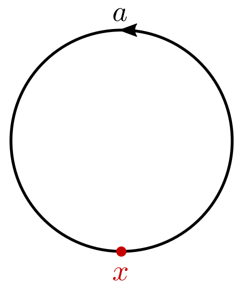
Example 2.10: Simplicial Homology examples

We give a few examples for Δ-complexes and the resulting simplicial homology.

(a) Let $X = \mathrm{S}^1$. For the first Δ-complex structure, we use a 0-simplex and a 1-simplex. This is indicated in the picture



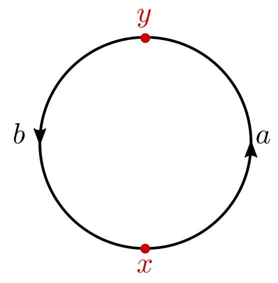
The map σ_x is obvious, while for σ_a we just indicate by the arrow how the standard 1-simplex $[v_0, v_1]$ gets mapped onto a. The exact definition of σ_a is not important, the key features are that it maps the faces of the standard 1-simplex Δ^1 to x and σ_a is injective on the interior of Δ^1. The only non-trivial chain groups are

$$\Delta_0(X) = \mathbb{Z}\sigma_x \text{ and } \Delta_1(X) = \mathbb{Z}\sigma_a.$$

and the only (potentially) non-trivial map $\partial_1 : \Delta_1(X) \to \Delta_0(X)$ is given by $\partial_1(\sigma_a) = \sigma_x - \sigma_x = 0$, hence is also trivial. Thus we see that for the simplicial homology we obtain

$$H_0^\Delta(X) = \mathbb{Z}, \quad H_1^\Delta(X) = \mathbb{Z}, \text{ and } H_n^\Delta(X) = \{0\} \text{ for } n \geq 2.$$

We want to see in this trivial example what happens if we change the Δ-complex structure of X. For this we choose a structure as indicated in the picture

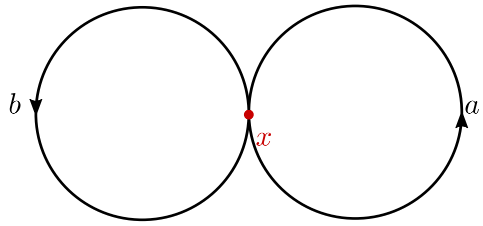


Now $\Delta_0(X) = \mathbb{Z}\sigma_x \oplus \mathbb{Z}\sigma_y$ and $\Delta_1(X) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b$ and the map ∂_1 is defined via $\partial_1(\sigma_a) = \sigma_y - \sigma_x$ and $\partial_1(\sigma_b) = \sigma_x - \sigma_y$. In this case $\mathrm{Ker}(\partial_1) = \langle \sigma_a + \sigma_b \rangle$ and $\mathrm{Im}(\partial_1) = \langle \sigma_x - \sigma_y \rangle$. Hence we obtain

$$\begin{aligned}
&H_0^\Delta(X) = \Delta_0(X)/\langle \sigma_x - \sigma_y \rangle \cong \mathbb{Z}, \\
&H_1^\Delta(X) = \mathrm{Ker}(\partial_1) = \langle \sigma_a + \sigma_b \rangle \cong \mathbb{Z}, \text{ and} \\
&H_n^\Delta(X) = \{0\} \text{ for } n \geq 2.
\end{aligned}$$

We see in this example that, up to isomorphism, the simplicial homology groups do not change if we change the Δ-complex structure, this is something that we want to prove in general.

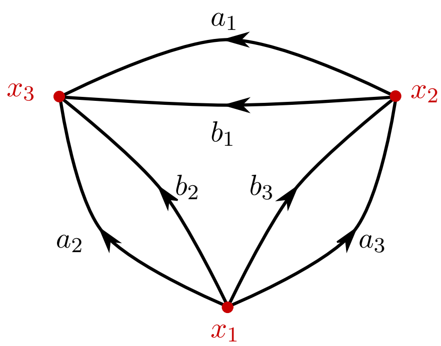
(b) As for the fundamental group, the next space that had an interesting structure was $X = \mathrm{S}^1 \vee \mathrm{S}^1$, we embed this space into $\mathbb{R}^2$ as a figure 8 and use the Δ-complex structure as indicated in the picture:



