CHAPTER 5

Section 5.3

1. (a) From the given end points $(0,0)$, $(2,2)$ it follows that we can represent the curve C in the form $y = x$, $0 \le x \le 2$. Hence, by (5.6) we find

$$\int_{(0,0)}^{(2,2)} y^2\, dx = \int_0^2 x^2\, dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

(b) Given the end points $(2,1)$, $(1,2)$ we will parameterise the curve C according to: $x = 2 - t$, $y = 1 + t$, $0 \le t \le 1$. Then by (5.4) we find

$$\int_{(2,1)}^{(1,2)} y\, dx = -\int_0^1 (1+t)\; dt = -\left[t + \frac{t^2}{2}\right]_0^1 = -\frac{3}{2}$$

(c) Given the end points $(1,1)$, $(2,1)$ we will parameterise the curve C according to $x = 1 + t$, $y = 1$, $0 \le t \le 1$. Then by (5.5) we find

$$\int_{(1,1)}^{(2,1)} x\, dy = \int_0^1 (1+t)\,(0)\; dt = 0$$

2. (a) Let us represent the curve $C : x = \sqrt{1 - y^2}$ in the form $x = \cos t$, $y = \sin t$, $-\pi/2 \le t \le \pi/2$. Then by (5.4) and (5.5)

$$\begin{aligned}
\int_{(0,-1)}^{(0,1)} y^2\, dx + x^2\, dy &= \int_{-\pi/2}^{\pi/2} -\sin^3 t\, dt + \cos^3 t\, dt \\
&= \int_{-\pi/2}^{\pi/2} -\left(1 - \cos^2 t\right) \sin t + \left(1 - \sin^2 t\right) \cos t\, dt \\
&= \left[\cos t - \frac{\cos^3 t}{3} + \sin t - \frac{\sin^3 t}{3}\right]_{-\pi/2}^{\pi/2} = \frac{4}{3}
\end{aligned}$$

(b) Let C be the parabola $y = x^2$. Then by (5.6) and (5.7) we find

$$\int_{(0,0)}^{(2,4)} y\, dx + x\, dy = \int_0^2 \left(x^2 + 2x^2\right)\; dx = \left[\frac{x^3}{3} + \frac{2}{3}x^3\right]_0^2 = 8$$

(c) Let C be the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi/2$ and let us use the substitution $u = \tan^3 t$. Then by (5.4) and (5.5) we can rewrite the integral as

$$\begin{aligned}
\int_{(1,0)}^{(0,1)} \frac{y\, dx - x\, dy}{x^2 + y^2} &= -3\int_0^{\pi/2} \frac{\sin^4 t \cos^2 t + \sin^2 t \cos^4 t}{\cos^6 t + \sin^6 t}\, dt = \int_0^{\pi/2} \frac{-3 \sin^2 t \cos^2 t}{\cos^6 t + \sin^6 t}\, dt \\
&= -\int_0^{\infty} \frac{\cos^6 t}{\cos^6 t + \sin^6 t}\, du = -\int_0^{\infty} \frac{du}{1 + u^2} = \lim_{b \to \infty} -\int_0^b \frac{du}{1+u^2} \\
&= \lim_{b\to\infty} -\tan^{-1} u\Big|_0^b = \lim_{b \to \infty} -\tan^{-1} b = -\frac{\pi}{2}
\end{aligned}$$

3. (a) Let C be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Then the integral

$$\oint_C y^2\,dx + xy\,dy$$

can be evaluated by computing the sum of the four integrals

$$\underbrace{\int_{(1,1)}^{(-1,1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(-1,1)}^{(-1,-1)} xy\,dy}_{dx=0} \qquad \underbrace{\int_{(-1,-1)}^{(1,-1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(1,-1)}^{(1,1)} xy\,dy}_{dx=0}$$

Hence,

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy &= \int_1^{-1} dx - \int_1^{-1} y\,dy + \int_{-1}^{1} dx + \int_{-1}^{1} y\,dy \\ &= x\Big|_1^{-1} - \frac{y^2}{2}\Big|_1^{-1} + x\Big|_{-1}^{1} + \frac{y^2}{2}\Big|_{-1}^{1} = 0\end{aligned}$$

(b) Let C be the circle $x^2+y^2=1$. Using the parameterization $x=\cos t$, $y=\sin t$ where $0 \le t \le 2\pi$, then by (5.4) and (5.5) the integral

$$\oint_C y\,dx - x\,dy$$

may be written as

$$\begin{aligned}\oint_C y\,dx - x\,dy = \int_0^{2\pi} -\sin^2 t\,dt - \cos^2 t\,dt = -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\,dt &= -\int_0^{2\pi} dt \\ &= -2\pi\end{aligned}$$

(c) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Then the integral

$$\oint_C x^2y^2\,dx - xy^3\,dy$$

can be evaluated by computing the sum of the three integrals

$$\underbrace{\int_{(0,0)}^{(1,0)} x^2y^2\,dx = 0}_{dy=0} \qquad \underbrace{-\int_{(1,0)}^{(1,1)} xy^3\,dy}_{dx=0} \qquad \int_{(1,1)}^{(0,0)} x^2y^2\,dx - xy^3\,dy$$

Hence,

$$\begin{aligned}\oint_C x^2y^2\,dx - xy^3\,dy &= -\int_0^1 y^3\,dy + \int_0^1 x^4\,dx - \int_0^1 y^4\,dy \\ &= -\frac{y^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 - \frac{y^5}{5}\Big|_0^1 = -\frac{1}{4}\end{aligned}$$

4. (a) Let C be the circle $x^2 + y^2 = 4$. Then using the parametrisation $x = 4\cos t$, $y = 4\sin t$, where $0 \leq t \leq 2\pi$ and (5.12) the integral

$$\oint_C \left(x^2 - y^2\right) \, ds$$

may be written as

$$\oint_C \left(x^2 - y^2\right) \, ds = 64 \int_0^{2\pi} \left(\cos^2 t - \sin^2 t\right) \, dt = 64 \int_0^{2\pi} \cos 2t \, dt = 32 \sin 2t\Big|_0^{2\pi} = 0$$

(b) Let C be the line $y = x$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) the integral

$$\int_{(0,0)}^{(1,1)} x \, ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} x \, ds = \sqrt{2} \int_0^1 x \, dx = \frac{\sqrt{2}}{2} x^2 \Big|_0^1 = \frac{1}{\sqrt{2}}$$

(c) Let C be the parabola $y = x^2$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) and using the substitution $x = (1/2)\tan u$, such that $dx = (1/2)\sec^2 u \, du$ the integral

$$\int_{(0,0)}^{(1,1)} ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1 + 4x^2} \, dx = \frac{1}{2} \int_0^{\tan^{-1} 2} \sec^3 u \, du$$

In order to solve the integral on the right hand side, let us solve the indefinite integral

$$\begin{aligned}
\int \sec^3 x \, dx = \int_0 \sec^2 x \sec x \, dx &= \sec x \tan x - \int \sec x \tan^2 x \, dx + C \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) \, dx + C \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx + C
\end{aligned}$$

Adding the term $\int \sec^3 x \, dx$ to both sides and dividing by two then gives

$$\begin{aligned}
\int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx + C \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C
\end{aligned}$$

Substituting in the original equation then gives

$$\begin{aligned}\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1+4x^2}\,dx &= \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3 u\,du \\ &= \frac{1}{4}\sec u\tan u\Big|_0^{\tan^{-1}2} + \frac{1}{4}\ln|\sec u + \tan u|\Big|_0^{\tan^{-1}2} \\ &= \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5}+2\right)}{4}\end{aligned}$$

