^2$, which satisfies the following conditions:

i. $f(x)$ is defined and continuous for $c \leq x < \infty$ for $c \neq 0$
ii. $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$
iii. $f(n) = b_n$

Then by the integral test the series $\sum_{n=1}^{\infty} b_n$ converges or diverges according to whether the improper integral $\int_c^{\infty} f(x)\,dx$ converges or diverges. As such, we evaluate

$$\int_1^{\infty} \frac{5}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{5}{x^2}\,dx = \lim_{b\to\infty} -\frac{5}{x}\bigg|_1^b = 5 - \lim_{b\to\infty} \frac{5}{b} = 5 - \frac{5}{\infty} = 5$$

Hence, since the improper integral $\int_c^{\infty} f(x)\,dx$ converges, so do the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$.

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n-5}{n2^n}$$

be given. Since $a_n \neq 0$ for $n = 1, 2, \ldots$ we can try the ratio test in order to determine convergence or divergence:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty} \left|\frac{3(n+1)-5}{(n+1)2^{n+1}} \frac{n2^n}{3n-5}\right| &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{3(n+1)-5}{n+1} \frac{n}{3n-5}\right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left|\frac{3(1+1/n)-5/n}{1+1/n} \frac{1}{3-5/n}\right| \\
&= \frac{1}{2} \frac{3+0-0}{1+0} \frac{1}{3-0} = \frac{1}{2}
\end{aligned}$$

Hence, since $L = 1/2 < 1$ the series $\sum_{n+1}^{\infty} a_n$ is absolutely convergent.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n}{n+1}$$

be given. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n+1} = \lim_{n\to\infty} e^n = \infty$$

using *L'Hospital's rule*. Hence, it follows from the nth term test that the series $\sum_{n=1}^{\infty} a_n$ diverges.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n!+1}$$

be given. Since

$$|a_n| = \frac{n^2}{n!+1} < \frac{n^2}{n!} = b_n \quad \text{for } n = 1, 2, \ldots$$

the comparison test for convergence tells us that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Since $b_n \neq 0$ for $n = 1, 2, \ldots$ we can use the ratio test in order to determine if $\sum_{n=1}^{\infty} b_n$ converges:

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2(n+1)} = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{1+1/n}{n}$$
$$= \frac{1+0}{\infty} = 0$$

Hence, since $L = 0 < 1$ we may conclude that $\sum_{n=1}^{\infty} b_n$ is absolutely convergent by the ratio test and thus, that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent by the comparison test for convergence.

(e) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{3 \cdot 5 \cdots (2n+3)}$$

be given. Since $a_n \neq 0$ for $n = 1, 2, \ldots$ we can use the ratio test to determine convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L &= \lim_{n\to\infty} \frac{(n+1)!}{3 \cdot 5 \cdots [2(n+1)+3]} \frac{3 \cdot 5 \cdots (2n+3)}{n!} \\
&= \lim_{n\to\infty} \frac{n}{2(n+1)+3} \\
&= \lim_{n\to\infty} \frac{1}{2(1+1/n)+3/n} = \frac{1}{2(1+0)+0} = \frac{1}{2}
\end{aligned}$$

Hence, since $L = 1/2 < 1$ we may conclude that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(f) Let the series
$$\sum_{n=1}^{\infty}(-1)^n a_n=\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{2n+3}$$
be given. This is an alternating series. Note that for $n=1,2,3,4$ its terms are actually increasing (i.e. $a_{n+1}>a_n$) in absolute value and $a_{n+1}\leq a_n$ only becomes true when $n=5,6,\ldots$. This is not a problem for the alternating series test to be valid however. Furthermore
$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\ln n}{2n+3}=\lim_{n\to\infty}\frac{1/n}{2}=\frac{0}{2}=0$$
using *L'Hospital's rule*. Hence, the alternating series converges.

(g) Let the series
$$\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{1+\ln^2 n}{n\ln^2 n}$$
be given. As such, let us define the function $y=f(x)=(1+\ln^2 x)/n\ln^2 x$. Then
$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1+\ln^2 x}{x\ln^2 x}=\lim_{x\to\infty}\left(\frac{1}{x\ln^2 x}+\frac{1}{x}\right)=\frac{1}{\infty}+\frac{1}{\infty}=0$$
Furthermore, $f(x)$ satisfies the following conditions:

i. $f(x)$ is defined and continuous for $c\leq x<\infty$
ii. $f(x)$ decreases as x increases for $x\geq 2$ and $\lim_{x\to\infty}f(x)=0$
iii. $f(n)=a_n$

Hence, we can use the integral test to determine whether the series $\sum_{n=2}^{\infty}a_n$ converges or diverges:
$$\begin{aligned}\int_2^{\infty}\frac{1+\ln^2 x}{x\ln^2 x}\,dx=\lim_{b\to\infty}\int_2^{b}\frac{1+\ln^2 x}{x\ln^2 x}\,dx&=\lim_{b\to\infty}\int_2^{b}\left(\frac{1}{x\ln^2 x}+\frac{1}{x}\right)dx\\&=\lim_{b\to\infty}\int_2^{b}\frac{dx}{x\ln^2 x}+\lim_{b\to\infty}\int_2^{b}\frac{dx}{x}\\&=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u^2}+\lim_{b\to\infty}\ln|x|\Big|_2^b\\&=\lim_{b\to\infty}-\frac{1}{u}\Big|_{\ln 2}^{\ln b}+\lim_{b\to\infty}(\ln b-\ln 2)\\&=\lim_{b\to\infty}\left(-\frac{1}{\ln b}+\frac{1}{\ln 2}+\ln b-\ln 2\right)=\infty\end{aligned}$$
In conclusion, since the improper integral $\int_c^{\infty}f(x)\,dx$ diverges, so will the series $\sum_{n=2}^{\infty}a_n$.

(h) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n+2} \equiv \sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$$

be given. $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series with terms that are decreasing in absolute value: $b_{n+1} < b_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} b_n = 0$. Hence, by the alternating series test the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges and thus, so will the series $\sum_{n=1}^{\infty} a_n$.

(i) Let the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + \ln n}$$

be given. Now since $a \geq 0$ and $n + \ln n < 2n$ for $n = 1, 2, \dots$ we can define $b_n = \ln n/2n$ such that $a_n > b_n \geq 0$. Then by the comparison test for divergence if $\sum_{n=1}^{\infty} b_n$ diverges so will $\sum_{n=1}^{\infty} a_n$. To this end, let us define the function $y = f(x) = \ln x/2x$. Now since $\ln x < 2x$ for $1 \leq x < \infty$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{2x} = \lim_{x\to\infty} \frac{1/x}{2} = 0$$

using *L'Hospital's rule*, we find that

i. $f(x)$ is defined and continuous for $c \leq x < \infty$, where $c = 1$
ii. $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$
iii. $f(n) = a_n$

Then the series $\sum_{n=1}^{\infty} b_n$ converges or diverges according to whether the improper integral $\int_c^{\infty} f(x)\, dx$ converges or diverges:

$$\int_1^{\infty} \frac{\ln x}{2x}\, dx = \frac{1}{2} \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x}\, dx = \frac{1}{2} \lim_{b\to\infty} \int_0^{\ln b} u\, du = \lim_{b\to\infty} \left. \frac{u^2}{4} \right|_0^{\ln b} = \lim_{b\to\infty} \frac{\ln^2 b}{4} = \infty$$

where we have used the substitution $u = \ln x$. Hence, by the integral test the series $\sum_{n=1}^{\infty} b_n$ diverges and so by the comparison test for divergence the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

(j) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{n+1}{2n}\right)^n$$

be given. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{2n}\right)^n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}$$

Then by the root test, since $R = 1/2 < 1$ the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

13. Let $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \dots$ and let the sequence a_n/b_n have limit k, possibly infinite.

   (a) Suppose $0 < k < \infty$, i.e. $\lim_{n\to\infty} a_n/b_n = k$ is some positive number. Then for some $\epsilon > 0$ we know that there must exist a positive integer N such that for all $n > N$ it is true that

   $$\left|\frac{a_n}{b_n} - k\right| < \epsilon \iff (k-\epsilon)\, b_n < a_n < (k+\epsilon)\, b_n$$

   As $k > 0$ we can choose ϵ sufficiently small so that $k - \epsilon > 0$. Hence,

   $$b_n < \frac{a_n}{k-\epsilon}$$

   As such, by the comparison test for convergence, if $\sum a_n$ converges then so must $\sum b_n$. Similarly $a_n < (k+\epsilon)b_n$. Hence, if $\sum a_n$ diverges then by the comparison test for divergence so will $\sum b_n$. In conclusion, both series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

   (b) Suppose $k = 0$. Then for some $\epsilon > 0$ there must exist a positive integer N such that for all $n > N$ it is true that

   $$\frac{a_n}{b_n} < \epsilon \iff a_n < \epsilon b_n$$

   Hence, by the comparison test for convergence, if $\sum b_n$ converges then so must $\sum a_n$. Additionally, as long as $\sum a_n$ converges the inequality can still be satisfied if $\sum b_n$ diverges by choosing ϵ sufficiently small.

   (c) Suppose $k = \infty$. Then for some $\epsilon > 0$ we know that there must exist a positive integer N such that for all $n > N$ it is true that

   $$(k-\epsilon)\, b_n < a_n < (k+\epsilon)\, b_n$$

   From the first inequality we see that

   $$a_n > (k-\epsilon)\, b_n$$

   from which we may gather that $\sum a_n$ may diverge while $\sum b_n$ converges, since $k = \infty$. Similarly, since $a_n < (k+\epsilon)b_n$ then the comparison test for divergence tells us that divergence of $\sum a_n$ implies divergence of $\sum b_n$.

14. (a) Let the series

   $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{3n^2+n+1}$$

   be given and let $b_n = 1/n$. Using Problem 13 we thus find

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{3n^2+n+1} = \lim_{n\to\infty} \frac{2+1/n}{3+1/n+1/n^2} = \frac{2}{3}$$

and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ diverges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must diverge as well.

(b) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^3 - 3n^2 + 5}{n^5 + n + 1}$$

be given and let $b_n = 1/n^2$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^5 - 3n^4 + 5n^2}{n^5 + n + 1} = \lim_{n\to\infty} \frac{1 - 3/n + 5/n^3}{1 + 1/n^4 + 1/n^5} = 1$$

and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must converge as well.

(c) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin\frac{1}{n}$$

be given and let $b_n = 1/n$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(1/n)/n^2}{1/n^2} = \lim_{n\to\infty} \cos\frac{1}{n} = \cos\frac{1}{\infty} = 1$$

using *L'Hospital's rule* and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ diverges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must diverge as well.

(d) Let the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right)$$

be given and let $b_n = 1/n^2$. Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 - \cos(1/n)}{1/n^2} = \frac{1}{2}\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \frac{1}{2}$$

using *L'Hospital's rule* and so the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges we conclude that the series $\sum_{n=1}^{\infty} a_n$ must converge as well.

Section 6.9

1. (a) Let the sum $\sum_{n=1}^{\infty} 1/n^2$ be given and let us define the allowed error as $\epsilon = 1$. We know from the previous section that this series converges by the integral test of

Theorem 14. Hence, by Theorem 23 we find

$$|R_n| = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

and so the condition $T_n \leq \epsilon$ then translates to the inequality $n \geq 1$, which is satisfied for $n = 1$. Hence, one term is sufficient to compute the sum with given allowed error $\epsilon = 1$ and so $S_1 = 1$.

(b) Let the sum $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ be given and let us define the allowed error as $\epsilon = 1/10$. Now since this series converges by the alternating series test then by Theorem 26

$$|R_n| < a_{n+1} = T_n$$

Hence, we end up with the inequality $a_{n+1} \leq \epsilon$ or $1/(n+1)^2 \leq 1/10 \iff (n+1)^2 \geq 10$, which is satisfied for $n = 3$. Hence, three terms is sufficient to compute the sum with the given allowed error $\epsilon = 1/10$ and so $S_3 \approxeq 0.86$.

(c) Let the sum $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n/(n^3+5)$ be given and let us define the allowed error as $\epsilon = 1/5$. It is true that $n^3 + 5 > n^3$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 5$. Hence, five terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/5$ and so $S_5 \approxeq 0.51$.

(d) Let the sum $\sum n = 1^{\infty} 1/(n^2+1)$ be given and let us define the allowed error as $\epsilon = 1/2$. It is true that $n^2 + 1 > n^2$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m =\leq \sum_{m=n+1}^{\infty} b_m \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 2$. Hence, two terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_2 = 0.7$.

(e) Let the sum $\sum_{n=1}^{\infty} 1/n^n$ be given and let us define the allowed error as $\epsilon = 1/100$. Then

$$\sqrt[n]{|a_n|} = \frac{1}{n} \leq r < 1$$

for $n \geq 2$, so that the series $\sum a_n$ converges by the root test. Hence, by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = T_n \implies \frac{1}{(n+1)^{n+1}} \cdot \frac{1}{1 - \dfrac{1}{n+1}} = \frac{1}{n\,(n+1)^n} \leq \epsilon$$

for $n \geq 2$. In other words, we are looking for the smallest integer $n \geq 2$ such that $n(n+1)^n \geq 100$, which is satisfied for $n = 3$. Hence, three terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/100$ and so $S_3 \approxeq 1.287$.

(f) Let the sum $\sum_{n+1}^{\infty} 1/n!$ be given and let us define the allowed error as $\epsilon = 1/100$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq r < 1$$

for $n \geq 1$, so that the series $\sum a_n$ converges by the ratio test. Hence, by Theorem 24

$$|R_n| \leq \frac{|a_{n+1}|}{1-r} = T_n \implies \frac{1}{(n+1)!} \cdot \frac{1}{1 - \dfrac{1}{n+2}} = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1}\right) \leq \epsilon$$

for $n \geq 1$. In other words, we are looking for the smallest integer $n \geq 1$ such that $T_n \leq \epsilon$, which is satisfied for $n = 4$. Hence, four terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/100$ and so $S_4 \approxeq 1.708$.

(g) Let the sum $\sum_{n+1}^{\infty}(-1)^{n+1}/(2n-1)!$ be given and let us define the allowed error as $\epsilon = 1/1000$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n-1)!}{(2n+1)!} < 1 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$$

the series $\sum a_n$ converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(2n+1)!} = T_n \implies \frac{1}{(2n+1)!} \leq \epsilon$$

and so we are looking for the smallest integer such that $(2n+1)! \geq 1000$, which is satisfied for $n = 3$. Hence, three terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/1000$ and so $S_3 \approxeq 0.8417$.

(h) Let the sum $\sum_{n+2}^{\infty}(-1)^n/(n \ln n)$ be given and let us define the allowed error as $\epsilon = 1/2$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n \ln n}{(n+1)\ln(n+1)} < 1 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

the series $\sum a_n$ converges by the alternating series test. Hence, by Theorem 26

$$|R_n| < a_{n+1} = \frac{1}{(n+1)\ln(n+1)} = T_n \implies \frac{1}{(n+1)\ln(n+1)} \leq \epsilon$$

and so we are looking for the smallest integer such that $(n+1)\ln(n+1) \geq 2$, which is satisfied for $n = 2$. Hence, one term is sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_1 \approxeq 0.72$.

(i) Let the sum $\sum_{n=2}^{\infty} 1/(n^3 \ln n)$ be given and let us define the allowed error as $\epsilon = 1/2$. It is true that $n^3 \ln n > n^2$ for $n \geq 2$ and so we can define $b_n = 1/n^2$ such that $|a_n| < b_n$ for $n \geq n_1 = 2$. Now since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} 1/n^2$ converges then by Theorem 22 and Theorem 23 it follows that

$$|R_n| \leq \sum_{m=n+1}^{\infty} b_m = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \int_n^{\infty} f(x)\, dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n} = T_n$$

Taking a hint from (a) we find $T_n \leq \epsilon \implies n \geq 2$. Hence, one term is sufficient to compute the sum with the given allowed error $\epsilon = 1/2$ and so $S_1 \approxeq 0.18$.

(j) Let the sum $\sum_{n=1}^{\infty} 2^n/(3^n + 1)$ be given and let us define the allowed error as $\epsilon = 1/10$. It is true that $3^n + 1 > 3^n$ for $n \geq 1$ and so we can define $b_n = 2^n/3^n$ such that $|a_n| < b_n$ for $n \geq n_1 = 1$. Now since $\sqrt[n]{|b_n|} = \sqrt[n]{2^n/3^n} = 2/3 \leq r < 1$ for $n \geq 1$ we may conclude that the series $\sum b_n$ converges by the root test. Hence, choosing $r = 2/3$ then by Theorem 25

$$|R_n| \leq \frac{r^{n+1}}{1-r} = \frac{2^{n+1}}{3^n} = T_n \implies \frac{2^{n+1}}{3^n} \leq \epsilon$$

and so we are looking for the smallest integer such that $3^n/2^{n+1} \geq 10$, which is satisfied for $n = 8$. Hence, eight terms are sufficient to compute the sum with the given allowed error $\epsilon = 1/10$ and so $S_8 \approxeq 1.697$.

2. Let $\sum a_n$ be the geometric series $1 + r + r^2 + \cdots = \sum_{n=0}^{\infty} r^n$. By Theorem 16 this series converges for $-1 < r < 1$. Hence, by Theorem 23

$$|R_n| = \sum_{m=n+1}^{\infty} r^m < \int_n^{\infty} r^x\, dx = T_n$$

Or

$$\begin{aligned} T_n = \int_n^{\infty} r^x\, dx = \lim_{b\to\infty} \int_n^b r^x\, dx = \lim_{b\to\infty} \int_n^b e^{x \ln r}\, dx &= \lim_{b\to\infty} \int_{n \ln r}^{b \ln r} \frac{e^u}{\ln r}\, du \\ &= \lim_{b\to\infty} \left. \frac{e^u}{\ln r} \right|_{n \ln r}^{b \ln r} \\ &= \lim_{b\to\infty} \frac{e^{b \ln r}}{\ln r} - \frac{e^{n \ln r}}{\ln r} \\ &= -\frac{e^{n \ln r}}{\ln r} = -\frac{r^n}{\ln r} \end{aligned}$$

assuming $0 < r < 1$.

(a) let the given allowed error $\epsilon = 1/100$. In order to determine how many terms are needed to compute the sum with error less than ϵ we require $T_n < \epsilon$. For $r = 1/2$ this results in

$$-\frac{1}{2^n \ln 2^{-1}} < \frac{1}{100} \iff n > \frac{\ln(100/\ln 2)}{\ln 2}$$

which is satisfied for $n = 8$. Hence, when $r = 1/2$, 8 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$. For $r = 0.9 = 9/10$ we get

$$-\frac{1}{\ln(9/10)}\left(\frac{9}{10}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 9/10)}{\ln 10/9}$$

which is satisfied for $n = 66$. Hence, when $r = 0.9$, 66 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$. For $r = 0.99 = 99/100$ we get

$$-\frac{1}{\ln(99/100)}\left(\frac{99}{100}\right)^n < \frac{1}{100} \iff n > \frac{\ln 100 - \ln(-\ln 99/100)}{\ln 100/99}$$

which is satisfied for $n = 916$. Hence, when $r = 0.99$, 916 terms are sufficient to compute the sum with error less than $\epsilon = 1/100$.

(b) The closed form formula (6.17) for a geometric series $1 + ar + ar^2 + \cdots$, with $a = 1$ and $-1 < r < 1$ is given by $S = 1/(1-r)$. Likewise, the closed form formula for the partial sum of the same geometric series is given by $S_n = (1-r^n)/(1-r)$. The remainder R_n after n terms thus can be defined as

$$|R_n| = |S_n - S| = \left|\frac{1-r^n}{1-r} - \frac{1}{1-r}\right| = \left|\frac{-r^n}{1-r}\right| < \epsilon \iff -\epsilon < -\frac{r^2}{1-r} < \epsilon$$

The inequality on the right hand side can be further manipulated to finally get

$$\begin{aligned}
-\frac{r^n}{1-r} &< \epsilon \\
r^n &> -\epsilon(1-r) \\
\ln|r|^n &> \ln|-\epsilon(1-r)| \\
n &> \frac{\ln \epsilon(1-r)}{\ln|r|}
\end{aligned}$$

where $-1 < r < 1$.

(c) When r approaches 1 from the left we note that

$$\lim_{r\to 1^-} \frac{\ln \epsilon(1-r)}{\ln|r|} = \lim_{r\to 1^-} \frac{\ln \epsilon(1-r)}{\ln r} = \lim_{r\to 1^-} \ln \epsilon(1-r) \cdot \lim_{r\to 1^-} \frac{1}{\ln r} = -\infty \cdot -\infty = \infty$$

Hence, it follows from (b) that $n \to \infty$ when $r \to 1^-$, or in other words; that the number of terms needed to compute the sum with error less than a fixed ϵ becomes infinite.

3. Let the series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $p > 0$ be given. As such, $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$ and so $S_1 = a_1 = 1$, $S_2 = a_1 - a_2 = 1 - 2^{-p}$ so that $0 < S_2 < S_1$, $S_3 = S_1 - (a_2 - a_3) = 1 - 2^{-p} + 3^{-p}$ so that $0 < S_3 < S_1$ and $S_2 < S_3 < S_1$. Reasoning in this way, we conclude that

$$S_1 > S_3 > S_5 > S_7 > \cdots > S_6 > S_4 > S_2$$

Hence, the smallest partial sum is S_2, but we just established that $S_2 = 1 - 2^{-p} > 0$. Hence, it follows that the sum $S = \lim_{n\to\infty} S_n$ must be positive whenever $p > 0$.

Section 6.10

1. Let the following relations be given:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

Then by (6.15)

(a)

$$\sum_{n=1}^{\infty} \frac{6}{n^2} = 6\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90} =$$

(c)

$$\sum_{n=1}^{\infty} \frac{2n^2-3}{n^4} = \sum_{n=1}^{\infty} \frac{2}{n^2} - \sum_{n=1}^{\infty} \frac{3}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 3\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{3} - \frac{\pi^4}{30}$$

(d)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{9+3n^2+5n^4}{n^6} &= \sum_{n=1}^{\infty} \frac{9}{n^6} + \sum_{n=1}^{\infty} \frac{3}{n^4} + \sum_{n=1}^{\infty} \frac{5}{n^2} = 9\sum_{n=1}^{\infty} \frac{1}{n^6} + 3\sum_{n=1}^{\infty} \frac{1}{n^4} + 5\sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{5\pi^2}{6} + \frac{\pi^4}{30} + \frac{\pi^6}{105}\end{aligned}$$

(e)

$$\sum_{n=3}^{\infty} \frac{n^4-1}{n^6} = \sum_{n=3}^{\infty} \frac{1}{n^2} - \sum_{n=3}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{5}{4} - \sum_{n=1}^{\infty} \frac{1}{n^6} + \frac{65}{64} = \frac{\pi^2}{6} - \frac{\pi^6}{945} - \frac{15}{64}$$

(f)

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n^2+1}{\left(n^2-1\right)^2} = \sum_{n=2}^{\infty}\left[\frac{1}{2\left(n+1\right)^2}+\frac{1}{2\left(n-1\right)^2}\right] &= \sum_{n=2}^{\infty}\frac{1}{2\left(n+1\right)^2}+\sum_{n=2}^{\infty}\frac{1}{2\left(n-1\right)^2} \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{\left(n+1\right)^2}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{\left(n-1\right)^2} \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2}-\frac{1}{8}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2}+\frac{1}{2} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{8}-\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{2}-\frac{1}{2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{5}{8}=\frac{\pi^2}{6}-\frac{5}{8}
\end{aligned}$$

2. (a)

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}+1-1=\sum_{n=2}^{\infty}\frac{1}{n^3}+1=\sum_{n=2}^{\infty}\frac{1}{\left(n-1\right)^3}$$

(b)

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[f\left(n+1\right)-f\left(n\right)\right] &= \sum_{n=1}^{\infty}f\left(n+1\right)-\sum_{n=1}^{\infty}f\left(n\right) \\
&= \sum_{n=1}^{\infty}f\left(n\right)+\lim_{n\to\infty}f\left(n\right)-f\left(1\right)-\sum_{n=1}^{\infty}f\left(n\right) \\
&= \lim_{n\to\infty}f\left(n\right)-f\left(1\right)
\end{aligned}$$

if the limit exists.

