CHAPTER 5

Section 5.3

1. (a) From the given end points $(0,0)$, $(2,2)$ it follows that we can represent the curve C in the form $y = x$, $0 \le x \le 2$. Hence, by (5.6) we find

$$\int_{(0,0)}^{(2,2)} y^2\, dx = \int_0^2 x^2\, dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

(b) Given the end points $(2,1)$, $(1,2)$ we will parameterise the curve C according to: $x = 2 - t$, $y = 1 + t$, $0 \le t \le 1$. Then by (5.4) we find

$$\int_{(2,1)}^{(1,2)} y\, dx = -\int_0^1 (1+t)\; dt = -\left[t + \frac{t^2}{2}\right]_0^1 = -\frac{3}{2}$$

(c) Given the end points $(1,1)$, $(2,1)$ we will parameterise the curve C according to $x = 1 + t$, $y = 1$, $0 \le t \le 1$. Then by (5.5) we find

$$\int_{(1,1)}^{(2,1)} x\, dy = \int_0^1 (1+t)\,(0)\; dt = 0$$

2. (a) Let us represent the curve $C : x = \sqrt{1-y^2}$ in the form $x = \cos t$, $y = \sin t$, $-\pi/2 \le t \le \pi/2$. Then by (5.4) and (5.5)

$$\begin{aligned}\int_{(0,-1)}^{(0,1)} y^2\, dx + x^2\, dy &= \int_{-\pi/2}^{\pi/2} -\sin^3 t\, dt + \cos^3 t\, dt \\ &= \int_{-\pi/2}^{\pi/2} -\left(1-\cos^2 t\right)\sin t + \left(1 - \sin^2 t\right)\cos t\, dt \\ &= \left[\cos t - \frac{\cos^3 t}{3} + \sin t - \frac{\sin^3 t}{3}\right]_{-\pi/2}^{\pi/2} = \frac{4}{3}\end{aligned}$$

(b) Let C be the parabola $y = x^2$. Then by (5.6) and (5.7) we find

$$\int_{(0,0)}^{(2,4)} y\, dx + x\, dy = \int_0^2 \left(x^2 + 2x^2\right)\; dx = \left[\frac{x^3}{3} + \frac{2}{3}x^3\right]_0^2 = 8$$

(c) Let C be the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi/2$ and let us use the substitution $u = \tan^3 t$. Then by (5.4) and (5.5) we can rewrite the integral as

$$\begin{aligned}\int_{(1,0)}^{(0,1)} \frac{y\, dx - x\, dy}{x^2 + y^2} &= -3\int_0^{\pi/2} \frac{\sin^4 t\cos^2 t + \sin^2 t\cos^4 t}{\cos^6 t + \sin^6 t}\, dt = \int_0^{\pi/2} \frac{-3\sin^2 t\cos^2 t}{\cos^6 t + \sin^6 t}\, dt \\ &= -\int_0^{\infty} \frac{\cos^6 t}{\cos^6 t + \sin^6 t}\, du = -\int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} -\int_0^b \frac{du}{1+u^2} \\ &= \lim_{b\to\infty} -\tan^{-1} u\Big|_0^b = \lim_{b\to\infty} -\tan^{-1} b = -\frac{\pi}{2}\end{aligned}$$

3. (a) Let C be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Then the integral

$$\oint_C y^2\,dx + xy\,dy$$

can be evaluated by computing the sum of the four integrals

$$\underbrace{\int_{(1,1)}^{(-1,1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(-1,1)}^{(-1,-1)} xy\,dy}_{dx=0} \qquad \underbrace{\int_{(-1,-1)}^{(1,-1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(1,-1)}^{(1,1)} xy\,dy}_{dx=0}$$

Hence,

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy &= \int_1^{-1} dx - \int_1^{-1} y\,dy + \int_{-1}^{1} dx + \int_{-1}^{1} y\,dy \\ &= x\big|_1^{-1} - \frac{y^2}{2}\bigg|_1^{-1} + x\big|_{-1}^{1} + \frac{y^2}{2}\bigg|_{-1}^{1} = 0\end{aligned}$$

(b) Let C be the circle $x^2 + y^2 = 1$. Using the parameterization $x = \cos t$, $y = \sin t$ where $0 \le t \le 2\pi$, then by (5.4) and (5.5) the integral

$$\oint_C y\,dx - x\,dy$$

may be written as

$$\begin{aligned}\oint_C y\,dx - x\,dy = \int_0^{2\pi} -\sin^2 t\,dt - \cos^2 t\,dt = -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\,dt &= -\int_0^{2\pi} dt \\ &= -2\pi\end{aligned}$$

(c) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Then the integral

$$\oint_C x^2y^2\,dx - xy^3\,dy$$

can be evaluated by computing the sum of the three integrals

$$\underbrace{\int_{(0,0)}^{(1,0)} x^2y^2\,dx = 0}_{dy=0} \qquad \underbrace{-\int_{(1,0)}^{(1,1)} xy^3\,dy}_{dx=0} \qquad \int_{(1,1)}^{(0,0)} x^2y^2\,dx - xy^3\,dy$$

Hence,

$$\begin{aligned}\oint_C x^2y^2\,dx - xy^3\,dy &= -\int_0^1 y^3\,dy + \int_0^1 x^4\,dx - \int_0^1 y^4\,dy \\ &= -\frac{y^4}{4}\bigg|_0^1 + \frac{x^5}{5}\bigg|_0^1 - \frac{y^5}{5}\bigg|_0^1 = -\frac{1}{4}\end{aligned}$$

4. (a) Let C be the circle $x^2 + y^2 = 4$. Then using the parametrisation $x = 4\cos t$, $y = 4\sin t$, where $0 \leq t \leq 2\pi$ and (5.12) the integral

$$\oint_C \left(x^2 - y^2\right) \, ds$$

may be written as

$$\oint_C \left(x^2 - y^2\right) \, ds = 64 \int_0^{2\pi} \left(\cos^2 t - \sin^2 t\right) \, dt = 64 \int_0^{2\pi} \cos 2t \, dt = 32 \sin 2t\Big|_0^{2\pi} = 0$$

(b) Let C be the line $y = x$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) the integral

$$\int_{(0,0)}^{(1,1)} x \, ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} x \, ds = \sqrt{2} \int_0^1 x \, dx = \frac{\sqrt{2}}{2} x^2 \Big|_0^1 = \frac{1}{\sqrt{2}}$$

(c) Let C be the parabola $y = x^2$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) and using the substitution $x = (1/2)\tan u$, such that $dx = (1/2)\sec^2 u \, du$ the integral

$$\int_{(0,0)}^{(1,1)} ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1 + 4x^2} \, dx = \frac{1}{2} \int_0^{\tan^{-1} 2} \sec^3 u \, du$$

In order to solve the integral on the right hand side, let us solve the indefinite integral

$$\begin{aligned}
\int \sec^3 x \, dx = \int_0 \sec^2 x \sec x \, dx &= \sec x \tan x - \int \sec x \tan^2 x \, dx + C \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) \, dx + C \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx + C
\end{aligned}$$

Adding the term $\int \sec^3 x \, dx$ to both sides and dividing by two then gives

$$\begin{aligned}
\int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx + C \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C
\end{aligned}$$

Substituting in the original equation then gives

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3 u\,du$$
$$= \frac{1}{4}\sec u \tan u\Big|_0^{\tan^{-1}2} + \frac{1}{4}\ln|\sec u + \tan u|\Big|_0^{\tan^{-1}2}$$
$$= \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5}+2\right)}{4}$$

5. Let a path $x = \phi(t)$, $y = \psi(t)$, $h \le t \le k$, where x and y are continuous and have continuous derivatives for $h \le t \le k$ like (5.1) be given. Next, let us make a change of parameter by the equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. Then by (5.4) the line integral $\int_C f(x,y)\,dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$, such that $dx = (d/d\tau)\phi(g(\tau))\,d\tau$, is given by

$$\int_C f(x,y)\,dx = \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d}{d\tau}\phi\left(g\left(\tau\right)\right)\,d\tau$$
$$= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d\phi}{dt}\frac{d}{d\tau}g\left(\tau\right)\,d\tau$$
$$= \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right] \frac{d\phi}{dt}\frac{dt}{d\tau}\,d\tau = \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right]\phi'\left(t\right)\,dt$$

6. (a) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFG$ may be approximated as

$$\int_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right]$$
$$+ \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] = 7$$

(b) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFGKH$ may be approximated as

$$\int_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(1+4)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right]$$
$$+ \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+9)\cdot 1\right] + \left[\frac{1}{2}(0+2)\cdot 1 + \frac{1}{2}(9+8)\cdot -1\right]$$
$$= 5$$

(c) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABCDHLSONMIEA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+8)\cdot 1 + \frac{1}{2}(2+3)\cdot 0\right] \\
& + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+4)\cdot 0\right] + \left[\frac{1}{2}(5+2)\cdot 0 + \frac{1}{2}(4+8)\cdot 1\right] \\
& + \left[\frac{1}{2}(2+1)\cdot 0 + \frac{1}{2}(8+2)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 0 + \frac{1}{2}(2+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(4+3)\cdot -1 + \frac{1}{2}(6+2)\cdot 0\right] + \left[\frac{1}{2}(3+7)\cdot -1 + \frac{1}{2}(2+8)\cdot 0\right] \\
& + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] \\
& + \left[\frac{1}{2}(8+3)\cdot 0 + \frac{1}{2}(3+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
= & \, 8
\end{aligned}$$

(d) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFJNMIJFA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(4+1)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 0 + \frac{1}{2}(4+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(5+7)\cdot 0 + \frac{1}{2}(6+8)\cdot 1\right] + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] \\
& + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+6)\cdot 0\right] \\
& + \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+4)\cdot -1\right] + \left[\frac{1}{2}(0+0)\cdot -1 + \frac{1}{2}(4+1)\cdot -1\right] \\
= & \, \frac{11}{2}
\end{aligned}$$

(e) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFEAEFBA$ may be approximated as

$$\begin{aligned}
\int_C P\,dx + Q\,dy \sim &\left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right] \\
&+ \left[\frac{1}{2}(0+3)\cdot -1 + \frac{1}{2}(4+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
&+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(1+2)\cdot 1\right] + \left[\frac{1}{2}(3+0)\cdot 1 + \frac{1}{2}(2+4)\cdot 0\right] \\
&+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(4+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot -1 + \frac{1}{2}(2+1)\cdot 0\right] \\
= 0 &
\end{aligned}$$

7. Let C be a smooth curve in the xy-plane and let $f(x,y) > 0$ be a continuous function defined over a region of the xy-plane containing the curve C. The equation $z = f(x,y)$ then is the equation of a surface that lies above the region of the xy-plane containing the curve C. Next, we imagine moving a straight line along C perpendicular to the xy-plane, effectively tracing out a "wall" standing on C, orthogonal to the xy-plane. This "wall" cuts the surface $z = f(x,y)$, forming a curve on it that lies above the curve C. In fact, the curve C may be interpreted as the projection of the surface curve onto the xy-plane. Using (5.11), the line integral

$$\int_C f(x,y)\,ds = \lim_{\substack{n\to\infty \\ \max\Delta_i s\to 0}} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta_i s$$

then may be interpreted as an infinite sum of the length of each straight line directed from C to the surface curve lying above it in the limit where the distance Δs between each subsequent line becomes infinitely small, effectively tracing out a "wall" with height at each point (x,y) given by $f(x,y)$. This may be interpreted the as the area of the cylindrical surface $0 \le z \le f(x,y)$, (x,y) on C.

Section 5.5

1. Let the vector $v = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ be given. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_T\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.9)}{=} \int_0^1 2x^2\,dx + \int_0^1 2y^2\,dy = \frac{4}{3}$$

   (b) The integral $\int_C v_T\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.7)}{=} \int_0^1 \left(x^2+5x^4\right)\,dx = \frac{4}{3}$$

(c) The integral $\int_C v_T\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_T\,ds &= \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_{(0,0)}^{(1,0)} \left(x^2+y^2\right)\,dx + 2xy\,dy + \int_{(1,0)}^{(1,1)} \left(x^2+y^2\right)\,dx + 2xy\,dy \\ &= \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{4}{3}\end{aligned}$$

2. Let $\mathbf{v} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be the same vector as given in Problem 1, and let $\mathbf{n}$ be the unit normal vector 90° behind the tangent vector $\mathbf{T}$ as given by (5.37). Then the normal component of $\mathbf{v}$ is given by $v_n = \mathbf{v}\cdot\mathbf{n} = (P\mathbf{i}+Q\mathbf{j})\cdot(y_s\mathbf{i} - x_s\mathbf{j}) = -Qx_s + Py_s$. Then by (5.25) and (5.29)

(a) The integral $\int_C v_n\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.9)}{=} \int_0^1 -2x^2\,dx + \int_0^1 2y^2\,dy = 0$$

(b) The integral $\int_C v_n\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.7)}{=} \int_0^1 2x^5\,dx = \frac{1}{3}$$

(c) The integral $\int_C v_n\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_n\,ds &= \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_{(0,0)}^{(1,0)} -2xy\,dx + \left(x^2+y^2\right)\,dy + \int_{(1,0)}^{(1,1)} -2xy\,dx + \left(x^2+y^2\right)\,dy \\ &= \int_0^1 \left(1+y^2\right)\,dy = \frac{4}{3}\end{aligned}$$

3. Let the gravitational force near a point on the earth's surface be represented approximately by the vector $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = -mg\mathbf{j}$, where the y-axis points upwards. Then by (5.29) and the fact that $P(x,y) = 0$ the work done by the force $\mathbf{F}$ on a body moving in a vertical plane from height h_1 to height h_2 along any path is equal to

$$\int\limits_C F_T\,ds = \int\limits_C \left(P\cos\alpha + Q\sin\alpha\right)\,ds = \int\limits_C Q\,dy = -\int_{h_1}^{h_2} mg\,dy = -mgy\Big|_{h_1}^{h_2} = mg\left(h_1 - h_2\right)$$

4. Let the gravitational force $\mathbf{F}$ be given by $\mathbf{F} = -(kMm/r^2)(\mathbf{r}/r)$. Then in order to compute the work by the gravitational force $\mathbf{F}$ in bringing a particle to its present position r from infinite distance along the ray through the earth's center, we will represents the curve C in terms of parameter t and then use (5.34) to get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\infty^r \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r \left(-\frac{kMm}{t^2}\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r -\frac{kMm}{t^2}\, dt = \left.\frac{kMm}{t}\right|_\infty^r$$
$$= kMm\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{kMm}{r}$$
$$= -U$$

where $(d\mathbf{r}/dr) \cdot (d\mathbf{r}/dr) = 1$ follows from the fact that $d\mathbf{r}/dr$ is a unit vector.