5. Let a path $x = \phi(t)$, $y = \psi(t)$, $h \le t \le k$, where x and y are continuous and have continuous derivatives for $h \le t \le k$ like (5.1) be given. Next, let us make a change of parameter by the equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. Then by (5.4) the line integral $\int_C f(x,y)\,dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$, such that $dx = (d/d\tau)\phi(g(\tau))\,d\tau$, is given by

$$\begin{aligned}\int\limits_C f(x,y)\,dx &= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right),\psi\left(g\left(\tau\right)\right)\right]\frac{d}{d\tau}\phi\left(g\left(\tau\right)\right)\,d\tau \\ &= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right),\psi\left(g\left(\tau\right)\right)\right]\frac{d\phi}{dt}\frac{d}{d\tau}g\left(\tau\right)\,d\tau \\ &= \int_h^k f\left[\phi\left(t\right),\psi\left(t\right)\right]\frac{d\phi}{dt}\frac{dt}{d\tau}\,d\tau = \int_h^k f\left[\phi\left(t\right),\psi\left(t\right)\right]\phi'\left(t\right)\,dt\end{aligned}$$

6. (a) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFG$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy \sim &\left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right] \\ &+ \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] = 7\end{aligned}$$

(b) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFGKH$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy \sim &\left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(1+4)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] \\ &+ \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+9)\cdot 1\right] + \left[\frac{1}{2}(0+2)\cdot 1 + \frac{1}{2}(9+8)\cdot -1\right] \\ = 5\end{aligned}$$

(c) Using (a), the integral $\int P\,dx+Q\,dy$ along the path $C \to ABCDHLSONMIEA$ may be approximated as

$$\begin{aligned}
\int_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+8)\cdot 1 + \frac{1}{2}(2+3)\cdot 0\right] \\
& + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+4)\cdot 0\right] + \left[\frac{1}{2}(5+2)\cdot 0 + \frac{1}{2}(4+8)\cdot 1\right] \\
& + \left[\frac{1}{2}(2+1)\cdot 0 + \frac{1}{2}(8+2)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 0 + \frac{1}{2}(2+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(4+3)\cdot -1 + \frac{1}{2}(6+2)\cdot 0\right] + \left[\frac{1}{2}(3+7)\cdot -1 + \frac{1}{2}(2+8)\cdot 0\right] \\
& + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] \\
& + \left[\frac{1}{2}(8+3)\cdot 0 + \frac{1}{2}(3+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
= & \ 8
\end{aligned}$$

(d) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFJNMIJFA$ may be approximated as

$$\begin{aligned}
\int_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(4+1)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 0 + \frac{1}{2}(4+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(5+7)\cdot 0 + \frac{1}{2}(6+8)\cdot 1\right] + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] \\
& + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+6)\cdot 0\right] \\
& + \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+4)\cdot -1\right] + \left[\frac{1}{2}(0+0)\cdot -1 + \frac{1}{2}(4+1)\cdot -1\right] \\
= & \ \frac{11}{2}
\end{aligned}$$

(e) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFEAEFBA$ may be approximated as

$$\int_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right]$$
$$+ \left[\frac{1}{2}(0+3)\cdot -1 + \frac{1}{2}(4+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right]$$
$$+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(1+2)\cdot 1\right] + \left[\frac{1}{2}(3+0)\cdot 1 + \frac{1}{2}(2+4)\cdot 0\right]$$
$$+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(4+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot -1 + \frac{1}{2}(2+1)\cdot 0\right]$$
$$= 0$$

7. Let C be a smooth curve in the xy-plane and let $f(x,y) > 0$ be a continuous function defined over a region of the xy-plane containing the curve C. The equation $z = f(x,y)$ then is the equation of a surface that lies above the region of the xy-plane containing the curve C. Next, we imagine moving a straight line along C perpendicular to the xy-plane, effectively tracing out a "wall" standing on C, orthogonal to the xy-plane. This "wall" cuts the surface $z = f(x,y)$, forming a curve on it that lies above the curve C. In fact, the curve C may be interpreted as the projection of the surface curve onto the xy-plane. Using (5.11), the line integral

$$\int_C f(x,y)\;ds = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta_i s$$

then may be interpreted as an infinite sum of the length of each straight line directed from C to the surface curve lying above it in the limit where the distance Δs between each subsequent line becomes infinitely small, effectively tracing out a "wall" with height at each point (x,y) given by $f(x,y)$. This may be interpreted the as the area of the cylindrical surface $0 \le z \le f(x,y)$, (x,y) on C.

Section 5.5

1. Let the vector $v = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ be given. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_T\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.9)}{=} \int_0^1 2x^2\,dx + \int_0^1 2y^2\,dy = \frac{4}{3}$$

   (b) The integral $\int_C v_T\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.7)}{=} \int_0^1 \left(x^2+5x^4\right)\,dx = \frac{4}{3}$$

(c) The integral $\int_C v_T\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_T\,ds &= \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_{(0,0)}^{(1,0)} \left(x^2+y^2\right)\,dx + 2xy\,dy + \int_{(1,0)}^{(1,1)} \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{4}{3}\end{aligned}$$

2. Let $\mathbf{v} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be the same vector as given in Problem 1, and let $\mathbf{n}$ be the unit normal vector 90° behind the tangent vector $\mathbf{T}$ as given by (5.37). Then the normal component of $\mathbf{v}$ is given by $v_n = \mathbf{v}\cdot\mathbf{n} = (P\mathbf{i}+Q\mathbf{j})\cdot(y_s\mathbf{i} - x_s\mathbf{j}) = -Qx_s + Py_s$. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_n\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.9)}{=} \int_0^1 -2x^2\,dx + \int_0^1 2y^2\,dy = 0$$

   (b) The integral $\int_C v_n\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.7)}{=} \int_0^1 2x^5\,dx = \frac{1}{3}$$

   (c) The integral $\int_C v_n\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_n\,ds &= \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_{(0,0)}^{(1,0)} -2xy\,dx + \left(x^2+y^2\right)\,dy + \int_{(1,0)}^{(1,1)} -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_0^1 \left(1+y^2\right)\,dy = \frac{4}{3}\end{aligned}$$

3. Let the gravitational force near a point on the earth's surface be represented approximately by the vector $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = -mg\mathbf{j}$, where the y-axis points upwards. Then by (5.29) and the fact that $P(x,y) = 0$ the work done by the force $\mathbf{F}$ on a body moving in a vertical plane from height h_1 to height h_2 along any path is equal to

$$\int\limits_C F_T\,ds = \int\limits_C \left(P\cos\alpha + Q\sin\alpha\right)\,ds = \int\limits_C Q\,dy = -\int_{h_1}^{h_2} mg\,dy = -mgy\Big|_{h_1}^{h_2} = mg\left(h_1 - h_2\right)$$

4. Let the gravitational force $\mathbf{F}$ be given by $\mathbf{F} = -(kMm/r^2)(\mathbf{r}/r)$. Then in order to compute the work by the gravitational force $\mathbf{F}$ in bringing a particle to its present position r from infinite distance along the ray through the earth's center, we will represents the curve C in terms of parameter t and then use (5.34) to get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\infty^r \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r \left(-\frac{kMm}{t^2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r -\frac{kMm}{t^2}\,dt &= \left.\frac{kMm}{t}\right|_\infty^r \\ &= kMm\left(\frac{1}{r}-\frac{1}{\infty}\right) = \frac{kMm}{r} \\ &= -U\end{aligned}$$

where $(d\mathbf{r}/dr)\cdot(d\mathbf{r}/dr) = 1$ follows from the fact that $d\mathbf{r}/dr$ is a unit vector.

5. (a) By (5.40) the integral $\oint_C ay\,dx + bx\,dy$ may be written as

$$\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\ dx\,dy = (b-a)\,A$$

where A is the area enclosed by the curve C.