(c)

$$\begin{aligned}
\sum_{n=2}^{\infty}\left[f\left(n+1\right)-f\left(n-1\right)\right] &= \sum_{n=2}^{\infty}f\left(n+1\right)-\sum_{n=2}^{\infty}f\left(n-1\right) \\
&= \sum_{n=1}^{\infty}f\left(n+1\right)+\lim_{n\to\infty}f\left(n+1\right)-f\left(2\right)-\sum_{n=1}^{\infty}f\left(n\right) \\
&= \sum_{n=1}^{\infty}f\left(n\right)-f\left(1\right)+\lim_{n\to\infty}f\left(n\right)+\lim_{n\to\infty}f\left(n+1\right)-f\left(2\right) \\
&\quad -\sum_{n=1}^{\infty}f\left(n\right) \\
&= \lim_{n\to\infty}\left[f\left(n\right)+f\left(n+1\right)\right]-f\left(1\right)-f\left(2\right)
\end{aligned}$$

if the limit exists.

3. (a) Let $f(n) = 1/n^2$. Then using 2(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n\to\infty} f(n) - f(1) \\
\sum_{n=1}^{\infty} \left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] &= \lim_{n\to\infty} \frac{1}{n^2} - 1 \\
\sum_{n=1}^{\infty} -\frac{2n+1}{n^2(n+1)^2} &= 0 - 1 \\
\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} &= 1
\end{aligned}$$

(b) Let $f(n) = 1/n$. Then using 2(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} [f(n+1) - f(n)] &= \lim_{n\to\infty} f(n) - f(1) \\
\sum_{n=1}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n}\right] &= \lim_{n\to\infty} \frac{1}{n} - 1 \\
\sum_{n=1}^{\infty} -\frac{1}{n(n+1)} &= 0 - 1 \\
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= 1
\end{aligned}$$

(c) Let $f(n) = 1/n$. Then using 2(c)

$$\begin{aligned}
\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] &= \lim_{n\to\infty} [f(n) + f(n+1)] - f(1) - f(2) \\
\sum_{n=2}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n-1}\right] &= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1}\right) - 1 - \frac{1}{2} \\
\sum_{n=2}^{\infty} -\frac{2}{n^2-1} &= 0 + 0 - 1 - \frac{1}{2} \\
-2\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= -\frac{3}{2} \\
\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \frac{3}{4}
\end{aligned}$$

(d) Let $f(n) = 1/n^2$. Then using 2(c)

$$\sum_{n=2}^{\infty} [f(n+1) - f(n-1)] = \lim_{n\to\infty} [f(n) + f(n+1)] - f(1) - f(2)$$
$$\sum_{n=2}^{\infty} \left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right] = \lim_{n\to\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right] - 1 - \frac{1}{4}$$
$$\sum_{n=2}^{\infty} -\frac{4n}{(n^2-1)^2} = 0 + 0 - 1 - \frac{1}{4}$$
$$\sum_{n=2}^{\infty} \frac{4n}{(n^2-1)^2} = \frac{5}{4}$$

4. Let the relation

$$\frac{1}{1-r} = 1 + r + \cdots + r^n + \cdots = \sum_{n=0}^{\infty} r^n \qquad -1 < r < 1$$

be given.

(a) Using the *Cauchy product* as illustrated in Fig. 6.6 we can thus write

$$\begin{aligned}
\frac{1}{(1-r)^2} &= \frac{1}{1-r} \cdot \frac{1}{1-r} \\
&= (1 + r + \cdots + r^n + \cdots) \cdot (1 + r + \cdots + r^n + \cdots) \\
&= 1 + (1 \cdot r + r \cdot 1) + \left(1 \cdot r^2 + r \cdot r + r^2 \cdot 1\right) + \cdots \\
&\quad + \left(1 \cdot r^n + r \cdot r^{n-1} + \cdots + r^n \cdot 1\right) + \cdots \\
&= 1 + 2r + 3r^2 + \cdots + (n+1) r^n + \cdots
\end{aligned}$$

(b) Firstly, we will derive the formula for a sum of an arithmetic sequence $a_m = a_1 + (m-1)d$, where d denotes the common difference between successive terms. We will start by expressing the arithmetic series in two different ways:

$$S_m = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (m-2)\,d] + [a_1 + (m-1)\,d]$$
$$S_m = [a_m - (m-1)\,d] + [a_m - (m-2)\,d] + \cdots + (a_m - 2d) + (a_m - d) + a_m$$

Adding both equations, we find that all terms involving d cancel and so we are left with

$$2S_m = m\,(a_1 + a_m) \iff S_m = \frac{m\,(a_1 + a_m)}{2}$$

Now, again using the *Cauchy product*

$$
\begin{aligned}
\frac{1}{(1-r)^3} &= \frac{1}{(1-r)^2}\cdot\frac{1}{1-r} \\
&= \left[1+2r+3r^2+\cdots+(n+1)\,r^n+\cdots\right]\cdot\left(1+r+\cdots+r^n+\cdots\right) \\
&= 1+(1\cdot r+2r\cdot 1)+\cdots+\left[1\cdot r^n+2r\cdot r^{n-1}+\cdots+(n+1)\cdot r^n\right]+\cdots \\
&= 1+3r+\cdots+\left[1+2+\cdots+(n+1)\right]r^n+\cdots \\
&= 1+3r+\cdots+\frac{(n+2)\,(n+1)}{2}r^n+\cdots
\end{aligned}
$$

where we have used the fact that the arithmetic sequence $1+2+\cdots+(n+1)$ can be written as $(n+2)(n+1)/2$ using the derived formula above.

5. We want to prove that

$$
(1-r)^{-k} = 1+kr+\frac{k\,(k+1)}{1\cdot 2}r^2+\cdots+\frac{k\,(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}r^n+\cdots
$$

for $-1<r<1$, $k=1,2,\ldots$. Using the solutions to 4(a) and 4(b) we can confirm the above equation is true for $k=1,2,3$. It remains to be proven that the equation is true for $k=1,2,\ldots$. In order to simplify the discussion we will write the coefficients appearing in the equation above as binomial coefficients and also make use of *Pascal's identity*:

$$
\binom{k}{n} = \frac{k!}{n!\,(k-n)!} \qquad\qquad \binom{k}{n} = \binom{k-1}{n-1}+\binom{k-1}{n}
$$

Next, we assume the equation is true for some positive integer $k\geq 2$ and consider the expansion

$$
\begin{aligned}
(1-r)^{-k-1} &= (1-r)^{-1}\left[1+kr+\frac{k\,(k+1)}{1\cdot 2}r^2+\cdots+\frac{k\,(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}r^n+\cdots\right] \\
&= (1-r)^{-1}\left[1+\binom{k}{1}r+\binom{k+1}{2}r^2+\cdots+\binom{k+n-1}{n}r^n+\cdots\right] \\
&= \left(1+r+r^2+\cdots+r^n\right)\left[1+\binom{k}{1}r+\binom{k+1}{2}r^2+\cdots+\binom{k+n-1}{n}r^n+\cdots\right] \\
&= 1+\left[1+\binom{k}{1}\right]r+\left[1+\binom{k}{1}+\binom{k+1}{2}\right]r^2+\cdots \\
&\quad+\left[1+\binom{k}{1}+\binom{k+1}{2}+\cdots+\binom{k+n-2}{n-1}+\binom{k+n-1}{n}\right]r^n+\cdots \\
&= 1+\binom{k+1}{1}r+\binom{k+2}{2}r^2+\cdots+\binom{k+n}{n}r^n+\cdots \\
&= 1+(k+1)\,r+\frac{(k+1)\,(k+2)}{1\cdot 2}r^2+\cdots+\frac{(k+1)\,(k+2)\cdots(k+n)}{1\cdot 2\cdots n}r^n+\cdots
\end{aligned}
$$

Hence, the equation is true for $k+1$ and so by induction the equation must be true for any positive integer $k \geq 1$.

6. Let $\sin x$ and $\cos x$ be represented for all x by the absolutely convergent series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n+1}\,\frac{x^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} a_n$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n}\,\frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} b_n$$

Then by (6.24) it follows that

$$\begin{aligned}
\sin x \cos x &= \sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n \\
&= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\,\frac{x^{2n+1}}{(2n+1)!} + \cdots\right] \\
&\quad \times \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\,\frac{x^{2n}}{(2n)!} + \cdots\right] \\
&= x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots + x^{2n+1} \sum_{k=0}^{m} \frac{(-1)^k\,(-1)^{m-k}}{(2k+1)!\,(2m-2k)!} + \cdots
\end{aligned}$$

and

$$\frac{1}{2}\sin 2x = x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \cdots + (-1)^n\,\frac{2^{2n}x^{2n+1}}{(2n+1)!} + \cdots$$

Hence, we need to prove that

$$\sum_{k=0}^{m} \frac{1}{(2m+1)!\,(2m-2k)!} = \frac{2^{2m}}{(2k+1)!} \iff \sum_{k=0}^{m} \frac{(2m+1)!}{(2k+1)!\,(2m-2k)!} = 2^{2m}$$

To this end (making use of *Pascal's identity*)

$$\begin{aligned}
\sum_{k=0}^{m} \frac{(2m+1)!}{(2k+1)!\,(2m-2k)!} &= \sum_{k=0}^{m} \binom{2m+1}{2k+1} \\
&= \sum_{k=0}^{m} \left[\binom{2m}{2k} + \binom{2m}{2k+1}\right] \\
&= \sum_{k=0}^{m} \left[\frac{(2m)!}{(2k)!\,(2m-2k)!} + \frac{(2m)!}{(2k+1)!\,(2m-2k-1)!}\right] \\
&= \sum_{k=0}^{2m} \frac{(2m)!}{k!\,(2m-k)!} \\
&= \sum_{k=0}^{2m} \binom{2m}{k}
\end{aligned}$$

Now from the definition of the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

it then finally follows that

$$\sum_{k=0}^{2m} \binom{2m}{k} = (1+1)^{2m} = 2^{2m}$$

which thus completes the proof.

7. Let the sequence a_n be close to the sequence b_n and let $\sum_{n=1}^{\infty} b_n$ be known. We can write $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty}(b_n - a_n)$.

   (a) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(2^n+1)^{-1}$ and let us choose $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2^{-n} = 1$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n)\\ \sum_{n=1}^{\infty} (2^n+1)^{-1} &= \sum_{n=1}^{\infty} 2^{-n} - \sum_{n=1}^{\infty} (4^n+2^n)^{-1}\\ &= 1 - \sum_{n=1}^{\infty} (4^n+2^n)^{-1}\end{aligned}$$

   As such, we find that for $n \geq 7$ the expression $\sum_{n=1}^{\infty} a_n = 1 - \sum_{n=1}^{\infty}(4^n+2^n)^{-1} \approxeq 0.7645$, whereas we require $n \geq 15$ for $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(2^n+1)^{-1} \approxeq 0.7645$.

   (b) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(2^n+9^{-1})^{-1}$ and let us choose $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2^{-n} = 1$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} (b_n - a_n)\\ \sum_{n=1}^{\infty} \left(2^n+9^{-1}\right)^{-1} &= \sum_{n=1}^{\infty} 2^{-n} - \sum_{n=1}^{\infty} (4^n 9+2^n)^{-1}\\ &= 1 - \sum_{n=1}^{\infty} (4^n 9+2^n)^{-1}\end{aligned}$$

   As such, we find that for $n \geq 6$ the expression $\sum_{n=1}^{\infty} a_n = 1 - \sum_{n=1}^{\infty}(4^n 9+2^n)^{-1} \approxeq 0.9646$, whereas we require $n \geq 14$ for $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(2^n+9^{-1})^{-1} \approxeq 0.9646$.

(c) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(n^2+1)^{-1}$ and let us choose $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty}(b_n - a_n)\\ \sum_{n=1}^{\infty}\left(n^2+1\right)^{-1} &= \sum_{n=1}^{\infty} n^{-2} - \sum_{n=1}^{\infty}\left(n^4+n^2\right)^{-1}\\ &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\left(n^4+n^2\right)^{-1}\end{aligned}$$

For $n \geq 16$ we then find $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - \sum_{n=1}^{\infty}(n^4+n^2)^{-1} \approxeq 1.0767$. Next, we use $b_n = n^{-4}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\left(n^4+n^2\right)^{-1}\\ &= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} n^{-4} + \sum_{n=1}^{\infty}\left(n^6+n^4\right)^{-1}\\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty}\left(n^6+n^4\right)^{-1}\end{aligned}$$

For $n \geq 6$ we then find $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - (\pi^4/90) + \sum_{n=1}^{\infty}(n^6+n^4)^{-1} \approxeq 1.0767$. Lastly, we use $b_n = n^{-6}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty}\left(n^6+n^4\right)^{-1}\\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \sum_{n=1}^{\infty} n^{-6} - \sum_{n=1}^{\infty}\left(n^8+n^6\right)^{-1}\\ &= \frac{\pi^2}{6} - \frac{\pi^4}{90} + \frac{\pi^6}{945} - \sum_{n=1}^{\infty}\left(n^8+n^6\right)^{-1}\end{aligned}$$

For $n \geq 3$ we then find $\sum_{n=1}^{\infty} a_n = (\pi^2/6) - (\pi^4/90) + (\pi^6/945) - \sum_{n=1}^{\infty}(n^8+n^6)^{-1} \approxeq 1.0767$.

Section 6.13

1. (a) Let the series $\sum_{n=1}^{\infty} x^n/(2n^2-n)$ be given. By the ratio test we find

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2\left(n+1\right)^2-\left(n+1\right)}\frac{2n^2-n}{x^n}\right| &= |x|\lim_{n\to\infty}\frac{2n^2-n}{2n^2+3n+1}\\ &= |x|\lim_{n\to\infty}\frac{2-1/n}{2+3/n+1/n^2} = |x|\end{aligned}$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty} |a_{n+1}/a_n| = L = |x| < 1$ or $-1 < x < 1$. To test for convergence when $x = \pm 1$ we employ the integral test:

$$\begin{aligned}
\int_1^\infty \frac{(\pm 1)}{2y^2 - y}\, dy = (\pm 1) \lim_{b\to\infty} \int_1^b \left(\frac{2}{2y-1} - \frac{1}{y} \right) dy &= (\pm 1) \lim_{b\to\infty} \left(\int_1^{2b-1} \frac{du}{u} - \int_1^b \frac{dy}{y} \right) \\
&= (\pm 1) \lim_{b\to\infty} \left(\ln|2b-1| - \ln|b| \right) \\
&= (\pm 1) \lim_{b\to\infty} \ln \frac{2b-1}{b} \\
&= (\pm 1) \lim_{b\to\infty} \ln \left(2 - \frac{1}{b} \right) = (\pm 1) \ln 2
\end{aligned}$$

Since the improper integral $\int_c^\infty f(y)\, dy$ converges, so will the series $\sum_{n=1}^\infty (\pm 1)/(2n^2 - n)$. Hence, the series $\sum_{n=1}^\infty x^n/(2n^2 - n)$ converges for $-1 \le x \le 1$.

(b) Let the series $\sum_{n=1}^\infty nx^n/2^n$ be given. By the ratio test we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\, x^{n+1}}{2^{n+1}} \frac{2^n}{nx^n} \right| = \frac{|x|}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = \frac{|x|}{2}$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty} |a_{n+1}/a_n| = L = |x|/2 < 1$ or $-2 < x < 2$.

(c) Let the series $\sum_{n=1}^\infty 1/nx^{2n}$ be given. By the ratio test we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{nx^{2n}}{(n+1)\, x^{2(n+1)}} \right| = \frac{1}{x^2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{x^2} \lim_{n\to\infty} \frac{1}{1 + 1/n} = \frac{1}{x^2}$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty} |a_{n+1}/a_n| = L = 1/x^2 < 1$ or $|x| > 1 \iff x > 1,\ x < -1$.

(d) Let the series $\sum_{n=0}^\infty 1/2^{nx}$ be given. By the ratio test we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{nx}}{2^{(n+1)x}} \right| = \frac{1}{|2^x|}$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty} |a_{n+1}/a_n| = L = 1/2^x < 1$ or $2^x > 2^0 \implies x > 0$.

(e) Let the series $\sum_{n=1}^\infty x^n/(1-x)^n$ be given. By the ratio test we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(1-x)^{n+1}} \frac{(1-x)^n}{x^n} \right| = \left| \frac{x}{1-x} \right|$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty} |a_{n+1}/a_n| = L = |x/(1-x)| < 1$ or $x - 1 < x < 1 - x \implies x < 1/2$.

(f) Let the series $\sum_{n=1}^{\infty} 2^n \sin^n x/n^2$ be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\sin^{n+1}x}{(n+1)^2}\frac{n^2}{2^n\sin^n x}\right| = 2|\sin x|\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}$$
$$= 2|\sin x|\lim_{n\to\infty}\frac{1}{1+2/n+1/n^2} = 2|\sin x|$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty}|a_{n+1}/a_n| = L = 2|\sin x| < 1$ or $-1/2 < \sin x < 1/2 \iff \sin^{-1}(-1/2) < x < \sin^{-1}(1/2)$, which is satisfied when $(-\pi/6)+n\pi < x < (\pi/6)+n\pi$ for $n = 0, \pm1, \pm2, \ldots$.

(g) Let the series $\sum_{n=1}^{\infty}(x-1)^n/n^2$ be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)^2}\frac{n^2}{(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}$$
$$= |x-1|\lim_{n\to\infty}\frac{1}{1+2/n+1/n^2} = |x-1|$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty}|a_{n+1}/a_n| = L = |x-1| < 1$ or $-1 < x-1 < 1 \iff 0 < x < 2$. For $x = 0$ and $x = 2$ the series converges by comparison with the harmonic series of order 2:

$$\left|\frac{(\pm1)^n}{n^2}\right| \le \frac{1}{n^2}$$

Hence, the series converges for $0 \le x \le 2$.

(h) Let the series $\sum_{n=1}^{\infty} 1/x^n \ln(n+1)$ be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^n\ln(n+1)}{x^{n+1}\ln(n+2)}\right| = \frac{1}{|x|}\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n+2)} = \frac{1}{|x|}\lim_{n\to\infty}\frac{1/(n+1)}{1/(n+2)}$$
$$= \frac{1}{|x|}\lim_{n\to\infty}\frac{1+2/n}{1+1/n} = \frac{1}{|x|}$$

Hence, by the ratio test the series converges when $\lim_{n\to\infty}|a_{n+1}/a_n| = L = 1/|x| < 1$ or $|x| > 1 \iff x > 1,\ x \le -1$.

(i) Let the series $\sum_{n=1}^{\infty}(x-2)^{3n}/n!$ be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{3(n+1)}}{(n+1)!}\frac{n!}{(x-2)^{3n}}\right| = |(x-2)^3|\lim_{n\to\infty}\frac{1}{n+1} = 0$$

Hence, the series converges for all x.

(j) Let the series $\sum_{n=2}^{\infty} x^n/\ln^n n$ be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln^{n+1}n}\frac{\ln^n n}{x^n}\right| = |x|\lim_{n\to\infty}\frac{1}{\ln n} = 0$$

Hence, the series converges for all x.

2. (a) Let the series $\sum_{n=1}^{\infty} x^n/n^3$, where $-1 \leq x \leq 1$ be given. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^3}\frac{n^3}{x^n}\right| = |x|\lim_{n\to\infty}\frac{n^3}{n^3+3n^2+3n+1}$$
$$= |x|\lim_{n\to\infty}\frac{1}{1+3/n+3/n^2+1/n^3} = |x|$$

Hence, the series converges for $|x| < 1 \iff -1 < x < 1$. For $x = \pm 1$ the series converges by comparison with the harmonic series of order 2:

$$\left|\frac{(\pm 1)^n}{n^3}\right| \leq \frac{1}{n^3} \leq \frac{1}{n^2}$$

Hence, the series converges for $-1 \leq x \leq 1$. The convergence is uniform for this range, since the comparison

$$\left|\frac{x^n}{n^3}\right| \leq \frac{1}{n^2} = M_n$$

holds for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} 1/n^2$ converges.

(b) Let the series $\sum_{n=1}^{\infty} \tanh^n x/n!$, where x is any real number be given. This series converges uniformly for all x, since

$$\left|\frac{\tanh^n x}{n!}\right| \leq \frac{1}{n!} = M_n$$

holds for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} 1/n!$ converges.

(c) Let the series $\sum_{n=1}^{\infty} \sin nx/(n^2+1)$, where x is any real number be given. This series converges uniformly for all x, since

$$\left|\frac{\sin nx}{n^2+1}\right| \leq \frac{1}{n^2+1} < \frac{1}{n^2} = M_n$$

holds for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} 1/n^2$ converges.

(d) Let the series $\sum_{n=1}^{\infty} e^{nx}/2^n$, where $x \leq \ln(3/2)$ be given. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{(n+1)x}}{2^{n+1}}\frac{2^n}{e^{nx}} = \lim_{n\to\infty}\frac{e^x}{2} = \frac{e^x}{2}$$

Hence, the series converges for $e^x/2 < 1 \iff x < \ln 2$. Because $\ln 3/2 < \ln 2$ the series converges uniformly, since

$$\frac{e^{nx}}{2^n} \leq \frac{e^{n\ln 3/2}}{2^n} = \frac{3^n}{4^n} = M_n$$

holds for all $x < \ln 3/2$ and the series $\sum M_n = \sum_{n=1}^{\infty}(3/4)^n$ converges according to the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{4^{n+1}}\frac{4^n}{3^n} = \frac{3}{4} = L < 1$$

(e) Let the series $\sum_{n=0}^{\infty} x^n/n! = \sum_{n=1}^{\infty} x^n/n! + 1$, where $-1 \le x \le 1$ be given. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1}$$

Hence, the series $\sum_{n=1}^{\infty} x^n/n!$ converges for $|x| < 1 \iff -1 < x < 1$. For $x = \pm 1$ the series converges by comparison with:

$$\left|\frac{(\pm 1)^n}{n!}\right| \le \frac{1}{n!}$$

Hence, the series converges for $-1 \le x \le 1$. The convergence is uniform for this range, since the comparison

$$\left|\frac{x^n}{n!}\right| \le \frac{1}{n!} = M_n$$

holds for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} 1/n!$ converges.

(f) Let the series $\sum_{n=1}^{\infty} nx^n$, where $-1/2 \le x \le 1/2$ be given. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x^{n+1}}{nx^n}\right| = |x|\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = |x|$$

Hence, the series converges for $|x| < 1 \iff -1 < x < 1$. This series converges uniformly for $-1/2 \le x \le 1/2$, since

$$|nx^n| \le \frac{n}{2^n} = M_n$$

for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} n/2^n$ converges according to the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{2^{n+1}}\frac{2^n}{n} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2} = L < 1$$

(g) Let the series $\sum_{n=1}^{\infty} nx^n$, where $-0.9 \le x \le 0.9$ be given. From (f) it follows that the series converges for $|x| < 1 \iff -1 < x < 1$. The series converges uniformly for $-0.9 \le x \le 0.9$, since

$$|nx^n| \le n\left(\frac{9}{10}\right)^n = M_n$$

for all x of the range and the series $\sum M_n = \sum_{n=1}^{\infty} n(9/10)^n$ converges according to the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{9}{10}\right)^{n+1}\left(\frac{10}{9}\right)^n = \frac{9}{10}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 0.9 = L < 1$$

(h) Let the series $\sum_{n=1}^{\infty} nx^n$, where $-a \leq x \leq a$, $a < 1$ be given. From (f) it follows that the series converges for $|x| < 1 \iff -1 < x < 1$. The series converges uniformly for $-a \leq x \leq a$, since

$$|nx^n| \leq na^n = M_n < n^n$$

and the series $\sum M_n = \sum_{n=1}^{\infty} na^n$ converges according to the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)\, a^{n+1}}{na^n} = a \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = a = L < 1$$

3. Let $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent for the interval $a \leq x \leq b$. In other words, some convergent series of constants $\sum_{n=1}^{\infty} M_n$ exists such that

$$|u_n\,(x)\,| \leq M_n \quad a \leq x \leq b$$

Note that each constant M_n is the *same* for all $x \in [a, b]$. Hence, it must be the same for any smaller interval contained in $a \leq x \leq b$, since this smaller interval is just some subset E_1 that is part of the set E of values of x that represents the interval $a \leq x \leq b$. As such, the series must be uniformly convergent in each smaller interval contained in $a \leq x \leq b$ as well.

