SINGULAR TQFTS, FOAMS AND TYPE D ARC ALGEBRAS

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ABSTRACT. We combinatorially describe the 2-category of singular cobordisms, called (rank one) foams, which governs the functorial version of Khovanov homology. As an application we topologically realize the type D arc algebra (which e.g. controls parabolic category $\mathcal{O}$ of type D with respect to a maximal parabolic of type A) using this singular cobordism construction.

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1. INTRODUCTION

In this paper we study the *web algebra* $\mathfrak{W}$ attached to $\mathfrak{gl}_2$, cf. Remark 1.1.

The main results in a few words. The algebra $\mathfrak{W}$ naturally appears in the setup of singular TQFTs in the sense that its bimodule 2-category $\mathfrak{W}$**-biMod** is equivalent to the 2-category $\mathfrak{F}$ of certain singular surfaces à la Blanchet [2], called *foams*, and $\mathfrak{W}$ algebraically controls the functorial version of Khovanov's link homology.

The 2-category $\mathfrak{F}$ is a sign modified version of Bar-Natan's [1] original cobordism ("$\mathfrak{sl}_2$-foam") 2-category $\mathfrak{F}^{\mathfrak{sl}_2}$ attached to Khovanov's link and tangle invariant. The signs are crucial for making Khovanov's link homology functorial, but very delicate to compute in practice. Our first main result can be seen as a combinatorial way to compute these signs, i.e. we define a *combinatorial, planar model* $c\mathfrak{W}$ of $\mathfrak{W}$:

Theorem A. There is an isomorphism of graded algebras

$$\texttt{comb}\colon c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}.$$

(Consequently, we obtain a combinatorial model of the foam 2-category $\mathfrak{F}$.) $\square$

Note the difference to [10] and [9]: The web algebra $\mathfrak{W}$ is the algebra presenting the (entire) 2-category of foams $\mathfrak{F}$ and not just the "highest weight 2-subcategories" of $\mathfrak{F}$ studied in [10] and [9]. (Again, cf. Remark 1.1.)

Next, we know that $\mathfrak{W}$ topologically controls $\mathcal{O}_0^{\mathrm{A}_p\times\mathrm{A}_q}(\mathfrak{gl}_m(\mathbb{C}))$, i.e. the principal block of the parabolic BGG category $\mathcal{O}$ of type A_m with parabolic of type $\mathrm{A}_p\times\mathrm{A}_q$ for $p+q=m$. This is proven by realizing Khovanov's (type A) arc algebra $\mathfrak{A}^{\mathrm{A}}$, which controls $\mathcal{O}_0^{\mathrm{A}_p\times\mathrm{A}_q}(\mathfrak{gl}_m(\mathbb{C}))$ by [5], as a subalgebra of $\mathfrak{W}$. (Even stronger: it is an idempotent truncation giving an embedding of the associated bimodule 2-categories.)

In joint work with Stroppel [8], the first author has defined a type D generalization $\mathfrak{A}=\mathfrak{A}^{\mathrm{D}}$ of Khovanov's arc algebra, which we call the *type* D *arc algebra*. By results in [8], the algebra $\mathfrak{A}$ controls $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$, i.e. the principal block of the parabolic BGG category $\mathcal{O}$ of type D_n with parabolic of type A_{n-1}.

Our second main result is then that, surprisingly, the type D arc algebra is also a subalgebra of $\mathfrak{W}$:

Theorem B. There is an embedding of graded algebras

$$\mathtt{top}\colon \mathfrak{A} \hookrightarrow \mathfrak{W}.$$

(Again, $\mathfrak{A}$ is an idempotent truncation of $\mathfrak{W}$ giving an embedding between the associated bimodule 2-categories.) $\square$

Thus, the representation theory of the web algebra $\mathfrak{W}$ (and hence, the cobordism 2-category $\mathfrak{F}$) controls $\mathcal{O}_0^{\mathrm{A}_p\times\mathrm{A}_q}(\mathfrak{gl}_m(\mathbb{C}))$ as well as $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$.

The picture the reader should keep in mind how these three "worlds", represented by elements from the algebras $\mathfrak{W}$, $c\mathfrak{W}$ and $\mathfrak{A}$, are connected is given in Figure 1.

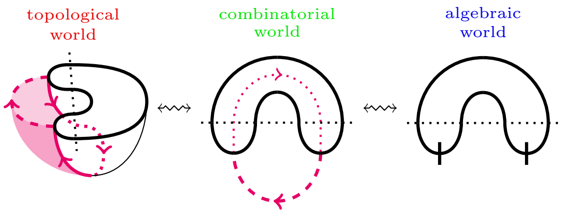

FIGURE 1. From foams to dotted webs to arc diagrams.

The construction of `top`. As an intermediate step on our way to prove Theorem B, we obtain a sign adjusted version $\overline{\mathfrak{A}}$ of $\mathfrak{A}$, and an isomorphism of graded algebras

$$\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}.$$

The definition of $\overline{\mathfrak{A}}$, although we have directly obtained it from the "foamy side", does not require any knowledge of singular TQFTs or foams, and comes with a simpler sign placement than the original type D arc algebra $\mathfrak{A}$. (And, due to its "foamy multiplication", clears some other "defects" as for example non-locality of the multiplication as well, cf. Remark 5.3.)

Then `top` is given by assembling the pieces, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\mathtt{sign}"] \arrow[rrr, bend right, "\mathtt{top}"'] & \overline{\mathfrak{A}} \arrow[r, "\overline{\mathtt{top}}"] & c\mathfrak{W} \arrow[r, "\mathtt{comb}"] & \mathfrak{W}, \end{tikzcd}$$

where $\overline{\mathtt{top}}\colon \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W}$ is the "foamy interpretation" of $\overline{\mathfrak{A}}$, cf. Figure 1.

Now, we give a rough sketch of the main constructions of this paper.

The papers content in a nutshell. In Section 2 we explain webs and foams and their origin in singular topological quantum field theories (*singular TQFTs*). All the reader needs to know at the moment is that webs are certain trivalent graphs and foams are certain singular surfaces whose boundary are webs.

These are pieced together into the web algebra $\mathfrak{W}$ and its representation theory in Section 3. Again, all the reader needs to know at the moment is that both, $\mathfrak{W}$ and $\mathfrak{W}$-**biMod**, are topologically in nature with multiplication and action given by gluing of singular surfaces.

In Section 4 we give a combinatorial model of these. The combinatorial model is given by certain decorated webs called *dotted webs*.

In Section 5 we recall the notions underlying the arc algebras, which is given by putting an algebraic multiplication structure on *arc diagrams*.

Thus, in total, our two main theorems connect three worlds: foams, their combinatorial model and (type A and D) arc algebras. The mnemonic to keep in mind is given in Figure 1, with a singular cobordisms on the left, a combinatorial, 2-dimensional, version of it in the middle and an arc diagram on the right.

Mathematically speaking, what we do in this paper is to make this precise.

The main step to make the left arrow in Figure 1 rigorous is to find a "good" basis of the foam spaces, which we call the *cup foam basis*, on which we can calculate the multiplication explicitly. The main step for making the right arrow accurate is to "sign adjust" the type D arc algebra very much in the spirit of [9].

Although the connection in Figure 1 is basically as neat as sketched therein, the related proofs require some quite involved combinatorics and calculations. For readability we moved all these proofs to the end, see Section 6, which is the technical heart of the paper. (But it can easily be skipped on the first reading.)

Remark 1.1. (The following is not important to understand the context and the results of the paper, but we still think it is worthwhile to be mentioned. The reader unfamiliar with the translation from webs to intertwiners is referred to [15].)

We stress that the webs we use in this paper do not carry orientations on *ordinary* (black) edges. In contrast, *phantom* (reddish, dashed) edges carry orientations, cf. Figure 1. If one sees the ordinary edges as corresponding to the vector representation $V_{\mathfrak{g}}$ of an associated Lie (quantum) algebra $\mathfrak{g}$ and phantom edges corresponding to the second exterior power $\Lambda^2 V_{\mathfrak{g}}$ of it, then this translates to $(V_{\mathfrak{g}})^* \cong V_{\mathfrak{g}}$ as $\mathfrak{g}$-modules, but $(\Lambda^2 V_{\mathfrak{g}})^* \not\cong \Lambda^2 V_{\mathfrak{g}}$. Thus, if we see webs as $\mathfrak{g}$-intertwiners with "empty" corresponding to the ground field $\mathbb{K}$, then we have the situation as in Figure 2.

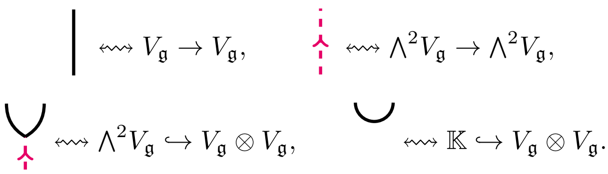

FIGURE 2. Examples of $\mathfrak{g}$-webs as $\mathfrak{g}$-intertwiners. The two bottom $\mathfrak{g}$-intertwiners are given by inclusion, the others are identities.

For $\mathfrak{sl}_2$-webs (Temperley-Lieb diagrams) one does not need orientations since

$$(V_{\mathfrak{sl}_2})^* \cong V_{\mathfrak{sl}_2} \quad \text{and} \quad (\Lambda^2 V_{\mathfrak{sl}_2})^* \cong \Lambda^2 V_{\mathfrak{sl}_2}.$$

Since we basically use phantom edges to encode signs, repeating all constructions from Sections 2, 3 and 4 for $\mathfrak{sl}_2$ is very easy and one obtains Bar-Natan's $\mathfrak{sl}_2$-foams $\mathfrak{F}^{\mathfrak{sl}_2}$ and the associated web and arc algebras as in [1] and [11].

For $\mathfrak{gl}_2$-webs one would have to orient ordinary edges as well, since

$$(V_{\mathfrak{gl}_2})^* \ncong V_{\mathfrak{gl}_2} \quad \text{and} \quad (\Lambda^2 V_{\mathfrak{gl}_2})^* \ncong \Lambda^2 V_{\mathfrak{gl}_2}.$$

Again, "copying" Sections 2, 3 and 4 would give a $\mathfrak{gl}_2$-foam 2-category $\mathfrak{F}^{\mathfrak{gl}_2}$ as in [10] or [9]. Note that ordinary circles in such $\mathfrak{gl}_2$-webs are all isomorphic as morphisms of $\mathfrak{F}^{\mathfrak{gl}_2}$, no matter of their orientation. Moreover, the isomorphisms between these are 'canonical' (the reader is encouraged to find them, cf. (5) and below Lemma 6.1) in the sense that they do not introduce any signs. Thus, for a lot of application as e.g. functorial link homologies, $\mathfrak{F}$ and $\mathfrak{F}^{\mathfrak{gl}_2}$ are "the same". In fact, we do not have a representation theoretical interpretation of $\mathfrak{F}$, but it is the 2-category which we can connect to the type D arc algebras. (For more on the relation between $\mathfrak{sl}_2$- and $\mathfrak{gl}_2$-web categories see e.g. [20, Remark 1.1]). ▲

Some more details regarding Theorem A. Arc algebras originally appeared in work of Khovanov [11] on the extension of his celebrated link homology to tangles. These algebras were obtained using a certain TQFT (which also reappeared later in work of Bar-Natan [1]) showing up in the construction of his link homology, and can be seen as a categorification of the $\mathfrak{sl}_2$-*web category* a.k.a. the Temperley-Lieb category.

Web algebras $\mathfrak{W}^{\mathfrak{g}}$ (for some Lie algebra $\mathfrak{g}$) are a generalization of this viewpoint on arc algebras: they naturally arise as categorifications of $\mathfrak{g}$-*web categories* in the sense of Kuperberg [15]. Although the details are quite delicate in general, these are in principle constructed by using singular TQFTs showing up in the construction of the link homologies which categorify the link polynomials associated to the underlying $\mathfrak{g}$-web categories. The main example for our purposes is the case $\mathfrak{g} = \mathfrak{gl}_2$. Here the algebra $\mathfrak{W}^{\mathfrak{gl}_2}$ consists of $\mathfrak{gl}_2$-foams in the spirit of Blanchet [2] using certain singular TQFTs, see e.g. [10] or [9]. Consequently, the web algebras we consider in this paper are intrinsically connected to the categories of representations associated to $\mathfrak{gl}_2$, as well as to the functorial version of Khovanov's link homology.

Indeed, $\mathfrak{W}$ has a direct, topological interpretation. Let $\mathfrak{W}$**-biMod** be a certain 2-category of $\mathfrak{W}$-bimodules (with details given in Definition 3.10). Further, let $\mathfrak{F}$ denote the foam 2-category obtained from the corresponding singular TQFT. Then there is a (structure preserving) equivalence of 2-categories, see Proposition 3.11,

$$\mathfrak{F} \xrightarrow{\cong} \mathfrak{W}\textbf{-biMod}.$$

The first main purpose of our paper is to give a purely *combinatorial model* $c\mathfrak{W}$ of $\mathfrak{W}$. We stress a few things:

▷ Extending our combinatorial model to the 2-category $\mathfrak{W}$**-biMod** is doable (following [9, Subsections 4.3 and 4.4]), but also lengthy. For brevity, we do not do it explicitly in this paper.

▷ The functorial version of Khovanov's link homology can be realized using $\mathfrak{W}$**-biMod**, cf. Remark 1.1. Hence, our combinatorial model can be used to calculate signs turning up in functorial properties of this link homology.

Since our combinatorial model works globally, there might be further possible implications for these link homologies.

Some more details regarding Theorem B. Another viewpoint on Khovanov's arc algebras was taken by Brundan-Stroppel in their influential work on an algebraic model $\mathfrak{A}^{\mathrm{A}}$ and a generalization $\mathfrak{C}^{\mathrm{A}}$ of Khovanov's arc algebra (its quasi-hereditary cover), see e.g. [4] or [5]. In their construction, the arc algebra was reinterpreted in an algebraic way which eased to show connections to more algebraic realms of mathematics. In particular, $\mathfrak{A}^{\mathrm{A}}$ and $\mathfrak{C}^{\mathrm{A}}$ are intrinsically connected to classical (type A) Lie theory in the sense that their categories of representations give graded versions of $\mathcal{O}_0^{\mathrm{A}_p \times \mathrm{A}_q}(\mathfrak{gl}_m(\mathbb{C}))$ for any $p + q = m$.

The work of Brundan and Stroppel is nowadays extended in various direction. The main example for us is the D arc algebra $\mathfrak{A}$ and its generalization $\mathfrak{C} = \mathfrak{C}^{\mathrm{D}}$ introduced by the first author in joint work with Stroppel [8]. Again, $\mathfrak{A}$ and $\mathfrak{C}$ are intrinsically connected to classical (type D) Lie theory in the sense that their categories of representations give graded versions of $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$ for all n. (The reader unfamiliar with these notations and statements might want to check [4] or [8]. But for this paper it is enough to keep in mind that the representation theory of $\mathfrak{A}^{\mathrm{A}}$ and $\mathfrak{C}^{\mathrm{A}}$ respectively $\mathfrak{A}$ and $\mathfrak{C}$ encodes type A respectively D Lie theory.)

Following this viewpoint, Stroppel-Webster [17] constructed the algebras $\mathfrak{A}^{\mathrm{A}}$ and $\mathfrak{C}^{\mathrm{A}}$ as convolution algebras using 2-block Springer fibers of type A. Similarly, $\mathfrak{A}$ and $\mathfrak{C}$ can be constructed using 2-block Springer fibers of type D, see [7] or [21].

The second main purpose of our paper is to "reconnect" the two viewpoints on Khovanov's construction by realizing $\mathfrak{A}$ as a subalgebra of $\mathfrak{W}$ (which we call a *topological model*). Again, we stress a few things:

- ▷ Using the subquotient construction explained e.g. in [9, Subsection 5.1], one can immediately generalize the web algebra and one obtains an algebra $g\mathfrak{W}$ in which $\mathfrak{C}$ embeds, cf. Remark 5.20.
- ▷ Since these embeddings are actually idempotent truncations, we get an associated embedding of their bimodule 2-categories:

$$\mathfrak{A}\textbf{-biMod} \hookrightarrow \mathfrak{W}\textbf{-biMod}, \qquad \mathfrak{C}\textbf{-biMod} \hookrightarrow g\mathfrak{W}\textbf{-biMod}.$$

- ▷ It will be clear that $\mathfrak{A}^{\mathrm{A}} \hookrightarrow \mathfrak{W}$. Thus, we have the type A analog of this:

$$\mathfrak{A}^{\mathrm{A}}\textbf{-biMod} \hookrightarrow \mathfrak{W}\textbf{-biMod}, \qquad \mathfrak{C}^{\mathrm{A}}\textbf{-biMod} \hookrightarrow g\mathfrak{W}\textbf{-biMod}.$$

- ▷ Hence, one can say that we get a topological model of $\mathcal{O}_0^{\mathrm{A}_p\times\mathrm{A}_q}(\mathfrak{gl}_m(\mathbb{C}))$ and $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$ with the foam 2-category $\mathfrak{F}$ controlling both at once.

This might have several implications. One direct upshot however is that the associativity and the well-definedness of the action on bimodules, which are very involved to prove for the type D (generalized) arc algebras, are immediate from the topological model, cf. Corollary 5.2.

A brief summary. To summarize, our "story" is as in Figure 3.

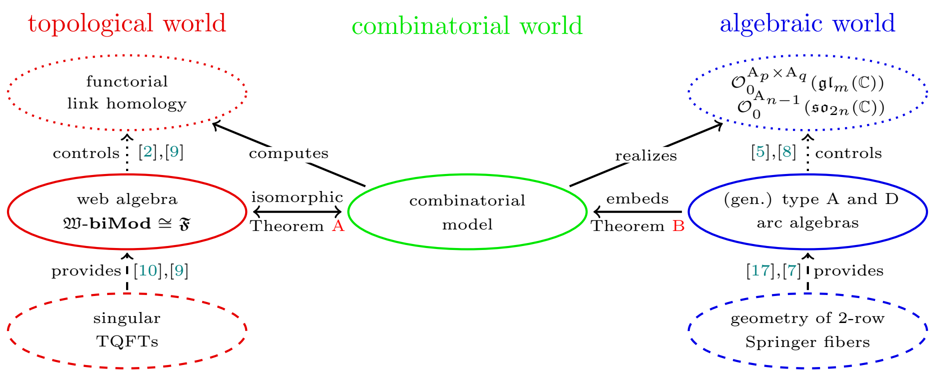

FIGURE 3. Our "story" in a nutshell.

Basic conventions. Throughout we work over a field $\mathbb{K}$ of arbitrary characteristic, and "dimension" is always meant with respect to $\mathbb{K}$. There are two exceptions: our proof of Proposition 3.11 requires $\mathbb{K} = \overline{\mathbb{K}}$ (this can be avoided, but extends the proof considerably), while all connections to category $\mathcal{O}$ work over $\mathbb{K} = \mathbb{C}$ only. Apart from these instances, working over $\mathbb{Z}$ instead is entirely possible.

For us algebras are $\mathbb{K}$-algebras which are not necessarily associative nor finite-dimensional nor unital. (All our algebras are associative, but this is, except for the web algebra, a non-trivial fact.)

By a graded algebra we always mean a $\mathbb{Z}$-graded algebra, where we use the same conventions as in [10, Conventions 1.1 and 1.2] for its graded (finite-dimensional) representation theory. In particular, graded biprojective means graded left and right projective, and $\{\cdot\}$ denotes grading shifts (with conventions as fixed below).

Let us stress our grading conventions for 2-categories:

Convention 1.2. An additive, graded, $\mathbb{K}$-linear 2-category is a category enriched over the category of additive, $\mathbb{Z}$-graded, $\mathbb{K}$-linear categories. Additionally, in our setup, the morphisms of such a 2-category admit grading shifts. That is, given any morphism X and any $s \in \mathbb{Z}$, there is a morphism $X\{s\}$ such that the identity 2-morphism on X gives rise to a degree s homogeneous 2-isomorphism from X to $X\{s\}$. General 2-morphisms in such 2-categories are $\mathbb{K}$-linear combinations of homogeneous ones. Hereby, any 2-morphism of degree d between X and Y becomes a 2-morphism of degree $d + t - s$ when seen between $X\{s\}$ and $Y\{t\}$. ▲

A remark about colors and diagrams 1.3. We read all diagrams from bottom to top and from left to right, and we often illustrate only local pieces.

Regarding colors: The important colors are the reddish phantom parts of webs and foams. In a black-and-white version, these can be distinguished by noting that phantom edges are dashed and phantom facets are shaded. ▲

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2. Singular TQFTs and foams

In the present section we briefly recall the topological construction of foams via the singular TQFT approach outlined in [10, Section 2] and [9, Section 2]. (The reader is referred to these two papers for more details.)

2.1. **Webs and prefoams.** First, we recall the construction of prefoams.