(b) By (5.40) the integral $\oint e^x \sin y\,dx + e^x\cos y\,dy$ around the rectangle with vertices $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$ may be written as

$$\oint e^x\sin y\,dx + e^x\cos y\,dy = \int_0^{\pi/2}\int_0^1 (e^x\cos y - e^x\cos y)\ dx\,dy = 0$$

(c) By (5.40) and (4.61) the integral $\oint(2x^3-y^3)\,dx + (x^3+y^3)\,dy$ around the circle $x^2+y^2=1$ may be written as

$$\oint\left(2x^3-y^3\right)\ dx + \left(x^3+y^3\right)\ dy = 3\int_0^1\int_0^{2\pi} r^3\,d\theta\,dr = 6\pi\int_0^1 r^3\,dr = \frac{3\pi}{2}$$

(d) By (5.43) and (3.31) the integral $\oint_C u_T\,ds$, where $\mathbf{u} = \operatorname{grad}(x^2y)$ and C is the circle $x^2+y^2=1$ may be written as

$$\oint_C u_T\,ds = \iint_R \operatorname{curl}_z\mathbf{u}\,dx\,dy = \iint_R \operatorname{curl}_z \operatorname{grad}\left(x^2y\right)\ dx\,dy = 0$$

(e) By (5.44) the integral $\oint_C v_n\,ds$, where $\mathbf{v} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j}$ and C is the circle $x^2+y^2=1$ ($\mathbf{n}$ being the outer normal) may be written as

$$\begin{aligned}\oint_C v_n\,ds = \iint_R \operatorname{div}\mathbf{v}\,dx\,dy = \iint_R \operatorname{div}\ \left[\left(x^2+y^2\right)\mathbf{i} - 2xy\mathbf{j}\right]\ dx\,dy = \iint_R (2x-2x)\ dx\,dy \\ = 0\end{aligned}$$

(f) Let $F = (x-2)^2 + y^2$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44) and (4.64)

$$\oint_C v_n\, ds = \iint_R \text{div } (\nabla F)\ dx\, dy = \iint_R \nabla \cdot \nabla F\, dx\, dy = \iint_R \nabla^2 F\, dx\, dy$$
$$= \iint_R \nabla^2 \left[(x-2)^2 + y^2\right]\ dx\, dy$$
$$= 4\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = 4\pi$$

(g) Let $F = \ln[(x-2)^2 + y^2]^{-1}$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44)

$$\oint_C v_n\, ds = \iint_R \text{div } (\nabla F)\ dx\, dy = \iint_R \nabla \cdot \nabla F\, dx\, dy = \iint \nabla^2 F\, dx\, dy$$
$$= \iint_R \nabla \ln \frac{1}{(x-2)^2 + y^2}\, dx\, dy$$
$$= 2\iint_R \frac{x^2 - 4x + 4 - y^2 - (x-2)^2 + y^2}{\left[(x-2)^2 + y^2\right]^2}\, dx\, dxy = 0$$

(h) By (5.40) the integral $\oint_C f(x)\, dx + g(y)\, dy$ may be written as

$$\oint_C f\left(x\right)\ dx + g\left(y\right)\ dy = \iint_R \left[\frac{\partial}{\partial x} g\left(y\right) - \frac{\partial}{\partial y} f\left(x\right)\right]\ dx\, dy = 0$$

6. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of an arbitrary point (x, y) and let $\mathbf{n}$ be the outer normal to some arbitrary closed curve C. Then by (5.44)

$$\frac{1}{2}\oint_C r_n\, ds = \frac{1}{2}\oint_C \mathbf{r} \cdot \mathbf{n}\, ds = \frac{1}{2}\iint_R \text{div}\, \mathbf{r}\, dx\, dy = \frac{1}{2}\iint_R \nabla \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy$$
$$= \frac{1}{2}\iint_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy$$
$$= \iint_R dx\, dy = A$$

7. As for Problem 2(a), let the line integral $\int_{(0,-1)}^{(0,1)} y^2\, dx + x^2\, dy$, where C is the semi-circle

$x = \sqrt{1-y^2}$ be given. Then by (5.40) and (4.64)

$$\begin{aligned}
\oint_C y^2\,dx + x^2\,dy = \iint_R \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right)\,dx\,dy &= 2\iint_R (x-y)\,dx\,dy \\
&= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (x-y)\,dx\,dy \\
&= 2\int_{-\pi/2}^{\pi/2}\int_{0}^{1} (\cos\theta - \sin\theta)\,r^2\,dr\,d\theta \\
&= \frac{2}{3}\int_{\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\,d\theta = \frac{4}{3}
\end{aligned}$$

As for Problem 3(a), let the line integral $\oint_C y^2\,dx + xy\,dy$, where C is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ be given. Then by (5.40)

$$\begin{aligned}
\oint_C y^2\,dx + xy\,dy = \iint_R \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}y^2\right]\,dx\,dy = -\iint_R y\,dx\,dy &= -\int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy \\
&= -\int_{-1}^{1} xy\big|_{-1}^{1}\,dy = -2\int_{-1}^{1} y\,dy \\
&= -y^2\big|_{-1}^{1} = 0
\end{aligned}$$

As for Problem 3(b), let the line integral $\oint_C y\,dx - x\,dy$, where C is the circle $x^2+y^2=1$ be given. Then by (5.40) and (4.64)

$$\begin{aligned}
\oint_C y\,dx - x\,dy = -\iint_R \left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\,dx\,dy = -2\iint_R dx\,dy &= -2\int_0^{2\pi}\int_0^1 r\,dr\,d\theta \\
&= -\int_0^{2\pi} d\theta = -2\pi
\end{aligned}$$

As for Problem 3(c), let the line integral $\oint_C x^2y^2\,dx - xy^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ be given. Then by (5.40)

$$\begin{aligned}
\oint_C x^2y^2\,dx - xy^3\,dy &= -\iint_R \left[\frac{\partial}{\partial x}\left(xy^3\right) + \frac{\partial}{\partial y}\left(x^2y^2\right)\right]\,dx\,dy \\
&= -\iint_R \left(y^3 + 2x^2y\right)\,dx\,dy = -\int_0^1\int_0^x \left(y^3+2x^2y\right)\,dy\,dx \\
&= -\int_0^1 \left[\frac{y^4}{4} + x^2y^2\right]_0^x\,dx = -\frac{5}{4}\int_0^1 x^4\,dx = -\frac{1}{4}
\end{aligned}$$

As for Problem 4(a), let the line integral $\oint_C (x^2 - y^2)\, ds$, where C is the circle $x^2 + y^2 = 4$ be given. Then by (5.44) and the fact that $\mathbf{n}$ may be written as $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/|x + y|$

$$\oint_C (x^2 - y^2)\, ds = a \iint_R \text{div } (x\mathbf{i} - y\mathbf{j})\ dx\, dy = a \iint_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) \cdot (x\mathbf{i} - y\mathbf{j})\ dx\, dy = 0$$

Section 5.7

1. (a) Let

$$dF = 2xy\, dx + x^2\, dy \qquad\qquad \int_C {}_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy$$

where C is the curve $y = x^{3/2}$. To determine the function $F(x, y)$ we firstly note that

$$dF = 2xy\, dx + x^2\, dy = P\, dx + Q\, dy$$

where the functions $P(x, y)$ and $Q(x, y)$ are defined and continuous in the domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$. From inspection it then follows that

$$F\,(x, y) = x^2 y + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = 2xy \qquad\qquad \frac{\partial F}{\partial y} = x^2$$

from which follows

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 2xy\, dx + x^2\, dy = P\, dx + Q\, dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy$$

is independent of path (and hence curve C given by $y = x^{3/2}$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy = F\,(1, 1) - F\,(0, 0) = 1$$

(b) Let

$$dF = ye^{xy}\,dx + xe^{xy}\,dy \qquad\qquad \int_C\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

where C is the curve $y = \sin^3 x$. From inspection it follows that

$$F(x,y) = e^{xy} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = ye^{xy} \qquad\qquad \frac{\partial F}{\partial y} = xe^{xy}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = ye^{xy}\,dx + xe^{xy}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

is independent of path (and hence curve C given by $y = \sin^3 x$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy = F(\pi,0) - F(0,0) = 0$$

(c) Let

$$dF = \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} \qquad\qquad \int_C\int_{(1,0)}^{(e^{2\pi},0)} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}}$$

where C is the curve $x = e^t \cos t\, y = e^t \sin t$. From inspection it follows that

$$F(x,y) = -\frac{1}{\sqrt{x^2+y^2}} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \qquad\qquad \frac{\partial F}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy$$

is independent of path (and hence curve C given by $x = e^t \cos t\, y = e^t \sin t$) and can easily be evaluated by (5.48) to have the value

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}} = F\left(e^{2\pi},0\right) - F\left(1,0\right) = 1 - e^{-2\pi}$$

2. (a) Let

$$\int_{C\ (1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2} = \int_{C\ (1,-2)}^{(3,4)} P\,dx + Q\,dy$$

where C is the line $y = 3x - 5$ be given. From inspection we can define the function $F(x,y) = -(y/x) + D$, where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{1}{x} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $x = 0$. And so by (5.48) the integral has the value $F(3,4) - F(1,-2) = -10/3$.

