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# Assignment 2 - Solutions

**Exercise 1.** Let $B = \mathrm{S}^1$, $C = \mathrm{D}^2$ and denote by $\partial C = \mathrm{S}^1$ the circle contained in $C$. For a fixed integer $k \geq 1$ we define the map $\phi_k : \partial C \to B$ via $\phi_k(e^{2\pi \mathrm{i} s}) = e^{2\pi \mathrm{i} k s}$. We then define the following space

$$X_k = (B \coprod C)/_\sim,$$

where $z \sim \varphi_k(z)$ for $z \in \partial C$.

(a) Show that the space $X_k$ is path-connected.

(b) Use the van Kampen theorem to determine $\pi_1(X_k)$.

*Hint:* The calculation is easier with the choice of a basepoint in $C \setminus \partial C$.

**Solutions:** Denote by $\pi$ the quotient map from $B \coprod C$ to $X_k$. Note that $\pi$ restricted to $C \setminus \partial C$ is a homeomorphism and $\pi$ restricted to $B$ is a homeomorphism.

**(a)** Let $x, y \in X_k$. If $x, y \in \pi(B)$ respectively $x, y \in \pi(C)$ then we can take a path $f$ from $x' \in \pi^{-1}(x)$ and $y' \in \pi^{-1}(y)$ in $B$ respectively $C$, and $\pi \circ f$ is a path in $X_k$ from $x$ to $y$. Thus we can assume that $x \in \pi(B)$ and $y \in \pi(C)$. Note that the map $\phi_k$ is surjective, hence there is $z \in \partial C$ such that $\phi_k(z) = x$ in $X_k$, hence by the previous discussion $y$ is path connected to $z$ which gets identified with $x$, hence $X_k$ is path-connected.

**(b)** Fix $1 > \varepsilon' > \varepsilon > 0$.

Define $U' = \{x \in C \mid |x| > \varepsilon\}$, which is open in $C$ and path-connected as the image of a path-connected space. Set $U = \pi(U')$. Then $\pi^{-1}(U) = U' \coprod B$, which is open, hence $U$ is open in $X_k$ by definition of the quotient topology.

Define $V' = \{x \in C \mid |x| < \varepsilon'\}$, which is open in $C$ as well. Set $V = \pi(V')$. Since $V' \subset C \setminus \partial C$ and $\pi$ is a homeomorphism when restricted to $C \setminus \partial C$, $V$ is open in $X_k$. Note that $V \cap \pi(B) = \varnothing$ and it is path-connected as the image of a path-connected space.

By construction $U \cup V = X_k$ and $U \cap V \cong \{x \in C \mid \varepsilon' > |x| > \varepsilon\}$, since the intersection is contained in $\pi(C \setminus \partial C)$, where $\pi$ is a homeomorphism. Especially $U \cap V$ is path-connected.

Fix $x_0 \in V \cap U$. Then the open cover $\{U, V\}$ satisfies the assumption of the van Kampen theorem, both sets are open, path-connected and contain $x_0$. Since the intersection $U \cap V$ is also path-connected we know that the map given in the van Kampen theorem is surjective and we can deduce the kernel.

By definition $V$ is contractible, as an open ball. For $U$, extend the retraction $r : (\mathrm{D}^2 \setminus \{(0,0)\}) \to \mathrm{S}^1$ (from the course) to $B$ by using the identity on $B$. In this way we obtain a retraction $r' : (C \setminus \{(0,0)\}) \coprod B \to \partial C \coprod B$. This descents to a retraction $\overline{r}$ on the quotient and so $U$ deformation retracts to $(\partial C \coprod B)/_\sim \cong \mathrm{S}^1$ via $\overline{r}$. Finally $U \cap V$ deformation retracts to a circle with radius $\tau$ for $\tau = |x_0'|$. Hence $\pi_1(U \cap V, x_0) \cong \mathbb{Z} = \langle [\omega] \rangle$, where we denote by $\omega$ a loop around the circle of radius $\tau$ based at $x_0$. Applying the van Kampen theorem we obtain

$$\pi_1(X_k, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0)) / \langle \varphi_{UV}([\omega]) \rangle \cong \pi_1(V, x_0)) / \langle \varphi_{UV}([\omega]) \rangle,$$

where $\varphi_{UV} : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is induced by the inclusion. We omit the term $\varphi_{VU}([\omega])^{-1}$ since $\pi_1(V, x_0)$ is trivial since $V$ is contractible, hence the image is trivial anyway.