Then we see that the map ∂_1 is again trivial as for the circle and we obtain

$$H_0^\Delta(X) = \mathbb{Z}, \quad H_1^\Delta(X) = \mathbb{Z}^2, \text{ and } H_n^\Delta(X) = \{0\} \text{ for } n \geq 2.$$

This is in line with what happens for the fundamental group. But we want to look at a covering space of X, the space $\widetilde{X}$ as given in the following picture:



The covering space map $p : \widetilde{X} \to X$ is given by sending a point x_i to x and an edge a_i, respectively b_i, to a, respectively b, according to the orientation. Using the Δ-complex structure indicated in the picture above we see that there is one non-trivial chain map, given by

$$\partial_1(\sigma_{a_1}) = \partial_1(\sigma_{b_1}) = \sigma_{x_3} - \sigma_{x_2}, \quad \partial_1(\sigma_{a_2}) = \partial_1(\sigma_{b_2}) = \sigma_{x_2} - \sigma_{x_1},$$
$$\partial_1(\sigma_{a_3}) = \partial_1(\sigma_{b_3}) = \sigma_{x_1} - \sigma_{x_3}.$$

The image of ∂_1 has rank 2 inside $\Delta_0(\widetilde{X}) \cong \mathbb{Z}^3$ and we see that $H_0^\Delta(\widetilde{X}) \cong \mathbb{Z}$ as expected. While the kernel of ∂_1 is equal to

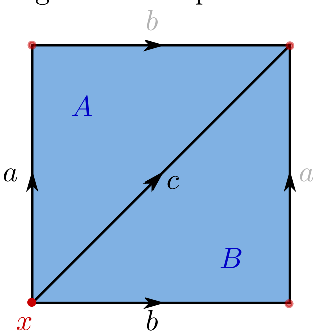
$$\mathrm{Ker}(\partial_1) = \langle \sigma_{a_1} - \sigma_{b_1}, \sigma_{a_2} - \sigma_{b_2}, \sigma_{a_3} - \sigma_{b_3}, \sigma_{a_1} + \sigma_{a_2} - \sigma_{a_3} \rangle ,$$

hence $H_0^\Delta(\widetilde{X}) \cong \mathbb{Z}^4$.
We want to compare this to the fundamental group of $\widetilde{X}$. We know that $\pi_1(\widetilde{X}, x_1)$ is a subgroup of $\pi_1(X, x)$, since $\widetilde{X}$ is a covering space of X. Since $\widetilde{X}$ is a graph, applying the van Kampen theorem is quite easy and one obtains $\pi_1(\widetilde{X}, x_1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Applying the map p_* one gets that

$$p_*(\pi_1(\widetilde{X}, x_1)) = \left\langle [a^3], [a^2 b], [a^2 b \overline{a}], [a \overline{b}] \right\rangle .$$

From this presentation it is not even immediately clear that the subgroup is isomorphic to $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, which of course it is since p_* is injective. This gives an example for the free group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ realized as a subgroup of the free group $\mathbb{Z} * \mathbb{Z}$.

(c) We now want to look at surfaces. The most basic one being the torus $X = \mathrm{T}^2$. We use the Δ-complex structure of $\mathrm{I} \times \mathrm{I}$ as given in the picture below:



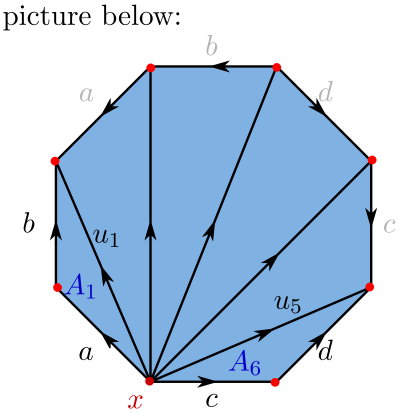
The edges with grey edge labels are identified with their opposite edges in the quotient, hence they do not give any additional 1-simplices in X. Similarly all vertices get identified in the quotient, hence there is only a single 0-simplex. In this case there are three chain groups that are non-trivial

$$\Delta_0(X) = \mathbb{Z}\sigma_x, \quad \Delta_1(X) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c, \text{ and } \Delta_2(X) = \mathbb{Z}\sigma_A \oplus \mathbb{Z}\sigma_B.$$

