# Fundamental group of the circle

This is an extended version of the calculation for the fundamental group of the circle. The proof itself is the same as in the course itself. It includes some additional details and explanations that were only given verbally in the lecture.

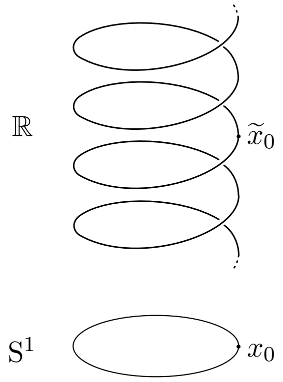
The theorem we want to prove is the following:

**Theorem 1.21.** *Define* $x_0 = (1, 0) \in \mathrm{S}^1$ *and* $\omega_n : \mathrm{I} \to \mathrm{S}^1$ *via* $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ *for* $n \in \mathbb{Z}$*. The map* $\Phi : \mathbb{Z} \to \pi_1(\mathrm{S}^1, x_0)$ *given by* $\Phi(n) = [\omega_n]$ *is an isomorphism.*

Before we start with the proof we need some preparations to set up the notations.

**Preparations:** Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection given by $p(x, y, z) = (x, y)$. Inside $\mathbb{R}^2$ we have $\mathrm{X} = \mathrm{S}^1$. We want to view $\widetilde{\mathrm{X}} = \mathbb{R}$ as being embedded in $\mathbb{R}^3$ in such a way that it gets mapped to X via $p$. For this we use the embedding $s \mapsto (\cos(2\pi s), \sin(2\pi \mathrm{s}), \mathrm{s})$. Using this embedding followed by the projection $p$ gives us a map from $\widetilde{\mathrm{X}}$ to X. By abuse of notation we call this map $p$ as well, hence it is given by $p(s) = (\cos(2\pi s), \sin(2\pi s))$ for $s \in \widetilde{\mathrm{X}}$.

In pictures this is shown to the right as a "spiral" around the $z$-axis. Together with the choice of our basepoint $x_0$ as well as the image of 0 under the embedding of $\mathbb{R}$ into $\mathbb{R}^3$, called $\widetilde{x}_0$ here.




*Note:* Even though the map $p$ is given in the usual real coordinate, it is better to visually think of $\widetilde{\mathrm{X}}$ as being the spiral. It makes the constructions we will use appear much more logical and intuitive. In the following we usually depict everything on the spiral, even though we are really working in the usual coordinates of $\mathbb{R}$.

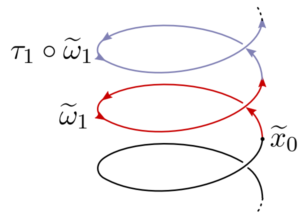
Next we want to "lift" the loops $\omega_n$ to $\widetilde{\mathrm{X}}$. Specifically we want the following. Define $\widetilde{\omega}_n : \mathrm{I} \to \widetilde{\mathrm{X}}$ via $\widetilde{\omega}_n(s) = ns$, $n \in \mathbb{Z}$, hence the most obvious path from 0 to $n$. Comparing the definition we see immediately that $p \circ \widetilde{\omega}_n = \omega_n$. This is why we call $\widetilde{\omega}_n$ a lift of $\omega_n$.

**Notation:** For any map $F : \mathrm{Y} \to \mathrm{X}$ from a space Y to X, we call a map $\widetilde{F} : \mathrm{Y} \to \widetilde{\mathrm{X}}$ such that $p \circ \widetilde{F} = F$ a lift of $F$. Such a lift is not necessarily unique and it is an important part of the proof to see for which kinds of maps such a lift exists and is unique.

Since $\widetilde{\mathrm{X}}$ is simply-connected all paths from 0 to $n$ are homotopic, hence $[\omega_n] = [p \circ \widetilde{\omega}_n] = [p \circ f]$ where $f$ is any path in $\widetilde{\mathrm{X}}$ from 0 to $n$. This is a useful feature of $\widetilde{\mathrm{X}}$ that we will make use of in the proof.

*Proof.* (of Theorem 1.21) We first check that $\Phi$ is a group homomorphism. Define $\tau_n : \widetilde{\mathrm{X}} \to \widetilde{\mathrm{X}}$ via $\tau_n(x) = n + x$ for $n \in \mathbb{Z}$, the translation by the integer $n$. Due to our definition of $p$ it then holds that $p \circ \tau_n = p$.