4. Let $\sum_{n=1}^{\infty} v_n(x)$ be uniformly convergent for a set E of values of x. Hence, some convergent series of constants $\sum_{n=1}^{\infty} M_n$ exists such that

$$|v_n(x)| \leq M_n \quad \text{for all } x \text{ in } E$$

Furthermore, let $|u_n(x)| \leq v_n(x)$ for $x \in E$. In other words, for each fixed x, each term of the series $\sum_{n=1}^{\infty} |u_n(x)|$ is less than or equal to the nth term $v_n(x)$ of the uniformly convergent series $\sum_{n=1}^{\infty} v_n(x)$. Hence, by the comparison test (Section 6.6, Theorem 12) the series $\sum_{n=1}^{\infty} u_n(x)$ is absolutely convergent for $x \in E$ and since

$$|u_n\,(x)\,| \leq |v_n\,(x)\,| \leq M_n \quad \text{for all } x \text{ in } E$$

it follows that $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent for $x \in E$.

5. Let $0 < u_n(x) < 1/n$ (which implies that $\lim_{n\to\infty} u_n(x) = 0$) and $u_{n+1}(x) \leq u_n(x)$ for $a \leq x \leq b$. Hence, by the alternating series test (Section 6.6, Theorem 18) the series $\sum_{n=1}^{\infty}(-1)^n u_n(x)$ converges. Furthermore, $|u_n(x)| < 1/n = M_n$ for all x of the range considered and hence, the alternating series converges uniformly for $a \leq x \leq b$.

6. Let a convergent series $\sum_{n=1}^{\infty} M_n$ of constants $M_n > 0$ be given. Hence, for some $\epsilon > 0$ and N can be found such that $|M_{n+1} + M_{n+2} + \cdots + M_m| \leq \epsilon$ for $m > n > N$ (Section 6.5, Theorem 9). Next, let a sequence $f_n(x)$ be given such that $|f_{n+1}(x) - f_n(x)| \leq M_n$ for all $x \in E$. Since $M_{n+1} \leq \epsilon$ for $n > N$ it is true (after relabeling) that $|f_{n+1}(x) - f_n(x)| \leq \epsilon$. In other words, there exists some $n > N$ such that the difference between $f_{n+1}(x)$ and $f_n(x)$ is not greater than $\epsilon > 0$ (which can be chosen arbitrarily small) for each $x \in E$. Hence, the sequence $f_n(x)$ is uniformly convergent for all $x \in E$.

7. (a) Let the sequence $(n+x)/x$, where $0 \le x \le 1$ be given. This sequence converges uniformly for the range of x given, since

$$|f_{n+1}(x) - f_n(x)| = \left|\frac{n+1+x}{n+1} - \frac{n+x}{n}\right| = \left|\frac{-x}{n(n+1)}\right| \le \frac{1}{n^2} = M_n$$

and the series of constants $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n^2$ converges.

(b) Let the sequence $x^n/n!$, where $-1 \le x \le 1$ be given. This sequence converges uniformly for the range of x given, since

$$|f_{n+1}(x) - f_n(x)| = \left|\frac{x^{n+1}}{(n+1)!} - \frac{x^n}{n!}\right| = \frac{|x^n|}{n!}\left|\frac{x}{n+1} - 1\right| \le \frac{3}{2}\frac{1}{n!} = M_n$$

and the series of constants $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n!$ converges. The constant $3/2$ is justified by noting that

$$\max\left|\frac{x}{n+1} - 1\right| = \max(a)$$

in the interval $-1 \le x \le 1$ occurs when $x = -1$, $n = 1$. Furthermore, as $n \to \infty$ we see that $a \to 1$.

(c) Let the sequence $f_n(x) = \ln(1+nx)/n$, where $1 \le x \le 2$ be given. Firstly, we note that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\ln(1+nx)}{n} = \lim_{n\to\infty} \frac{x}{1+nx} = 0$$

and since $f_n(x) > 0$ for $1 \le x \le 2$ this implies $f_{n+1}(x) < f_n(x)$. As such

$$\frac{\ln(1+nx)}{n+1} < \frac{\ln[1+(n+1)x]}{n+1} < \frac{\ln(1+nx)}{n}$$

or equivalently

$$\frac{\ln(1+nx)}{n+1} - \frac{\ln(1+nx)}{n} < \frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n} < 0$$

And so we learn that

$$\left|\frac{\ln(1+nx)}{n+1} - \frac{\ln(1+nx)}{n}\right| > \left|\frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n}\right|$$

Hence, for $1 \le x \le 2$ we find

$$\begin{aligned} |f_{n+1}(x) - f_n(x)| = \left|\frac{\ln[1+(n+1)x]}{n+1} - \frac{\ln(1+nx)}{n}\right| &< \ln(1+nx)\left|\frac{1}{n+1} - \frac{1}{n}\right| \\ &= \frac{\ln(1+nx)}{n(n+1)} \\ &< \frac{\ln(1+nx)}{n^2} \\ &\le \frac{\ln(1+2n)}{n^2} = M_n \end{aligned}$$

It remains to be shown that the series of constants $\sum_{n=1}^{\infty} M_n$ converges. To this end we employ the *integral test*:

$$
\begin{aligned}
\int_1^{\infty} \frac{\ln(1+2x)}{x^2}\,dx &= \lim_{b\to\infty}\int_1^{b} \frac{\ln(1+2x)}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{\ln(1+2x)}{x}\bigg|_1^b + \lim_{b\to\infty}\int_1^b \frac{2}{x(1+2x)}\,dx \\
&= \ln 3 - \lim_{b\to\infty}\frac{\ln(1+2b)}{b} + 2\lim_{b\to\infty}\int_1^b \left(\frac{1}{x}-\frac{2}{1+2x}\right)dx \\
&= \ln 3 + 2\lim_{b\to\infty}\left[\ln|x| - \ln|1+2x|\right]_1^b \\
&= 3\ln 3 + 2\lim_{b\to\infty}\left[\ln b - \ln(1+2b)\right] \\
&= 3\ln 3 + 2\lim_{b\to\infty}\ln\frac{b}{1+2b} = 3\ln 3 + 2\ln\left(\lim_{b\to\infty}\frac{b}{1+2b}\right) \\
&= 3\ln 3 - 2\ln 2
\end{aligned}
$$

Since this integral converges and all the conditions of the integral test are satisfied, we may conclude that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \ln(1+2n)/n^2$ converges. Hence, the original sequence $f_n(x) = \ln(1+nx)/n$ converges uniformly for $1 \le x \le 2$.

(d) Let the sequence $f_n(x) = n/e^{nx^2}$, where $1/2 \le x \le 1$ be given. Firstly, we note that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{n}{e^{nx^2}} = \lim_{n\to\infty}\frac{1}{x^2 e^{nx^2}} = 0$$

and since $f_n(x) > 0$ for $1/2 \le x \le 1$ this implies $f_{n+1}(x) < f_n(x)$. However, if we plot $f_n(x)$ for various values of n (keeping x fixed) we see that $f_{n+1} < f_n(x)$ only is true for $n \ge 4$. As stated at the start of Section 6.6, the convergence or divergence of a series is unaffected if a finite number of terms of the series are discarded. Hence, in testing for convergence of $\sum M_n$ we can simply ignore the first four terms and aim to prove $\sum_{n=4}^{\infty} M_n$ does converge for a certain M_n yet to be determined. Continuing with our sequence, we conclude (for $n \ge 4$)

$$\frac{n}{e^{(n+1)x^2}} < \frac{n+1}{e^{(n+1)x^2}} < \frac{n}{e^{nx^2}}$$

or equivalently

$$\frac{n}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} < \frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} < 0$$

And so we learn that

$$\left|\frac{n}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}}\right| > \left|\frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}}\right|$$

Hence, for $1/2 \le x \le 1$, $n \ge 4$ we find

$$\begin{aligned}
|f_{n+1}(x) - f_n(x)| = \left| \frac{n+1}{e^{(n+1)x^2}} - \frac{n}{e^{nx^2}} \right| &< \frac{n}{e^{nx^2}} \left| \frac{1}{e^{x^2}} - 1 \right| \\
&= \frac{n}{e^{nx^2}} \left(1 - \frac{1}{e^{x^2}} \right) \\
&\le \max_{1/2 \le x \le 1} \left[\frac{n}{e^{nx^2}} \left(1 - \frac{1}{e^{x^2}} \right) \right] \\
&= \frac{n}{e^{n/4}} \left(1 - \frac{1}{e^{1/4}} \right) = M_n
\end{aligned}$$

It remains to be shown that the series of constants $\sum_{n=4}^{\infty} M_n$ converges. To this end we employ the *integral test*:

$$\begin{aligned}
\int_4^{\infty} \frac{x}{e^{x/4}}\,dx = \lim_{b\to\infty} \int_4^b \frac{x}{e^{x/4}}\,dx &= \lim_{b\to\infty} \int_4^b xe^{-x/4}\,dx \\
&= \lim_{b\to\infty} -4xe^{-x/4}|_4^b + \lim_{b\to\infty} \int_4^b 4e^{-x/4}\,dx \\
&= \lim_{b\to\infty} \left[-4xe^{-x/4} - 16e^{-x/4} \right]_4^b \\
&= \lim_{b\to\infty} \left(-4be^{-b/4} - 16e^{-b/4} \right) + 32e^{-1} = \frac{32}{e}
\end{aligned}$$

Since this integral converges and all the conditions of the integral test are satisfied, we may conclude that $\sum_{n=4}^{\infty} M_n = \sum_{n=4}^{\infty} ne^{-n/4}(1 - e^{-1/4})$ converges. Hence, the original sequence $f_n(x) = n/e^{nx^2}$ converges uniformly for $1/2 \le x \le 1$.

Section 6.16

1. (a) Let the relation

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

where $-1 < x < 1$ be given. Integrating both sides gives

$$\begin{aligned}
\int \frac{dx}{1-x} &= \sum_{n=0}^{\infty} \int x^n\,dx \\
\ln \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n} \\
&= x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots
\end{aligned}$$

To verify (6.43) we note that

$$f(a) = f(0) = \ln\frac{1}{1-0} = 0$$

This checks out, since $c_0 = 0$ for $\sum_{n=1}^{\infty} x^n/n$. Also

$$f'(a) = f'(0) = \left(\frac{d}{dx}\ln\frac{1}{1-x}\right)_{x=0} = \left.\frac{1}{1-x}\right|_{x=0} = 1$$

Again, this checks out, since $c_1 = 1$ for $\sum_{n=1}^{\infty} x^n/n$. And in general

$$f^{(n)}(a) = f^{(n)}(0) = \left(\frac{d^{(n)}}{dx^{(n)}}\ln\frac{1}{1-x}\right)_{x=0} = \left.\frac{(n-1)!}{(1-x)^n}\right|_{x=0} = (n-1)!$$

which checks out, since $c_n = f^{(n)}(a)/n! = (n-1)!/n! = 1/n$ for $\sum_{n=1}^{\infty} x^n/n$.

(b) For $x = -1$ the series $\sum_{n=1}^{\infty} x^n/n$ reduces to the alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}(-1)^n a_n$$

Since $a_{n+1} < a_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 0$ it follows from Theorem 18 that the alternating series converges. Hence,

$$\begin{aligned}\ln\frac{1}{1-(-1)} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\ -\ln 2 &= \\ \ln 2 &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\end{aligned}$$

2. From the relation

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

we obtain by successive differentiation the relations

$$\begin{aligned}\frac{1}{(1-x)^2} &= 1 + 2x + \cdots + nx^{n-1} + \cdots &&= \sum_{n=0}^{\infty}(n+1)\,x^n && -1 < x < 1\\ \frac{2}{(1-x)^3} &= 2 + 6x + \cdots + n(n-1)\,x^{n-2} + \cdots &&= \sum_{n=0}^{\infty}(n+2)(n+1)\,x^n && -1 < x < 1\end{aligned}$$

Hence, in general

$$\begin{aligned}\frac{1}{(1-x)^k} &= (1-x)^{-k} \\ &= 1 + \frac{kx}{1} + \frac{k(k+1)}{1\cdots 2}x^2 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}x^n + \cdots\end{aligned}$$

where $-1 < x < 1$ for some known $k = 1, 2, 3, \ldots$. Differentiating both sides of this equation gives

$$\begin{aligned}\frac{d}{dx}(1-x)^{-k} &= \frac{d}{dx}\left[1 + \frac{kx}{1} + \frac{k(k+1)}{1\cdots 2}x^2 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}x^n + \cdots\right] \\ k(1-x)^{-k-1} &= k + \frac{2k(k+1)}{1\cdot 2}x + \frac{3k(k+1)(k+2)}{1\cdot 2\cdot 3}x^2 + \cdots + \frac{nk(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}x^{n-1} \\ &\quad + \cdots \\ (1-x)^{-k-1} &= 1 + \frac{k(k+1)}{1}x + \frac{k(k+1)(k+2)}{1\cdot 2}x^2 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n-1}x^{n-1} + \cdots \\ &= 1 + \frac{k(k+1)}{1}x + \frac{k(k+1)(k+2)}{1\cdot 2}x^2 + \cdots + \frac{k(k+1)\cdots(k+n)}{1\cdot 2\cdots n}x^n + \cdots\end{aligned}$$

We see that this is none other than (6.41), i.e. the generalised relation we started with, but for $k+1$ instead of k. In other words, if we know that (6.41) is true for some known k we have established that it will be true for $k+1$ also and hence, by induction for any $k = 1, 2, 3, \ldots$.

3. From (6.41) we know that

$$\frac{1}{(1-r)^k} = 1 + \frac{kr}{1} + \frac{k(k+1)}{2}r^2 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}r^n + \cdots$$

for $-1 < r < 1$, $k = 1, 2, 3, \ldots$.

(a) Let the function $f(x) = 1/x$ be given. To expand this function in a Taylor series about $x = 1$ we note that

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \frac{1}{1-r}$$

using the substitution $r = 1 - x$. Hence, by (6.41) setting $k = 1$ we find

$$\begin{aligned}\frac{1}{x} = \frac{1}{1-r} &= 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^n + \cdots \\ &= 1 - (x-1) + (x-1)^2 + \cdots + (-1)^n (x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n\end{aligned}$$

for $-1 < r < 1 \implies 0 < x < 2$.

(b) Let the function $f(x) = 1/(x+2)$ be given. To expand this function in a Maclaurin series (i.e. expand it around $x = 0$) we note that

$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1-(-x/2)} = \frac{1}{2} \cdot \frac{1}{1-r}$$

using the substitution $r = -x/2$. Hence, by (6.41) setting $k = 1$ we find

$$\begin{aligned}\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1-r} &= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} + \cdots + (-1)^n \frac{x^n}{2^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}\end{aligned}$$

for $-1 < r < 1 \implies -2 < x < 2$.

(c) Let the function $f(x) = 1/(3x+5)$ be given. To expand this function in a Maclaurin series we employ (6.44) to get

$$\begin{aligned}f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ \frac{1}{3x+5} &= \frac{1}{5} - \frac{3}{5^2}x + \frac{9}{5^3}x^2 + \cdots + (-1)^n \frac{3^n}{5^{n+1}}x^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^{n+1}}x^n\end{aligned}$$

Since we require $3x+5 > 0$ in order for the function $f(x)$ to be differentiable the convergence interval is given by $-5/3 < x < 5/3$.

(d) Let the function $f(x) = 1/(3x+5)$ be given. To expand this function in a Taylor series about $x = 1$ we note that

$$\frac{1}{3x+5} = \frac{1}{3(x-1)+8} = \frac{1}{8} \cdot \frac{1}{1+3(x-1)/8} = \frac{1}{8} \cdot \frac{1}{1-r}$$

using the substitution $r = 3(1-x)/8$. Hence, by (6.41) setting $k = 1$ we find

$$\begin{aligned}\frac{1}{3x+5} = \frac{1}{8} \cdot \frac{1}{1-r} &= \frac{1}{8} + \frac{3}{8^2}(1-x) + \frac{9}{8^3}(1-x)^2 + \cdots + \frac{3^n}{8^{n+1}}(1-x)^n + \cdots \\ &= \frac{1}{8} - \frac{3}{8^2}(x-1) + \frac{9}{8^3}(x-1)^2 + \cdots + (-1)^n \frac{3^n}{8^{n+1}}(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{8^{n+1}}(x-1)^n\end{aligned}$$

for $-1 < r < 1 \implies -5/3 < x < 11/3$.

(e) Let the function $f(x) = 1/(ax+b)$ be given. To expand this function in a Taylor series about $x = c$ we employ (6.43) to get

$$\begin{aligned} f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\ \frac{1}{ax+b} &= \frac{1}{ac+b} - \frac{a}{(ac+b)^2}(x-c) + \cdots + (-1)^n \frac{a^n}{(ac+b)^{n+1}}(x-c)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{(ac+b)^{n+1}}(x-c)^n \end{aligned}$$

for

$$c - \left|\frac{ac+b}{a}\right| < x < c + \left|\frac{ac+b}{a}\right|$$

which follows from (6.38) and (6.42).

(f) Let the function $f(x) = 1/(1-x^2)$ be given. To expand this function in a Maclaurin series we employ (6.44) to get

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ \frac{1}{1-x^2} &= 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots \\ &= \sum_{n=0}^{\infty} x^{2n} \end{aligned}$$

for $-1 < x < 1$.

(g) Let the function $f(x) = 1/[(x-2)(x-3)]$ be given. To expand this function in a Maclaurin series we note that

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3} = \frac{1}{2}\cdot\frac{1}{1-x/2} - \frac{1}{3}\cdot\frac{1}{1-x/3}$$

Hence, by (6.41) setting $k = 1$ we find

$$\begin{aligned} \frac{1}{2}\cdot\frac{1}{1-x/2} &= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots + \frac{x^n}{2^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \end{aligned}$$

for $-2 < x < 2$ and

$$\begin{aligned} \frac{1}{3}\cdot\frac{1}{1-x/3} &= \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \cdots + \frac{x^n}{3^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \end{aligned}$$

for $-3 < x < 3$. And so

$$\frac{1}{(x-2)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) x^n$$

for $-2 < x < 2$.

(h) Let the function $f(x) = 1/x^2$ be given. To expand this function in a Taylor series about $x = 1$ we note that

$$\frac{1}{x^2} = \frac{1}{[1-(1-x)]^2} = \frac{1}{(1-r)^2}$$

using the substitution $r = 1 - x$. Hence, by (6.41) setting $k = 2$ we find

$$\begin{aligned} \frac{1}{x^2} = \frac{1}{(1-r)^2} &= 1 + 2(1-x) + 3(1-x)^2 + \cdots + (n+1)(1-x)^n + \cdots \\ &= 1 - 2(x-1) + 3(x-1)^2 + \cdots + (-1)^n (n+1)(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n \end{aligned}$$

for $-1 < r < 1 \implies 0 < x < 2$.

(i) Let the function $f(x) = 1/(3x+5)^2$ be given. To expand this function in a Taylor series about $x = 1$ we note that

$$\frac{1}{(3x+5)^2} = \frac{1}{[3(x-1)+8]^2} = \frac{1}{8^2} \cdot \frac{1}{[1+3(x-1)/8]^2} = \frac{1}{8^2} \cdot \frac{1}{(1-r)^2}$$

using the substitution $r = 3(1-x)/8$. Hence, by (6.41) setting $k = 2$ we find

$$\begin{aligned} \frac{1}{(3x+5)^2} &= \frac{1}{8^2} \cdot \frac{1}{(1-r)^2} \\ &= \frac{1}{8^2} + \frac{2\cdot 3}{8^3}(1-x) + \frac{2\cdot 3\cdot 3^2}{8^4 \cdot 1 \cdot 2}(1-x)^2 + \cdots + \frac{3^n (n+1)}{8^{n+2}}(1-x)^n + \cdots \\ &= \frac{1}{8^2} - \frac{2\cdot 3}{8^3}(x-1) + \frac{2\cdot 3\cdot 3^2}{8^4 \cdot 1 \cdot 2}(x-1)^2 + \cdots + (-1)^n \frac{3^n (n+1)}{8^{n+2}}(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^n (n+1)}{8^{n+2}} (x-1)^n \end{aligned}$$

for $-1 < r < 1 \implies -5/3 < x < 11/3$.

(j) Let the function $f(x) = 1/(ax+b)^k$ be given. To expand this function in a Taylor series about $x = c$ we employ (6.43) to get

$$
\begin{aligned}
f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\
\frac{1}{(ax+b)^k} &= \frac{1}{(ac+b)^k} - \frac{k}{1}\frac{a}{(ax+b)^{k+1}}(x-c) + \frac{k(k+1)}{1\cdot 2}\frac{a^2}{(ax+b)^{k+2}}(x-c)^2 + \cdots \\
&\quad + (-1)^n \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}\frac{a^n}{(ax+b)^{k+n}}(x-c)^n + \cdots \\
&= \frac{1}{(ac+b)^k} + \sum_{n=1}^{\infty}(-1)^n \frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n}\frac{a^n}{(ax+b)^{k+n}}(x-c)^n
\end{aligned}
$$

for

$$
c - \left|\frac{ac+b}{a}\right| < x < c + \left|\frac{ac+b}{a}\right|
$$

which follows from (6.38) and (6.42).

4. Let $f(x) = \sum_{n=1}^{\infty} x^n/n^n$.

   (a) From the formal definition of the limit: $\lim_{x\to x_1} f(x) = c$ means that given any $\epsilon > 0$, a $\delta > 0$ can be found such that for every x in a domain D where $|x - x_1| < \delta$ an ϵ can be found such that $|f(x) - c| < \epsilon$, follows that $\lim_{x\to x_1} k = k$ for some constant function $f(x) = k$ and $\lim_{x\to x_1} x = x_1$ for some linear function $f(x) = x$. Hence, both are continuous for any $x \in D : (-\infty, \infty)$. Now if both $f(x)$ and $g(x)$ are continuous in D then so will be $f(x)g(x)$. And thus we may conclude at once that $kx^n = kxx^{n-1} = kxxx^{n-2} = \cdots = kx^{n-1}x$ is continuous in D. Furthermore, if $f(x)$ and $g(x)$ are continuous in d so will be $f(x) + g(x)$. Hence $a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=1}^{\infty} a_nx^n$ is continuous in D. Lastly, choosing $a_n = 1/n^n$ for $n = 1, 2, \ldots$ then proves that $f(x) = \sum_{n=1}^{\infty} x^n/n^n$ is continuous (and hence, defined) for all $x \in D : (-\infty, \infty)$.