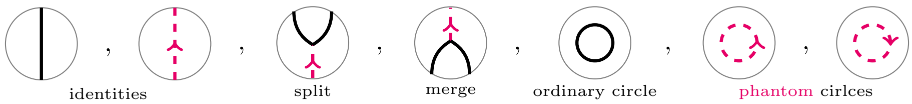
The boundary of foams. We start by recalling two definitions of webs, which are slightly different than the ones considered in [10] or [9], cf. Remark 1.1, but the construction of (pre)foams will be, mutatis mutandis, the same as therein.

Definition 2.1. A *web* is a labeled, piecewise linear, one-dimensional CW complex (a graph with vertices and edges) embedded in $\mathbb{R}^2 \times \{z\} \subset \mathbb{R}^3$ for some fixed $z \in \mathbb{R}$ with boundary supported in two horizontal lines, such that all horizontal slices consists only of a finite number of points. (Hence, it makes sense to talk about the bottom and top boundary of such webs.) Each vertex is either internal and of valency three, or a boundary vertex of valency one.

We assume that each edge carries a label from $\{\mathtt{o}, \mathtt{p}\}$ (we say they are *colored* by $\mathtt{o}$ or $\mathtt{p}$). Moreover, the $\mathtt{p}$-colored edges are assumed to be oriented, and each internal vertex has precisely one attached edge which is $\mathtt{p}$-colored. By convention, the empty web $\varnothing$ is also a web, and we allow circle components which consist of edges only. Webs are considered modulo boundary preserving isotopies in $\mathbb{R}^2 \times \{z\}$. ▲

Throughout we will, not just for webs, consider labelings with $\mathtt{o}$ or $\mathtt{p}$ and always illustrate them directly as colors using the convention that "$\mathtt{p}$=reddish". Moreover, both, webs and (pre)foams as defined below, contain $\mathtt{p}$-colored edges/facets. We call, everything related to these $\mathtt{p}$-colored edges/facets *phantom*, anything else *ordinary*.

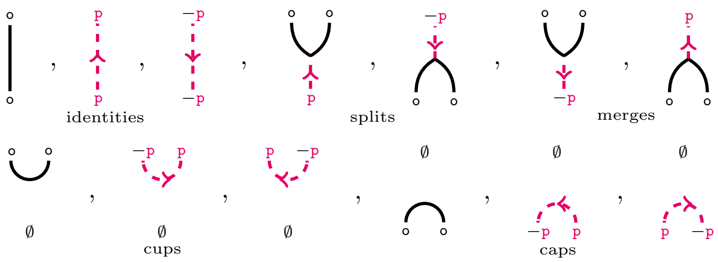
Example 2.2. When we depict webs we omit the edge labels and color the edges instead. Furthermore, for readability, we draw $\mathtt{p}$-colored edges dashed. Using these conventions, such webs are for example locally of the form:



Here the outer circle indicates that these are local pictures. (We omit it in what follows and hope no confusion can arise.) ▲

Definition 2.3. Let $\mathbf{W}$ be the *monoidal category of webs* given as follows:

- ▷ Objects are finite words $\vec{k}$ in the symbols $\mathtt{o}$, $\mathtt{p}$ and $-\mathtt{p}$. (The empty word $\emptyset$ is also allowed.)
- ▷ The morphisms spaces $\mathrm{Hom}_{\mathbf{W}}(\vec{k}, \vec{l})$ are given by all webs with bottom boundary $\vec{k}$ and top boundary $\vec{l}$ using the following local conventions (read from bottom to top) for identities, splits, merges, cups and caps:

(1)


- ▷ The composition $uv = v \circ u$ is the evident gluing of v on top of u, and monoidal product $\vec{k} \otimes \vec{l}$ or $u \otimes v$ given by putting $\vec{k}$ or u to the left of $\vec{l}$ or v.

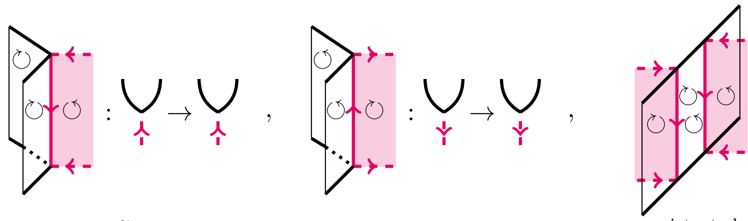
A *closed* web $\overline{w}$ is an endomorphism of the empty word $\emptyset$, i.e. $\overline{w} \in \mathrm{End}_{\mathbf{W}}(\emptyset)$. ▲

We will use the topological and the algebraic notion of webs interchangeable (e.g. the generators from (1) are allowed to have their boundary points "far apart").

For later use, we denote by * the involution that mirrors a web along the top horizontal line and reverses orientations. Moreover, since the objects of $\mathbf{W}$ can be read off from the webs, we omit to indicate them.

Prefoams. Instead of giving the details of the construction of prefoams here (the reader is referred to [10, Section 2] and [9, Section 2]), we only stress the important points which we need in this paper. For our purpose it will be enough to see prefoams as certain singular cobordism whose boundary are webs (including the empty web).

Remark 2.4. *Closed prefoams* $\overline{f}$ are certain singular CW complexes embedded in $\mathbb{R}^3$ whose singularities, called *singular seams*, are locally of the form

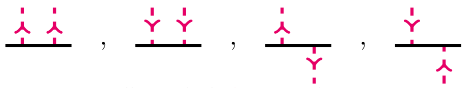
(2) 

split   merge   no consistent choice

Hereby we stress that we only consider those prefoams which can be embedded into $\mathbb{R}^3$ such that there is a choice of orientation of its facets as illustrated in (2) (we fix this orientation); this choice of orientation is consistent in the sense that it induces orientations on the singular seams.

Such prefoams also carry a finite number of markers per connected component which we call *dots* $\bullet$ and which we illustrate as in (4). ▲

Remark 2.5. Due to these orientation conventions, there are no prefoams bounding closed webs with an odd number of trivalent vertices. There are also no prefoams bounding a local situation which has *ill-attached* phantom edges (cf. (2)), i.e.:



ill-attached phantom edges

All other local situations are said to have *well-attached* phantom edges (see (66)). In contrast, their might be closed webs with an odd number of trivalent vertices or ill-attached phantom edges, but these will play no role for us. ▲

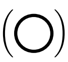
Remark 2.6. Let $P_{xy}^{\pm 1}$ be the plane spanned by the first two coordinates in $\mathbb{R}^3$, embedded such that the third coordinate is ± 1. A (non-necessarily closed) *prefoam* f is the intersection of $\mathbb{R}^2 \times [-1,+1]$ with some closed prefoam $\overline{f}$ such that $P_{xy}^{\pm 1}$ intersects $\overline{f}$ generically, now with orientation on its boundary induced as in (2): the orientation on the phantom facets agrees with the orientation on the phantom edges of the webs which we view as being the target sitting at the top and disagrees at the bottom. Clearly it suffices to indicate the orientations of the singular seams and we do so in the following. In particular, the orientation of the singular seams point into splits and out of merges at the bottom of a prefoam. ▲

2.2. **Obtaining relations via singular TQFTs.** Next, we recall and state the relations which we need, and we sketch where they come from.

Singular TQFTs. The bottom and top of a prefoam f are webs $\overline{w}_b$ and $\overline{w}_t$, and we see f as a cobordism from $\overline{w}_b$ to $\overline{w}_t$, as indicated in (2). Using the cobordisms description, the whole data assembles into a symmetric monoidal category which we denote by $\mathbf{pF}$. (Objects are closed webs $\overline{w}_b$ and $\overline{w}_t$, and morphisms are prefoams $f\colon \overline{w}_b \to \overline{w}_t$ between these. Composition is the evident gluing, the monoidal structure is given by juxtaposition.)

Let us denote by $\mathbb{K}$-$\mathbf{VS}$ the symmetric monoidal category of finite-dimensional $\mathbb{K}$-vector spaces. The point now is the existence of a symmetric monoidal functor:

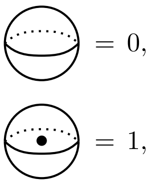
Theorem 2.7. (See e.g. [10, Theorem 2.11] or [9, Theorem 2.10].) There exists a symmetric monoidal functor $\mathcal{T}\colon \mathbf{pF} \to \mathbb{K}$-$\mathbf{VS}$ such that

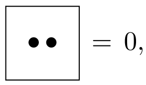
$\mathcal{T}\Big($  $\Big) \cong \mathcal{T}\Big($  $\Big) \cong \mathcal{T}\Big($  $\Big) \cong \mathbb{K}[X]/(X^2)$, $\qquad \mathcal{T}\Big($  $\Big) \cong \mathcal{T}\Big($  $\Big) \cong \mathbb{K}$. ■

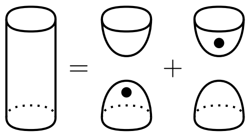
The symmetric monoidal functor $\mathcal{T}$ is called a *singular TQFT*, and its construction is based on ideas from [3] ("the universal construction") as well as [2].

Various foamy relations. A principal objective in the "singular TQFT game" is to identify sufficiently many relations "in the kernel". Hereby we say that a relation $a_1 f_1 + \cdots + a_k f_k = b_1 g_1 + \cdots + b_l g_l$ between formal, finite, $\mathbb{K}$-linear combinations of prefoams lies *in the kernel of* $\mathcal{T}$, if $a_1\mathcal{T}(f_1) + \cdots + a_k\mathcal{T}(f_k) = b_1\mathcal{T}(g_1) + \cdots + b_l\mathcal{T}(g_l)$ holds as $\mathbb{K}$-linear maps. Here are the first examples:

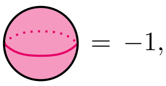
Lemma 2.8. (See [10, Lemmas 2.9 and 2.13].) The following relations

(3) 

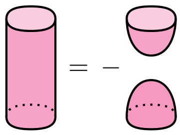
(4) 

(5) 

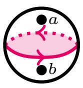
called (from left to right) *ordinary sphere, dot and neck cut* relations, as well as the *phantom sphere, dot and neck cut* relations

(6) 

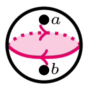
(7)

(8) 

are in the kernel of $\mathcal{T}$. Similarly, if one considers ordinary and phantom parts separately, then all the usual TQFT relations ("Frobenius isotopies") as e.g. illustrated in [14, Section 2.3] are in the kernel of $\mathcal{T}$. Furthermore, the *theta foam* relations

(9) 

$$= \begin{cases} +1, & \text{if } a = 1, b = 0, \\ -1, & \text{if } a = 0, b = 1, \\ 0, & \text{otherwise,} \end{cases}$$

(10) 

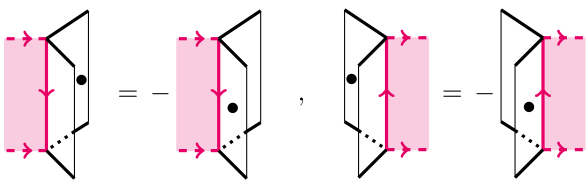
$$= \begin{cases} -1, & \text{if } a = 1, b = 0, \\ +1, & \text{if } a = 0, b = 1, \\ 0, & \text{otherwise,} \end{cases}$$

are also in the kernel of $\mathcal{T}$. ■

Note that (9) and (10) are actually the same relation (the reader might want to turn her/his head), but when reading from bottom to top the orientation of the singular seam is reversed when comparing (9) to (10), which gives an asymmetry.

Moreover, we have the following local relations involving phantom facets.

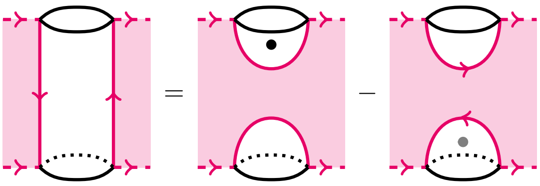
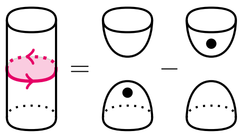
Lemma 2.9. (See [10, Lemma 2.14] and [9, (19)].) The following relations are in the kernel of $\mathcal{T}$. The *dot moving* relations

(11) 

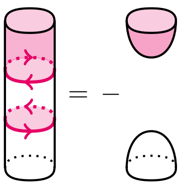
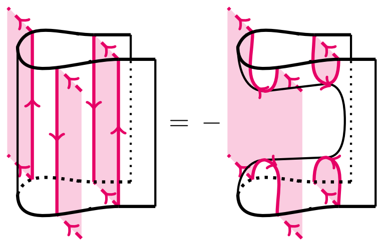
the *singular sphere removal* relations

(12) 

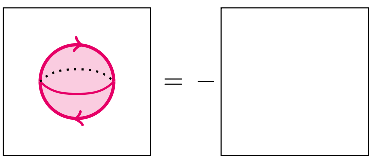
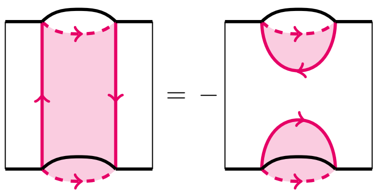
the *singular neck cutting* and the *closed seam removal* relations (the shaded dots in (12) and (13) sit on the facets in the back)

(13)  , (14) 

the *ordinary-to-phantom neck cutting* and the *ordinary squeeze* relations

(15)  , (16) 

Further, the *phantom cup removal* and *phantom squeeze* relations are also in the kernel of $\mathcal{T}$ (the facets facing towards the reader are phantom facets):

(17)  , (18) 

(These do not directly appear in [10] or [9] since they are not used in the "highest weight" setup. We leave the verification that they are also in the kernel of $\mathcal{T}$ to the reader which can be done, mutatis mutandis, as in the cited lemmas.) ■

Remark 2.10. The relations from Lemma 2.9 exist in various differently oriented versions as well, as the reader is encouraged to check (see also [10, Lemma 2.12]). It is crucial that the sign difference in the theta foam relations (9) and (10) give opposite signs for the "reoriented" relations (12), (13), (14), (17) and (18). ▲

Gradings. Note that all finite-dimensional, commutative Frobenius algebras used in the construction of $\mathcal{T}$ carry a grading. In particular, the functor $\mathcal{T}$ from Theorem 2.7 can be regarded as a functor to graded, finite-dimensional $\mathbb{K}$-vector spaces. Pulling this degree back to **pF** leads to the following definition.

Definition 2.11. Let χ denote the topological Euler characteristic, and let #dots denote the number of dots on some prefoam f. Let $\hat{f}$ be the CW complex obtained from f by removing the phantom edges/facets. Define

$$\deg(f) = -\chi(\hat{f}) + 2 \cdot \#\text{dots}. \tag{19}$$

(The empty prefoam $f(\varnothing)$ has, by definition, Euler characteristic zero.) This gives **pF** the structure of a graded category. ▲

By the above, we see that $\mathcal{T}$ is actually a graded, symmetric monoidal functor.

2.3. **Lineralization of the foam 2-category.** Next, we need the notion of an *open prefoam*. These are constructed similarly to closed prefoams, but are embedded in $\mathbb{R}\times[-1,+1]^2 \subset \mathbb{R}^3$, such that its vertical (second coordinate) boundary components are straight lines in $\mathbb{R} \times \{\pm 1\} \times [-1,+1]$, and its horizontal (third coordinate) boundary components are webs embedded in $\mathbb{R} \times [-1,+1] \times \{\pm 1\}$ (using the same conventions for orientation etc. as before, see e.g. (2)). Again, we can see these as cobordisms between the (non-necessarily closed) webs u and v. This gives rise to a vertical composition $\circ$ via gluing (and rescaling), as well as a horizontal composition $\otimes$ via juxtaposition (and rescaling). We consider such open prefoams modulo isotopies in $\mathbb{R} \times [-1,+1]^2$ which fix the vertical and horizontal boundary, and the condition that generic slices are webs.

Let $\#$vbound the number of vertical boundary components of some open prefoam f. We extend Definition 2.11 to open prefoams f via:

$$\deg(f) = -\chi(\hat{f}) + 2 \cdot \#\text{dots} + \tfrac{1}{2} \cdot \#\text{vbound}. \tag{20}$$

(The reader should check that this definition is additive under composition.)

From prefoams to foams. We are now ready to define *foams*, which, in contrast to prefoams, "live" in a $\mathbb{K}$-linear setup and all the relations from Subsection 2.2 directly make sense.

Definition 2.12. Let $\mathfrak{F}$ denote the additive closure of the graded, $\mathbb{K}$-linear 2-category given by:

- ▷ The underlying structure of objects and morphisms is given by the category **W** of webs from Definition 2.3.
- ▷ The space of 2-morphisms between two webs u and v is a quotient of the graded, free $\mathbb{K}$-vector space on basis given by all open prefoams from u to v. The grading is hereby defined to be induced by (20).
- ▷ The quotient is obtained by modding out the relations from Subsection 2.2 as well as all relations they induce by closing prefoams.
- ▷ The vertical and the horizontal compositions are $\circ$ and $\otimes$ from above.

Note that these relations are homogeneous which endows $\mathfrak{F}$ with the structure of a graded 2-category (in the sense of Convention 1.2). ▲

We call $\mathfrak{F}$ the *(full) foam 2-category*. The 2-morphisms from $\mathfrak{F}$ are called *foams*. (We also use the same notions as we had for prefoams for foams.)

Remark 2.13. The 2-category $\mathfrak{F}$ can also be defined as a canopolises in the sense of Bar-Natan [1, Subsection 8.2]. We stay here with the 2-categorical formulation since in this setup we obtain an equivalent, algebraic, description in Proposition 3.11. ▲

The cup foam basis. Next, we have a *cup foam basis*:

Proposition 2.14. Given $u, v \in \mathrm{Hom}_{\mathfrak{F}}(\vec{k}, \vec{l})$. There is a finite, homogeneous cup foam basis for $2\mathrm{Hom}_{\mathfrak{F}}(u, v)$ in the sense of [10, Definition 4.12]. □

The proof of Proposition 2.14 and the construction of the cup foam basis is given in Section 6. (The construction is provided by a "cookie-cutter strategy".) As we will see, up to signs, the construction is essentially dictated by our desire to have "dotted cups" as our basis elements, see e.g. in Example 6.5. (For details see Subsection 6.1.) We also write ${}_u\mathbb{B}_v$ and $\mathbb{B}(\overline{w}) = {}_{\varnothing}\mathbb{B}_{\overline{w}}$ (for closed webs $\overline{w}$) whenever we mean the fixed cup foam bases given in Definition 6.11.

3. The (entire) web algebra

We aim to define the algebra presenting $\mathfrak{F}$. This follows a similar strategy as for more general foam 2-categories.

3.1. **The algebra presenting foams.** Recall that $\vec{k}, \vec{l}$ etc. denote finite words in the symbols $\mathtt{o}$ and $\mathtt{p}, -\mathtt{p}$. We call these *balanced* in case they have an even number of symbols $\mathtt{o}$. The set of such balanced words is denoted by $\mathtt{bl}^{\circ}$. Furthermore, we write $\mathtt{o}_{\vec{k}}$ to denote the total number of $\mathtt{o}$'s in $\vec{k}$. For later use: A *(balanced) block* $\vec{K}$ is a set consisting of all words $\vec{k}$ with $\mathtt{o}_{\vec{k}} = K$, for some fixed, even, non-negative integer K, called the *rank of* $\vec{K}$. (Note that there is only one block of a fixed rank, and we always match this block and its rank notation-wise.) The set of these blocks is denoted by $\mathtt{Bl}^{\circ}$.

Further, denote by $\mathrm{CUP}^{\vec{k}} = \mathrm{Hom}_{\mathbf{W}}(\emptyset, \vec{k})$, whose elements are called *cup webs*. Having two cup webs $u, v \in \mathrm{CUP}^{\vec{k}}$, one obtains a closed web $uv^* = v^* \circ u$ with composition $\circ$ as in Definition 2.3. (In words: we glue v^* on top of u.)

Convention 3.1. Whenever we work with cup webs $u, v \in \mathrm{CUP}^{\vec{k}}$ or closed webs of the form uv^* we fix a line (which we will illustrate as a dotted line, cf. (22)) on which $\vec{k}$ lives. This is the monoidal view on webs as in Definition 2.3 and breaks some symmetry, which is important to define some notions later. (For example, the notions of a $\mathtt{Ↄ}$ shape and a $\mathtt{C}$ shape make sense.) ▲

Following the terminology of [12, Section 3] (and abusing notation a bit), we define the *web homology* $\mathcal{T}(\overline{w}) = 2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$, i.e. the graded, $\mathbb{K}$-linear vector space given by all foams from the empty web $\varnothing$ into the closed web $\overline{w}$.

The web algebra as a $\mathbb{K}$*-vector space.* Let $d_{\vec{k}} = \frac{1}{2}\mathtt{o}_{\vec{k}}$.