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Note that in the quotient

$$\overline{r} \circ \pi(e^{(2\pi \mathrm{i} l)/k} x_0') \sim \overline{r}(x_0) \text{ for } 0 \leq l < k.$$

Thus $\overline{r} \circ \omega$ is a loop at $\overline{r}(x_0)$ that passes $\overline{r}(x_0)$ a total of $k+1$ times. Hence generates the subgroup $k\mathbb{Z}$ inside $\pi_1(U, x_0) \cong \pi_1(\mathrm{S}^1) \cong \mathbb{Z}$. Hence we obtain $\pi_1(X_k, x_0) \cong \mathbb{Z}/k\mathbb{Z}$.

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**Exercise 2.** Let $X$ be a space with $X = U \cup V$ for $U$, $V$, and $U \cap V$ all open, non-empty and path-connected.

(a) Show that $X$ is path-connected.

(b) Assume that $V \cap U$ is simply-connected and show that $\pi(X) \cong \pi_1(U) * \pi_1(V)$.

**Solutions:** **(a)** Since $U \cap V \neq \varnothing$ we can choose $x_0 \in U \cap V$. By assumption $x_0$ is path-connected to any point in $U$ and to any point in $V$, since $U$ and $V$ are both path-connected. Hence $x_0$ is path-connected to any point in $X$ and so $X$ is path-connected.

**(b)** We already know that $X$ is path-connected. Assume now that also $V \cap U$ is simply-connected and let $x_0$ be as in part (a). We can apply the van Kampen theorem to the open subsets $U$ and $V$, since they are both open, path-connected, contain $x_0$ and $V \cap U$ is path-connected. Hence by the van Kampen theorem

$$\pi_1(X, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0))/N,$$

where $N$ is given as in the van Kampen theorem. But by assumption $\pi_1(U \cap V, x_0) = \{0\}$, hence the maps induced from the embedding of $U \cap V$ into $U$ respectively $V$ are both trivial and thus $N = \{0\}$. Hence the claim follows.

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**Exercise 3.**

(a) Let $[v_0, v_1, v_2, v_3]$ be a 3-simplex. Define $X = [v_0, v_1, v_2, v_3]/_\sim$ with the equivalence relation given by $v_0 \sim v_1 \sim v_2 \sim v_3$. Compute the simplicial homology of $X$.

(b) Let $[v_0, v_1, \ldots, v_n]$ be an $n$-simplex. For $1 \leq k \leq n$ and $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$ denote by $\varphi_{\underline{i}} : \Delta^k \to [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ the canonical homeomorphism from the standard $(k-1)$-simplex to $[v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ (as defined in the course).

Now define $X = [v_0, v_1, \ldots, v_n]/_\sim$ where for any $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$, $\underline{j} = (0 \leq j_1 < j_2 < \ldots < j_k \leq n)$, and $x \in [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ we set $x \sim \varphi_{\underline{j}} \circ \varphi_{\underline{i}}^{-1}(x)$, i.e. we identify all $(k-1)$-simplices contained as iterative faces in $[v_0, v_1, \ldots, v_n]$ via the canonical homeomorphisms. Compute the simplicial homology of $X$.

**Solutions:** For any simplex $[v_0, \ldots, v_n]$ and $1 \leq k \leq n$, we denote by $\sigma_{[v_{i_1}, \ldots, v_{i_k}]}$ the canonical homeomorphism from the standard $(k-1)$-simplex to the simplex $[v_{i_1}, \ldots, v_{i_k}] \subset [v_0, \ldots, v_n]$.
**(a)** As the $\Delta$-complex structure on $[v_0, v_1, v_2, v_3]$, we choose

$$\begin{aligned}\Sigma = \{&\sigma_{[v_0,v_1,v_2,v_3]}, \sigma_{[v_0,v_1,v_2]}, \sigma_{[v_0,v_1,v_3]}, \sigma_{[v_0,v_2,v_3]}, \sigma_{[v_1,v_2,v_3]},\\ &\sigma_{[v_0,v_1]}, \sigma_{[v_0,v_2]}, \sigma_{[v_0,v_3]}, \sigma_{[v_1,v_2]}, \sigma_{[v_1,v_3]}, \sigma_{[v_2,v_3]},\\ &\sigma_{[v_0]}, \sigma_{[v_1]}, \sigma_{[v_2]}, \sigma_{[v_3]}\}.\end{aligned}$$