Since x is start and end point of every 1-simplices, it is obvious that ∂_1 is the trivial map, while for ∂_2 we have $\partial_2(\sigma_A) = \sigma_b - \sigma_c + \sigma_a$ and $\partial_2(\sigma_B) = \sigma_a - \sigma_c + \sigma_b$ by Definition 2.6. We see that $\mathrm{Ker}(\partial_2) = \langle \sigma_A - \sigma_B \rangle$ and $\mathrm{Im}(\partial_2) = \langle \sigma_a - \sigma_c + \sigma_b \rangle$. This gives us

$$\begin{aligned}
&H_0^\Delta(X) = \langle \sigma_x \rangle \cong \mathbb{Z},\\
&H_1^\Delta(X) = \Delta_1(X)/\mathrm{Im}(\partial_2) \cong (\langle \sigma_a, \sigma_b \rangle \oplus \mathrm{Im}(\partial_2))/\mathrm{Im}(\partial_2) \cong \mathbb{Z}^2,\\
&H_2^\Delta(X) = \langle \sigma_A - \sigma_B \rangle \cong \mathbb{Z}, \text{ and } H_n^\Delta(X) = \{0\} \text{ for } n \geq 3.
\end{aligned}$$

This procedure generalizes to the surface M_g of genus g we defined in Chapter 0. We illustrate the Δ-complex for M_2 in the picture below:

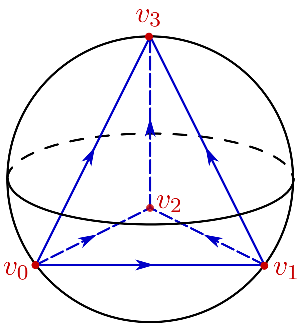


As indicated in the picture, we label, from left to right, the edges in the interior of the octagon by $u_1, \ldots, u_5$ and the enclosed triangular areas by $A_1, \ldots, A_6$. The simplicial homology is then given by

$$\begin{aligned}
&H_0^\Delta(M_2) = \langle \sigma_x \rangle \cong \mathbb{Z},\\
&H_1^\Delta(M_2) = \Delta_1(X)/\mathrm{Im}(\partial_2) \cong (\langle \sigma_{u_1}, \sigma_{u_2}, \sigma_{u_3}, \sigma_{u_4} \rangle \oplus \mathrm{Im}(\partial_2))/\mathrm{Im}(\partial_2) \cong \mathbb{Z}^4,\\
&H_2^\Delta(M_2) = \langle \sigma_{A_1} - \sigma_{A_2} - \sigma_{A_3} + \sigma_{A_4} + \sigma_{A_5} - \sigma_{A_6} \rangle \cong \mathbb{Z}, \text{ and } H_n^\Delta(M_2) = \{0\} \text{ for } n \geq 3.
\end{aligned}$$

In general $H_1^\Delta(M_g) \cong \mathbb{Z}^{2g}$ and $H_0^\Delta(M_g) \cong \mathbb{Z} \cong H_2^\Delta(M_g)$. This can be done by just generalizing the Δ-complex structure we defined here for M_2 and $M_1 = \mathrm{T}^2$ to M_g.

(d) We have already seen that different Δ-complex structure gave the same homology in an example, we want to quickly illustrate how this can also affect calculation efficiency. For this we take $X = \mathrm{S}^2$, then we can choose 4 distinct points on X, denoted by v_0, v_1, v_2, and v_3 and form the corresponding tetrahedron in $\mathbb{R}^3$ such that the origin of $\mathbb{R}^3$ is contained in the interior of the tetrahedron. Using the retraction from $\mathrm{D}^3 \setminus \{(0,0,0)\}$ onto S^2 we obtain a homeomorphism between the tetrahedron and X. Hence, up to this homeomorphism, we use the Δ-complex structure of the tetrahedron for X:



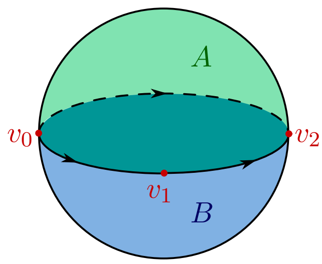
Thus we have a 3-simplex $[v_0, v_1, v_2, v_3]$, its faces as 2-simplices, their faces as 1-simplices and so on, thus

$$\Delta_0(X) \cong \mathbb{Z}^4, \quad \Delta_1(X) \cong \mathbb{Z}^6, \quad \Delta_2(X) \cong \mathbb{Z}^4, \text{ and } \Delta_n(X) = \{0\} \text{ for } n \geq 3.$$

