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拓扑学（模块2）

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Assignment 1 - Solutions

Exercise 1. Let $\mathcal{I} = \{0, \frac{1}{2}, 1\}$ and fix the collection of subsets $\mathcal{B} = \{\{0\}, \{1\}, \mathcal{I}\}$.

(a) Show that $\mathcal{B}$ is a basis for a topology of $\mathcal{I}$.

(b) Determine all open subsets of $\mathcal{I}$ with respect to the topology $\mathcal{U}_\mathcal{B}$ generated by $\mathcal{B}$.

(c) Show that $\mathcal{I}$ is path-connected and contractible.

Solutions: **(a)** Any finite intersection of elements of $\mathcal{B}$ need to be unions of elements in $\mathcal{B}$. Since $\{0\} \subset \mathcal{I}$ and $\{1\} \subset \mathcal{I}$, the only intersections that are not automatically again elements in $\mathcal{B}$ is the intersection of no elements in $\mathcal{B}$, i.e. the empty intersection, which is equal to the whole set $\mathcal{I} \in \mathcal{B}$, and the intersection $\{0\} \cap \{1\} = \varnothing$, which is equal to the union of no elements in $\mathcal{B}$, i.e. the empty union. Thus $\mathcal{B}$ is a basis of a topology.

(b) By definition $\mathcal{U}_\mathcal{B}$ is given by forming arbitrary unions of elements in $\mathcal{B}$. Thus up to the empty union the only open subset that is not already an element in $\mathcal{B}$ is $\{0, 1\}$ and we obtain $\mathcal{U}_\mathcal{B} = \{\varnothing, \{0\}, \{1\}, \{0, 1\}, \mathcal{I}\}$.

(c) We define a path from 0 to 1 in $\mathcal{I}$ first, for this let

$$\gamma : \mathrm{I} \to \mathcal{I}, \text{ for } \gamma(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}), \\ 1 & s \in (\frac{1}{2}, 1], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

For continuity, we need to look at preimages of open subsets. Since taking preimages commutes with unions, it is enough to check the preimages of elements in the basis $\mathcal{B}$ (every other open set is a union of such elements). We see that $\gamma^{-1}(\{0\}) = [0, \frac{1}{2})$, $\gamma^{-1}(\{1\}) = (\frac{1}{2}, 1]$, and $\gamma^{-1}(\mathcal{I}) = \mathrm{I}$. These are all open in I, hence γ is continuous and thus a path from 0 to 1. To obtain a path from 0 to $\frac{1}{2}$ we precompose γ with the continuous map $h : \mathrm{I} \to \mathrm{I}$ given by $h(s) = \frac{s}{2}$. Then $\gamma \circ h$ is a path from 0 to $\frac{1}{2}$. Since path-connectedness is an equivalence relation it thus follows that all points in $\mathcal{I}$ are in the same path-component and so $\mathcal{I}$ is path-connected.

To see that $\mathcal{I}$ is contractible we define a homotopy F between the identity map of $\mathcal{I}$ and the map $c : \mathcal{I} \to \mathcal{I}$ given by $c(s) = \frac{1}{2}$ for all $s \in \mathcal{I}$. The map c is continuous since $c^{-1}(\{\frac{1}{2}\}) = \mathcal{I}$ is open. We define the homotopy F as

$$F : \mathcal{I} \times \mathrm{I} \to \mathcal{I}, \text{ for } F(x, t) = \begin{cases} 0 & (x, t) \in \{0\} \times [0, \frac{1}{2}), \\ 1 & (x, t) \in \{1\} \times [0, \frac{1}{2}), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

To see that F is continuous, we check again the preimages of elements in $\mathcal{B}$. We have $F^{-1}(\{0\}) = \{0\} \times [0, \frac{1}{2})$, since the first factor is open in $\mathcal{I}$ and the second is open in I, this is open in $\mathcal{I} \times \mathrm{I}$. The same argument shows that $F^{-1}(\{1\})$ is open and $F^{-1}(\mathcal{I})$ is the whole space anyway since F is obviously surjective. Finally, $F(-, 0)$ is the identity on $\mathcal{I}$, while $F(-, 1) = c$. Hence $\mathcal{I}$ deformation retracts to the subset $\{\frac{1}{2}\}$. Thus $\mathcal{I}$ is homotopy equivalent to a point, i.e. contractible as a space.

