Numerical Sound Synthesis

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1 Preliminaries

1.1 (Product) identities and inequalities

Listed below are various (product) identities which are of use in the energetic analysis of finite difference schemes.

$$\mu_{t\cdot} = 1 + \frac{k^2}{2}\delta_{tt} \tag{1a}$$
$$\delta_{t\cdot} = \delta_{t+}\mu_{t-} = \delta_{t-}\mu_{t+} \tag{1b}$$
$$\delta_{tt} = \frac{1}{k}\left(\delta_{t+} - \delta_{t-}\right) \tag{1c}$$
$$1 = \mu_{t\pm} \mp \frac{k}{2}\delta_{t\pm} \tag{1d}$$
$$e_{t\pm} = \mu_{t\pm} \pm \frac{k}{2}\delta_{t\pm} \tag{1e}$$
$$e_{t\pm} = 1 \pm k\delta_{t\pm} \tag{1f}$$
$$\delta_{tt} = \frac{2}{k}\left(\delta_{t\cdot} - \delta_{t-}\right) \tag{1g}$$
$$\mu_{t\cdot} = k\delta_{t\cdot} + e_{t-} \tag{1h}$$

$$\langle \delta_{t\cdot}u, \delta_{tt}u\rangle_{\mathcal{D}} = \delta_{t+}\left(\frac{1}{2}\|\delta_{t-}u\|^2_{\mathcal{D}}\right) \tag{2a}$$
$$\langle \delta_{t\cdot}u, u\rangle_{\mathcal{D}} = \delta_{t+}\left(\frac{1}{2}\langle u, e_{t-}u\rangle_{\mathcal{D}}\right) \tag{2b}$$
$$\langle u, e_{t-}u\rangle_{\mathcal{D}} = \|\mu_{t-}u\|^2_{\mathcal{D}} - \frac{k^2}{4}\|\delta_{t-}u\|^2_{\mathcal{D}} \tag{2c}$$

1.1.1 The discrete Laplacian and biharmonic operators

A discrete five point approximation to the Laplacian operator, making use of points adjacent to the center point, is of the form

$$\delta_{\Delta\boxplus} = \delta_{xx} + \delta_{yy} \tag{3}$$

Also important, in the case of the vibrating stiff plate, is the discrete biharmonic operator, or bi-Laplacian, which consists of the composition of the Laplacian with itself A simple approximation may be given as

$$\delta_{\Delta\boxplus,\Delta\boxplus} \triangleq \delta_{\Delta\boxplus}\delta_{\Delta\boxplus} = \delta_{xxxx} + \delta_{yyyy} + 2\delta_{xxyy} \tag{4}$$

1.2 Inner products and identities

An l_2 spatial inner product of two 1D grid functions, f_l^n and g_l^n, over the interval $l \in \mathcal{D}$, may be defined as

$$\langle f^n, g^n \rangle_{\mathcal{D}} = \sum_{l \in \mathcal{D}} h f_l^n g_l^n = \sum_{l=d_-}^{d_+} h f_l^n g_l^n \tag{5}$$

From this, an l_2 norm follows as

$$\|f^n\|_{\mathcal{D}} = \sqrt{\langle f^n, f^n \rangle_{\mathcal{D}}} \geq 0 \tag{6}$$

The discrete inner product and norm over a Cartesian grid in 2D are a direct extension of those given by (5) and (6). For an arbitrary domain $\mathcal{D}$, the simplest definition, for two grid function $f_{l,m}$ and $g_{l,m}$ defined over a grid of spacing h_x in the x and h_y in the y direction, is

$$\langle f^n, g^n \rangle_{\mathcal{D}} = \sum_{(l,m) \in \mathcal{D}} h_x h_y f_{l,m}^n g_{l,m}^n = \sum_{l=x_-}^{x_+} \sum_{m=y_-}^{y_+} h_x h_y f_{l,m}^n g_{l,m}^n \quad \|f^n\|_{\mathcal{D}} = \langle f^n, f^n \rangle_{\mathcal{D}} \geq 0 \tag{7}$$