(b) Let

$$\int_{C\ (0,2)}^{(1,3)} \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy$$

where C is the parabola $y = 2 + x^2$ be given. From inspection we can define the function $F(x,y) = x^3/y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $y = 0$. And so by (5.48) the integral has the value $F(1,3) - F(0,2) = 1/3$.

(c) Let

$$\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy$$

where C is the circular arc $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$ be given. From inspection it follows that we cannot define a function $F(x,y)$ such that (5.46) holds. Hence, the given integral is not independent of path. Instead we use (5.6) and (5.7) to find

$$\begin{aligned}\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy &= \int_1^{-1} \left(2x\sqrt{1-x^2} - 1 - \frac{x^3}{\sqrt{1-x^2}} - 3x\right)\ dx \\ &= 2\end{aligned}$$

where we have made use of the fact the first, third and fourth term in the integral on the right hand side are odd and hence, will be zero when integrated from $x = 1$ to $x = -1$.

(d) Let

$$\int_{C\ (0,0)}^{(\pi/4,\pi/4)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$$

where C is the curve $y = 16x^3/\pi^2$. From inspection we can define the function $F(x,y) = \tan x \tan y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \sec^2 x \tan y = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = \sec^2 y \tan x = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\{x, y \mid x, y \neq k\pi/2\}$ for $k = \pm 1, 3, 5, \ldots$. And so by (5.48) the integral has the value $F(\pi/4, \pi/4) - F(0,0) = 1$.

3. (a) Let

$$\oint_C \left[\sin\left(xy\right) + xy\cos\left(xy\right)\right]\ dx + x^2\cos\left(xy\right)\ dy = \oint_C P\, dx + Q\, dy$$

where C is the circle $x^2 + y^2 = 1$. Now since $P(x,y)$ and $Q(x,y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x\cos\left(xy\right) - x^2 y\sin\left(xy\right)$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\, dx + Q\, dy$ when integrated on the circle $x^2 + y^2 = 1$ is equal to 0.

(b) Let

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2 + y^2 = 4$. Furthermore, we note that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(x-1)^2 - y^2}{\left[(x-1)^2 + y^2\right]^2}$$

in the doubly connected region D with hole A at point $(1,0)$. Hence, according to the discussion in Section 5.7 and (5.57) the integral is equal to some constant k, which is the same for all curves enclosing the hole A. We will use this fact to simplify the integral $\oint P\,dx + Q\,dy$ by evaluating it on the circle $(x-1)^2 + y^2 = 1$ instead. This is permitted since both the circles $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ enclose the hole A at point $(1,0)$. Thus we can write

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C y\,dx - (x-1)\;dy \underset{(5.40)}{=} \iint_R -2\,dx\,dy = -2\pi$$

where C now denotes the circle $(x-1)^2 + y^2 = 1$ and the last step follows from Example 1 following Section 5.7.

(c) Let

$$\oint_C y^3\,dx - x^3\,dy = \oint_C P\,dx + Q\,dy$$

where C is the square $|x| + |y| = 1$. Using (5.6) and (5.7) with $x = x$ and $y = 1 - x$ for the first quadrant, $y = 1 + x$ for the second quadrant, $y = -1 - x$ for the third quadrant and $y = -1 + x$ for the fourth quadrant, this integral can be written as

$$\oint_C y^3\,dx - x^3\,dy = \int_1^0 \left(1 - 3x + 3x^2\right)\,dx + \int_0^{-1} \left(1 + 3x + 3x^2\right)\,dx$$

$$+ \int_{-1}^0 \left(-1 - 3x - 3x^2\right)\,dx + \int_0^1 \left(-1 + 3x - 3x^2\right)\,dx = -2$$

(d) Let

$$\oint_C xy^6\,dx + \left(3x^2y^5 + 6x\right)\;dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $x^2 + 4y^2 = 4$. Next, let us define the parametrisation $x = 2\cos t$, $y = \sin t$ such that the integral on the ellipse becomes

$$\int_0^{2\pi} \left(-4\sin^7 t\cos t + 12\sin^5 t\cos^3 t + 12\cos^2 t\right)\;dt$$

Using integration by parts on the first term gives

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t\Big|_0^{2\pi} - \int_0^{2\pi} 7\sin^7 t\cos t\,dt\right]$$

from which follows

$$\int_0^{2\pi} 7\sin^7 t\cos t\,dt = \frac{7}{8}\sin^8 t\Big|_0^{2\pi}$$

and so

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t - \frac{7}{8}\sin^8 t\right]_0^{2\pi} = -\frac{\sin^8 t}{2}\Bigg|_0^{2\pi} = 0$$

Evaluating the second term gives

$$\int_0^{2\pi} 12\sin^5 t\cos^3 t = 12\int_0^{2\pi} \sin^4 t\sin t\cos^3 t\,dt = 12\int_0^{2\pi} \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt$$

Now applying the substitution $u = \cos t$, such that $du = -\sin t\,dt$ the integral becomes

$$\begin{aligned}
12\int \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt &= -12\int u^3\left(1-u^2\right)^2\,du \\
&= -12\int \left(u^3 - 2u^5 + u^7\right)\,du \\
&= -12\left[\frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8}\right] + C \\
&= -12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right] + C
\end{aligned}$$

Evaluating at the endpoints then gives

$$-12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right]_0^{2\pi} = -\frac{1}{2} + \frac{1}{2} = 0$$

Lastly, evaluating the third term gives

$$\int_0^{2\pi} 12\cos^2 t\,dt = 12\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = 6\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 12\pi$$

(e) Let

$$\oint_C (7x - 3y + 2)\,dx + (4y - 3x - 5)\,dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $2x^2+3y^2=1$. Now since $P(x,y)$ and $Q(x,y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -3$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\,dx + Q\,dy$ when integrated on the ellipse $2x^2+3y^2$ is equal to 0.

(f) Let

$$\oint_C \frac{(e^x \cos y - 1)\ dx + e^x \sin y\,dy}{e^{2x} - 2e^x \cos y + 1} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2+y^2=1$. The denominator may also be written as $e^{2x} - 2e^x \cos y + 1 = (e^x-1)^2 + 2e^x(1-\cos y)$, from which follows that the denominator is equal to 0 only for $x=0$ and $y = 2n\pi$, $n = \pm 0, 1, 2, \ldots$. Hence, according to (5.53), the integral $\int P\,dx + Q\,dy$ is independent of path in any simply connected domain D not containing the points $(0, 2n\pi)$, $n = \pm 0, 1, 2, \ldots$ for

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(e^x - e^{3x})\sin y}{(e^{2x} - 2e^x \cos y + 1)^2}$$

except at the aforementioned points. However, since the circle $x^2+y^2=1$ encloses the point $(0,0)$ we are dealing with a doubly connected domain and hence, Theorem IV does not hold. As such, let us make the substitution $x = \cos t$, $y = \sin t$ where $-\pi \le t \le \pi$ so that the integral can be written as

$$\int_{-\pi}^{\pi} \frac{[1 - e^{\cos t}\cos(\sin t)]\sin t + e^{\cos t}\sin(\sin t)\cos t}{e^{2\cos t} - 2e^{\cos t}\cos(\sin t) + 1}\,dt = \int_{-\pi}^{\pi} \frac{f(t)}{g(t)}\,dt = \int_{-\pi}^{\pi} h(t)\ dt$$

Now since $f(-t) = -f(t)$ and $g(-t) = g(t)$ it follows that $h(-t) = -h(t)$ and hence, that $h(t)$ is an odd function. Using the fact that integrating an odd function on an interval $[-a, a]$ always gives 0, we may finally conclude that the integral $\int P\,dx + Q\,dy$ evaluated on the circle x^2+y^2 is in fact 0.

