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Assignment 3 - Solutions

Exercise 1.

(a) Let $[v_0, v_1, v_2]$ be a 2-simplex. Define $X = [v_0, v_1, v_2]/_\sim$ with the equivalence relation given by $v_0 \sim v_1 \sim v_2$. Compute the simplicial homology of X.

(b) Let $[v_0, v_1, \ldots, v_n]$ be an n-simplex. For $1 \leq k \leq n$ and $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$ denote by $\varphi_{\underline{i}} : \Delta^k \to [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ the canonical homeomorphism from the standard k-simplex to $[v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ (as defined in the course).

Now define $X = [v_0, v_1, \ldots, v_n]/_\sim$ where for any $\underline{i} = (0 \leq i_1 < i_2 < \ldots < i_k \leq n)$, $\underline{j} = (0 \leq j_1 < j_2 < \ldots < j_k \leq n)$, and $x \in [v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$ we set $x \sim \varphi_{\underline{j}} \circ \varphi_{\underline{i}}^{-1}(x)$, i.e. we identify all k-simplices contained as iterative faces in $[v_0, v_1, \ldots, v_n]$ via the canonical homeomorphisms. Compute the simplicial homology of X.

Solutions: For any simplex $[v_0, \ldots, v_n]$ and $k \leq n$, we denote by $\sigma_{[v_{i_1}, \ldots, v_{i_k}]}$ the canonical homeomorphism from the standard k-simplex to the simplex $[v_{i_1}, \ldots, v_{i_k}] \subset [v_0, \ldots, v_n]$.
(a) As the Δ-complex structure of $[v_0, v_1, v_2]$, we choose

$$\Sigma = \{\sigma_{[v_0,v_1,v_2]}, \sigma_{[v_0,v_1]}, \sigma_{[v_0,v_2]}, \sigma_{[v_1,v_2]}, \sigma_{[v_0]}, \sigma_{[v_1]}, \sigma_{[v_2]}\}.$$

For the Δ-complex structure on X we use $\widetilde{\Sigma} = \{\pi \circ \sigma \mid \sigma \in \Sigma\}$ where $\pi : [v_0, v_1, v_2] \to X$ is the natural quotient map. Note that $\widetilde{\sigma}_{[v_0]} = \widetilde{\sigma}_{[v_1]} = \widetilde{\sigma}_{[v_2]}$, hence there is only a single generator for the 0-chains. To see that this is a Δ-complex structure, we need to check part (1), (2) and (3) from the definition:

(1) Since π is a homeomorphism outside of $\{v_0, v_1, v_2\}$, any $\widetilde{\sigma} \in \widetilde{\Sigma}$ is injective when restricted to the interior to the interior of the standard simplex.

(2) For (2) we only need to check if we restrict a $\widetilde{\sigma}_{[v_i,v_j]}$ $(i < j)$ to one of the faces of the standard 1-simplex. But such a face is a 0-simplex, for which we have a unique map in our Δ-complex structure, hence (2) is automatically full-filled.

(3) Since our Δ-complex structure is obtained from the one of $[v_0, v_1, v_2]$ via a quotient map, part (3) is automatic.

We now have the following non-trivial chain groups

$$\Delta_2(X) = \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1,v_2]},\ \Delta_1(X) = \mathbb{Z}\widetilde{\sigma}_{[v_0,v_1]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_0,v_2]} \oplus \mathbb{Z}\widetilde{\sigma}_{[v_1,v_2]},\ \text{and}\ \Delta_0(X) = \mathbb{Z}\widetilde{\sigma}_{[v_0]}.$$