The Cauchy-Schwartz and triangle inequalities extend directly to such discrete inner products as

$$|\langle f^n, g^n \rangle_{\mathcal{D}}| \leq \|f^n\|_{\mathcal{D}} \|g^n\|_{\mathcal{D}} \tag{8a}$$

$$\|f^n + g^n\|_{\mathcal{D}} \leq \|f^n\|_{\mathcal{D}} + \|g^n\|_{\mathcal{D}} \tag{8b}$$

Other types of inner products may be defined:

$$\langle f^n, g^n \rangle'_{\mathcal{D}} = \sum_{l=d_-+1}^{d_+-1} h f_l^n g_l^n + \frac{h}{2} f_{d_-}^n g_{d_-}^n + \frac{h}{2} f_{d_+} g_{d_+}^n \tag{9a}$$

$$\langle f^n, g^n \rangle_{\underline{\mathcal{D}}} = \sum_{l=d_-}^{d_+-1} h f_l^n g_l^n \tag{9b}$$

$$\langle f^n, g^n \rangle_{\overline{\mathcal{D}}} = \sum_{l=d_-+1}^{d_+} h f_l^n g_l^n \tag{9c}$$

$$\langle f^n, g^n \rangle_{\underline{\overline{\mathcal{D}}}} = \sum_{l=d_-+1}^{d_+-1} h f_l^n g_l^n \tag{9d}$$

which extend naturally to 2D:

$$\begin{aligned}\langle f^n, g^n \rangle'_{\mathcal{D}} = &\sum_{l=x_-+1}^{x_+-1} \sum_{m=y_-+1}^{y_+-1} h_x h_y f_{l,m}^n g_{l,m}^n + \frac{h_x h_y}{2} \sum_{l=x_-+1}^{x_+-1} f_{l,y_-} g_{l,y_-} + \frac{h_x h_y}{2} \sum_{m=y_-+1}^{y_+-1} f_{x_-,m} g_{x_-,m} \\ &+ \frac{h_x h_y}{2} \sum_{l=x_-+1}^{x_+-1} f_{l,y_+} g_{l,y_+} + \frac{h_x h_y}{2} \sum_{m=y_-+1}^{y_+-1} f_{x_+,m} g_{x_+,m} + \frac{h_x h_y}{4} f_{x_-,y_-} g_{x_-,y_-} \\ &+ \frac{h_x h_y}{4} f_{x_-,y_+} g_{x_-,y_+} + \frac{h_x h_y}{4} f_{x_+,y_-} g_{x_+,y_-} + \frac{h_x h_y}{4} f_{x_+,y_+} g_{x_+,y_+}\end{aligned} \tag{10a}$$

$$\langle f^n, g^n \rangle_{\underline{\mathcal{D}}} = \sum_{l=x_-}^{x_+-1} \sum_{m=y_-}^{y_+-1} h_x h_y f_{l,m}^n g_{l,m}^n \tag{10b}$$

$$\langle f^n, g^n \rangle_{\overline{\mathcal{D}}} = \sum_{l=x_-+1}^{x_+} \sum_{m=y_-+1}^{y_+} h_x h_y f_{l,m}^n g_{l,m}^n \tag{10c}$$

$$\langle f^n, g^n \rangle_{\underline{\overline{\mathcal{D}}}} = \sum_{l=x_-+1}^{x_+-1} \sum_{m=y_-+1}^{y_+-1} h_x h_y f_{l,m}^n g_{l,m}^n \tag{10d}$$

$$\{f^n, g^n\}_{(x_-,\mathcal{D})} = \sum_{m=y_-}^{y_+} h_y f_{x_-,m}^n g_{x_-,m}^n \tag{10e}$$

1.3 Summation by parts

The following series manipulations are of use in the energetic analysis of finite difference schemes.