5. (a) By (5.40) the integral $\oint_C ay\, dx + bx\, dy$ may be written as

$$\oint_C ay\, dx + bx\, dy = \iint_R (b-a)\ dx\, dy = (b-a)\, A$$

where A is the area enclosed by the curve C.

(b) By (5.40) the integral $\oint e^x \sin y\, dx + e^x \cos y\, dy$ around the rectangle with vertices $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$ may be written as

$$\oint e^x \sin y\, dx + e^x \cos y\, dy = \int_0^{\pi/2}\int_0^1 (e^x\cos y - e^x \cos y)\ dx\, dy = 0$$

(c) By (5.40) and (4.61) the integral $\oint (2x^3 - y^3)\, dx + (x^3+y^3)\, dy$ around the circle $x^2+y^2=1$ may be written as

$$\oint \left(2x^3 - y^3\right)\ dx + \left(x^3+y^3\right)\ dy = 3\int_0^1\int_0^{2\pi} r^3\, d\theta\, dr = 6\pi\int_0^1 r^3\, dr = \frac{3\pi}{2}$$

(d) By (5.43) and (3.31) the integral $\oint_C u_T\, ds$, where $\mathbf{u} = \operatorname{grad}(x^2y)$ and C is the circle $x^2+y^2=1$ may be written as

$$\oint_C u_T\, ds = \iint_R \operatorname{curl}_z \mathbf{u}\, dx\, dy = \iint_R \operatorname{curl}_z \operatorname{grad}\left(x^2y\right)\ dx\, dy = 0$$

(e) By (5.44) the integral $\oint_C v_n\, ds$, where $\mathbf{v} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j}$ and C is the circle $x^2+y^2=1$ ($\mathbf{n}$ being the outer normal) may be written as

$$\oint_C v_n\, ds = \iint_R \operatorname{div}\mathbf{v}\, dx\, dy = \iint_R \operatorname{div}\ \left[\left(x^2+y^2\right)\mathbf{i} - 2xy\mathbf{j}\right]\ dx\, dy = \iint_R (2x-2x)\ dx\, dy$$
$$= 0$$

(f) Let $F = (x-2)^2 + y^2$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44) and (4.64)

$$\begin{aligned}\oint\limits_C v_n\, ds = \iint\limits_R \text{div } (\nabla F)\ dx\, dy = \iint\limits_R \nabla \cdot \nabla F\, dx\, dy &= \iint\limits_R \nabla^2 F\, dx\, dy \\ &= \iint\limits_R \nabla^2 \left[(x-2)^2 + y^2\right]\, dx\, dy \\ &= 4\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = 4\pi\end{aligned}$$

(g) Let $F = \ln[(x-2)^2 + y^2]^{-1}$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ and since $\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C v_n\, ds$ we find by (5.44)

$$\begin{aligned}\oint\limits_C v_n\, ds = \iint\limits_R \text{div } (\nabla F)\ dx\, dy &= \iint\limits_R \nabla \cdot \nabla F\, dx\, dy = \iint \nabla^2 F\, dx\, dy \\ &= \iint\limits_R \nabla \ln \frac{1}{(x-2)^2 + y^2}\, dx\, dy \\ &= 2\iint\limits_R \frac{x^2 - 4x + 4 - y^2 - (x-2)^2 + y^2}{\left[(x-2)^2 + y^2\right]^2}\, dx\, dxy = 0\end{aligned}$$

(h) By (5.40) the integral $\oint_C f(x)\, dx + g(y)\, dy$ may be written as

$$\oint\limits_C f(x)\ dx + g(y)\ dy = \iint\limits_R \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right]\, dx\, dy = 0$$

6. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of an arbitrary point (x, y) and let $\mathbf{n}$ be the outer normal to some arbitrary closed curve C. Then by (5.44)

$$\begin{aligned}\frac{1}{2}\oint\limits_C r_n\, ds = \frac{1}{2}\oint\limits_C \mathbf{r} \cdot \mathbf{n}\, ds = \frac{1}{2}\iint\limits_R \text{div}\, \mathbf{r}\, dx\, dy &= \frac{1}{2}\iint\limits_R \nabla \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy \\ &= \frac{1}{2}\iint\limits_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot (x\mathbf{i} + y\mathbf{j})\ dx\, dy \\ &= \iint\limits_R dx\, dy = A\end{aligned}$$

7. As for Problem 2(a), let the line integral $\int_{(0,-1)}^{(0,1)} y^2\, dx + x^2\, dy$, where C is the semi-circle

$x = \sqrt{1-y^2}$ be given. Then by (5.40) and (4.64)

$$\begin{aligned}
\oint\limits_C y^2\,dx + x^2\,dy = \iint\limits_R \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right)\,dx\,dy &= 2\iint\limits_R (x-y)\,dx\,dy \\
&= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (x-y)\,dx\,dy \\
&= 2\int_{-\pi/2}^{\pi/2}\int_{0}^{1} (\cos\theta - \sin\theta)\,r^2\,dr\,d\theta \\
&= \frac{2}{3}\int_{\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\,d\theta = \frac{4}{3}
\end{aligned}$$

As for Problem 3(a), let the line integral $\oint_C y^2\,dx + xy\,dy$, where C is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ be given. Then by (5.40)

$$\begin{aligned}
\oint\limits_C y^2\,dx + xy\,dy = \iint\limits_R \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}y^2\right]\,dx\,dy = -\iint\limits_R y\,dx\,dy &= -\int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy \\
&= -\int_{-1}^{1} xy\big|_{-1}^{1}\,dy = -2\int_{-1}^{1} y\,dy \\
&= -y^2\big|_{-1}^{1} = 0
\end{aligned}$$

As for Problem 3(b), let the line integral $\oint_C y\,dx - x\,dy$, where C is the circle $x^2+y^2=1$ be given. Then by (5.40) and (4.64)

$$\begin{aligned}
\oint\limits_C y\,dx - x\,dy = -\iint\limits_R \left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\,dx\,dy = -2\iint\limits_R dx\,dy &= -2\int_0^{2\pi}\int_0^1 r\,dr\,d\theta \\
&= -\int_0^{2\pi} d\theta = -2\pi
\end{aligned}$$

As for Problem 3(c), let the line integral $\oint_C x^2y^2\,dx - xy^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ be given. Then by (5.40)

$$\begin{aligned}
\oint\limits_C x^2y^2\,dx - xy^3\,dy &= -\iint\limits_R \left[\frac{\partial}{\partial x}\left(xy^3\right) + \frac{\partial}{\partial y}\left(x^2y^2\right)\right]\,dx\,dy \\
&= -\iint\limits_R \left(y^3 + 2x^2y\right)\,dx\,dy = -\int_0^1\int_0^x \left(y^3+2x^2y\right)\,dy\,dx \\
&= -\int_0^1 \left[\frac{y^4}{4} + x^2y^2\right]_0^x\,dx = -\frac{5}{4}\int_0^1 x^4\,dx = -\frac{1}{4}
\end{aligned}$$

As for Problem 4(a), let the line integral $\oint_C (x^2 - y^2)\, ds$, where C is the circle $x^2 + y^2 = 4$ be given. Then by (5.44) and the fact that $\mathbf{n}$ may be written as $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/|x + y|$

$$\oint\limits_C (x^2 - y^2)\, ds = a \iint\limits_R \text{div } (x\mathbf{i} - y\mathbf{j})\ dx\, dy = a \iint\limits_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) \cdot (x\mathbf{i} - y\mathbf{j})\ dx\, dy = 0$$

Section 5.7

1. (a) Let

$$dF = 2xy\, dx + x^2\, dy \qquad\qquad \int\limits_C \int_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy$$

where C is the curve $y = x^{3/2}$. To determine the function $F(x, y)$ we firstly note that

$$dF = 2xy\, dx + x^2\, dy = P\, dx + Q\, dy$$

where the functions $P(x, y)$ and $Q(x, y)$ are defined and continuous in the domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$. From inspection it then follows that

$$F\,(x, y) = x^2 y + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = 2xy \qquad\qquad \frac{\partial F}{\partial y} = x^2$$

from which follows

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 2xy\, dx + x^2\, dy = P\, dx + Q\, dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy$$

is independent of path (and hence curve C given by $y = x^{3/2}$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(1,1)} 2xy\, dx + x^2\, dy = F\,(1, 1) - F\,(0, 0) = 1$$

(b) Let

$$dF = ye^{xy}\,dx + xe^{xy}\,dy \qquad\qquad \int_C \int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

where C is the curve $y = \sin^3 x$. From inspection it follows that

$$F(x,y) = e^{xy} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = ye^{xy} \qquad\qquad \frac{\partial F}{\partial y} = xe^{xy}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = ye^{xy}\,dx + xe^{xy}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

is independent of path (and hence curve C given by $y = \sin^3 x$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy = F(\pi,0) - F(0,0) = 0$$

(c) Let

$$dF = \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} \qquad\qquad \int_C \int_{(1,0)}^{(e^{2\pi},0)} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}}$$

where C is the curve $x = e^t \cos t\, y = e^t \sin t$. From inspection it follows that

$$F(x,y) = -\frac{1}{\sqrt{x^2+y^2}} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \qquad\qquad \frac{\partial F}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy$$

is independent of path (and hence curve C given by $x = e^t \cos t\, y = e^t \sin t$) and can easily be evaluated by (5.48) to have the value

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}} = F\left(e^{2\pi},0\right) - F\left(1,0\right) = 1 - e^{-2\pi}$$

2. (a) Let

$$\int_{C\ (1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2} = \int_{C\ (1,-2)}^{(3,4)} P\,dx + Q\,dy$$

where C is the line $y = 3x - 5$ be given. From inspection we can define the function $F(x,y) = -(y/x) + D$, where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{1}{x} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $x = 0$. And so by (5.48) the integral has the value $F(3,4) - F(1,-2) = -10/3$.

(b) Let

$$\int_{C\ (0,2)}^{(1,3)} \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy$$

where C is the parabola $y = 2 + x^2$ be given. From inspection we can define the function $F(x,y) = x^3/y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} = P\left(x,y\right) \qquad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $y = 0$. And so by (5.48) the integral has the value $F(1,3) - F(0,2) = 1/3$.

(c) Let

$$\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy$$

where C is the circular arc $y=\sqrt{1-x^2}$, $-1 \le x \le 1$ be given. From inspection it follows that we cannot define a function $F(x,y)$ such that (5.46) holds. Hence, the given integral is not independent of path. Instead we use (5.6) and (5.7) to find

$$\begin{aligned}\int_{C\ (1,0)}^{(-1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy &= \int_1^{-1} \left(2x\sqrt{1-x^2} - 1 - \frac{x^3}{\sqrt{1-x^2}} - 3x\right)\ dx \\ &= 2\end{aligned}$$

where we have made use of the fact the first, third and fourth term in the integral on the right hand side are odd and hence, will be zero when integrated from $x=1$ to $x=-1$.

(d) Let

$$\int_{C\ (0,0)}^{(\pi/4,\pi/4)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$$

where C is the curve $y = 16x^3/\pi^2$. From inspection we can define the function $F(x,y) = \tan x \tan y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \sec^2 x \tan y = P\left(x,y\right) \qquad\qquad \frac{\partial F}{\partial y} = \sec^2 y \tan x = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\{x, y \mid x, y \neq k\pi/2\}$ for $k = \pm 1, 3, 5, \ldots$. And so by (5.48) the integral has the value $F(\pi/4, \pi/4) - F(0,0) = 1$.

