# 拓扑学（模块2）

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## Long exact sequence of homology

We start with the following situation:

Let $C^1_\bullet$, $C^2_\bullet$, and $C^3_\bullet$ be complexes, and $f : C^1_\bullet \to C^2_\bullet$ and $g : C^2_\bullet \to C^3_\bullet$ chain maps. Assume that for every $i \in \mathbb{Z}$ it holds that:

- $f_i : C^1_i \to C^2_i$ is injective,
- $g_i : C^2_i \to C^3_i$ is surjective, and
- $\mathrm{Im}(f_i) = \mathrm{Ker}(g_i)$.

In diagrams this looks as follows for the indices $n+1$, $n$, $n-1$, and $n-2$:

$$\begin{tikzcd}
 & 0 \arrow[d, "0"] & 0 \arrow[d, "0"] & 0 \arrow[d, "0"] & 0 \arrow[d, "0"] & \\
\cdots \arrow[r, "\partial^1_{n+2}"] & C^1_{n+1} \arrow[r, "\partial^1_{n+1}"] \arrow[d, "f_{n+1}"] & C^1_n \arrow[r, "\partial^1_n"] \arrow[d, "f_n"] & C^1_{n-1} \arrow[r, "\partial^1_{n-1}"] \arrow[d, "f_{n-1}"] & C^1_{n-2} \arrow[r, "\partial^1_{n-2}"] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r, "\partial^2_{n+2}"] & C^2_{n+1} \arrow[r, "\partial^2_{n+1}"] \arrow[d, "g_{n+1}"] & C^2_n \arrow[r, "\partial^2_n"] \arrow[d, "g_n"] & C^2_{n-1} \arrow[r, "\partial^2_{n-1}"] \arrow[d, "g_{n-1}"] & C^2_{n-2} \arrow[r, "\partial^2_{n-2}"] \arrow[d, "g_{n-2}"] & \cdots \\
\cdots \arrow[r, "\partial^3_{n+2}"] & C^3_{n+1} \arrow[r, "\partial^3_{n+1}"] \arrow[d, "0"] & C^3_n \arrow[r, "\partial^3_n"] \arrow[d, "0"] & C^3_{n-1} \arrow[r, "\partial^3_{n-1}"] \arrow[d, "0"] & C^3_{n-2} \arrow[r, "\partial^3_{n-2}"] \arrow[d, "0"] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where each row is a complex, i.e. the image of each horizontal map is contained in the kernel of the map to the right, while every column is exact, i.e. the image of each vertical map is equal to the kernel of the map below it. Furthermore every square that we see commutes.

**Theorem: Long exact sequence of homology** In the situation above there exists an exact complex of the form

$$\cdots \to H_n(C^1_\bullet) \xrightarrow{f_*} H_n(C^2_\bullet) \xrightarrow{g_*} H_n(C^3_\bullet) \xrightarrow{\partial} H_{n-1}(C^1_\bullet) \xrightarrow{f_*} H_{n-1}(C^2_\bullet) \xrightarrow{g_*} H_{n-1}(C^3_\bullet) \to \cdots$$

*Proof.* We focus on the columns for index $n$ and $n-1$, hence we look at the following diagram:

$$\begin{tikzcd}
0 \arrow[d, "0"] & 0 \arrow[d, "0"] \\
C_n^1 \arrow[r, "\partial_n^1"] \arrow[d, "f_n"] & C_{n-1}^1 \arrow[d, "f_{n-1}"] \\
C_n^2 \arrow[r, "\partial_n^2"] \arrow[d, "g_n"] & C_{n-1}^2 \arrow[d, "g_{n-1}"] \\
C_n^3 \arrow[r, "\partial_n^3"] \arrow[d, "0"] & C_{n-1}^3 \arrow[d, "0"] \\
0 & 0
\end{tikzcd}$$

As the first step, we know that $\mathrm{Im}(\partial_n^i) \subset \mathrm{Ker}(\partial_{n-1}^i)$. Hence we can replace $C_{n-1}^i$ in the diagram by $\mathrm{Ker}(\partial_{n-1}^i)$ for each $i = 1, 2, 3$. Thus we obtain

$$\begin{tikzcd}
0 \arrow[d, "0"] & 0 \arrow[d, "0"] \\
C_n^1 \arrow[r, "\partial_n^1"] \arrow[d, "f_n"] & \mathrm{Ker}(\partial_{n-1}^1) \arrow[d, "f'_{n-1}"] \\
C_n^2 \arrow[r, "\partial_n^2"] \arrow[d, "g_n"] & \mathrm{Ker}(\partial_{n-1}^2) \arrow[d, "g'_{n-1}"] \\
C_n^3 \arrow[r, "\partial_n^3"] \arrow[d, "0"] & \mathrm{Ker}(\partial_{n-1}^3) \\
0 &
\end{tikzcd}$$

where $f'_{n-1}$ and $g'_{n-1}$ are just the restrictions of $f_{n-1}$ and $g_{n-1}$ to the kernels. We have seen that they give well defined maps, because $f$ and $g$ are chain maps. Furthermore $f'_{n-1}$ is still injective, but in generel $g'_{n-1}$ is not surjective anymore (hence we loose the zero on the lower right).