Definition 3.2. Given $u, v \in \mathrm{CUP}^{\vec{k}}$ we set

$$ {}_u(\mathfrak{W}_{\vec{k}})_v = \mathcal{T}(uv^*)\{d_{\vec{k}}\}. $$

The *web algebra* $\mathfrak{W}_{\vec{k}}$ *for* $\vec{k} \in \mathtt{bl}^{\circ}$ is the graded $\mathbb{K}$-vector space

$$ \mathfrak{W}_{\vec{k}} = \bigoplus_{u,v \in \mathrm{CUP}^{\vec{k}}} {}_u(\mathfrak{W}_{\vec{k}})_v $$

and the *(full) web algebra* $\mathfrak{W}$ is the direct sum of all $\mathfrak{W}_{\vec{k}}$ for $\vec{k} \in \mathtt{bl}^{\circ}$. These $\mathbb{K}$-vector spaces are equipped with the multiplication recalled below. ▲

We also define $\mathfrak{W}_{\vec{K}} = \bigoplus_{\vec{k} \in \vec{K}} \mathfrak{W}_{\vec{k}}$ for all $\vec{K} \in \mathtt{Bl}^{\circ}$ which we will use in Section 5.

By Proposition 2.14, ${}_u(\mathfrak{W}_{\vec{k}})_v$ has a (fixed) cup foam basis which we denote by ${}_u\mathbb{B}(\vec{k})_v = \mathbb{B}(uv^*)$. We also use the evident notation ${}_u\mathbb{B}(\vec{K})_v$ later on.

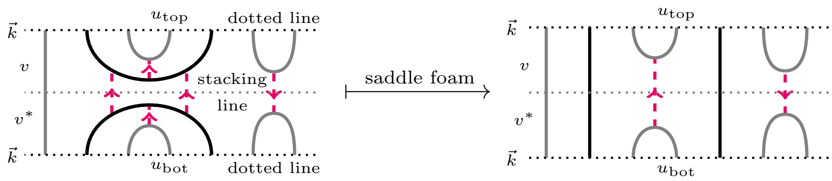
The web algebra as an algebra. We sketch the multiplication structure. Details (which are easily adapted to our setup) can be found in [16, Section 3].

Convention 3.3. We sometimes need more general webs than webs of the form uv^* for $u, v \in \mathrm{CUP}^{\vec{k}}$. Namely, all possible webs which can turn up in multiplication steps which we recall below. We call such webs *stacked webs*, and will use the evident notions of *stacked dotted webs* and *stacked (circle) diagrams* for the two calculi in Sections 4 and 5 as well. The illustrative example to keep in mind is given in (22), where also some terminology (*dotted and stacking line*) is fixed. Note that, as stacked webs, uv^* has a middle part consisting of identity strands only. ▲

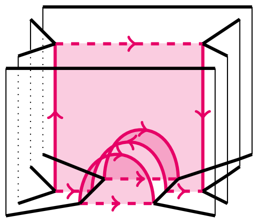
The multiplication

(21) $$\mathbf{Mult}^{\mathfrak{W}}_{\vec{k}} \colon \mathfrak{W}_{\vec{k}} \otimes \mathfrak{W}_{\vec{k}} \to \mathfrak{W}_{\vec{k}}, \qquad f \otimes g \mapsto \mathbf{Mult}^{\mathfrak{W}}_{\vec{k}}(f, g)$$

is defined using the *surgery rules*. That is, the multiplication of $f \in {}_{u_{\mathrm{bot}}}(\mathfrak{W}_{\vec{k}})_v$ and $g \in {}_{v'}(\mathfrak{W}_{\vec{k}})_{u_{\mathrm{top}}}$ is zero if $v \neq v'$. An example in case $v = v'$ is:

(22) 

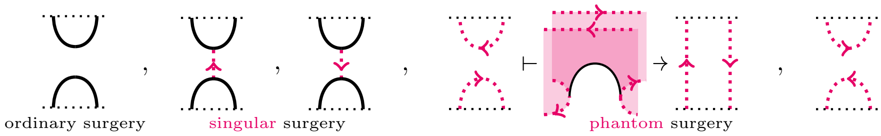
(the stacking line in (22) will be omitted in the following) where the *saddle foam* locally looks as follows (and is the identity elsewhere)

(23) 

See e.g. [16, Definition 3.3] or [10, Definition 2.24] for a detailed account.

Taking direct sums then defines $\mathbf{Mult}^{\mathfrak{W}}$.

Remark 3.4. We stress that the multiplication with a web $u_{\mathrm{bot}}v^*$ at the bottom and a web $v'u^*_{\mathrm{top}}$ at the top is zero in case $v \neq v'$. In particular, one has locally (read as in (22)) only the following non-zero surgery configurations:

(24) 

Hereby the multiplication foams are "saddles": either ordinary, in the first case, singular saddles in the second and third cases (as illustrated in (23), which is a composition of two saddles as in (24)), or phantom in the final two cases (one of which we have illustrated in (24)). ▲

By identifying the multiplication in $\mathfrak{W}$ with the composition in $\mathfrak{F}$ (which can be done analogously as in [16, Lemma 3.7] via "clapping") we obtain:

Proposition 3.5. The multiplication $\mathbf{Mult}^{\mathfrak{W}}$ is independent of the order in which the surgeries are performed, which turns $\mathfrak{W}$ into an associative, graded algebra. ■

Remark 3.6. The "highest weight" web algebras studied in [10] and [9] (with the "Blanchet choice" of parameters) are subalgebras of the web algebra. This can be seen by closing the "highest weight" diagrams in [10] and [9] in a "braid closure fashion". (Hereby, the reader should keep Remark 1.1 in mind.) Consequently, the signs that turn up in the combinatorial model presented in Section 4 are more sophisticated versions of the ones from e.g. [10, Section 3].

Moreover, Khovanov's original arc algebra from [11] is a subalgebra of $\mathfrak{W}$ in at least two seemingly different ways: first by direct embedding, second by using one of the main results from [9] and the "highest weight embedding" sketched above. ▲

3.2. **Its bimodule 2-category.** Fix $\vec{k},\vec{l}\in\mathtt{bl}^{\circ}$.

Definition 3.7. Given $u\in\mathrm{Hom}_{\mathbf{W}}(\vec{k},\vec{l})$, we consider the graded $\mathbb{K}$-vector space

$$\mathbf{W}(u)=\bigoplus_{v_{\mathrm{bot}},v_{\mathrm{top}}}\mathcal{T}(v_{\mathrm{bot}}uv_{\mathrm{top}}^{*}),$$

with the sum running over all $v_{\mathrm{bot}}\in\mathrm{CUP}^{\vec{k}}, v_{\mathrm{top}}\in\mathrm{CUP}^{\vec{l}}$. We endow $\mathbf{W}(u)$ with a left and a right action of $\mathfrak{W}$ as in Definition 3.2. ▲

Noting that the left and right actions are "far apart", we see that all $\mathbf{W}(u)$'s are graded $\mathfrak{W}$-bimodules referred to as *web bimodules*. In fact:

Proposition 3.8. The web bimodules $\mathbf{W}(u)$ are graded biprojective $\mathfrak{W}$-bimodules with finite-dimensional subspaces for all pairs $v_{\mathrm{bot}}\in\mathrm{CUP}^{\vec{k}}, v_{\mathrm{top}}\in\mathrm{CUP}^{\vec{l}}$. □

Proof. They are clearly graded. The finite-dimensionality follows from the existence of an explicit cup foam basis, see Proposition 2.14. They are biprojective, because they are direct summands of some $\mathfrak{W}_{\vec{k}}$ (of some $\mathfrak{W}_{\vec{l}}$) as left (right) modules for suitable $\vec{k}\in\mathtt{bl}^{\circ}$ (or $\vec{l}\in\mathtt{bl}^{\circ}$), see also [16, Proposition 5.11]. ■

Remark 3.9. The web bimodules as well as the web algebras (all of them, i.e. $\mathfrak{W}_{\vec{k}},\mathfrak{W}_{\vec{K}}$ and $\mathfrak{W}$) are infinite-dimensional in a "naive" way, since one takes huge direct sums within their definition. Most of the summands are actually unnecessary since the webs are isomorphic as morphisms of $\mathfrak{F}$, cf. Lemma 6.1. If one wants to work with finite-dimensional modules and algebras, then one could choose a "basis" of the web spaces (as e.g. in [16]). ▲

Taking everything together, we can define:

Definition 3.10. Let $\mathfrak{W}$**-biMod** be the following 2-category.

- Objects are the various balanced words $\vec{k}\in\mathtt{bl}^{\circ}$.
- The morphisms are finite sums and tensor products (taken over the algebra $\mathfrak{W}$) of $\mathfrak{W}$-bimodules $\mathbf{W}(u)$, with composition given by tensoring (over $\mathfrak{W}$).
- The 2-morphisms are $\mathfrak{W}$-bimodule homomorphisms, with vertical and horizontal composition given by composition and by tensoring (over $\mathfrak{W}$).

We consider $\mathfrak{W}$**-biMod** as a graded 2-category as in Convention 1.2. ▲

For the next proposition we assume $\mathbb{K}=\overline{\mathbb{K}}$. This can be avoided, but additional care needs to be taken in the proof.

Proposition 3.11. There is an equivalence of additive, graded, $\mathbb{K}$-linear 2-categories

$$\Upsilon\colon\mathfrak{F}\xrightarrow{\cong}\mathfrak{W}\textbf{-biMod},$$

which is bijective on objects and essential surjective on morphisms. □

The proof is, as usual, given in Subsection 6.1.

4. The combinatorial model

Foams and the web algebra carry information about two-dimensional topological spaces sitting in three-space. This makes direct (non-local) computations quite involved. The aim of this section is to define a stricter version of the web algebra given by web-like objects sitting in the plane, called the *combinatorial* model. That is, we are going to define an algebra $c\mathfrak{W}$ with multiplication $\mathbf{Mult}^{c\mathfrak{W}}$ and show:

Theorem 4.1. There is an isomorphism of graded algebras

$$\mathtt{comb}\colon c\mathfrak{W}\xrightarrow{\cong}\mathfrak{W}.$$

(Similarly, denoted by $\mathtt{comb}_{\vec{k}}$ or $\mathtt{comb}_{\vec{K}}$, on summands.) □

(The proof of Theorem 4.1 is given in Subsection 6.2.) Note that Theorem 4.1 immediately gives the following, which would otherwise be rather involved to prove:

Corollary 4.2. The multiplication $\mathbf{Mult}^{c\mathfrak{W}}$ is independent of the order in which the surgeries are performed, turning $c\mathfrak{W}$ into an associative, graded algebra. ■

In order to define $c\mathfrak{W}$ we first need to introduce several combinatorial notions, all of which are dictated by our desire to see $c\mathfrak{W}$ as a "projection" of $\mathfrak{W}$, cf. Figure 4.

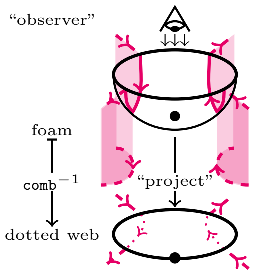


FIGURE 4. From foams to dotted webs: looking from the top to the bottom, a dotted web is obtained from a foam by projection.

These combinatorial notions will assemble into what we call *dotted webs.* The algebra $c\mathfrak{W}$ is then defined very much in the spirit of arc algebras: It has an underlying $\mathbb{K}$-linear structure given by dotted (basis) webs, and its multiplicative structure is inductively defined using a combinatorially defined surgery procedure (in contrast to the topologically defined surgery for web algebras). As usual, the signs turning up are intricate and a major part of this section is just devoted to define combinatorial ways to calculate them. The definition of the mapping $\mathtt{comb}\colon c\mathfrak{W} \to \mathfrak{W}$ is then, up to details which we have migrated to Section 6, the inverse of the one from Figure 4.

4.1. **Basic notions.** The first step toward the definition of a combinatorial model for foams is to replace foams by a decoration on webs.

Dotted webs. Since it follows from the existence of the cup foam basis, cf. Proposition 2.14, that there is a foam basis given by "(potentially) dotted cups", such a decoration for us will be a dot $\bullet$ on some component of a web, as well as certain lines keeping track of the singular seams attached to cup foams basis elements.

Remark 4.3. For more general web algebras the situation is more delicate, cf. Remark 6.7, and combinatorial models are missing for most of them at the moment. We prefer the combinatorially easier model using dots on webs in this paper, but for e.g. web algebras as in [16] one would need more sophisticated notions as e.g. "flows" in the sense of [13] as decorations. ▲

Hereby, and throughout, a *component* of a web is meant as a topological space after erasing all phantom edges. Moreover, by our definition of webs, connected components are circles, and we call them *circles* for short. In this spirit (and recalling Convention 3.1), we also say *cup* and *cap* in a web meaning the evident notion obtained by erasing phantom edges, while a *phantom cup/cap* are also to be understood in the evident way, cf. (31) where several cups and caps appear. Having a circle, we can speak about its *internal/external* by ignoring all other circles.

Convention 4.4. Webs can have circles with an odd number of trivalent vertices or ill-attached phantom edges, but their associated foam spaces are zero, cf. Remark 2.5.

We call such webs *ill-oriented*, all others *well-oriented*. Henceforth, if not stated otherwise, we consider only well-oriented webs (webs for short) with an even number of trivalent vertices and well-attached phantom edges. ▲

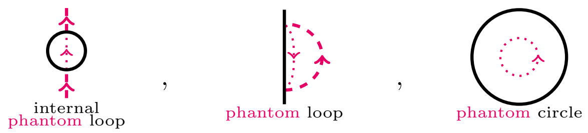
A *path* in a web u is an embedding of $[0,1]$ into the CW complex given by u after erasing all phantom edges.

Given a point i on a web u, then the *segment containing* i is the maximal path containing i which does not cross any phantom edges. Recalling that webs are embedded in $\mathbb{R}\times[-1,+1]$, we make the following definition. Hereby and throughout, *points on* u are always meant to be on ordinary parts of the web u, and are always contained in some segment (which one will be evident).

Definition 4.5. Given a web u and a circle C of it. Then the *base point* $\mathtt{B}(C)$ on it is defined to be any point in the bottom right segment of C. ▲

As in [10, Subsection 3.1], $\mathtt{B}(C)$ is a "choice of a rightmost point". We also write $\mathtt{B}=\mathtt{B}(C)$ for short if no confusion can arise.

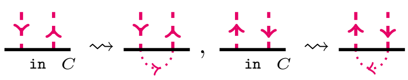
Definition 4.6. Given a web u, then a *phantom seam* is a decoration of u with an extra edge starting and ending at some trivalent vertices of u which is oriented in the direction of the adjacent phantom edges of u, e.g. (we illustrate phantom seams dotted and slightly thinner than the other phantom edges):

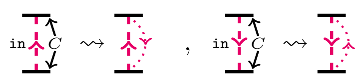
(25) 

Hereby we also allow phantom circles, which are always assumed to be in some circle of a web, as on the right above in (25). Moreover, the phantom seams have to be attached to a web such that the result does not have any intersections, and no trivalent vertex has more than one attached phantom seam. ▲

We are quite free to decorate webs. In order to match decorated webs with cup foam basis elements, we have to chose a decoration. This corresponds to choosing a cup foam basis as we will see in Subsection 6.1. In particular, it will depend on a choice of a point for each circle in question. Choosing such a point i, we call a phantom edge i-*closest* if its the first phantom edge one passes when going around anticlockwise, starting at i. (Similarly for other notions.) If these points are the base points, then we call such a decoration $\mathtt{B}$-admissible.

In order to define such decorations, we fix the following choices of how to put phantom seams locally on webs, fixing a circle C of it:

(26) 

(27) 

where $\mathtt{in}$ denotes the interior of C. (We do not distinguish between putting the phantom seams to the bottom or top in (26), or right or left in (27), cf. (28)).

Now, a $\mathtt{B}$-*admissible* decoration is one obtained by applying Algorithm 1 to a circle C in the web u with chosen base points $i=\mathtt{B}$.

Then we piece everything together as in Subsection 6.1. (Detail are not important for the moment and will follow in Subsection 6.1, e.g. phantom digon is defined therein. Furthermore, for the time being, we ignore questions regarding well-definedness etc. The only thing the reader need to know at this point is that the $\mathtt{B}$-admissible decoration are the ones turning up as in Figure 4.)

input : a web u, a circle C in it and a chosen point i of it;
output : an i-admissible decoration C_{dec} of C;

initialization, let C_{dec} be the circle without decorations;
while *C has attached phantom edges* **do**
  if *C contains a pair as in (26)* **then**
    apply (26) to the i-closest such pair;
    add the corresponding phantom seam to C_{dec};
    remove the corresponding pair from C;
  else
    apply (27) to any phantom digon not containing i;
    add the corresponding phantom seam to C_{dec};
    remove the corresponding phantom digon from C;
  end
end

Algorithm 1: The i-admissible decoration algorithm.

Definition 4.7. Given a web u, we allow each circle of it to be decorated by a dot $\bullet$, where we assume that the dot is on a segment of u. Moreover, each trivalent vertex of u is decorated by an attached phantom seam (there can be any finite number of phantom circles), which are not allowed to cross each other. We call such a web with decorations a *dotted web*.

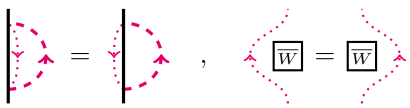
We call a dotted web a *dotted basis web* in case the following are satisfied.

- ▷ All dots are on the segments of the base points B for all circles C in u.
- ▷ All phantom seams decorations are B-admissible.

(If some circle C is not dotted at all, then the first condition is satisfied for it.) ▲

We stress that dotted basis webs will never have phantom circles.

We denote dotted webs using capital letters as e.g. $U, V, \overline{W}$ etc., and we say they are of *shape* $u, v, \overline{w}$ etc. In the following we consider dotted (basis) webs up to isotopies of these seen as decorated (by dots) planar graphs, as well as the relations

(28) 

where $\overline{W}$ is some dotted web not connected to the displayed phantom seam. (Similar for all versions of these with different orientations.)

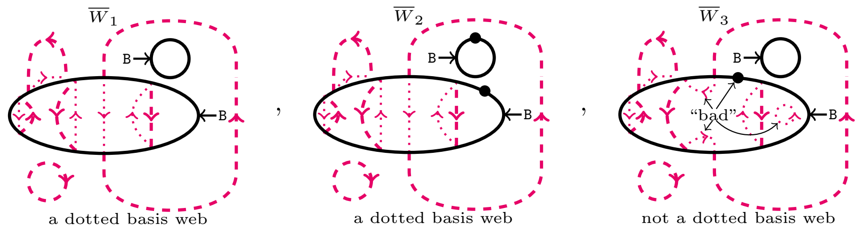
Definition 4.8. The *degree* of a dotted web U is defined as

$$\deg(U) = -\#C + 2 \cdot \#\mathrm{dots}, \tag{29}$$

where $\#C$ is the number of circles and $\#$dots the number of dots in U. ▲

Example 4.9. Below we have illustrated three examples $\overline{W}_1, \overline{W}_2$ and $\overline{W}_3$ of possible decorations of a web $\overline{w}$. The dotted web $\overline{W}_3$ is not a dotted basis web: the dot is not on the B-segment, there is a phantom circle and the two phantom seams are not

B-admissible. (We have indicated all of them using the word “bad”.)



Note hereby that we do not allow more than one dot per circle. From left to right, these are of degree -2, 2 and 0. ▲

Moreover, we define:

Definition 4.10. Given a dotted web U we define $\mathtt{np}\mathrm{circ}(U)$ to be the total number of anticlockwise (“negative”) oriented phantom circles. ▲

Example 4.11. For the three dotted webs displayed in Example 4.9 one has $\mathtt{np}\mathrm{circ}(\overline{W}_1) = \mathtt{np}\mathrm{circ}(\overline{W}_2) = 0$, but $\mathtt{np}\mathrm{circ}(\overline{W}_3) = 1$. ▲

Keeping track of the dot moving signs. In the following path from a point i to a point j are denoted by $i \to j$.

Definition 4.12. Given web u and two fixed points i, j on u which are connected by a path $i \to j$, we define

$$\mathrm{pedge}(i \to j) = \text{number of phantom edges attached to } i \to j.$$

We extend $\mathrm{pedge}(\square \to \square)$ additively for concatenations of distinct path. (Here and in the following $\square$ plays the role of a place holder.) ▲

Example 4.13. A blueprint example is provided by the web from Example 4.16, using the same choice of a circle C and points as therein. If we choose the corresponding path going around anticlockwise, then we have

$$\mathrm{pedge}(i \to j) = 3, \qquad \mathrm{pedge}(i \to k) = 5, \qquad \mathrm{pedge}(i \to l) = 8. \qquad ▲$$

In general, $\mathrm{pedge}(i \to j)$ depends on which path connecting i and j is chosen. But we note the following lemma which we need to make the sign assignment below in Subsection 4.2 well defined, and whose (very easy) proof is left to the reader (keeping Convention 4.4 in mind):

Lemma 4.14. The statistics defined in Definition 4.12 taken modulo 2 do not depend on the chosen path. ■

Keeping track of the topological signs. We call a situation as in the middle of (25) a *phantom loop*, no matter how many other phantom edges are in between the two trivalent vertices in this illustration. In particular, phantom loops in closed webs are always associated to a circle, namely the one they start/end. Thus, we can say whether they are *internal* or *external*, cf. (25).