   (b)

$$
f(0) = \sum_{n=1}^{\infty}\frac{0^n}{n^n} = \frac{0^1}{1^1} + \frac{0^2}{2^2} + \cdots + \frac{0^n}{n^n} + \cdots = 0
$$

$$
\begin{aligned}
f(1) = \sum_{n=1}^{\infty}\frac{1^n}{n^n} &= \frac{1^1}{1^1} + \frac{1^2}{2^2} + \cdots + \frac{1^n}{n^n} + \cdots \\
&= 1 + \frac{1}{4} + \cdots + \frac{1}{n^n} + \cdots \approxeq 1.29
\end{aligned}
$$

$$
f'(0) = \sum_{n=1}^{\infty}\frac{0^{n-1}}{n^{n-1}} = \frac{0^0}{1^0} + \frac{0^1}{2^1} + \cdots + \frac{0^{n-1}}{n^{n-1}} + \cdots = 1
$$

$$f'(1) = \sum_{n=1}^{\infty} \frac{1^{n-1}}{n^{n-1}} = \frac{1^0}{1^0} + \frac{1^1}{2^1} + \cdots + \frac{1^{n-1}}{n^{n-1}} + \cdots$$
$$= 1 + \frac{1}{2} + \cdots + \frac{1^{n-1}}{n^{n-1}} + \cdots \approxeq 1.63$$
$$f''(0) = \sum_{n=1}^{\infty} \frac{(n-1)\,0^{n-2}}{n^{n-1}} = \frac{0 \cdot 0^{-1}}{1^0} + \frac{1 \cdot 0^0}{2^1} + \cdots + \frac{(n-1)\,0^{n-2}}{n^{n-1}} + \cdots = \frac{1}{2}$$

(c) Using (6.44) we find

$$\begin{aligned} f'(x) &= f'(0) + \frac{f''(0)}{1!}x + \frac{f'''(0)}{2!}x^2 + \cdots + \frac{f^{(n+1)}(0)}{n!}x^n + \cdots \\ &= 1 + \frac{x}{2^1} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{(n+1)^n} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^n} \end{aligned}$$

and

$$\begin{aligned} f''(x) &= f''(0) + \frac{f'''(0)}{1!}x + \frac{f''''(0)}{2!}x^2 + \cdots + \frac{f^{(n+2)(0)}}{n!}x^n + \cdots \\ &= \frac{1}{2^1} + \frac{2x}{3^2} + \frac{3x^2}{4^3} + \cdots + \frac{(n+1)\,x^n}{(n+2)^{n+1}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(n+1)\,x^n}{(n+2)^{n+1}} \end{aligned}$$

5. Let the function

$$y = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

be given. From Problem 4(a) it follows that this function is defined for all $x \in D : (-\infty, \infty)$. Also, we may conclude at once that

$$f(0) = 1 + 0 + \frac{0^2}{2!} + \cdots + \frac{0^n}{n!} + \cdots = 1$$

Furthermore, it is easy to show that

$$\begin{aligned} \frac{dy}{dx} = f'(x) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = y = f(x) \end{aligned}$$

and that generally

$$\frac{d^{(n)}y}{dx^{(n)}} = f^{(n)}(x) = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right) = y = f(x) \implies f(0) = f'(0) = \cdots = f^{(n)}(0) = 1$$

Hence,

$$
\begin{aligned}
y = f(x) &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots
\end{aligned}
$$

is a Maclaurin series valid for all x.

Section 6.18

1. (a) Since $\sinh x = (1/2)(e^x - e^{-x})$ then by (6.46) we find

$$
\begin{aligned}
\sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right) \\
&= \frac{1}{2}\left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) - \left(1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots\right)\right] \\
&= \frac{x}{1!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}
\end{aligned}
$$

for $-\infty < x < \infty$.

(b) Since $\cos^2 x = (1 + \cos 2x)/2$ then by (6.48) we find

$$
\begin{aligned}
\cos^2 x &= \frac{1 + \cos 2x}{2} \\
&= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} + \cdots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} + \cdots\right) \\
&= 1 - \frac{2^1 x^2}{2!} + \frac{2^3 x^4}{4!} + \cdots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}
\end{aligned}
$$

for $-\infty < x < \infty$.

(c) Since $\sin^2 x = (1-\cos 2x)/2$ then by (6.47) we find

$$\begin{aligned}
\sin^2 x &= \frac{1-\cos 2x}{2} \\
&= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} + \cdots + \frac{(-1)^n\, 2^{2n}x^{2n}}{(2n)!} + \cdots\right) \\
&= \frac{2^1x^2}{2!} - \frac{2^3x^4}{4!} + \cdots + \frac{(-1)\, 2^{2n-1}x^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^{2n-1}x^{2n}}{(2n)!}
\end{aligned}$$

for $-\infty < x < \infty$.

(d) Since $\ln x = \int_1^x du/u$ and using the substitution $x' = u - 1$ so that

$$\frac{1}{u} = \frac{1}{1+x'}$$

then by (6.49) (recognising that $m = -1$) we find

$$\begin{aligned}
\ln x = \int_1^x \frac{du}{u} &= \int_0^{x-1} \frac{dx'}{1+x'} \\
&= \int_0^{x-1} \left[1 - x' + (x')^2 + \cdots + (-1)^n (x')^n + \cdots\right] dx' \\
&= \left[x' - \frac{(x')^2}{2} + \frac{(x')^3}{3} + \cdots + \frac{(-1)^{n+1}(x')^n}{n} + \cdots\right]_0^{x-1} \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n+1}(x-1)^n}{n} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}
\end{aligned}$$

(e) Since $\sqrt{1-x} = (1+x')^m$ for $|x| < 1$, using the substitutions $x = -x'$ and $m = 1/2$ then by (6.49) we find

$$\begin{aligned}
\sqrt{1-x} &= (1+x')^m \\
&= 1 + \frac{m}{1!}(x') + \frac{m(m-1)}{2!}(x')^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}(x')^n + \cdots \\
&= 1 - \frac{1}{2}x - \frac{1}{2^2 2!}x^2 - \frac{1\cdot 3}{2^3 3!}x^3 - \cdots
\end{aligned}$$

(f) Since $(1-x^2)^{-1/2} = (1+x')^m$ for $|x| < 1$, using the substitutions $x' = -x^2$ and

$m = -1/2$ then by (6.49) we find

$$\begin{aligned}
\left(1 - x^2\right)^{-1/2} &= (1 + x')^m \\
&= 1 + \frac{m}{1!}(x') + \frac{m(m-1)}{2!}(x')^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}(x')^n + \cdots \\
&= 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2^2 2!}x^4 + \frac{1 \cdot 3 \cdot 5}{2^3 2!}x^6 + \cdots
\end{aligned}$$

(g) Since $d(\sin^{-1} x)/dx = (1 - x^2)^{-1/2}$ for $|x| < 1$ we can find the Taylor series expansion for $\sin^{-1} x$ by integrating the series of Problem 1(f):

$$\begin{aligned}
\sin^{-1} x = \int_0^x \frac{du}{\sqrt{1 - u^2}} &= \int_0^x \left(1 + \frac{1}{2}u^2 + \frac{1 \cdot 3}{2^2 2!}u^4 + \frac{1 \cdot 3 \cdot 5}{2^3 2!}u^6 + \cdots\right) du \\
&= \left[u + \frac{1}{2}\frac{u^3}{3} + \frac{1 \cdot 3}{2^2 2!}\frac{u^5}{5} + \frac{1 \cdot 3 \cdot 5}{2^3 2!}\frac{u^7}{7} + \cdots\right]_0^x \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2^2 2!}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2^3 2!}\frac{x^7}{7} + \cdots
\end{aligned}$$

2. (a) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned}
f(x) &= e^x \sin x \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= e^0 \sin(0) + \left[e^0 \sin(0) + e^0 \cos(0)\right] x + e^0 \cos(0)\, x^2 + \frac{e^0 \cos(0) - e^0 \sin(0)}{3}x^3 + \cdots \\
&= x + x^2 + \frac{x^3}{3}
\end{aligned}$$

(b) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned}
f(x) &= \tan x \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= \tan(0) + \sec^2(0)\, x + \sec^2(0)\tan(0)\, x^2 + \left[\frac{-2}{3}\sec^2(0) + \sec^4(0)\right] x^3 \\
&\quad + \frac{\sec^2(0)\tan(0)\left[3\sec^2(0) - 1\right]}{3}x^4 \\
&\quad + \frac{12\sec^4(0)\tan^2(0) - 2\sec^2(0)\tan^2(0) + 3\sec^6(0) - \sec^4(0)}{15}x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5
\end{aligned}$$

(c) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= \ln^2(1+x) \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \ln^2(1) + 2\ln(1)\,x + [1 - \ln(1)]\,x^2 + \left[\frac{2}{3}\ln(1) - 1\right]x^3 + \left[\frac{11}{12} - \frac{\ln(1)}{2}\right]x^4 + \cdots \\ &= x^2 - x^3 + \frac{11}{12}x^4 \end{aligned}$$

(d) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= \ln\left(1 - x^2\right) \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \ln(1) - x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 + \cdots \\ &= -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 \end{aligned}$$

(e) Using (6.43) we find that the first three non-zero terms of the Taylor series about $x = 2$ of $f(x)$ are given by

$$\begin{aligned} f(x) &= x^3 + 3x + 1 \\ &= f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\ &= 15 + 15(x-2) + 6(x-2)^2 \end{aligned}$$

(f) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= e^{\tan x} \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + e^{\tan 0}\sec^2(0)\,x + \left[\frac{e^{\tan 0}}{2}\sec^4(0) + e^{\tan 0}\sec^2(0)\tan(0)\right]x^2 + \cdots \\ &= 1 + x + \frac{x^2}{2} \end{aligned}$$

(g) Using (6.44) we find that the first three non-zero terms of the Taylor series about

$x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= \sinh^{-1} x \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \sinh^{-1}(0) + x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots \\ &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 \end{aligned}$$

(h) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= \tanh x \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \tanh(0) + \operatorname{sech}^2(0)\, x - \operatorname{sech}^2(0)\tanh(0)\, x^2 + \frac{2\operatorname{sech}^2(0)\tanh^2(0) - \operatorname{sech}^4(0)}{3}x^3 \\ &\quad + \frac{\operatorname{sech}^2(0)\tanh(0)\left[2\operatorname{sech}^2(0) - \tanh^2(0)\right]}{3}x^4 \\ &\quad \frac{2\tanh^4(0)\operatorname{sech}^2(0) - 11\operatorname{sech}^4(0)\tanh^2(0) + 2\operatorname{sech}^6(0)}{15}x^5 + \cdots \\ &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \end{aligned}$$

(i) Using (6.44) we find that the first three non-zero terms of the Taylor series about $x = 0$ (Maclaurin series) of $f(x)$ are given by

$$\begin{aligned} f(x) &= \tanh^{-1} x \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \tanh^{-1}(0) + x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} \end{aligned}$$

(j) Using (6.44) we find that the first three non-zero terms of the Taylor series about

$x = 0$ (Maclaurin series) of $f(x)$ are given by

$$
\begin{aligned}
f(x) &= \ln \sec x \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= \ln \sec(0) + \tan(0)\, x + \frac{\sec^2(0)}{2}x^2 + \frac{\sec^2(0)\tan(0)}{3}x^3 + \frac{3\sec^4(0) - 2\sec^2(0)}{12}x^4 \\
&\quad \frac{\sec^2(0)\tan(0)\left[3\sec^2(0) - 1\right]}{15}x^5 + \\
&\quad \frac{12\sec^4(0)\tan^2(0) - 2\sec^2(0)\tan^2(0) + 3\sec^6(0) - \sec^4(0)}{90}x^6 + \cdots \\
&= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}
\end{aligned}
$$

3. Let $n \geq 1$ be a positive integer and x_2 be a fixed number in the interval $a - r_0 < x < a + r_0$, $x_2 \neq a$. Furthermore, let

$$\bar{G}(x) = G(x) - \left(\frac{x_2 - x}{x_2 - a}\right)^n G(a)$$

where

$$G(x) = f(x_2) - f(x) - (x_2 - x) f'(x) - \cdots - \frac{(x_2 - x)^{n-1}}{(n-1)!} f^{(n-1)}(x)$$

We assume that $f(x)$ is defined and continuous and has continuous derivatives up to the $(n+1)^{st}$ order in the given interval, which implies $\bar{G}(x)$ is defined and continuous for x in the same interval. Also, note that

$$\bar{G}(a) = G(a) - \left(\frac{x_2 - a}{x_2 - a}\right)^n G(a) = 0$$

and

$$
\begin{aligned}
\bar{G}(x_2) &= G(x_2) - \left(\frac{x_2 - x_2}{x_2 - a}\right)^n G(a) \\
&= G(x_2) \\
&= f(x_2) - f(x_2) - (x_2 - x_2) f'(x_2) - \cdots - \frac{(x_2 - x_2)^{n-1}}{(n-1)!} f^{(n-1)}(x_2) \\
&= 0
\end{aligned}
$$

Hence, by the Mean Value theorem, $\bar{G}'(x) = 0$ for some x_1 between a and x_2. Now

$$\begin{aligned}
\bar{G}'(x) &= G'(x) + \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} G(a) \\
&= -\frac{(x_2 - x)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} G(a) \\
&= \frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} \left[G(a) - \frac{1}{n!} f^{(n)}(x)(x_2 - a)^n \right]
\end{aligned}$$

and so the equation $\bar{G}'(x_1) = 0$ thus becomes the equation

$$\begin{aligned}
\bar{G}'(x_1) &= 0 \\
\frac{n(x_2 - x)^{n-1}}{(x_2 - a)^n} \left[G(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) \right] &= \\
G(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= \\
f(x_2) - f(a) - (x_2 - a) f'(a) - \cdots - \frac{(x_2 - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= \\
f(a) + (x_2 - a) f'(a) + \cdots + \frac{(x_2 - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x_2 - a)^n}{n!} f^{(n)}(x_1) &= f(x_2)
\end{aligned}$$

If x_2 is now replaced by a variable x, we get the desired result:

$$f(x) = f(a) + (x - a) f'(a) + \cdots + \frac{(x - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x - a)^n}{n!} f^{(n)}(x_1)$$

4. Starting from the premise that

$$R_n(x) = f(x) - f(a) - \left[\frac{f'(a)}{1!}(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \right]$$

we see that for $n = 0$ the term R_0, using the *fundamental theorem of calculus*, can be written as

$$R_0 = f(x) - f(a) = \int_a^x f'(t)\, dt$$

For $n = 1$ we find (using integration by parts)

$$\begin{aligned}
R_1 = f(x) - f(a) - \frac{f'(a)}{1!}(x - a) &= R_0 - (x - a) f'(a) \\
&= -(x - a) f'(a) + \int_a^x f'(t)\, dt \\
&= \underbrace{(x - t) f'(t)}_{uv} \Big|_a^x - \int_a^x \underbrace{-f'(t)\, dt}_{v\,du} \\
&= \int_a^x (x - t) f''(t)\, dt
\end{aligned}$$

Likewise, for $n=2$ we find

$$\begin{aligned}
R_2 &= f(x) - f(a) - \frac{f'(a)}{1!}(x-a) - \frac{f''(a)}{2!}(x-a)^2 \\
&= R_1 - (x-a)^2 \frac{f''(a)}{2} \\
&= -(x-a)^2 \frac{f''(a)}{2} + \int_a^x (x-t) f''(t)\, dt \\
&= \underbrace{\frac{(x-t)^2}{2} f''(t)}_{uv} \Bigg|_a^x - \int_a^x \underbrace{-f''(t)(x-t)\, dt}_{v\, du} = \int_a^x \frac{(x-t)^2}{2} f^{(3)}(t)\, dt
\end{aligned}$$

and $n=3$:

$$\begin{aligned}
R_3 &= f(x) - f(a) - \frac{f'(a)}{1!}(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \frac{f'''(a)}{3!}(x-a)^3 \\
&= R_2 - \frac{f'''(a)}{3!}(x-a)^3 \\
&= -(x-a)^3 \frac{f^{(3)}(a)}{3!} + \int_a^x \frac{(x-t)^2}{2!} f^{(3)}(t)\, dt \\
&= \underbrace{\frac{(x-t)^3}{3!} f^{(3)}(t)}_{uv} \Bigg|_a^x - \int_a^x \underbrace{-f^{(3)}(t) \frac{(x-t)^2}{2!}\, dt}_{v\, du} = \int_a^x \frac{(x-t)^3}{3!} f^{(4)}\, dt
\end{aligned}$$

As such, we consider the formula

$$R_m = \int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)\, dt$$

to be true for some known, positive integer $m \geq 0$. Next, using integration by parts with $u = f^{(m+1)}(t)$ and $dv = (x-t)^m\, dt/m!$ we can write

$$\begin{aligned}
R_m &= \int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)\, dt \\
&= \int_a^x u\, dv \\
&= uv\big|_a^x - \int_a^x v\, du \\
&= -\frac{(x-t)^{m+1}}{(m+1)!} f^{(m+1)}(t) \Bigg|_a^x + \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t)\, dt \\
&= \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) + \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t)\, dt \\
R_m - \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) &= \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t)\, dt
\end{aligned}$$

From (6.45) it follows that the last expression can be written as

$$R_m - \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(a) = R_{m+1} = \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t)\ dt$$

Thus by induction, since the integral formula for R_n is true for an some fixed positive integer m and $m+1$ it must be true for any arbitrary, positive integer $n \geq 0$.

5. (a) To evaluate the integral $\int_0^1 e^{-x^2}\, dx$ to three decimal places we will expand the integrand in a Maclaurin series using (6.44) and integrate term by term:

$$\begin{aligned}\int_0^1 e^{-x^2}\, dx &= \int_0^1 \left[1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right] dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!} + \cdots\right]_0^1 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!}\Bigg|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, n!}\end{aligned}$$

Using (6.45) we learn that in order to evaluate the integral to three decimal places we need to add the first five terms of the Maclaurin series since:

$$R_4 = \frac{1}{(2\cdot 5+1)\cdot 5!} \approxeq 0.00076$$

Hence,

$$\int_0^1 e^{-x^2}\, dx \approxeq \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)\, n!} \approxeq 0.747$$

(b) To evaluate the integral $\int_0^{1/2} (1+x^4)^{-1/2}\, dx$ to three decimal places we will use (6.41) to expand the integrand in a power series:

$$\begin{aligned}\frac{1}{\sqrt{1+x^4}} &= \left(1+x^4\right)^{-1/2} \\ &= 1 + \frac{1/2}{1}\left(-x^4\right) + \frac{(1/2)(3/2)}{1\cdot 2}\left(-x^4\right)^2 + \frac{(1/2)(3/2)(5/2)}{1\cdot 2\cdot 3}\left(-x^4\right)^3 \\ &\quad + \cdots + \frac{(1/2)(3/2)\cdots(1/2+n-1)}{1\cdot 2\cdots n}\left(-x^4\right)^n + \cdots\end{aligned}$$

Focusing on the n^{th} for now, this can be written as:

$$\begin{aligned}\frac{(1/2)(3/2)\cdots(1/2+n-1)}{1\cdot 2\cdots n}\left(-x^4\right)^n &= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\,(1\cdot 2\cdot 3\cdots n)} x^{4n} \\ &= (-1)^n \frac{1\cdot 2\cdot 3\cdots 2n}{2^{2n}\,(1\cdot 2\cdot 3\cdots n)^2} x^{4n} = (-1)^n \frac{(2n)!}{2^{2n}\,(n!)^2} x^{4n}\end{aligned}$$

Hence,

$$\int_0^{1/2} \frac{dx}{\sqrt{1+x^4}} = \int_0^{1/2} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^{4n}\,dx = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \int_0^{1/2} x^{4n}\,dx$$
$$= (-1)^n \frac{(2n)!}{2^{2n}(4n+1)(n!)^2} x^{4n+1}\Big|_0^{1/2}$$
$$= (-1)^n \frac{(2n)!}{2^{6n+1}(4n+1)(n!)^2}$$

Using (6.45) we learn that in order to evaluate the integral to three decimal places we need to add the first two terms of the power series since:

$$|R_1| = \frac{3!}{2^{13}\cdot 9} \approxeq 0.00008$$

And so

$$\int_0^{1/2} \frac{dx}{\sqrt{1+x^4}} \approxeq \sum_{n=0}^{1} (-1)^n \frac{(2n)!}{2^{6n+1}(4n+1)(n!)^2} \approxeq 0.497$$

6. Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.

   (a) A differentiable function is continuous by definition. Hence, since the derivative of $f(x)$:

   $$f'(x) = \frac{df}{dx} = \frac{d}{dx}\left(e^{-1/x^2}\right) = \frac{d}{dx}\left(-\frac{1}{x^2}\right) e^{-1/x^2} = \frac{2}{x^3} e^{-1/x^2} = P_1(x)\, f(x)$$

   is defined for all $x \neq 0$ it follows that $f(x)$ is continuous for all $x \neq 0$. Furthermore, since we have explicitly defined the value of $f(x)$ at $x = 0$ as $f(0) = 0$ and

   $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-1/x^2} = \lim_{x\to -\infty} e^{x^2} = 0 = \lim_{x\to 0^+} f(x)$$

   we may conclude that $f(x)$ is continuous at $x = 0$ and hence, that $f(x)$ is continuous for all x.

   (b) Since $f'(x) = P_1(x)f(x)$, i.e. the product of a polynomial and the original function $f(x)$ and the product of two continuous functions is itself continuous it follows at once that $f'(x)$ is continuous for all $x \neq 0$. Next, using the definition of the derivative, we find

   $$\lim_{x\to 0} f'(x) = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$$
   $$= \lim_{h\to 0} \frac{h^{-1}}{e^{1/h^2}}$$
   $$= \lim_{h\to 0} \frac{-h^{-2}}{-2h^{-3}e^{1/h^2}}$$
   $$= \lim_{h\to 0} \frac{h}{2e^{1/h^2}} = 0$$

Hence, $f'(x)$ is continuous for all x.

(c) To prove that $f^{(n)}(x)$ is continuous for all x and $f^{(n)}(0) = 0$ we will use induction. First, let us assume that $f^{(k)}(x) = P_k(x)f(x)$ for some fixed, positive integer $k \geq 0$ and some polynomial $P_k(x)$, $P_0(x) = 1$. For $k+1$ we thus have

$$\begin{aligned} f^{(k+1)}(x) = (P_k(x) f(x))' = P_k(x) f'(x) + f(x) P_k'(x) &= \frac{2}{x^3} P_k(x) f(x) + P_k'(x) f(x) \\ &= (P_1(x) P_k(x) + P_k'(x)) f(x) \\ &= P_{k+1}(x) f(x) \end{aligned}$$

Thus by induction, since the formula for $f^{(n)}(x)$ is true for fixed, positive integers k and $k+1$ it must be true for any arbitrary positive integer $n \geq 0$. Hence, since $f^{(n)}(x)$ exists for all $x \neq 0$ it follows that $f^{(n)}(x)$ is continuous for all $x \neq 0$.

