拓扑学（模块2）

Michael Ehrig
micha.ehrig@outlook.com
michael.ehrig@bit.edu.cn

BIT, Fall 2020

Assignment 1 - Solutions

Exercise 1. Let X be a space.

(a) Show that X is contractible if and only if id_X is nullhomotopic.

(b) Assume that X is contractible and there exists $A \subset X$ together with a retraction $r : X \to A$. Show that A is contractible.

(c) Show that the following are equivalent

(i) X is contractible,
(ii) every map $f : X \to Y$ (for a space Y) is nullhomotopic,
(iii) every map $f : Y \to X$ (for a space Y) is nullhomotopic.

Solutions: We denote by pt the space consisting of a single point.

(a) Assume that X is contractible with homotopy equivalence $f : X \to \mathrm{pt}$ and a homotopy inverse $g : \mathrm{pt} \to X$. Then $g \circ f \simeq \mathrm{id}_X$ by assumption and $g \circ f$ is the constant map with image $g(\mathrm{pt})$. Thus id_X is nullhomotopic.

Assume that id_X is nullhomotopic with a homotopy $F : X \times I \to X$ such that $f_0 = \mathrm{id}_X$ and f_1 is a constant map, with image $x_0 \in X$. Define $p : X \to \{x_0\}$ via $p(x) = x_0$ and $i : \{x_0\} \to X$ the inclusion. Then $i \circ p = f_0 \simeq \mathrm{id}_X$ by assumption and $p \circ i = \mathrm{id}_{\{x_0\}}$ by construction. Hence X is contractible.

(b) Assume that X is contractible with homotopy equivalence $f : X \to \mathrm{pt}$ and $g : \mathrm{pt} \to X$ a homotopy inverse. Then it holds

$$r = r \circ \mathrm{id}_X \simeq r \circ (g \circ f),$$

where the latter is a constant map as it factors through pt. Hence r is nullhomotopic and thus also $\mathrm{id}_A = r \circ i$, for $i : A \to X$ the inclusion, is nullhomotopic. Thus by part (a), A is contractible.

(c) *(i) ⇒ (ii)* Let $g : X \to \mathrm{pt}$ be a homotopy equivalence with homotopy inverse $h : \mathrm{pt} \to X$. For a map $f : X \to Y$ it holds $f = f \circ \mathrm{id}_X \simeq f \circ (h \circ g)$. But the latter is a constant map with image $(f \circ h)(\mathrm{pt})$.

(i) ⇒ (iii) For g and h as above and $f : Y \to X$ it holds $f = \mathrm{id}_X \circ f \simeq (h \circ g) \circ f$. Again the latter is a constant map with image $h(\mathrm{pt})$.

(ii) or (ii) ⇒ (i) In either case choose $Y = X$ and $f = \mathrm{id}_X$, then by part (a), X is contractible.

Michael Ehrig
micha.ehrig@outlook.com
michael.ehrig@bit.edu.cn

Exercise 2. Let X be a space. Show that the following are equivalent

(a) Every map $f : \mathrm{S}^1 \to X$ is nullhomotopic,

(b) every map $f : \mathrm{S}^1 \to X$ can be extended to a map $\widetilde{f} : \mathrm{D}^2 \to X$,

(c) for any $x_0 \in X$ it holds $\pi_1(X, x_0)$ is trivial.

Solutions: *(a) ⇒ (b)* For $f : \mathrm{S}^1 \to X$ let $F : \mathrm{S}^1 \times I \to X$ be the homotopy between $f_1 = f$ and a constant map f_0 with image $x_0 \in X$. Define $\widetilde{f} : \mathrm{D}^2 \to X$ via $\widetilde{f}(r \cdot e^{2\pi \mathrm{i} s}) = F(e^{2\pi \mathrm{i} s}, r)$ for $0 \leq r, s \leq 1$. Then by construction $\widetilde{f}$ is continuous and for $r = 1$ it agrees with f, hence is an extension of f.

(b) ⇒ (c) Let $f : I \to X$ be a loop in X at x_0. Since $f(0) = f(1)$ we can regard f as map from S^1 to X. By assumption there exists an extension $\widetilde{f} : \mathrm{D}^2 \to X$. Let $i : \mathrm{S}^1 \to \mathrm{D}^2$ be the inclusion, then $f = \widetilde{f} \circ i$. Then i is a loop in D^2, but D^2 is simply-connected, hence i is homotopic to the constant loop $\mathbb{1}_{(1,0)}$. Then $f = \widetilde{f} \circ i \simeq \widetilde{f} \circ \mathbb{1}_{(1,0)} = \mathbb{1}_{x_0}$ and this is a homotopy of path (i.e. relative to the basepoint).