For $n, m \in \mathbb{Z}$, it holds $\widetilde{\omega}_{n+m} \simeq \widetilde{\omega}_n \cdot (\tau_n \circ \widetilde{\omega}_m)$, since both are paths from 0 to $n+m$. We use again that $\widetilde{\mathrm{X}}$ is simply-connected here. The picture to the right depicts $\widetilde{\omega}_1 \cdot (\tau_1 \circ \widetilde{\omega}_1)$, each of them going up by one "level" in the spiral.




Using this we obtain

$$\begin{aligned}\Phi(n+m) &= [\omega_{n+m}] = [p \circ \widetilde{\omega}_{n+m}] \\ &= [p \circ (\widetilde{\omega}_n \cdot (\tau_n \circ \widetilde{\omega}_m))] = [p \circ \widetilde{\omega}_n] \cdot [p \circ \tau_n \circ \widetilde{\omega}_m] \\ &= [p \circ \widetilde{\omega}_n] \cdot [p \circ \widetilde{\omega}_m] = [\omega_n] \cdot [\omega_m] = \Phi(n) \cdot \Phi(m).\end{aligned}$$

Thus we obtain that $\Phi$ is a group homomorphism.

We will make two assumptions on our situation now that we will later have to prove. For any choice of $x_0 \in \mathrm{X}$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. We assume that the following holds

(a) For every path $f : \mathrm{I} \to \mathrm{X}$ starting at $x_0$ there is a unique lift $\widetilde{f} : \mathrm{I} \to \widetilde{\mathrm{X}}$ starting at $\widetilde{x}_0$.

(b) For any homotopy $F$ of paths in X starting at $x_0$ there is a unique homotopy $\widetilde{F}$ of paths in $\widetilde{\mathrm{X}}$ starting at $\widetilde{x}_0$ that is a lift of $F$.

We will first show why our statement follows from these two assumptions. Of course we choose $x_0 = (1,0)$ in X and $\widetilde{x}_0 = 0$ in $\widetilde{\mathrm{X}}$.

*Surjectivity of* $\Phi$*:* Let $f$ be a loop in X at $(1,0)$. By (a), we know that there is a unique lift $\widetilde{f}$, i.e. a path in $\widetilde{\mathrm{X}}$ starting at 0. Then $\widetilde{f}$ ends at some $n$, since $p^{-1}((1,0)) = \mathbb{Z}$, hence $[f] = [p \circ \widetilde{f}] = [p \circ \widetilde{\omega}_n] = [\omega_n] = \Phi(n)$. Hence $\Phi$ is surjective.

*Injectivity of* $\Phi$*:* Suppose that $\Phi(n) = \Phi(m)$, i.e. $\omega_n \simeq \omega_m$. Let $F$ be a homotopy from $\omega_m$ to $\omega_n$. Then, by (b), this homotopy lifts to a homotopy of paths $\widetilde{F}$. It is a homotopy between $\widetilde{\omega}_m$ and $\widetilde{\omega}_n$, since by (a) these are the unique lifts of $\omega_m$ and $\omega_n$ that start at 0. Since two homotopic paths have the same endpoint it thus means that $m = n$. Hence $\Phi$ is injective.

Thus we see that assumptions (a) and (b) proof the statement. Instead of proving that (a) and (b) hold, we will show that a more general property holds that itself implies (a) and (b).

(c) Let $F : \mathrm{Y} \times \mathrm{I} \to \mathrm{X}$ be a map for a space Y and $\widetilde{F} : \mathrm{Y} \times \{0\} \to \widetilde{\mathrm{X}}$ such that $\widetilde{F}$ is a lift of $F$ restricted to $\mathrm{Y} \times \{0\}$. Then $\widetilde{F}$ can be uniquely extended to a map $\widetilde{F} : \mathrm{Y} \times \mathrm{I} \to \widetilde{\mathrm{X}}$ that is a lift of $F$ on all of $\mathrm{Y} \times \mathrm{I}$.

We will show that (c) implies (a) and (b).

*For (a):* Choose $Y = \{\mathrm{pt}\}$ to be a point. The map $F$ is then just a path in X starting at some point $x_0$. Since $\mathrm{Y} \times \{0\}$ is just a point, the map The map $\widetilde{F}$ is precisely the choice of a single point $\widetilde{x}_0$ in the fibre over $x_0$. Thus the statement of (c) is the existence of unique lift of the whole path.