3. (a) Let

$$\oint_C \left[\sin\left(xy\right) + xy\cos\left(xy\right)\right]\ dx + x^2\cos\left(xy\right)\ dy = \oint_C P\, dx + Q\, dy$$

where C is the circle $x^2+y^2=1$. Now since $P(x,y)$ and $Q(x,y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x\cos\left(xy\right) - x^2 y \sin\left(xy\right)$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\, dx + Q\, dy$ when integrated on the circle $x^2+y^2=1$ is equal to 0.

(b) Let

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2 + y^2 = 4$. Furthermore, we note that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(x-1)^2 - y^2}{\left[(x-1)^2 + y^2\right]^2}$$

in the doubly connected region D with hole A at point $(1, 0)$. Hence, according to the discussion in Section 5.7 and (5.57) the integral is equal to some constant k, which is the same for all curves enclosing the hole A. We will use this fact to simplify the integral $\oint P\,dx + Q\,dy$ by evaluating it on the circle $(x-1)^2 + y^2 = 1$ instead. This is permitted since both the circles $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ enclose the hole A at point $(1, 0)$. Thus we can write

$$\oint_C \frac{y\,dx - (x-1)\;dy}{(x-1)^2 + y^2} = \oint_C y\,dx - (x-1)\;dy \underset{(5.40)}{=} \iint_R -2\,dx\,dy = -2\pi$$

where C now denotes the circle $(x-1)^2 + y^2 = 1$ and the last step follows from Example 1 following Section 5.7.

(c) Let

$$\oint_C y^3\,dx - x^3\,dy = \oint_C P\,dx + Q\,dy$$

where C is the square $|x| + |y| = 1$. Using (5.6) and (5.7) with $x = x$ and $y = 1 - x$ for the first quadrant, $y = 1 + x$ for the second quadrant, $y = -1 - x$ for the third quadrant and $y = -1 + x$ for the fourth quadrant, this integral can be written as

$$\oint_C y^3\,dx - x^3\,dy = \int_1^0 \left(1 - 3x + 3x^2\right)\,dx + \int_0^{-1} \left(1 + 3x + 3x^2\right)\,dx$$
$$+ \int_{-1}^0 \left(-1 - 3x - 3x^2\right)\,dx + \int_0^1 \left(-1 + 3x - 3x^2\right)\,dx = -2$$

(d) Let

$$\oint_C xy^6\,dx + \left(3x^2y^5 + 6x\right)\,dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $x^2 + 4y^2 = 4$. Next, let us define the parametrisation $x = 2\cos t$, $y = \sin t$ such that the integral on the ellipse becomes

$$\int_0^{2\pi} \left(-4\sin^7 t\cos t + 12\sin^5 t\cos^3 t + 12\cos^2 t\right)\,dt$$

Using integration by parts on the first term gives

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t\Big|_0^{2\pi} - \int_0^{2\pi} 7\sin^7 t\cos t\,dt\right]$$

from which follows

$$\int_0^{2\pi} 7\sin^7 t\cos t\,dt = \frac{7}{8}\sin^8 t\Big|_0^{2\pi}$$

and so

$$\int_0^{2\pi} -4\sin^7 t\cos t\,dt = -4\left[\sin^8 t - \frac{7}{8}\sin^8 t\right]_0^{2\pi} = -\frac{\sin^8 t}{2}\Bigg|_0^{2\pi} = 0$$

Evaluating the second term gives

$$\int_0^{2\pi} 12\sin^5 t\cos^3 t = 12\int_0^{2\pi} \sin^4 t\sin t\cos^3 t\,dt = 12\int_0^{2\pi} \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt$$

Now applying the substitution $u = \cos t$, such that $du = -\sin t\,dt$ the integral becomes

$$\begin{aligned}
12\int \left(1-\cos^2 t\right)^2 \sin t\cos^3 t\,dt &= -12\int u^3\left(1-u^2\right)^2\,du \\
&= -12\int \left(u^3 - 2u^5 + u^7\right)\,du \\
&= -12\left[\frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8}\right] + C \\
&= -12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right] + C
\end{aligned}$$

Evaluating at the endpoints then gives

$$-12\left[\frac{\cos^4 t}{4} - \frac{\cos^6 t}{3} + \frac{\cos^8 t}{8}\right]_0^{2\pi} = -\frac{1}{2} + \frac{1}{2} = 0$$

Lastly, evaluating the third term gives

$$\int_0^{2\pi} 12\cos^2 t\,dt = 12\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = 6\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 12\pi$$

(e) Let

$$\oint_C (7x - 3y + 2)\,dx + (4y - 3x - 5)\,dy = \oint_C P\,dx + Q\,dy$$

where C is the ellipse $2x^2+3y^2 = 1$. Now since $P(x,y)$ and $Q(x,y)$ have continuous derivatives in domain D given by $\{x, y \in \mathbb{R} \mid -\infty < x, y < \infty\}$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -3$$

Theorem IV and (5.53) tells us that the integral is independent of path in D and so according to (5.51) the value of the integral $\int P\,dx + Q\,dy$ when integrated on the ellipse $2x^2 + 3y^2$ is equal to 0.

(f) Let

$$\oint_C \frac{(e^x \cos y - 1)\ dx + e^x \sin y\, dy}{e^{2x} - 2e^x \cos y + 1} = \oint_C P\,dx + Q\,dy$$

where C is the circle $x^2 + y^2 = 1$. The denominator may also be written as $e^{2x} - 2e^x \cos y + 1 = (e^x - 1)^2 + 2e^x(1 - \cos y)$, from which follows that the denominator is equal to 0 only for $x = 0$ and $y = 2n\pi$, $n = \pm 0, 1, 2, \ldots$. Hence, according to (5.53), the integral $\int P\,dx + Q\,dy$ is independent of path in any simply connected domain D not containing the points $(0, 2n\pi)$, $n = \pm 0, 1, 2, \ldots$ for

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{(e^x - e^{3x}) \sin y}{(e^{2x} - 2e^x \cos y + 1)^2}$$

except at the aforementioned points. However, since the circle $x^2+y^2 = 1$ encloses the point $(0,0)$ we are dealing with a doubly connected domain and hence, Theorem IV does not hold. As such, let us make the substitution $x = \cos t$, $y = \sin t$ where $-\pi \le t \le \pi$ so that the integral can be written as

$$\int_{-\pi}^{\pi} \frac{[1 - e^{\cos t} \cos(\sin t)] \sin t + e^{\cos t} \sin(\sin t) \cos t}{e^{2\cos t} - 2e^{\cos t} \cos(\sin t) + 1}\, dt = \int_{-\pi}^{\pi} \frac{f(t)}{g(t)}\, dt = \int_{-\pi}^{\pi} h(t)\ dt$$

Now since $f(-t) = -f(t)$ and $g(-t) = g(t)$ it follows that $h(-t) = -h(t)$ and hence, that $h(t)$ is an odd function. Using the fact that integrating an odd function on an interval $[-a, a]$ always gives 0, we may finally conclude that the integral $\int P\,dx + Q\,dy$ evaluated on the circle $x^2 + y^2$ is in fact 0.

4. Let

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_{(1,0)}^{(2,2)} P\,dx + Q\,dy$$

where C is an arbitrary path connecting the points $(1,0)$ and $(2,2)$ not passing through the origin. As stated in Example 2 of Section 2, $P\,dx + Q\,dy$ is a familiar differential, namely that of the polar coordinate angle θ:

$$d\theta = d\left(\tan^{-1} \frac{y}{x}\right) = \frac{-y\,dx + x\,dx}{x^2 + y^2}$$

and so we can write

$$\int_A^B \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_A^B d\theta = \theta_B - \theta_A = \text{total increase in } \theta \text{ from } A \text{ to } B$$

as θ varies continuously on the path C. The integral is thus not independent of path, but depends on the number of times C goes around the origin. As such we find that

$$\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2+y^2} = \tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{0}{1}\right) + 2n\pi = \frac{\pi}{4} + 2n\pi$$

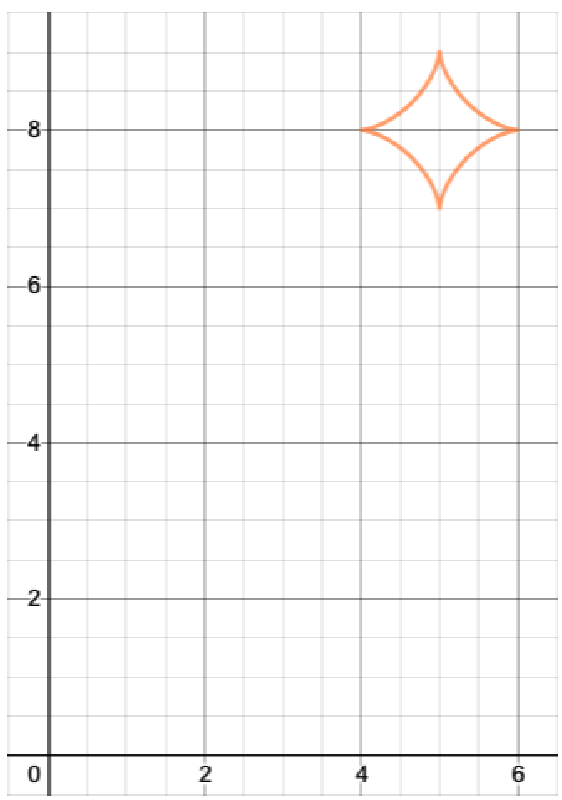
where $n = \pm 0, 1, 2, \ldots$.

5. Let

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_C P\,dx + Q\,dy$$

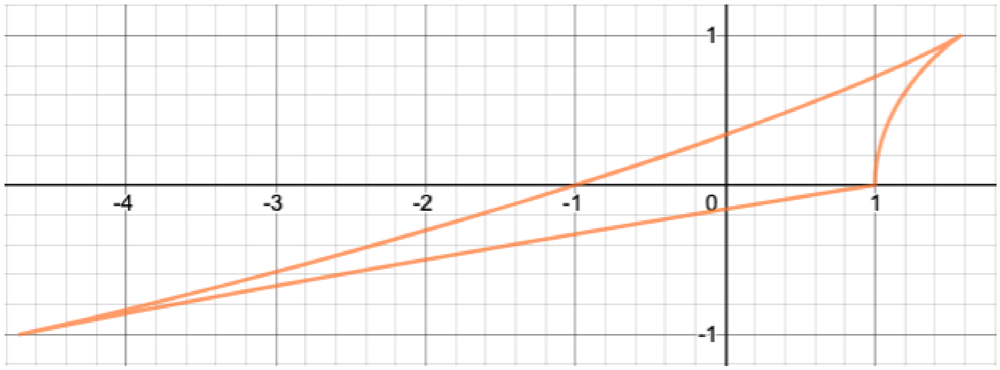
where C is a path given by $x = f(t)$, $y = g(t)$, $a \le t \le b$ not passing through the origin, for which $f(a) = f(b)$, $g(a) = g(b)$. The analysis of Section 5.6 shows that $\int_C P\,dx + Q\,dy$ equals $n \cdot 2\pi$, where n is the number of times C encircles the origin. The value of n can be determined from plotting the path.

(a) From a plot of the path $C:\ x = 5 + \cos^3 t$, $y = 8 + \sin^3 t$, $0 \le t \le 2\pi$



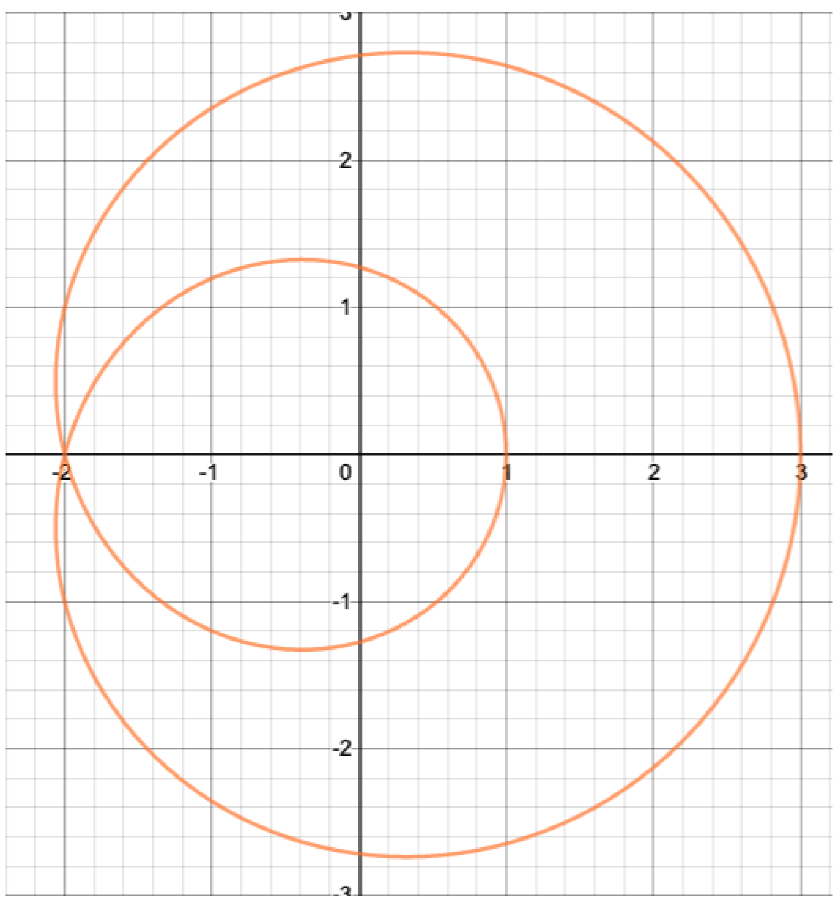
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