For the first column, note that $\mathrm{Im}(\partial^i_{n+1}) \subset \mathrm{Ker}(\partial^i_n)$, hence we get an induced map $\overline{\partial}^i_n$ if we replace $C^i_n$ by $C^i_n/\mathrm{Im}(\partial^i_{n+1})$:

$$\begin{tikzcd}
& 0 \arrow[d, "0"] \\
C^1_n/\mathrm{Im}(\partial^1_{n+1}) \arrow[r, "\overline{\partial}^1_n"] \arrow[d, "\overline{f}_n"] & \mathrm{Ker}(\partial^1_{n-1}) \arrow[d, "f'_{n-1}"] \\
C^2_n/\mathrm{Im}(\partial^2_{n+1}) \arrow[r, "\overline{\partial}^2_n"] \arrow[d, "\overline{g}_n"] & \mathrm{Ker}(\partial^2_{n-1}) \arrow[d, "g'_{n-1}"] \\
C^3_n/\mathrm{Im}(\partial^3_{n+1}) \arrow[r, "\overline{\partial}^3_n"] \arrow[d, "0"] & \mathrm{Ker}(\partial^3_{n-1}) \\
0 &
\end{tikzcd}$$

As in the previous diagram we still have that $\overline{g}_n$ is surjective, but we loose that $\overline{f}_n$ is injective, hence we loose the zero at the upper left. By construction we still have $\mathrm{Im}(f'_{n-1}) \subset \mathrm{Ker}(g'_{n-1})$ and $\mathrm{Im}(\overline{f}_n) \subset \mathrm{Ker}(\overline{g}_n)$. But we claim that we still have equality in both cases:

Case $\mathrm{Im}(f'_{n-1}) = \mathrm{Ker}(g'_{n-1})$: Let $x \in \mathrm{Ker}(\partial^2_{n-1})$ such that $g'_{n-1}(x) = 0$. Thus $g_{n-1}(x) = 0$. Then we use exactness in the original diagram and there exists $y \in C^1_{n-1}$ such that $f_{n-1}(y) = x$. Thus $0 = \partial^2_{n-1} \circ f_{n-1}(y) = f_{n-2} \circ \partial^1_{n-1}(y)$, but $f_{n-2}$ is injective, hence $y \in \mathrm{Ker}(\partial^1_{n-1})$ and so $x \in \mathrm{Im}(f'_{n-1})$.

Case $\mathrm{Im}(\overline{f}_n) = \mathrm{Ker}(\overline{g}_n)$: Let $x + \mathrm{Im}(\partial^2_{n+1}) \in \mathrm{Ker}(\overline{g}_n)$. Thus $g_n(x) \in \mathrm{Im}(\partial^3_{n+1})$, hence there exists $y \in C^3_{n+1}$ such that $\partial^3_{n+1}(y) = g_n(x)$. Now we use that $g_{n+1}$ is surjective, hence there exists $z \in C^2_{n+1}$ such that $g_{n+1}(z) = y$. Then

$$g_n(x - \partial^2_{n+1}(z)) = g_n(x) - g_n \circ \partial^2_{n+1}(z) = \partial^3_{n+1}(y) - \partial^3_{n+1} \circ g_{n+1}(z) = \partial^3_{n+1}(y) - \partial^3_{n+1}(y) = 0,$$

hence $x - \partial^2_{n+1}(z) \in \mathrm{Ker}(g_n)$. Thus there exists $u \in C^1_n$ such that $f_n(u) = x - \partial^2_{n+1}(z)$ by exactness in the original diagram. Then $\overline{f}_n(u + \mathrm{Im}(\partial^1_{n+1})) = x + \mathrm{Im}(\partial^2_{n+1})$ and so $x + \mathrm{Im}(\partial^2_{n+1}) \in \mathrm{Im}(\overline{f}_n)$.

This is now the situation we are looking at, the columns are exact in the positions that are left, i.e. we do not have injectivity for the first map in the first column and do not have surjectivity for the second map in the second column. We now apply the Snake Lemma from homological algebra to this. We will explain how this works.

