Algebraic Geometry

Beijing Institute of Technology, 2019

Exercises for chapter 4: Affine algebraic groups

Exercise Let G be an affine algebraic group over an algebraically closed field $\mathbb{K}$ and A a, not necessarily commutative, $\mathbb{K}$-algebra.

(a) Show that $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[G], A)$, the space of $\mathbb{K}$-linear maps from $\mathbb{K}[G]$ to A, is a $\mathbb{K}$-algebra.

(b) Show that for A commutative, $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{K}[G], A)$, the set of $\mathbb{K}$-algebra homomorphisms from $\mathbb{K}[G]$ to A, is a group.

(c) Show that for $G = GL_n(\mathbb{K})$ and A being commutative, $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{K}[GL_n(\mathbb{K})], A)$ is isomorphic to $GL_n(A)$, the group of invertible $n \times n$-matrices with entries in A.

(d) Show that for A commutative and finite dimensional one can give A the structure of an affine algebraic variety such that the multiplication $m : A \times A \to A$ is a morphism of affine varieties.

(e) Show that for A commutative and finite dimensional one can give $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{K}[G], A)$ the structure of an affine algebraic group over $\mathbb{K}$.

Remark

(a) Let C be a coalgebra, i.e. a $\mathbb{K}$ vector space with maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ that satisfy the coassociativity and counital conditions from the lecture, and A an algebra, then $\mathrm{Hom}_{\mathbb{K}}(C, A)$ is a $\mathbb{K}$-algebra. Thus part a) above only needs the co-structure of $\mathbb{K}[G]$ and neither the algebra nor the Hopf algebra structure. The product structure for part a) is called the "convolution product".

(b) For part b) one can always show that for a Hopf algebra H, $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ consists of elements that are invertible with respect to the multiplication from part a). Thus one can look at the subgroup generated by $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ inside the group of units $\mathrm{Hom}_{\mathbb{K}}(H, A)^{\times}$, but in general this subgroup will be strictly bigger than $\mathrm{Hom}_{\mathrm{alg}}(H, A)$.