
ALGEBRAIC GEOMETRY
BEIJING INSTITUTE OF TECHNOLOGY, 2019

Exercises for chapter 2: Algebraic sets

We use the notations from the course, i.e. \mathbb{K} is a field, $\overline{\mathbb{K}}$ its algebraic closure, and so on.

Exercise 1. Let T and T' be topological spaces. Show the following

- (a) $\emptyset \neq S \subset T$ is irreducible if and only if $\overline{S} \subset T$ is irreducible,
- (b) T is irreducible if and only if $\overline{U} = T$ for all open subsets $\emptyset \neq U \subset T$,
- (c) and for $f : T \rightarrow T'$ continuous it holds that if T is irreducible then $f(T) \subset T'$ is irreducible.

Exercise 2. Let $f : R \rightarrow R'$ be a surjective map of rings and

$$\left\{ \text{ideals of } R' \right\} \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{array} \left\{ \begin{array}{c} \text{ideals of } R \\ \text{containing } \ker(f) \end{array} \right\},$$

where $\Phi(I) = f^{-1}(I)$ and $\Psi(J) = f(J)$. Show that

- (a) Φ and Ψ define inclusion preserving bijections that are mutually inverse to each other,
- (b) that this can be restricted to prime ideals of R' on the left hand side and prime ideals of R containing $\ker(f)$ on the right hand side,
- (c) and that this can be restricted to maximal ideals of R' on the left hand side and maximal ideals of R containing $\ker(f)$ on the right hand side.

Exercise 3. Let $V \subset \overline{\mathbb{K}}^n$ be a \mathbb{K} -algebraic set. Show that

- (a) for $I \subset \mathbb{K}[V]$ an ideal it holds $\mathcal{I}_V(\mathcal{V}_V(I)) = \text{Rad}(I)$,
- (b) the assignment $W \mapsto \mathcal{I}_V(W)$ defines a bijection between the set of \mathbb{K} -closed subset of $\overline{\mathbb{K}}^n$ contained in V and the set of radical ideals of $\mathbb{K}[V]$,
- (c) and that the assignment from part (b) can be restricted to irreducible \mathbb{K} -closed subsets of V on the left hand side and to prime ideals in $\mathbb{K}[V]$ on the right hand side.