We first enlarge our diagram a bit:

$$
\begin{tikzcd}
 & & 0 \arrow[d, "0"] & \\
\mathrm{Ker}(\overline{\partial}_n^1) \arrow[r] & C_n^1/\mathrm{Im}(\partial_{n+1}^1) \arrow[r, "\overline{\partial}_n^1"] \arrow[d, "\overline{f}_n"] & \mathrm{Ker}(\partial_{n-1}^1) \arrow[r] \arrow[d, "f'_{n-1}"] & \mathrm{Ker}(\partial_{n-1}^1)/\mathrm{Im}(\overline{\partial}_n^1) \\
\mathrm{Ker}(\overline{\partial}_n^2) \arrow[r] & C_n^2/\mathrm{Im}(\partial_{n+1}^2) \arrow[r, "\overline{\partial}_n^2"] \arrow[d, "\overline{g}_n"] & \mathrm{Ker}(\partial_{n-1}^2) \arrow[r] \arrow[d, "g'_{n-1}"] & \mathrm{Ker}(\partial_{n-1}^2)/\mathrm{Im}(\overline{\partial}_n^2) \\
\mathrm{Ker}(\overline{\partial}_n^3) \arrow[r] & C_n^3/\mathrm{Im}(\partial_{n+1}^3) \arrow[r, "\overline{\partial}_n^3"] \arrow[d, "0"] & \mathrm{Ker}(\partial_{n-1}^3) \arrow[r] & \mathrm{Ker}(\partial_{n-1}^3)/\mathrm{Im}(\overline{\partial}_n^3) \\
 & 0 & &
\end{tikzcd}
$$

Hence we added the kernel and the so-called cokernel in each row of the diagram with the canonical inclusion respectively projection map. We now apply the Snake Lemma. This yields the following diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d, "0"] & \\
\mathrm{Ker}(\overline{\partial}_n^1) \arrow[r] \arrow[d, "\overline{f}_*"] & C_n^1/\mathrm{Im}(\partial_{n+1}^1) \arrow[r, "\overline{\partial}_n^1"] \arrow[d, "\overline{f}_n"] & \mathrm{Ker}(\partial_{n-1}^1) \arrow[r] \arrow[d, "f'_{n-1}"] & \mathrm{Ker}(\partial_{n-1}^1)/\mathrm{Im}(\overline{\partial}_n^1) \arrow[d, "f'_*"] \\
\mathrm{Ker}(\overline{\partial}_n^2) \arrow[r] \arrow[d, "\overline{g}_*"] & C_n^2/\mathrm{Im}(\partial_{n+1}^2) \arrow[r, "\overline{\partial}_n^2"] \arrow[d, "\overline{g}_n"] & \mathrm{Ker}(\partial_{n-1}^2) \arrow[r] \arrow[d, "g'_{n-1}"] & \mathrm{Ker}(\partial_{n-1}^2)/\mathrm{Im}(\overline{\partial}_n^2) \arrow[d, "g'_*"] \\
\mathrm{Ker}(\overline{\partial}_n^3) \arrow[r] \arrow[uurrr, dashed, "\partial"] & C_n^3/\mathrm{Im}(\partial_{n+1}^3) \arrow[r, "\overline{\partial}_n^3"] \arrow[d, "0"] & \mathrm{Ker}(\partial_{n-1}^3) \arrow[r] & \mathrm{Ker}(\partial_{n-1}^1)/\mathrm{Im}(\overline{\partial}_n^3) \\
 & 0 & &
\end{tikzcd}
$$

The statements of the Snake Lemma are the following:

(a) There are well-defined homomorphisms $\overline{f}_*$, $\overline{g}_*$, $f'_*$ and $g'_*$
*Idea:* This is done in exactly the same way as for maps between homology groups coming from chain maps in the course.

(b) It holds that $\mathrm{Im}(\overline{f}_*) = \mathrm{Ker}(\overline{g}_*)$.
*Idea:* This is a short calculation very similar to the check that $\mathrm{Im}(\overline{f}_n) = \mathrm{Ker}(\overline{g}_n)$.

(c) It holds that $\mathrm{Im}(f'_*) = \mathrm{Ker}(g'_*)$.
*Idea:* This is a short calculation very similar to the check that $\mathrm{Im}(f'_{n-1}) = \mathrm{Ker}(g'_{n-1})$.

(d) There exists a well-defined homomorphism $\partial : \mathrm{Ker}(\overline{\partial}^3_n) \to \mathrm{Ker}(\partial^1_{n-1})/\mathrm{Im}(\overline{\partial}^1_n)$.
*Idea:* We give a quick construction of the map $\partial$. Let $x \in \mathrm{Ker}(\overline{\partial}^3_n) \subset C^3_n/\mathrm{Im}(\partial^3_{n+1})$. Then we use that $\overline{g}_n$ is surjective, hence there exists $y \in C^2_n/\mathrm{Im}(\partial^2_{n+1})$ such that $\overline{g}_n(y) = x$. Then set $z = \overline{\partial}^2_n(y)$. Then by commutativity of the middle bottom square $g'_{n-1}(z) = 0$, hence by exactness there exists $u \in \mathrm{Ker}(\partial^1_{n-1})$ such that $f'_{n-1}(u) = z$. Then we define $\partial(x) = u + \mathrm{Im}(\overline{\partial}^1_n)$. Now one needs to check that the definition is independent of the two choices ($y$ and $u$).