$$\langle f, \delta_{x+} g\rangle_{\underline{\mathcal{D}}} = -\langle \delta_{x-} f, g\rangle_{\overline{\mathcal{D}}} + f_{d_+} g_{d_+} - f_{d_-} g_{d_-} \tag{11a}$$
$$\langle f, \delta_{x-} g\rangle_{\overline{\mathcal{D}}} = -\langle \delta_{x+} f, g\rangle_{\underline{\mathcal{D}}} + f_{d_+} g_{d_+} - f_{d_-} g_{d_-} \tag{11b}$$
$$\langle f, \delta_{xx} g\rangle_{\mathcal{D}} = -\langle \delta_{x-} f, \delta_{x-} g\rangle_{\overline{\mathcal{D}}} + f_{d_+} \delta_{x+} g_{d_+} - f_{d_-} \delta_{x-} g_{d_-} \tag{11c}$$
$$\langle f, \delta_{xx} g\rangle_{\mathcal{D}} = -\langle \delta_{x+} f, \delta_{x+} g\rangle_{\underline{\mathcal{D}}} + f_{d_+} \delta_{x+} g_{d_+} - f_{d_-} \delta_{x-} g_{d_-} \tag{11d}$$
$$\langle f, \delta_{xx} g\rangle'_{\mathcal{D}} = -\langle \delta_{x-} f, \delta_{x-} g\rangle_{\overline{\mathcal{D}}} + f_{d_+} \delta_{x\cdot} g_{d_+} - f_{d_-} \delta_{x\cdot} g_{d_-} \tag{11e}$$
$$\langle f, \delta_{xx} g\rangle'_{\mathcal{D}} = -\langle \delta_{x+} f, \delta_{x+} g\rangle_{\underline{\mathcal{D}}} + f_{d_+} \delta_{x\cdot} g_{d_+} - f_{d_-} \delta_{x\cdot} g_{d_-} \tag{11f}$$
$$\langle f, \delta_{xx} g\rangle_{\mathcal{D}} = \langle \delta_{xx} f, g\rangle_{\mathcal{D}} + f_{d_+} \delta_{x+} g_{d_+} - f_{d_-} \delta_{x-} g_{d_-} - g_{d_+} \delta_{x+} f_{d_+} + g_{d_-} \delta_{x-} f_{d_-} \tag{11g}$$
$$\langle f, \delta_{xx} g\rangle_{\mathcal{D}} = \langle \delta_{xx} f, g\rangle_{\overline{\underline{\mathcal{D}}}} + f_{d_+} \delta_{x+} g_{d_+} - f_{d_-} \delta_{x-} g_{d_-} - g_{d_+} \delta_{x-} f_{d_+} + g_{d_-} \delta_{x+} f_{d_-} \tag{11h}$$
$$\langle f, \delta_{xx} g\rangle'_{\mathcal{D}} = \langle _{xx} f, g\rangle_{\overline{\underline{\mathcal{D}}}} + f_{d_+} \delta_{x\cdot} g_{d_+} - f_{d_-} \delta_{x\cdot} g_{d_-} - g_{d_+} \delta_{x\cdot} f_{d_+} + g_{d_-} \delta_{x\cdot} f_{d_-} \tag{11i}$$

$$\begin{aligned}\langle f, \delta_{\Delta\boxplus} g\rangle_{\mathcal{D}} = &-\langle \delta_{x+} f, \delta_{x+} g\rangle_{\underline{\mathcal{D}}} - \langle \delta_{y+} f, \delta_{y+} g\rangle_{\underline{\mathcal{D}}} + \{f, \delta_{x+} g\}_{(x_+,\mathcal{D})} + \{f, \delta_{y+} g\}_{(\mathcal{D},y_+)} \\ &- \{f, \delta_{x-} g\}_{(x_-,\mathcal{D})} - \{f, \delta_{y-} g\}_{(\mathcal{D},y_-)}\end{aligned} \tag{12a}$$
$$\begin{aligned}\langle f, \delta_{\Delta\boxplus} g\rangle_{\mathcal{D}} = &\langle \delta_{\Delta\boxplus} f, g\rangle_{\mathcal{D}} + \{f, \delta_{x+} g\}_{(x_+,\mathcal{D})} + \{f, \delta_{y+} g\}_{(\mathcal{D},y_+)} - \{f, \delta_{x-} g\}_{(x_-,\mathcal{D})} - \{f, \delta_{y-} g\}_{(\mathcal{D},y_-)} \\ &- \{g, \delta_{x+} f\}_{(x_+,\mathcal{D})} - \{g, \delta_{y+} f\}_{(\mathcal{D},y_+)} + \{g, \delta_{x-} f\}_{(x_-,\mathcal{D})} + \{g, \delta_{y-} f\}_{(\mathcal{D},y_-)}\end{aligned} \tag{12b}$$