For the simplicial homology we thus obtain

- Since $\partial_2(\widetilde{\sigma}_{[v_0,v_1,v_2]}) = \widetilde{\sigma}_{[v_1,v_2]} - \widetilde{\sigma}_{[v_0,v_2]} + \widetilde{\sigma}_{[v_0,v_1]}$, ∂_2 is injective and so $H_2^\Delta(X) = 0$.
- Note that $\partial_1(\widetilde{\sigma}_{[v_i,v_j]}) = 0$ for all $i < j$, hence the kernel of ∂_1 is all of $\Delta_1(X)$ and so

$$H_1^\Delta(X) = \Delta_1(X)/\left\langle \widetilde{\sigma}_{[v_1,v_2]} - \widetilde{\sigma}_{[v_0,v_2]} + \widetilde{\sigma}_{[v_0,v_1]} \right\rangle \cong \mathbb{Z}^2.$$

- As already mentioned, the image of ∂_1 is trivial, hence $H_0^\Delta(X) = \Delta_0(X) \cong \mathbb{Z}$.

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(b) As a Δ-complex structure on X we use

$$\widetilde{\Sigma} = \{\pi \circ \sigma_{[v_0,\dots,v_k]} \mid 0 \leq k \leq n\},$$

with π being the quotient map. This forms a Δ-complex structure

(1) For property (1), we note that for every $0 \leq k \leq n$, no two points in the interior of $[v_{i_1}, \dots, v_{i_k}]$ get identified in the quotient. Via the equivalence relation, they are only equivalent to a unique other point in any other k-simplex. Hence we still have injectivity.

(2) For (2) we note that all k-simplices get identified, hence the face of a $k+1$-simplex is always the unique k-simplex. Since they are all identified via the canonical homeomorphism, the trianlge in the definition part (2) commutes as well.

(3) As before, since we are going to a quotient, part (3) is automatic.

As chain groups we thus get $\Delta_k(X) = \mathbb{Z}\widetilde{\sigma}_{[v_0,\dots,v_k]}$ for $0 \leq k \leq n$, all other ones are trivial. For the image ∂_k we just check

$$\partial_k(\widetilde{\sigma}_{[v_0,\dots,v_k]}) = \sum_{i=0}^{k} (-1)^i \widetilde{\sigma}_{[v_0,\dots,v_{k-1}]} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \widetilde{\sigma}_{[v_0,\dots,v_{k-1}]} & \text{if } k \text{ is even.} \end{cases}$$

Hence we see that ∂_k is an isomorphism when k is even and the zero map if k is odd. Hence we obtain for $0 < k < n$ we get

$$H_k^{\Delta}(X) \cong \begin{cases} \{0\}/\{0\} \cong \{0\} & \text{for } k \text{ even} \\ \mathbb{Z}/\mathbb{Z} \cong \{0\} & \text{for } k \text{ odd.} \end{cases}$$

In the two extreme cases we get $H_0^{\Delta}(X) = \mathbb{Z}$, since the image of ∂_1 is trivial, and finally, since there are no $n+1$-simplices

$$H_n^{\Delta}(X) = \mathrm{Ker}(\partial_n) \cong \begin{cases} \{0\} & \text{for } k \text{ even.} \\ \mathbb{Z} & \text{for } k \text{ odd} \end{cases}$$

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Exercise 2. Let $r : X \to A$ be a retraction of a space X to a subspace A and $i : A \to X$ the inclusion. Show that $i_* : H_n(A) \to H_n(X)$ is injective and $r_* : H_n(X) \to H_n(A)$ is surjective for all $n \geq 0$.

Solutions: By definition of a retraction we have $r \circ i = \mathrm{id}_A$. Applying now homology to this we get, for any n,

$$r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_A)_* = \mathrm{id}_{H_n(A)}.$$

Hence the map r_* needs to be surjective as it has a right inverse and the map i_* needs to be injective, since it has a left inverse.

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Exercise 3. Let $A = \mathrm{S}^2$ and $D = \{(x,0,0) \mid -1 \leq x \leq 1\}$ inside $\mathbb{R}^3$.

(a) Compute the simplicial homology of $X = A \cup D$.

(b) Compute the simplicial homology of the simply-connected covering space $\widetilde{X}$ of X.