Definition 4.15. Given a circle C in some web u and a fixed point i on it. Let L be an internal phantom loop attached to C. Then:

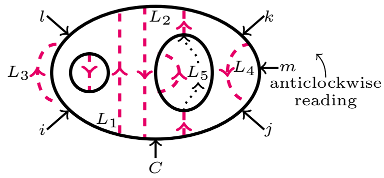
- ▷ The internal phantom loop L is said to be *i-positive*, if it points out of C first when reading from i anticlockwise.
- ▷ The internal phantom loop L is said to be *i-negative*, if it points into C first when reading from i anticlockwise. ▲

(Note that this is an asymmetric property heavily depending on i.)

We sometimes need to consider internal phantom loops attached to some circle C after removing all circles nested in C, cf. Example 4.16.

The notion of an *outgoing phantom edge* of some circle C of some web u is, by definition, a phantom edge in the exterior of C, counting the phantom loops in the exterior twice. (For example, the right nested circle in the web from Example 4.16 has two outgoing phantom edges and one phantom loop; the circle C in the very same example has two outgoing phantom edges given by the phantom loop L_3.)

Example 4.16. Consider the following web with five fixed points on a circle of it.

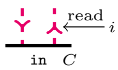
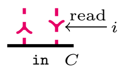


The circle C has four attached loops L_1, L_2, L_3 and L_4 as illustrated. If one removes all circles in C, then there is an extra loop formed by the two outgoing edges of the rightmost nested circle, which is L_5.

With respect to positive or negative phantom loops we have to read anticlockwise starting from the varies points. (Since we always read anticlockwise we just write "read" in the following.) One sees that L_1 is positive with respect to i and l, but negative with respect to j, k and m. For L_2 the situation is vice versa, and L_4 is m-positive, but negative for all other points. The phantom loop L_3 is exterior and we do not need to check whether its $\square$-positive or $\square$-negative.

The reader is encouraged to work out the situation for L_5. ▲

Definition 4.17. A local situation (local in the sense that such edges might close to exterior phantom loops, cf. (67)) of the following form

(30) i-positive:  , i-negative: 

is called an *outgoing phantom edge pair of* C. Hereby, `in` denotes the interior of C. The notion of these being i-positive respectively i-negative is defined by reading from a point i on C in the anticlockwise fashion, and then seeing if the i-closest of the two outgoing phantom edges points outwards or inwards, see (30). ▲

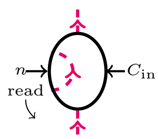
Definition 4.18. Given a web u and a circle C in u, and fix a point i on it, and ignore all of its nested circles. With respect to the chosen point i we define:

$$\mathtt{nploop}(C,i) = \text{number of } i\text{-}\mathtt{negative} \text{ internal phantom loops attached to } C$$
$$+ \text{ number of } i\text{-}\mathtt{negative} \text{ outgoing phantom edge pairs of } C.$$

(Again, this heavily depends on i.) ▲

Example 4.19. Back to Example 4.16, one has $\mathtt{nploop}(C,i) = \mathtt{nploop}(C,l) = 3$, $\mathtt{nploop}(C,j) = \mathtt{nploop}(C,k) = 4$ and $\mathtt{nploop}(C,m) = 3$. (The external phantom loop L_3 contributes to all of these as a negative outgoing phantom edge pair.)

Next, for the right nested circle C_{in} of the web in the very same example, i.e.



we get $\mathtt{nploop}(C_{\text{in}}, n) = 1$, since the two outgoing phantom edges are n-negative in the sense of (30) and the internal phantom loop is n-positive. ▲

Keeping track of the saddle signs. Let u be a stacked web. All three definitions below are with respect to the stacked web u, for which we assume that we are in the situation of a cup-cap pair involved in an ordinary or singular surgery (as e.g. in (22)) with fixed points i and j on them.

Definition 4.20. A phantom edge attached to the cup of the cup-cap pair is said to be *i-positive* respectively *i-negative* in cases

i-positive:  i-negative: 

For j instead of i we swap positive ones with negative ones. ▲

Definition 4.21. Then the *saddle type* $\mathtt{s}$type is defined to be

$$\mathtt{s}\mathrm{type}(i,j)=\mathtt{s}\mathrm{type}(j,i)=\begin{cases}0, & \text{if } \mathtt{p}\mathrm{edge}(i\to j) \text{ is even,}\\ 1, & \text{if } \mathtt{p}\mathrm{edge}(i\to j) \text{ is odd.}\end{cases}$$

(We write $\mathtt{s}\mathrm{type}=\mathtt{s}\mathrm{type}(i,j)=\mathtt{s}\mathrm{type}(j,i)$ for short.) ▲

Definition 4.22. The *i-saddle width* is defined to be

$$\mathtt{np}\mathrm{sad}(i)=\text{number of } i\text{-}\mathtt{negative} \text{ phantom edges attached to the cup.}$$

The *j-saddle width* is defined similarly, but using j-negative phantom edges. ▲

Remark 4.23. The asymmetries in Definitions 4.10, 4.18 and 4.22 comes from our "choice" of evaluation in (9) and (10). We stress this using $\mathtt{np}$ as a prefix. ▲

All of the above can be used for dotted webs as well, and we do so in the following.

Example 4.24. Consider the surgery from (22). Then one has $\mathtt{np}\mathrm{sad}(i)=2$ for a point i to the left of it, $\mathtt{np}\mathrm{sad}(j)=1$ for a point j to the right of it and $\mathtt{s}\mathrm{type}=1$. More general, the saddle type can be thought of as being 0 (usual) or 1 (singular) with the convention as in (31). ▲

4.2. **Combinatorics of foams.** Given $\vec{k}$, and two webs $u,v\in\mathrm{CUP}^{\vec{k}}$, then the set

$${}_u\mathbb{B}(\vec{k})_v=\mathbb{B}(uv^*)=\{\text{all dotted basis webs of shape } uv^*\}$$

will play the role of a combinatorial version of the cup foam basis.

The $\mathbb{K}$-linear structure. We start by defining the graded $\mathbb{K}$-vector space structure of the combinatorial model of the web algebra. Recall that $d_{\vec{k}}=\frac{1}{2}\mathtt{o}_{\vec{k}}$.

Definition 4.25. Given $u,v\in\mathrm{CUP}^{\vec{k}}$ for $\vec{k}\in\mathtt{bl}^{\circ}$ we set

$${}_u(c\mathfrak{W}_{\vec{k}})_v=\langle{}_u\mathbb{B}(\vec{k})_v\rangle_{\mathbb{K}}\{d_{\vec{k}}\},$$

that is the free $\mathbb{K}$-vector space on basis ${}_u\mathbb{B}(\vec{k})_v$. The *combinatorial web algebra $c\mathfrak{W}_{\vec{k}}$ for $\vec{k}\in\mathtt{bl}^{\circ}$* is the graded $\mathbb{K}$-vector space

$$c\mathfrak{W}_{\vec{k}}=\bigoplus_{u,v\in\mathrm{CUP}^{\vec{k}}}{}_u(c\mathfrak{W}_{\vec{k}})_v$$

with grading given on dotted basis webs via (29). The *combinatorial (full) web algebra $c\mathfrak{W}$* is the direct sum of all $c\mathfrak{W}_{\vec{k}}$ for $\vec{k}\in\mathtt{bl}^{\circ}$. These $\mathbb{K}$-vector spaces are equipped with the multiplication we define below. ▲

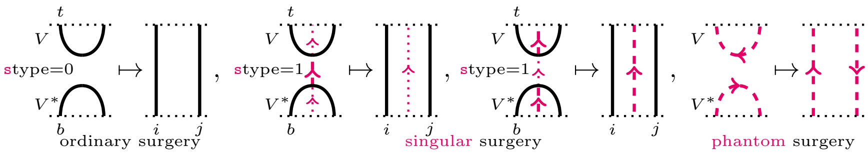
For later use in Section 5, we also define $c\mathfrak{W}_{\vec{K}}=\bigoplus_{\vec{k}\in\vec{K}}c\mathfrak{W}_{\vec{k}}$ for all $\vec{K}\in\mathtt{Bl}^{\circ}$. Clearly, a basis of $c\mathfrak{W}_{\vec{K}}$ is given by ${}_u\mathbb{B}(\vec{K})_v=\coprod_{\vec{k}\in\vec{K}}{}_u\mathbb{B}(\vec{k})_v$.

Note the crucial difference to Definition 3.2: The multiplicative structure of $\mathfrak{W}$ was naturally given by the foam 2-category $\mathfrak{F}$, but we had (or we will) to construct a basis for the algebra. In contrast, the basis of $c\mathfrak{W}$ is given, but we have to construct the multiplicative structure. That is what we are going to do next.

The multiplication without signs. We define again $\mathbf{Mult}^{c\mathfrak{W}}_{\vec{k}}$ and then take direct sums to obtain $\mathbf{Mult}^{c\mathfrak{W}}$. (We use notation similar to Subsection 3.1.)

To define $\mathbf{Mult}^{c\mathfrak{W}}_{\vec{k}}$ as in (21) we have to assign to each pair of dotted basis webs $U_{\mathrm{bot}}V^*$ and VU^*_{top} a sum of dotted basis webs of shape $u_{\mathrm{bot}}u^*_{\mathrm{top}}$. We do so by the usual inductive surgery process, where we first only change the shape (very similar to [10, Subsection 3.3]) and reconnect phantom seams.

Now, for any (ordinary, singular or phantom) cup-cap pair in the middle section V^*V we want to model the situation from (24) and we do the following local replacements, where we also fix four points on the webs in question, e.g.:

(31) 

If we are not in a situation as in exemplified in (31), then the multiplication is defined to be zero. (See also (24).)

Remark 4.26. The rules in (31) should be read locally in the sense that there might be several unaffected components "in between" as e.g. in (22). These do not matter for what happens to the shape, but the scalars will depend on the precise form as we will see below. ▲

We assume that we perform the local rules from (31) for the leftmost available cup-cap pair. Using these conventions, one directly checks that the rules presented below turn $c\mathfrak{W}$ into a graded algebra (not necessarily associative at this point).

Remark 4.27. There is a $\mathtt{Ↄ}$ shape within the multiplication procedure, cf. Example 4.30 (which essentially defines the notion of a $\mathtt{Ↄ}$ shape). Its mirror, the $\mathtt{C}$ shape, is ruled out by choosing the leftmost available cup-cap pair. Still, below we give the rule for this case as well, since it will follow that one can actually choose any cup-cap pair, see Corollary 4.2. This is in contrast to the type D situation as we will see later in Section 5. ▲

We are now ready to define the multiplication without signs. Hereby we say for short that we *put a dot on a circle C* and we mean that we put it on the segment of its base point $\mathtt{B}$. Clearly, the procedure from (31) in the ordinary or singular case either merges two circles into one, or it splits one into two.

The multiplication without signs is defined as follows, where we always perform the local procedure from (31). Hereby, if we write e.g. $C_{\square}$, then the corresponding circle should contain the point $\square$.

Merge. Assume two circles C_b and C_t are merged into a circle C_{af}.

▷ If both are undotted, then nothing additionally needs to be done.
▷ If one is undotted and the other one dotted, then put a dot on C_{af}.
▷ If both circles are dotted, then the result is zero.

Split. Assume a circle C_{be} splits into C_i and C_j.

▷ If C_{be} is undotted, then take the sum of two copies of the result, in one put a dot on C_i, in the other on C_j.
▷ If C_{be} is dotted, then put a dot on either C_i or C_j such that both are dotted.

In case of a $\supset$ or $\subset$ shape, remove all phantom circles from the result.

Phantom surgery. In this case nothing additionally needs to be done.

"Turning inside out". In the nested case (which we will meet below) the interior of some circle turns into the exterior of another circle after surgery and vice versa. In those cases reconnect the phantom seams until they are $\mathtt{B}$-admissible, cf. Example 4.29. (We will show in Subsection 6.1 that this can always be done by reconnection locally as illustrated in (59).)

The multiplication with signs. We have to define several notions to fix the signs for the multiplication. The signs will crucially depend on the number and the positions of the phantom edges. As in [10, Subsection 3.3] there will be *dot moving signs*, *topological signs*, *saddle signs* and, a new type, *phantom circle signs*. All of the "old" signs are more general notions than the ones in [10, Subsection 3.3] (due to the fact that we deal here with more flexible notions).

Following [10, Subsection 3.3], there are now several cases for the ordinary and singular surgeries depending on whether a merge or a split involves nested circles or not. In contrast, the phantom surgery only depends on whether the phantom cup-cap pair involved in the surgery form a closed circle.

Then the multiplication result from above is modified as follows. (We use the notation from above. Moreover, the meticulous reader might note that we have to use Lemma 4.14 to make sure that the signs are well-defined.) Below all points b, t, i, j are as in (31), and we write $\mathtt{B}_\square = \mathtt{B}(C_\square)$ for short.

Non-nested merge. In this case only one modification is made:

▷ If C_b is dotted and C_t undotted, then we multiply by

$$(-1)^{\mathtt{p}\mathrm{edge}(\mathtt{B}_b \to \mathtt{B}_{\mathrm{af}})}. \tag{32}$$

This sign is called the *(existing) dot moving sign*, and works in the same way if we exchange the roles of b and t.

Nested merge. Denote the inner of the two circles C_b and C_t by C_{in}. Then this case is modified by *(existing) dot moving signs*, *topological signs* and *saddle signs*:

▷ If both circles are undotted, then we multiply the result by

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}. \tag{33}$$

▷ If one of them is dotted, say C_b, then we multiply the result by

$$(-1)^{\mathtt{p}\mathrm{edge}(\mathtt{B}_b \to \mathtt{B}_{\mathrm{af}})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}. \tag{34}$$

Similarly for exchanged roles of C_b and C_t.

Non-nested split. Here, both cases are modified by *(new and existing) dot moving signs* and *saddle signs*:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathtt{p}\mathrm{edge}(i \to \mathtt{B}_i)}(-1)^{\mathtt{np}\mathrm{sad}(i)}, \tag{35}$$

and the one where C_j is dotted by

$$(-1)^{\mathtt{p}\mathrm{edge}(j \to \mathtt{B}_j)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}, \tag{36}$$

▷ If C_{be} is dotted, then we multiply the result by

$$(-1)^{\mathtt{p}\mathrm{edge}(\square \to \mathtt{B}_\square)}(-1)^{\mathtt{np}\mathrm{sad}(\square)}. \tag{37}$$

Here $\square \in \{i, j\}$ is such that $C_\square$ does not contain $\mathtt{B}_{\mathrm{be}}$.

Nested split, Ↄ shape. Let $\overline{W}$ denotes the dotted web after the surgery and before removing the phantom circles. Both cases are modified by *(new and existing) dot moving signs*, *topological signs* and *phantom circle sign*:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}, \tag{38}$$

and the one where C_j is dotted by

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}. \tag{39}$$

▷ If C_{be} is dotted, then we multiply with

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}. \tag{40}$$

Nested split, C shape. This is slightly different from the Ↄ shape:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}, \tag{41}$$

and the one where C_j is dotted by

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}. \tag{42}$$

▷ If C_{be} is dotted, then we multiply with

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}. \tag{43}$$

Phantom surgery. Only one modification has to be made, i.e.:

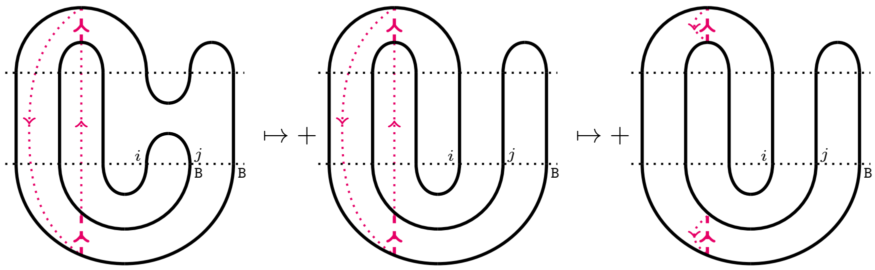
▷ If the phantom cup-cap pair forms a circle, then we multiply by -1.

"Turning inside out". No additional changes need to be done.

Remark 4.28. The careful reader might wonder why "turning inside out" does not give any signs. It does, but the signs are collected in what we call topological signs. In fact, the phantom seams need to be reconnected precisely because some non-trivial manipulation needs to be done as we will see in Subsection 6.2. ▲

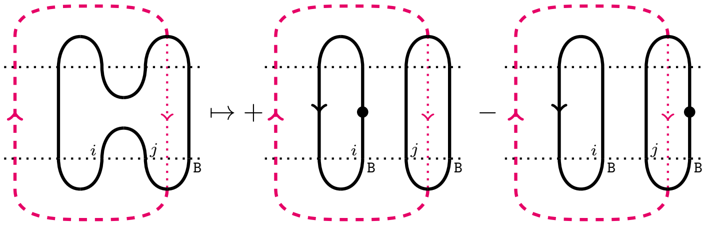
Examples of the multiplication. Next, let us give some examples. Note that we always omit the step called *collapsing*, cf. [10, (27)]. Moreover, the reader can find several examples in [10, Examples 3.15 and 3.16] of which we encourage her/him to convert to our situation here, see also Remark 3.6.

Example 4.29. As already in the "highest weight" setup, the most involved example is a nested merge, which now comes in plenty of varieties. Here is one:

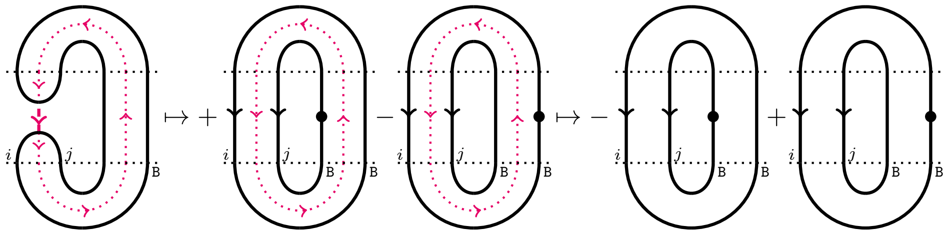


We have $\mathtt{s}\mathrm{type} = 0$, $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i) = 0$ and $\mathtt{np}\mathrm{sad}(i) = 0$, giving a positive sign. The last move, which never gives any signs, cf. Remark 4.28, reconnects the phantom seams to fit our choice of basis (which is formally defined in Subsection 6.1). ▲

Example 4.30. The basic splitting situation are the H and the Ↄ, which come in different flavors (depending on the various attached phantom edges). Here are two small blueprint examples illustrating some new phenomena which do not appear in the "highest weight" setup. First, an H:



Note that stype $= 0$ and the saddle sign npsad(i) is trivial in this case, but the rightmost summand acquires a dot moving sign. Next, a Ↄ:



In the first step the only non-trivial sign comes from stype $= 1$ (which gives the minus sign for the element in the middle). Furthermore, in the last step we have removed the phantom circle at the cost of an overall minus sign. (Again, the phantom seams show us what kind of non-trivial manipulation we need to do to bring the multiplication result in the form of our chosen basis.) ▲

4.3. The combinatorial realization.

Next, we define the combinatorial isomorphism comb. Morally it is given as in Figure 4.

Formally it is given by using our algorithmic construction (where we use subscripts to distinguish between the two "cup foam bases"):

Definition 4.31. Given $\overline{w} = uv^*$ with $u, v \in \mathrm{CUP}^{\vec{k}}$, we define a $\mathbb{K}$-linear map

$$\mathtt{comb}_u^v \colon \langle \mathbb{B}(\overline{w})_{c\mathfrak{W}} \rangle_{\mathbb{K}} \to \langle \mathbb{B}(\overline{w})_{\mathfrak{W}} \rangle_{\mathbb{K}}, \quad \overline{W} \mapsto f(\overline{W}),$$

by sending a dotted basis web $\overline{W}$ with phantom seam structure as obtained from Algorithm 1 (pieced together as in Subsection 6.1) to the foam $f(\overline{W})$ of the shape as obtained by using Algorithm 5 with the dot placement matched in the sense that $f(\overline{W})$ has a dot on the facet attached a segment which carries a base point B if and only if $\overline{W}$ has a dot on the very same segment.

Similarly, by taking direct sums, we define $\mathtt{comb}_{\vec{k}}$, $\mathtt{comb}_{\vec{K}}$ and comb. ▲

We will see in Section 6 that this $\mathbb{K}$-linear map extends to the isomorphisms of algebras as in Theorem 4.1.

Remark 4.32. We point out that one could upgrade Theorem 4.1 to include combinatorial description for the web bimodules $\mathbf{W}(u)$ as well. In principal, the steps one has to do are the same as for the algebras, but more different local situations as in (31) have to be considered, cf. [9, Subsections 4.2, 4.3 and 4.4] where the same was done in the "highest weight" setup. In order to keep the length of this paper in reasonable boundaries, we omit the rather involved details. ▲

5. Foams and the type D arc algebra

The purpose of this section is to give a "foamy presentation" of the type D arc algebra. If $\mathfrak{A}$ denotes the type D arc algebra with multiplication $\mathbf{Mult}^{\mathfrak{A}}$, then:

Theorem 5.1. There is an embedding of graded algebras

$$\mathtt{top}\colon \mathfrak{A} \hookrightarrow \mathfrak{W}.$$

(Similarly, denoted by $\mathtt{top}_\Lambda$, on each summand.) □

(Again, the all proofs are given in Subsection 6.3.)

