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# Assignment 2

We use the notations and conventions from the course, e,g. a map refers to a continuous map, a space refers to a topological space, $\mathrm{S}^1$ denotes the unit circle, and $\mathrm{D}^2$ the unit disc.

**Exercise 1.** Let $B = \mathrm{S}^1$, $C = \mathrm{D}^2$ and denote by $\partial C = \mathrm{S}^1$ the circle contained in $C$. For a fixed integer $k \geq 1$ we define the map $\phi_k : \partial C \to B$ via $\phi_k(e^{2\pi \mathrm{i} s}) = e^{2\pi \mathrm{i} ks}$. We then define the following space

$$X_k = (B \coprod C)/_{\sim},$$

where $z \sim \varphi_k(z)$ for $z \in \partial C$.

(a) Show that the space $X_k$ is path-connected.

(b) Use the van Kampen theorem to determine $\pi_1(X_k)$.

*Hint:* The calculation is easier with the choice of a basepoint in $C \setminus \partial C$.

**Exercise 2.** Let $X$ be a space with $X = U \cup V$ for $U$, $V$, and $U \cap V$ all open, non-empty and path-connected.

(a) Show that $X$ is path-connected.

(b) Assume that $V \cap U$ is simply-connected and show that $\pi(X) \cong \pi_1(U) * \pi_1(V)$.

**Exercise 3.**

(a) Let $[v_0, v_1, v_2, v_3]$ be a 3-simplex. Define $X = [v_0, v_1, v_2, v_3]/_{\sim}$ with the equivalence relation given by $v_0 \sim v_1 \sim v_2 \sim v_3$. Compute the simplicial homology of $X$.

(b) Let $[v_0, v_1, \ldots, v_n]$ be an $n$-simplex. For $1 \leq k \leq n$ and $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$ denote by $\varphi_{\underline{i}} : \Delta^k \to [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ the canonical homeomorphism from the standard $k$-simplex to $[v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ (as defined in the course).

Now define $X = [v_0, v_1, \ldots, v_n]/_{\sim}$ where for any $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$, $\underline{j} = (0 \leq j_1 < j_2 < \ldots < j_k \leq n)$, and $x \in [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ we set $x \sim \varphi_{\underline{j}} \circ \varphi_{\underline{i}}^{-1}(x)$, i.e. we identify all $k$-simplices contained as iterative faces in $[v_0, v_1, \ldots, v_n]$ via the canonical homeomorphisms. Compute the simplicial homology of $X$.

**Exercise 4.** Let $r : X \to A$ be a retraction of a space $X$ to a subspace $A$ and $i : A \to X$ the inclusion. Show that $i_* : H_n(A) \to H_n(X)$ is injective and $r_* : H_n(X) \to H_n(A)$ is surjective for all $n \geq 0$.