To show $f^{(n)}(0) = 0$ we will again use induction. Let us assume that $f^{(k)}(0) = 0$ for some fixed, positive integer $k \geq 0$. For $k+1$ we thus have

$$\begin{aligned} \lim_{x\to 0} f^{(k+1)}(x) = f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(0+h) - f^{(k)}(0)}{h} &= \lim_{h\to 0} \frac{P_k(h) f(h) - 0}{h} \\ &= \lim_{h\to 0} \frac{P_k(h) e^{-1/h^2}}{h} \end{aligned}$$

Before we continue with attempting to evaluate the limit let us first rewrite our polynomial $P_k(x)$ as

$$P_k(x) = \sum_{n=1}^{3k} \frac{a_n}{x^n}$$

Here a_n denotes the n^{th} polynomial coefficient, which will be equal to zero in most cases and x^n is simply x raised to the n^{th} power. For example: when $k = 2$ then

$$P_2(x) = \sum_{n=1}^{6} \frac{a_n}{x^n} = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_6}{x^6} = -\frac{6}{x^4} + \frac{4}{x^6}$$

Thus, we see that in this case $a_4 = -6$, $a_6 = 4$ and $a_1 = a_2 = a_3 = a_5 = 0$. Furthermore, it should be noted that $P_k(x)$ is finite. Hence,

$$\begin{aligned} \lim_{h\to 0} \frac{P_k(h) e^{-1/h^2}}{h} &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} \sum_{n=1}^{3k} \frac{a_n}{h^n} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} \left(\frac{a_1}{h} + \frac{a_2}{h^2} + \cdots + \frac{a_{3k}}{h^{3k}} \right) \\ &= a_1 \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2} + a_2 \lim_{h\to 0} \frac{e^{-1/h^2}}{h^3} + \cdots + a_{3k} \lim_{h\to 0} \frac{e^{-1/h^2}}{h^{3k+1}} \end{aligned}$$

Let us consider the arbitrary j^{th} term of the last expression, since, if we can prove that

$$\lim_{h\to 0}\frac{e^{-1/h^2}}{h^j}=0$$

then by the additive law of limits it will follow that

$$\lim_{h\to 0}\frac{e^{-1/h^2}}{h}\sum_{n=1}^{3k}\frac{a_n}{h^n}=0$$

also. In order to prove that the limit above is equal to zero we need to consider the following one sided limits:

$$\lim_{h\to 0^-}\frac{e^{-1/h^2}}{h^j} \qquad \lim_{h\to 0^+}\frac{e^{-1/h^2}}{h^j}$$

Now instead of attempting to evaluate the above two limits it will be more convenient to evaluate the following two equivalent limits:

$$\lim_{h\to\infty}\frac{h^j}{e^{h^2}} \qquad \lim_{h\to-\infty}\frac{h^j}{e^{h^2}}$$

Because $f(x)=e^x$ is a strictly increasing function for $x>1$ it follows that $e^x<e^{x^2}$. Hence, if we can prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty \implies \lim_{x\to\infty}\frac{e^{x^2}}{x^n}=\infty$$

Expanding e^x in a Maclaurin series we note that

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\cdots+\frac{x^{n+1}}{(n+1)!}+\cdots>\frac{x^{n+1}}{(n+1)!}$$

and so

$$\frac{e^x}{x^n}>\frac{x}{(n+1)!}$$

Now since it is trivial to see that since

$$\lim_{x\to\infty}\frac{x}{(n+1)!}=\infty \implies \lim_{x\to\infty}\frac{e^x}{x^n}=\infty \implies \lim_{x\to\infty}\frac{e^{x^2}}{x^n}=\infty$$

Of course the inverse of the last expression must be equal to zero then, i.e.

$$\lim_{x\to\infty}\frac{e^{x^2}}{x^n}=\infty \implies \lim_{x\to\infty}\frac{x^n}{e^{x^2}}=0$$

Hence, we may conclude that

$$\lim_{h\to\infty}\frac{h^j}{e^{h^2}}=0$$

Now if j is even we find that

$$\lim_{h\to-\infty}\frac{h^j}{e^{h^2}}=\lim_{h\to\infty}\frac{h^j}{e^{h^2}}=0$$

If j is odd then

$$\lim_{h\to-\infty}\frac{h^j}{e^{h^2}}=-\lim_{h\to\infty}\frac{h^j}{e^{h^2}}=-1\times 0=0$$

It then follows that

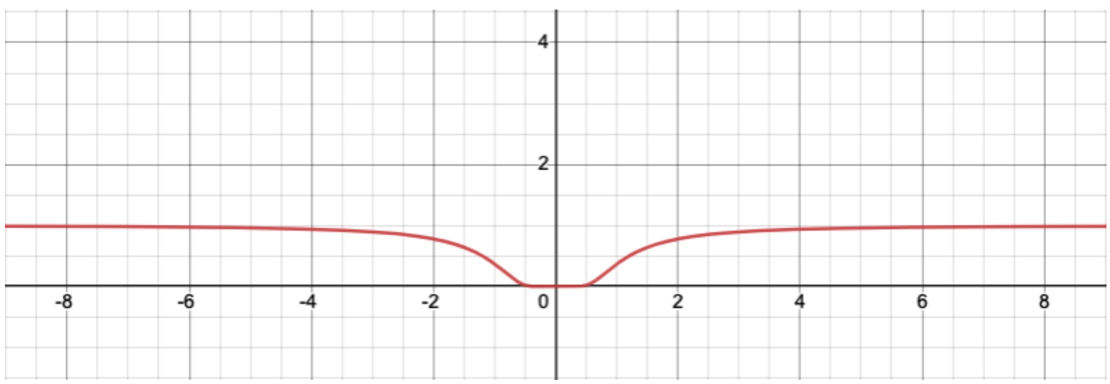
$$\lim_{h\to\infty}\frac{h^j}{e^{h^2}}=0 \implies \lim_{h\to 0}\frac{1/h^j}{e^{1/h^2}}=\lim_{h\to 0}\frac{e^{-1/h^2}}{h^j}=0$$

And so

$$\lim_{h\to 0}\frac{P_k\left(h\right)e^{-1/h^2}}{h}=\lim_{x\to 0}f^{(k+1)}\left(x\right)=f^{(k+1)}\left(0\right)=0$$

Concluding, since we have shown that $f^{(n)}(x)=P_k(x)f(x)$ is continuous for all $x\neq 0$ and $f^{(n)}(0)=0$ and $f(0)=0$, it follows that $f^{(n)}(x)$ is continuous for all x.

(d)



7. (a) Let

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}$$

be given. Furthermore, we assume that it is known that $e<3$. Then by (6.45) and (6.54), setting $a=0$, $x=1$ and acknowledging that $|f^{(6)}(1)|=e\leq M_6 \implies M_6=3$, an estimate for the error is given by

$$|R_5|<\frac{M_6|x-a|^6}{6!}=\frac{3}{6!}\approxeq 0.0042$$

(b) Let

$$\sin 1=1-\frac{1}{3!}+\frac{1}{5!}$$

be given. Furthermore, we know that $\cos(1) < 1$. Then by (6.45) and (6.54), setting $a = 0$, $x = 1$ and acknowledging that $|f^{(7)}(x)| = |-\cos(x)| = \cos(x) \leq M_7 \implies M_7 = 1$, an estimate for the error is given by

$$|R_6| < \frac{M_7|x-a|^7}{7!} = \frac{1}{7!} \approxeq 0.000198$$

(c) Let

$$\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3}$$

be given. The expression above is equal to the Maclaurin series of $\ln(1+x)$ evaluated at $x = 1/2$:

$$\begin{aligned}\ln(1+x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 0 + \frac{x}{1} - \frac{x^2}{2} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots\end{aligned}$$

Acknowledging that $|f^{(4)}(x)| = |-6/(1+x)^4| \leq M_4 \implies M_4 = 6$, an estimate for the error is given by

$$|R_3| < \frac{M_4|x-a|^4}{4!} = \frac{6}{2^4\cdot 4!} \approxeq 0.0156$$

8. The problem statement is a little screwy, as Theorem 41 clearly states the function $f(x)$ only has continuous derivatives up to the $(n+1)^{\text{st}}$ order and nothing is being said about any derivatives beyond the $(n+1)^{\text{st}}$ order. In any case, we will instead provide a proof of the rule deduced in Section 2.19: *Let $f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0$, but $f^{(n+1)}(a) \neq 0$. Then $f(x)$ has a relative maximum at $x = a$ if n is odd and $f^{(n+1)}(a) < 0$; $f(x)$ has a relative minimum at $x = a$ if n is odd and $f^{(n+1)}(a) > 0$; $f(x)$ has neither relative maximum nor relative minimum at $x = a$, but a horizontal inflection point at $x = a$ if n is even* in order to avoid any further confusion.

Let a function $f(x)$ be defined and continuous and have continuous derivatives up the $(n+1)^{\text{st}}$ order for $a - r_0 < x < a + r_0$. Then for each x of this interval except $x = a$:

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(x_1)}{(n+1)!}(x-a)^{n+1}$$

for some x_1 such that $a < x_1 < x$ or, if $x < a$, $x < x_1 < a$. Furthermore, let $f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0$, but $f^{(n+1)}(a) \neq 0$. Hence, the expression for $f(x)$ reduces to

$$f(x) = f(a) + \frac{f^{(n+1)}(x_1)}{(n+1)!}(x-a)^{n+1} \iff f(x) - f(a) = \frac{f^{(n+1)}(x_1)}{(n+1)!}(x-a)^{n+1}$$

Let us first consider the case where n is odd, such that $(x-a)^{n+1} > 0$ always. Then $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$ for $x_1 > a$ and x_1 sufficiently close to a. Hence, $f(x) - f(a) > 0$ for $x > a$ and x sufficiently close to a. Similarly, $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$ for $x_1 < a$ and x_1 sufficiently close to a. Again, $f(x) - f(a) > 0$. However, now for $x < a$ and x sufficiently close to a. Combined with the knowledge that $f'(a) = 0$ we thus may conclude that $f(x)$ has a relative minimum at the point $x = a$. Still assuming n is odd, then when $f^{(n+1)}(a) < 0 \implies f^{(n+1)}(x_1) < 0$ for $x_1 > a$ and x_1 sufficiently close to a. Hence, $f(x) - f(a) < 0$ for $x > a$ and x sufficiently close to a. Similarly, $f^{(n+1)}(a) < 0 \implies f^{(n+1)}(x_1) < 0$ for $x_1 < a$ and x_1 sufficiently close to a. Again, $f(x) - f(a) < 0$, However, now for $x < a$ and x sufficiently close to a. Combined with the knowledge that $f'(a) = 0$ we thus may conclude that $f(x)$ has a relative maximum at the point $x = a$.

Lastly, we consider the case where n is even. Now the sign of $(x-a)^{n+1}$ will depend on whether $x > a$ or $x < a$. Assuming that $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$ for $x_1 > a$ and x_1 sufficiently close to a. Hence, $f(x) - f(a) > 0$ for $x > a$ and x sufficiently close to a because $(x-a)^{n+1} > 0$. So far there is no difference with the previously analysed case when n is odd and $x > a$. However, when $x_1 < a$ for x_1 sufficiently close to a we still have $f^{(n+1)}(a) > 0 \implies f^{(n+1)}(x_1) > 0$, but now $f(x) - f(a) < 0$ for $x < a$ and x sufficiently close to a because $(x-a)^{n+1} < 0$. Thus the function $f(x)$ has a horizontal inflection point at a.

9. (a) Let

$$g_1(x,h) = \frac{f(x+h) - f(x)}{h}$$

be given, where it is assumed h is a small, positive number. Taylor's formula with remainder gives the series expansion

$$f(x+h) = f(x) + h\frac{f'(x)}{1!} + h^2\frac{f''(x_1)}{2!}$$

for some x_1 such that $x < x_1 < x + h$. Rearranging gives

$$\frac{f(x+h) - f(x)}{h} - f'(x) = g_1(x,h) - f'(x) = h\frac{f''(x_1)}{2!}$$

And so

$$\lim_{h\to 0} h\frac{f''(x_1)}{2!} = 0 \implies \lim_{h\to 0} g_1(x,h) - f'(x) = 0$$

In a similar way we find that

$$\frac{g_1(x,h) - f'(x)}{h} = \frac{f''(x_1)}{2!}$$

Hence,

$$\lim_{h\to 0}\frac{f''(x_1)}{2!} = \frac{f''(x_1)}{2} \implies \lim_{h\to 0}\frac{g_1(x,h) - f'(x)}{h} = \frac{f''(x_1)}{2}$$

(b) Let

$$g_2\left(x,h\right) = \frac{f\left(x+h\right) - f\left(x-h\right)}{2h}$$

be given. Taylor's formula with remainder gives the series expansions

$$\begin{aligned}
f\left(x+h\right) &= f\left(x\right) + h\frac{f'\left(x\right)}{1!} + h^2\frac{f''\left(x\right)}{2!} + h^3\frac{f'''\left(x_1\right)}{3!} \\
f\left(x-h\right) &= f\left(x\right) - h\frac{f'\left(x\right)}{1!} + h^2\frac{f''\left(x\right)}{2!} - h^3\frac{f'''\left(x_2\right)}{3!}
\end{aligned}$$

for some x_1 such that $x < x_1 < x+h$ and some x_2 such that $x-h < x_2 < x$. Rearranging, subtracting both equations and finally dividing by a factor of $2h$ gives

$$\frac{f\left(x+h\right) - f\left(x-h\right)}{2h} - f'\left(x\right) = g_2\left(x,h\right) - f'\left(x\right) = h^2\left(\frac{f'''\left(x_1\right)}{12} + \frac{f'''\left(x_2\right)}{12}\right)$$

And so

$$\lim_{h\to 0} h^2\left(\frac{f'''\left(x_1\right)}{12} + \frac{f'''\left(x_2\right)}{12}\right) = 0 \implies \lim_{h\to 0} g_2\left(x,h\right) - f'\left(x\right) = 0$$

Similarly, we find that

$$\frac{g_2\left(x,h\right) - f'\left(x\right)}{h^2} = \frac{f'''\left(x_1\right)}{12} + \frac{f'''\left(x_2\right)}{12}$$

Hence,

$$\lim_{h\to 0} \frac{f'''\left(x_1\right)}{12} + \frac{f'''\left(x_2\right)}{12} \implies \lim_{h\to 0} \frac{g_2\left(x,h\right) - f'\left(x\right)}{h^2} = \frac{f'''\left(x_1\right)}{12} + \frac{f'''\left(x_2\right)}{12}$$

(c) Let

$$g_3\left(x,h\right) = \frac{f\left(x+h\right) - 2f\left(x\right) + f\left(x-h\right)}{h^2}$$

be given. Taylor's formula with remainder gives the series expansions

$$\begin{aligned}
f\left(x+h\right) &= f\left(x\right) + h\frac{f'\left(x\right)}{1!} + h^2\frac{f''\left(x\right)}{2!} + h^3\frac{f'''\left(x_1\right)}{3!} + h^4\frac{f''''\left(x_1\right)}{4!} \\
f\left(x-h\right) &= f\left(x\right) - h\frac{f'\left(x\right)}{1!} + h^2\frac{f''\left(x\right)}{2!} - h^3\frac{f'''\left(x_1\right)}{3!} + h^4\frac{f''''\left(x_2\right)}{4!}
\end{aligned}$$

for some x_1 such that $x < x_1 < x+h$ and some x_2 such that $x-h < x_2 < x$. Rearranging, adding both equations and finally dividing by a factor of h^2 gives

$$\frac{f\left(x+h\right) - 2f\left(x\right) + f\left(x-h\right)}{h^2} - f''\left(x\right) = g_3\left(x,h\right) - f''\left(x\right) = h^2\left(\frac{f''''\left(x_1\right)}{24} + \frac{f''''\left(x_2\right)}{24}\right)$$

And so

$$\lim_{h\to 0} h^2 \left(\frac{f''''(x_1)}{24} + \frac{f''''(x_2)}{24} \right) = 0 \implies \lim_{h\to 0} g_3(x,h) - f''(x) = 0$$

Note that the book divides by a factor of h instead of h^2. This would *not* result in a finite limit when $h \to 0$ for the expression $[g_3(x,h) - f''(x)]/h$. This is most likely a mistake and we will instead divide by a factor of h^2. And so we find that

$$\frac{g_3(x,h) - f''(x)}{h^2} = \frac{f''''(x_1)}{24} + \frac{f''''(x_2)}{24}$$

Hence,

$$\lim_{h\to 0} \frac{f''''(x_1)}{24} + \frac{f''''(x_2)}{24} \implies \lim_{h\to 0} \frac{g_3(x,h) - f''(x)}{h^2} = \frac{f''''(x_1)}{24} + \frac{f''''(x_2)}{24}$$

Thus, the error in approximating $f''(x)$ by $g_3(x,h)$ is *not* of order h, but of order h^2.

Section 6.19

1. (a) Let $z = i$ be given. Then

$$\lim_{n\to\infty} \frac{z^n}{n} = \lim_{n\to\infty} \frac{i^n}{n}$$

Now

$$\lim_{n\to\infty} i^n$$

is undefined, since i^n keeps cycling between the values $i, -1, -i$ and 1. However, it is true that

$$\lim_{n\to\infty} \left| \frac{z^n}{n} \right| = \lim_{n\to\infty} \frac{|z^n|}{n} = \lim_{n\to\infty} \frac{|z|^n}{n} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

Can we thus conclude that $\lim_{n\to\infty} z^n/n = \lim_{n\to\infty} i^n/n = 0$? The answer is yes, since Theorem 46 is satisfied when taking $z_n = i^n/n$ and $z_0 = 0$ for then there will exists an N such that when $n > N$ for some $\epsilon > 0$ then $|z_n - z_0| = |z_n| = 1/n < \epsilon \implies n > 1/\epsilon$.

(b)

$$\begin{aligned}\lim_{n\to\infty} \frac{(1+i)\,n^3 - 2in + 3}{in^3 - 1} = \lim_{n\to\infty} \frac{1 + i - 2i/n^2 + 3/n^3}{i - 1/n^3} = \frac{1+i}{i} = \frac{1}{i} + 1 &= \frac{i}{i^2} + 1 \\ &= 1 - i\end{aligned}$$

2. (a) Let the series

$$\frac{1+i}{2}+\left(\frac{1+i}{2}\right)^2+\cdots+\left(\frac{1+i}{2}\right)^n+\cdots=\sum_{n=1}^{\infty}\left(\frac{1+i}{2}\right)^n$$

be given. Employing the root test gives

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{1+i}{2}\right)^n\right|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{1+i}{2}\right|^n}=\left|\frac{1+i}{2}\right|=\frac{|1+i|}{2}=\frac{1}{\sqrt{2}}<1$$

Hence, the series is absolutely convergent.

(b) Let the series

$$\sum_{n=1}^{\infty} ni^n$$

be given. Employing the n^{th} term test gives

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty} ni^n=\infty\cdot i^{\infty}\neq 0$$

Hence, the series diverges.

(c) Let the series

$$\sum_{n=1}^{\infty}\frac{ni^n}{n^2+1}$$

be given. Now

$$|a_n|=\left|\frac{ni^n}{n^2+1}\right|=\frac{n}{n^2+1}|i^n|=\frac{n}{n^2+1}|i|^n=\frac{n}{n^2+1}<\frac{1}{n}=b_n$$

for $n=1,2,\ldots$. The series $\sum_{n=1}^{\infty} 1/n$ diverges and hence, the series $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent. However, the series of real parts is

$$0-\frac{1}{2}+0+\frac{1}{4}+0-\frac{1}{6}+\cdots$$

and the series of imaginary parts is

$$1+0-\frac{1}{3}+0+\frac{1}{5}+\cdots$$

If the zeros are disregarded, these are convergent alternating series. Hence, $\sum_{n=1}^{\infty} i^n/n$ converges and since

$$\frac{ni^n}{n^2+1}<\frac{i^n}{n}$$

the series $\sum_{n=1}^{\infty} a_n$ converges.

(d) Let the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)^2}$$

be given. Since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{1}{(n+i)^2} \right| = \sum_{n=1}^{\infty} \frac{1}{|\,(n+i)^2\,|} = \sum_{n=1}^{\infty} \frac{1}{|n+i|^2} = \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} 1/n^2$ converges implies the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

3. Let the series

$$e^z = 1 + z + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

be given. Employing the ratio test gives

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = |z| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Hence, the series converges for all z.

Let the series

$$\sin z = z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$$

be given. Employing the ratio test gives

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{z^{2n+1}}{(2n+1)!} \frac{(2n-1)!}{z^{2n-1}} \right| &= \lim_{n\to\infty} \left| \frac{z^2}{2n\,(2n+1)} \right| \\ &= |z|^2 \lim_{n\to\infty} \frac{1}{2n\,(2n+1)} \\ &= 0 < 1 \end{aligned}$$

Hence, the series converges for all z.

Let the series

$$\cos z = 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

be given. Employing the ratio test gives

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{z^{2n+2}}{(2n+2)!} \frac{(2n)!}{z^{2n}} \right| &= \lim_{n\to\infty} \left| \frac{z^2}{(2n+2)\,(2n+1)} \right| \\ &= |z|^2 \lim_{n\to\infty} \frac{1}{(2n+2)\,(2n+1)} \\ &= 0 < 1 \end{aligned}$$

Hence, the series converges for all z.

4. (a) Setting $z = iy$ in (6.57) we find

$$
\begin{aligned}
e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots + \frac{(iy)^n}{n!} + \cdots \\
&= \left(1 - \frac{y^2}{2!} + \cdots + (-1)^n \frac{y^{2n}}{(2n)!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \cdots + (-1)^{n-1} \frac{y^{2n-1}}{(2n-1)!} + \cdots\right) \\
&= \cos y + i \sin y
\end{aligned}
$$

where the last equality follows from (6.59) and (6.58).

(b)

$$
e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)
$$

where the last expression follows from (a) directly.
Setting $z = x + iy$ in (6.57) we find

$$
e^{x+iy} = 1 + (x+iy) + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \cdots + \frac{(x+iy)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!}
$$

The term $(x + iy)^n$ can be expanded using the Binomial formula:

$$
(x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (iy)^{n-k}
$$

And so

$$
\begin{aligned}
e^{x+iy} = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k (iy)^{n-k} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k (iy)^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=k}^{\infty} \frac{(iy)^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\
&= e^x e^{iy} = e^x (\cos y + i \sin y)
\end{aligned}
$$

5. (a) By (6.57)

$$
\begin{aligned}
\frac{e^{iz}+e^{-iz}}{2} &= \frac{1}{2}\left(1+iz+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\cdots+\frac{(iz)^n}{n!}+\cdots\right) \\
&\quad+\frac{1}{2}\left(1-iz+\frac{(-iz)^2}{2!}+\frac{(-iz)^3}{3!}+\cdots+\frac{(-iz)^n}{n!}+\cdots\right) \\
&= 1+\frac{(iz)^2}{2\cdot 2!}+\frac{(-iz)^2}{2\cdot 2!}+\cdots+\frac{(iz)^n}{2\cdot n!}+\frac{(-iz)^n}{2\cdot n!}+\cdots \\
&= 1-\frac{z^2}{2!}+\cdots+(-1)^n\,\frac{z^{2n}}{(2n)!}+\cdots \\
&= \cos z
\end{aligned}
$$

(b) By (6.57)

$$
\begin{aligned}
\frac{e^{iz}-e^{-iz}}{2i} &= \frac{1}{2i}\left(1+iz+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\cdots+\frac{(iz)^n}{n!}+\cdots\right) \\
&\quad-\frac{1}{2i}\left(1-iz+\frac{(-iz)^2}{2!}+\frac{(-iz)^3}{3!}+\cdots+\frac{(-iz)^n}{n!}+\cdots\right) \\
&= z+\frac{(iz)^2}{2\cdot 2!}+\frac{(-iz)^2}{2\cdot 2!}+\cdots+\frac{(iz)^n}{2\cdot n!}+\frac{(-iz)^n}{2\cdot n!}+\cdots \\
&= z-\frac{z^3}{3!}+\cdots+(-1)^{n-1}\,\frac{z^{2n-1}}{(2n-1)!}+\cdots \\
&= \sin z
\end{aligned}
$$

(c) By (6.57)

$$
\begin{aligned}
e^{z_1+z_2} = 1 + (z_1+z_2) + \cdots + \frac{(z_1+z_2)^n}{n!} + \cdots &= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{n=k}^{\infty} \frac{z_2^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{n=0}^{\infty} \frac{z_2^n}{n!} = e^{z_1} e^{z_2}
\end{aligned}
$$

(d) By (6.58)

$$
\begin{aligned}
\sin(-z) &= (-z) - \frac{(-z)^3}{3!} + \cdots + (-1)^{n-1} \frac{(-z)^{2n-1}}{(2n-1)!} + \cdots \\
&= -z + \frac{z^3}{3!} - \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots \\
&= -\left(z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right) = -\sin z
\end{aligned}
$$

(e) By (6.59)

$$
\begin{aligned}
\cos(-z) &= 1 - \frac{(-z)^2}{2!} + \cdots + (-1)^n \frac{(-z)^{2n}}{(2n)!} + \cdots \\
&= 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots = \cos z
\end{aligned}
$$

(f) By (6.58) and (6.59) and using the Cauchy product

$$
\left(\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \right)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^k}{k!} (-1)^{n-k} \frac{z^{n-k}}{(n-k)!}
$$

$$
\begin{aligned}
\sin^2 z + \cos^2 z &= \left(z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \right)^2 \\
&\quad + \left(1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \right)^2 \\
&= \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \right)^2 + \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right)^2 + \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right)^2 \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{z^{2(n-k)+1}}{[2(n-k)+1]!} \\
&\quad + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^{2k}}{(2k)!} (-1)^{n-k} \frac{z^{2(n-k)}}{[2(n-k)]!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n+2}{2k+1} \frac{z^{2n+2}}{(2n+2)!} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-1} \binom{2n}{2k+1} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \left[\sum_{k=0}^{n} \binom{2n}{2k} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \right] \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n (1-1)^{2n} \frac{z^{2n}}{(2n)!} = 1
\end{aligned}
$$

where in the last step we have made use of the Binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

with $x = 1$ and $y = -1$.