(b) From a plot of the path $C:\ x = \cos t + t\sin t,\ y = \sin t,\ 0 \le t \le 2\pi$



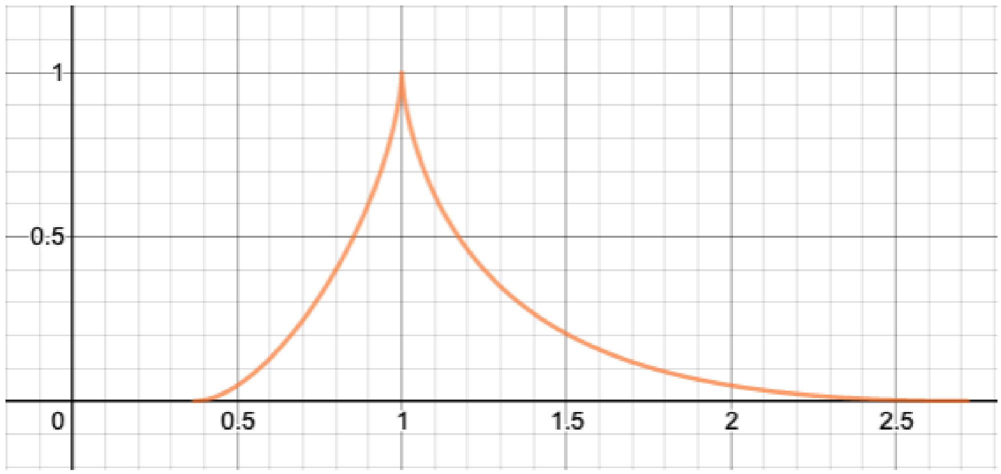
we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 2π, since $n = 1$ as the path encircles the origin once.

(c) From a plot of the path $C:\ x = 2\cos 2t - \cos t,\ y = 2\sin 2t - \sin t,\ 0 \le t \le 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 4π, since $n = 2$ as the path encircles the origin twice.

(d) From a plot of the path $C: \, x = e^{\cos^3 t}$, $y = \sin^4 t$, $0 \le t \le 2\pi$



we may conclude that the line integral $\int_C P\,dx + Q\,dy$ evaluated on the path C is equal to 0, since $n = 0$ as the path does not encircle the origin once.

6. (a) Let

$$\int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 - y^2\right)\,dy = \int_{(1,1)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(1,1)$ to $(x,1)$ and $(x,1)$ to (x,y) such that the integral becomes

$$\begin{aligned} F &= \int_{(1,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_{(1,1)}^{(x,1)} 2xy\,dx + \left(x^2 + y^2\right)\,dy + \int_{(x,1)}^{(x,y)} 2xy\,dx + \left(x^2 + y^2\right)\,dy \\ &= \int_1^x 2x\,dx + \int_1^y \left(x^2 + y^2\right)\,dy = x^2\Big|_1^x + \left[x^2 y + \frac{y^3}{3}\right]_1^y = x^2 y - \frac{1}{3}\left(y^3 + 2\right) \end{aligned}$$

(b) Let

$$\int_{(0,0)}^{(x,y)} \sin y\,dx + x\cos y\,dy = \int_{(0,0)}^{(x,y)} P\,dx + Q\,dy$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \cos y$$

then by Theorem IV and (5.53) the integral $\int P\,dx + Q\,dy$ is independent of path in the xy-plane. To find a function for which $\nabla F = P\mathbf{i} + Q\mathbf{j}$ we use a broken line path consisting of the two segments running from $(0,0)$ to $(x,0)$ and $(x,0)$ to (x,y) such that the integral becomes

$$
\begin{aligned}
F &= \int_{(0,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_{(0,0)}^{(x,0)} \sin y\,dy + x\cos y\,dy + \int_{(x,0)}^{(x,y)} \sin y\,dy + x\cos y\,dy \\
&= \int_0^y x\cos y\,dy = x\sin y\big|_0^y = x\sin y
\end{aligned}
$$

7. The integral

$$
\oint \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} = \oint P\,dx + Q\,dy
$$

is independent of path in any simply connected domain D not containing the origin, for

$$
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{x^2\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}
$$

except at the origin. Hence, the integral is 0 for any path not enclosing the origin. For the square with vertices $(\pm 1, \pm 1)$ however, the integral has a certain value k. To find k, we thus have to evaluate the integral

$$
\begin{aligned}
k &= \int_{(1,1)}^{(-1,1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} + \int_{(-1,1)}^{(-1,-1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} + \int_{(-1,-1)}^{(1,-1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} \\
&\quad + \int_{(1,-1)}^{(1,1)} \frac{x^2y\,dx - x^3\,dy}{\left(x^2+y^2\right)^2} \\
&= \int_1^{-1} \frac{x^2\,dx}{\left(x^2+1\right)^2} + \int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} - \int_{-1}^1 \frac{x^2\,dx}{\left(x^2+1\right)^2} - \int_{-1}^1 \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\int_1^{-1} \frac{x^2\,dx}{\left(x^2+1\right)^2} + 2\int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\int_1^{-1} \left[\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right] dx + 2\int_1^{-1} \frac{dy}{\left(y^2+1\right)^2} \\
&= 2\tan^{-1} x\big|_1^{-1} - 2\int_{\pi/4}^{-\pi/4} du + 2\int_{\pi/4}^{-\pi/4} dv = 2\tan^{-1} x\big|_1^{-1} = -\pi
\end{aligned}
$$

where we have used the substitution $x = \tan u$, $y = \tan v$ such that $dx = \sec^2 u\,du$, $dy = \sec^2 v\,dv$.

8. Let D be a domain with a finite number of holes at points A_1, $A_2, \dots$, A_k so that D is $(k+1)$-tuply connected as in Figure 5.23. Let P and Q be continuous and have continuous derivatives in D and let $\partial P/\partial y = \partial Q/\partial x$ in D. Let C_1 denote a circle around the point A_1 in D, enclosing none of the other A's. Let C_2 be chosen similar for A_2 and so on. Futhermore, let

$$\oint_{C_1} P\,dx + Q\,dy = \alpha_1,\ \oint_{C_2} P\,dx + Q\,dy = \alpha_2, \dots,\ \oint_{C_k} P\,dx + Q\,dy = \alpha_k$$

(a) Let C be an arbitrary simple closed path in D enclosing A_1, $A_2, \dots$, A_k. Furthermore, we assume that the circles C_1, $C_2, \dots$, C_k do not intersect C at any point. Let us also define the closed region R in D whose boundaries are given by the simply closed path C and all of the circles C_1, $C_2, \dots$, C_k that are interior to C. Next, let us introduce auxiliary arcs from C to C_1, C to $C_2, \dots$, two to each so that we end up with a figure similar to Figure 5.21. These decompose the region R into $k+1$ smaller regions, each of which is simply connected (i.e. does not contain any holes in its interior). If we integrate in a positive direction around the boundary of each sub region and then add the results we find that the integrals along the auxiliary arcs cancel out, leaving just the integral around C in the positive direction plus the integrals around C_1, $C_2, \dots$, C_k in the negative direction. On the other hand, the line integral around the boundary of each sub region can be expressed as a double integral

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$

over the sub region by Green's theorem. Hence, the sum of the line integrals is equal to the double integral over R:

$$\oint_{C} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$

However, since $\partial P/\partial y = \partial Q/\partial x$ in D the integral on the right hand side is equal to zero and hence, we end up with

$$\oint_{C} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy + \cdots + \oint_{C_k} P\,dx + Q\,dy$$
$$= \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

(b) Let

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P\,dx + Q\,dy$$

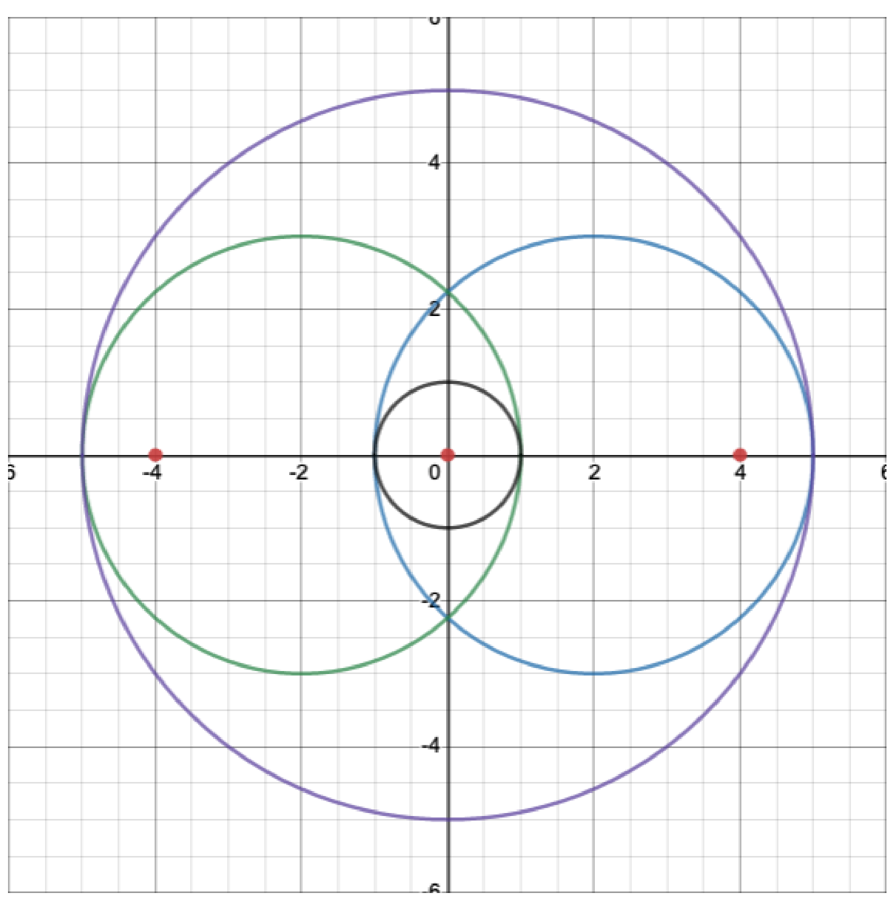
where (x_1, y_1), (x_2, y_2) are two fixed points in D and let this integral have the value K for one particular path. All possible values of the integral are then given by

$$K + n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k$$

where $n_1, \ldots, n_k$ are positive or negative integers or 0. The exact value of $n_1, \ldots, n_k$ depends on how many times a chosen path between (x_1, y_1), (x_2, y_2) encircles one of the holes at A_1, $A_2, \ldots, A_k$ and in which direction (i.e. negative for clockwise or positive for anti-clockwise).

9. Let D be a domain with holes at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$ and let P and Q be continuous and have continuous derivatives in D, with $\partial P/\partial y = \partial Q/\partial x$ except at the points $(4, 0)$, $(0, 0)$, $(-4, 0)$. Let C_1 denote the circle $(x-2)^2 + y^2 = 9$; let C_2 denote the circle $(x+2)^2 + y^2 = 9$; let C_3 denote the circle $x^2 + y^2 = 25$. Furthermore, let it be given that

$$\oint_{C_1} P\,dx + Q\,dy = 11, \qquad \oint_{C_2} P\,dx + Q\,dy = 9, \qquad \oint_{C_3} P\,dx + Q\,dy = 13$$



From inspection of the figure it follows that

$$\oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy$$

and so

$$\oint_{C_4} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy - \oint_{C_3} P\,dx + Q\,dy = 11 + 9 - 13 = 7$$

10. Let $F(x, y) = x^2 - y^2$

    (a) By Theorem I of Section 5.6 the integral

    $$\int_{(0,0)}^{(2,8)} \nabla F \cdot d\mathbf{r} = \int_{(0,0)}^{(2,8)} 2x\,dx - 2y\,dy = \int_{(0,0)}^{(2,8)} \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \int_{(0,0)}^{(2,8)} P\,dx + Q\,dy$$

    is independent of path in D, where D is the xy-plane. Hence, since the points $(0, 0)$ and $(2, 8)$ both lie on the curve $y = x^3$ the integral on the curve $y = x^3$ is equal to $F(2, 8) - F(0, 0) = 4 - 60 = -60$.