1.4 Some bounds

The following relations between norms of grid function under spatial difference operations to norms of the grid function themselves are of use in the energetic analysis of finite difference schemes.

$$\begin{aligned}\|\delta_{x+} u\|^2_{\underline{\mathcal{D}}} = \sum_{l=d_-}^{d_+-1} h\left(\delta_{x+} u_l\right)^2 = \sum_{l=d_-}^{d_+-1} \frac{1}{h}\left(u_{l+1} - u_l\right)^2 &= \sum_{l=d_-}^{d_+-1} \frac{1}{h}\left(u_{l+1}^2 + u_l^2 - 2u_{l+1}u_l\right) \\ &\leq \sum_{l=d_-}^{d_+-1} \frac{2}{h}\left(u_{l+1}^2 + u_l^2\right) \\ &= \sum_{l=d_-+1}^{d_+-1} \frac{4}{h} u_l^2 + \frac{2}{h} u_{d_-}^2 + \frac{2}{h} u_{d_+}^2 \\ &= \left(\frac{2}{h}\|u\|'_{\mathcal{D}}\right)^2 \\ &\leq \frac{4}{h^2}\|u\|^2_{\mathcal{D}}\end{aligned} \tag{13a}$$

$$
\begin{aligned}
\|\delta_{x-}u\|_{\underline{\mathcal{D}}}^2 = \sum_{l=d_-+1}^{d_+} h\left(\delta_{x-}u_l\right)^2 = \sum_{l=d_-+1}^{d_+} \frac{1}{h}\left(u_l - u_{l-1}\right)^2 &= \sum_{l=d_-+1}^{d_+} \frac{1}{h}\left(u_l^2 + u_{l-1}^2 - 2u_l u_{l-1}\right) \\
&\leq \sum_{l=d_-+1}^{d_+} \frac{2}{h}\left(u_l^2 + u_{l-1}^2\right) \\
&= \sum_{l=d_-+1}^{d_+-1} \frac{4}{h}u_l^2 + \frac{2}{h}u_{d_-}^2 + \frac{2}{h}u_{d_+}^2 \\
&= \left(\frac{2}{h}\|u\|_{\mathcal{D}}'\right)^2 \\
&\leq \frac{4}{h^2}\|u\|_{\mathcal{D}}^2
\end{aligned}
\tag{13b}
$$

$$
\|\delta_{xx}u\|_{\underline{\mathcal{D}}}^2 = \|\delta_{x+}\delta_{x-}u\|_{\underline{\mathcal{D}}}^2 \leq \frac{4}{h^2}\|\delta_{x-}u\|_{\underline{\mathcal{D}}}^2 \leq \frac{16}{h^4}\|u\|_{\mathcal{D}}^2 \tag{13c}
$$

1.5 Matrix interpretation of difference operators

Assuming the domain $\mathcal{D} = \mathbb{Z}$ for now, so $l = -\infty, \dots, \infty$, the grid function u_l may be arranged in an infinite column vector $\mathbf{u} = [\dots, u_{-1}, u_0, u_1, \dots]^\top$. This allows us to express the operators δ_{x-}, δ_{x+}, δ_{xx} and δ_{xxxx} as the matrices $h^{-1}\mathbf{D}_{x-}^{(1)}$, $h^{-1}\mathbf{D}_{x+}^{(1)}$, $h^{-2}\mathbf{D}_{xx}^{(1)}$ and $h^{-4}\mathbf{D}_{xxxx}^{(1)}$ as follows:

$$
\mathbf{D}^{(1)}_{x-} = \begin{bmatrix} \ddots & & & & & 0 \\ \ddots & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ 0 & & & & \ddots & \ddots \end{bmatrix} \quad \mathbf{D}^{(1)}_{x+} = \begin{bmatrix} \ddots & \ddots & & & & 0 \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & \ddots \\ 0 & & & & & \ddots \end{bmatrix}
$$

$$
\mathbf{D}^{(1)}_{xx} = \begin{bmatrix} \ddots & \ddots & & & & 0 \\ \ddots & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix} \quad \mathbf{D}^{(1)}_{xxxx} = \begin{bmatrix} \ddots & \ddots & \ddots & & & & 0 \\ \ddots & 6 & -4 & 1 & & & \\ \ddots & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & 1 & -4 & 6 & -4 & \ddots \\ & & & 1 & -4 & 6 & \ddots \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{14}
$$

2 Finite difference scheme for linear stiff strings

An explicit finite difference scheme for a stiff string with frequency dependent and independent loss is given by

$$
\delta_{tt} u = \gamma^2 \delta_{xx} u - \kappa^2 \delta_{xxxx} u - 2\sigma_0 \delta_{t\cdot} u + 2\sigma_1 \delta_{t-} \delta_{xx} u \tag{15}
$$

2.1 Energy and boundary conditions

Taking a primed l_2 inner product of (15) with $\delta_{t\cdot} u$ over the domain $\mathcal{D} = \mathbb{U}_N = \{l \in \mathbb{R} \mid 0 \leq l \leq N\}$ and rearranging terms results in

$$
\langle \delta_{t\cdot} u, \delta_{tt} u \rangle'_{\mathbb{U}_N} - \gamma^2 \langle \delta_{t\cdot} u, \delta_{xx} u \rangle'_{\mathbb{U}_N} + \kappa^2 \langle \delta_{t\cdot} u, \delta_{xxxx} u \rangle'_{\mathbb{U}_N} = -2\sigma_0 \left(\| \delta_{t\cdot} u \|'_{\mathbb{U}_N} \right)^2 + 2\sigma_1 \langle \delta_{t\cdot} u, \delta_{t-} \delta_{xx} u \rangle'_{\mathbb{U}_N}
$$

After employing summation by parts (11f), (11i) one has

$$
\langle \delta_{t\cdot} u, \delta_{tt} u \rangle'_{\mathbb{U}_N} + \gamma^2 \langle \delta_{t\cdot} \delta_{x+} u, \delta_{x+} u \rangle_{\underline{\mathbb{U}_N}} + \kappa^2 \langle \delta_{t\cdot} \delta_{xx} u, \delta_{xx} u \rangle_{\overline{\underline{\mathbb{U}_N}}} = \mathfrak{b} - \mathfrak{q}
$$

where the boundary terms denoted by $\mathfrak{b}$ are given by

$$
\begin{aligned}
\mathfrak{b} \triangleq \gamma^2 \left(\delta_{t\cdot} u_N \right) \left(\delta_{x\cdot} u_N \right) - \gamma^2 \left(\delta_{t\cdot} u_0 \right) \left(\delta_{x\cdot} u_0 \right) - \kappa^2 \left(\delta_{t\cdot} u_N \right) \left(\delta_{x\cdot} \delta_{xx} u_N \right) \\
+ \kappa^2 \left(\delta_{t\cdot} u_0 \right) \left(\delta_{x\cdot} \delta_{xx} u_0 \right) + \kappa^2 \left(\delta_{xx} u_N \right) \left(\delta_{t\cdot} \delta_{x\cdot} u_N \right) - \kappa^2 \left(\delta_{xx} u_0 \right) \left(\delta_{t\cdot} \delta_{x\cdot} u_0 \right) \\
+ 2\sigma_1 \left(\delta_{t\cdot} u_N \right) \left(\delta_{t-} \delta_{x\cdot} u_N \right) - 2\sigma_1 \left(\delta_{t\cdot} u_0 \right) \left(\delta_{t-} \delta_{x\cdot} u_0 \right)
\end{aligned}
$$

and the loss terms denoted by $\mathfrak{q}$ are given by

$$\mathfrak{q} = 2\sigma_0 \left(\|\delta_{t\cdot} u\|'_{\mathbb{U}_N}\right)^2 + 2\sigma_1 \langle \delta_{t\cdot}\delta_{x+}u, \delta_{t-}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}}$$