Defining the differentials according to Definition 2.6 one sees

$$H_0^\Delta(X) \cong \mathbb{Z}, \quad H_1^\Delta(X) = \{0\}, \quad H_2^\Delta(X) \cong \mathbb{Z}, \text{ and } H_n^\Delta(X) = \{0\} \text{ for } n \geq 3.$$

But calculation-wise we made our life difficult, since we could have also use:

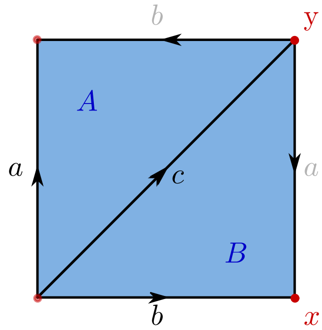


Hence we use only two 2-simplices, A and B, being the upper and lower half-spheres. They have the same boundary, namely the equator, which is divided into three 1-simplices, with the labelling of the vertices indicating how the map from the standard 2-simplex onto A respectively B is defined. In this case

$$\Delta_0(X) \cong \mathbb{Z}^3, \quad \Delta_1(X) \cong \mathbb{Z}^3, \quad \Delta_2(X) \cong \mathbb{Z}^2, \text{ and } \Delta_n(X) = \{0\} \text{ for } n \geq 3,$$

which are a lot lower ranks for such a small example. Of course the calculation gives, up to isomorphism, the same homology groups.

(e) We now look at a modification of the torus example, by using a different quotient of $\mathrm{I} \times \mathrm{I}$. We indicate the different orientations in the picture below:

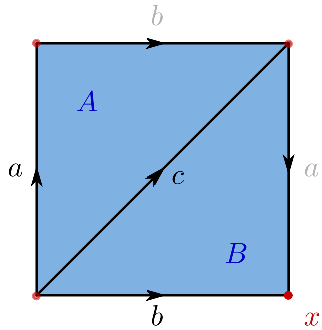


The resulting space is the real projective plane $\mathbb{R}P^2$. It is an example of a non-orientable real manifold and we want to see if we can see a difference in the corresponding homology groups. Note that there are two 0-simplices in X. With the notations as in the picture, one deduces that the simplicial homology groups are

$$\begin{aligned}
H_0^{\Delta}(X) &= \Delta_0(X)/\left\langle \sigma_x - \sigma_y \right\rangle \cong \mathbb{Z}, \\
H_1^{\Delta}(X) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \langle \sigma_a - \sigma_b + \sigma_c, \sigma_c \rangle \,/\, \langle \sigma_a - \sigma_b + \sigma_c, \sigma_b - \sigma_a + \sigma_c \rangle \\
&= \langle \sigma_a - \sigma_b + \sigma_c, \sigma_c \rangle \,/\, \langle \sigma_a - \sigma_b + \sigma_c, 2\sigma_c \rangle \cong \mathbb{Z}/2\mathbb{Z}, \\
H_2^{\Delta}(X) &= \{0\} \text{ since } \partial_2 \text{ is injective, and } H_n^{\Delta}(X) = \{0\} \text{ for } n \geq 3.
\end{aligned}$$

We see here a first example where a group with finite order shows up.

(f) Very similar to the previous example is the Klein bottle, with the quotient construction and Δ-complex structure for X shown in the picture below:



In this case the resulting homology groups are

$$\begin{aligned}
H_0^{\Delta}(X) &= \Delta_0(X)/\{0\} \cong \mathbb{Z}, \\
H_1^{\Delta}(X) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \Delta_1(X)/\left\langle \sigma_b - \sigma_c + \sigma_a, \sigma_a - \sigma_b + \sigma_c \right\rangle \\
&= \Delta_1(X)/\left\langle \sigma_a - \sigma_b + \sigma_c, 2\sigma_a \right\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\
H_2^{\Delta}(X) &= \{0\} \text{ since } \partial_2 \text{ is injective, and } H_n^{\Delta}(X) = \{0\} \text{ for } n \geq 3.
\end{aligned}$$

Note that there is only one edge with a different orientation compared to the torus case. The homology group $H_1^{\Delta}(X)$ is in a sense in between the one for the torus and the real projective plane.