    (b) By (5.37) and (5.38) the integral

    $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C (Q\mathbf{i} - P\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right) ds = \oint_C \left(Q\frac{dy}{ds} + P\frac{dx}{ds}\right) ds$$
    $$= \oint_C P\,dx + Q\,dy$$

    where C is the circle $x^2 + y^2 = 1$, $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the outer normal to C and $\partial F/\partial n = \nabla F \cdot \mathbf{n}$ is the directional derivative of F in the direction of $\mathbf{n}$ (see Section 2.4). Since

    $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2$$

    then by Theorem III of Section 5.6 the integral is independent of path in D and hence, by Theorem II of Section 5.6 it follows that

    $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C \nabla F \cdot \mathbf{n}\,ds = \oint_C P\,dx + Q\,dy = 0$$

11. Let $F(x, y)$ and $G(x, y)$ be continuous and have continuous derivatives in a domain D and let R be a closed region in D with directed boundary B_R consisting of closed curves $C_1, \ldots, C_n$ as in Figure 5.21. Let $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ be the outer normal of R and let $\partial F/\partial n$ and $\partial G/\partial n$ denote the directional derivatives of F and G in the direction of $\mathbf{n}$: $\partial F/\partial n = \nabla F \cdot \mathbf{n}$, $\partial G/\partial n = \nabla G \cdot \mathbf{n}$.

(a) From (5.37), (5.38), (5.44) and (5.56) it follows that

$$\int\limits_{B_R} \frac{\partial F}{\partial n}\,ds = \int\limits_{B_R} \nabla F\cdot\mathbf{n}\,ds = \int\limits_{B_R} (Q\mathbf{i}-P\mathbf{j})\cdot\left(\frac{dy}{ds}\mathbf{i}-\frac{dx}{ds}\mathbf{j}\right)\,ds$$

$$= \int\limits_{B_R} P\,dx + Q\,dy = \iint\limits_{R}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,dx\,dy$$

$$= \iint\limits_{R}\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}\right)\cdot(Q\mathbf{i}-P\mathbf{j})\,dx\,dy$$

$$= \iint\limits_{R}\nabla\cdot\nabla F\,dx\,dy = \iint\limits_{R}\nabla^2 F\,dx\,dy$$

(b) By (5.56) and (2.124)

$$\int\limits_{B_R}\nabla F\cdot d\mathbf{r} = \int\limits_{B_R}\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \iint\limits_{B_R}\underbrace{\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial}{\partial y}\frac{\partial F}{\partial x}\right)}_{0}\,dx\,dy = 0$$

(c) By (2.126) if a function $z=f(x,y)$ has continuous second derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be harmonic in D. Hence, it follows automatically from Problem 11(a) that

$$\int\limits_{B_R}\frac{\partial F}{\partial n}\,ds = \iint\limits_{R}\underbrace{\nabla^2 F}_{0}\,dx\,dy = 0$$

(d) Using the identity $\nabla\cdot(f\mathbf{u}) = f\nabla\cdot\mathbf{u}+\nabla f\cdot\mathbf{u}$ and the solution to Problem 11(a) we can write

$$\int\limits_{B_R} F\frac{\partial G}{\partial n}\,ds = \int\limits_{B_R} F\,(\nabla G\cdot\mathbf{n})\,ds = \iint\limits_{R}\nabla\cdot(F\nabla G)\,dx\,dy$$

$$= \iint\limits_{R}(F\nabla\cdot\nabla G+\nabla F\cdot\nabla G)\,dx\,dy$$

$$= \iint\limits_{R} F\nabla^2 G\,dx\,dy + \iint\limits_{R}(\nabla F\cdot\nabla G)\,dx\,dy$$

12. (a) Using the solution to Problem 11(d) we find

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R F\nabla^2 G \, dx \, dy + \iint\limits_R (\nabla F \cdot \nabla G) \, dx \, dy - \iint\limits_R G\nabla^2 F \, dx \, dy$$

$$- \iint\limits_R (\nabla G \cdot \nabla F) \, dx \, dy$$

$$= \iint\limits_R F\nabla^2 G \, dx \, dy - \iint\limits_R G\nabla^2 F \, dx \, dy$$

$$= \iint\limits_R \left(F\nabla^2 G - G\nabla^2 F \right) dx \, dy$$

where we have utilised the fact that $\nabla F \cdot \nabla G = \nabla G \cdot \nabla F$.

(b) If F and G are harmonic in R, i.e. when $\nabla^2 F = 0$, $\nabla^2 G = 0$ in R, then

$$\int\limits_{B_R} \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds = \iint\limits_R \left(F \underbrace{\nabla^2 G}_{0} - G \underbrace{\nabla^2 F}_{0} \right) dx \, dy = 0$$

Section 5.10

1. (a)

$$\int\limits_C \int_{(1,0,0)}^{(1,0,2\pi)} z \, dx + x \, dy + y \, dz = \int_0^{2\pi} \left(\omega \frac{d\phi}{dt} + \phi \frac{d\psi}{dt} + \psi \frac{d\omega}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(-t \sin t + \cos^2 t + \sin t \right) dt$$

$$= -\int_0^{2\pi} t \sin t \, dt + \int_0^{2\pi} \cos^2 t \, dt + \int_0^{2\pi} \sin t \, dt$$

$$= t \cos t \Big|_0^{2\pi} + \int_0^{2\pi} \cos t \, dt + \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt - \cos t \Big|_0^{2\pi}$$

$$= 2\pi + \pi = 3\pi$$

where C is the curve $x = \phi(t) = \cos t$, $y = \psi(t) = \sin t$, $z = \omega(t) = t$, $0 \leq t \leq 2\pi$.

(b) Let

$$\int_{(1,0,1)}^{(2,3,2)} x^2 \, dx - xz \, dy + y^2 \, dz$$

To evaluate this integral on the straight line joining the two points $(1, 0, 1)$, $(2, 3, 2)$ we use the parametrisation $x = \phi(t) = z = \omega(t) = 1 + t$, $y = \psi(t) = 3t$, $0 \leq t \leq 1$, so that

$$\int_{(1,0,1)}^{(2,3,2)} x^2\,dx - xz\,dy + y^2\,dz = \int_0^1 \left(\phi^2\frac{d\phi}{dt} - \phi\omega\frac{d\psi}{dt} + \psi^2\frac{d\omega}{dt}\right)\,dt$$

$$= \int_0^1 \left(-2-4t+7t^2\right)\,dt = \left[-2t-2t^2+\frac{7t^3}{3}\right]_0^1 = -\frac{5}{3}$$

(c)

$$\int_{C\ (1,1,0)}^{\left(0,0,\sqrt{2}\right)} x^2yz\,ds = \int_0^{\pi/2} \phi^2\psi\omega\sqrt{\left(\frac{d\phi}{dt}\right)^2+\left(\frac{d\psi}{dt}\right)^2+\left(\frac{d\omega}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} 2\sin t\cos^3 t\,dt = \int_{-1}^{0} -2\theta^3\,d\theta = -\left.\frac{\theta^4}{2}\right|_{-1}^{0} = \frac{1}{2}$$

where C is the curve $x=\phi(t)=\cos t$, $y=\psi(t)=\cos t$, $z=\omega(t)=\sqrt{2}\sin t$, $0\le t\le\pi/2$ and we have used the substitution $\theta=-\cos t$.

(d) Using (5.65) and (5.34)

$$\int_C u_T\,ds = \int_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi}\left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right)\,dt$$

$$= \int_0^{2\pi}\left(2xy^2z\mathbf{i}+2x^2yz\mathbf{j}+x^2y^2\mathbf{k}\right)\cdot\left(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}\right)\,dt$$

$$= \int_0^{2\pi}\left(4\sin^2 t\cos t\mathbf{i}+4\sin t\cos^2 t\mathbf{j}+\sin^2 t\cos^2 t\mathbf{k}\right)\cdot\left(-\sin t\mathbf{i}+\cos t\mathbf{j}\right)\,dt$$

$$= 4\int_0^{2\pi}\left(-\sin^3 t\cos t+\sin t\cos^3 t\right)\,dt$$

$$= 4\int_0^{2\pi} -\sin^3 t\cos t\,dt + 4\int_0^{2\pi}\sin t\cos^3 t\,dt = -4\int_{u_0}^{u_1} u^3\,du - 4\int_{v_0}^{v_1} v^3\,dv$$

$$= -\sin^4 t\Big|_0^{2\pi} - \cos^4 t\Big|_0^{2\pi} = 0$$

where C is the circle $x=\cos t$, $y=\sin t$, $z=2$, $0\le t\le 2\pi$ and we have used the substitution $u=\sin t$, $v=-\cos t$.

(e) Using (5.65), (5.34) and (3.23)

$$\int\limits_C u_T\, ds = \int\limits_C \mathbf{u}\cdot d\mathbf{r} = \int_0^1 \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 \left(\nabla\times\mathbf{v}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$

$$= \int_0^1 -2\,(z\mathbf{i}+x\mathbf{j}+y\mathbf{k})\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right)\, dt$$

$$= -2\int_0^1 \left[\left(1+t^3\right)\mathbf{i}+(2t+1)\,\mathbf{j}+t^2\mathbf{k}\right]\cdot\left(2\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right)\, dt$$

$$= -2\int_0^1 \left(2+2t+4t^2+2t^3+3t^4\right)\, dt = -2\left[2t+t^2+\frac{4t^3}{3}+\frac{t^4}{2}+\frac{3t^5}{5}\right]_0^1$$

$$= -\frac{163}{15}$$

2. Let $\mathbf{u} = \nabla F$ in a domain D.

(a) By (5.65) and the chain rule (see Section 2.8) we find

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} u_T\, ds = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \mathbf{u}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \nabla F\cdot d\mathbf{r}$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}\right) dt$$

$$= \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = F\Big|_{t=t_2} - F\Big|_{t=t_1}$$

$$= F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right)$$

where we have used the parametrisation $x=\phi(t)$, $y=\psi(t)$, $z=\omega(t)$, such that $(x_1,y_1,z_1)=(\phi_1,\psi_1,\omega_1)$, $(x_2,y_2,z_2)=(\phi_2,\psi_2,\omega_2)$ for $t_1\le t\le t_2$ and the integral is along any path in D joining the two points.

(b) On any closed path in D it follows that $(x_1,y_1,z_1)=(x_2,y_2,z_2)$ and so

$$\int\limits_C u_T\, ds = F\left(x_2,y_2,z_2\right) - F\left(x_1,y_1,z_1\right) = 0$$

3. Let a curve C in space represent a wire and let its density (mass per unit length) be given by $\delta=\delta(x,y,z)$, where (x,y,z) is a variable point on C.

(a) For a smooth or piecewise smooth path C arc length s in well defined, i.e. as the distance traversed from some initial point $t=h$ up to a general t:

$$s = \int_h^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve is directed with increasing t, then s also increases in the direction of motion, going from 0 up to the length L of C. We can subdivide C so that $\Delta_i s$ denotes the increment in s from t_{i-1} to t_i, that is, the distance moved in this interval, which leads to the definition

$$\lim_{\substack{n\to\infty \\ \max\Delta_i s\to 0}} \sum_{i=1}^{n} \Delta_i s = \int\limits_C ds = L = \text{length of wire}$$

(b) Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the total mass of the wire can be computed by summing over all of the products of the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max\Delta_i s \to 0$, or put more succinctly:

$$\lim_{\substack{n\to\infty \\ \max\Delta_i s\to 0}} \sum_{i=1}^{n} \delta\left(x_i^*, y_i^*, z_i^*\right)\Delta_i s = \int\limits_C \delta\left(x, y, z\right)\, ds = M = \text{total mass of wire}$$

(c) Let the center of mass of the wire be given by point $(\bar{x}, \bar{y}, \bar{z})$. Furthermore, it is a given that the first moment (i.e. expected value) in any direction about the center of mass is equal to zero. As such, this implies

$$\int\limits_C (x - \bar{x})\,\delta\, ds = 0 \qquad \int\limits_C (y - \bar{y})\,\delta\, ds = 0 \qquad \int\limits_C (z - \bar{z})\,\delta\, ds = 0$$

Focusing on the x-coordinate for now, we thus find

$$\int\limits_C (x - \bar{x})\,\delta\, ds = 0 \iff \bar{x}\underbrace{\int\limits_C \delta\, ds}_{M} = \int\limits_C x\delta\, ds$$

and so

$$M\bar{x} = \int\limits_C x\delta\, ds \qquad M\bar{y} = \int\limits_C y\delta\, ds \qquad M\bar{z} = \int\limits_C z\delta\, ds$$

(d) The moment of inertia can be defined as a measure of the resistance of an object to a change in its rotational motion. Because it has to do with rotational motion, the moment of inertia is always measured about a reference line, i.e. the axis of rotation. For a point mass m the moment of inertia about the z-axis is defined as $I_z = md^2$, where d is the distance of the mass m to the z-axis. Given that the density (mass per unit length) of the wire is $\delta = \delta(x, y, z)$ for some point (x, y, z) on the wire, the moment of inertia about the z-axis of the wire can be computed by summing over all of the products of the distance to the z-axis at a given point (x_i^*, y_i^*, z_i^*), the density at a given point (x_i^*, y_i^*, z_i^*) and corresponding $\Delta_i s$ (i.e. the increment in s from t_{i-1} to t_i) while s goes from 0 up to the length L of C in the limit that the number of segments $n \to \infty$ and $\max \Delta_i s \to 0$, or put more succinctly:

$$I_z = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} \left[(x_i^*)^2 + (y_i^*)^2 \right] \delta\left(x_i^*, y_i^*, z_i^*\right) \Delta_i s = \int_C \left(x^2 + y^2\right) \delta \, ds$$

4. Let a smooth surface S in space represent a thin curved sheet of metal and let its density (mass per unit area) be given by $\mu = \mu(x, y, z)$, where (x, y, z) is a variable point on S. The surface may be represented in any of the forms given by (5.66), (5.67) or (5.68).