In order to define $\mathtt{top}$, we have to sign adjust the multiplication structure of $\mathfrak{A}$, denoted by $\overline{\mathfrak{A}}$, which we do in Subsection 5.2, where we also give an isomorphism $\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}$. In fact, up to signs, the algebra $\overline{\mathfrak{A}}$ is defined almost verbatim as $\mathfrak{A}$, namely in the usual spirit of arc algebras as a $\mathbb{K}$-linear vector space on certain diagrams called *(marked) arc or circle diagrams*. (With markers displayed as ▎.)

Having $\overline{\mathfrak{A}}$ and $\mathtt{comb}\colon c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}$ from Section 4, it is almost a tautology to define an embedding $\overline{\mathtt{top}}\colon \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W}$. We do the latter in Subsection 5.3, but the picture to keep in mind how to go from $\overline{\mathfrak{A}}$ to $c\mathfrak{W}$ is provided by (another) "cookie-cutter strategy" as in Figure 5. This gives $\mathtt{top} = \mathtt{comb} \circ \overline{\mathtt{top}} \circ \mathtt{sign}$.

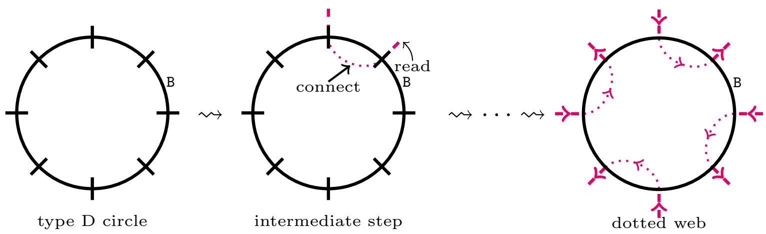


FIGURE 5. From cup diagrams to dotted webs: see a marked circle as a topological space, "cut" it from the rest, start at a fixed point B and go around anticlockwise, and connect neighboring markers via phantom seams, choosing the first to be oriented outwards.

Note hereby that we will not assume $\mathfrak{A}$ or $\overline{\mathfrak{A}}$ to be associative, and associativity follows immediately from Theorem 5.1 and Proposition 3.5:

Corollary 5.2. The multiplication $\mathbf{Mult}^{\overline{\mathfrak{A}}}$ is independent of the order in which the surgeries are performed, turning $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ into associative, graded algebras. ■

Remark 5.3. By [8, Example 6.7], the order of surgeries is important for $\mathfrak{A}$. In contrast, the order is not important for $\overline{\mathfrak{A}}$, cf. Example 5.18. The reason is that, by Theorem 5.1 and the above, $\overline{\mathfrak{A}}$ has a "foamy" multiplication rule which is "correct" in some sense. In fact, the signs are "easier" for $\overline{\mathfrak{A}}$ than for $\mathfrak{A}$. ▲

We conclude this section by sketching how our results extend to generalized arc and web algebras and the associated bimodule categories (which includes the connection to category $\mathcal{O}$).

5.1. **Recalling the type D arc algebra.** First, we briefly recall the definition of the type D arc algebra $\mathfrak{A}$. All of this closely follows [8], where the reader can find more details (and examples). Note that we simplify the notation here and focus on the type D arc algebra for two types of regular blocks in [8] and only allow cup and cap diagrams without rays. The general case can be recovered from this construction, see Remarks 5.19 and 5.20.

Weight combinatorics. We call $\mathbb{R} \times \{0\}$ the *dotted line*, cf. (44). We identify $\mathbb{Z}$ with the integral points on the dotted line, called *vertices*. (Hence, in contrast to the more flexible setup for the web algebra $\mathfrak{W}$, "points" i, j etc. are integers.)

A labeling of the vertices $\{1, \dots, 2K\}$ by the symbols in the set $\{\wedge, \vee\}$ is called a *weight* (of *rank* K). We identify weights of rank K with $2K$-tuples $\lambda = (\lambda_i)_{1 \leq i \leq 2K}$ with entries $\lambda_i \in \{\wedge, \vee\}$. We say that two weights λ, μ are in the same *(balanced) block* Λ if μ is obtained from λ by finitely many combinations of basic linkage moves, i.e. of swapping neighbored labels $\wedge$ and $\vee$, or swapping the pairs $\wedge\wedge$ and $\vee\vee$ at the first two positions. Thus, a block is fixed by the number K and the parity of the occurrence of $\wedge$ in the weight. We denote by $\mathtt{Bl}^{\circ}$ the set of blocks and denote by the rank of a block the rank of the weights in the block.

Diagram combinatorics. Following [8, Subsection 3.1] we define *cup diagrams* of rank K. This is a collection of crossingless arcs $\{\gamma_1, \dots, \gamma_K\}$, i.e. embeddings of the interval $[0, 1]$ into $\mathbb{R} \times [-1, 0]$, such that the collection of endpoints of all arcs coincides one-to-one with the set $\{1, \dots, 2K\}$. As in [8, Definition 3.5] we allow arcs whose interior can be connected to $(0, 0)$ in $\mathbb{R} \times [-1, 0]$ without crossing any other arcs to carry a decoration (this condition is called *admissibility*), in which case we call the arc *marked* and otherwise *unmarked*.

For short, we simply say *diagram* for any kind of cup, cap or circle diagram (where we also allow stacked circle diagrams, cf. Convention 3.3). Moreover, we use the evident notion of a circle C in a diagram D in what follows.

Example 5.4. Examples for admissible and non-admissible diagrams can be found in [8, Section 3]. We stress additionally that admissible diagrams will never have circles with markers that are nested in other circles. ▲

Beware 5.5. For illustration, we decorate marked arcs by **|**, which are the same as the dots in [8]. But since dots • already turn up in the foam picture (as e.g. illustrated in (4)), this notation had to be altered - our deepest apologies. Also note the difference in terminology, our marked arcs are called dotted in [8]. •

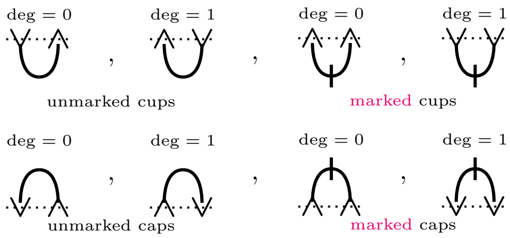
Reflecting a cup diagram d along the horizontal axis produces a *cap diagram* d^*, and putting such a cap diagram d^* on top of a cup diagram c of the same rank produces a *circle diagram*, denoted by cd^*, of the corresponding rank. As mentioned above, this is a special case of [8, Definition 3.2]. In all three cases we do not distinguish diagrams whose arcs connect the same points.

For a block Λ of rank K, a triple $c\lambda d^*$ consisting of two cup diagrams c, d of rank K and a weight $\lambda \in \Lambda$ is called an *oriented circle diagram* if all unmarked arcs connect an $\wedge$ and a $\vee$ in λ, while all marked arcs either connect two $\wedge$'s or two $\vee$'s, see e.g. (44). In this case we call λ the *orientation* of the diagram cd^*.

By $\mathbb{B}(\Lambda)$ we denote the set of all oriented circle diagrams (with orientations from Λ). Similarly, for cup diagrams c, d of rank K, we denote by ${}_c\mathbb{B}(\Lambda)_d$ the set of all oriented circle diagrams of the form $c\lambda d^*$ with $\lambda \in \Lambda$. In case ${}_c\mathbb{B}(\Lambda)_d = \emptyset$ we say that cd^* is *non-orientable* (by weights in Λ), otherwise it is called *orientable*.

Remark 5.6. By direct observation one sees that a circle diagram cd^* is orientable if and only if all of its circles have an even number of decorations on them. ▲

Further, we equip the elements of these sets with a degree by declaring that arcs have the degrees given locally via (which are added globally):

(44) 

Here, as in the following, the dotted line indicates $\mathbb{R} \times \{0\}$.

The *degree* of an oriented circle diagram is then in turn the sum of the degrees of all arcs contained in it, both in the cup and the cap diagram.

The type D *arc algebra as a* $\mathbb{K}$*-vector space.* Very similar as before we define:

Definition 5.7. Given a block Λ of rank K and cup diagrams c, d of rank K, we define the graded $\mathbb{K}$-vector space

$$ {}_c(\mathfrak{A}_\Lambda)_d = \langle {}_c\mathbb{B}(\Lambda)_d \rangle_{\mathbb{K}}, $$

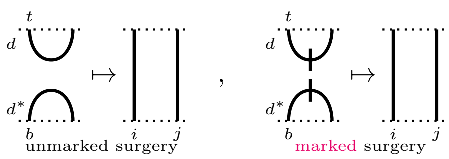
that is, the free $\mathbb{K}$-vector space on basis given by all oriented circle diagrams $c\lambda d^*$ with $\lambda \in \Lambda$. (The grading is hereby defined to be the one induced via (44).) The *type* D *arc algebra* $\mathfrak{A}_\Lambda$ *for* $\Lambda \in \mathtt{Bl}^\circ$ is the graded $\mathbb{K}$-vector space

$$ \mathfrak{A}_\Lambda = \bigoplus_{c,d} {}_c(\mathfrak{A}_\Lambda)_d, $$

with the sum running over all pairs of cup diagrams of rank K. Finally the *(full) type* D *arc algebra* $\mathfrak{A}$ is the direct sum of all $\mathfrak{A}_\Lambda$, where Λ varies over all blocks. The multiplication turning $\mathfrak{A}$ into a graded algebra is described in [8, Subsection 4.3] and we summarize it below. ▲

The type D *arc algebra as an algebra.* As usual, to define $\mathbf{Mult}^{\mathfrak{A}} \colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, we do it for fixed Λ of rank K. Hereby, the product of two basis elements $c_b\lambda d^*$ and $d'\mu c_t^*$ in $\mathfrak{A}_\Lambda$ is declared to be zero unless $d = d'$. Otherwise, to obtain the product of $c_b\lambda d^*$ and $d\mu c_t^*$, put $d\mu c_t^*$ on top of $c_b\lambda d^*$, producing a diagram that has d^*d as its middle piece. (In this notation, c_b, d, c_t are cup diagrams of rank K.)

For a cup-cap pair (possibly marked) in the middle section d^*d, which can be connected without crossing any arcs and such that to the right of this pair there are no marked arcs, we replace the cup and cap by using the (un)marked surgery:

(45) 

To avoid questions of well-definedness, we assume that we always pick the leftmost available cup-cap pair as above in what follows. One easily checks that this turns $\mathfrak{A}$ into a graded algebra (not necessarily associative at this point).

The surgery procedure itself is, as usual, performed inductively until there are no cup-cap pairs left in the middle section of the diagram. The final result is a $\mathbb{K}$-linear combination of oriented circle diagrams, all of which have $c_b c_t^*$ as the underlying circle diagram. The result in each step depends on the local situation, i.e. whether two components are merged together, or one is split into two. One then has to add orientations and scalars to the corresponding diagrams. Before we discuss how to obtain these, we need some notions to define the scalars for the multiplication.

For the next few definitions fix a cup or cap $\gamma = i \to j$ in a cup, cap, or circle diagram connecting vertex i and j.

Definition 5.8. With the notation as above define

$$\mathtt{utype}(\gamma) = \begin{cases} 0, & \text{if } \gamma \text{ is } \mathtt{marked}, \\ 1, & \text{if } \gamma \text{ is } \mathtt{unmarked}, \end{cases} \qquad \mathtt{mtype}(\gamma) = \begin{cases} 0, & \text{if } \gamma \text{ is unmarked}, \\ 1, & \text{if } \gamma \text{ is } \mathtt{marked}, \end{cases}$$

which we call the *unmarked* respectively *marked saddle type*. ▲

Definition 5.9. We define the *unmarked* and *marked distance* of $\gamma = i \to j$ by

$$\mathtt{ulen}_\Lambda(i \to j) = \mathtt{utype}(\gamma) \cdot |i - j|, \qquad \mathtt{mlen}_\Lambda(i \to j) = \mathtt{mtype}(\gamma) \cdot |i - j|.$$

We extend both additively for sequences of distinct cups and caps. ▲

Similar to Definition 4.5 we define:

Definition 5.10. The *base point* $\mathtt{B}(C)$ is the rightmost vertex on a circle C inside a circle diagram. ▲

We usually omit the subscript Λ in the following, and we also write $\mathtt{B} = \mathtt{B}(C)$ and $\mathtt{utype} = \mathtt{utype}(\gamma)$ etc. for short. The same works for diagrams D as well.

Let $i < j$ denote the left respectively right vertex for the cup-cap pair where the surgery is performed, see (45). Below D_{be} will denote the diagram before the surgery à la (45), while D_{af} will denote the diagram after the surgery. Moreover, a circle C is said to be (oriented) clockwise if the rightmost vertex $\mathtt{B}(C)$ it contains is labeled with $\vee$; otherwise it is said to be (oriented) anticlockwise.

Then the multiplication result is defined as follows. (If we write e.g. $C_\square$, then the corresponding circle should contain the vertex $\square$ as in (45). Moreover, as in Subsection 4.2, we write $\mathtt{B}_\square = \mathtt{B}(C_\square)$ for short.)

Merge. Assume two circles C_b and C_t are merged into a circle C_{af}.

- ▷ If both are anticlockwise, then apply (45) and orient the result anticlockwise.
- ▷ If one circle is anticlockwise and one is clockwise, then apply (45), orient the result clockwise and also multiply with

$$(-1)^{\mathtt{ulen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})}, \tag{46}$$

  where $C_\square$ (for $\square \in \{b, t\}$) is the clockwise circle, and $\mathtt{B}_r \to \mathtt{B}_{\mathrm{af}}$ is some concatenation of cups and caps connecting $\mathtt{B}_\square$ and $\mathtt{B}_{\mathrm{af}}$.
- ▷ If both circles are clockwise, then the result is zero.

Split. Assume a circle C_{be} splits into C_i and C_j. If, after applying (45), the resulting diagram is non-orientable, the result is zero. Otherwise:

- ▷ If C_{be} is anticlockwise, then apply (45) and take two copies of the result. In one copy orient C_i clockwise and C_j anticlockwise, in the other vice versa. Multiply the summand where C_i is oriented clockwise by

$$(-1)^{\mathtt{ulen}(i \to \mathtt{B}_i)}(-1)^{\mathtt{utype}}(-1)^i, \tag{47}$$

  and the one where C_j is oriented clockwise by

$$(-1)^{\mathtt{ulen}(j \to \mathtt{B}_j)}(-1)^i, \tag{48}$$

  using a notation similar to (46) with $i \to \mathtt{B}_i$ and $j \to \mathtt{B}_j$ appropriately chosen sequences of cups and caps connecting the indicated points.
- ▷ If C_{be} is clockwise, then apply (45) and orient C_i and C_j clockwise. Finally multiply with

$$(-1)^{\mathtt{ulen}(\mathtt{B}_{\mathrm{be}} \to \mathtt{B}_j)}(-1)^{\mathtt{ulen}(i \to \mathtt{B}_i)}(-1)^{\mathtt{utype}}(-1)^i. \tag{49}$$

Again, $\mathtt{B}_{\mathrm{be}} \to \mathtt{B}_j$ and $i \to \mathtt{B}_i$ are appropriate concatenations of cups and caps connecting the indicated points.

5.2. **A sign adjusted version.** The construction of the embedding from Theorem 5.1 splits into two pieces. First we define a *sign adjusted version* $\overline{\mathfrak{A}}$ of the type D arc algebra $\mathfrak{A}$ and show (the proof is again given in Subsection 6.3):

Proposition 5.11. There is an isomorphism of graded algebras

$$\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}.$$

(Similarly, denoted by $\mathtt{sign}_{\Lambda}$, on each summand.) □

The sign adjusted type D arc algebra $\overline{\mathfrak{A}}$ is then “easy” to embed into the web algebra $\mathfrak{W}$, which will be the purpose of the next subsection.

By definition, the algebra $\overline{\mathfrak{A}}$ has the same graded $\mathbb{K}$-vector space structure as given in Definition 5.7, but a multiplication modeled on the one from Section 4.

The sign adjusted type D *arc algebra as an algebra.* By definition, up to signs, the surgery procedures for both multiplications $\mathbf{Mult}^{\mathfrak{A}}$ and $\mathbf{Mult}^{\overline{\mathfrak{A}}}$ coincide. The multiplication procedure in contrast follows closely the one from Subsection 4.2. (As we will see in Subsection 5.3, it is the one from Subsection 4.2 specialized to the more “rigid” setup of the type D arc algebra.) That is, we change the steps in the multiplication as follows. As usual, all vertices b, t, i, j are as in (45). (And we also use the same notations and conventions as in 4.2 adjusted in the evident way.) Moreover, as in Remark 4.27, we give the rules for the $\mathtt{Ↄ}$ and the $\mathtt{C}$ shapes.

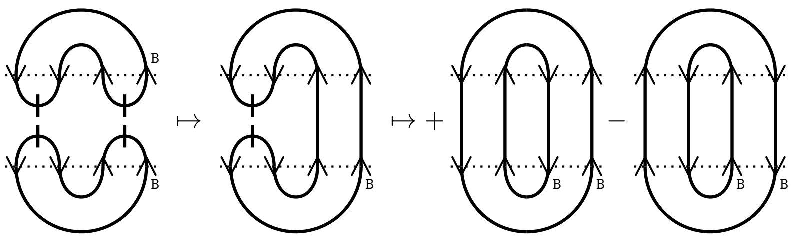
Non-nested merge, nested merge and non-nested split. Take the signs as in (32) to (37), and change:

$$(50)\qquad \begin{aligned} (-1)^{\mathtt{pedge}(\square\to\square)} &\rightsquigarrow (-1)^{\mathtt{mlen}(\square\to\square)}, \quad \text{(keep dot moving signs)} \\ \text{Rest} &\rightsquigarrow +1, \quad \text{(trivial other signs)}. \end{aligned}$$

Nested split, $\mathtt{Ↄ}$ and $\mathtt{C}$ shape. These cases are the most elaborate. That is, take the signs as in (38) to (43), and change:

$$(51)\qquad \begin{aligned} (-1)^{\mathtt{pedge}(\square\to\square)} &\rightsquigarrow (-1)^{\mathtt{mlen}(\square\to\square)}, \quad \text{(keep dot moving signs)} \\ (-1)^{\mathtt{stype}} &\rightsquigarrow (-1)^{\mathtt{mtype}}, \quad \text{(keep the saddle type)} \\ \text{Rest} &\rightsquigarrow +1, \quad \text{(trivial other signs)}. \end{aligned}$$

Example 5.12. One of the key examples why one needs to be careful with the multiplication in the (original) type D arc algebra $\mathfrak{A}$ is [8, Example 6.7], which is the case of the following $\mathtt{Ↄ}$ shape:

(52) 

Here we have used the sign adjusted multiplication. The reader should check that doing the $\mathtt{C}$ gives the same result for $\overline{\mathfrak{A}}$, but not for $\mathfrak{A}$. ▲

The isomorphism $\mathtt{sign}$. Next, we define the isomorphism $\mathtt{sign}$ from Proposition 5.11. We stress that $\mathtt{sign}$ is, surprisingly, quite easy.

To this end, recall that $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ have basis given by orientations on certain diagrams. The isomorphism $\mathtt{sign}$, seen as a $\mathbb{K}$-linear map $\mathtt{sign}\colon \mathfrak{A} \to \overline{\mathfrak{A}}$, will be given by rescaling each of these basis elements. In order to give the scalar, we first fix some diagram D and let $\mathbb{B}(D)$ denote the set of all possible orientations of D.

We will write coeff_D^C to indicate the contribution of a circle C inside D to the coefficient, which we define to be

$$\mathrm{coeff}_D^C(D^{\mathrm{or}}) = \begin{cases} 1, & \text{if } C \text{ is anticlockwise in } D^{\mathrm{or}}, \\ -(-1)^{\mathtt{B}(C)}, & \text{if } C \text{ is clockwise in } D^{\mathrm{or}}. \end{cases}$$

Here D^{or} denotes D together with a choice of orientation, which induces a orientation for the circle C.

Definition 5.13. We define a $\mathbb{K}$-linear map via:

$$\mathrm{coeff}_D\colon \langle \mathbb{B}(D)\rangle_{\mathbb{K}} \longrightarrow \langle \mathbb{B}(D)\rangle_{\mathbb{K}}, \qquad D^{\mathrm{or}} \longmapsto \left(\prod_{\text{circles } C \text{ in } D} \mathrm{coeff}_D^C(D^{\mathrm{or}}) \right) D^{\mathrm{or}}.$$

(With $\mathrm{coeff}_D^C(D^{\mathrm{or}})$ as above.) ▲

Thus, we can use Definition 5.13 to define $\mathbb{K}$-linear maps

$$(53) \qquad \mathtt{sign}_c^d\colon {}_c(\mathfrak{A}_\Lambda)_d \to {}_c(\overline{\mathfrak{A}}_\Lambda)_d, \qquad \mathtt{sign}_\Lambda\colon \mathfrak{A}_\Lambda \to \overline{\mathfrak{A}}_\Lambda, \qquad \mathtt{sign}\colon \mathfrak{A} \to \overline{\mathfrak{A}},$$

for every blocks Λ of some rank K and all cup and cap diagrams c, d of rank K.

