Algebraic Geometry

Beijing Institute of Technology, 2019

Exercises for chapter 1: Commutative algebra

We use the notation from the course, i.e. $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ its algebraic closure, $\mathbb{K}[X_1, \ldots, X_n]$ denotes the polynomial ring in n variables and R is a commutative ring.

Exercise 1. For $(a_1, \ldots, a_n) \in \mathbb{K}^n$ define the ideal $I(a_1, \ldots, a_n)$ in $\mathbb{K}[X_1, \ldots, X_n]$ generated by $X_i - a_i$ for $1 \leq i \leq n$.

(a) Find an example for a maximal ideal $\mathfrak{m}$ in $\mathbb{K}[X_1, \ldots, X_n]$ for some n such that $\mathfrak{m} \neq I(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in \mathbb{K}^n$.

(b) Show that $I(a_1, \ldots, a_n) = I(b_1, \ldots, b_n)$ if and only if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$.

(c) Assume that $\mathbb{K}$ has infinitely many elements. Show that for any $f \in \mathbb{K}[X_1, \ldots, X_n]$ such that $f \neq 0$ there are infinitely many $(a_1, \ldots, a_n) \in \mathbb{K}^n$ such that $f \notin I(a_1, \ldots, a_n)$.

Exercise 2. Show the following alternative definitions of R being noetherian

(a) R is noetherian if and only if any ascending chain $I_1 \subset I_2 \subset I_3 \subset \ldots$, of ideals $\{I_j \mid j \in \mathbb{Z}_{\geq 1}\}$ in R, is stationary, i.e. there exists a $n \in \mathbb{Z}_{\geq 1}$ such that $I_n = I_{n+k}$ for all $k \in \mathbb{Z}_{\geq 0}$.

(b) R is noetherian if and only if any collection $\{I_j \mid j \in J\}$ of ideals in R has a maximal element with respect to the inclusion of ideals.

Exercise 3. For a subset $S \subset \mathbb{K}[X_1, \ldots, X_n]$ denote by $\mathcal{V}(S)$ the vanishing set of S in $\overline{\mathbb{K}}$ and for a subset $V \subset \overline{\mathbb{K}}$ denote by $\mathcal{I}(V)$ the vanishing ideal of V in $\mathbb{K}[X_1, \ldots, X_n]$.

(a) Show the following properties of $\mathcal{V}$:

(i) $S \subset S' \subset \mathbb{K}[X_1, \ldots, X_n]$ then $\mathcal{V}(S') \subset \mathcal{V}(S)$,

(ii) $\mathcal{V}(S) = \mathcal{V}(I)$ for I the ideal generated by the set S,

(iii) for $\{I_j \mid j \in J\}$ an arbitrary collection of ideals in $\mathbb{K}[X_1, \ldots, X_n]$ it holds

$$\bigcap_{j \in J} \mathcal{V}(I_j) = \mathcal{V}\left(\sum_{j \in J} I_j\right),$$

(iv) and for I, I' ideals in $\mathbb{K}[X_1, \ldots, X_n]$ then $\mathcal{V}(I) \cup \mathcal{V}(I') = \mathcal{V}(I \cap I') = \mathcal{V}(I \cdot I')$.

(b) Show the following properties of $\mathcal{I}$:

(i) $\mathcal{I}(\emptyset) = \mathbb{K}[X_1, \ldots, X_n]$ and $\mathcal{I}(\overline{\mathbb{K}}^n) = (0)$,

(ii) for $V \subset \overline{\mathbb{K}}^n$ $\mathbb{K}$-algebraic it holds $\mathcal{V}(\mathcal{I}) = V$,

(iii) and for $V, W \subset \overline{\mathbb{K}}^n$ $\mathbb{K}$-algebraic, it holds $\mathcal{I}(V \cup W) = \mathcal{I}(V) \cap \mathcal{I}(W)$.