   (a) We assume the surface S is cut into n pieces as in Fig. 5.25. The quantity $\Delta_i \sigma$ then denotes the area of the ith piece and it is assumed that the ith piece shrinks to a point as $n \to \infty$ in an appropriate manner. The total surface area of S then can be obtained by summing over all $\Delta_i \sigma$ in the limit that $n \to \infty$ and $\max \Delta_i \sigma \to 0$, which leads to the definition

   $$\lim_{\substack{n\to\infty \\ \max \Delta_i \sigma \to 0}} \sum_{i=1}^{n} \Delta_i \sigma = \iint_S d\sigma = \text{surface area of sheet}$$

   (b) Given that the density (mass per unit area) of the thin metal sheet is $\mu = \mu(x, y, z)$ for some point (x, y, z) on C, the total mass of the sheet can be computed by summing over the product of all $\Delta_i \sigma$'s and the density at the corresponding point (x_i^*, y_i^*, z_i^*) in the limit that the number of area segments $n \to \infty$ and the area of the ith piece $\Delta_i \sigma \to 0$, or put more succinctly:

   $$\lim_{\substack{n\to\infty \\ \max \Delta_i \sigma \to 0}} \sum_{i=1}^{n} \mu\left(x_i^*, y_i^*, z_i^*\right) \Delta_i \sigma = \iint_S \mu\left(x, y, z\right) \, d\sigma = M = \text{total mass of sheet}$$

   (c) Let the center of mass of the sheet be given by point $(\bar{x}, \bar{y}, \bar{z})$. Furthermore, it is a given that the first moment (i.e. expected value) in any direction about the center of mass is equal to zero. As such, this implies

   $$\iint_S (x - \bar{x}) \, \mu \, d\sigma = 0 \qquad \iint_S (y - \bar{y}) \, \mu \, d\sigma = 0 \qquad \iint_S (z - \bar{z}) \, \mu \, d\sigma = 0$$

   Focusing on the x-coordinate for now, we thus find

   $$\iint_S (x - \bar{x}) \, \mu \, d\sigma = 0 \iff \bar{x} \underbrace{\iint_S \mu \, d\sigma}_{M} = \iint_S x\mu \, d\sigma$$

and so

$$M\bar{x} = \iint\limits_S x\mu \, d\sigma \qquad M\bar{y} = \iint\limits_S y\mu \, d\sigma \qquad M\bar{z} = \iint\limits_S z\mu \, d\sigma$$

(d) Given that the density (mass per unit area) of the sheet is $\mu = \mu(x, y, z)$ for some point (x, y, z) on the surface S, the moment of inertia about the z-axis of the sheet can be computed by summing over the product of all $\Delta_i\sigma$'s, the density at the corresponding point (x_i^*, y_i^*, z_i^*) and the distance to the z-axis at (x_i^*, y_i^*, z_i^*) in the limit that the number of area segments $n \to \infty$ and the area of the ith piece $\Delta_i\sigma \to 0$, or put more succinctly:

$$I_z = \lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \left[(x_i^*)^2 + (y_i^*)^2\right] \mu\left(x_i^*, y_i^*, z_i^*\right) \Delta_i\sigma = \iint\limits_S \left(x^2 + y^2\right) \mu \, d\sigma$$

5. (a) Let

$$\iint\limits_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy$$

where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and the normal is pointing away from $(0, 0, 0)$. From geometry it follows that the normal $\mathbf{n}$ to our triangle S pointing away from the origin is given by $\sqrt{3}\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Using (1.23) and (1.24) we then can formulate the equation for the plane in space that coincides with S: $x + y + z = 1$, and so $z = f(x, y) = 1 - x - y$. The region R_{xy} may be expressed as $y = 1 - x$, where $0 \le x \le 1$. Finally, we can use (5.80) to evaluate the surface integral to get

$$\begin{aligned}
\iint\limits_S L \, dy \, dz + M \, dz \, dx + N \, dx \, dy &= \iint\limits_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy \\
&= \iint\limits_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N\right) dx \, dy \\
&= \int_0^1 \int_0^{1-x} (x + y + z) \, dy \, dx = \int_0^1 \int_0^{1-x} dy \, dx \\
&= \int_0^1 (1 - x) \, dx = \frac{1}{2}
\end{aligned}$$

(b) Let

$$\iint\limits_S dy \, dz + dz \, dx + dx \, dy$$

where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \leq 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Next, let us introduce the parametrisation $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{1 - r^2}$, $r^2 \leq 1$, where $(r, \theta) \in R_{r\theta}$. Then by (5.81)

$$\begin{aligned}
\iint_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy &= \iint_S dy\,dz + dz\,dx + dx\,dy \\
&= \iint_{R_{r\theta}} \left[L\frac{\partial(y,z)}{\partial(r,\theta)} + M\frac{\partial(z,x)}{\partial(r,\theta)} + N\frac{\partial(x,y)}{\partial(r,\theta)} \right] dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 \left[\frac{r^2}{\sqrt{1-r^2}} (\sin\theta + \cos\theta) + r \right] dr\,d\theta \\
&= \underbrace{\int_0^{2\pi} (\sin\theta + \cos\theta)\,d\theta}_{0} \int_0^1 \frac{r^2\,dr}{\sqrt{1-r^2}} + \int_0^{2\pi} \int_0^1 r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2}\bigg|_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \frac{\theta}{2}\bigg|_0^{2\pi} = \pi
\end{aligned}$$

(c) Let

$$\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma$$

where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \leq 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Again, we introduce the parametrisation $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{1 - r^2}$, $r^2 \leq 1$, where $(r, \theta) \in R_{r\theta}$. Then by (5.78) and (5.81)

$$\begin{aligned}
\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma &= \iint_S (L\cos\alpha + M\cos\beta + N\cos\gamma)\,d\sigma \\
&= \iint_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy
\end{aligned}$$

$$= \iint\limits_{R_{r\theta}} \left[L\frac{\partial(y,z)}{\partial(r,\theta)} + M\frac{\partial(z,x)}{\partial(r,\theta)} + N\frac{\partial(x,y)}{\partial(r,\theta)} \right] dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2\sqrt{u}}\,du\,d\theta$$
$$= \int_0^{2\pi} \sqrt{u}\Big|_0^1\,d\theta = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

where we have used the substitution $u = 1 - r^2$ such that $du = -2r\,dr$.

(d) Let

$$\iint\limits_S x^2 z\,d\sigma$$

where S is the cylindrical surface $x^2 + y^2 = 1$, $0 \le z \le 1$. Using the parametrisation $x = \cos\theta$, $y \sin\theta$, $z = z$, the surface area element $d\sigma$ can be represented by the product $d\theta\,dz$ and so the integral can be evaluated as

$$\iint\limits_S x^2 z\,d\sigma = \int_0^{2\pi}\int_0^1 z\cos^2\theta\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \frac{1}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{\pi}{2}$$

6. (a) Let

$$\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and the normal is pointing away from $(0,0,0)$. Furthermore, let the parametrisation $x = u + v$, $y = u - v$, $z = 1 - 2u$, where $0 \le u \le 1/2$, $0 \le v \le 1/2$ be a given. From Problem 5(a) we know that $\sqrt{3}\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence, utilising (5.82) we find that

$$\mathbf{n} = -\frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

where

$$\mathbf{P}_1 = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{P}_2 = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

and so by (5.81)

$$\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy = \iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$
$$= -\iint\limits_{R_{uv}} \left[L\frac{\partial\,(y,z)}{\partial\,(u,v)} + M\frac{\partial\,(z,x)}{\partial\,(u,v)} + N\frac{\partial\,(x,y)}{\partial\,(u,v)} \right] du\,dv$$
$$= 2\int_0^{1/2}\int_0^{1/2} du\,dv = \frac{1}{2}$$

(b) Let

$$\iint\limits_S dy\,dz + dz\,dx + dx\,dy$$

where S is the hemisphere $z = \sqrt{1-x^2-y^2}$, $x^2+y^2 \le 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

Furthermore, let the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$, where $0 \le u \le \pi/2$, $0 \le v \le 2\pi$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.81)

$$\iint\limits_S L\,dy\,dz + M\,dz\,dx + N\,dx\,dy = \iint\limits_S dy\,dz + dz\,dx + dx\,dy$$
$$= \iint\limits_{R_{uv}} \left[L\frac{\partial\,(y,z)}{\partial\,(u,v)} + M\frac{\partial\,(z,x)}{\partial\,(u,v)} + N\frac{\partial\,(x,y)}{\partial\,(u,v)} \right] du\,dv$$
$$= \int_0^{2\pi}\int_0^{\pi/2} \left[\sin^2 u\,(\sin v + \cos v) + \sin u \cos u\right] du\,dv$$
$$= \underbrace{\int_0^{2\pi} (\sin v + \cos v)\,dv}_{0} \int_0^{\pi/2} \sin^2 u\,du$$
$$+ \int_0^{2\pi}\int_0^{\pi/2} \sin u \cos u\,du\,dv$$
$$= \int_0^{2\pi}\int_0^{1} w\,dw\,dv = \frac{1}{2}\int_0^{2\pi} dv = \pi$$

(c) Let

$$\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\; d\sigma$$

where S is the hemisphere $z = \sqrt{1-x^2-y^2}$, $x^2+y^2 \le 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

Furthermore, let the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$, where $0 \le u \le \pi/2$, $0 \le v \le 2\pi$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.78) and (5.81)

$$\begin{aligned}
\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\; d\sigma &= \iint_S (L\cos\alpha + M\cos\beta + N\cos\gamma)\; d\sigma \\
&= \iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)} \right] du\, dv \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left[\sin u + \sin^2 u \cos u\,(\sin v + \cos v)\right] du\, dv \\
&= \int_0^{2\pi}\int_0^{\pi/2} \sin u\, du\, dv \\
&\quad + \underbrace{\int_0^{2\pi} (\sin v + \cos v)\; dv}_{0} \int_0^{\pi/2} \sin^2 u \cos u\, du \\
&= \int_0^{2\pi} dv = 2\pi
\end{aligned}$$

(d) Let

$$\iint_S x^2 z\, d\sigma$$

where S is the cylindrical surface $x^2+y^2 = 1$, $0 \le z \le 1$ and the normal is the upper / outer normal given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$

Furthermore, let the parametrisation $x = \cos u$, $y = \sin u$, $z = v$, where $0 \le u \le 2\pi$, $0 \le v \le 1$ be a given. By (5.82) we find that

$$\mathbf{n} = \frac{\mathbf{P}_1 \times \mathbf{P}_2}{|\mathbf{P}_1 \times \mathbf{P}_2|}$$

and so by (5.78) and (5.81)

$$\begin{aligned}
\iint\limits_S x^2 z \, d\sigma = \iint\limits_S L \cos\alpha \, d\sigma = \iint\limits_S L \, dy \, dz &= \iint\limits_{R_{uv}} L \frac{\partial (y, z)}{\partial (u, v)} \, du \, dv \\
&= \int_0^1 \int_0^{2\pi} v \cos^2 u \, du \, dv \\
&= \int_0^1 v \, dv \int_0^{2\pi} \frac{1 + \cos 2u}{2} \, du \\
&= \pi \int_0^1 v \, dv = \frac{\pi}{2}
\end{aligned}$$

7. (a) Let

$$\iint\limits_S \mathbf{w} \cdot \mathbf{n} \, d\sigma$$

where $\mathbf{w} = xy^2 z \mathbf{i} - 2x^3 \mathbf{j} + yz^2 \mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$ and $\mathbf{n}$ is the upper / outer normal. Furthermore, let us introduce the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos^2 u$, where $0 \le u \le \pi/2$, $0 \le v \le 2\pi$. Then by (5.79) and (5.81)

$$\begin{aligned}
\iint\limits_S \mathbf{w} \cdot \mathbf{n} \, d\sigma &= \iint\limits_S L \, dy \, dz + M \, dz \, dx + N \, dx \, dy \\
&= \iint\limits_S xy^2 z \, dy \, dz - 2x^3 \, dz \, dx + yz^2 \, dx \, dy \\
&= \iint\limits_{R_{uv}} \left[L \frac{\partial (y, z)}{\partial (u, v)} + M \frac{\partial (z, x)}{\partial (u, v)} + N \frac{\partial (x, y)}{\partial (u, v)} \right] du \, dv \\
&= 2 \int_0^{2\pi} \int_0^{\pi/2} \cos^3 u \sin^5 u \sin^2 v \cos^2 v \, du \, dv \\
&\quad - 4 \int_0^{2\pi} \int_0^{\pi/2} \cos u \sin^5 u \cos^3 v \sin v \, du \, dv + \int_0^{2\pi} \int_0^{\pi/2} \cos^5 u \sin^2 u \sin v \, du \, dv
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{2\pi} \sin^2 v \cos^2 v\, dv \int_0^{\pi/2} \cos^3 u \sin^5 u\, du - 4\underbrace{\int_0^{2\pi} \cos^3 v \sin v\, dv}_{0} \\
&\quad \times \int_0^{\pi/2} \cos u \sin^5 u\, du + \underbrace{\int_0^{2\pi} \sin v\, dv}_{0} \int_0^{\pi/2} \cos^5 u \sin^2 u\, du \\
&= 2\int_0^{2\pi} \sin^2 v \cos^2 v\, dv \int_0^{\pi/2} \cos^3 u \sin^5 u\, du \\
&= \frac{1}{4}\int_0^{2\pi} (1-\cos 4v)\, dv \int_0^{\pi/2} \cos^3 u \left(1-\cos^2 u\right)^2 \sin u\, du \\
&= \frac{1}{4}\left[v - \frac{\sin 4v}{4}\right]_0^{2\pi} \times \int_0^1 w^3\left(1-w^2\right)^2\, dw = \frac{\pi}{2}\int_0^1 \left(w^3 - 2w^5 + w^7\right)\, dw \\
&= \frac{\pi}{2}\left[\frac{w^4}{4} - \frac{w^6}{3} + \frac{w^8}{8}\right]_0^1 = \frac{\pi}{48}
\end{aligned}
$$

where we have used the substitution $w = \cos u$ so that $dw = -\sin u\, du$.

