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Assignment 2

We use the notations and conventions from the course, e,g. a map refers to a continuous map, a space refers to a topological space, S^1 denotes the unit circle, and D^2 the unit disc.

Exercise 1. Let $B = \mathrm{S}^1$, $C = \mathrm{D}^2$ and denote by $\partial C = \mathrm{S}^1$ the circle contained in C. For a fixed integer $k \geq 1$ we define the map $\phi_k : \partial C \to B$ via $\phi_k(e^{2\pi \mathrm{i} s}) = e^{2\pi \mathrm{i} ks}$. We then define the following space

$$X_k = (B \coprod C)/_{\sim},$$

where $z \sim \varphi_k(z)$ for $z \in \partial C$.

(a) Show that the space X_k is path-connected.

(b) Use the van Kampen theorem to determine $\pi_1(X_k)$.

Hint: The calculation is easier with the choice of a basepoint in $C \setminus \partial C$.

Exercise 2. Let $X \subset \mathbb{R}^n$ and assume that $X = \bigcup_{i=1}^{m} X_i$ where each X_i is an open convex subset of $\mathbb{R}^n$. Assume that any triple intersection $X_i \cap X_j \cap X_k \neq \varnothing$ for $1 \leq i, j, k \leq m$. Show that X is simply-connected.

Exercise 3. Let X be a space with $X = U \cup V$ for U, V, and $U \cap V$ all open, non-empty and path-connected.

(a) Show that X is path-connected.

(b) Assume that $V \cap U$ is simply-connected and show that $\pi(X) \cong \pi_1(U) * \pi_1(V)$.

Exercise 4.

(a) Let $p : \widetilde{X} \to X$ be a covering space. Let $A \subset X$ be a subspace and set $\widetilde{A} = p^{-1}(A)$. Show that the restriction of p to $\widetilde{A}$ is a covering space.

(b) Let $p_1 : \widetilde{X}_1 \to X_1$ and $p_2 : \widetilde{X}_2 \to X_2$ be covering spaces. Show that the product $\widetilde{X}_1 \times \widetilde{X}_2$ can be made into a covering space for $X_1 \times X_2$.

(c) Let $X = \mathrm{S}^2 \cup D$, where $D = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$, i.e. the 2-sphere together with a line segment connecting two anti-podal points. Construct a simply-connected covering space $\widetilde{X}_1$ of X and a non simply-connected covering space $\widetilde{X}_2 \neq X$ of X.