CHAPTER 6

Section 6.4

1. (a)

$$\lim_{n\to\infty}\frac{n^2+1}{n^3+1}=\lim_{n\to\infty}\frac{(1/n^2)+1/n^3}{1+1/n^3}=\frac{0}{1}=0$$

(b)

$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{d(\ln n)/dn}{d(n)/dn}=\lim_{n\to\infty}\frac{1/n}{1}=\frac{0}{1}=0$$

(c)

$$\lim_{n\to\infty}\frac{n}{2^n}=\lim_{n\to\infty}\frac{1}{2^n\ln 2}=\frac{1}{\infty}=0$$

(d)

$$\begin{aligned}\lim_{n\to\infty}n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+1/n)}{n^{-1}}&=\lim_{n\to\infty}\frac{d[\ln(1+1/n)]/dn}{d(n^{-1})/dn}\\&=\lim_{n\to\infty}\frac{1/(n^2+n)}{1/n^2}\\&=\lim_{n\to\infty}\frac{1}{1+1/n}\\&=\frac{1}{1+0}=1\end{aligned}$$

(e)

$$\lim_{n\to\infty}s_n=1\text{ for }n=1,2,3,\cdots\implies\lim_{n\to\infty}s_n=1$$

2. (a)

$$\overline{\lim}_{n\to\infty}\cos n\pi=1\qquad\qquad\underline{\lim}_{n\to\infty}\cos n\pi=-1\tag{1}$$

(b)

$$\overline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx 0.951\qquad\qquad\underline{\lim}_{n\to\infty}\sin\frac{1}{5}n\pi\approx -0.951$$

(c)

$$\overline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=\infty\qquad\qquad\underline{\lim}_{n\to\infty}n\sin\frac{1}{2}n\pi=-\infty$$

3. (a) A sequence

$$s_n=1+\cos n\pi$$

has limits

$$\overline{\lim}_{n\to\infty}s_n=2\qquad\qquad\underline{\lim}_{n\to\infty}s_n=0$$

(b) A sequence

$$s_n = -n^2 \sin^2\left(\frac{1}{2}n\pi\right)$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = 0 \qquad \underline{\lim}_{n\to\infty} s_n = -\infty$$

(c) A sequence

$$s_n = n$$

has limits

$$\overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n = \infty$$

4. Let a sequence $s_n = 1/n$ be given. Now this sequence converges, since

$$s = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, for every $\epsilon > 0$ an N can be found such that

$$|s_n - s| < \frac{\epsilon}{2}$$

for all $n > N$. Hence, for all $m, n > N$

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so condition (6.10) is satisfied.

5. In order to define e to 2 decimal places from its definition

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

we let $\epsilon = 0.00828$ in order to find a value N such that (6.5)

$$|s_n - s| < \epsilon \quad \text{for } n > N$$

Inserting in the condition above then gives

$$\left|\left(1 + \frac{1}{n}\right)^n - e\right| < 0.00828$$

which is satisfied for $n = 164$. Hence,

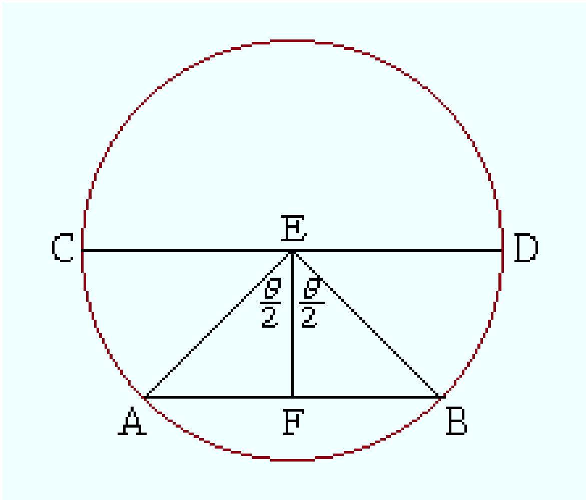
$$e \approx \left(1 + \frac{1}{164}\right)^{164} \approx 2.71$$

6.

$$\overline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } |x| > 1$$
$$\overline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = \pm 1$$
$$\overline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$

$$\underline{\lim}_{n\to\infty} x^n = -\infty \qquad \text{for } x < -1$$
$$\underline{\lim}_{n\to\infty} x^n = -1 \qquad \text{for } x = -1$$
$$\underline{\lim}_{n\to\infty} x^n = 0 \qquad \text{for } |x| < 1$$
$$\underline{\lim}_{n\to\infty} x^n = 1 \qquad \text{for } x = 1$$
$$\underline{\lim}_{n\to\infty} x^n = \infty \qquad \text{for } x > 1$$

7.