(b) Let

$$
\iint\limits_S \mathbf{w} \cdot \mathbf{n}\, d\sigma
$$

where $\mathbf{w} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, S is the surface $x = e^u \cos v,\ y = e^u \sin v,\ z = \cos v \sin v,\ 0 \leq u \leq 1,\ 0 \leq v \leq \pi/2$ and $\mathbf{n}$ is given by (5.82) with the $+$ sign. Then by (5.79) and (5.81)

$$
\begin{aligned}
\iint\limits_S \mathbf{w} \cdot \mathbf{n}\, d\sigma &= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S dy\, dz + 2\, dz\, dx + 3\, dx\, dy \\
&= \iint\limits_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right]\, du\, dv \\
&= \int_0^{\pi/2}\int_0^1 e^u \sin v\left(\cos^2 v - \sin^2 v\right)\, du\, dv \\
&\quad - \int_0^{\pi/2}\int_0^1 2e^u \cos v\left(\cos^2 v - \sin^2 v\right)\, du\, dv + \int_0^{\pi/2}\int_0^1 3e^{2u}\, du\, dv \\
&= \int_0^{\pi/2}\int_0^1 e^u \sin v\left(2\cos^2 v - 1\right)\, du\, dv \\
&\quad - \int_0^{\pi/2}\int_0^1 2e^u \cos v\left(1 - 2\sin^2 v\right)\, du\, dv + \int_0^{\pi/2}\int_0^1 3e^{2u}\, du\, dv
\end{aligned}
$$

$$= \int_0^{\pi/2} \sin v \left(2\cos^2 v - 1\right)\, dv \int_0^1 e^u\, du - \int_0^{\pi/2} \cos v \left(1 - 2\sin^2 v\right)\, dv \int_0^1 2e^u\, du$$

$$+ \frac{3\left(e^2 - 1\right)}{2} \int_0^{\pi/2} dv$$

$$= (e-1) \int_0^1 \left(2p^2 - 1\right)\, dp - 2(e-1) \int_0^1 \left(1 - 2q^2\right)\, dq + \frac{3\pi\left(e^2-1\right)}{4}$$

$$= (e-1)\left[\frac{2p^3}{3} - p\right]_0^1 - 2(e-1)\left[q - \frac{2q^3}{3}\right]_0^1 + \frac{3\pi\left(e^2-1\right)}{4}$$

$$= -(e-1) + \frac{3\pi\left(e^2-1\right)}{4}$$

(c) Let

$$\iint_S \frac{\partial w}{\partial n}\, d\sigma$$

where $w = x^2y^2z$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$ and $\mathbf{n}$ is the upper / outer normal. Furthermore, let us introduce the parametrisation $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos^2 u$, where $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$. Then by (5.79), (5.81) and (2.113)

$$\iint_S \frac{\partial w}{\partial n}\, d\sigma = \iint_S \nabla w \cdot \mathbf{n}\, d\sigma$$

$$= \iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy$$

$$= \iint_S 2xy^2z\, dy\, dz + 2x^2yz\, dz\, dx + x^2y^2\, dx\, dy$$

$$= \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right] du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(8\cos^3 u \sin^5 u + \cos u \sin^5 u\right) \cos^2 v \sin^2 v\, du\, dv$$

$$= \int_0^{2\pi} \cos^2 v \sin^2 v\, dv \int_0^{\pi/2} \left(8\cos^3 u \sin^5 u + \cos u \sin^5 u\right)\, du$$

$$= \int_0^{2\pi} \frac{1 - \cos 4v}{8}\, dv \left[8\int_0^1 p^3\left(1-p^2\right)^2\, dp + \int_0^1 q^5\, dq\right]$$

$$= \frac{1}{8}\left[v - \frac{\sin 4v}{4}\right]_0^{2\pi} \left\{8\left[\frac{p^4}{4} - \frac{p^6}{3} + \frac{p^8}{8}\right]_0^1 + \left.\frac{q^6}{6}\right|_0^1\right\} = \frac{\pi}{4}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{\pi}{8}$$

where we have used the substitutions $p = \cos u$ so that $dp = -\sin u\, du$ and $q = \sin u$ so that $dq = \cos u\, du$.

(d) Let

$$\iint\limits_S \frac{\partial w}{\partial n}\, d\sigma$$

where $w = x^2 - y^2 + z^2$, S is the surface $x = e^u \cos v,\ y = e^u \sin v,\ z = \cos v \sin v,\ 0 \leq u \leq 1,\ 0 \leq v \leq \pi/2$ and $\mathbf{n}$ is given by (5.82) with the + sign. Then by (5.79), (5.81) and (2.113)

$$\begin{aligned}
\iint\limits_S \frac{\partial w}{\partial n}\, d\sigma &= \iint\limits_S \nabla w \cdot \mathbf{n}\, d\sigma \\
&= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S 2x\, dy\, dz - 2y\, dz\, dx + 2z\, dx\, dy \\
&= \iint\limits_{R_{uv}} \left[L\frac{\partial\,(y,z)}{\partial\,(u,v)} + M\frac{\partial\,(z,x)}{\partial\,(u,v)} + N\frac{\partial\,(x,y)}{\partial\,(u,v)} \right] du\, dv \\
&= \int_0^{\pi/2} \int_0^1 \left[4e^u \cos v \sin v \left(\cos^2 v - \sin^2 v \right) + 2e^{2u} \cos v \sin v \right] du\, dv \\
&= \int_0^{\pi/2} \cos v \sin v \left(\cos^2 v - \sin^2 v \right)\, dv \int_0^1 4e^u\, du + \int_0^{\pi/2} \cos v \sin v\, dv \int_0^1 2e^{2u}\, du \\
&= 2\,(e-1) \int_0^{\pi/2} \sin 2v \cos 2v\, dv + \frac{e^2-1}{2} \int_0^{\pi/2} \sin 2v\, dv \\
&= (e-1) \int_0^{\pi/2} \sin 4v\, dv - \frac{e^2-1}{4} \cos 2v\Big|_0^{\pi/2} = \underbrace{-\frac{e-1}{4} \cos 4v\Big|_0^{\pi/2}}_{0} + \frac{e^2-1}{2} \\
&= \frac{e^2-1}{2}
\end{aligned}$$

(e) Let

$$\iint\limits_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{u} = yz\mathbf{i} - xz\mathbf{j} + xz\mathbf{k}$, S is the triangle with vertices $P_1 : (1,2,8)$, $P_2 : (3,1,9)$, $P_3 : (2,1,7)$ and $\mathbf{n}$ is the upper / outer normal. To find $\mathbf{n}$ we utilise the fact that a vector $\mathbf{v} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is by definition perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. As such, we find

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

and note that $\mathbf{n} = a\mathbf{v}$, where a is an arbitrary positive scalar. Furthermore, in Section 1.3 it was stated that if $\mathbf{n}$ is a nonzero normal vector and $P_1 : (x_1, y_1, z_1)$ is a point of a plane, then $P : (x, y, z)$ is in the plane precisely when $\mathbf{v} \cdot \overrightarrow{P_1 P} = 0$. Hence, we find

$$2(x-1) + 3(y-2) - (z-8) = 0 \iff z = 2x + 3y$$

Then by (5.79), (5.80) and (3.23)

$$\begin{aligned}
\iint\limits_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, d\sigma &= \iint\limits_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy \\
&= \iint\limits_S x\, dy\, dz + (y - z)\, dz\, dx - 2z\, dx\, dy \\
&= \iint\limits_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N \right) dx\, dy \\
&= \iint\limits_{R_{xy}} \underbrace{[-2x - 3(y - z) - 2z]}_{0}\, dx\, dy = 0
\end{aligned}$$

8. (a) Let a surface $S : z = f(x, y)$ be defined by an implicit equation $F(x, y, z) = 0$. Assuming F is differentiable we may write

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy + \frac{\partial F}{\partial z}\, dz \iff dz = -\frac{F_x}{F_z}\, dx - \frac{F_y}{F_z}\, dy$$

and hence,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

From (5.70) and (5.75) it then follows that

$$\begin{aligned}
\iint\limits_S H\, d\sigma = \iint\limits_{R_{xy}} H[x, y, f(x, y)] \sec\gamma\, dx\, dy &= \iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, H\, dx\, dy \\
&= \iint\limits_{R_{xy}} \sqrt{1 + \left(-\frac{F_x}{F_z}\right)^2 + \left(-\frac{F_y}{F_z}\right)^2}\, H\, dx\, dy \\
&= \iint\limits_{R_{xy}} \sqrt{F_x^2 + F_y^2 + F_z^2}\, \frac{H}{|F_z|}\, dx\, dy \\
&= \iint\limits_{R_{xy}} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, \frac{H}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy
\end{aligned}$$

provided that $\partial F / \partial z \neq 0$.

(b) Let $\mathbf{n} = \nabla F / |\nabla F|$. Then using the result to part (a), (1.4) and (3.8) we find

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma = \iint_S H\, d\sigma = \iint_{R_{xy}} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, \frac{H}{\left|\dfrac{\partial F}{\partial z}\right|}\, dx\, dy$$

$$= \iint_{R_{xy}} |\nabla F|\, (\mathbf{v}\cdot\mathbf{n})\, \frac{1}{\left|\dfrac{\partial F}{\partial z}\right|}\, dx\, dy = \iint_{R_{xy}} |\nabla F| \left(\mathbf{v}\cdot\frac{\nabla F}{|\nabla F|}\right) \frac{1}{\left|\dfrac{\partial F}{\partial z}\right|}\, dx\, dy$$

$$= \iint_{R_{xy}} (\mathbf{v}\cdot\nabla F)\, \frac{1}{\left|\dfrac{\partial F}{\partial z}\right|}\, dx\, dy$$

(c) Using (5.82), (5.72), (5.73) and (5.79) we find

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma = \iint_{R_{uv}} \left(\mathbf{v}\cdot \pm\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right) d\sigma = \pm\iint_S \left(\mathbf{v}\cdot\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right)\sqrt{EG-F^2}\, du\, dv$$

$$= \pm\iint_{R_{uv}} \left(\mathbf{v}\cdot\frac{\mathbf{P}_1\times\mathbf{P}_2}{|\mathbf{P}_1\times\mathbf{P}_2|}\right)|\mathbf{P}_1\times\mathbf{P}_2|\, du\, dv$$

$$= \pm\iint_{R_{uv}} (\mathbf{v}\cdot\mathbf{P}_1\times\mathbf{P}_2)\, du\, dv$$

$$= \pm\iint_{R_{uv}} (L\mathbf{i}+M\mathbf{j}+N\mathbf{k})\cdot\left[\frac{\partial(y,z)}{\partial(u,v)}\mathbf{i} + \frac{\partial(z,x)}{\partial(u,v)}\mathbf{j} + \frac{\partial(x,y)}{(u,v)}\mathbf{k}\right] du\, dv$$

$$= \pm\iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)}\right] du\, dv$$

(d) When $x = u$, $y = v$, $z = f(u,v)$

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial u} = -\frac{\partial z}{\partial x}\frac{dx}{du} = -\frac{\partial z}{\partial x}$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial y}\frac{dy}{dv} = -\frac{\partial z}{\partial y}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = 1$$