As a reminder: There is an explicit construction of $\widetilde{X}$ in Assignment 2 - Solutions, Exercise 4(c). Let $\Theta_k : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation defined by $\Theta_k(x,y,z) = (x+k,y,z)$. Define $A_k = \Theta_{2k}(A)$, for $k \in \mathbb{Z}$ even, and $A_k = \Theta_{2k}(D)$, for k odd, and set

$$\widetilde{X} = \bigcup_{k\in\mathbb{Z}} A_k \subset \mathbb{R}^3.$$

Solutions: We fix the following Δ-complex structure on A and D.
For A, we fix the following points on the (x,y)-plane equator of the sphere: $v_{-1} = (-1,0,0)$, $v_1 = (1,0,0)$ and $v_0 = (0,1,0)$. Then let $\sigma^+_{[v_{-1},v_0,v_1]}$ be the homeomorphism between Δ^2 and the upper half-sphere (i.e. z-coordinate positive) sending the standard vertices of Δ^2 to v_{-1}, v_0 and v_1 (in this order). Let $\sigma^-_{[v_{-1},v_0,v_1]}$ be the same homeomorphism but we negate the z-coordinate (i.e. $\sigma^-_{[v_{-1},v_0,v_1]}(x) = (a,b,-c)$ for $\sigma^+_{[v_{-1},v_0,v_1]}(x) = (a,b,c)$. We define $\sigma_{[v_{-1},v_0]}$, $\sigma_{[v_{-1},v_1]}$, and $\sigma_{[v_0,v_1]}$ to be restrictions of $\sigma^+_{[v_{-1},v_0,v_1]}$ (or $\sigma^-_{[v_{-1},v_0,v_1]}$, since they agree on points mapped to the equator) to the corresponding face pre-composed with the canonical homeomorphism from the standard 1-simplex to the face. Finally $\sigma_{[v_{-1}]}$, $\sigma_{[v_0]}$, and $\sigma_{[v_1]}$ are mapping the standard 0-simplex to the corresponding point. These form the Δ-complex structure Σ_A.
For D we use the Δ-complex structure consisting of $\sigma'_{[v_{-1},v_1]}$ that maps the standard 1-simplex homeomorphically onto D and the maps $\sigma_{[v_{-1}]}$ and $\sigma_{[v_1]}$. These form the Δ-complex structure Σ_D.
For both cases part (1) of the definition of a Δ-complex structure is obvious by construction, all the maps are even homeomorphisms. Part (2) of the definition only needs to be checked for the faces for the simplicial 2-simplices for A. But there (2) holds by construction. As already noted, all maps are homeomorphisms onto their image, hence (3) holds automatically as well.

(a) For the Δ-complex of X we take the union $\Sigma_A \cup \Sigma_D$. Note that $\sigma'_{[v_{-1},v_1]}$ does not appear as the face of any of the simplicial 2-simplexes.
Since it holds $\partial_2(\sigma^+_{[v_{-1},v_0,v_1]}) = \partial_2(\sigma^-_{[v_{-1},v_0,v_1]})$, the kernel of ∂_2 is generated by $\sigma^+_{[-1,v_0,v_1]} - \sigma^-_{[v_{-1},v_0,v_1]}$, hence $H_2^\Delta(X) \cong \mathbb{Z}$.
For ∂_1 it holds that $\mathrm{Ker}(\partial_1) = \left\langle \sigma_{[v_{-1},v_0]} + \sigma_{[v_0,v_1]} - \sigma_{[v_{-1},v_1]}, \sigma_{[v_{-1},v_0]} + \sigma_{[v_0,v_1]} - \sigma'_{[v_{-1},v_1]} \right\rangle$. Since we have $\partial_2(\sigma^+_{[v_{-1},v_0,v_1]}) = \sigma_{[v_0,v_1]} - \sigma_{[v_{-1},v_1]} + \sigma_{[v_0,v_1]}$, we see that the quotient has rank 1 and so $H_1^\Delta(X) \cong \mathbb{Z}$.
Finally $\mathrm{Im}(\partial_1) = \langle \sigma_{[v_1]} - \sigma_{[v_{-1}]}, \sigma_{[v_0]} - \sigma_{[v_{-1}]}, \sigma_{[v_0]} - \sigma_{[v_1]} \rangle$, which together with $\mathrm{Ker}(\partial_0) = \Delta_0(X)$ implies that $H_0^\Delta(X) \cong \mathbb{Z}$.