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Remark for (c) We did not need to show separately that $\mathcal{I}$ is path-connected, because $F(x,-)$ from above gives a path from x to $\frac{1}{2}$ for every $x \in \mathcal{I}$ anyway. So the proof of $\mathcal{I}$ to be contractible also shows that $\mathcal{I}$ is path-connected.

This argument of course works for any homotopy between the identity of a space X and a constant map, i.e. by Exercise 3(a) below any contractible space is automatically path-connected with the exact same proof as just outlined here.

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Exercise 2.

(a) Let $f : \mathrm{D}^2 \to \mathrm{D}^2$ be a map such that $f(x) = x$ for $x \in \mathrm{S}^1$. Show that f is surjective.
Hint: This can be done very similar to the proof of the Brouwer fixed point theorem in the course.

(b) Show that a loop $f : \mathrm{I} \to \mathrm{S}^1$ with the property $f(s + \frac{1}{2}) = -f(s)$ for $0 \leq s \leq \frac{1}{2}$ cannot be nullhomotopic, i.e. $f \not\simeq \omega_0$.
Hint: We used a loop with this property in a proof in the course.

(c) Show that there exists no map $g : \mathrm{S}^2 \to \mathrm{S}^1$ such that $g(-x) = -g(x)$.
Hint: Using such a map g, construct a suitable loop to apply part (b).

Solutions: **(a)** Assume that f is not surjective and fix $x_0 \in \mathrm{D}^2$ that is not contained in the image of f. For any $x \in \mathrm{D}^2$ we can define the following map $g_x(t) = x_0 + t(f(x) - x_0)$ for $t \in \mathbb{R}_{\geq 0}$. This is a ray that starts at $g_x(0) = x_0$ and goes through $g_x(1) = f(x)$. Since $f(x) \neq x_0$ by assumption g_x is not constant for any x. Hence there exists a unique $t_x \in \mathbb{R}_{\geq 0}$ such that $g_x(t_x) \in \mathrm{S}^1$. Now we define $r : \mathrm{D}^2 \to \mathrm{S}^1$ via $r(x) = g_x(t_x)$. If $x \in \mathrm{S}^1$ then $g_x(1) = f(x) = x \in \mathrm{S}^1$, so $r(x) = x$. Thus r is a retraction from D^2 to S^1. We showed in the course that no such retraction can exist. Thus our assumption on f is wrong and so f is surjective.

(b) Let f be a loop at x_0 in S^1 such that $f(s + \frac{1}{2}) = -f(s)$ for $s \in [0, \frac{1}{2}]$. Replacing f by a loop $\rho \circ f$ where ρ is the rotation of the circle by an arbitrary angle, keeps all of the required properties. Without loss of generality, we can thus assume that $x_0 = (1, 0)$, by replacing f by a suitable $\rho \circ f$.
We use the map $p : \mathbb{R} \to \mathrm{S}^1$ from the lecture (i.e. $p(s) = (\cos(2\pi s), \sin(2\pi s))$) and lift the loop f to a path $\widetilde{f}$ at 0. Since $f(s + \frac{1}{2}) = -f(s)$ it has to hold that $\widetilde{f}(s + \frac{1}{2}) = \widetilde{f}(s) + q_s/2$ where q_s is an odd integer. This is simply the fact that the map p has the property that $p(x+n) = p(x)$ holds exactly when $n \in \mathbb{Z}$ and $p(x + n) = -p(x)$ holds exactly when $n \in \mathbb{Z} + \frac{1}{2}$. Thus the assignment $s \mapsto q_s$ defines a continuous map from $[0, \frac{1}{2}]$ to $\mathbb{Z}$, which has to have a connected image (since $[0, \frac{1}{2}]$ is connected) and so is constant. We denote this constant value by q. Then $\widetilde{f}(1) = \widetilde{f}(\frac{1}{2}) + q/2 = \widetilde{f}(0) + q = q$. Since $\mathbb{R}$ is simply-connected we thus obtain that $\widetilde{f}$ is homotopic to $\widetilde{\omega}_q$ for q an odd integer. Thus f is homotopic to ω_q and since q is odd we know that $q \neq 0$ and so f is not nullhomotopic.