*For (b):* Choose $Y = \mathrm{I}$. The map $F$ is then a homotopy, so we choose a homotopy of paths like in (b). Then $f_0(s) = F(s,0)$ is a path in X, which can be lifted to a path $\widetilde{f}_0$ in $\widetilde{\mathrm{X}}$ starting at $\widetilde{x}_0$ by (a). This $\widetilde{f}_0$ is the map $\widetilde{F}$ restricted to $\mathrm{I} \times \{0\}$ in (c). Then (c) gives the unique existence of the lift $\widetilde{F} : \mathrm{I} \times \mathrm{I} \to \widetilde{\mathrm{X}}$ of $F$ on all of $\mathrm{I} \times \mathrm{I}$. To check that this is indeed a homotopy of paths, we look at the path $\widetilde{F}(0, -)$ (i.e. we fix the first coordinate in contrast to the usual situation). Then $\widetilde{F}(0, -)$ is a lift of the path $F(0, -) = \mathbb{1}_{x_0}$. But by (a) we know that this lift is unique and a possible choice for this lift is $\mathbb{1}_{\widetilde{x}_0}$. Hence it follows that $\widetilde{F}(0, -) = \mathbb{1}_{\widetilde{x}_0}$. The same holds for $\widetilde{F}(1, -)$ which is thus also constant, hence $\widetilde{F}$ is a homotopy of paths. Hence $\widetilde{F}$ is a homotopy of paths.

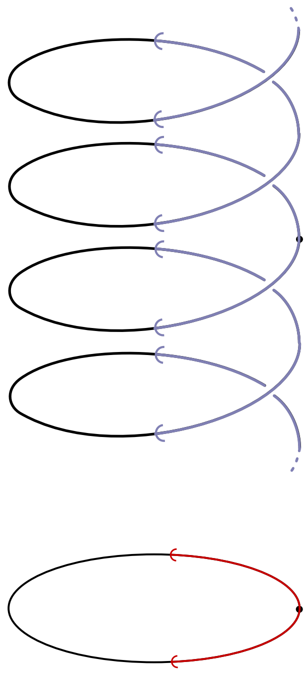
Thus we see that (c) implies (a) and (b) and thus the statement that $\Phi$ is a group isomorphism. That (c) holds will be proven in Lemma 1.22 below. □

Before we start with the proof of assumption (c) in Theorem 1.21 we record an important property of our map $p$.

**Covering space property of $p$:** There is an open cover $\{\mathrm{U}_\alpha\}_\alpha$ of X such that $p^{-1}(\mathrm{U}_\alpha) = \coprod_\beta \mathrm{U}_{\alpha,\beta}$ is a disjoint union of open subsets of $\widetilde{\mathrm{X}}$ such that $p : \mathrm{U}_{\alpha,\beta} \to \mathrm{U}_\alpha$ is a homeomorphism for all $\alpha$ and $\beta$. (For ease of notation we do not specify the indexing sets for $\alpha$ and $\beta$ as they are not important for the proof. They can be finite or infinite.)

To obtain that $p$ has the Covering space property, we look at the preimage of an open interval on the circle as depicted to the right. The interval is depicted in red, while the preimage is depicted in blue above it. As one can easily check in the formulas, the preimage its just a disjoint copy of lots of open intervals in $\widetilde{\mathrm{X}}$, each one mapping homeomorphically onto the red interval. The depicted blue part has five disjoint components, but this of course continues infinitely to the top and bottom.

Thus one can take any cover of open intervals $\mathrm{U}_\alpha$ on the circle, as the one depicted. The only condition is that one is not allowed to take the whole circle as an open set, but every honest open interval has the property that the fibre nicely decomposes into disjoint open pieces.



**Lemma 1.22.** *Let $p : \widetilde{\mathrm{X}} \to \mathrm{X}$ as in our preparation above, $\mathrm{Y}$ a topological space, $F : \mathrm{Y} \times \mathrm{I} \to \mathrm{X}$ a map and $\widetilde{F} : \mathrm{Y} \times \{0\} \to \widetilde{\mathrm{X}}$ a lift of $F$ restricted to $\mathrm{Y} \times \{0\}$. Then $\widetilde{F}$ can be uniquely extended to a lift $\widetilde{F} : \mathrm{Y} \times \mathrm{I} \to \widetilde{\mathrm{X}}$ of $F$.*

*Proof.* We will construct the lift $\widetilde{F}$ in two steps.

