PMH2: Commutative Algebra

University of Sydney, 2018

Assignment 1

Let R denote an associative, commutative, and unital ring.

Exercise 1. (4 points) Let S be a ring, $f : R \longrightarrow S$ a map of rings and $I < R$ respectively $J < S$ ideals. We denote by $I^{\mathrm{e}} = (f(I)) < S$ the *extension* of I (along f) and by $J^{\mathrm{c}} = f^{-1}(J) < R$ the *contraction* of J (along f). Show the following statements for extensions and contractions of ideals along the map f.

(a) $I \subset (I^{\mathrm{e}})^{\mathrm{c}}$ and $(J^{\mathrm{c}})^{\mathrm{e}} \subset J$.

(b) $I^{\mathrm{e}} = ((I^{\mathrm{e}})^{\mathrm{c}})^{\mathrm{e}}$ and $J^{\mathrm{c}} = ((J^{\mathrm{c}})^{\mathrm{e}})^{\mathrm{c}}$.

(c) Denote by $\mathcal{C}$ the set of ideals in R obtained as contractions of ideals in S and by $\mathcal{E}$ the set of ideals in S obtained as extensions of ideals in R. Then it holds

$$\mathcal{C} = \{I < R \mid I = (I^{\mathrm{e}})^{\mathrm{c}}\} \text{ and } \mathcal{E} = \{J < S \mid J = (J^{\mathrm{c}})^{\mathrm{e}}\}$$

and

$$\begin{array}{ccc} \mathcal{C} & \longleftrightarrow & \mathcal{E} \\ I & \longmapsto & I^{\mathrm{e}} \\ J^{\mathrm{c}} & \longleftarrow\!\shortmid & J \end{array}$$

is a one-to-one correspondence.

Exercise 2. (4 points) Denote by $R[x]$ the polynomial ring in one indeterminant and coefficients in R. Let $f = r_0 + r_1 x + r_2 x^2 + \ldots + r_n x^n \in R[x]$. Show that

(a) f is a unit in $R[x]$ if and only if r_0 is a unit in R and $r_1, \ldots, r_n$ are nilpotent in R.

(b) f is nilpotent in $R[x]$ if and only if $r_0, \ldots, r_n$ are nilpotent in R.

(c) f is a zero-divisor in $R[x]$ if and only if there exists $r \in R \setminus \{0\}$ such that $rf = 0$.

Exercise 3. (4 points) Let R be an associative and unital ring, but not necessarily commutative, such that $x^2 = x$ for every $x \in R$.

(a) Show that $x + x = 0$ for all $x \in R$.

(b) Show that R is commutative.

(c) Show that any finitely generated ideal in R is a principal ideal.

(d) Show that any prime ideal of R is a maximal ideal.

Exercise 4. (4 points) Let $f : R \to S$ be a surjective map of rings and

$$\left\{ \text{ ideals of } S \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{ideals of } R \\ \text{containing } \ker(f) \end{array} \right\}.$$

, where $\Phi(I) = f^{-1}(I)$ and $\Psi(J) = f(J)$.

(a) Show that Φ and Ψ define inclusion preserving bijections that are mutually inverse to each other.

(b) Show that this can be restricted to prime ideals in S on the left hand side and prime ideals containing $\ker(f)$ in R on the right hand side.

Exercise 5. (4 points)

(a) Let $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ be families of R-modules and $\{f_i : M_i \to N_i\}$ a family of R-module maps. Show that this naturally determines maps

$$f_\oplus : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} N_i \quad \text{and} \quad f_\Pi : \prod_{i\in I} M_i \to \prod_{i\in I} N_i.$$

(b) Let $\{M_i\}_{i\in I}$ be a family of R-modules and N an R-module. Show that

$$\begin{aligned} \operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) &\cong \prod_{i\in I} \operatorname{Hom}_R\left(M_i, N\right) \text{ and} \\ \operatorname{Hom}_R\left(N, \prod_{i\in I} M_i\right) &\cong \prod_{i\in I} \operatorname{Hom}_R\left(N, M_i\right) \end{aligned}$$

Hint: Use the universal properties of direct sum and direct product to show the existence of maps in a suitable direction as well as their injectivity and surjectivity.