Assuming the figure above represents the unit circle, it follows that $AE = BE = 1$ and that the area of the polygon AEB is given by

$$A(AEB) = \frac{1}{2} AB \times EF = AF \times EF = AE \sin\frac{\theta}{2} \times AE \cos\frac{\theta}{2} = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\theta}{2}$$

The area of the unit circle may then be approximated as the sum of the areas of n such polygons in the limit $n \to \infty$:

$$A_{S_1} = s_n = \lim_{n\to\infty} \frac{n}{2} \sin\frac{2\pi}{n} = \lim_{n\to\infty} \pi \frac{n}{2\pi} \sin\frac{2\pi}{n}$$

Now using the fact that $\lim_{x\to 0} \sin(x)/x = 1$ and setting $x = 2\pi/n$ we find

$$A_{S_1} = s_n = \lim_{n\to\infty} \pi \frac{n}{2\pi} \sin\frac{2\pi}{n} = \lim_{x\to 0} \pi \frac{\sin x}{x} = \pi$$

Hence, since the sequence s_n is bounded and has limit π, it is monotone increasing.

Section 6.7

1. (a) Since

$$\overline{\lim}_{n\to\infty} \sin\left(\frac{n^2\pi}{2}\right) = 1 \neq 0$$

then by the nth term test $\sum_{n=1}^{\infty} \sin(n^2\pi/2)$ diverges.

(b) Since

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^{n-1}}{3n} = \lim_{n\to\infty} \frac{(n-1)\,2^{n-2}}{3} = \infty \neq 0$$

employing *L'Hospital's rule*, then by the nth term test $\sum_{n=1}^{\infty} 2^n/n^3$ diverges.

2. (a) Since $n^3 > n$ for $n > 0$ it follows that

$$\frac{1}{n^3-1} = \left|\frac{1}{n^3-1}\right| < \frac{1}{n-1}$$

for $n = 2, 3, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \frac{1/n}{1-(1/n)} = 0$$

then $\sum_{n=2}^{\infty} 1/(n-1)$ converges and hence, by the comparison test for convergence $\sum_{n=2}^{\infty} 1/(n^3-1)$ is absolutely convergent.

(b) Since $|\sin n| < 1$ for $n \geq 1$ it follows that

$$\left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$$

for $n = 1, 2, \ldots$. Now since

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

then $\sum_{n=1}^{\infty} 1/n^2$ converges and hence, by the comparison test for convergence $\sum_{n=1}^{\infty} \sin(n)/n^2$ is absolutely convergent.

3. (a) Since $n+5 > n$ and $n^2 - 3n - 5 < n^2$ for $n \geq 1$ it follows that

$$\frac{n+5}{n^2-3n-5} > \frac{n}{n^2} = \frac{1}{n}$$

for $n = 1, 2, \ldots$. Now since $\sum_{n=1}^{\infty} 1/n$ is the *harmonic series*, which diverges, it follows by the comparison test for divergence that $\sum_{n=1}^{\infty}(n+5)/(n^2-3n-5)$ diverges as well.

(b) Since $\sqrt{n}\ln n < n\ln n$ for $n \geq 2$ it follows that

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n\ln n}$$

for $n = 2, 3, \ldots$. Using the inequality $\ln(1+x) \leq x$ we may continue to write

$$\frac{1}{n\ln n} \geq \frac{\ln(1+1/n)}{\ln n} \geq \ln\left(1 + \frac{\ln(1+1/n)}{\ln n}\right) \geq \ln\frac{\ln(1+n)}{\ln n}$$

In summary, we find

$$\frac{1}{\sqrt{n}\ln n} > \ln\frac{\ln(1+n)}{\ln n} = \ln\ln(1+n) - \ln\ln n$$

Now let us consider the series

$$\sum_{n=2}^{N} \ln\ln(1+n) - \ln\ln n = \ln\ln(1+N) - \ln\ln 2$$

Hence, when $N \to \infty$

$$\sum_{n=2}^{\infty} \ln\ln(1+n) - \ln\ln n = \lim_{N\to\infty} \ln\ln(1+N) - \ln\ln 2 = \infty$$

And so by the comparison test for divergence we may conclude that $\sum_{n=2}^{\infty} 1/(\sqrt{n}\ln n)$ diverges as well.

4. (a) Let $y = f(x) = 1/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 1$ then evaluates to

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_{\pi/4}^{\tan^{-1} b} du = \lim_{b\to\infty} u\Big|_{\pi/4}^{\tan^{-1} b} &= \lim_{b\to\infty} \tan^{-1} b - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

where we have used the substitution $x = \tan u$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=1}^{\infty} 1/(n^2+1)$.

(b) let $y = f(x) = 1/(x\ln^2 x)$. As such, $f(x)$ is defined and continuous for $c \leq x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 2$ then evaluates to

$$\begin{aligned}\int_2^\infty \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln^2 x} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln b} &= \frac{1}{\ln 2} - \lim_{b\to\infty}\frac{1}{\ln b} \\ &= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}\end{aligned}$$

where we have used the substitution $u = \ln x$. Hence, by the integral test, since the improper integral $\int_2^\infty f(x)\,dx$ converges so will the series $a_n = f(n) = \sum_{n=2}^\infty 1/(n\ln^2 n)$.