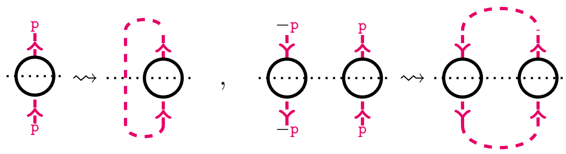
Note that, by construction, the maps in (53) are isomorphisms of graded $\mathbb{K}$-vector spaces. We will prove in Section 6 that the (two right) isomorphisms of graded $\mathbb{K}$-vector spaces from (53) are actually isomorphisms of graded algebras.

5.3. **The "foamy" embedding.** To define $\mathtt{top}$, we first define $\overline{\mathtt{top}}\colon \overline{\mathfrak{A}} \to c\mathfrak{W}$. Morally, it is defined as in Figure 5. The formal definition is not much more elaborate and is obtained from an algorithm, cf. Algorithm 2.

Before we give Algorithm 2, we need to close up the result from Figure 5:

Definition 5.14. Let $\vec{\mathtt{p}}$ denote a finite word in the symbols $\mathtt{p}$ and $-\mathtt{p}$ only, which alternates in these. Give two webs u, v such that $uv^* \in \mathrm{End}_{\mathbf{W}}(\vec{\mathtt{p}})$. Then we obtain from it a closed web by first connecting neighboring (counting from right to left) outgoing phantom edges of u and v separately, and then finally the remaining outgoing phantom edge of u with the one of v to the very left of uv^*. ▲

Example 5.15. Two basic examples of Definition 5.14 are:



Observe that one has a phantom edge passing from u to v if and only if $\vec{\mathtt{p}}$ has an odd number of symbols in total. Note also, as in particular the right example illustrates, it is crucial for this to work that $\vec{\mathtt{p}}$ alternates in the symbols $\mathtt{p}$ and $-\mathtt{p}$. ▲

For any circle diagram cd^* of rank K we obtain via Algorithm 2 webs

$$(54) \qquad \overline{w}(cd^*) \quad \text{and} \quad u(c), u(d) \in \mathrm{CUP}^{\vec{k}}, \quad \vec{k} \in \vec{K}$$

by considering the shape of $\overline{W}(cd^*)$. Hence, for an oriented circle diagram $c\lambda d^*$ with $\lambda \in \Lambda$ for a block Λ of rank K, we obtain a dotted basis web

$$(55) \qquad \overline{W}(c\lambda d^*) \in {}_{u(c)}\mathbb{B}(\vec{K})_{u(d)}$$

by putting a dot on each circle in $\overline{W}(cd^*)$ for which the corresponding circle in $c\lambda d^*$ is oriented clockwise. We call the dotted basis web from (55) *the dotted basis web*

input : a diagram D;
output : a dotted web $\overline{W}(D)$;

initialization, let $\overline{W}(D)$ be the empty web;
for *circles C in D* **do**
  if *C is marked* **then**
    run the procedure from Figure 5;
    add the corresponding web to $\overline{W}(D)$;
    remove the corresponding circle from D;
  else
    add C as a circle in a web to $\overline{W}(D)$;
    remove the corresponding circle from D;
  end
end
close the phantom edges as in Definition 5.14;

Algorithm 2: Turning a marked circle into a web.

associated to $c\lambda d^*$. (The careful reader might want to check that this is actually well-defined by observing that Algorithm 2 gives a well-defined result.)

This almost concludes the definition of $\overline{\texttt{top}}$, but we also need a certain sign which corrects the sign turning up for the nested splits, see Example 4.30.

Definition 5.16. For a stacked dotted web $\overline{W}$ we define

$$\texttt{n}\mathrm{pesci}(\overline{W}) = \text{number of } \texttt{anticlockwise} \text{ phantom (edge+seam) cirlces}$$
$$\text{touching the top dotted line of } \overline{W},$$

where phantom (edge+seam) circles are the circles obtained by considering phantom edges and seams. (These have a well-defined notion of being anticlockwise.) ▲

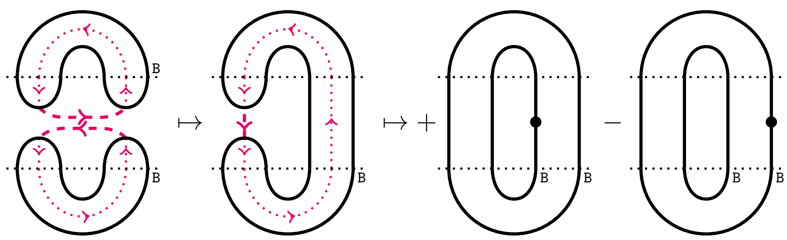
(Note that Definition 5.16 is again asymmetric in the sense that we only count anticlockwise phantom (edge+seam) circles.)

Definition 5.17. We define a $\mathbb{K}$-linear map via:

$$\overline{\texttt{top}}_c^d\colon \langle {}_c\mathbb{B}(\Lambda)_d\rangle_{\mathbb{K}} \to \langle {}_{u(c)}\mathbb{B}(\vec{K})_{u(d)}\rangle_{\mathbb{K}}, \quad c\lambda d^* \mapsto (-1)^{\texttt{n}\mathrm{pesci}(\overline{W}(c\lambda d^*))}\cdot \overline{W}(c\lambda d^*),$$

by using the notion from (55). Taking direct sums then defines $\overline{\texttt{top}}_\Lambda$ and $\overline{\texttt{top}}$. ▲

Example 5.18. Again, back to [8, Example 6.7]. The diagrams in (52) are sent to the following dotted basis webs:



Note the difference to the result calculated in Example 4.30, i.e. there is a phantom circle sign turning up, which is corrected by $\overline{\texttt{top}}$: For the leftmost stacked dotted basis web $\overline{W}_1$ we have $\texttt{n}\mathrm{pesci}(\overline{W}_1) = 1$ (since it has an anticlockwise phantom edge-seam circle at the top), the middle stacked dotted basis web $\overline{W}_2$ also has $\texttt{n}\mathrm{pesci}(\overline{W}_2) = 1$. In contrast, for the leftmost stacked dotted basis web $\overline{W}_3$ one has $\texttt{n}\mathrm{pesci}(\overline{W}_3) = 0$, and the last step is where the sign "goes wrong". ▲

The definition of $\mathtt{top}$ is now dictated:

$$\mathtt{top} = \mathtt{comb} \circ \overline{\mathtt{top}} \circ \mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}, \tag{56}$$

and similarly for $\mathtt{top}_c^d$ and $\mathtt{top}_\Lambda$. As usual, we show in Section 6 that $\mathtt{top}_\Lambda$ and $\mathtt{top}$ are isomorphisms of graded algebras.

5.4. **Its bimodules, category $\mathcal{O}$ and foams.** Comparing with [8] there are two generalizations that needs to be addressed from the point of view of this section.

Remark 5.19. The first generalization is towards more general blocks as they are defined in [8, Section 2.2]. This includes defining weights supported on the positive integers with allowed symbols from the set $\{\circ, \wedge, \vee, \times\}$. As long as one restricts to the situation of balanced blocks, i.e. blocks where the total combined number of $\wedge$ and $\vee$ symbols in a weights is even, the whole construction presented in this section can be used with one key difference: whenever a formula in any of the multiplications (or the isomorphism from Section 6.3) includes a power $(-1)^{\square}$ where $\square$ is some index of a vertex this must be interchanged with $(-1)^{\mathrm{p}(\square)}$ (with p as defined in [8, (3.12)]). The rest works verbatim. ▲

Remark 5.20. The second generalization is towards the *generalized type* D *arc algebra* $\mathfrak{C}$, and a *generalized web algebra* $g\mathfrak{W}$ topologically presenting it. The algebra $\mathfrak{C}$ is the algebra as defined in [8, Section 5] including rays in addition to cups and caps, while, as we explain now, the algebra $g\mathfrak{W}$ can be thought of as a "foamy type D version" of the algebra defined by Chen-Khovanov [6]:

The discussion in [8, Subsection 5.3] is the analog of the (type A) *subquotient construction* from [9, Subsection 5.1], and the analog of [9, Theorem 5.8] holds in the type D setup as well (by using [8, Theorem A.1]). Hereby, the main difference to [9, Subsection 5.1] lies in the fact that for the closure of a weight (as defined in [9, Definition 5.1]) one only uses additional symbols $\wedge$ to the right of the non-trivial vertices of the weight, similar to the type A situation, and one adds a total number of $\wedge$'s equal to the combined number of $\wedge$ and $\vee$ occurring in the weight.

Copying the same subquotient construction on the side of the web algebra (as done in the type A situation in [9, Subsection 5.1]) defines $g\mathfrak{W}$, which then can be seen to be similar to Chen-Khovanov's (type A) construction, cf. [9, Remark 5.7]. The corresponding *generalized foam* 2*-category* $g\mathfrak{F}$ can then, keeping Proposition 3.11 in mind, defined to be $g\mathfrak{W}$-**biMod**. ▲

Using Remarks 5.19 and 5.20, we see that everything from above generalizes to $\mathfrak{C}$, $g\mathfrak{W}$, $g\mathfrak{F}$ etc. In particular, one gets an embedding

$$g\mathtt{top}\colon \mathfrak{C} \hookrightarrow g\mathfrak{W}.$$

In fact, $g\mathtt{top}(\mathfrak{C})$ is an idempotent truncation of $g\mathfrak{W}$. Hence, we can actually define *type* D *arc bimodules* in the spirit of Definition 3.7 for any stacked circle diagram.

Taking everything together and recalling that $\mathfrak{C}$ is the algebra presenting the category $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$ (see [8, Theorem 9.1]), we can say that we get a "foamy presentation" of $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$. This is in fact a graded presentation with the grading being basically the Euler characteristic of foams, cf. (20).

6. Proofs

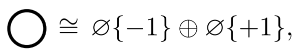
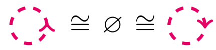
In this section we give all intricate proofs. There are essentially three things to prove: in the first subsection we construct the cup foam basis, in the second we show that $c\mathfrak{W}$ is a combinatorial model of the web algebra, and in the last we prove that the type D arc algebra embeds into $c\mathfrak{W}$.

Let us stress that we only consider (well-oriented) webs as in Convention 4.4, if not stated otherwise. For ill-oriented webs all foam spaces are zero and these also do not show up in the translation from type D to the foam setting. (Hence, there is no harm in ignoring them.)

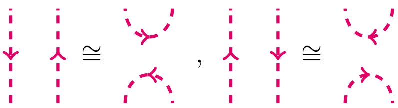
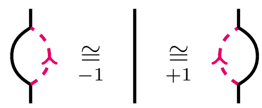
6.1. **Proofs of Propositions 2.14 and 3.11.** In this subsection we construct the cup foam basis and prove all the consequences of its existence/construction.

Proof of the existence of the cup foam basis. Our next goal is to describe isomorphisms among the morphisms of $\mathfrak{F}$ which we call *relations among webs.*

Lemma 6.1. There exist isomorphisms in $\mathfrak{F}$ realizing the following relations among webs. First, the *ordinary* and *phantom circle removals*:

(57)  (58) 

Second, the *phantom saddles* and the *phantom digon removal*:

(59)  (60) 

(The signs indicated in (60) are related to our choice of foams lifting these, see below.) There are isotopy relations of webs as well. □

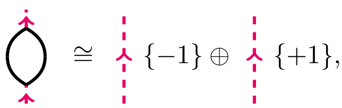
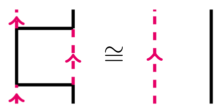
Note that each phantom digon is a phantom loop, but not vice versa since a phantom loop might have additional phantom edges in between its trivalent vertices.

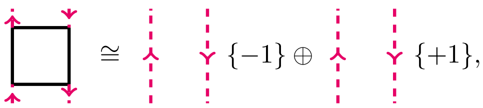
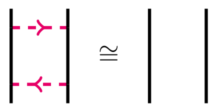
Proof. All of these can be proven in the usual fashion, i.e. by using the corresponding relation of foams and "cutting the pictures in half", see e.g. [10, Lemma 4.3].

For the relations among webs the corresponding relations of foams are:

▷ The foams corresponding to (57) are the ones in (3) and (5).
▷ The foams corresponding to (58) are the ones in (6) and (8).
▷ The relations (59) among webs are, by (8), lifted by phantom saddle foams.
▷ The foams corresponding to (60) are the ones in (17) and (18) (as well as their orientation reversed counterparts). ■

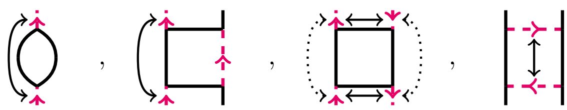
Lemma 6.2. The *digon* and *square removals*

(61)  (62) 

(63)  (64) 

are consequences of the relations among webs from Lemma 6.1. (There are various reoriented versions as well.) □

Proof. We indicate where we can apply phantom saddle relations (59):



(For (63), there is a choice where to apply the phantom saddles, cf. Example 6.6.) One can then continue using the phantom digon (60), and removing the circle (57) in case of (63). The corresponding foams inducing the relations from Lemma 6.1 then induce the isomorphisms in $\mathfrak{F}$ realizing the above relations among webs. ■

When referring to these relations among webs we fix the isomorphisms that we have chosen in the proof of Lemma 6.1 realizing these relations. (These induce the corresponding isomorphisms lifting the relations from Lemma 6.2, except for (63) where there is no preferred choice where to apply the phantom saddles.) We call these *evaluation foams*. Note hereby, as indicated in (60), the foams realizing the phantom digon removal might come with a plus or a minus sign, cf. Remark 2.10.

The point of the relation among webs is that they "evaluate closed webs":

Lemma 6.3. For closed web $\overline{w}$ there exists a sequence $(\phi_1, \dots, \phi_r)$ of relations among webs and some shifts $s \in \mathbb{Z}$ such that

$$\overline{w} \overset{\phi_1}{\cong} \cdots \overset{\phi_r}{\cong} \bigoplus_s \varnothing\{s\} \qquad (\text{in } \mathfrak{F}).$$

Such a sequence is called *an evaluation of* $\overline{w}$. □

Proof. By induction on the number n of vertices of $\overline{w}$.

If $n \leq 4$, the statement is clear by Lemmas 6.1 and 6.2. (Recall that we consider well-oriented webs only.) So assume that $n > 4$.

First, we can view a closed (well-oriented) web $\overline{w}$ as a planar, trivalent graph in $\mathbb{R}^2$ with all faces having an even number of adjacent vertices. Thus, by Euler characteristic arguments, $\overline{w}$ must contain at least a circle face (zero adjacent vertices), a digon face (two adjacent vertices) or a square face (four adjacent vertices). By (57) and (58) we can assume that $\overline{w}$ does not have circle faces. Hence, we are done by induction, since using (60), (61), (62), (63) or (64) reduces n. (Observe that these are all possibilities of what such digon or square faces could look like.) ■

We are now ready to prove Proposition 2.14. The main ingredient is the *cup foam basis algorithm* as provided by Algorithm 3.

input : a closed web $\overline{w}$ and an evaluation $(\phi_1, \dots, \phi_r)$ of it;
output : a sum of evaluation foams in $\mathcal{T}(\overline{w}) = 2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$;

initialization, let f_0 be the identity foam in $2\mathrm{End}_{\mathfrak{F}}(\overline{w})$;
for $k = 1$ **to** r **do**
  apply the isomorphism lifting ϕ_k to the bottom of f_{k-1} and obtain f_k;
end

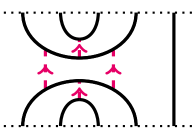
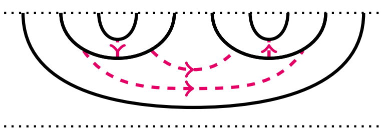
Algorithm 3: The cup foam basis algorithm.

(Hereby, if $\overline{w}$ has more than one connected component, it is important to evaluate nested components first and we do so without saying.)

Proof of Proposition 2.14. Given a closed web $\overline{w}$, by Lemma 6.3, there exists some evaluation of it which we fix.

Hence, using Algorithm 3, we get a sum of evaluation foams, all of which are $\mathbb{K}$-linear independent by construction. Thus, by taking the set of all summands produced this way, one gets a basis of $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$ by Lemmas 6.1 and 6.2.

For general webs u, v, we use the "bending trick". Define $b(u)$ to be

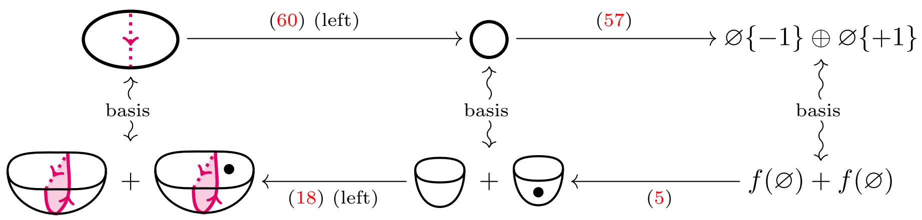
(65) $u =$  , $b(u) =$ 

Similarly for $b(v)$. Next, using the very same arguments as above, we can write down a basis for $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, b(u)b(v)^*)$. Bending this basis back proves the statement.

Scrutiny in the above process (keeping track of grading shifts) actually shows that everything works graded as well and the resulting basis is homogeneous. ■

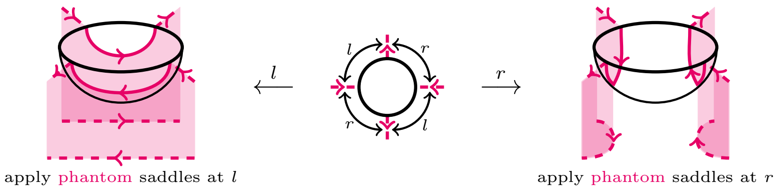
Remark 6.4. Indeed, almost verbatim, we also get a dual cup foam basis, called *cap foam basis*, i.e. a basis of $2\mathrm{Hom}(\overline{w}, \varnothing)$, which is dual in the sense that the evident pairing given by stacking a cap foam basis element onto a cup foam basis element gives ± 1 for precisely one pair, and zero else. ▲

Example 6.5. Let us consider an easy example, namely:



(Here we apply (60) to the left face.) Each summand is a basis element in $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$ with the signs depending on whether we apply (60) on the left or right face of $\overline{w}$. (Note that the lift of (60) gives an overall plus sign in this case.) ▲

Example 6.6. The following (local) example illustrates the choice we have to make with respect to the topological shape of our cup foam basis elements:



The two possible cup foam basis elements illustrated above (obtained by applying the phantom saddle relation among webs (59) to either the pair indicated by l=left or r=right, and then by applying (60) to remove the phantom digons) differ in shape, but after gluing an additional phantom saddle to the bottom of the left (or right) foam, they are the same up to a minus sign. ▲

Remark 6.7. Our proof of the existence of a cup foam basis using Algorithm 3 works more general for any kind of web algebra (as e.g. the one studied in [16]).

Note that Algorithm 3 heavily depends on the choice of an evaluation and it is already quite delicate to choose an evaluation such that one can control the structure constants within the multiplication. For general web algebras this is very complicated and basically unknown at the time of writing this paper, cf. [18] and [19]. This in turn makes e.g. the proof of the analog of Proposition 3.11 (as given below) much more elaborate for general web algebras. ▲

Presenting foam 2-categories. Next, we prove Proposition 3.11.

Proof of Proposition 3.11. Similar to [9, Proof of Proposition 2.43], we can define the 2-functor Υ, which is given by sending a web u to the $\mathfrak{W}$-bimodule $\mathbf{W}(u)$. Moreover, by following [9, Proposition 2.43], one can see that Υ is bijective on objects, essential surjective on morphisms and faithful on 2-morphisms.

To see fullness, fix two webs u, v. We need to compare $\dim(2\mathrm{Hom}_{\mathfrak{F}}(u, v))$ with $\dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u), \mathbf{W}(v)))$. The former is easy to compute using bending, since we already know that it has a cup foam basis by Proposition 2.14. In order to compute

$\dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u),\mathbf{W}(v)))$ we need to find the filtrations of the $\mathfrak{W}$-bimodules $\mathbf{W}(u)$ and $\mathbf{W}(v)$ by simple $\mathfrak{W}$-bimodules. (Here we need $\mathbb{K}=\overline{\mathbb{K}}$.)