4. Let

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_{(1,0)}^{(2,2)} P\,dx + Q\,dy$$

where C is an arbitrary path connecting the points $(1,0)$ and $(2,2)$ not passing through the origin. As stated in Example 2 of Section 2, $P\,dx + Q\,dy$ is a familiar differential, namely that of the polar coordinate angle θ:

$$d\theta = d\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y\,dx + x\,dx}{x^2+y^2}$$

and so we can write

$$\int_A^B \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_A^B d\theta = \theta_B - \theta_A = \text{total increase in } \theta \text{ from } A \text{ to } B$$

as θ varies continuously on the path C. The integral is thus not independent of path, but depends on the number of times C goes around the origin. As such we find that

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2+y^2} = \tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{0}{1}\right) + 2n\pi = \frac{\pi}{4} + 2n\pi$$

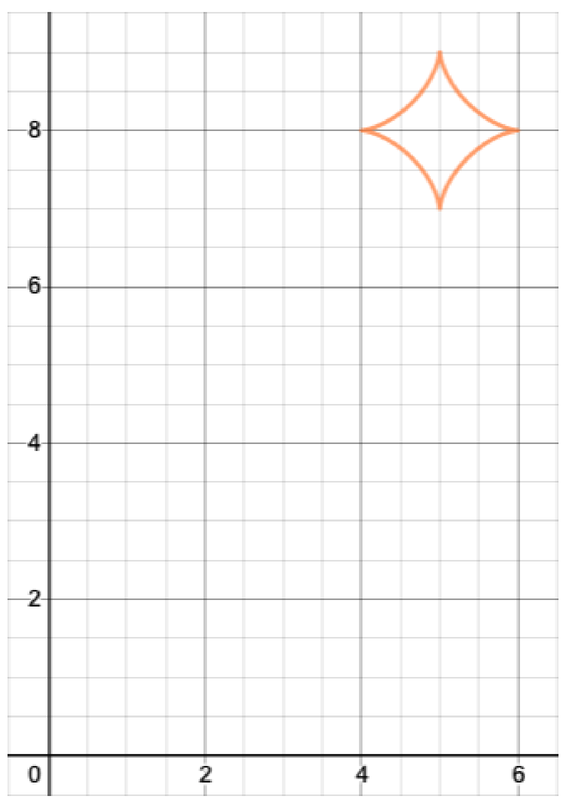
where $n = \pm 0, 1, 2, \ldots$.

5. Let

$$\int\limits_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int\limits_C P\,dx + Q\,dy$$

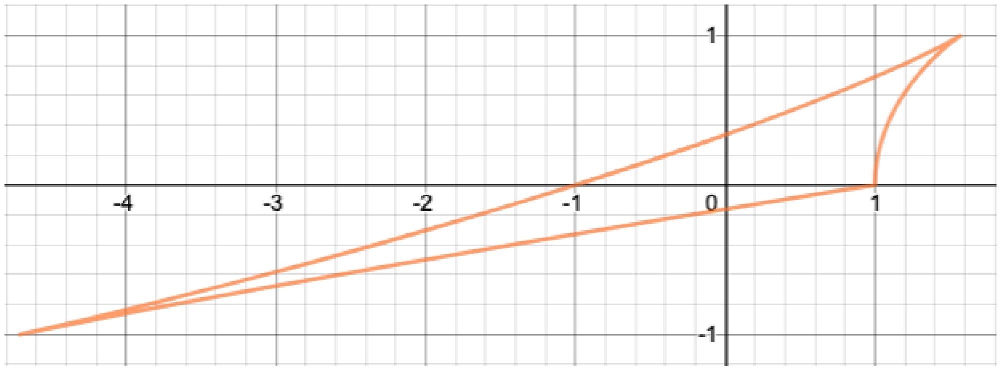
where C is a path given by $x = f(t)$, $y = g(t)$, $a \le t \le b$ not passing through the origin, for which $f(a) = f(b)$, $g(a) = g(b)$. The analysis of Section 5.6 shows that $\int_C P\,dx + Q\,dy$ equals $n \cdot 2\pi$, where n is the number of times C encircles the origin. The value of n can be determined from plotting the path.

(a) From a plot of the path $C:\ x = 5 + \cos^3 t$, $y = 8 + \sin^3 t$, $0 \le t \le 2\pi$



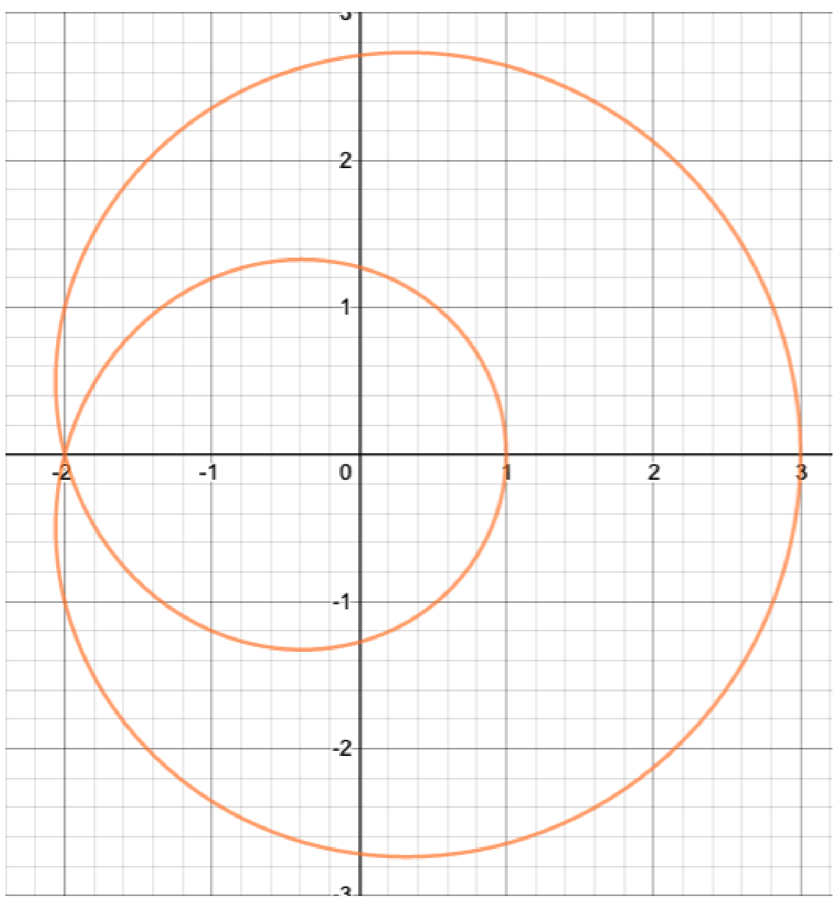
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

(b) From a plot of the path $C:\ x = \cos t + t\sin t,\ y = \sin t,\ 0 \le t \le 2\pi$



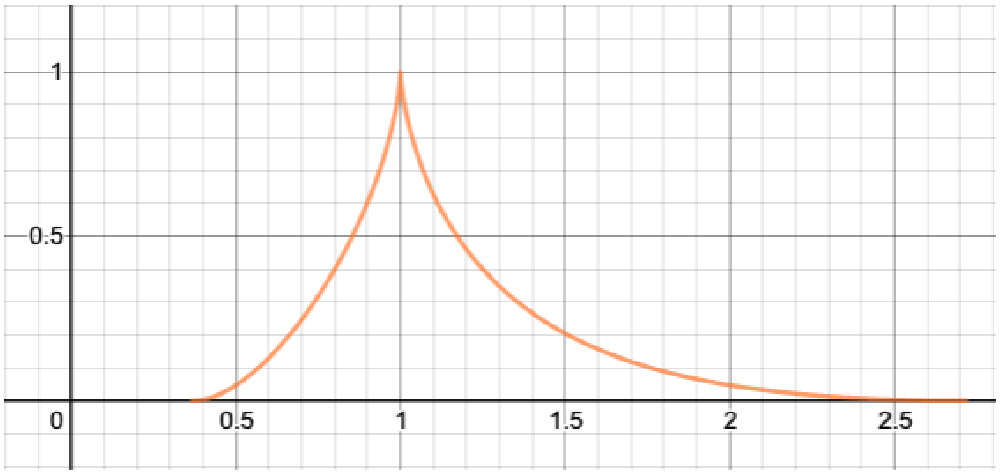
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 2π, since $n = 1$ as the path encircles the origin once.