The term involving frequency dependent loss may, using identity (1g), be written as

$$2\sigma_1 \langle \delta_{t\cdot}\delta_{x+}u, \delta_{t-}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}} = 2\sigma_1 \|\delta_{t\cdot}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} - \sigma_1 k \langle \delta_{t\cdot}\delta_{x+}u, \delta_{tt}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}}$$

Using (2a) and (2b) the left-hand side of the inner product together with the frequency dependent term involving the forward difference may be written as the total difference

$$\delta_{t+}\left(\frac{1}{2}\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2 + \frac{\gamma^2}{2}\langle \delta_{x+}u, e_{t-}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}} + \frac{\kappa^2}{2}\langle \delta_{xx}u, e_{t-}\delta_{xx}u\rangle_{\overline{\underline{\mathbb{U}_N}}} - \frac{\sigma_1 k}{2}\|\delta_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}}\right) = \delta_{t+}\mathfrak{h} \tag{16}$$

where

$$\mathfrak{t} = \frac{1}{2}\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2 \tag{17a}$$

$$\mathfrak{v} = \frac{\gamma^2}{2}\langle \delta_{x+}u, e_{t-}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}} + \frac{\kappa^2}{2}\langle \delta_{xx}u, e_{t-}\delta_{xx}u\rangle_{\overline{\underline{\mathbb{U}_N}}} - \frac{\sigma_1 k}{2}\|\delta_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} \tag{17b}$$

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \tag{17c}$$

are the kinetic, potential and numerical energy respectively. *Note that for the explicit scheme* (15) *the potential energy includes a term related to the original frequency dependent loss term.*

To determine the conditions under which the numerical energy $\mathfrak{h}$ is positive definite, we can write, using (2c), (13a) and (13c), for the potential energy

$$\begin{aligned}
\mathfrak{v} &= \frac{\gamma^2}{2}\langle \delta_{x+}u, e_{t-}\delta_{x+}u\rangle_{\underline{\mathbb{U}_N}} + \frac{\kappa^2}{2}\langle \delta_{xx}u, e_{t-}\delta_{xx}u\rangle_{\overline{\underline{\mathbb{U}_N}}} - \frac{\sigma_1 k}{2}\|\delta_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} \\
&= \frac{\gamma^2}{2}\left(\|\mu_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} - \frac{k^2}{4}\|\delta_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}}\right) + \frac{\kappa^2}{2}\left(\|\mu_{t-}\delta_{xx}u\|^2_{\overline{\underline{\mathbb{U}_N}}} - \frac{k^2}{4}\|\delta_{t-}\delta_{xx}u\|^2_{\overline{\underline{\mathbb{U}_N}}}\right) - \frac{\sigma_1 k}{2}\|\delta_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} \\
&\geq \frac{\gamma^2}{2}\left(\|\mu_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} - \frac{k^2}{h^2}\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2\right) + \frac{\kappa^2}{2}\left(\|\mu_{t-}\delta_{xx}u\|^2_{\overline{\underline{\mathbb{U}_N}}} - \frac{4k^2}{h^4}\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2\right) - \frac{2\sigma_1 k}{h^2}\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2
\end{aligned}$$

Thus, we have for the total energy

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \geq \frac{1}{2}\left(1 - \lambda^2 - 4\mu^2 - \frac{4\sigma_1 k}{h^2}\right)\left(\|\delta_{t-}u\|'_{\mathbb{U}_N}\right)^2 + \frac{\gamma^2}{2}\|\mu_{t-}\delta_{x+}u\|^2_{\underline{\mathbb{U}_N}} + \frac{\kappa^2}{2}\|\mu_{t-}\delta_{xx}u\|^2_{\overline{\underline{\mathbb{U}_N}}}$$

where

$$\lambda = \frac{\gamma k}{h} \qquad\qquad \mu = \frac{\kappa k}{h^2} \tag{18}$$