(c) Assume that $g : \mathrm{S}^2 \to \mathrm{S}^1$ with $g(-x) = -g(x)$ exists. Let $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in \mathrm{I}$ be the loop at $(1, 0, 0)$ that goes around an equator of S^2. Then $g \circ \eta(s + \frac{1}{2}) = (-\cos(2\pi s), -\sin(2\pi s), 0)$ and so $g \circ]eta$ is a loop in S^1 that one can apply (b) to. Hence $g \circ \eta$ is not nullhomotopic. But we checked in the lecture that S^2 is simply-connected, hence there exists a homotopy F between η and the constant path $\mathbb{1}_{(1,0,0)}$. Then $g \circ F$ is a homotopy between $g \circ \eta$ and $\mathbb{1}_{g(1,0,0)}$, which is a contradiction. Thus no g as assumed exists.

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Addition to (c) We do not need to know the exact form of the homotopy F, since S^2 is simply-connected. But for completeness we can write down an explicit homotopy between η and $\mathbb{1}_{(1,0,0)}$ here.

$$\begin{aligned} F : \mathrm{I} \times \mathrm{I} &\longrightarrow \mathrm{S}^2 \\ (s,t) &\mapsto (\cos(\theta)\sin(\rho), \sin(\theta)\sin(\rho), \cos(\rho)), \end{aligned}$$

where θ and ρ are given as follows

$$(\theta,\rho) = \begin{cases} (2\pi s, \pi(\frac{1}{2} - st)) & s \in [0,\frac{1}{2}],\, t \in [0,\frac{1}{2}], \\ (2\pi s, \pi(\frac{1}{2} - (1-s)t)) & s \in [\frac{1}{2},1],\, t \in [0,\frac{1}{2}], \\ (4\pi s(1-t), \pi(\frac{1}{2} + 2s(t^2 - t))) & s \in [0,\frac{1}{2}],\, t \in [\frac{1}{2},1], \\ (4\pi s + 4\pi(1-s)t, \pi(\frac{1}{2} + 2(1-s)(t^2-t))) & s \in [\frac{1}{2},1],\, t \in [\frac{1}{2},1]. \end{cases}$$

Obviously all of the assignments define continuous maps, we just need to check they all agree when multiple of them are defined. We need to check that the first and second case agree for $s = \frac{1}{2}$, which is obvious, and that the third and fourth agree for $s = \frac{1}{2}$. Similarly one needs to check that the first and third case agree for $t = \frac{1}{2}$ as well as the second and fourth case for $t = \frac{1}{2}$. Then we obtain that F is continuous. For the choice of $t = 0$, we obtain $F(s,0) = (\cos(2\pi s), \sin(2\pi s), 0) = \eta(s)$ and for the choice of $t = 1$ we obtain $F(s,1) = (1,0,0)$. For all of these checks, one has to remember that one can add integral multiples of 2π in the coordinated θ and ρ without changing the value of F, since afterwards cos and sin are applied, which are invariant under adding integral multiples of 2π. Thus one obtains that F is a homotopy (of maps) from η to $\mathbb{1}_{(1,0,0)}$. Finally one needs to check that F is really a homotopy of path, hence $F(0,t)$ and $F(1,t)$ need to be independent of t, and thus equal to $(1,0,0)$. This is done directly for $s = 0$ using the first and third case, and for $s = 1$ using the second and fourth case. Thus F is a homotopy of path and η is homotopic to the constant path $\mathbb{1}_{(1,0,0)}$ as paths.
Note that this is just one possible example for such a homotopy that one can choose. This is also a good example that homotopies can get quite complicated quickly, even though the space and the two loops that are homotopic are all of a very simple form.