(c) ⇒ (a) Let $f : \mathrm{S}^1 \to X$. Then f is a loop in X at $x_0 = f((1,0))$. By assumption $f \simeq \mathbb{1}_{x_0}$ and so f is nullhomotopic.

Michael Ehrig
micha.ehrig@outlook.com
michael.ehrig@bit.edu.cn

Exercise 3. Show that there exists no retraction $r : X \to A$ for the following situations

(a) $X = \mathbb{R}^n$ and A an arbitrary subspace homeomorphic to S^1.

(b) $X = \mathrm{S}^1 \times \mathrm{D}^2$ and $A = \mathrm{S}^1 \times \mathrm{S}^1$ the boundary of X.

(c) $X = \mathrm{D}^2 \vee \mathrm{D}^2$ and $A = \mathrm{S}^1 \vee \mathrm{S}^1$ for the pointed spaces $(\mathrm{D}^2, (1, 0))$ and $(\mathrm{S}^1, (1, 0))$.
Hint: One can argue with the methods from the course why A has a non-trivial fundamental group without the use of the van Kampen theorem.

(d) $X = \mathrm{D}^2/_\sim$, for $(1, 0) \sim (-1, 0)$, and $A = \pi(\mathrm{S}^1)$ the image of the boundary of D^2 under the quotient map π.
Hint: There is a natural way to identify A with $\mathrm{S}^1 \vee \mathrm{S}^1$.

(e) X the Möbius strip from the course, i.e. $X = I \times I/_\sim$ where $(0, s) \sim (1, 1 - s)$, and $A = \pi(I \times \{0\} \cup I \times \{1\})$ for π the quotient map.
Hint: Find a subspace B in X such that $B \cong \mathrm{S}^1$ and B is a deformation retract of X to obtain $\pi_1(X)$.

Solutions: We will start each case by assuming that such a retraction r exists together with the inclusion of the subspace i.

(a) There is an injective map from $\pi_1(A, x_0) \cong \mathbb{Z}$ to $\pi_1(X, x_0) \cong \{0\}$. This is a contradiction.

(b) There is an injective map from $\pi_1(A, x_0) \cong \pi_1(\mathrm{S}^1, x_0) \times \pi_1(\mathrm{S}^1, x_0) \cong \mathbb{Z}^2$ to $\pi_1(X, x_0) \cong \pi_1(\mathrm{S}^1, x_0) \times \pi_1(\mathrm{D}^2, x_0) \cong \mathbb{Z} \times \{0\} \cong \mathbb{Z}$. This is a contradiction.

(c) There exists a homotopy equivalence from D^2 to pt, using this on both copies of D^2 inside $\mathrm{D}^2 \vee \mathrm{D}^2$ we obtain that $\mathrm{D}^2 \vee \mathrm{D}^2$ is contractible and so $\pi_1(X, x_0) = \{0\}$. On the other hand using the constant map to $(1, 0)$ on the first factor of $\mathrm{S}^1 \vee \mathrm{S}^1$ we obtain a retraction from $\mathrm{S}^1 \vee \mathrm{S}^1$ to S^1, hence $\pi_1(A, x_0)$ surjects onto $\pi_1(\mathrm{S}^1, x_0) \cong \mathbb{Z}$. Thus $\pi_1(A, x_0)$ is non-trivial, again giving a contradiction to the injective map from $\pi_1(A, x_0)$ to $\pi_1(X, x_0)$.

(d) Let $C = \{(x, 0) \mid -1 \leq x \leq 1\}$ and define $F : \mathrm{D}^2 \times I \to \mathrm{D}^2$ via $F((x, y), t) = (x, (1 - t)y)$. Then F is continuous, $f_0 = \mathrm{id}_{\mathrm{D}^2}$ and the image of f_1 is in C and f_1 restricted to C is the identity of C, hence f_1 is a retraction. Thus C is a deformation retract of D^2. Since $(-1, 0)$ and $(1, 0)$ are in C and F is independent of its second argument on either point, we get a well-defined deformation retract from X to $C/_\sim \cong \mathrm{S}^1$ (where we identify the same two points in C as we did in D^2).