For the $\Delta$-complex structure on $X$ we use $\widetilde{\Sigma} = \{\pi \circ \sigma \mid \sigma \in \Sigma\}$ where $\pi : [v_0, v_1, v_2, v_3] \to X$ is the natural quotient map. Note that $\widetilde{\sigma}_{[v_0]} = \widetilde{\sigma}_{[v_1]} = \widetilde{\sigma}_{[v_2]} = \widetilde{\sigma}_{[v_3]}$, hence there is only a single generator for the 0-chains. To see that this is a $\Delta$-complex structure, we need to check part (1), (2) and (3) from the definition:

(1) Since $\pi$ is a homeomorphism outside of $\{v_0, v_1, v_2, v_3\}$, any $\widetilde{\sigma} \in \widetilde{\Sigma}$ is injective when restricted to the interior of the standard simplex.

(2) For (2) we only need to check when we restrict a $\widetilde{\sigma}_{[v_i,v_j]}$ $(i < j)$ to one of the faces of the standard 1-simplex. But such a face is a 0-simplex, for which we have a unique map in our $\Delta$-complex structure, hence (2) is automatically full-filled.

(3) Since our $\Delta$-complex structure is obtained from the one of $[v_0, v_1, v_2, v_3]$ via a quotient map, part (3) is automatic.

We now have the following non-trivial chain groups

$$\begin{aligned}\Delta_3(X) &= \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1,v_2,v_3]},\\ \Delta_2(X) &= \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1,v_2]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1,v_3]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_0,v_2,v_3]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_1,v_2,v_3]},\\ \Delta_1(X) &= \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_0,v_2]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_0,v_3]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_1,v_2]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_1,v_3]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_2,v_3]}, \text{ and}\\ \Delta_0(X) &= \mathbb{Z}\widetilde{\sigma}_{[v_0]}.\end{aligned}$$

To compute simplicial homology we proceed as follows:

- It holds $\partial_3(\widetilde{\sigma}_{[v_0,v_1,v_2,v_3]}) = \widetilde{\sigma}_{[v_1,v_2,v_3]} - \widetilde{\sigma}_{[v_0,v_2,v_3]} + \widetilde{\sigma}_{[v_0,v_1,v_3]} - \widetilde{\sigma}_{[v_0,v_1,v_2]}$. Since $\partial_3$ is injective and $\partial_4$ is the zero map, we have $H_3^\Delta(X) = 0$.

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- The most complicated map is $\partial_2$. We need to deduce the kernel and the image of it. We first note that the images of the generators are as follows

$$\begin{aligned}
\partial_2(\widetilde{\sigma}_{[v_0,v_1,v_2]}) &= \widetilde{\sigma}_{[v_1,v_2]} - \widetilde{\sigma}_{[v_0,v_2]} + \widetilde{\sigma}_{[v_0,v_1]},\\
\partial_2(\widetilde{\sigma}_{[v_0,v_1,v_3]}) &= \widetilde{\sigma}_{[v_1,v_3]} - \widetilde{\sigma}_{[v_0,v_3]} + \widetilde{\sigma}_{[v_0,v_1]},\\
\partial_2(\widetilde{\sigma}_{[v_0,v_2,v_3]}) &= \widetilde{\sigma}_{[v_2,v_3]} - \widetilde{\sigma}_{[v_0,v_3]} + \widetilde{\sigma}_{[v_0,v_2]}, \text{and}\\
\partial_2(\widetilde{\sigma}_{[v_1,v_2,v_3]}) &= \widetilde{\sigma}_{[v_2,v_3]} - \widetilde{\sigma}_{[v_1,v_3]} + \widetilde{\sigma}_{[v_1,v_2]}.
\end{aligned}$$

  Note that each of the six generators of $\Delta_1(X)$ appears as a summand in the images of exactly two generators of $\Delta_2(X)$ and each pair of images has exactly one summand in common. Hence any subset of three images is linearly independent. But $\partial_2(\widetilde{\sigma}_{[v_0,v_1,v_2]}) - \partial_2(\widetilde{\sigma}_{[v_0,v_1,v_3]}) + \partial_2(\widetilde{\sigma}_{[v_0,v_2,v_3]}) = \partial_2(\widetilde{\sigma}_{[v_1,v_2,v_3]})$, hence the kernel of $\partial_2$ is equal to

$$\left\langle \widetilde{\sigma}_{[v_0,v_1,v_2]} - \widetilde{\sigma}_{[v_0,v_1,v_3]} + \widetilde{\sigma}_{[v_0,v_2,v_3]} - \widetilde{\sigma}_{[v_1,v_2,v_3]} \right\rangle,$$

  which we calculated in the previous point is equal to the image of $\partial_3$ and so $H_2^\Delta(X) = \{0\}$.