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Exercise 3. Let X be a space.

(a) Show that X is contractible if and only if id_X is nullhomotopic.

(b) Assume that X is contractible and there exists $A \subset X$ together with a retraction $r : X \to A$. Show that A is contractible.

(c) Show that the following are equivalent

(i) X is contractible,

(ii) every map $f : X \to Y$ (for a space Y) is nullhomotopic,

(iii) every map $f : Y \to X$ (for a space Y) is nullhomotopic.

Solutions: We denote by pt the space consisting of a single point.

(a) Assume that X is contractible with homotopy equivalence $f : X \to \mathrm{pt}$ and a homotopy inverse $g : \mathrm{pt} \to X$. Then $g \circ f \simeq \mathrm{id}_X$ by assumption and $g \circ f$ is the constant map with image $g(\mathrm{pt})$. Thus id_X is nullhomotopic.

Assume that id_X is nullhomotopic with a homotopy $F : X \times I \to X$ such that $f_0 = \mathrm{id}_X$ and f_1 is a constant map, with image $x_0 \in X$. Define $p : X \to \{x_0\}$ via $p(x) = x_0$ and $i : \{x_0\} \to X$ the inclusion. Then $i \circ p = f_0 \simeq \mathrm{id}_X$ by assumption and $p \circ i = \mathrm{id}_{\{x_0\}}$ by construction. Hence X is contractible.

(b) Assume that X is contractible with homotopy equivalence $f : X \to \mathrm{pt}$ and $g : \mathrm{pt} \to X$ a homotopy inverse. Then it holds

$$r = r \circ \mathrm{id}_X \simeq r \circ (g \circ f),$$

where the latter is a constant map as it factors through pt. Hence r is nullhomotopic and thus also $\mathrm{id}_A = r \circ i$, for $i : A \to X$ the inclusion, is nullhomotopic. Thus by part (a), A is contractible.

(c) *(i) ⇒ (ii)* Let $g : X \to \mathrm{pt}$ be a homotopy equivalence with homotopy inverse $h : \mathrm{pt} \to X$. For a map $f : X \to Y$ it holds $f = f \circ \mathrm{id}_X \simeq f \circ (h \circ g)$. But the latter is a constant map with image $(f \circ h)(\mathrm{pt})$.

(i) ⇒ (iii) For g and h as above and $f : Y \to X$ it holds $f = \mathrm{id}_X \circ f \simeq (h \circ g) \circ f$. Again the latter is a constant map with image $h(\mathrm{pt})$.

(ii) or (iii) ⇒ (i) In either case choose $Y = X$ and $f = \mathrm{id}_X$, then by part (a), X is contractible.

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Exercise 4. Show that there exists no retraction $r : X \to A$ for the following situations

(a) $X = \mathbb{R}^n$ and A an arbitrary subspace homeomorphic to S^1.

(b) $X = \mathrm{S}^1 \times \mathrm{D}^2$ and $A = \mathrm{S}^1 \times \mathrm{S}^1$ the boundary of X.

(c) $X = \mathrm{D}^2 \vee \mathrm{D}^2$ and $A = \mathrm{S}^1 \vee \mathrm{S}^1$ for the pointed spaces $(\mathrm{D}^2, (1, 0))$ and $(\mathrm{S}^1, (1, 0))$.
Hint: One can argue with the methods from the course why A has a non-trivial fundamental group without specifically calculating it.

(d) $X = \mathrm{D}^2/_\sim$, for $(1, 0) \sim (-1, 0)$, and $A = \pi(\mathrm{S}^1)$ the image of the boundary of D^2 under the quotient map π.
Hint: There is a natural way to identify A with $\mathrm{S}^1 \vee \mathrm{S}^1$.

Solutions: We will start each case by assuming that such a retraction r exists together with the inclusion of the subspace i.

(a) Then $i_\star$ is an injective map from $\pi_1(A, x_0) \cong \mathbb{Z}$ to $\pi_1(X, x_0) \cong \{0\}$. This is a contradiction.