5. (a) Let $y = f(x) = x/(x^2+1)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, (fx) decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 1$ then evaluates to

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\frac{1}{2}\int_2^{b^2+1}\frac{du}{u} &= \lim_{b\to\infty}\left.\frac{\ln u}{2}\right|_2^{b^2+1} \\ &= \lim_{b\to\infty}\frac{\ln|b^2+1| - \ln 2}{2} \\ &= \infty - \frac{\ln 2}{2} = \infty\end{aligned}$$

where we have used the substitution $u = x^2+1$. Hence, by the integral test, since the improper integral $\int_1^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=1}^\infty n/(n^2+1)$.

(b) Let $y = f(x) = 1/(x\ln x\ln\ln x)$. As such, $f(x)$ is defined and continuous for $c \le x < \infty$, $f(x)$ decreases as x increases and $\lim_{x\to\infty} f(x) = 0$. The improper integral $\int_c^\infty f(x)\,dx$ with $c = 10$ then evaluates to

$$\begin{aligned}\int_{10}^\infty \frac{dx}{x\ln x\ln\ln x} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{x\ln x\ln\ln x} &= \lim_{b\to\infty}\int_{\ln 10}^{\ln b}\frac{du}{u\ln u} \\ &= \lim_{b\to\infty}\int_{\ln\ln 10}^{\ln\ln b}\frac{dv}{v} \\ &= \lim_{b\to\infty}\ln v\Big|_{\ln\ln 10}^{\ln\ln b} \\ &= \lim_{b\to\infty}\ln\ln\ln b - \ln\ln\ln 10 \\ &= \infty - \ln\ln\ln 10 = \infty\end{aligned}$$

where we have used the substitutions $u = \ln x$ and $v = \ln u$. Hence, by the integral test, since the improper integral $\int_{10}^\infty f(x)\,dx$ diverges so will the series $a_n = f(n) = \sum_{n=10}^\infty 1/(n\ln n\ln\ln n)$.

6. (a) Let $a_n = (-1)^n/n!$. As such we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty}\left|\frac{n!}{(-1)^n}\frac{(-1)^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|-\frac{1}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^\infty (-1)^n/n!$ is absolutely convergent.

(b) Let $a_n = 2^n + 1/(3^n + n)$. As such we find

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L &= \lim_{n\to\infty}\left|\frac{3^n+n}{2^n+1}\frac{2^{n+1}+1}{3^{n+1}+n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{3^n\left(1+n/3^n\right)}{2^n\left(1+1/2^n\right)}\frac{2^{n+1}\left(1+1/2^{n+1}\right)}{3^{n+1}\left[1+(n/3^{n+1})+(1/3^{n+1})\right]}\right| \\
&= \frac{2}{3}\lim_{n\to\infty}\left|\frac{\left(1+n/3^n\right)\left(1+1/2^{n+1}\right)}{\left(1+1/2^n\right)\left(1+n/3^{n+1}+1/3^{n+1}\right)}\right| \\
&= \frac{2}{3}\frac{(1+0)\cdot(1+0)}{(1+0)\cdot(1+0+0)} = \frac{2}{3}
\end{aligned}$$

where we have used the fact that

$$\lim_{x\to\infty}\frac{x}{a^x} = \lim_{x\to\infty}\frac{1}{xa^{x-1}} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. Hence, $L < 1$ and so according to the ratio test the series $\sum_{n+1}^{\infty} 2^n + 1/(3^n + n)$ is absolutely convergent.

7. (a) Let $a_n = 1/\ln n$. Then for $2 \le n < \infty$ we find

$$a_n = \frac{(-1)^n}{\ln n} = \frac{1}{\ln n}$$

Now since $\ln n$ is monotonically increasing for $2 \le n < \infty$ we may conclude that $a_n = 1/\ln n$ is monotonically decreasing for $2 \le n < \infty$ and so $a_{n+1} \le a_n$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\ln n} = \frac{1}{\infty} = 0$$

provided $n \ge 2$ and so by the alternating series test we may conclude that the series

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$$

converges.

(b) Let $f(x) = \ln x/x$. Hence,

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2}$$

Note that the derivative of $f(x)$ becomes negative when $x > e \approxeq 2.71828$ and hence, that $f(x)$ becomes monotonically decreasing when $e < x < \infty$. As such, the terms of the sequence $a_n = f(n) = \ln n/n$ are decreasing (i.e. $a_{n+1} \le a_n$) when $3 \le n < \infty$. Furthermore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0$$

using *L'Hospital's rule*. As such, by the alternating series test we may conclude that the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$$

converges.

8. (a)