(g) By (6.58) and (6.59)

$$
\begin{aligned}
\cos^2 z - \sin^2 z &= \left(1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots\right)^2 \\
&\quad - \left(z - \frac{z^3}{3!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots\right)^2 \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right)^2 - \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}\right)^2 \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right)^2 - \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right)^2 \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^{2k}}{(2k)!} (-1)^{n-k} \frac{z^{2(n-k)}}{[2(n-k)]!} \\
&\quad - \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{z^{2(n-k)+1}}{[2(n-k)+1]!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n+2}{2k+1} \frac{z^{2n+2}}{(2n+2)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{2n}{2k} \frac{z^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-1} \binom{2n}{2k+1} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \left[\sum_{k=0}^{n} \binom{2n}{2k} + \sum_{k=0}^{n-1} \binom{2n}{2k+1}\right] \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{2n} \binom{2n}{k} \frac{z^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^n (1+1)^{2n} \frac{z^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!} \\
&= 1 - \frac{(2z)^2}{2!} + \cdots + (-1)^n \frac{(2z)^{2n}}{(2n)!} + \cdots = \cos 2z
\end{aligned}
$$

(h) By (6.58) and (6.59)

$$\begin{aligned}
2\sin z\cos z &= 2\left(z-\frac{z^3}{3!}+\cdots+(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+\cdots\right)\\
&\quad\times\left(1-\frac{z^2}{2!}+\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots\right)\\
&= 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}\\
&= 2\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}\\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{z^{2k+1}}{(2k+1)!}(-1)^{n-k}\frac{z^{2(n-k)}}{[2(n-k)]!}\\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n\binom{2n+1}{2k+1}\frac{z^{2n+1}}{(2n+1)!}\\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n\left[\binom{2n}{2k}+\binom{2n}{2k+1}\right]\frac{z^{2n+1}}{(2n+1)!}\\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{2n}(-1)^n\binom{2n}{k}\frac{z^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty}(-1)^n(1+1)^{2n}\frac{z^{2n+1}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2z)^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2z)^{2n-1}}{(2n-1)!}\\
&= 2z-\frac{(2z)^3}{3!}+\cdots+(-1)^{n-1}\frac{(2z)^{2n-1}}{(2n-1)!}+\cdots = \sin 2z
\end{aligned}$$

where we have made use of the relation

$$\binom{n}{k} = \binom{n-1}{k-1}+\binom{n-1}{k}$$

6. (a) Let the series

$$z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}+\cdots = \sum_{n=1}^{\infty}\frac{z^n}{n}$$

be given. By the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{n+1}\frac{n}{z^n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|\lim_{n\to\infty}\frac{n}{n+1} = |z|\lim_{n\to\infty}\frac{1}{1+1/n} = |z|$$

Hence, the series converges for $|z|<1$ and diverges for $|z|>1$.

(b) Let the series

$$1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n$$

where $z = x + iy$ be given. By the extended ratio test we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{n+1}}{z^n}\right| = |z| = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+y^2}$$

Hence, the series converges for $|z| = \sqrt{x^2+y^2} < 1$ and diverges for $|z| > 1$.

Section 6.21

1. (a) Let $F(x,y) = e^{x^2-y^2}$. Now let us make the substitution $t = x^2-y^2 \implies F(t) = e^t$. Expanding $F(t)$ in a Taylor series about the point $(0,0)$, using (6.46), and finally substituting back for t in terms of x, y gives

$$\begin{aligned} F(t) &= 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots \\ &= 1 + \left(x^2-y^2\right) + \frac{1}{2!}\left(x^2-y^2\right)^2 + \cdots + \frac{1}{n!}\left(x^2-y^2\right)^n + \cdots \end{aligned}$$

Since (6.46) converges for all x it follows that $F(x,y) = F(t)$ converges for all t and hence, $(x,y) \in \mathbb{R}$.

(b) Let $F(x,y) = \sin(xy)$. Again, let us make a substitution, this time of the form $t = xy$. Expanding $F(t)$ in a Taylor series about the point $(0,0)$, using (6.47), and finally substituting back for t in terms of x, y gives

$$\begin{aligned} F(t) &= t - \frac{t^3}{3!} + \cdots + (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!} + \cdots \\ &= xy - \frac{(xy)^3}{3!} + \cdots + (-1)^{n-1}\frac{(xy)^{2n-1}}{(2n-1)!} + \cdots \end{aligned}$$

Since (6.47) converges for all x it follows that $F(x,y) = F(t)$ converges for all t and hence, $(x,y) \in \mathbb{R}$.

(c) Let $F(x,y) = 1/(1-x-y)$. Let us make the substitution $t = x+y$. Using (6.17), setting $a = 1$ and $r = t$, and finally substituting back for t in terms of x, y we thus find

$$\begin{aligned} F(t) &= 1 + t + t^2 + \cdots + t^n + \cdots \\ &= 1 + (x+y) + (x+y)^2 + \cdots + (x+y)^n + \cdots \end{aligned}$$

Again, from (6.17) it follows that this series converges for $-1 < t < 1 \implies -1 < x+y < 1$.

(d) Let $F(x,y) = 1/(1-x-y-z)$. Let us make the substitution $t = x+y+z$. Using (6.17), setting $a = 1$ and $r = t$, and finally substituting back for t in terms of x, y, z we thus find

$$\begin{aligned} F(t) &= 1 + t + t^2 + \cdots + t^n + \cdots \\ &= 1 + (x+y+z) + (x+y+z)^2 + \cdots + (x+y+z)^n + \cdots \end{aligned}$$

Again, from (6.17) it follows that this series converges for $-1 < t < 1 \implies -1 < x+y+z < 1$.

2. Let

$$\phi(t) = F\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right]$$

where $0 \leq t \leq 1$. Hence, as stated in Section 6.21, it follows that

$$\begin{aligned} \phi'(t) &= (x - x_1) F_x \left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] + (y - y_1) F_y \left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \\ &= dF\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \end{aligned}$$

As such, we assume that in general

$$\begin{aligned} \phi^{(k)}(t) &= (x - x_1)^k \frac{\partial^k F}{\partial x^k}\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] + \cdots \\ &= \sum_{r=0}^{k} \binom{k}{r} (x - x_1)^r (y - y_1)^{k-r} \frac{\partial^k F}{\partial x^r \partial y^{k-r}} \left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \\ &= d^k F\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \end{aligned}$$

for some arbitrary, fixed integer $k \geq 0$. For $k+1$ we find

$$\begin{aligned} \phi^{(k+1)}(t) &= \left[\phi^{(k)}(t)\right]' \\ &= \left[(x - x_1)^k \frac{\partial^k F}{\partial x^k}\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] + \cdots\right]' \\ &= (x - x_1)^{k+1} \frac{\partial^{k+1} F}{\partial x^{k+1}}\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \\ &\quad + (x - x_1)^k (y - y_1) \frac{\partial^{k+1} F}{\partial x^k \partial y}\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] + \cdots \\ &= \sum_{r=0}^{k+1} \binom{k+1}{r} (x - x_1)^k (y - y_1)^{k+1-r} \frac{\partial^{k+1} F}{\partial x^r \partial y^k}\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \\ &= d^{k+1} F\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right] \end{aligned}$$

And so by induction the equation

$$\phi^{(n)}(t) = d^n F\left[x_1 + t(x - x_1), y_1 + t(y - y_1)\right]$$

must be true for any positive integer n.

3. Let it be a given that a power series

$$c_{0,0} + (c_{1,0}x + c_{1,1}y) + \left(c_{2,0}x^2 + c_{2,1}xy + c_{2,2}y^2\right) + \cdots$$

converges absolutely at a point (x_0, y_0).

Note: the problem statement only speaks of convergence of a power series at the point (x_0, y_0). However, in itself this is not a sufficient enough condition to prove convergence at every point $(\lambda x_0, \lambda y_0)$, for $|\lambda < 1|$. We require absolute convergence, since absolute convergence implies convergence, but the inverse is not necessarily true.

The n^{th} term of this power series may be written as

$$u_n(x, y) = \sum_{r=0}^{n} \binom{n}{r} c_{n,r} x^{n-r} y^r$$

Then it follows that if $|\lambda| < 1$ it must be true that

$$\begin{aligned}
|u_n(\lambda x, \lambda y)| = \left| \sum_{r=0}^{n} \binom{n}{r} c_{n,r} (\lambda x)^{n-r} (\lambda y)^r \right| &= \left| \sum_{r=0}^{n} \binom{n}{r} c_{n,r} \lambda^n x^{n-r} y^r \right| \\
&= \sum_{r=0}^{n} \binom{n}{r} |\lambda|^n |c_{n,r} x^{n-r} y^r| \\
&< \sum_{r=0}^{n} \binom{n}{r} |c_{n,r} x^{n-r} y^r| \\
&= \left| \sum_{r=0}^{n} \binom{n}{r} c_{n,r} x^{n-r} y^r \right| = |u_n(x, y)|
\end{aligned}$$

for all x. Hence, since $u_n(x, y)$ is absolutely convergent at the point (x_0, y_0) and we have shown that $|u_n(\lambda x_0, \lambda y_0)| < |u_n(x_0, y_0)|$, for $|\lambda| < 1$, it follows at once that it is also absolutely convergent (and hence, convergent) at every point $(\lambda x_0, \lambda y_0)$, for $|\lambda| < 1$.

4. From Problem 1(b) it follows that we can write the integral $\int_0^1 \int_0^1 \sin(xy)\, dxdy$ as

$$\begin{aligned}
\int_0^1 \int_0^1 \sin(xy)\, dxdy &= \int_0^1 \int_0^1 \left[xy - \frac{(xy)^3}{3!} + \cdots + (-1)^{n-1} \frac{(xy)^{2n-1}}{(2n-1)!} + \cdots \right] dxdy \\
&= \int_0^1 \left[\frac{y}{2} - \frac{y^3}{4 \cdot 3!} + \cdots + (-1)^{n-1} \frac{y^{2n-1}}{2n(2n-1)} + \cdots \right] dy \\
&= \frac{1}{4} - \frac{1}{16 \cdot 3!} + \cdots + (-1)^{n-1} \frac{1}{4n^2(2n-1)} + \cdots \approxeq 0.240
\end{aligned}$$

5. (a) Taylor's formula with remainder for functions of several variables (6.67) gives the series expansions

$$\begin{aligned}
f(x+h,y+h) &= f(x,y) + h\left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\right) \\
&\quad + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_1^*,y_1^*) + 2\frac{\partial^2 f}{\partial x \partial y}(x_1^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_1^*)\right) \\
f(x+h,y-h) &= f(x,y) + h\left(\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right) \\
&\quad + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_1^*,y_2^*) - 2\frac{\partial^2 f}{\partial x \partial y}(x_1^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_2^*)\right) \\
f(x-h,y+h) &= f(x,y) - h\left(\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right) \\
&\quad + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_2^*,y_1^*) - 2\frac{\partial^2 f}{\partial x \partial y}(x_2^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_1^*)\right) \\
f(x-h,y-h) &= f(x,y) - h\left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\right) \\
&\quad + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_2^*,y_2^*) + 2\frac{\partial^2 f}{\partial x \partial y}(x_2^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_2^*)\right)
\end{aligned}$$

where $x_1^* = x + t^*h$, $y_1^* = y + t^*h$ and $x_2^* = x - t^*h$, $y_2^* = y - t^*h$ respectively for $0 < t^* < 1$. Adding these 4 equations, rearranging and dividing by a factor of h^2 gives

$$\begin{aligned}
&\frac{f(x+h,y+h) + f(x+h,y-h)}{h^2} + \frac{f(x-h,y+h) + f(x-h,y-h) - 4f(x,y)}{h^2} = \\
&g_1(x,y,h) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_1^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_1^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_2^*)\right) + \\
&\qquad \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_2^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_2^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_2^*)\right)
\end{aligned}$$

Taking the limit when $h \to 0$ of the right-hand side results in

$$\begin{aligned}
&\frac{1}{2}\lim_{h\to 0}\left(\frac{\partial^2 f}{\partial x^2}(x_1^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_1^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_1^*,y_2^*)\right) + \\
&\quad \frac{1}{2}\lim_{h\to 0}\left(\frac{\partial^2 f}{\partial x^2}(x_2^*,y_1^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_1^*) + \frac{\partial^2 f}{\partial x^2}(x_2^*,y_2^*) + \frac{\partial^2 f}{\partial y^2}(x_2^*,y_2^*)\right) = \\
&\qquad 2\left(\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right)
\end{aligned}$$

since $\lim_{h\to 0} x_1^* = \lim_{h\to 0} x_2^* = x$ and $\lim_{h\to 0} y_1^* = \lim_{h\to 0} y_2^* = y$. Hence, we find that

$$\frac{1}{2}\lim_{h\to 0} g_1(x,y,h) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \nabla^2 f(x,y)$$

(b) Adding the first and last equations and subtracting the second and third equations from (a), rearranging and dividing by a factor of $4h^2$ gives

$$\frac{f(x+h,y+h)-f(x+h,y-h)-f(x-h,y+h)+f(x-h,y-h)}{4h^2}=g_2(x,y,h)=$$
$$\frac{1}{4}\left(\frac{\partial^2 f}{\partial x\partial y}(x_1^*,y_1^*)+\frac{\partial^2 f}{\partial x\partial y}(x_1^*,y_2^*)+\frac{\partial^2 f}{\partial x\partial y}(x_2^*,y_1^*)+\frac{\partial^2 f}{\partial x\partial y}(x_2^*,y_2^*)\right)$$

Taking the limit when $h\to 0$ of the right-hand side results in

$$\frac{1}{4}\lim_{h\to 0}\left(\frac{\partial^2 f}{\partial x\partial y}(x_1^*,y_1^*)+\frac{\partial^2 f}{\partial x\partial y}(x_1^*,y_2^*)+\frac{\partial^2 f}{\partial x\partial y}(x_2^*,y_1^*)+\frac{\partial^2 f}{\partial x\partial y}(x_2^*,y_2^*)\right)=\frac{\partial^2 f}{\partial x\partial y}(x,y)$$

since $\lim_{h\to 0}x_1^*=\lim_{h\to 0}x_2^*=x$ and $\lim_{h\to 0}y_1^*=\lim_{h\to 0}y_2^*=y$. Hence, we find that

$$\lim_{h\to 0}g_2(x,y,h)=\frac{\partial^2 f}{\partial x\partial y}(x,y)$$

6. We will start by writing

$$d^2F=F_{uu}(x(u,v),y(u,v))\,du^2+2F_{uv}(x(u,v),y(u,v))\,dudv+F_{vv}(x(u,v),y(u,v))\,dv^2$$

Next, we will make use of (2.133) and (2.132) in order to write out each term explicitly:

$$F_{uu}(x(u,v),y(u,v))=F_x\frac{\partial^2 x}{\partial u^2}+F_{xx}\left(\frac{\partial x}{\partial u}\right)^2+2F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}+F_{yy}\left(\frac{\partial y}{\partial u}\right)^2+F_y\frac{\partial^2 y}{\partial u^2}$$
$$F_{vv}(x(u,v),y(u,v))=F_x\frac{\partial^2 x}{\partial v^2}+F_{xx}\left(\frac{\partial x}{\partial v}\right)^2+2F_{xy}\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}+F_{yy}\left(\frac{\partial y}{\partial v}\right)^2+F_y\frac{\partial^2 y}{\partial v^2}$$
$$F_{uv}(x(u,v),y(u,v))=F_{xx}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+F_{xy}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}+F_x\frac{\partial^2 x}{\partial u\partial v}+F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}+F_{yy}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+F_y\frac{\partial^2 y}{\partial u\partial v}$$

Now at each point (u_1,v_1) at which $x=x_1$, $y=y_1$ and $F_x(x_1,y_1)=F_y(x_1,y_1)=0$ the terms above will reduce to

$$F_{uu}(x(u,v),y(u,v))=F_{xx}\left(\frac{\partial x}{\partial u}\right)^2+2F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}+F_{yy}\left(\frac{\partial y}{\partial u}\right)^2$$
$$F_{vv}(x(u,v),y(u,v))=F_{xx}\left(\frac{\partial x}{\partial v}\right)^2+2F_{xy}\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}+F_{yy}\left(\frac{\partial y}{\partial v}\right)^2$$
$$F_{uv}(x(u,v),y(u,v))=F_{xx}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+F_{xy}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}+F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}+F_{yy}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}$$

And so we end up with

$$
\begin{aligned}
d^2F &= F_{uu}\left(x\left(u,v\right),y\left(u,v\right)\right)\,du^2 + 2F_{uv}\left(x\left(u,v\right),y\left(u,v\right)\right)\,dudv + F_{vv}\left(x\left(u,v\right),y\left(u,v\right)\right)\,dv^2 \\
&= \left[F_{xx}\left(\frac{\partial x}{\partial u}\right)^2 + 2F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} + F_{yy}\left(\frac{\partial y}{\partial u}\right)^2\right]\,du^2 \\
&\quad + 2\left[F_{xx}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + F_{xy}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} + F_{xy}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + F_{yy}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right]\,dudv \\
&\quad + \left[F_{xx}\left(\frac{\partial x}{\partial v}\right)^2 + 2F_{xy}\frac{\partial x}{\partial v}\frac{\partial y}{\partial v} + F_{yy}\left(\frac{\partial y}{\partial v}\right)^2\right]\,dv^2 \\
&= F_{xx}\left(x\left(u,v\right),y\left(u,v\right)\right)\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right)^2 + F_{yy}\left(x\left(u,v\right),y\left(u,v\right)\right)\left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right)^2 \\
&\quad + 2F_{xy}\left(x\left(u,v\right),y\left(u,v\right)\right)\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right)\left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right) \\
&= F_{xx}\left(x,y\right)\,dx^2 + 2F_{xy}\left(x,y\right)\,dxdy + F_{yy}\left(x,y\right)\,dv^2
\end{aligned}
$$

Section 6.25

1. (a) Let the integral $\int_1^\infty f(x)\,dx = \int_1^\infty e^{\sin x}/x\,dx$ be given. Let us define $g(x) = e^{-1}/x = 1/ex$ such that $0 \leq g(x) \leq f(x)$. Now we will employ Theorem 54 (Comparison Test) to check for divergence. Since

$$\int_1^\infty g\left(x\right)\,dx = \frac{1}{e}\int_1^\infty \frac{1}{x}\,dx = \frac{1}{e}\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \frac{1}{e}\lim_{b\to\infty}\ln|b| = \infty$$

it follows that the integral $\int_1^\infty f(x)\,dx$ diverges as well.

(b) Let the integral $\int_1^\infty f(x)\,dx = \int_1^\infty dx/(\ln x)^x$ be given. Since $f(x) = |f(x)|$ for $1 < x < \infty$ and it is true that $\ln x \geq 2$ whenever $x \geq e^2$ it is also true that

$$\int_1^\infty \frac{dx}{\left(\ln x\right)^x} < \int_1^{e^2} \frac{dx}{\left(\ln x\right)^x} + \int_{e^2}^\infty \frac{dx}{2^x} = C + \int_{e^2}^\infty \frac{dx}{2^x} = C + \int_{e^2}^\infty g\left(x\right)\,dx$$

where C is a finite constant. Next, we find that

$$
\begin{aligned}
\int_{e^2}^\infty g\left(x\right)\,dx = \int_{e^2}^\infty \frac{dx}{2^x} = \lim_{b\to\infty}\int_{e^2}^b \frac{dx}{e^{x\ln 2}} = \frac{1}{\ln 2}\lim_{b\to\infty}\int_{e^2\ln 2}^b \frac{du}{e^u} &= \frac{1}{\ln 2}\lim_{b\to\infty}\left.\frac{1}{e^u}\right|_{e^2\ln 2}^b \\
&= \frac{1}{\ln 2}\lim_{b\to\infty}\left(\frac{1}{e^b} - \frac{1}{e^{e^2\ln 2}}\right) \\
&= \frac{1}{2^{e^2}\ln 2}
\end{aligned}
$$

Hence, by Theorem 54 we may conclude that $\int_{e^2}^\infty f(x)\,dx$ is absolutely convergent and since

$$\int_1^\infty \frac{dx}{\left(\ln x\right)^x} < C + \frac{1}{2^{e^2}\ln 2}$$

the original integral $\int_1^\infty f(x)\,dx$ converges as well.

(c) Let the integral $\int_1^\infty f(x)\,dx = \int_1^\infty dx/x^x$ be given, where we note that $f(x) = |f(x)|$ for $1 < x < \infty$. Now since $x > \ln x$ it follows that $x^x > (\ln x)^x$ for $1 < x < \infty$. Defining $g(x) = 1/(\ln x)^x$ we thus find

$$\int_1^\infty \frac{dx}{x^x} < \int_1^\infty \frac{dx}{(\ln x)^x}$$

We know from (b) that $\int_1^\infty g(x)\,dx = \int_1^\infty 1/(\ln x)^x$ converges and hence, by Theorem 54 we may conclude that the integral $\int_1^\infty f(x)\,dx = \int_1^\infty dx/x^x$ converges as well.

(d) Let the integral $\int_0^\infty t^k e^{-st}\,dt$, $k > -1$ be given. Firstly, we note that when $s < 0$ the integral clearly diverges. To check whether it converges for $s > 0$ we continue by re-writing the integral as

$$\int_0^\infty t^k e^{-st}\,dt = \int_0^N t^k e^{-st}\,dt + \int_N^\infty t^k e^{-st}\,dt$$

where N is some arbitrary, finite real number. For the first integral we note that when $0 \le t \le N \implies |f(t,s)| = t^k e^{-st} \le t^k = M_1(t)$. Now since

$$\int_0^N M_1(t)\;dt = \int_0^N t^k\,dt = \frac{N^{k+1}}{k+1}$$

then by Theorem 55 the first integral converges as well. For the second integral we note that

$$\lim_{t\to\infty} \frac{t^k e^{-st}}{1/t^2} = \lim_{t\to\infty} \frac{t^{k+2}}{e^{st}} = \lim_{t\to\infty} \frac{(k+2)\,t^{k+1}}{se^{st}} = \cdots = 0$$

following from the continued application of *L'Hospital's rule*. In other words, there exists some $t \ge N$ such that $|f(t,s)| = t^k e^{-st} \le t^{-2} = M_2(t)$. Since

$$\int_N^\infty M_2(t)\;dt = \int_N^\infty t^{-2}\,dt = \lim_{b\to\infty}\int_N^b t^{-2}\,dt = \lim_{b\to\infty}\left(\frac{1}{N} - \frac{1}{b}\right) = \frac{1}{N}$$

then by Theorem 55 the second integral converges as well. In conclusion, the original integral $\int_0^\infty t^k e^{-st}\,dt$ converges for $k > -1$, $s > 0$.

(e) Let the integral $\int_1^\infty \sin x^2\,dx$ be given. Let us make the substitution $u = x^2 \implies$

$x = \sqrt{u}$, $dx = du/(2\sqrt{u})$ so that we can write the integral as

$$\begin{aligned}
\int_1^\infty \sin x^2\,dx = \lim_{b\to\infty}\int_1^b \sin x^2\,dx &= \lim_{b\to\infty}\int_1^{b^2} \frac{\sin u}{2\sqrt{u}}\,du \\
&= \lim_{b\to\infty} -\frac{\cos u}{2\sqrt{u}}\bigg|_1^{b^2} - \frac{1}{4}\lim_{b\to\infty}\int_1^{b^2} u^{-3/2}\cos u\,du \\
&= \frac{\cos 1}{2} - \lim_{b\to\infty}\frac{\cos b^2}{2b} - \frac{1}{4}\lim_{b\to\infty}\int_1^{b^2} u^{-3/2}\cos u\,du \\
&= \frac{\cos 1}{2} - \frac{1}{4}\lim_{b\to\infty}\int_1^{b^2} u^{-3/2}\cos u\,du \\
&= \frac{\cos 1}{2} - \frac{1}{4}\lim_{b\to\infty}\int_1^{b^2} f\left(u\right)\cos u\,du
\end{aligned}$$

We note that $f(x)$ decreases as x increases and that $\lim_{u\to\infty} f(u) = \lim_{u\to\infty} u^{-3/2} = 0$. Hence, by the corollary to Theorem 51 the integral

$$\lim_{b\to\infty}\int_1^{b^2} f\left(u\right)\cos u\,du$$

converges and consequently, so will the integral $\int_1^\infty \sin x^2\,dx$.