(b) Then $i_\star$ is an injective map from $\pi_1(A, x_0) \cong \pi_1(\mathrm{S}^1, x_0) \times \pi_1(\mathrm{S}^1, x_0) \cong \mathbb{Z}^2$ to $\pi_1(X, x_0) \cong \pi_1(\mathrm{S}^1, x_0) \times \pi_1(\mathrm{D}^2, x_0) \cong \mathbb{Z} \times \{0\} \cong \mathbb{Z}$. This is a contradiction.

(c) There exists a homotopy equivalence from D^2 to pt, using this on both copies of D^2 inside $\mathrm{D}^2 \vee \mathrm{D}^2$ we obtain that $\mathrm{D}^2 \vee \mathrm{D}^2$ is contractible and so $\pi_1(X, x_0) = \{0\}$. On the other hand using the constant map to $(1, 0)$ on the first factor of $\mathrm{S}^1 \vee \mathrm{S}^1$ we obtain a retraction from $\mathrm{S}^1 \vee \mathrm{S}^1$ to S^1, hence $\pi_1(A, x_0)$ surjects onto $\pi_1(\mathrm{S}^1, x_0) \cong \mathbb{Z}$. Thus $\pi_1(A, x_0)$ is non-trivial, again giving a contradiction to the injective map from $\pi_1(A, x_0)$ to $\pi_1(X, x_0)$.

(d) Let $C = \{(x, 0) \mid -1 \le x \le 1\}$ and define $F' : \mathrm{D}^2 \times I \to \mathrm{D}^2$ via $F'((x, y), t) = (x, (1-t)y)$. Then F' is continuous, $f'_0 = \mathrm{id}_{\mathrm{D}^2}$ and the image of f'_1 is in C and f'_1 restricted to C is the identity of C, hence f'_1 is a retraction. Thus C is a deformation retract of D^2. Since $F'((-1, 0), t) = F'((1, 0), t)$, we get a well-defined map $F : X \times I \to X$, via $F([x], t) = [F'(x, t)]$, that is a homotopy between the identity on X and a retraction $f = F(-, 1)$ from X to $C/_\sim$ (where we identify the same two points in C as we did in D^2). Since $C/_\sim \cong \mathrm{S}^1$ we thus see that X is homotopic to S^1 via the deformation retraction F.

Furthermore we can look at the subsets $I_+ = \{(x, y) \mid |(x, y)| = 1, y \ge 0\}$ and $I_- = \{(x, y) \mid |(x, y)| = 1, y \le 0\}$. Then both I_+ and I_- are homeomorphic to the interval $[-1, 1]$ (via the projection on the first coordinate) and $\pi(I_+) \cong \mathrm{S}^1 \cong \pi(I_-)$. Since they have exactly one point in common we thus get that $A = \pi(\mathrm{S}^1) = \pi(I_+ \cup I_-) \cong \mathrm{S}^1 \vee \mathrm{S}^1$.

Let a_+ be the image of a path from $(1, 0)$ to $(-1, 0)$ in I_+ under π and a_- the image of a path from $(1, 0)$ to $(-1, 0)$ in I_-. Then using the retraction $r' : \mathrm{S}^1 \vee \mathrm{S}^1 \to \mathrm{S}^1$ onto the first factor (as discussed in part (c)), we see that $r'_*([a_+]) = [\omega_1]$ in S^1, while $r'_*([a_-]) = 0$. Hence in $\pi_1(A, x_0)$ it holds $[a_+] \ne [a_-]$. Then $i_\star[a_+] \ne i_\star[a_-]$ by assumption on injectivity of $i_\star$. But applying the deformation retract from X to S^1, we obtain that

$$f_\star \circ i_\star[a_+] = [f \circ i \circ a_+] = [\omega] = [f \circ i \circ a_-] = f_\star \circ i_\star[a_-],$$

where $[\omega]$ is the class of the image of any path from $(1, 0)$ to $(-1, 0)$ in C under π. Since f was a retraction, $f_\star$ is injective and can be cancelled on both sides and we obtain the contradiction. Which is a contradiction to the injectivity of $i_\star$.