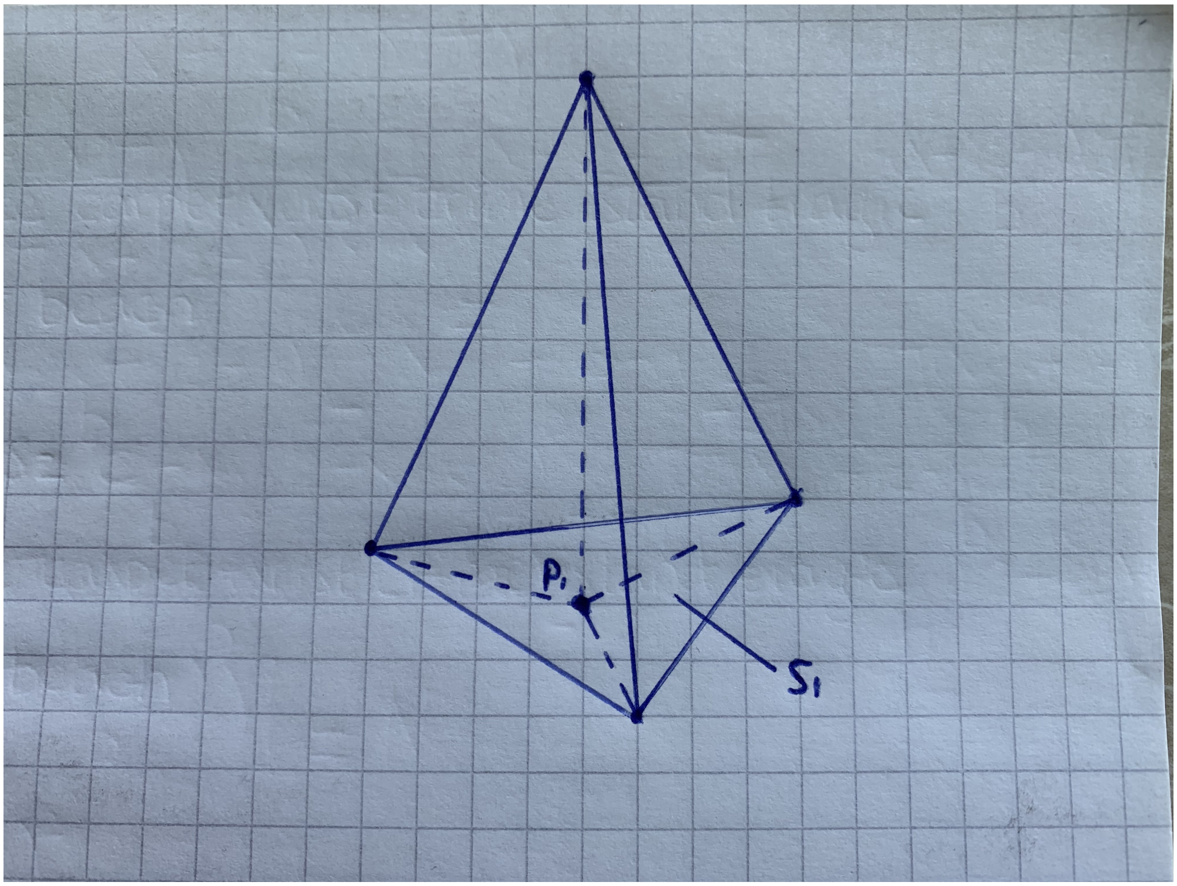
and $dx = du$, $dy = dv$. Hence, (5.81) reduces to

$$\pm\iint_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N\right) dx\, dy = \iint_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma$$

$$= \iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy$$

which is none other than (5.80)

9. Let S be an oriented surface in space that is planar. With S one can associate the vector $\mathbf{S}$, which has the direction of the normal chosen on S and has a length equal to the area of S.



(a) Let S_1, S_2, S_3, S_4 be the faces of a tetrahedron, oriented so that the normal is the exterior normal. With each $S_1, \ldots, S_4$ we can then associate the vector $\mathbf{S}_i = A_i \mathbf{n}_i \, (A_i > 0)$ for $i = 1, \ldots, 4$, where A_i is the area of face S_i and $\mathbf{n}_i$ is the exterior normal to S_i. Next, let the point p_1 be the foot of the altitude on face S_1. Then we can join p_1 to the vertices of S_1 to form three triangles of areas A_{12}, A_{13}, A_{14}, such that $A_{12} + A_{13} + A_{14} = A_2$ (see the above figure). Analogous to (4.79), it follows readily from geometry that $A_{1j} = \pm A_j \mathbf{n}_j \cdot \mathbf{n}_1$, with $+$ when $\mathbf{n}_j \cdot \mathbf{n}_1 > 0$ or $-$ when $\mathbf{n}_j \cdot \mathbf{n} < 0$ and $A_{1j} = 0$ if $\mathbf{n}_j \cdot \mathbf{n}_1 = 0 \, (j = 2, 3, 4)$, i.e. $A_{1j} = 0$ when $\mathbf{n}_j$ and $\mathbf{n}_1$ are perpendicular. In other words, the area of each triangle A_{12}, A_{13}, A_{14} is equal to the projection of S_2, S_3, S_4 onto S_1. Let us now introduce the vector $\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{b}$ and consider the dot product $\mathbf{b} \cdot \mathbf{n}_j$

for $j = 1, 2, 3, 4$. Fixing $j = 2$ for example, we find

$$\begin{aligned}
\mathbf{b} \cdot \mathbf{n}_2 &= \mathbf{S}_1 \cdot \mathbf{n}_2 + \mathbf{S}_2 \cdot \mathbf{n}_2 + \mathbf{S}_3 \cdot \mathbf{n}_2 + \mathbf{S}_4 \cdot \mathbf{n}_2 \\
&= A_1 \mathbf{n}_1 \cdot \mathbf{n}_2 + A_2 \mathbf{n}_2 \cdot \mathbf{n}_2 + A_3 \mathbf{n}_3 \cdot \mathbf{n}_2 + A_4 \mathbf{n}_4 \cdot \mathbf{n}_2 \\
&= A_1 \mathbf{n}_1 \cdot \mathbf{n}_2 + A_2 + A_3 \mathbf{n}_3 \cdot \mathbf{n}_2 + A_4 \mathbf{n}_4 \cdot \mathbf{n}_2 \\
&= -A_{21} + A_2 - A_{23} - A_{24} = 0
\end{aligned}$$

where the minus sign for the first, third and fourth term on the right hand side follows from the fact that the angle between the normal $\mathbf{n}_2$ to face S_2 and any of the other normal vectors $\mathbf{n}_1$, $\mathbf{n}_3$, $\mathbf{n}_4$ associated with faces S_1, S_3, S_4 is always at least equal to or larger than $\pi/2$ and hence, $\mathbf{n}_i \cdot \mathbf{n}_2 \leq 0 \, (i = 1, 3, 4)$. The same reasoning applies to the other faces and hence, we may conclude that $\mathbf{b} \cdot \mathbf{n}_j = 0$ and so clearly $\mathbf{b} \cdot A_j \mathbf{n}_j = 0$ as well. Summing over j then gives

$$\begin{aligned}
\mathbf{b} \cdot A_1 \mathbf{n}_1 + \mathbf{b} \cdot A_2 \mathbf{n}_2 + \mathbf{b} \cdot A_3 \mathbf{n}_3 + \mathbf{b} \cdot A_4 \mathbf{n}_4 &= \mathbf{b} \cdot (A_1 \mathbf{n}_1 + A_2 \mathbf{n}_2 + A_3 \mathbf{n}_3 + A_4 \mathbf{n}_4) \\
&= \mathbf{b} \cdot (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4) \\
&= \mathbf{b} \cdot \mathbf{b} \\
&= 0
\end{aligned}$$

By (1.102) $\mathbf{b} \cdot \mathbf{b} = 0$ if only if $\mathbf{b} = \mathbf{0}$ and hence, this proves that

$$\mathbf{b} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}$$

(b) Let an arbitrary convex polyhedron with faces $S_1, \ldots, S_n$, oriented so that the normal is the exterior normal, be given. With each $S_1, \ldots, S_n$ we can again associate the vector $\mathbf{S}_i = A_i \mathbf{n}_i \, (A_i > 0)$ for $i = 1, 2, \ldots, n$, where A_i is the area of face S_i and $\mathbf{n}_i$ is the exterior normal to S_i. Just like an arbitrary polygon can be subdivided into a finite number of triangles, an arbitrary (convex) polyhedron can be subdivided into a finite number of tetrahedra. Let $\mathbf{S}_1^{'}$ and $\mathbf{S}_1^{''}$ be the vectors associated with sides $S_1^{'}$ and $S_1^{''}$ of two such tetrahedra that are glued together. By definition $\mathbf{S}_1^{'}$ and $\mathbf{S}_1^{''}$ will have the same magnitude, but point in opposite directions and so $\mathbf{S}_1^{'} + \mathbf{S}_1^{''} = \mathbf{0}$. From part (a) we know that

$$\mathbf{S}_1^{'} = -\left(\mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}\right) \qquad \mathbf{S}_1^{''} = -\left(\mathbf{S}_2^{''} + \mathbf{S}_3^{''} + \mathbf{S}_4^{''}\right)$$

Hence,

$$\mathbf{S}_1^{'} + \mathbf{S}_1^{''} = -\left(\mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}\right) - \left(\mathbf{S}_2^{''} + \mathbf{S}_3^{''} + \mathbf{S}_4^{''}\right) = \mathbf{0}$$

The thing to take away from this exercise is that the vector associated with the face that is hidden when gluing two tetrahedra together on one tetrahedron is equal to the sum of the remaining vectors on the other tetrahedron, i.e.

$$\mathbf{S}_1^{''} = \mathbf{S}_2^{'} + \mathbf{S}_3^{'} + \mathbf{S}_4^{'}$$

Hence, starting with the fact that for a single tetrahedron we proved that $\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}$, we find that as we build up our arbitrary convex polyhedron by gluing tetrahedra together we can always express the vector associated with one of its connecting faces in terms of the outward pointing vectors associated with the remaining faces on the tetrahedron that is glued on next. As we continue this operation, the vectors associated with the faces that connect to another tetrahedron will cancel each other out, leaving only the n vectors associated with the faces that do not connect to the face of another tetrahedron, and so we may conclude that

$$\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n = \mathbf{0}$$

(c) Using the definition

$$\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \Delta_i\sigma = \iint\limits_S d\sigma$$

then, provided that $\mathbf{v}$ is a constant vector and so will have the same magnitude and direction at any point (x, y, z) in 3D space, we can write

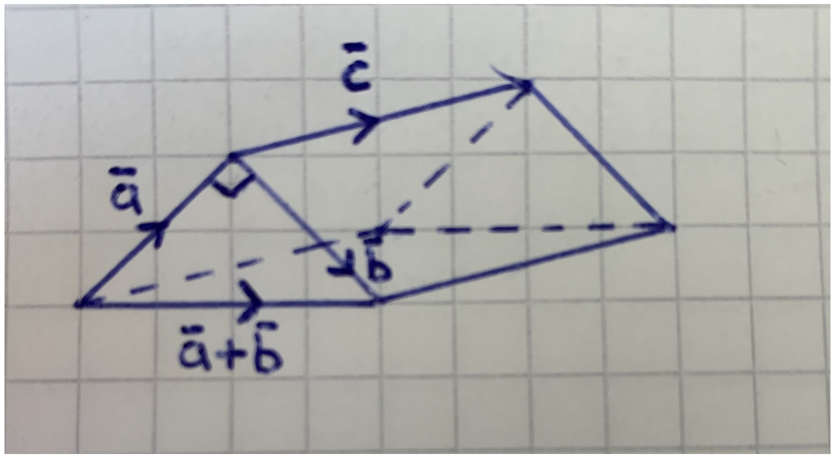
$$\iint\limits_S \mathbf{v} \cdot d\boldsymbol{\sigma} = \iint\limits_S \mathbf{v} \cdot \mathbf{n}\, d\sigma = \lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \mathbf{v} \cdot \mathbf{n}\, \Delta_i\sigma = \mathbf{v} \cdot \left(\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \Delta_i\sigma \right)$$

Now from part (a) and (b) we know that any arbitrary convex polyhedron can be build up from simple tetrahedra and that the sum of their face vectors, i.e. $\mathbf{S}_i = A_i\mathbf{n}_i\,(A_i > 0)$ for $i = 1, 2, \ldots, n$, adds up to zero. An arbitrary convex closed surface S (such as the surface of a sphere or ellipsoid) can be though of as a convex polyhedron in the limit that the number of faces $n \to \infty$, such that $\max A_i \to 0$. Identifying that $A_i = \Delta_i\sigma$ we thus may conclude that

$$\iint\limits_S \mathbf{v} \cdot d\boldsymbol{\sigma} = \Delta_i\sigma = \mathbf{v} \cdot \left(\lim_{\substack{n\to\infty \\ \max \Delta_i\sigma \to 0}} \sum_{i=1}^{n} \Delta_i\sigma \right) = \mathbf{v} \cdot \mathbf{0} = 0$$

provided, as stated before, that $\mathbf{v}$ is a constant vector and S is any convex closed surface.

(d)



Let the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}+\mathbf{b}$ and $\mathbf{c}$ represent the edges of a triangular prism (see figure above). We can then introduce the vector $\mathbf{c}\times\mathbf{a}$, which is perpendicular to and pointing away from the face formed by vectors $\mathbf{a}$ and $\mathbf{c}$. Similarly, we can define the vector $\mathbf{c}\times\mathbf{b}$, which is perpendicular to and pointing away from the face formed by vectors $\mathbf{b}$ and $\mathbf{c}$. Lastly, we can define the vector $\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ which is pointing towards the face formed by vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{c}$. Moreover, from inspection of the figure above and visualising these 3 newly defined vectors in space it follows readily that

$$\mathbf{c}\times(\mathbf{a}+\mathbf{b}) = \mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}$$

Section 5.11

1. (a)