Algebraic Geometry

Beijing Institute of Technology, 2019

Exercises for chapter 2: Algebraic sets

We use the notations from the course, i.e. $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ its algebraic closure, and so on.

Exercise 1. Let T and T' be topological spaces. Show the following

(a) $\varnothing \neq S \subset T$ is irreducible if and only if $\overline{S} \subset T$ is irreducible,

(b) T is irreducible if and only if $\overline{U} = T$ for all open subsets $\varnothing \neq U \subset T$,

(c) and for $f : T \to T'$ continuous it holds that if T is irreducible then $f(T) \subset T'$ is irreducible.

Exercise 2. Let $f : R \to R'$ be a surjective map of rings and

$$\left\{ \text{ ideals of } R' \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{ideals of } R \\ \text{containing } \ker(f) \end{array} \right\},$$

where $\Phi(I) = f^{-1}(I)$ and $\Psi(J) = f(J)$. Show that

(a) Φ and Ψ define inclusion preserving bijections that are mutually inverse to each other,

(b) that this can be restricted to prime ideals of R' on the left hand side and prime ideals of R containing $\ker(f)$ on the right hand side,

(c) and that this can be restricted to maximal ideals of R' on the left hand side and maximal ideals of R containing $\ker(f)$ on the right hand side.

Exercise 3. Let $V \subset \overleftarrow{\mathbb{K}}^n$ be a $\mathbb{K}$-algebraic set. Show that

(a) for $I \subset \mathbb{K}[V]$ an ideal it holds $\mathcal{I}_V(\mathcal{V}_V(I)) = \mathrm{Rad}(I)$,

(b) the assignment $W \mapsto \mathcal{I}_V(W)$ defines a bijection between the set of $\mathbb{K}$-closed subset of $\overleftarrow{\mathbb{K}}^n$ contained in V and the set of radical ideals of $\mathbb{K}[V]$,

(c) and that the assignment from part (b) can be restricted to irreducible $\mathbb{K}$-closed subsets of V on the left hand side and to prime ideals in $\mathbb{K}[V]$ on the right hand side.