This is done as follows. By using the cup foam basis for $\mathbf{W}(u)$, we see that $\mathbf{W}(u)$ has one simple $\mathfrak{W}$-sub-bimodule L_1 spanned by the cup foam basis element with a dot on each component corresponding to a circle in $\mathbf{W}(u)$ (called maximally dotted). Then $\mathbf{W}(u)/L_1$ has one $\mathfrak{W}$-sub-bimodule given as the $\mathbb{K}$-linear span of all cup foam basis elements with one dot less than the maximally dotted cup foam basis element. Continue this way computes the filtration of $\mathbf{W}(u)$ by simple $\mathfrak{W}$-bimodules. The same works verbatim for $\mathbf{W}(v)$ which in the end shows that

$$\dim(2\mathrm{Hom}_{\mathfrak{F}}(u,v))=\dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u),\mathbf{W}(v))).$$

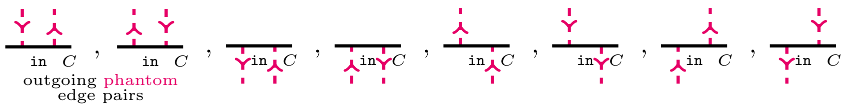
We already know faithfulness and Υ is, by birth, a structure preserving 2-functor, which finishes the proof. ■

Choosing a cup foam basis. Up to this point, having some basis was enough. For all further applications, e.g. for computing the multiplication explicitly, we have to fix a basis. That is what we are going to do next.

Note hereby, that, as illustrated in Example 6.6, the cup foam basis algorithm depends on the choice of an evaluation. Hence, what we have to do is to choose an evaluation for every closed web $\overline{w}$. Then, by choosing to bend to the left as in (65), we also get a fixed cup foam basis for $2\mathrm{Hom}_{\mathfrak{F}}(u,v)$ for all webs u,v.

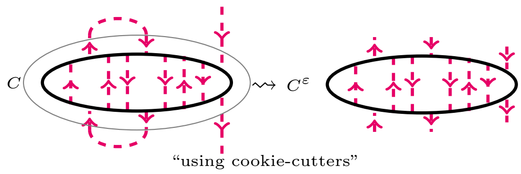
We start by giving an algorithm how to "evaluate" a fixed circle C in a web u into an "empty web", i.e. a web without ordinary edges. This will depend on a choice of a point i on C.

Before giving this algorithm, which we call the *circle evaluation algorithm*, note that one is locally always in one of the following situations (cf. Remark 2.5 and (30)):

(66) 

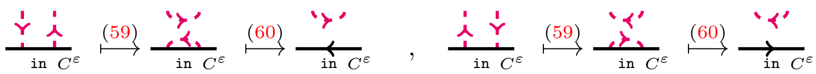
Again, `in` denotes the interior of the circle C. The two leftmost situations are called *outgoing phantom edge pairs*. We say, such a pair is closest to the point i, or *i-closest*, if it is the first such pair reading anticlockwise starting from i.

An ε-neighborhood C^{ε} of a circle C is called a *local neighborhood* if C^{ε} contains the whole interior of C and C^{ε} has no phantom loops in the exterior, e.g.

(67) 

Lemma 6.8. Let C be a circle with a point i on it. There is a way to evaluate C^{ε} while keeping i fixed till the end, using only (62) followed by (60) to detach outgoing phantom edges, and removing all internal phantom edges using (60) only. □

Proof. Local situations of the following forms



can always be simplified as indicated above. Thus, we can assume that C^{ε} does not have outgoing phantom edge pairs. But this means that C^{ε} is of the form as in (67) (right side), which than can be evaluated recursively using (60) only. ■

To summarize, we have two basic situations for C^{ε}'s:

▷ Outgoing phantom edge pairs, cf. (66) (left two pictures).
▷ Phantom digons, cf. (60).

Now, the *circle evaluation algorithm* is defined in Algorithm 4.

input : a circle C in a web u and a point i on it;
output : an evaluation $\phi = (\phi_1, \dots, \phi_r)$ of the circle C;

initialization; let $\phi = ()$;
while C^ε *contains two ordinary edges* **do**
  if C^ε *contains an outgoing phantom edge pair* **then**
    apply (59) to the i-closest such pair;
    add the corresponding relation among webs to ϕ;
  else
    remove any phantom digon not containing i using (60);
    add the corresponding relation among webs to ϕ;
  end
end
remove the circle containing i using (57) and all phantom circles using (58);
add the corresponding relations among webs to ϕ;

Algorithm 4: The circle evaluation algorithm.

Lemma 6.9. Algorithm 4 terminates and is well-defined. □

Proof. That it terminates follows by its very definition via Lemma 6.8.

To see well-definedness, observe that the used phantom digon removals (60) are "far apart" and hence, the corresponding foams realizing these commute by height reasons. Similarly, for the relations (57) and (58). This in total shows that the resulting evaluation foams are the same (as 2-morphisms in $\mathfrak{F}$). ■

Before we can finally define our choice of a cup foam basis, we need to piece Algorithms 3 and 4 together to the *evaluation algorithm*, see Algorithm 5.

input : a closed web $\overline{w}$ and a fixed point on each of its circles;
output : an evaluation $\phi = (\phi_1, \dots, \phi_r)$ of $\overline{w}$;

initialization; let $\phi = ()$;
while $\overline{w}$ *contains a circle* **do**
  if C *does not contain a nested circle* **then**
    take the circle C with its fixed point and apply Algorithm 4;
    add the result to ϕ;
    remove C from $\overline{w}$;
  else
    remove all remaining phantom circles using (58);
    add the corresponding relation among webs to ϕ;
  end
end

Algorithm 5: The evaluation algorithm.

Lemma 6.10. Algorithm 5 terminates and is well-defined. □

Proof. That the algorithm terminates is clear. That it is well-defined (i.e. that the resulting evaluation foams do not depend on the choice of which circles are taken first to be evaluated) follows because of the "cookie-cutter strategy" (cf. Example 6.13) taken within the algorithm which ensures that the resulting foam parts are "far apart" and thus, height commute. ■

Armed with these notions, we are ready to fix a cup foam basis.

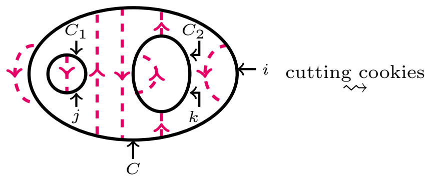
Definition 6.11. For any closed web $\overline{w}$ together with a fixed choice of a base point for each of its circles, we define the cup foam basis $\mathbb{B}(\overline{w})$ attached to it to be the evaluation foams turning up by applying Algorithm 3 to the evaluation of $\overline{w}$ obtained by applying Algorithm 5 to $\overline{w}$.

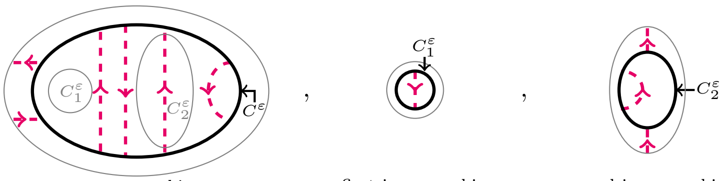
More generally, by choosing to bend to the left as in (65), we also fix a cup foam basis ${}_u\mathbb{B}_v$ for any two webs u, v. ▲

Note that, by Lemmas 6.9 and 6.10, the notion of $\mathbb{B}(\overline{w})$ is well-defined, while Proposition 2.14 guarantees that $\mathbb{B}(\overline{w})$ is a basis of $\mathcal{T}(\overline{w}) = 2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$.

Example 6.12. Depending on the choice of a base point, the cup foam basis attached to the local situation as in Example 6.6 gives either of the two results. ▲

Example 6.13. Our construction follows a "cookie-cutter strategy":





To this web the algorithm applies the "cookie-cutter strategy" by first cutting out C_1^{ε} and C_2^{ε} and evaluate them using Algorithm 4. (The resulting evaluation foams in the first case are as in Example 6.5; the reader is encouraged to work out the resulting evaluation foams in the second case.) Then its cuts out C^{ε} (with C_1 and C_2 already removed) and applies Algorithm 4 again. The resulting cup foam basis elements are then obtained by piecing everything back together. ▲

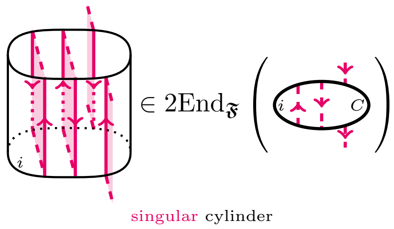
6.2. **Proof of Theorem 4.1.** The aim is to show that the combinatorial algebra defined in Section 4 gives a model for the web algebra. To this end, we follow the ideas from [10, proof of Theorem 4.18], but carefully treat the more flexible situation we are in.

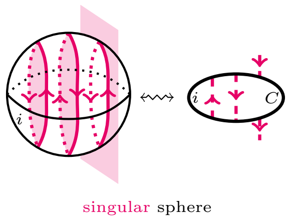
In particular, it is very important to keep in mind that we have fixed a cup foam basis, and we say a foam is (locally) of *cup foam basis shape* if it is topologically as the corresponding foam showing up in our choice of the cup foam basis.

Simplifying foams. First, we give three useful lemmas how to simplify foams. Before we state and prove these lemmas, we need some terminology.

Take a web u, a circle C in it and a local neighborhood C^{ε} of it, and consider the identity foam in $2\mathrm{End}_{\mathfrak{F}}(u)$. Then $C^{\varepsilon} \times [-1, +1]$ is called a *singular cylinder*. Blueprint examples are the foams in (5) or (13) (seeing bottom/top as webs containing C^{ε}), but also the situation in (68).

Similarly, a *singular sphere* in a foam is a part of it that is a sphere after removing all phantom edges/facets, cf. (3), (12) or (69).

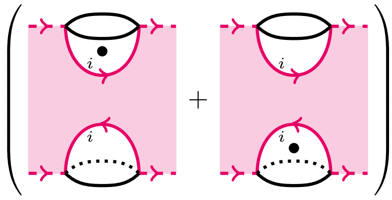
(68) 

(69) 

Next, the local situation (69) has an associated web with an associated circle given by "cutting the pictures in halves". (This is exemplified in (69), i.e. cutting the singular sphere around the equator gives the web on the right side.) Hence, from the bottom/top web for singular cylinders, and the webs associated to singular spheres we obtain the numbers as defined in Subsection 4.1. Hereby we use the points indicated above, which we also fix for Lemmas 6.14 and 6.15.

Now we can state the three main lemmas on our way to prove Theorem 4.1, namely the signs turning up by simplifying singular cylinders and spheres. For short, we say □-facets for foam facets touching the web segments containing the point □.

Lemma 6.14. Given a singular cylinder. Then we can simplify it to

$(-1)^{\mathtt{nploop}(C,i)} \cdot$ 

(There might be more or fewer attached phantom edges/facets as well - depending on the starting configuration.) Both dots sit on the i-facets and the coefficients are obtained from the associated circle C and base point i on it in the bottom/top web, and the cup and cap are in cup foam basis shape in case $i = \mathtt{B}$. □

Proof. This follows by a recursive "squeezing" procedure lowering the number of trivalent vertices attached to the circle C in question.

This recursive squeezing procedure should be read as starting from bottom/top of the singular cylinder, applying some foam relations giving a "thinner" singular cylinder on the next level of the recursion until one ends with a usual cylinder which we can cut using (5). (The pictures to keep in mind are (16) and (18).)

The main technical point is that we want to end with a cup and a cap of cup foam basis shape with respect to the point $\mathtt{B}$. Thus, the squeezing process depends on the particular way how to squeeze the singular cylinder.

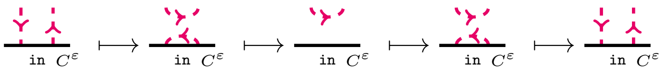
Luckily, an easy trick enables us to always end up with cup foam basis shapes with respect to the point $\mathtt{B}$. Namely, we squeeze the cylinder by first evaluating the bottom circle using Algorithm 4 and then "anti"evaluate the result by reading Algorithm 4 backwards. We obtain from this a sequence of relations among webs and a foam lifting them which correspond to a situation as in the lemma:

$$(\phi_1, \dots, \phi_r, \phi_r^{-1}, \dots, \phi_1^{-1}) \rightsquigarrow f \in 2\mathrm{End}_{\mathfrak{F}}(C^{\varepsilon}).$$

(Here ϕ_k^{-1} means the "other halves" of the foams chosen in Subsection 6.1.) By construction, the foam f is the identity and it remains to analyze the signs turning up by the foams lifting the concatenations $\phi_k \phi_k^{-1}$ of the relations among webs.

Now, Algorithm 4 gives us the following:

▷ Outgoing phantom edge pairs are squeezed using

(70) (8) and (18) $\overset{\text{lift}}{\rightsquigarrow}$ 

(Or its reoriented version.)

▷ Internal phantom loops are squeezed as in (70), but using (18) only.

Note that we only use (18), which gives a minus or plus sign depending on the local situation (cf. (60)), and (8), which always gives a minus sign. Carefully keeping track of these signs (e.g. the sign turns around for outgoing edge pairs compared to internal phantom loops since we use both, (18) and (8)) shows that we get the claimed coefficients.

By construction, the dots sit at the i-facet (since the facet with the point i is the last one to remove in Algorithm 4), and the cup and cap are of cup foam basis shape in case $i = \mathtt{B}$. In total, the lemma follows. ∎

Lemma 6.15. Given a singular sphere with a dot sitting on some i-facet. Then this singular sphere evaluated to

$$(-1)^{\mathrm{nploop}(C,i)}.$$

The coefficient is obtained from the associated web and its statistics. In case the singular sphere has not precisely one dot, then it evaluates to zero. □

Proof. In fact, the steps for the evaluation of singular spheres are the inverses of the steps for recursively squeezing singular cylinder. Thus, the first statement follows, mutatis mutandis, as in Lemma 6.14. The second statement is evident by the described squeezing procedure and (3). ∎

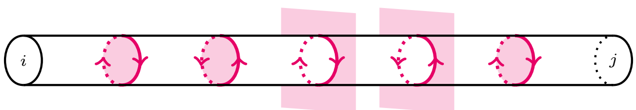
Lemma 6.16. In the setting of Lemma 6.14: if $f_{\square}$ denote the foams obtained by cutting the singular cylinder with respect to the points $\square \in \{i, j\}$, then

$$f_i = (-1)^{\mathrm{pedge}(i \to j)} \cdot f_j.$$

(Note that f_i and f_j have their dots on different facets and are of different shape.) □

Proof. Take the foam f_i and close its cap/cup at bottom/top such that the result are two singular spheres as in Lemma 6.15, that is, with dots on the i-facets. Hence, by Lemmas 6.14 and 6.15, the result is $+1$ times a foam which consists of parallel phantom facets only. Applying the same to f_j also gives a foam which consists of parallel phantom facets only but with a different sign: The bottom/top singular sphere are topologically equal (not necessarily to the ones for f_i, but equal to each other), but they have a different dot placement. One of them has a dot on the i-facet, one of them on the j-facet. Thus, after moving the dot from the i-facet to the j-facet (giving the claimed sign), the two created singular spheres can be evaluated in the same way and all other appearing signs cancel. ∎

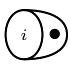
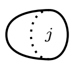
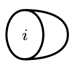
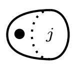
Next, a *singular neck* is a local situation of the form

(71) 

singular neck

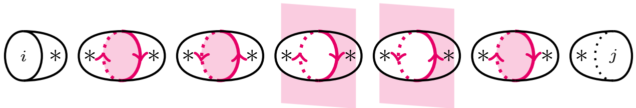
Again, for (71) one has an associated web, cup-cap-pair and points, and we get:

Lemma 6.17. Given a singular neck. Then we can simplify it to

$(-1)^{\mathrm{npsad}(i)} \cdot$   $+ (-1)^{\mathrm{stype}}(-1)^{\mathrm{npsad}(i)} \cdot$  

(There might be phantom facets in between as well - depending on the starting configuration.) The coefficients are obtained from the associated web. □

Proof. Assume that the singular neck has n singular seams in total. By using neck cutting (5) in between all of these (cutting to the left and the right of the two outermost singular seams as well) we obtain 2^{n+1} summands of the form



with the $*$ indicating that there might be a dot. By (9), (10), (12) as well as the second, reoriented, version of (12) we see that all of them but two die. The two remaining summands have a dot on i and j, respectively. The other dots coming from neck cutting (5) for these two are always placed on the opposite side of the singular seams in question (looking form i respectively j). So we are left with the foam we want plus a bunch of dotted theta foams and dotted singular spheres.

Next, removing now the theta foams and the singular spheres (using again the relations (9), (10), (12) as well as the second, reoriented, version of (12)) gives signs depending on the orientations of the singular seams. In total, the sign for the i-dotted respectively j-dotted component is given by $(-1)^{\mathtt{np}\mathrm{sad}(i)}$ respectively by $(-1)^{\mathtt{np}\mathrm{sad}(j)}$. But, clearly, $\mathtt{np}\mathrm{sad}(i) + \mathtt{np}\mathrm{sad}(j) = \mathtt{s}\mathrm{type}$ and we are done. ■

We stress that we abuse language: singular cylinders, spheres and necks might contain no phantom facets at all. The above lemma still work and all appearing coefficients are $+1$.

The combinatorics of the multiplication. First, we complete the definition of dotted basis webs. This is easy: copy almost word-by-word Algorithm 5 and then Definition 6.11. The resulting dotted basis webs correspond to our choice of cup foam basis from Definition 6.11. Now we prove Theorem 4.1:

Proof of Theorem 4.1. First note that the $\mathbb{K}$-linear maps $\mathtt{comb}_u^v$ defined in Definition 4.31 are isomorphisms of $\mathbb{K}$-vector spaces because dotted basis webs of shape uv^* are clearly in bijection with the cup foam basis elements in $\mathbb{B}(uv^*)$, and the latter is a basis of ${}_u(\mathfrak{W}_{\vec{k}})_v$.

These isomorphisms are homogeneous which basically follows by definition. That is, a cup foam basis element with some dots is, after forgetting phantom edges/facets, topologically just a bunch of dotted cups. Thus, by direct comparison of (4.8) and (19), we see that all these isomorphisms are homogeneous.

Hence, it remains to show that they intertwine the inductively given multiplication. To this end, similar to [10, Subsection 4.5], we distinguish some cases, with some new cases turning up due to our more flexible setting:

(i) **Non-nested merge**. Two non-nested components are merged.
(ii) **Nested merge**. Two nested components are merged.
(iii) **Non-nested split**. One component splits into two non-nested components.
(iv) **Nested split**. One component splits into two nested components.
(v) **Phantom surgery**. We are in the phantom surgery situation.
(vi) **"Turning inside out"**. Reconnection of phantom seams.

The cases (i) to (iv) are the main cases, and we will start with these. The other cases follow almost directly by construction (as we will see below).

We follow [10, Proof of Theorem 4.18] or [9, Proof of Theorem 4.7]: First, one observes that all components of the webs which are not involved in the multiplication step under consideration can be moved "far away" (and, consequently, can be ignored). Second, there will be three circles involved in the multiplication. After the multiplication process the resulting foam might not be of the topological form of a basis cup foam and some non-trivial manipulation has to be done:

(I) In all of the main cases, it might be necessary to move existing or newly created dots to the adjacent facets of the chosen base points.
(II) The sign $(-1)^{\mathrm{npcirc}(\overline{W})}$ only appears in the nested split case and comes precisely as stated.
(III) In all of the main cases, we cut (one or two) singular cylinders and remove (one or two) singular spheres.

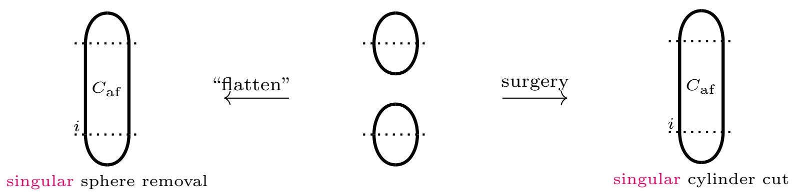
Note that the manipulation that we need to do in (I) is, on the side of foams, given by the dot moving relations (11). Clearly, these are combinatorially modeled by the (old and new) dot moving signs, and we ignore these in the following.

Regarding (II): Phantom circles correspond to singular phantom cups which one creates at the bottom of a cup foam basis element and needs to be removed. By (17) and its reoriented counterpart, we see that only anticlockwise oriented phantom seams contribute while removing it, giving precisely $(-1)^{\mathrm{npcirc}(\overline{W})}$. This sign can only turn up for the nested split, since the corresponding phantom circles have to in the inside of the circle resulting from the surgery (which rules out the non-nested cases as well as the nested merge).

Note also that the operation from (III) is more complicated than the corresponding ones in [10, Proof of Theorem 4.18] or [9, Proof of Theorem 4.7], but ensures that the resulting foam is of cup foam basis shape.

Hence, it remains to analyze what happens case-by-case. The procedure we are going to describe in detail will always basically be the same for all cases. Namely, in order to ensure that the result is of cup foam basis shape, we cut singular cylinders which correspond to circles after the surgery in the way described in Lemma 6.14. Since we started already with a foam which is of cup foam basis shape, this creates singular spheres corresponding to circles before the surgery. We will call both of these simplification moves. The total sign will depend on the difference between the signs picked up from the simplification moves.