*Step (1):* First we will show that for any point $y \in \mathrm{Y}$ there exists a open neighbourhood $N$ of $y$ such that $\widetilde{F}$ can be defined on $N \times I$. Since we already know that it exists on $N \times \{0\}$ by assumption we will need to extend it to the whole interval in the second factor.

*Step (2):* In the second step we need to see that if we have an extension $\widetilde{F}$ defined on some $N \times I$ and $\widetilde{F}'$ defined on some $N' \times I$ using Step (1), then the two maps agree on $(N \cap N') \times I$.

Hence they can be glued together to form a continuous map.

With these two steps we have then defined an extension on all of $\mathrm{Y} \times I$.

*Proof of Step (1):* Fix $y \in \mathrm{Y}$. For $(y,t) \in \mathrm{Y} \times \mathrm{I}$ there is a product neighbourhood $N_t \times \mathrm{I}_t$ such that $F(N_t \times \mathrm{I}_t) \subset \mathrm{U}_\alpha$ for some $\alpha$ and $\mathrm{I}_t$ is an open interval contained in I. That an open neighbourhood of this form exists is due to $F$ being continuous and since there is an open set $\mathrm{U}_\alpha$ such that $F(y,t) \in \mathrm{U}_\alpha$. That the neighbourhood can be chosen of this form is due to the definition of the product topology. Since $\{y\} \times \mathrm{I}$ is compact, there are points $t'_1, \ldots, t'_n$ such that $\{N_{t'_i} \times \mathrm{I}_{t'_i} | 1 \leq i \leq n\}$ already covers $\{y\} \times \mathrm{I}$. Set $N = \cap N_{t'_i}$, which is again an open subset as a finite intersection. Now choose $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $\{N \times [t_{i-1}, t_i] \mid 1 \leq i \leq m\}$ covers $\{y_0\} \times \mathrm{I}$ and for each $i$ it holds $F(N \times [t_{i-1}, t_i]) \subset \mathrm{U}_i = \mathrm{U}_{\alpha(i)}$ for some $\alpha(i)$.

Assume now that the lift $\widetilde{F}$ is already constructed on $N \times [0, t_i]$ (for $i = 0$ this is part of the assumption). We want to extend this to $N \times [0, t_{i+1}]$. We know that $F(y, t_i) \in F(N \times [t_i, t_{i+1}]) \subset \mathrm{U}_{i+1}$, hence there exists an open subset $\widetilde{\mathrm{U}}_{i+1} = \mathrm{U}_{\alpha(i+1),\beta}$ of $\widetilde{\mathrm{X}}$ for some $\beta$ such that $\widetilde{F}(y, t_i) \in \widetilde{\mathrm{U}}_{i+1}$ and $\widetilde{\mathrm{U}}_{i+1}$ maps homeomorphically onto $\mathrm{U}_{i+1}$ via $p$ by assumption. By replacing $N$ with a smaller open neighbourhood of $y$, we can already assume that $\widetilde{F}(N \times \{t_i\}) \subset \widetilde{\mathrm{U}}_{i+1}$ since $\widetilde{F}$ is continuous. Then we define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ as $p^{-1} \circ F$, where $p^{-1} : \mathrm{U}_{i+1} \to \widetilde{\mathrm{U}}_{i+1}$ is the inverse of $p$ restricted to $\widetilde{\mathrm{U}}_{i+1}$. On $N \times \{t_i\}$ this agrees with our given $\widetilde{F}$ so it defines a new extension $\widetilde{F}$ that is now defined on $N \times [0, t_{i+1}]$. After $m$ steps we have a neighbourhood $N$ of $y$ and an extension $\widetilde{F} : N \times \mathrm{I} \to \widetilde{\mathrm{X}}$ of $F$ restricted to $N \times \mathrm{I}$. Note that with each step it can happen that we have to shrink the neighbourhood $N$, but since this is only done finitely many times, it is not a problem.