This discrete energy is thus non-negative under the condition

$$1 - \lambda^2 - 4\mu^2 - \frac{4\sigma_1 k}{h^2} \geq 0$$

or

$$h \geq \sqrt{\frac{1}{2}\left(\gamma^2 k^2 + 4\sigma_1 k + \sqrt{\left(\gamma^2 k^2 + 4\sigma_1 k\right)^2 + 16\kappa^2 k^2}\right)} \tag{19}$$

The boundary terms denoted by $\mathfrak{b}$ vanish under the following choices of numerical boundary condition at the left and right endpoints of a domain:

$$\begin{aligned} u_0 = \delta_{x+} u_0 &= 0 \\ u_N = \delta_{x-} u_N &= 0 \end{aligned} \quad \text{clamped} \tag{20a}$$

$$\begin{aligned} u_0 = \delta_{xx} u_0 &= 0 \\ u_N = \delta_{xx} u_N &= 0 \end{aligned} \quad \text{simply supported} \tag{20b}$$

$$\begin{aligned} \delta_{xx} u_0 = \kappa^2 \delta_{x-}\delta_{xx} u_0 - \gamma^2 \delta_{x-} u_0 - 2\sigma_1 \delta_{t-}\delta_{x-} u_0 &= 0 \\ \delta_{xx} u_N = \kappa^2 \delta_{x+}\delta_{xx} u_N - \gamma^2 \delta_{x+} u_0 - 2\sigma_1 \delta_{t-}\delta_{x+} u_N &= 0 \end{aligned} \quad \text{free} \tag{20c}$$

2.2 Matrix form

If we define the column state vector $\mathbf{u}^n$ as

$$\mathbf{u}^n = \left[u_0^n, \ldots, u_N^n\right]^\top \tag{21}$$

where $N = \lfloor 1/h \rfloor$, scheme (15) may be rewritten in vector-matrix form as

$$\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1} \tag{22}$$

where the matrices $\mathbf{B}$ and $\mathbf{C}$ are defined as

$$\mathbf{B} = \frac{1}{1+\sigma_0 k}\left[2\mathbf{I} + \left(\frac{\gamma^2 k^2}{h^2} + \frac{2\sigma_1 k}{h^2}\right)\mathbf{D}_{xx}^{(1)} - \frac{\kappa^2 k^2}{h^4}\mathbf{D}_{xxxx}^{(1)}\right] \tag{23a}$$

$$\mathbf{C} = -\frac{1}{1+\sigma_0 k}\left[\left(1-\sigma_0 k\right)\mathbf{I} + \frac{2\sigma_1 k}{h^2}\mathbf{D}_{xx}^{(1)}\right] \tag{23b}$$

and the matrices $\mathbf{D}_{xx}^{(1)}$ and $\mathbf{D}_{xxxx}^{(1)}$ are the finite versions of the ones defined by (14) with the relevant boundary conditions applied. Under clamped (20a) or simply supported (20b) conditions at both ends they take the form

$$\mathbf{D}_{xx}^{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & & & & & 0 \\ 0 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 0 \\ 0 & & & & & 0 & 0 \end{bmatrix}}_{\text{clamped or simply supported}}$$

$$\mathbf{D}_{xxxx}^{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & & & & & & & 0 \\ 0 & 7 & -4 & 1 & & & & & \\ 0 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 0 \\ & & & & & 1 & -4 & 7 & 0 \\ 0 & & & & & & & 0 & 0 \end{bmatrix}}_{\text{clamped}}$$

$$\mathbf{D}_{xxxx}^{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & & & & & & & 0 \\ 0 & 5 & -4 & 1 & & & & & \\ 0 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 0 \\ & & & & & 1 & -4 & 5 & 0 \\ 0 & & & & & & & 0 & 0 \end{bmatrix}}_{\text{simply supported}}$$

