Algebraic Geometry
Beijing Institute of Technology, 2019

Exercises for chapter 2: Algebraic sets

We use the notations from the course, i.e. $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ its algebraic closure, $\mathbb{K}[X_1, \ldots, X_n]$.

Exercise 1. Let T and T' be topological spaces. Show the following

(a) $\varnothing \neq S \subset T$ is irreducible if and only if $\overline{S} \subset T$ is irreducible,

(b) T is irreducible if and only if $\overline{U} = T$ for all open subsets $\varnothing \neq U \subset T$,

(c) and for $f : T \to T'$ continuous it holds, if T is irreducible, then $f(T) \subset T'$ is irreducible.

Exercise 2. Let $f : R \to R'$ be a surjective map of rings and

$$\left\{ \text{ ideals of } R' \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{ideals of } R \\ \text{containing } \ker(f) \end{array} \right\}.$$

, where $\Phi(I) = f^{-1}(I)$ and $\Psi(J) = f(J)$. Show that

(a) Φ and Ψ define inclusion preserving and mutually inverse bijections,

(b) this can be restricted to prime ideals in R' on the left hand side and prime ideals containing $\ker(f)$ in R on the right hand side,

(c) and that this can be restricted to maximal ideals in R' on the left hand side and maximal ideals containing $\ker(f)$ in R on the right hand side.

Exercise 3. Let $V \subset \overline{\mathbb{K}}^n$ be a $\mathbb{K}$-algebraic set. Show that

(a) for $I \subset \mathbb{K}[V]$ an ideal $\mathcal{I}_V(\mathcal{V}_V(I)) = \mathrm{Rad}(I)$,

(b) the assignment $W \mapsto \mathcal{I}_V(W)$ defines a bijection between the set of $\mathbb{K}$-closed subsets $W \subset V$ and the set of radical ideals of $\mathbb{K}[V]$,

(c) and that the assignment from part (b) can be restricted to an assignment between irreducible closed subsets of V and prime ideals in $\mathbb{K}[V]$.