*Proof of Step (2)* Assume now that there are two extensions $\widetilde{F}$ and $\widetilde{F}'$ that are both defined on $\{y\} \times \mathrm{I}$ for some point $y$ (We are checking the equality for every point in the intersection). By assumption $\widetilde{F}(y, 0) = \widetilde{F}'(y, 0)$. Now let $0 = t_0 < t_1 < \ldots < t_m = 1$ be a partition such that $F(\{y\} \times [t_{i-1}, t_i]) \subset \mathrm{U}_i$, this is done in exactly the same way as in the proof of Step (1). Assume now inductively that $\widetilde{F}$ and $\widetilde{F}'$ agree on $\{y\} \times [0, t_i]$ (for $i = 0$ this is the assumption already). Then there exists an open subset $\widetilde{\mathrm{U}}_{i+1}$ such that $\widetilde{F}(y, t_i) = \widetilde{F}'(y, t_i) \in \widetilde{\mathrm{U}}_{i+1}$. Since $\{y\} \times [t_i, t_{i+1}]$ is connected, so is $\widetilde{F}(\{y\} \times [t_i, t_{i+1}])$, hence it is also fully contained in $\widetilde{\mathrm{U}}_{i+1}$ and similarly $\widetilde{F}'(\{y\} \times [t_i, t_{i+1}]) \subset \widetilde{\mathrm{U}}_{i+1}$. But $p \circ \widetilde{F} = p \circ \widetilde{F}'$ and $p$ is a homeomorphism when restricted to $\widetilde{U}_{i+1}$, hence the extensions agree on $\{y\} \times [0, t_{i+1}]$. Again after finitely many steps we obtain that they agree on $\{y\} \times \mathrm{I}$.

*Definition of the extension:* For the definition of the complete extension $\widetilde{F}$ we can just say how it is defined on any point $(y,t)$. By Step (1) we can choose a neighbourhood $N$ of $y$ such that there exists an extension of $N \times \mathrm{I}$ and we use that one. By Step (2) we do not get into trouble by doing this because whenever these extensions are defined on the same point, they agree.

Note that the basic idea of the proof is quite simple. One has the map $\widetilde{F}$ defined on a part of the space $\mathrm{Y} \times \mathrm{I}$ and takes a part of the space that is adjacent, i.e. in the proof this was going from $N \times [0, t_i]$ and glueing on $N \times [t_i, t_{i+1}]$. This extra piece is chosen such that it gets mapped into an open subset $\mathrm{U}_\alpha$ by $F$ and extends $F$ by choosing an according open subset $\mathrm{U}_{\alpha,\beta}$ in $\widetilde{\mathrm{X}}$ by the fact that one already knows $\widetilde{F}$ on a part of the new subset, i.e. in our case the subset $N \times \{t_i\}$. Then one uses that for every $\alpha$ and $\beta$, $\mathrm{U}_\alpha$ and $\mathrm{U}_{\alpha,\beta}$ are homeomorphic, so one can define $\widetilde{F}$ by just taking $F$ and composing with the according homeomorphism. $\square$

Note that in the whole proof above, we never used any properties of the spaces $\mathrm{X}$ and $\widetilde{\mathrm{X}}$ themselves, we only used the covering space property of the map $p$. Hence we can use word for word the same proof for the following generalization.

**Proposition 1.23.** *Let* $\mathrm{X}$ *and* $\widetilde{\mathrm{X}}$ *be topological spaces and* $p : \widetilde{\mathrm{X}} \to \mathrm{X}$ *a map. Assume that there exists an open covering* $\{U_\alpha\}$ *of* $\mathrm{X}$ *with* $p^{-1}(U_\alpha) = \coprod_\beta \mathrm{U}_{\alpha,\beta}$ *is a disjoint union of open subsets of* $\widetilde{\mathrm{X}}$*, and* $p : \mathrm{U}_{\alpha,\beta} \to \mathrm{U}_\alpha$ *is a homeomorphism for all* $\alpha$ *and* $\beta$*.*

*If* $F : \mathrm{Y} \times \mathrm{I} \to \mathrm{X}$ *is a map from a space* $\mathrm{Y}$ *and* $\widetilde{F} : \mathrm{Y} \times \{0\} \to \widetilde{\mathrm{X}}$ *a lift of* $F$ *restricted to* $\mathrm{Y} \times \{0\}$*, then* $\widetilde{F}$ *can be uniquely extended to a lift* $\widetilde{F} : \mathrm{Y} \times \mathrm{I} \to \widetilde{\mathrm{X}}$ *of* $F$*.*