(c) From a plot of the path $C:\ x = 2\cos 2t - \cos t,\ y = 2\sin 2t - \sin t,\ 0 \le t \le 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 4π, since $n = 2$ as the path encircles the origin twice.

(d) From a plot of the path $C : x = e^{\cos^3 t}$, $y = \sin^4 t$, $0 \le t \le 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

6. (a) Let

$$\int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 - y^2\right)\,dy = \int_{(1,1)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(1,1)$ to $(x,1)$ and $(x,1)$ to (x,y) such that the integral becomes

$$\begin{aligned} F &= \int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_{(1,1)}^{(x,1)} 2xy\,dx + \left(x^2 + y^2\right)\,dy + \int_{(x,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_1^x 2x\,dx + \int_1^y \left(x^2 + y^2\right)\,dy = x^2\Big|_1^x + \left[x^2 y + \frac{y^3}{3}\right]_1^y = x^2 y - \frac{1}{3}\left(y^3 + 2\right) \end{aligned}$$

(b) Let

$$\int_{(0,0)}^{(x,y)} \sin y\,dx + x\cos y\,dy = \int_{(0,0)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \cos y$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(0,0)$ to $(x,0)$ and $(x,0)$ to (x,y) such that the integral becomes

$$\begin{aligned}
F &= \int_{(0,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_{(0,0)}^{(x,0)} \sin y\,dy + x\cos y\,dy + \int_{(x,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_0^y x\cos y\,dy = x\sin y\Big|_0^y = x\sin y
\end{aligned}$$

7. The integral

$$\oint \frac{x^2 y\,dx - x^3\,dy}{(x^2+y^2)^2} = \oint P\,dx + Q\,dy$$

is independent of path in any simply connected domain D not containing the origin, for

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{x^2\left(x^2 - 3y^2\right)}{(x^2+y^2)^3}$$

except at the origin. Hence, the integral is 0 for any path not enclosing the origin. For the square with vertices $(\pm 1, \pm 1)$ however, the integral has a certain value k. To find k, we thus have to evaluate the integral

$$\begin{aligned}
k &= \int_{(1,1)}^{(-1,1)} \frac{x^2 y\,dx - x^3\,dy}{(x^2+y^2)^2} + \int_{(-1,1)}^{(-1,-1)} \frac{x^2 y\,dx - x^3\,dy}{(x^2+y^2)^2} + \int_{(-1,-1)}^{(1,-1)} \frac{x^2 y\,dx - x^3\,dy}{(x^2+y^2)^2} \\
&\quad + \int_{(1,-1)}^{(1,1)} \frac{x^2 y\,dx - x^3\,dy}{(x^2+y^2)^2} \\
&= \int_1^{-1} \frac{x^2\,dx}{(x^2+1)^2} + \int_1^{-1} \frac{dy}{(y^2+1)^2} - \int_{-1}^{1} \frac{x^2\,dx}{(x^2+1)^2} - \int_{-1}^{1} \frac{dy}{(y^2+1)^2} \\
&= 2\int_1^{-1} \frac{x^2\,dx}{(x^2+1)^2} + 2\int_1^{-1} \frac{dy}{(y^2+1)^2} \\
&= 2\int_1^{-1} \left[\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right] dx + 2\int_1^{-1} \frac{dy}{(y^2+1)^2} \\
&= 2\tan^{-1} x\Big|_1^{-1} - 2\int_{\pi/4}^{-\pi/4} du + 2\int_{\pi/4}^{-\pi/4} dv = 2\tan^{-1} x\Big|_1^{-1} = -\pi
\end{aligned}$$

where we have used the substitution $x = \tan u$, $y = \tan v$ such that $dx = \sec^2 u\,du$, $dy = \sec^2 v\,dv$.

8. Let D be a domain with a finite number of holes at points $A_1, A_2, \ldots, A_k$ so that D is $(k+1)$-tuply connected as in Figure 5.23. Let P and Q be continuous and have continuous derivatives in D and let $\partial P/\partial y = \partial Q/\partial x$ in D. Let C_1 denote a circle around the point A_1 in D, enclosing none of the other A's. Let C_2 be chosen similar for A_2 and so on. Futhermore, let

$$\oint_{C_1} P\,dx + Q\,dy = \alpha_1,\ \oint_{C_2} P\,dx + Q\,dy = \alpha_2, \ldots,\ \oint_{C_k} P\,dx + Q\,dy = \alpha_k$$

(a) Let C be an arbitrary simple closed path in D enclosing $A_1, A_2, \ldots, A_k$. Furthermore, we assume that the circles $C_1, C_2, \ldots, C_k$ do not intersect C at any point. Let us also define the closed region R in D whose boundaries are given by the simply closed path C and all of the circles $C_1, C_2, \ldots, C_k$ that are interior to C. Next, let us introduce auxiliary arcs from C to C_1, C to $C_2, \ldots,$ two to each so that we end up with a figure similar to Figure 5.21. These decompose the region R into $k+1$ smaller regions, each of which is simply connected (i.e. does not contain any holes in its interior). If we integrate in a positive direction around the boundary of each sub region and then add the results we find that the integrals along the auxiliary arcs cancel out, leaving just the integral around C in the positive direction plus the integrals around $C_1, C_2, \ldots, C_k$ in the negative direction. On the other hand, the line integral around the boundary of each sub region can be expressed as a double integral

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

over the sub region by Green's theorem. Hence, the sum of the line integrals is equal to the double integral over R:

$$\oint_{C} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

However, since $\partial P/\partial y = \partial Q/\partial x$ in D the integral on the right hand side is equal to zero and hence, we end up with

