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Assignment 1

We use the notations and conventions from the course, e,g. a map refers to a continuous map, a space refers to a topological space, S^1 denotes the unit circle, D^2 the unit disc, etc.

Exercise 1. Let $\mathcal{I} = \{0, \frac{1}{2}, 1\}$ and fix the collection of subsets $\mathcal{B} = \{\{0\}, \{1\}, \mathcal{I}\}$.

(a) Show that $\mathcal{B}$ is a basis for a topology of $\mathcal{I}$.

(b) Determine all open subsets of $\mathcal{I}$ with respect to the topology $\mathcal{U}_{\mathcal{B}}$ generated by $\mathcal{B}$.

(c) Show that $\mathcal{I}$ is path-connected and contractible.

Exercise 2.

(a) Let $f : \mathrm{D}^2 \to \mathrm{D}^2$ be a map such that $f(x) = x$ for $x \in \mathrm{S}^1$. Show that f is surjective.
Hint: This can be done very similar to the proof of the Brouwer fixed point theorem in the course.

(b) Show that a loop $f : \mathrm{I} \to \mathrm{S}^1$ with the property $f(s + \frac{1}{2}) = -f(s)$ for $0 \leq s \leq \frac{1}{2}$ cannot be nullhomotopic, i.e. $f \not\simeq \omega_0$.
Hint: We used a loop with this property in a proof in the course.

(c) Show that there exists no map $g : \mathrm{S}^2 \to \mathrm{S}^1$ such that $g(-x) = -g(x)$.
Hint: Using such a map g, construct a suitable loop to apply part (b).

Exercise 3. Let X be a space.

(a) Show that X is contractible if and only if id_X is nullhomotopic.

(b) Assume that X is contractible and there exists $A \subset X$ together with a retraction $r : X \to A$. Show that A is contractible.

(c) Show that the following are equivalent

(i) X is contractible,
(ii) every map $f : X \to Y$ (for a space Y) is nullhomotopic,
(iii) every map $f : Y \to X$ (for a space Y) is nullhomotopic.

Exercise 4. Show that there exists no retraction $r : X \to A$ for the following situations

(a) $X = \mathbb{R}^n$ and A an arbitrary subspace homeomorphic to S^1.

(b) $X = \mathrm{S}^1 \times \mathrm{D}^2$ and $A = \mathrm{S}^1 \times \mathrm{S}^1$ the boundary of X.

(c) $X = \mathrm{D}^2 \vee \mathrm{D}^2$ and $A = \mathrm{S}^1 \vee \mathrm{S}^1$ for the pointed spaces $(\mathrm{D}^2, (1, 0))$ and $(\mathrm{S}^1, (1, 0))$.
Hint: One can argue with the methods from the course why A has a non-trivial fundamental group without specifically calculating it.

(d) $X = \mathrm{D}^2/_\sim$, for $(1, 0) \sim (-1, 0)$, and $A = \pi(\mathrm{S}^1)$ the image of the boundary of D^2 under the quotient map π.
Hint: There is a natural way to identify A with $\mathrm{S}^1 \vee \mathrm{S}^1$.

Due date: October 17, 2021
submission in English either by email or on paper, include name and student ID number