Free conditions are more complicated. Assuming free conditions at the left endpoint, then using the first condition we find

$$u_{-1}^n = 2u_0^n - u_1^n$$

Substituting this in the second condition we find

$$u_{-2}^n = 2\beta\left(u_0^n - u_1^n\right) + u_2^n - 2\alpha\left(u_0^{n-1} - u_1^{n-1}\right)$$

where

$$\alpha = \frac{2\sigma_1 h^2}{\kappa^2 k} \qquad\qquad \beta = 2 + \frac{\gamma^2 h^2}{\kappa^2} + \alpha$$

Now we can express the operator δ_{xxxx} acting on the grid function u_l^n at grid points $l = 0$ and $l = 1$ fully in terms of grid points lying in the domain interior:

$$\begin{aligned} h^4 \delta_{xxxx} u_0^n &= 2\left(\left(\beta - 1\right) u_0^n - \beta u_1^n + u_2^n - \alpha\left(u_0^{n-1} - u_1^{n-1}\right)\right) \\ h^4 \delta_{xxxx} u_1^n &= -2u_0^n + 5u_1^n - 4u_2^n + u_3^n \end{aligned}$$

As such, the matrix $\mathbf{D}_{xx}^{(1)}$ under free boundary conditions at both ends takes the form

$$\mathbf{D}_{xx}^{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 0 & 0 \end{bmatrix}}_{\text{free}}$$

Expressing the matrix $\mathbf{D}_{xxxx}^{(1)}$ under free boundary conditions is less straightforward though, due to the dependence of the grid function u_l at the point $l = 0$ (and / or $l = N$) on values computed at the previous time instant $n - 1$. One way to express this is by splitting $\mathbf{D}_{xxxx}$ into two matrices, one of which is acting on $\mathbf{u}^n$ and the other one on $\mathbf{u}^{n-1}$. Then we have, assuming free boundary conditions at both ends:

$$\begin{aligned} \mathbf{D}_{xxxx}^{(1)} &= \tilde{\mathbf{D}}_{xxxx}^{(1,n)} + \tilde{\mathbf{D}}_{xxxx}^{(1,n-1)} \\ &= \underbrace{\begin{bmatrix} 2\left(\beta-1\right) & -2\beta & 2 & & & & 0 \\ -2 & 5 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ 0 & & & & 2 & -2\beta & 2\left(\beta-1\right) \end{bmatrix}}_{\text{acting on } \mathbf{u}^n} + \underbrace{\begin{bmatrix} -2\alpha & 2\alpha & & & 0 \\ 0 & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 0 \\ 0 & & & 2\alpha & -2\alpha \end{bmatrix}}_{\text{acting on } \mathbf{u}^{n-1}} \end{aligned}$$

Hence, in the case of free boundary conditions at both ends equation (23a) and (23b) become

$$\begin{aligned} \mathbf{B} &= \frac{1}{1 + \sigma_0 k}\left[2\mathbf{I} + \left(\frac{\gamma^2 k^2}{h^2} + \frac{2\sigma_1 k}{h^2}\right)\mathbf{D}_{xx}^{(1)} - \frac{\kappa^2 k^2}{h^4}\tilde{\mathbf{D}}_{xxxx}^{(1,n)}\right] \\ \mathbf{C} &= -\frac{1}{1 + \sigma_0 k}\left[\left(1 - \sigma_0 k\right)\mathbf{I} + \frac{2\sigma_1 k}{h^2}\mathbf{D}_{xx}^{(1)} + \frac{\kappa^2 k^2}{h^4}\tilde{\mathbf{D}}_{xxxx}^{(1,n-1)}\right] \end{aligned}$$

3 Finite difference scheme for linear stiff membranes

An explicit finite difference scheme for a stiff membrane with frequency dependent and independent loss [Bil09] is given by

$$\delta_{tt}u = \gamma^2\delta_{\Delta\boxplus}u - \kappa^2\delta_{\Delta\boxplus,\Delta\boxplus}u - 2\sigma_0\delta_{t\cdot}u + 2\sigma_1\delta_{t-}\delta_{\Delta\boxplus}u \tag{24}$$

References

[Bil09] Stefan Bilbao. *Numerical Sound Synthesis: Finite Difference Schemes and Simulation in Musical Acoustics*. John Wiley & Sons Ltd, 2009. ISBN: 9780470510469.