$$\oint_{C} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

(b) Let

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P\,dx + Q\,dy$$

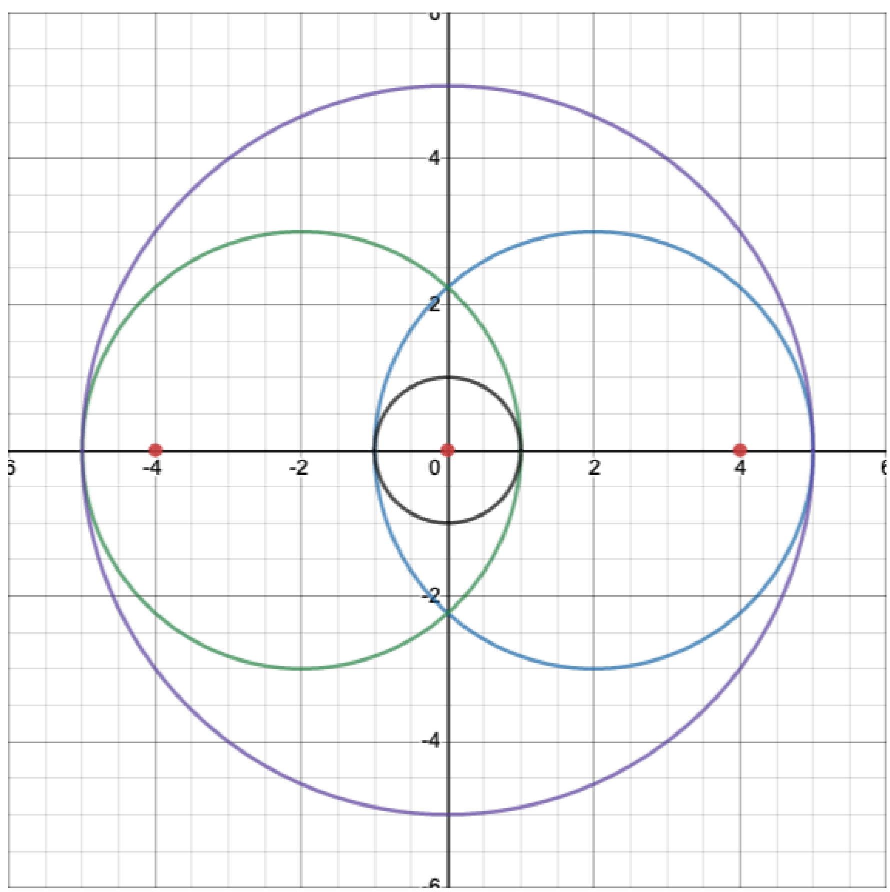
where (x_1, y_1), (x_2, y_2) are two fixed points in D and let this integral have the value K for one particular path. All possible values of the integral are then given by

$$K + n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k$$

where $n_1, \ldots, n_k$ are positive or negative integers or 0. The exact value of $n_1, \ldots, n_k$ depends on how many times a chosen path between (x_1, y_1), (x_2, y_2) encircles one of the holes at $A_1, A_2, \ldots, A_k$ and in which direction (i.e. negative for clockwise or positive for anti-clockwise).

9. Let D be a domain with holes at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$ and let P and Q be continuous and have continuous derivatives in D, with $\partial P/\partial y = \partial Q/\partial x$ except at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$. Let C_1 denote the circle $(x-2)^2 + y^2 = 9$; let C_2 denote the circle $(x+2)^2 + y^2 = 9$; let C_3 denote the circle $x^2 + y^2 = 25$. Furthermore, let it be given that

$$\oint_{C_1} P\,dx + Q\,dy = 11, \qquad \oint_{C_2} P\,dx + Q\,dy = 9, \qquad \oint_{C_3} P\,dx + Q\,dy = 13$$



From inspection of the figure it follows that

$$\oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy$$

and so

$$\oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy - \oint_{C_3} P\,dx + Q\,dy = 11 + 9 - 13 = 7$$

10. Let $F(x,y) = x^2 - y^2$

   (a) By Theorem I of Section 5.6 the integral

   $$\int_{(0,0)}^{(2,8)} \nabla F \cdot d\mathbf{r} = \int_{(0,0)}^{(2,8)} 2x\,dx - 2y\,dy = \int_{(0,0)}^{(2,8)} \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \int_{(0,0)}^{(2,8)} P\,dx + Q\,dy$$

   is independent of path in D, where D is the xy-plane. Hence, since the points $(0,0)$ and $(2,8)$ both lie on the curve $y = x^3$ the integral on the curve $y = x^3$ is equal to $F(2,8) - F(0,0) = 4 - 60 = -60$.

   (b) By (5.37) and (5.38) the integral

   $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C (Q\mathbf{i} - P\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right)\,ds = \oint_C \left(Q\frac{dy}{ds} + P\frac{dx}{ds}\right)\,ds$$
   $$= \oint_C P\,dx + Q\,dy$$

   where C is the circle $x^2 + y^2 = 1$, $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the outer normal to C and $\partial F/\partial n = \nabla F \cdot \mathbf{n}$ is the directional derivative of F in the direction of $\mathbf{n}$ (see Section 2.4). Since

   $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2$$

   then by Theorem III of Section 5.6 the integral is independent of path in D and hence, by Theorem II of Section 5.6 it follows that

   $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C P\,dx + Q\,dy = 0$$

11. Let $F(x,y)$ and $G(x,y)$ be continuous and have continuous derivatives in a domain D and let R be a closed region in D with directed boundary B_R consisting of closed curves $C_1, \ldots, C_n$ as in Figure 5.21. Let $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ be the outer normal of R and let $\partial F/\partial n$ and $\partial G/\partial n$ denote the directional derivatives of F and G in the direction of $\mathbf{n}$: $\partial F/\partial n = \nabla F \cdot \mathbf{n}$, $\partial G/\partial n = \nabla G \cdot \mathbf{n}$.

(a) From (5.37), (5.38), (5.44) and (5.56) it follows that

$$\begin{aligned}
\int\limits_{B_R} \frac{\partial F}{\partial n}\, ds = \int\limits_{B_R} \nabla F \cdot \mathbf{n}\, ds &= \int\limits_{B_R} (Q\mathbf{i} - P\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right)\, ds \\
&= \int\limits_{B_R} P\, dx + Q\, dy = \iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dx\, dy \\
&= \iint\limits_{R} \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot (Q\mathbf{i} - P\mathbf{j})\, dx\, dy \\
&= \iint\limits_{R} \nabla \cdot \nabla F\, dx\, dy = \iint\limits_{R} \nabla^2 F\, dx\, dy
\end{aligned}$$

(b) By (5.56) and (2.124)

$$\int\limits_{B_R} \nabla F \cdot d\mathbf{r} = \int\limits_{B_R} \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = \iint\limits_{B_R} \underbrace{\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\right)}_{0}\, dx\, dy = 0$$

(c) By (2.126) if a function $z = f(x, y)$ has continuous second derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be harmonic in D. Hence, it follows automatically from Problem 11(a) that

$$\int\limits_{B_R} \frac{\partial F}{\partial n}\, ds = \iint\limits_{R} \underbrace{\nabla^2 F}_{0}\, dx\, dy = 0$$

(d) Using the identity $\nabla \cdot (f\mathbf{u}) = f\nabla \cdot \mathbf{u} + \nabla f \cdot \mathbf{u}$ and the solution to Problem 11(a) we can write

$$\begin{aligned}
\int\limits_{B_R} F\frac{\partial G}{\partial n}\, ds = \int\limits_{B_R} F\,(\nabla G \cdot \mathbf{n})\, ds &= \iint\limits_{R} \nabla \cdot (F\nabla G)\, dx\, dy \\
&= \iint\limits_{R} (F\nabla \cdot \nabla G + \nabla F \cdot \nabla G)\, dx\, dy \\
&= \iint\limits_{R} F\nabla^2 G\, dx\, dy + \iint\limits_{R} (\nabla F \cdot \nabla G)\, dx\, dy
\end{aligned}$$

12. (a) Using the solution to Problem 11(d) we find

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R F\nabla^2 G \, dx \, dy + \iint\limits_R (\nabla F \cdot \nabla G) \, dx \, dy - \iint\limits_R G\nabla^2 F \, dx \, dy$$

$$- \iint\limits_R (\nabla G \cdot \nabla F) \, dx \, dy$$

$$= \iint\limits_R F\nabla^2 G \, dx \, dy - \iint\limits_R G\nabla^2 F \, dx \, dy$$

$$= \iint\limits_R \left(F\nabla^2 G - G\nabla^2 F \right) dx \, dy$$

where we have utilised the fact that $\nabla F \cdot \nabla G = \nabla G \cdot \nabla F$.

(b) If F and G are harmonic in R, i.e. when $\nabla^2 F = 0$, $\nabla^2 G = 0$ in R, then

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R \left(F \underbrace{\nabla^2 G}_{0} - G \underbrace{\nabla^2 F}_{0} \right) dx \, dy = 0$$

Section 5.10

1. (a)

$$\int_{C\ (1,0,0)}^{(1,0,2\pi)} z\,dx + x\,dy + y\,dz = \int_0^{2\pi} \left(\omega\frac{d\phi}{dt} + \phi\frac{d\psi}{dt} + \psi\frac{d\omega}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(-t\sin t + \cos^2 t + \sin t \right) dt$$

$$= -\int_0^{2\pi} t\sin t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} \sin t\,dt$$

$$= t\cos t\Big|_0^{2\pi} + \int_0^{2\pi} \cos t\,dt + \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt - \cos t\Big|_0^{2\pi}$$

$$= 2\pi + \pi = 3\pi$$

where C is the curve $x = \phi(t) = \cos t$, $y = \psi(t) = \sin t$, $z = \omega(t) = t$, $0 \le t \le 2\pi$.