2. Let the function $f(x) = \sin 2\pi x$ be given. Then the series

$$\begin{aligned}
\sum_{n=1}^\infty \int_{n-1}^n f\left(x\right)\,dx = \sum_{n=1}^\infty \int_{n-1}^n \sin 2\pi x\,dx &= \sum_{n=1}^\infty -\cos 2\pi x\Big|_{n-1}^n \\
&= \sum_{n=1}^\infty \left[\cos\left(2\pi\left(n-1\right)\right) - \cos 2\pi n\right] \\
&= \sum_{n=1}^\infty \cos\left(2\pi\left(n-1\right)\right) - \sum_{n=1}^\infty \cos 2\pi n \\
&= \cos 0 + \sum_{n=1}^\infty \cos 2\pi n - \sum_{n=1}^\infty \cos 2\pi n = 1
\end{aligned}$$

converges. However, the integral

$$\begin{aligned}
\int_0^\infty f\left(x\right)\,dx = \int_0^\infty \sin 2\pi x\,dx = \lim_{b\to\infty}\int_0^b \sin 2\pi x\,dx &= -\frac{1}{2\pi}\lim_{b\to\infty}\cos 2\pi x\Big|_0^b \\
&= -\frac{1}{2\pi}\lim_{b\to\infty}\cos 2\pi b + \frac{1}{2\pi}
\end{aligned}$$

diverges because $\lim_{b\to\infty}\cos 2\pi b$ doesn't converge to a specific value.

3. Let $f(x)$ be a continuous function for $a \leq x < \infty$ and

$$\lim_{x \to \infty} \left| \frac{f(x+1)}{f(x)} \right| = k < 1$$

Hence, there must exist some number B such that for sufficiently large $x > B \implies |f(x+1)| < r|f(x)|$ for $k < r < 1$. Since this must hold for any $x > B$ we may conclude that

$$|a_{n+1}| = \int_{a+n}^{a+n+1} |f(x)|\, dx < r \int_{a+n-1}^{a+n} |f(x)|\, dx = r|a_n| \qquad \text{for } n > N$$

where N is an integer chosen such that $x = a + n - 1 > B$ and we have made use of Theorem 50 that states we can write

$$\int_a^\infty f(x)\ dx = \sum_{n=1}^\infty \int_{a+n-1}^{a+n} f(x)\ dx = \sum_{n=1}^\infty a_n$$

Now we can write

$$\begin{aligned} |a_{N+1}| + |a_{N+2}| + \cdots + |a_{N+k}| + \cdots &= |a_{N+1}| \left(1 + \left| \frac{a_{N+2}}{a_{N+1}} \right| + \left| \frac{a_{N+2}}{a_{N+1}} \right| \left| \frac{a_{N+3}}{a_{N+2}} \right| + \cdots \right) \\ &< |a_{N+1}| \left(1 + r + r^2 + r^3 + \cdots \right) \end{aligned}$$

The series $1 + r + r^2 + r^3 + \cdots$ converges, by Theorem 16, since $r < 1$. Hence, by Theorem 12 the series $\sum_{n=N+1}^\infty |a_n|$ converges, which in turn implies that $\sum_{n=1}^\infty |a_n|$ converges and so the integral $\int_a^\infty f(x)\, dx$ is absolutely convergent.

4. (a) Let the integral $\int_1^\infty (x^2/e^x)\, dx$ be given. Applying the ratio test we find

$$\begin{aligned} \lim_{x \to \infty} \left| \frac{f(x+1)}{f(x)} \right| = \lim_{x \to \infty} \left| \frac{(x+1)^2}{e^{x+1}} \frac{e^x}{x^2} \right| &= \frac{1}{e} \lim_{x \to \infty} \frac{x^2 + 2x + 1}{x^2} \\ &= \frac{1}{e} \lim_{x \to \infty} \frac{1 + 2/x + 1/x^2}{1} \\ &= \frac{1}{e} = k < 1 \end{aligned}$$

Hence, the integral converges.

(b) Let the integral $\int_1^\infty (1/x^x)\, dx$ be given. Applying the ratio test we find

$$\begin{aligned} \lim_{x \to \infty} \left| \frac{f(x+1)}{f(x)} \right| = \lim_{x \to \infty} \left| \frac{x^x}{(x+1)^{x+1}} \right| &= \lim_{x \to \infty} \frac{1}{x+1} \left(\frac{x}{x+1} \right)^x \\ &= \left(\lim_{x \to \infty} \frac{1}{x+1} \right) \lim_{x \to \infty} \left(\frac{x}{x+1} \right)^x \\ &= \left(\lim_{x \to \infty} \frac{1/x}{1 + 1/x} \right) \lim_{x \to \infty} e^{x \ln \frac{x}{x+1}} \\ &= 0 \cdot e^{\lim_{x \to \infty} x \ln \frac{x}{x+1}} \end{aligned}$$

Using *L'Hospital's rule* we find

$$\lim_{x\to\infty} x\ln\frac{x}{x+1} = \lim_{x\to\infty}\frac{\ln\frac{x}{x+1}}{1/x} = \lim_{x\to\infty}\frac{\frac{x+1}{x}\frac{1}{(x+1)^2}}{-1/x^2} = \lim_{x\to\infty} -\frac{x}{x+1}$$
$$= \lim_{x\to\infty} -\frac{1}{1+1/x} = -1$$

and so

$$\lim_{x\to\infty}\frac{1}{x+1}\left(\frac{x}{x+1}\right)^x = e^{\lim_{x\to\infty} x\ln\frac{x}{x+1}} = e^{-1} = \frac{1}{e}$$

Putting this together we thus end up with

$$\lim_{x\to\infty}\left|\frac{f(x+1)}{f(x)}\right| = \lim_{x\to\infty}\left|\frac{x^x}{(x+1)^{x+1}}\right| = 0\cdot\frac{1}{e} = 0$$

Hence, the integral converges.

5. Let $f(x)$ be a continuous function for $a \leq x < \infty$ and

$$\lim_{x\to\infty}|f(x)|^{1/x} = k < 1$$

Hence, there must exist some number B such that for sufficiently large $x > B \implies |f(x)| < r^x$ for $k < r < 1$. Since this must hold for any $x > B$ we may conclude that

$$|a_n| = \int_{a+n-1}^{a+n}|f(x)|\,dx < \int_{a+n-1}^{a+n} r^x\,dx \qquad \text{for}\, n > N$$

where N is an integer chosen such that $x = a+n-1 > B$. Evaluating the integral on the right-hand side of the inequality gives

$$\int_{a+n-1}^{a+n} r^x\,dx = \int_{a+n-1}^{a+n} e^{x\ln r}\,dx = \frac{1}{\ln r}\int_{\ln r^{a+n-1}}^{\ln r^{a+n}} e^u\,du = \frac{r^a\left(1-r^{-1}\right)}{\ln r}r^n = r^* r^n \qquad \text{for}\, n > N$$

Clearly $0 < r^n < 1$ for $0 < r < 1$, $n \geq 1$. Furthermore, from (6.46) it follows that $e^x \geq 1+x \implies e^{-x} \geq 1-x$. Making the substitution $x = \ln r$ and rearranging we find

$$1-\frac{1}{r} \leq \ln r \implies 1-\frac{1}{r} < \ln r < 0 \implies \frac{1-r^{-1}}{\ln r} > 1 \qquad \text{for}\, 0 < r < 1$$

and $0 < r^a \leq 1$ for $0 < r < 1$, $a \geq 0$. As such we may conclude that $r^* > 0$. Furthermore, using *L'Hospital's rule* we find

$$\lim_{r\to 1^-} r^* = \lim_{r\to 1^-}\left(\frac{r^a}{\ln r} - \frac{r^{a-1}}{\ln r}\right) = \lim_{r\to 1^-}\left(\frac{ar^{a-1}}{1/r} - \frac{(a-1)\,r^{a-2}}{1/r}\right) = 1$$

which, along with the observation that both r^x and $1/\ln x$ are continuous, monotonic functions for $0 < x < \infty$, $0 < r < 1$, finally allows us to conclude that

$$|a_n| = \int_{a+n-1}^{a+n} |f(x)|\, dx < r^* r^n < 1 \qquad \text{for } n > N$$

Hence, the series $\sum_{n=1}^{\infty} |a_n|$ converges by comparison with the geometric series $r^* \sum_{n=1}^{\infty} r^n$ and so the integral $\int_a^{\infty} f(x)\, dx$ is absolutely convergent.

6. (a) Let the integral $\int_a^{\infty} e^{-x^2}\, dx$ be given. Applying the root test we find

$$\lim_{x\to\infty} |f(x)|^{1/x} = \lim_{x\to\infty} |e^{-x^2}|^{1/x} = \lim_{x\to\infty} e^{-x} = 0 = k < 1$$

Hence, the integral converges.

(b) Let the integral $\int_2^{\infty} dx/(\ln x)^x$ be given. Applying the root test we find

$$\lim_{x\to\infty} |f(x)|^{1/x} = \lim_{x\to\infty} \left|\frac{1}{(\ln x)^x}\right|^{1/x} = \lim_{x\to\infty} \frac{1}{\ln x} = \frac{1}{\infty} = 0 = k < 1$$

Hence, the integral converges.

7. (a) Let the integral $\int_a^{\infty} f(t,x)\, dt = \int_1^{\infty} dt/(x^2+t^2)^{5/2}$ be given. We find

$$|f(t,x)| = \left|\frac{1}{(x^2+t^2)^{5/2}}\right| \leq \frac{1}{t^5} = M(t) \qquad \text{for } 0 \leq x \leq 1, t > 0$$

Now

$$\int_a^{\infty} M(t)\, dt = \int_1^{\infty} t^{-5}\, dt = \lim_{b\to\infty} \int_1^b t^{-5}\, dt = \lim_{b\to\infty} -\frac{1}{4t^4}\bigg|_1^b = \frac{1}{4} - \lim_{b\to\infty} \frac{1}{4b^4} = \frac{1}{4}$$

Hence, since the integral $\int_a^{\infty} M(t)\, dt$ converges then by Theorem 55 the integral $\int_a^{\infty} f(t,x)\, dt = \int_1^{\infty} dt/(x^2+t^2)^{5/2}$ is uniformly and absolutely convergent for $0 \leq x \leq 1$.

(b) Let the integral $\int_a^{\infty} f(t,x)\, dt = \int_1^{\infty} \sin(t)\, dt/(x^2+t^2)$ be given. We find

$$|f(t,x)| = \left|\frac{\sin t}{x^2+t^2}\right| \leq \left|\frac{\sin t}{t^2}\right| \leq \frac{1}{t^2} = M(t) \qquad \text{for } 0 \leq x \leq 1$$

Now

$$\int_a^{\infty} M(t)\, dt = \int_1^{\infty} t^{-2}\, dt = \lim_{b\to\infty} \int_1^b t^{-2}\, dt = \lim_{b\to\infty} -\frac{1}{t}\bigg|_1^b = 1 - \lim_{b\to\infty} \frac{1}{b} = 1$$

Hence, since the integral $\int_a^{\infty} M(t)\, dt$ converges then by Theorem 55 the integral $\int_a^{\infty} f(t,x)\, dt = \int_1^{\infty} \sin t\, dt/(x^2+t^2)$ is uniformly and absolutely convergent for $0 \leq x \leq 1$.

8. (a) Let the integral $\int_0^\infty t^n e^{-xt^2}\,dt$, $n > 0$ be given. To show that this integral is uniformly convergent for $x \geq x_1$ assuming $x_1 > 0$ we will re-write the integral as

$$\int_0^\infty t^n e^{-xt^2}\,dt = \int_0^N t^n e^{-xt^2}\,dt + \int_N^\infty t^n e^{-xt^2}\,dt$$

where N is some arbitrary, but finite real number. For the first integral we note that when $0 \leq t \leq N \implies |f(t,x)| = t^n e^{-xt} \leq t^n = M_1(t)$. Now since

$$\int_0^N M_1(t)\,dt = \int_0^N t^n\,dt = \frac{N^{n+1}}{n+1}$$

then by Theorem 55 the first integral converges as well. For the second integral we note that

$$\lim_{t\to\infty} \frac{t^n e^{-xt^2}}{1/t^2} = \lim_{t\to\infty} \frac{t^{n+2}}{e^{xt^2}} = \lim_{t\to\infty} \frac{(n+2)\,t^n}{2xe^{xt^2}} = \cdots = 0$$

following from the continued application of *L'Hospital's rule*. In other words, there exists some $t \geq N$ such that $|f(t,x)| = t^n e^{-xt^2} \leq t^{-2} = M_2(t)$. It then follows from Problem 1(d) that the second integral converges as well. In conclusion, the original integral $\int_0^\infty t^n e^{-xt^2}\,dt$, $n > 0$ is uniformly convergent for $x \geq x_1$.

(b) For Problem 1 following Section 4.8 we made the substitution $u = -r^2 \implies du = -2r\,dr$ in order to solve the integral $(\int_0^\infty e^{-x^2}\,dx)^2 = \iint_R e^{-x^2-y^2}\,dx\,dy$ expressed in polar coordinates. To solve the integral $\int_0^\infty e^{-xt^2}\,dt$ we can follow the same steps, but instead use the substitution $u = -xr^2 \implies du = -2xr\,dr$, such that the integral to solve becomes

$$\begin{aligned}\left(\int_0^\infty e^{-xt^2}\,dt\right)^2 = -\frac{1}{2x}\int_0^{\pi/2}\int_0^\infty e^u\,du\,d\theta &= -\frac{1}{2x}\int_0^{\pi/2}\lim_{b\to-\infty}\int_0^b e^u\,du\,d\theta \\ &= \frac{1}{2x}\int_0^{\pi/2}\left(1 - \lim_{b\to-\infty} e^b\right)d\theta \\ &= \frac{\pi}{4x}\end{aligned}$$

And so

$$\int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}} \qquad x > 0$$

(c) Let the integral $\int_0^\infty t^{2n} e^{-xt^2}\,dt$, $x > 0$, $n = 1, 2, \ldots$ be given. Making the substitution $u = t^2 \implies du = 2t\,dt$ this integral can be written as

$$\int_0^\infty t^{2n} e^{-xt^2}\,dt = \frac{1}{2}\int_0^\infty \frac{u^n}{\sqrt{u}} e^{-xu}\,du = \frac{1}{2}\int_0^\infty u^{n-1/2} e^{-xu}\,du$$

Substituting back for t then gives

$$\int_0^\infty t^{2n} e^{-xt^2}\, dt = \frac{1}{2}\int_0^\infty t^{n-1/2} e^{-xt}\, dt$$

Next, let us define

$$F(x) = \int_0^\infty f(t,x)\, dt = \int_0^\infty e^{-xt^2}\, dt = \frac{1}{2}\sqrt{\frac{\pi}{x}} \qquad \text{for } x > 0$$

as follows from (b). Furthermore, we note that

$$\int_0^\infty \frac{\partial^n f}{\partial x^n}(t,x)\, dt = \int_0^\infty \frac{\partial}{\partial x^n}\left(e^{-xt^2}\right) dt = (-1)^n \int_0^\infty t^{2n} e^{-xt^2}\, dt$$

for $n = 1, 2, \ldots$. Since it follows from (a) that the integral $\int_0^\infty t^{2n} e^{-xt^2}\, dt$ is uniformly convergent for $x > 0$ we may conclude, by repeated application of Theorem 58, that

$$\begin{aligned} F^{(n)}(x) = \int_0^\infty \frac{\partial^n f}{\partial x^n}(t,x)\, dt = (-1)^n \int_0^\infty t^{2n} e^{-xt^2}\, dt &= \frac{(-1)^n}{2}\int_0^\infty t^{n-1/2} e^{-xt}\, dt \\ &= \frac{(-1)^n}{2}\Gamma\left(n+\frac{1}{2}\right) x^{-n-1/2} \end{aligned}$$

where the last step follows from (6.85). Ignoring the factor $(-1)^n$ we thus find

$$\int_0^\infty t^{2n} e^{-xt^2}\, dt = \int_0^\infty t^{n-1/2} e^{-xt}\, dt = \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right) x^{-n-1/2}$$

Now we also have

$$F(x) = \frac{\sqrt{\pi}}{2} x^{-1/2} \implies F^{(n)}(x) = (-1)^n \frac{\sqrt{\pi}}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} x^{-n-1/2}$$

and so we finally we may conclude that

$$\int_0^\infty t^{2n} e^{-xt^2}\, dt = \int_0^\infty t^{n-1/2} e^{-xt}\, dt = \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right) x^{-n-1/2} = \frac{\sqrt{\pi}}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n x^{n+1/2}}$$

Note: the problem statement contains an error. An equal sign is missing in the last term on the right-hand side as may be deduced from the worked out solution above.

9. (a) Let the integrals

$$\int_0^\infty t^n e^{-t^2}\cos(tx)\, dt \qquad\qquad \int_0^\infty t^n e^{-t^2}\sin(tx)\, dt$$

be given and let us assume that $n > 0$. Now since

$$0 \leq |t^n e^{-t^2} \cos(tx)| \leq t^n e^{-t^2} \qquad 0 \leq |t^n e^{-t^2} \sin(tx)| \leq t^n e^{-t^2}$$

for all x and we know from Problem 8(a) that the integral

$$\int_0^\infty M(t)\ dt = \int_0^\infty t^n e^{-t^2}\ dt$$

exists, we may, by Theorem 55, conclude that the original integrals are uniformly convergent for all x.

(b) Let

$$F(x) = \int_0^\infty e^{-t^2} \cos(tx)\ dt$$

From which it follows that

$$F'(x) = \frac{d}{dx}\int_0^\infty e^{-t^2} \cos(tx)\ dt = \int_0^\infty \frac{\partial}{\partial x}\left(e^{-t^2} \cos(tx)\right)\ dt = -\int_0^\infty te^{-t^2} \sin(tx)\ dt$$

Furthermore, employing integration by parts on $F(x)$ gives

$$\begin{aligned} F(x) = \int_0^\infty e^{-t^2} \cos(tx)\ dt &= \lim_{t\to\infty} \frac{e^{-t^2} \sin(tx)}{x} - \frac{e^0 \sin 0}{x} + \frac{2}{x}\int_0^\infty te^{-t^2} \sin(tx)\ dt \\ &= \frac{2}{x}\int_0^\infty te^{-t^2} \sin(tx)\ dt \end{aligned}$$

Hence, we find $F'(x) = -(1/2)xF(x) \iff F'(x)/F(x) = -(1/2)x$.

Note: the problem statement mistakenly multiplies the right-hand side of the equality above by a factor of $1/4$ *instead of* $1/2$.

Using the substitution $u = \ln F(x) \implies du = [F'(x)/F(x]\,dx$ allows us to write $du = -(1/2)x\,dx$ and so

$$\int du = -\frac{1}{2}\int x\,dx \implies \ln F(x) = -\frac{1}{4}x^2 + C \implies F(x) = ce^{-x^2/4}$$

Next, let $x = 0$ so that

$$F(0) = \int_0^\infty e^{-t^2}\ dt = \frac{\sqrt{\pi}}{2}$$

where the last equality follows from Problem 8(b). Hence, we find $c = \sqrt{\pi}/2$ and so

$$F(x) = \frac{\sqrt{\pi}}{2}e^{-x^2/4}$$

10. (a) Let the functions $u(x)$ and $v(x)$ be given. Furthermore, let us assume $u(x)$, $u'(x)$, $v(x)$, $v'(x)$ are continuous for $a \leq x < \infty$ and $\lim_{x\to\infty} u(x)v(x)$ exists. Next, let us consider the improper integral

$$\int_a^\infty u(x)\, v'(x)\; dx$$

which might or might not converge. Since $u(x)$ and $v(x)$ are continuous for $a \leq x < \infty$ they are differentiable. Likewise, because $u'(x)$ and $v'(x)$ are continuous for $a \leq x < \infty$ they are integrable. Hence, we can employ integration by parts (4.17) on the improper integral to get

$$\begin{aligned}\int_a^\infty u(x)\, v'(x)\; dx &= u(x)v(x)|_a^\infty - \int_a^\infty u'(x)v(x)\, dx \\ &= \lim_{x\to\infty} \left[u(x)\, v(x)\right] - u(a)\, v(a) - \int_a^\infty u'(x)v(x)\, dx\end{aligned}$$

Now from this equality and the assumption that $\lim_{x\to\infty} u(x)v(x)$ exists, we may conclude that if one of the improper integrals converges, so must the other.

(b) Let $u(x) = x^k$, $v'(x) = e^{-sx}$ and $s > 0$. Setting $a = 0$ and using the result from (a) we thus find

$$\begin{aligned}\int_a^\infty u(x)\, v'(x)\; dx = \int_0^\infty x^k e^{-sx}\, dx &= -\left.\frac{x^k e^{-sx}}{s}\right|_0^\infty + \frac{k}{s}\int_0^\infty x^{k-1}e^{-sx}\, dx \\ &= \lim_{x\to\infty}\left(\frac{x^k e^{-sx}}{s}\right) + \frac{k}{s}\int_0^\infty x^{k-1}e^{-sx}\, dx \\ &= \frac{k}{s}\int_0^\infty x^{k-1}e^{-sx}\, dx\end{aligned}$$

For proof that $\lim_{x\to\infty} x^k e^{-sx}/s = 0$ see Problem 1(d). Making the substitution $x = t$, we see that this justifies the derivation of (6.80).

(c) Let the sum $\sum_{k=1}^n u_k(v_{k+1} - v_k)$ be given. By manipulating indexes we can re-write this as

$$\begin{aligned}\sum_{k=1}^n u_k\left(v_{k+1} - v_k\right) &= \sum_{k=1}^n u_k v_{k+1} - \sum_{k=1}^n u_k v_k \\ &= \sum_{k=1}^n u_k v_{k+1} - \sum_{k=1}^{n-1} u_{k+1}v_{k+1} - u_1 v_1 \\ &= \sum_{k=1}^{n-1} u_k v_{k+1} + u_n v_{n+1} - \sum_{k=1}^{n-1} u_{k+1}v_{k+1} - u_1 v_1 \\ &= u_n v_{n+1} - u_1 v_1 - \sum_{k=1}^{n-1} v_{k+1}\left(u_{k+1} - u_k\right)\end{aligned}$$

Clearly, when $n \to \infty$ this becomes

$$\sum_{k=1}^{\infty} u_k \left(v_{k+1} - v_k\right) = \lim_{n\to\infty} \left(u_n v_{n+1}\right) - u_1 v_1 - \sum_{k=1}^{\infty} v_{k+1} \left(u_{k+1} - u_k\right)$$

11. Let us remind ourselves that an odd function is a function $f(x)$ such that $f(-x) = -f(x)$ for all x in the domain, and the graph of $f(x)$ is symmetric about the origin.