Furthermore we can look at the subsets $I_+ = \{(x, y) \mid |(x, y)| = 1, y \geq 0\}$ and $I_- = \{(x, y) \mid |(x, y)| = 1, y \leq 0\}$. Then both I_+ and I_- are homeomorphic to I and $\pi(I_+) \cong \mathrm{S}^1 \cong \pi(I_-)$. Since they have exactly one point in common we thus get that $A = \pi(\mathrm{S}^1) = \pi(I_+ \cup I_-) \cong \mathrm{S}^1 \vee \mathrm{S}^1$.

Let a_+ be the image of a path from $(1, 0)$ to $(-1, 0)$ in I_+ under π and a_- the image of a path from $(1, 0)$ to $(-1, 0)$ in I_-. Then using the retraction $r' : \mathrm{S}^1 \vee \mathrm{S}^1 \to \mathrm{S}^1$ onto the first factor (as discussed in part (c)), we see that $r'_*([a_+]) = [\omega_1]$ in S^1, while $r'_*([a_-]) = 0$. Hence in $\pi_1(A, x_0)$ it holds $[a_+] \neq [a_-]$. Then $[i \circ a_+] \neq [i \circ a_-]$ by assumption on injectivity of $i_\star$. But applying the deformation retract from X to S^1, we obtain that $[i \circ a_+] = [\omega_1] = [i \circ a_-]$, where $[\omega_1]$ is the image of the path from $(1, 0)$ to $(-1, 0)$ in C under π. Which is a contradiction to the injectivity of $i_\star$.

Michael Ehrig
micha.ehrig@outlook.com
michael.ehrig@bit.edu.cn

(e) We first choose three path inside $I \times I$. Define $a_0 : I \to I \times I$ via $a_0(s) = (s, 0)$, $a_1 : I \to I \times I$ via $a_1(s) = (s, 1)$, and $d : I \to I \times I$ via $d(s) = (s, s)$. Hence (in standard coordinates, i.e. first one horizontal, second one vertical) a_0 is a path connecting the bottom left to the bottom right corner of the rectangle $I \times I$, while a_1 connects the top left to the top right corner, and finally d is on the diagonal from bottom left to top right. Define now $x_0 = \pi((0, 0)) = \pi((1, 1))$, and $x_1 = \pi((0, 1)) = \pi((0, 1))$. Then $\overline{a}_0 = \pi \circ a_0$ is a path from x_0 to x_1, $\overline{a}_1 = \pi \circ a_1$ is a path from x_1 to x_0 and $\overline{d} = \pi \circ d$ is a loop at x_0. Since by construction A is two intervals identified at their respective endpoints, we see that $A \cong \mathrm{S}^1$ and $\overline{a}_0 \cdot \overline{a}_1$ is a loop in A at x_0 which is homotopic to ω_1 under the homeomorphism.

Let $B \cong \mathrm{S}^1$ be the image of $\overline{d}$, i.e. the image of the diagonal under the quotient map. Then define $F : (I \times I) \times I \to I \times I$ via $F((x, y), t) = (x, tx + (1-t)y)$. This is a continuous map that is the identity for $t = 0$ and for $t = 1$ is a retraction from $I \times I$ onto the image of d, the diagonal. Since $F((0, s), t) = (0, (1-t)s) \sim (1, 1-(1-t)s) = F((1, 1-s), t)$ we see that F induces a deformation retract f from X to B. We now look at what happens with $[\overline{a}_0 \cdot \overline{a}_1]$ under our maps. First $i_*([\overline{a}_0 \cdot \overline{a}_1])$ is a loop in X, if we apply the homotopy equivalence f to it we obtain $f_* \circ i_*([\overline{a}_0 \cdot \overline{a}_1])$ is a loop in B. But under f, x_1 is send to x_0 by construction. Hence $f_* \circ i_*([\overline{a}_0 \cdot \overline{a}_1]) = [\overline{d}] \cdot [\overline{d}]$. The homotopy inverse of f is just the inclusion of B into X, hence $i_*([\overline{a}_0 \cdot \overline{a}_1]) = [\overline{d}] \cdot [\overline{d}]$, and thus

$$[\overline{a}_0 \cdot \overline{a}_1] = (r \circ i)_*([\overline{a}_0 \cdot \overline{a}_1]) = r_* \circ i_*([\overline{a}_0 \cdot \overline{a}_1]) = [\overline{d}] \cdot [\overline{d}] = r_*([\overline{d}]) \cdot r_*([\overline{d}]).$$

But under our identification of $\pi_1(A, x_0)$ with $\mathbb{Z}$ we have that $[\overline{a}_0 \cdot \overline{a}_1]$ is send to 1, but there is no integer n such that $n + n = 1$, this is a contradiction.