(b) Let

$$\int_{(1,0,1)}^{(2,3,2)} x^2\,dx - xz\,dy + y^2\,dz$$

To evaluate this integral on the straight line joining the two points $(1,0,1)$, $(2,3,2)$ we use the parametrisation $x = \phi(t) = z = \omega(t) = 1+t$, $y = \psi(t) = 3t$, $0 \le t \le 1$, so that

$$\int_{(1,0,1)}^{(2,3,2)} x^2\,dx - xz\,dy + y^2\,dz = \int_0^1 \left(\phi^2\frac{d\phi}{dt} - \phi\omega\frac{d\psi}{dt} + \psi^2\frac{d\omega}{dt}\right)\,dt$$

$$= \int_0^1 \left(-2 - 4t + 7t^2\right)\,dt = \left[-2t - 2t^2 + \frac{7t^3}{3}\right]_0^1 = -\frac{5}{3}$$

(c)

$$\int_{C\ (1,1,0)}^{\left(0,0,\sqrt{2}\right)} x^2yz\,ds = \int_0^{\pi/2} \phi^2\psi\omega\sqrt{\left(\frac{d\phi}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\omega}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} 2\sin t\cos^3 t\,dt = \int_{-1}^0 -2\theta^3\,d\theta = -\frac{\theta^4}{2}\bigg|_{-1}^0 = \frac{1}{2}$$

where C is the curve $x = \phi(t) = \cos t$, $y = \psi(t) = \cos t$, $z = \omega(t) = \sqrt{2}\sin t$, $0 \le t \le \pi/2$ and we have used the substitution $\theta = -\cos t$.

(d) Using (5.65) and (5.34)

$$\int_C u_T\,ds = \int_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right)\,dt$$

$$= \int_0^{2\pi} \left(2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + x^2y^2\mathbf{k}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)\,dt$$

$$= \int_0^{2\pi} \left(4\sin^2 t\cos t\mathbf{i} + 4\sin t\cos^2 t\mathbf{j} + \sin^2 t\cos^2 t\mathbf{k}\right)\cdot\left(-\sin t\mathbf{i} + \cos t\mathbf{j}\right)\,dt$$

$$= 4\int_0^{2\pi} \left(-\sin^3 t\cos t + \sin t\cos^3 t\right)\,dt$$

$$= 4\int_0^{2\pi} -\sin^3 t\cos t\,dt + 4\int_0^{2\pi} \sin t\cos^3 t\,dt = -4\int_{u_0}^{u_1} u^3\,du - 4\int_{v_0}^{v_1} v^3\,dv$$

$$= -\sin^4 t\Big|_0^{2\pi} - \cos^4 t\Big|_0^{2\pi} = 0$$

where C is the circle $x = \cos t$, $y = \sin t$, $z = 2$, $0 \le t \le 2\pi$ and we have used the substitution $u = \sin t$, $v = -\cos t$.

(e) Using (5.65), (5.34) and (3.23)

$$\int_C u_T\, ds = \int_C \mathbf{u}\cdot d\mathbf{r} = \int_0^1 \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 \left(\nabla\times\mathbf{v}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$
$$= \int_0^1 -2\left(z\mathbf{i}+x\mathbf{j}+y\mathbf{k}\right)\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right) dt$$
$$= -2\int_0^1 \left[\left(1+t^3\right)\mathbf{i}+(2t+1)\,\mathbf{j}+t^2\mathbf{k}\right]\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right) dt$$
$$= -2\int_0^1 \left(2+2t+4t^2+2t^3+3t^4\right) dt = -2\left[2t+t^2+\frac{4t^3}{3}+\frac{t^4}{2}+\frac{3t^5}{5}\right]_0^1$$
$$= -\frac{163}{15}$$

2. Let $\mathbf{u}=\nabla F$ in a domain D.

(a) By (5.65) and the chain rule (see Section 2.8) we find

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} u_T\, ds = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \mathbf{u}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \nabla F\cdot d\mathbf{r}$$
$$= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}\right) dt$$
$$= \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = F\big|_{t=t_2} - F\big|_{t=t_1}$$
$$= F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right)$$

where we have used the parametrisation $x=\phi(t)$, $y=\psi(t)$, $z=\omega(t)$, such that $(x_1,y_1,z_1)=(\phi_1,\psi_1,\omega_1)$, $(x_2,y_2,z_2)=(\phi_2,\psi_2,\omega_2)$ for $t_1\le t\le t_2$ and the integral is along any path in D joining the two points.

(b) On any closed path in D it follows that $(x_1,y_1,z_1)=(x_2,y_2,z_2)$ and so

$$\int_C u_T\, ds = F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right) = 0$$

3. Let a curve C in space represent a wire and let its density be given by $\delta=\delta(x,y,z)$, where (x,y,z) is a variable point in C.

(a) For a smooth or piecewise smooth path C arc length s in well defined, i.e. as the distance traversed from some initial point $t=h$ up to a general t:

$$s = \int_h^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve is directed with increasing t, then s also increases in the direction of motion, going from 0 up to the length L of C. We can subdivide C so that $\Delta_i s$ denotes the increment in s from t_{i-1} to t_i, that is, the distance moved in this interval, which leads to the definition

$$\lim_{\substack{n\to\infty \\ \max \Delta_i s\to 0}} \sum_{i=1}^{n} \Delta_i s = \int_C ds = L = \text{length of wire}$$

(b) Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the total mass of the wire can be computed by summing over all of the products of the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max \Delta_i s \to 0$, or put more succinctly:

$$\lim_{\substack{n\to\infty \\ \max \Delta_i s\to 0}} \sum_{i=1}^{n} \delta\left(x_i^*, y_i^*, z_i^*\right) \Delta_i s = \int_C \delta\left(x, y, z\right)\, ds = M = \text{total mass of wire}$$

(c) Let the center of mass of the wire be given by point $(\bar{x}, \bar{y}, \bar{z})$. Furthermore, it is a given that the first moment (i.e. expected value) in any direction about the center of mass is equal to zero. As such, this implies

$$\int_C (x - \bar{x})\, \delta\, ds = 0 \qquad \int_C (y - \bar{y})\, \delta\, ds = 0 \qquad \int_C (z - \bar{z})\, \delta\, ds = 0$$

Focusing on the x-coordinate for now, we thus find

$$\int_C (x - \bar{x})\, \delta\, ds = 0 \iff \bar{x} \underbrace{\int_C \delta\, ds}_{M} = \int_C x\delta\, ds$$

and so

$$M\bar{x} = \int_C x\delta\, ds \qquad M\bar{y} = \int_C y\delta\, ds \qquad M\bar{z} = \int_C z\delta\, ds$$

(d) The moment of inertia can be defined as a measure of the resistance of an object to a change in its rotational motion. Because it has to do with rotational motion, the moment of inertia is always measured about a reference line, i.e. the axis of rotation. For a point mass m the moment of inertia about the z-axis is defined as $I_z = md^2$, where d is the distance of the mass m to the z-axis. Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the moment of inertia about the z-axis of the wire can be computed by summing over all of the products of the distance to the z-axis at a given point (x_i^*, y_i^*, z_i^*), the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max \Delta_i s \to 0$, or put more succinctly:

$$I_z = \lim_{\substack{n \to \infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} \left[(x_i^*)^2 + (y_i^*)^2 \right] \delta\left(x_i^*, y_i^*, z_i^*\right) \Delta_i s = \int_C \left(x^2 + y^2\right) \delta \, ds$$

4.