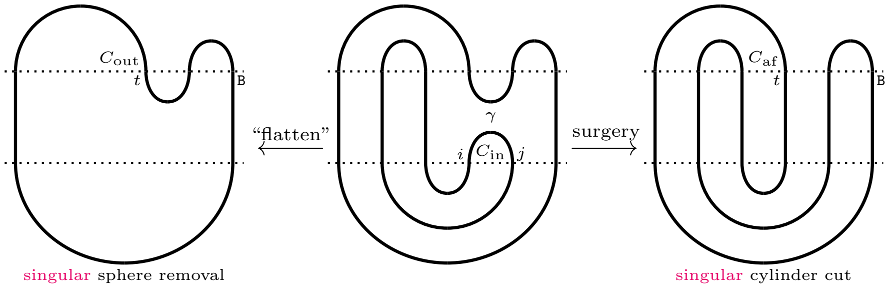
Non-nested merge. Here the picture (for arbitrary attached phantom edges, topological situations and orientations):

(72)


Above in (72), we have illustrated the circles (and points) where we perform the simplification moves. Note hereby that we can “flatten” the singular saddle and the singular sphere removal actually takes place in the left picture in (72).

One now directly observes that both simplification moves can be performed with respect to the same circle C_{af} and point i on it. Thus, in this case, by Lemmas 6.14 and 6.15, all obtained signs cancel and we are left with no signs at all (as claimed). ▶

Nested merge. The picture is as follows (again, for arbitrary attached phantom edges, topological situations and orientations. Note hereby that we can again “flatten” the situation (also vice versa as in the non-nested merge case) because we can grab the bottom of C_{in} in the created singular sphere and pull it straight to the top, and we get:

(73) 

(For later use, we have also illustrated the circle C_{in} and point i for which we read off the sign in the combinatorial model, as well as the cup-cap pair γ of the surgery and another point t which will play a role.) Thus, by using Lemmas 6.14 and 6.15, we end up with the sign

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},\mathtt{B})}. \tag{74}$$

Hence, it remains to rewrite the sign from (74) in terms of the circle C_{in} and the chosen point i as in (73).

Claim : In the setup from (73) one has

$$\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)+\mathtt{s}\mathrm{type}=\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)+\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)+\mathtt{np}\mathrm{sad}, \tag{75}$$

where $\mathtt{s}$type and $\mathtt{np}$sad are to be calculated with respect to the points i,j. ▷

Proof of the claim : We prove the claim inductively, where the basic case with no phantom edges whatsoever is clear.

If we attach a phantom edge to the situation from (73) which does not touch neither C_{in} nor γ, then, clearly, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)$ changes in the same way as $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)$ does, while everything else stays the same. Hence, the equation (75) stays true.

Similarly, if we attach a phantom edge which does not touch neither C_{out} nor γ, then, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)$ changes in the same way as $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)$ does. To see this, note the attached phantom edge is in the internal of C_{af} if and only if it is in the external of C_{in}. If its in the internal of C_{af}, then it is t-positive if and only if its corresponding outgoing phantom edge pair for C_{in} is i-negative. Similarly, when its in the external of C_{af}. Everything else stays the same and thus, (75) stays true.

Next, if we attach a phantom edge touching C_{out} and C_{in}, but not γ, then the equation (75) still stays true. To see this, note that such a phantom edge will form an outgoing pair for C_{in}, but an internal phantom loop for C_{out} and two internal phantom loops for C_{af}. We observe that precisely one of the new internal phantom loops for C_{af} are counted since they come by splitting the new internal phantom loop of C_{out} into two pieces, one pointing into C_{af}, one out. Hence, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)$ always grows by one. Because the new phantom loop for C_{out} is t-positive if and only if its corresponding outgoing phantom edge pair for C_{in} is i-negative, we see that either $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)$ or $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)$ grow by one. In total, (75) stays true.

Last, the remaining case where γ is affected can be, mutatis mutandis, treated as the preliminary case: Attaching a single phantom edge to C_{out}, we have that either $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)$ or $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i)$ gets one bigger, while $\mathtt{s}$type always gets one bigger. The difference to the preliminary case is that C_{af} now only gets one new internal phantom loop which contributes to $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)$ if and only if it does not contribute to $\mathtt{np}$sad. Again, the equation (75) stays true.

Thus, the claim is proven. ▶

Now, by Lemma 6.16, we have

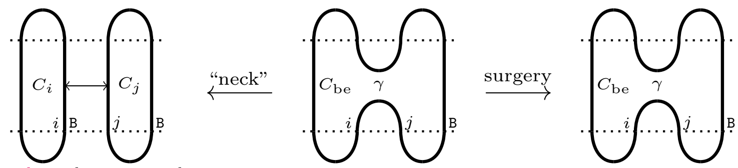
$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},\mathtt{B})} = (-1)^{\mathrm{pedge}(t\to\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)},$$

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},\mathtt{B})} = (-1)^{\mathrm{pedge}(t\to\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)}.$$

Hence, by the above claim, we get the same signs on both sides. Similar for the horizontal mirror of the situation from (73). ▶

Non-nested split. Now the situation looks as follows:

(76)



singular sphere removals

(Again, (76) should be seen as a dummy for the general case.) In contrast to the merges, we can not "flatten" the picture since there is a singular neck appearing around γ, and the singular sphere has to be removed in the leftmost situation in (76) by taking the singular neck into account using Lemma 6.17.

Moreover, we perform two singular cylinder cuts of which precisely two summands survive. Namely one with the dot on the i-facet, one with the dot on the j-facet. Moreover, the singular sphere in these two cases will have its dot on the j-facet respectively on the i-facet. By Lemmas 6.14 and 6.15 we obtain the two signs:

$$\begin{aligned}&(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},j)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)},\\ &(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}.\end{aligned}\tag{77}$$

In fact, by cutting the singular neck around γ we can rewrite

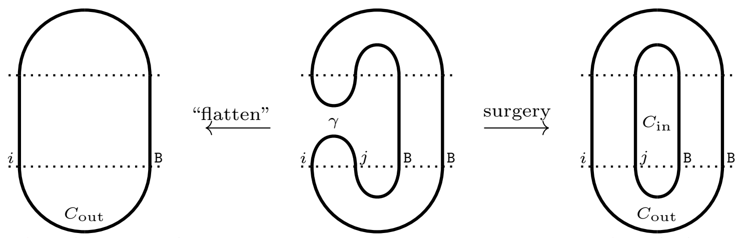
$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},j)} = (-1)^{\mathtt{np}\mathrm{sad}(i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)},$$

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},i)} = (-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}.$$

Now, rewriting this in terms of the basis points (using Lemma 6.16), and putting it together with (77), shows that this case works as claimed. ▶

Nested split. The $\mathtt{Ↄ}$ shape is (with "flatten" as for the non-nested merge):

(78)



singular sphere removals

singular cylinder cuts

Here we use a notation close to the one from the nested merge and the non-nested split. The difference to the nested merge is that our task is way easier now. In fact, by Lemmas 6.14 and 6.15 we are basically done since the contributions of the C_{out} related simplifications almost cancel. (We also use Lemma 6.16 to rewrite everything in terms of the point j. Note also that $(-1)^{\mathrm{pedge}(i\to j)} = (-1)^{\mathtt{s}\mathrm{type}}$ with $\mathtt{s}$type taken at γ.) The only thing which we can not see in the leftmost picture in (78) are phantom circles which contribute the factor $(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}$. The case of a $\mathtt{C}$ shape works similar and is omitted. ▶

It remains to check cases (v) and (vi). Case (v) is clear (the one special situation comes because we need to apply (8)). In the remaining case (vi) one would a priori expect some signs turning up, but these are already build into the four main cases.

The rest then works verbatim as in [10, Proof of Theorem 4.18]. ■

6.3. **Proofs of Proposition 5.11 and Theorem 5.1.** Last, we prove our foamy realization of the type D arc algebra $\mathfrak{A}$.

Adjusting signs. We need the following simple observations:

Lemma 6.18. Let $i \to j$ be a cup or cap connecting i and j, then

(79) $$i \equiv (j+1) \bmod 2.$$

Further, we also have

(80) $$\mathtt{ulen}(k \to l) + \mathtt{mlen}(k \to l) \equiv (k+l) \bmod 2,$$

where k and l are two points connected by a sequence $k \to l$ of cups and caps. □

Proof. The equation (79) is evident, while (80) follows by noting that summing up the length of all cups and caps in the sequence $\mathtt{ulen}_\Lambda$ will only contribute to the unmarked ones, while $\mathtt{mlen}$ only contributes to the marked ones. ■

Proof of Proposition 5.11. The maps coeff_D from 5.13 are, by birth, homogeneous and $\mathbb{K}$-linear for all diagrams D.

Hence, as in the proof for [9, Proposition 4.15], it remains to show that the maps coeff_D successively intertwine the two multiplication rules for $\mathfrak{A}$ and $\overline{\mathfrak{A}}$. Consequently, we compare two intermediate multiplications steps in the following fashion:

(81) $$\begin{tikzcd} D_m \arrow[r, "\mathbf{Mult}^{\mathfrak{A}}_{D_m,D_{m+1}}"] \arrow[d, "\mathrm{coeff}_{D_m}"'] & D_{m+1} \arrow[d, "\mathrm{coeff}_{D_{m+1}}"] \\ D_m \arrow[r, "\mathbf{Mult}^{\overline{\mathfrak{A}}}_{D_m,D_{m+1}}"'] & D_{m+1}, \end{tikzcd}$$

where we denote by $\mathbf{Mult}^{\mathfrak{A}}_{D_m,D_{m+1}}$ and $\mathbf{Mult}^{\overline{\mathfrak{A}}}_{D_m,D_{m+1}}$ the surgery procedure rules as indicated in the two multiplications. Thus, the goal is to show that each such diagrams, i.e. for each appearing D_m and D_{m+1}, commutes.

As usual, this will done by checking the four possible cases that appear in the surgery procedure. But before we start, note that (80) immediately implies

(82) $$(-1)^{\mathtt{ulen}(k\to l)} = (-1)^{\mathtt{mlen}(k\to l)} \cdot (-1)^k \cdot (-1)^l$$

which we will use throughout below.

Non-nested merge. Assume that circles C_b and C_t are merged into a circle C_{af}. If both circles are oriented anticlockwise, then both multiplication rules yield a factor of $+1$ and $\mathrm{coeff}^{C_b}_{D_m} = \mathrm{coeff}^{C_t}_{D_m} = \mathrm{coeff}^{C_{\mathrm{af}}}_{D_{m+1}} = +1$ as well. The claim follows.

Assume now that the circle $C_\square$ for $\square \in \{b,t\}$ is oriented clockwise and the other circle is oriented anticlockwise. The multiplication in $\mathfrak{A}$ gives the factor $(-1)^{\mathtt{ulen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})}$, while the multiplication in $\overline{\mathfrak{A}}$ yields $(-1)^{\mathtt{mlen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})}$. We check that

$$\mathrm{coeff}^{C_{\mathrm{af}}}_{D_{m+1}} \cdot (-1)^{\mathtt{ulen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})} = -(-1)^{\mathtt{B}_{\mathrm{af}}} \cdot (-1)^{\mathtt{ulen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})}$$
$$\overset{(82)}{=} -(-1)^{\mathtt{B}_\square} \cdot (-1)^{\mathtt{mlen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})} = \mathrm{coeff}^{C_b}_{D_m} \cdot \mathrm{coeff}^{C_t}_{D_m} \cdot (-1)^{\mathtt{mlen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})},$$

which proves the claim in this case.

If both circles are oriented clockwise, the both multiplications are zero. ▶

Nested merge. Due to the definition of $\overline{\mathfrak{A}}$, the signs in the nested merge are exactly as in the non-nested case. Thus, it is verbatim as the non-nested merge. ▶

Non-nested split. Assume that a circle C_{be} is split into circles C_i and C_j at a cup-cap pair connecting i and j.

Assume first that C_{be} is oriented anticlockwise. Note that, by admissibility, it must hold that $\mathtt{u}\mathrm{type} = 1$. Hence, the summand where C_i is oriented clockwise and C_j is oriented anticlockwise obtains a factor $(-1)^{\mathtt{u}\mathrm{len}(i \to \mathtt{B}_i)}(-1)(-1)^i$ for $\mathfrak{A}$. In contrast, the summand only gains the factor $(-1)^{\mathtt{m}\mathrm{len}(i \to \mathtt{B}_i)}$ in $\overline{\mathfrak{A}}$. Thus, we check

$$\begin{aligned} &\mathrm{coeff}^{C_i}_{D_{m+1}} \cdot \mathrm{coeff}^{C_j}_{D_{m+1}} \cdot (-1)^{\mathtt{u}\mathrm{len}(i \to \mathtt{B}_i)} \cdot (-1) \cdot (-1)^i \\ &= -(-1)^{\mathtt{B}_i} \cdot (-1)^{\mathtt{u}\mathrm{len}(i \to \mathtt{B}_i)} \cdot (-1) \cdot (-1)^i \\ &\overset{(82)}{=} (-1)^{\mathtt{m}\mathrm{len}(i \to \mathtt{B}_i)} = \mathrm{coeff}^{C_{\mathrm{be}}}_{D_m} \cdot (-1)^{\mathtt{m}\mathrm{len}(i \to \mathtt{B}_i)}, \end{aligned}$$

which proves the claim. The second summand is done completely analogous by using (79) to see that the factor in $\mathfrak{A}$ is equal to $(-1)(-1)^{\mathtt{u}\mathrm{len}(j \to \mathtt{B}_j)}(-1)^j$.

The clockwise case follows, mutatis mutandis, as the anticlockwise case by incorporating the two additional non-trivial coefficients. ▶

Nested split. Assume the same setup as in the non-nested split case. Note that, due to the definition of the multiplication in $\mathfrak{A}$, we are always looking at the situation of the $\supset$ shape here. Thus, if we assume that the circle C_{be} is oriented anticlockwise, the summand with C_i oriented clockwise and C_j oriented anticlockwise gains the factors $(-1)^{\mathtt{u}\mathrm{len}(i \to \mathtt{B}_i)}(-1)^{\mathtt{u}\mathrm{type}}(-1)^i$ in $\mathfrak{A}$ and $(-1)^{\mathtt{m}\mathrm{len}(i \to \mathtt{B}_i)}(-1)^{\mathtt{m}\mathrm{type}}$ in $\overline{\mathfrak{A}}$. Since

$$\mathtt{u}\mathrm{type} \equiv (\mathtt{m}\mathrm{type} + 1) \bmod 2$$

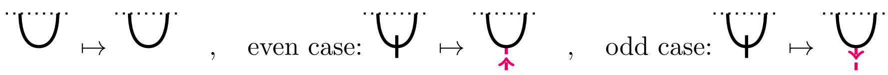
the claim follows by the same calculation as in the non-nested case.

For the other summand there is no difference to the non-nested split, and the case of C_{be} being oriented clockwise is also derived analogously. ▶

This in total proves the proposition. ■

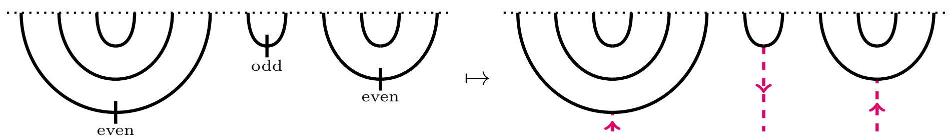
The embedding of the D *arc algebras into the web algebras.* Recall that for the type D arc algebra the multiplication is zero in case the result is non-orientable, i.e. has an odd number of markers on some component, see Remark 5.6. Hence, the first thing to make sure is that the isomorphism $\overline{\mathtt{top}}$ preserves this. This is the purpose of the following definition and two lemmas.

Definition 6.19. To a cup diagram c we associate a web $\mathtt{u}(c)$ using the rule



where we say a marked cup is even respectively odd if it has an even respectively odd number of marked cups to its right. ▲

Example 6.20. Here is one blueprint example:

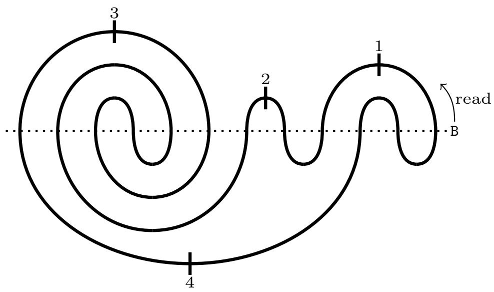


(Recall that we do not orient ordinary web components.) ▲

Lemma 6.21. Given two cups diagrams c, d. Then cd^* is orientable if and only if $\mathtt{u}(c)\mathtt{u}(d)^*$ is a (well-oriented) web (cf. Convention 4.4). □

Proof. If $\mathtt{u}(c)\mathtt{u}(d)^*$ is a web, then every face has an even number of adjacent trivalent vertices which translate to say that cd^* has an even number of markers per connected component, and we are done.

Conversely, assume (without loss of generality) that cd^* has only one circle C, and that C is orientable. If C is not marked, then we are done since the associated circle in $\mathtt{u}(c)\mathtt{u}(d)^*$ is an ordinary circle. Otherwise, follow C from $\mathtt{B}$ onwards in the anticlockwise fashion. By admissibility, going around C in this way will always pass the marked caps in d^* and then the marked cups in c. This can be best seen via example (we leave it to the reader to make this rigorous):

(83) 

Next, the number of markers on C is even since cd^* is orientable. This together with the above observation (and recalling that taking * on webs reverses the orientation of phantom edges) ensures that all neighboring phantom edge pairs of $\mathtt{u}(c)\mathtt{u}(d)^*$ are well-attached, and that $\mathtt{u}(c)\mathtt{u}(d)^*$ has an even number of trivalent vertices, i.e. $\mathtt{u}(c)\mathtt{u}(d)^*$ is well-oriented. ■

Lemma 6.22. Given two cups diagrams c, d. Then, up to closing of the phantom edges, we have: $\mathtt{u}(c) = u(c)$ and $\mathtt{u}(d) = u(d)$. (With $u(\square)$ as in (54).) □

Proof. By comparing (5) and (83). ■

Proof of Theorem 5.1. By Theorem 4.1, Proposition 5.11, and summation, it remains to show that $\overline{\mathtt{top}}_\Lambda$ is an embedding of graded algebras.

To this end, fix $\Lambda \in \mathtt{Bl}^\circ$ of rank K. There are four things to be checked, where c_b, d, d', c_t are always cup diagrams of rank K and λ, μ are in Λ:

(1) The $\mathbb{K}$-linear maps $\overline{\mathtt{top}}_{c_b}^{c_t}$ are homogeneous embeddings of $\mathbb{K}$-vector spaces.
(2) We have $\mathbf{Mult}^{\overline{\mathfrak{A}}}(c_b\lambda d^*, d'\mu c_t^*) = 0$ because one has $d \neq d'$ if and only if $\mathbf{Mult}^{c\mathfrak{W}}(\overline{\mathtt{top}}_\Lambda(c_b\lambda d^*), \overline{\mathtt{top}}_\Lambda(d\mu c_t^*)) = 0$ because $u(d) \neq u(d')$.
(3) We have $\mathbf{Mult}^{\overline{\mathfrak{A}}}(c_b\lambda d^*, d\mu c_t^*) = 0$ because $c_b c_t^*$ is not orientable if and only if $\mathbf{Mult}^{c\mathfrak{W}}(\overline{\mathtt{top}}_\Lambda(c_b\lambda d^*), \overline{\mathtt{top}}_\Lambda(d\mu c_t^*)) = 0$ because $u(c_b)u(c_t)^*$ is not a web.
(4) In case $d = d'$ and $c_b c_t^*$ is orientable, the usual diagram (the one very similar to (81), but with exchanged notation) commutes.

(1). Note that (44) sums up to 0 respectively 2 for anticlockwise respectively clockwise circles. Thus, $\overline{\mathtt{top}}_{c_b}^{c_t}$ is homogeneous by comparing (29) and (44), while keeping the shift $d(\vec{k})$ in mind. That $\overline{\mathtt{top}}_{c_b}^{c_t}$ is injective follows by definition. ▶

(2)+(3). Directly from Lemmas 6.21 and 6.22. ▶

(4). The signs for the multiplication process for $\overline{\mathfrak{A}}$ from (50) and (51) are specializations of the ones for $c\mathfrak{W}$ for dotted basis webs of shape $u(c)u(d)^*$ (with c, d standing for cup diagrams) - up to the phantom circle sign. Thus, it remains to show that the scaling factor $(-1)^{\mathtt{npesci}}$ accounts for this. To this end, one directly observes that only the phantom circle removal can change the number $\mathtt{npesci}$. Moreover, $\mathtt{npesci}$

is defined to count anticlockwise phantom circles, which is what $\mathtt{np}\mathrm{circ}(\overline{W})$ counts. ▶

Thus, the theorem is proven. ■

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