(a) Let $f(x)$ be a continuous, odd function for $-\infty < x < \infty$. Then we have

$$\begin{aligned}
(P)\int_{-\infty}^{\infty} f(x)\, dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)\, dx &= \lim_{a\to\infty} \left[\int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx\right] \\
&= \lim_{a\to\infty} \int_{-a}^{0} f(x)\, dx + \lim_{a\to\infty} \int_{0}^{a} f(x)\, dx \\
&= \lim_{a\to\infty} -\int_{a}^{0} f(-u)\, du + \lim_{a\to\infty} \int_{0}^{a} f(x)\, dx \\
&= \lim_{a\to\infty} \int_{0}^{a} f(-u)\, du + \lim_{a\to\infty} \int_{0}^{a} f(x)\, dx \\
&= \lim_{a\to\infty} -\int_{0}^{a} f(u)\, du + \lim_{a\to\infty} \int_{0}^{a} f(x)\, dx \\
&= \lim_{a\to\infty} \left[-\int_{0}^{a} f(u)\, du + \int_{0}^{a} f(x)\, dx\right] = 0
\end{aligned}$$

where we have used the substitution $x = -u$. Hence, the limit exists and so

$$(P)\int_{-\infty}^{\infty} f(x)\, dx = 0$$

(b) Let $f(x)$ be a continuous function for $-a \le x \le a$, except at $x = 0$ and let $f(x)$ be odd. Then we have

$$\begin{aligned}
(P)\int_{-a}^{a} f(x)\, dx &= \lim_{\epsilon\to 0+} \left[\int_{-a}^{-\epsilon} f(x)\, dx + \int_{\epsilon}^{a} f(x)\, dx\right] \\
&= \lim_{\epsilon\to 0+} \int_{-a}^{-\epsilon} f(x)\, dx + \lim_{\epsilon\to 0+} \int_{\epsilon}^{a} f(x)\, dx \\
&= \lim_{\epsilon\to 0+} -\int_{a}^{\epsilon} f(-u)\, du + \lim_{\epsilon\to 0+} \int_{\epsilon}^{a} f(x)\, dx \\
&= \lim_{\epsilon\to 0+} \int_{\epsilon}^{a} f(-u)\, du + \lim_{\epsilon\to 0+} \int_{\epsilon}^{a} f(x)\, dx \\
&= \lim_{\epsilon\to 0+} -\int_{\epsilon}^{a} f(u)\, du + \lim_{\epsilon\to 0+} \int_{\epsilon}^{a} f(x)\, dx \\
&= \lim_{\epsilon\to 0+} \left[-\int_{\epsilon}^{a} f(u)\, du + \int_{\epsilon}^{a} f(x)\, dx\right] = 0
\end{aligned}$$

where we have used the substitution $x = -u$. Hence, the limit exists and so

$$(P)\int_{-a}^{a} f(x)\ dx = 0$$

12. (a) Let the integral

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{\infty} \frac{x^3}{x^4+1}\ dx$$

be given. The function $f(x)$ is continuous for $-\infty < x < \infty$ and since

$$f(-x) = -\frac{x^3}{x^4+1} = -f(x)$$

it is odd as well. Hence, by Problem 11(a) the principal value is given by

$$(P)\int_{-\infty}^{\infty} f(x)\ dx = (P)\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\ dx = 0$$

The usual value is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{\infty} \frac{x^3}{x^4+1}\ dx &= \lim_{a\to\infty}\int_{-a}^{0} \frac{x^3}{x^4+1}\ dx + \lim_{a\to\infty}\int_{0}^{a} \frac{x^3}{x^4+1}\ dx \\
&= \frac{1}{4}\lim_{a\to\infty}\int_{a^4+1}^{1} \frac{du}{u} + \frac{1}{4}\lim_{a\to\infty}\int_{1}^{a^4+1} \frac{dv}{v} \\
&= -\frac{1}{4}\lim_{a\to\infty}\ln\left|a^4+1\right| + \frac{1}{4}\lim_{a\to\infty}\ln\left|a^4+1\right| = -\infty+\infty
\end{aligned}$$

where we have used the substitution $u = v = x^4+1 \implies du = dv = 4x^3\,dx$. Hence, the usual value does not exist.

(b) Let the integral

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1}\ dx$$

be given. The function $f(x)$ is continuous for $-\infty < x < \infty$ and since

$$f(-x) = -\frac{x}{x^4+1} = -f(x)$$

it is odd as well. Hence, by Problem 11(a) the principal value is given by

$$(P)\int_{-\infty}^{\infty} f(x)\ dx = (P)\int_{-\infty}^{\infty} \frac{x}{x^4+1}\ dx = 0$$

The usual value is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{\infty} \frac{x}{x^4+1}\ dx &= \lim_{a\to\infty}\int_{-a}^{0} \frac{x}{x^4+1}\ dx + \lim_{a\to\infty}\int_{0}^{a} \frac{x}{x^4+1}\ dx \\
&= \frac{1}{2}\lim_{a\to\infty}\int_{a^2}^{0} \frac{du}{u^2+1} + \frac{1}{2}\lim_{a\to\infty}\int_{0}^{a^2} \frac{dv}{v^2+1} \\
&= \frac{1}{2}\lim_{a\to\infty}\tan^{-1}a^2 - \frac{1}{2}\lim_{a\to\infty}\tan^{-1}a^2 = \frac{\pi}{4} - \frac{\pi}{4} = 0
\end{aligned}$$

where we have used the substitution $u = v = x^2 \implies du = 2x\,dx$. Hence, both the principal and usual value are equal to 0.

(c) Let the integral $\int_{-\infty}^{\infty} \sin x\,dx$ be given. The function $f(x)$ is continuous for $-\infty < x < \infty$ and since $f(-x) = \sin(-x) = -\sin x = -f(x)$ it is odd as well. Hence, by Problem 11(a) the principal value is given by $(P)\int_{-\infty}^{\infty} f(x)\,dx = (P)\int_{-\infty}^{\infty} \sin x\,dx = 0$. The usual value is given by

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \sin x\,dx &= \lim_{a\to\infty}\int_{-a}^{0} \sin x\,dx + \lim_{a\to\infty}\int_{0}^{a} \sin x\,dx \\ &= \lim_{a\to\infty}(-1+\cos a) - \lim_{a\to\infty}(\cos a - 1)\end{aligned}$$

Hence, the usual value does not exist.

(d) Let the integral

$$\int_{-\infty}^{\infty} \frac{x^3}{x^2+1}\,dx$$

be given. The function $f(x)$ is continuous for $-\infty < x < \infty$ and since

$$f(-x) = -\frac{x^3}{x^2+1} = -f(x)$$

it is odd as well. Hence, by Problem 11(a) the principal value is given by

$$(P)\int_{-\infty}^{\infty} f(x)\,dx = (P)\int_{-\infty}^{\infty} \frac{x^3}{x^2+1}\,dx = 0$$

The usual value is given by

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{x^3}{x^2+1}\,dx &= \lim_{a\to\infty}\int_{-a}^{0} \frac{x^3}{x^2+1}\,dx + \lim_{a\to\infty}\int_{0}^{a} \frac{x^3}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{a\to\infty}\int_{a^2+1}^{1} \frac{u-1}{u}\,du + \lim_{a\to\infty}\int_{1}^{a^2+1} \frac{v-1}{v}\,dv \\ &= \frac{1}{2}\lim_{a\to\infty}\ln|a^2+1| - \lim_{a\to\infty}\ln|a^2+1| = \infty - \infty\end{aligned}$$

Hence, the usual value does not exist.

(e) Let the integral $\int_{-1}^{1} dx/x$ be given. The function $f(x)$ is continuous for $-1 \le x \le 1$, except at $x = 0$ and since $f(-x) = -1/x = -f(x)$ it is odd as well. Hence, by Problem 11(b) the principal value is given by $(P)\int_{-1}^{1} f(x)\,dx = (P)\int_{-1}^{1} dx/x = 0$. The usual value is given by

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} = \infty - \infty$$

Hence, the usual value does not exist.

(f) Let the integral

$$\int_{-2}^{4} \frac{dx}{(x+1)^{1/3}}$$

be given. The function $f(x)$ is continuous for $-2 \leq x \leq 4$, except for $x = -1$. The principal value is given by

$$\begin{aligned}
(P)\int_{a}^{b} f(x)\,dx &= \lim_{\epsilon \to 0+} \left[\int_{a}^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx \right] \\
&= \lim_{\epsilon \to 0+} \left[\int_{-2}^{-1-\epsilon} \frac{dx}{(x+1)^{1/3}} + \int_{-1+\epsilon}^{4} \frac{dx}{(x+1)^{1/3}} \right] \\
&= \lim_{\epsilon \to 0+} \left[\int_{-1}^{-\epsilon} \frac{du}{u^{1/3}} + \int_{\epsilon}^{5} \frac{du}{u^{1/3}} \right] \\
&= \frac{3}{2} \lim_{\epsilon \to 0+} \left(\epsilon^{2/3} - 1 + 5^{2/3} - \epsilon^{2/3} \right) = \frac{3}{2} \left(5^{2/3} - 1 \right)
\end{aligned}$$

where we have used the substitution $u = x + 1 \implies du = dx$. The usual value is given by

$$\int_{-2}^{4} \frac{dx}{(x+1)^{1/3}} = \int_{-2}^{-1} \frac{dx}{(x+1)^{1/3}} + \int_{-1}^{4} \frac{dx}{(x+1)^{1/3}} = \frac{3}{2} \left(5^{2/3} - 1 \right)$$

(g) Let the integral $\int_{-\infty}^{\infty} dx/x$ be given. The function $f(x)$ is continuous for $-\infty < x < \infty$, except at $x = 0$ and since $f(-x) = -1/x = -f(x)$ it is odd as well. The principal value is given by

$$\begin{aligned}
(P)\int_{-\infty}^{\infty} f(x)\,dx = (P)\int_{-\infty}^{\infty} \frac{dx}{x} &= \lim_{a \to \infty} \lim_{\epsilon \to 0+} \left[\int_{-a}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{a} \frac{dx}{x} \right] \\
&= \lim_{a \to \infty} \lim_{\epsilon \to 0+} \left(\ln \epsilon - \ln a + \ln a - \ln \epsilon \right) = 0
\end{aligned}$$

The usual value is given by

$$\int_{-\infty}^{\infty} \frac{dx}{x} = \lim_{a \to \infty} \int_{-a}^{0} \frac{dx}{x} + \lim_{a \to \infty} \int_{0}^{a} \frac{dx}{x} = \lim_{a \to \infty} \left(\infty - \ln a \right) + \lim_{a \to \infty} \left(\ln a - \infty \right)$$

Hence, the usual value does not exist.

(h) Let the integral

$$\int_{-\infty}^{\infty} \frac{x}{(x-1)^2}\,dx$$

be given. The function $f(x)$ is continuous for $-\infty < x < \infty$, except at $x = 1$.

The principal value is given by

$$
\begin{aligned}
(P)\int_{-\infty}^{\infty} f(x)\, dx = (P)\int_{-\infty}^{\infty} \frac{x}{(x-1)^2}\, dx \\
&= \lim_{a\to\infty}\lim_{\epsilon\to 0+}\left[\int_{-a}^{1-\epsilon} \frac{x}{(x-1)^2}\, dx + \int_{1+\epsilon}^{a} \frac{x}{(x-1)^2}\, dx\right] \\
&= \lim_{a\to\infty}\lim_{\epsilon\to 0+}\left[\int_{-a-1}^{-\epsilon} \frac{u+1}{u^2}\, du + \int_{\epsilon}^{a-1} \frac{v+1}{v^2}\, dv\right] \\
&= \lim_{a\to\infty}\lim_{\epsilon\to 0+}\left(\frac{1}{\epsilon} - \ln|-a-1| - \frac{1}{a+1} + \ln|a-1| - \frac{1}{a-1} + \frac{1}{\epsilon}\right) \\
&= \lim_{a\to\infty}\lim_{\epsilon\to 0+}\left(\frac{1}{\epsilon} - \ln|a+1| - \frac{1}{a+1} + \ln|a-1| - \frac{1}{a-1} + \frac{1}{\epsilon}\right) = \infty
\end{aligned}
$$

where we have used the substitution $u = x - 1 \implies du = dx$. Since the principal value doesn't exist, neither will the usual value.

13. (a) The Gamma function $\Gamma(k)$ is defined by the equation

$$\Gamma(k) = \int_0^{\infty} t^{k-1}e^{-t}\, dt \qquad k > 0$$

Now let $f(t) = t^{k-1}$ and $g(t) = e^{-t}$. Then since $g'(t) = -g(t)$ we may conclude that $g(t) = e^{-t}$ is differentiable and therefore necessarily continuous. Next, because we can write $t^{k-1} = e^{(k-1)\ln t}$ it should be obvious to see that $f(t) = t^{k-1}$ is continuous too. Furthermore, it is trivial to see that for $t > 0$ we have $f(t), g(t) > 0$. Since the product of two continuous, positive functions is itself continuous and positive, and the integral of a continuous, positive function is continuous and positive we may conclude that $\Gamma(k)$ must be continuous and positive.

(b) From (6.86)

$$\Gamma(k+1) = k\Gamma(k)$$

we may deduce that

$$\lim_{k\to 0+}\Gamma(k) = \lim_{k\to 0+}\frac{\Gamma(k+1)}{k}$$

Now since it follows from (a) that $\Gamma(k)$ is continuous and positive for $k > 0$ we may safely assume that $\lim_{k\to 0+}\Gamma(k+1) = \Gamma(1) = 1$ and so we find

$$\lim_{k\to 0+}\Gamma(k) = \lim_{k\to 0+}\frac{\Gamma(k+1)}{k} = +\infty$$

(c) By repeatedly applying (6.86) it follows that for $n = 0, 1, 2\ldots$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 1\cdot\Gamma(1) = n!$$

Next, let k be negative and non-integral. Again, repeatedly applying (6.68) we find

$$\Gamma(k+n) = (k+n-1)\cdots(k+1)\,k\Gamma(k) \qquad k+n>0$$

Solving for $\Gamma(k)$ and taking the limit when $k \to -n$ gives

$$\lim_{k\to -n}\Gamma(k) = \lim_{k\to -n}\frac{\Gamma(k+n)}{(k+n-1)\cdots(k+1)\,k}$$

Now since $k+n>0$ it follows that $\lim_{k\to -n}\Gamma(k+n) \equiv \lim_{k\to 0+}\Gamma(k) = +\infty$, where the last equality follows from (b). Hence, we may conclude that

$$\lim_{k\to -n}|\Gamma(k)| = \lim_{k\to -n}\left|\frac{\Gamma(k+n)}{(k+n-1)\cdots(k+1)\,k}\right| = \left|\frac{+\infty}{(-1)\cdots(-n+1)(-n)}\right| = +\infty$$

14. Starting from (6.81)

$$\mathcal{L}\left[t^k\right] = \int_0^\infty t^k e^{-st}\,dt = \frac{k}{s}\frac{k-1}{s}\cdots\frac{1}{s}\frac{1}{s} = \frac{k!}{s^{k+1}} \qquad s>0$$

and then using (6.82) and (6.83) we can write this as

$$\mathcal{L}\left[t^k\right] = \int_0^\infty t^k e^{-st}\,dt = \frac{k!}{s^{k+1}} \overset{(6.82)}{=} \frac{\int_0^\infty t^k e^{-t}\,dt}{s^{k+1}} \overset{(6.83)}{=} \frac{\Gamma(k+1)}{s^{k+1}} \qquad s>0$$

which is none other than (6.85).

15. (a) The Euler-Mascheroni constant is given by

$$\gamma = 1 + \sum_{n=2}^{\infty}\left(\frac{1}{n} + \ln\frac{n-1}{n}\right)$$

Employing the integral test on the second term involving the infinite sum results in

$$\begin{aligned}
\int_2^\infty \left(\frac{1}{x} + \ln\frac{x-1}{x}\right) dx &= \lim_{b\to\infty} \int_2^b \left(\frac{1}{x} + \ln\frac{x-1}{x}\right) dx \\
&= \lim_{b\to\infty} \left[\int_2^b \frac{dx}{x} + \int_2^b \ln\frac{x-1}{x}\, dx\right] \\
&= \lim_{b\to\infty} \left[\ln x|_2^b + x\ln\frac{x-1}{x}\bigg|_2^b - \int_2^b \frac{dx}{x-1}\right] \\
&= \lim_{b\to\infty} \left[\ln x + x\ln\frac{x-1}{x} - \ln|x-1|\right]_2^b \\
&= \lim_{b\to\infty} \left(\ln b - \ln 2 + b\ln\frac{b-1}{b} + 2\ln 2 - \ln(b-1)\right) \\
&= \lim_{b\to\infty} \left(b\ln\frac{b-1}{b} - \ln\frac{b-1}{b} + \ln 2\right) \\
&= \lim_{b\to\infty} \left(\frac{\ln[(b-1)/b]}{1/b} - \ln\left(1-\frac{1}{b}\right) + \ln 2\right) \\
&= \lim_{b\to\infty} \frac{\ln[(b-1)/b]}{1/b} + \ln 2
\end{aligned}$$

where we have used integration by parts with $u = \ln[(x-1)/x]$, $v = x$. Employing *L'Hospital's rule* on the remaining limit gives

$$\lim_{b\to\infty} \frac{\ln[(b-1)/b]}{1/b} = \lim_{b\to\infty} -\frac{\overline{\frac{1}{b(b-1)}}}{1/b^2} = \lim_{b\to\infty} -\frac{b}{b-1} = \lim_{b\to\infty} -\frac{1}{1-1/b} = -1$$

Hence, we may conclude that γ converges.

(b) Using Theorem 23 we find

$$\begin{aligned}
|R_n| < \left|\int_n^\infty \left(\frac{1}{x} + \ln\frac{x-1}{x}\right) dx\right| &= \lim_{b\to\infty} \left|\ln|x| + x\ln\left(\frac{x-1}{x}\right) - \ln|x-1|\right|_n^b \\
&= \left|(1-n)\ln\left(1-\frac{1}{n}\right) - 1\right| = T_n
\end{aligned}$$

From this it follows we can choose $N(\epsilon)$ as the smallest integer n such that

$$\left|(1-n)\ln\left(1-\frac{1}{n}\right) - 1\right| < \epsilon$$

As such we find that in order to evaluate γ to one significant figure (i.e. choosing $\epsilon = 0.1$), six terms are sufficient:

$$\left|(1-6)\ln\left(1-\frac{1}{6}\right) - 1\right| \approxeq 0.09$$

implying

$$\gamma = 1 + \sum_{2}^{\infty}\left(\frac{1}{n} + \ln\frac{n-1}{n}\right) \approxeq 1 + \left(\frac{1}{2} - \ln\frac{1}{2}\right) + \cdots + \left(\frac{1}{7} - \ln\frac{6}{7}\right) \approxeq 0.6$$

16. Starting from (6.89)

$$\Gamma(k) = k^{k-1/2}e^{-k}\sqrt{2\pi}e^{\theta(k)/(12k)} \qquad k > 0$$

and using the relation $\Gamma(k+1) = k\Gamma(k)$ we find

$$\frac{\Gamma(k+1)}{k} = k^{k-1/2}e^{-k}\sqrt{2\pi}e^{\theta(k)/(12k)} \iff \frac{\Gamma(k+1)}{k^{k+1/2}\sqrt{2\pi}e^{-k}e^{\theta(k)/(12k)}} = 1$$

where $\theta(k)$ denotes a function of k such that $0 < \theta(k) < 1$. From this condition on $\theta(k)$ it should be obvious to see that $\lim_{k\to\infty}\theta(k) = a$ where a is some number such that $0 < a < 1$, and $\lim_{k\to\infty} 12k = \infty$. Hence, we may conclude that $\lim_{k\to\infty} e^{\theta(k)/(12k)} = e^0 = 1$ and so

$$\lim_{k\to\infty}\frac{\Gamma(k+1)}{k^{k+1/2}\sqrt{2\pi}e^{-k}e^{\theta(k)/(12k)}} = \lim_{k\to\infty}\frac{\Gamma(k+1)}{k^{k+1/2}\sqrt{2\pi}e^{-k}} = 1$$

17. Starting from (6.83)

$$\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt \qquad k > 0$$

and using the substitution $u = \sqrt{t} \implies du = (t^{-1/2}/2)\,dt$ the equation for the Gamma function $\Gamma(k)$ may be re-written as

$$\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt = 2\int_0^\infty u^{2k-1}e^{-u^2}\,du$$

Hence, we may define $\Gamma(p)$ and $\Gamma(q)$ as

$$\Gamma(p) = 2\int_0^\infty x^{2p-1}e^{-x^2}\,dx \qquad\qquad \Gamma(q) = 2\int_0^\infty y^{2q-1}e^{-y^2}\,dy$$

where $p > 0$, $q > 0$. And so

$$\Gamma(p)\,\Gamma(q) = \left(2\int_0^\infty x^{2p-1}e^{-x^2}\,dx\right)\left(2\int_0^\infty y^{2q-1}e^{-y^2}\,dy\right) = 4\int_0^\infty\int_0^\infty x^{2p-1}y^{2q-1}e^{-x^2-y^2}\,dx\,dy$$

Introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ then by (4.64) this may be written as

$$
\begin{aligned}
\Gamma(p)\,\Gamma(q) &= 4\int_0^\infty\int_0^\infty x^{2p-1}y^{2q-1}e^{-x^2-y^2}\,dx\,dy \\
&= 4\int_0^{\pi/2}\int_0^\infty \sin^{2p-1}\theta\cos^{2q-1}\theta r^{2p+2q-1}e^{-r^2}\,dr\,d\theta \\
&= \left(2\int_0^\infty r^{2p+2q-1}e^{-r^2}\,dr\right)\left(2\int_0^{\pi/2}\sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta\right) \\
&= \Gamma(p+q)\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \Gamma(p+q)\,\mathrm{B}(p,q)
\end{aligned}
$$

where we have used the substitution $x=\sin^2\theta \implies dx = 2\sin\theta\cos\theta\,d\theta$. In conclusion

$$
\mathrm{B}(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}
$$

which is none other than (6.93).

18. (a) Let the function $f(t)=e^{kt}$ be given. The *Laplace transform* of $f(t)$ is then given by

$$
\begin{aligned}
F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty e^{kt}e^{-st}\,dt = \int_0^\infty e^{-(s-k)t}\,dt &= \lim_{b\to\infty}\int_0^b e^{-(s-k)t}\,dt \\
&= \frac{1}{k-s}\lim_{b\to\infty}\int_0^{-(s-k)b} e^u\,du \\
&= \frac{1}{k-s}\lim_{b\to\infty}\left(e^{-(s-k)b}-1\right) \\
&= \frac{1}{s-k}
\end{aligned}
$$

where we have used the substitution $u=-(s-k)t \implies du = -(s-k)\,dt$ and it is assumed $s>k$.

(b) Let the function $f(t)=\sin kt$ be given. The *Laplace transform* of $f(t)$ is then given by

$$
\begin{aligned}
F(s) = \int_0^\infty f(t)\,e^{-st}\,dt &= \int_0^\infty \sin(kt)\,e^{-st}\,dt \\
&= \lim_{b\to\infty}\int_0^b \sin(kt)\,e^{-st}\,dt \\
&= \lim_{b\to\infty}\left[-\left.\frac{\cos(kt)\,e^{-st}}{k}\right|_0^b - s\int_0^b \frac{\cos(kt)\,e^{-st}}{k}\,dt\right] \\
&= \lim_{b\to\infty}\left[-\left.\frac{\cos(kt)\,e^{-st}}{k}\right|_0^b - \left.\frac{\sin(kt)\,se^{-st}}{k^2}\right|_0^b - \frac{s^2}{k^2}\int_0^b \sin(kt)\,e^{-st}\,dt\right]
\end{aligned}
$$

From which it follows that

$$\lim_{b\to\infty}\left(1+\frac{s^2}{k^2}\right)\int_0^b \sin(kt)\, e^{-st}\, dt = \lim_{b\to\infty}\left[-\frac{\cos(kt)\, e^{-st}}{k}-\frac{\sin(kt)\, se^{-st}}{k^2}\right]_0^b$$

and so

$$\begin{aligned}
F(s) &= \int_0^\infty \sin(kt)\, e^{-st}\, dt \\
&= \lim_{b\to\infty}\int_0^b \sin(kt)\, e^{-st}\, dt \\
&= \frac{1}{k^2+s^2}\lim_{b\to\infty}\left[-\cos(kt)\, ke^{-st}-\sin(kt)\, se^{-st}\right]_0^b \\
&= \frac{1}{k^2+s^2}\lim_{b\to\infty}\left[-\cos(kb)\, ke^{-sb}-\sin(kb)\, se^{-sb}+k\right] = \frac{k}{k^2+s^2}
\end{aligned}$$

where $\lim_{b\to\infty}\cos(kb)ke^{-sb} = \lim_{b\to\infty}\sin(kb)se^{-sb} = 0$ follows from applying the squeeze theorem and it is assumed $s > 0$.

(c) Let the function $f(t) = t^{k-1}/\Gamma(k)$ be given. The *Laplace transform* of $f(t)$ is then given by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \frac{t^{k-1}e^{-st}}{\Gamma(k)}\, dt = \frac{1}{\Gamma(k)}\int_0^\infty t^{k-1}e^{-st}\, dt \overset{(6.85)}{=} \frac{1}{s^k}$$

where it is assumed that $s > 0$, $k > 0$.

(d) Let the function $f(t) = \sum_{k=0}^\infty b_{k+1}t^k/k!$ be given. The *Laplace transform* of $f(t)$ is then given by

$$F(s) = \int_0^\infty \sum_{k=0}^\infty \frac{b_{k+1}}{k!}t^k e^{-st}\, dt = \sum_{k=0}^\infty \frac{b_{k+1}}{k!}\int_0^\infty t^k e^{-st}\, dt \overset{(6.81)}{=} \sum_{k=0}^\infty \frac{b_{k+1}}{s^{k+1}} = \sum_{k=1}^\infty \frac{b_k}{s^k}$$

where it is assumed that $s > s_1$ such that $\int_0^\infty f(t)e^{-st}\, dt$ converges.

Section 6.26

1.