  The image of $\partial_2$ is generated by the images of any three generators of $\Delta_2(X)$ as seen above, we will just take the first three from the list above.

- Since $\partial_1(\widetilde{\sigma}_{[v_i,v_j]}) = 0$ for all $i < j$, the kernel of $\partial_1$ is all of $\Delta_1(X)$ and so

$$H_1^\Delta(X) = \Delta_1(X)/\left\langle \partial_2(\widetilde{\sigma}_{[v_0,v_1,v_2]}), \partial_2(\widetilde{\sigma}_{[v_0,v_1,v_3]}), \partial_2(\widetilde{\sigma}_{[v_0,v_2,v_3]})\right\rangle \cong \mathbb{Z}^3.$$

  The most obvious choice for the isomorphism to $\mathbb{Z}^3$ is to record the coefficients of $\widetilde{\sigma}_{[v_0,v_2]}$, $\widetilde{\sigma}_{[v_0,v_1]}$, and $\widetilde{\sigma}_{[v_0,v_3]}$. The coefficients of the other three are then determined in the quotient.

- As seen above the image of $\partial_1$ is trivial, hence $H_0^\Delta(X) = \Delta_0(X) \cong \mathbb{Z}$.

**(b)** As a $\Delta$-complex structure on $X$ we use

$$\widetilde{\Sigma} = \{\pi \circ \sigma_{[v_0,\dots,v_k]} \mid 0 \le k \le n\},$$

with $\pi$ being the quotient map. This forms a $\Delta$-complex structure

(1) For property (1), we note that for every $0 \le k \le n$, no two points in the interior of $[v_{i_1}, \dots, v_{i_k}]$ get identified in the quotient. Via the equivalence relation, they are only equivalent to a unique point in every other $(k-1)$-simplex. Hence we still have injectivity.

(2) For (2) we note that all $(k-1)$-simplices get identified, hence the face of a $k$-simplex is always the unique $(k-1)$-simplex. Since they are all identified via the canonical homeomorphism, the trianlge in the definition part (2) commutes as well.

(3) As before, since we are going to a quotient, part (3) is automatic.

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As chain groups we thus get $\Delta_k(X) = \mathbb{Z}\widetilde{\sigma}_{[v_0,\dots,v_k]}$ for $0 \leq k \leq n$, all others are trivial. For the image $\partial_k$ we just check

$$\partial_k(\widetilde{\sigma}_{[v_0,\dots,v_k]}) = \sum_{i=0}^{k}(-1)^i\widetilde{\sigma}_{[v_0,\dots,v_{k-1}]} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \widetilde{\sigma}_{[v_0,\dots,v_{k-1}]} & \text{if } k \text{ is even.} \end{cases}$$

Hence we see that $\partial_k$ is an isomorphism when $k$ is even and the zero map if $k$ is odd. Hence we obtain for $0 < k < n$ we get

$$H_k^{\Delta}(X) \cong \begin{cases} \{0\}/\{0\} \cong \{0\} & \text{for } k \text{ even} \\ \mathbb{Z}/\mathbb{Z} \cong \{0\} & \text{for } k \text{ odd.} \end{cases}$$

In the two extreme cases we get $H_0^{\Delta}(X) = \mathbb{Z}$, since the image of $\partial_1$ is trivial, and finally, since there are no $n+1$-simplices

$$H_n^{\Delta}(X) = \mathrm{Ker}(\partial_n) \cong \begin{cases} \{0\} & \text{for } n \text{ even.} \\ \mathbb{Z} & \text{for } n \text{ odd} \end{cases}$$

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**Exercise 4.** Let $r : X \to A$ be a retraction of a space $X$ to a subspace $A$ and $i : A \to X$ the inclusion. Show that $i_* : H_n(A) \to H_n(X)$ is injective and $r_* : H_n(X) \to H_n(A)$ is surjective for all $n \geq 0$.

**Solutions:** By definition of a retraction we have $r \circ i = \mathrm{id}_A$. Applying now homology to this we get, for any n,

$$r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_A)_* = \mathrm{id}_{H_n(A)}.$$

Hence the map $r_*$ needs to be surjective as it has a right inverse and the map $i_*$ needs to be injective, since it has a left inverse.