(b) For the Δ-complex structure of $\widetilde{X}$ we compose the maps from part (a) with the translations Θ_{2k} and use

$$\Sigma = \{\Theta_{2k} \circ \sigma \mid \sigma \in \Sigma_A, k \text{ even }\} \cup \{\Theta_{2k} \circ \sigma \mid \sigma \in \Sigma_D, k \text{ odd }\}.$$

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Thus the translates of the form $\Theta_{2k}\circ\sigma$ for $\sigma \in \Sigma_A$ give a Δ-complex for all the translates of A, while the second set gives a Δ-complex structure for all the translates of D. By construction $H_n^\Delta(\widetilde{X}) = 0$ for $n > 2$. For the other cases we have

(a) For $n = 2$, note that the images of $\Theta_{2k} \circ \sigma$ and $\Theta_{2l} \circ \sigma'$ for $\sigma, \sigma' \in \Delta_2(A)$ are linearly independent for $k \neq l$ and both even. Hence the kernel of ∂_2 is generated by $\Theta_{2k} \circ \sigma^+_{[-1,v_0,v_1]} - \Theta_{2k} \circ \sigma^-_{[-1,v_0,v_1]}$ for all k even. Hence $H_2^\Delta(\widetilde{X}) \cong \mathbb{Z}^\infty$ (one copy of $\mathbb{Z}$ for each translate of A).

(b) For $n = 1$, note that the image of ∂_2 is generated by $\Theta_{2k} \circ \sigma_{[v_0,v_1]} - \Theta_{2k} \circ \sigma_{[v_{-1},v_1]} + \Theta_{2k} \circ \sigma_{[v_0,v_1]}$ for each k even. While the kernel of ∂_1 is generated by $\Theta_{2k} \circ \sigma_{[v_{-1},v_0]} + \Theta_{2k} \circ \sigma_{[v_0,v_1]} - \Theta_{2k} \circ \sigma_{[v_{-1},v_1]}$ for k even. Hence $H_1^\Delta(\widetilde{X}) \cong \{0\}$.

(c) Finally the image of ∂_1 contains all elements of the form $\Theta_{2k} \circ \sigma_{[v_1]} - \Theta_{2k} \circ \sigma_{[v_{-1}]}$, $\Theta_{2k} \circ \sigma_{[v_0]} - \Theta_{2k} \circ \sigma_{[v_{-1}]}$, and $\Theta_{2k} \circ \sigma_{[v_0]} - \Theta_{2k} \circ \sigma_{[v_1]}$ (for k even) coming from the 1-chains with images in the translated spheres. From the 1-chains in the translated copies of D we obtain the elements $\Theta_{2k} \circ \sigma_{[v_1]} - \Theta_{2k} \circ \sigma_{[v_{-1}]}$ (for k odd). Note that for even k it holds $\Theta_{2k} \circ \sigma_{[v_{-1}]} = \Theta_{2(k-1)} \circ \sigma_{[v_1]}$ and $\Theta_{2k} \circ \sigma_{[v_1]} = \Theta_{2(k+1)} \circ \sigma_{[v_{-1}]}$. Hence in the quotient any two generators of the 0-chains are in the same coset and so $H_0^\Delta(\widetilde{X}) \cong \mathbb{Z}$.

Alternatively: One can also use from the lecture that $\widetilde{X}$ is path-connected, which implies that $H_0(\widetilde{X}) \cong \mathbb{Z}$ and then use that simplicial and singular homology coincide.

To summarize we obtain

$$H_2^\Delta(\widetilde{X}) \cong \mathbb{Z}^\infty,\ H_1^\Delta(\widetilde{X}) \cong \{0\},\ \text{and } H_0^\Delta(\widetilde{X}) \cong \mathbb{Z}.$$