(e) It holds $\mathrm{Im}(\overline{g}_*) = \mathrm{Ker}(\partial)$ and $\mathrm{Im}(\partial) = \mathrm{Ker}(f'_*)$.
*Idea:* These are both calculations similar to parts (b) and (c), but a bit more involved. The inclusion $\subset$ is easy in both cases, while the inclusion $\supset$ is longer.

Hence we get a part of complex

$$\mathrm{Ker}(\overline{\partial}^1_n) \xrightarrow{\overline{f}_*} \mathrm{Ker}(\overline{\partial}^2_n) \xrightarrow{\overline{g}_*} \mathrm{Ker}(\overline{\partial}^3_n) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathrm{Ker}(\partial^1_{n-1})/\mathrm{Im}(\overline{\partial}^1_n) \xrightarrow{f'_*} \mathrm{Ker}(\partial^2_{n-1})/\mathrm{Im}(\overline{\partial}^2_n) \xrightarrow{g'_*} \mathrm{Ker}(\partial^3_{n-1})/\mathrm{Im}(\overline{\partial}^3_n)$$

where for each pair of neighbouring maps we have that the image of the left one is equal to the kernel of the right one, hence this part of the complex is exact.

Now we analyse all the terms in this complex:

(a) The kernel $\mathrm{Ker}(\overline{\partial}^i_n)$ is exactly the kernel of $\partial^i_n$ quotient by the image of $\partial^i_{n+1}$, hence it is by definition equal to $H_n(C^i_\bullet)$.

(b) The maps $\overline{f}_*$ and $\overline{g}_*$ are by construction equal to the maps $f_*$ and $g_*$ defined between homology groups as in the lecture.

(c) For the quotient $\mathrm{Ker}(\partial^i_{n-1})/\mathrm{Im}(\overline{\partial}^i_n)$ we note that by construction the image of $\overline{\partial}^i_n$ and $\partial^i_n$ agree, hence this term is equal to $H_{n-1}(C^i_\bullet)$.

(d) Again, the maps $f'_*$ and $g'_*$ are by construction equal to the maps $f_*$ and $g_*$ defined between homology groups as in the lecture

Thus we really have part of a complex

$$H_n(C^1_\bullet) \xrightarrow{f_*} H_n(C^2_\bullet) \xrightarrow{g_*} H_n(C^3_\bullet) \xrightarrow{\partial} H_{n-1}(C^1_\bullet) \xrightarrow{f_*} H_{n-1}(C^2_\bullet) \xrightarrow{g_*} H_{n-1}(C^3_\bullet)$$

Doing this now for every index $n$ and putting all the pieces together we get the statement. $\square$

## Applications

We can now apply this to our situation:

**Theorem 2.26** For a space $X$ and a subspace $A$, there exists an exact complex

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} H_{n-1}(X,A) \to \cdots .$$

*Proof.* Let $i : A \to X$ the inclusion. Then we use

$$C^1_\bullet = C_\bullet(A),\ C^2_\bullet = C_\bullet(X),\ \text{and } C^3_\bullet = C_\bullet(X,A).$$

For the maps we use $f = i_\#$ and $g = j$, where $j : C(X) \to C(X,A)$ is the natural quotient map. Then the statement follows by applying the long exact sequence of homology theorem. □

We can also use this in similar situations:

**Remarks: (a)** If we have $\varnothing \neq A \subset X$, we can look at reduced chain groups

$$\widetilde{C}_n(X) = \begin{cases} C_n(X) & n \neq -1 \\ \mathbb{Z} & n = -1 \end{cases},\ \widetilde{C}_n(A) = \begin{cases} C_n(A) & n \neq -1 \\ \mathbb{Z} & n = -1 \end{cases} \text{ and } \widetilde{C}_n(X,A) = \widetilde{C}_n(X)/\widetilde{C}_n(A).$$

Then again we can choose $C^1_\bullet = \widetilde{C}_\bullet(A)$, $C^2_\bullet = \widetilde{C}_\bullet(X)$, and $C^3_\bullet = \widetilde{C}_\bullet(X,A)$ with $f = i_\#$ coming from the inclusion and $g$ being the natural quotient map. Then applying the long exact sequence of homology theorem gives an exact complex

$$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X,A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \xrightarrow{j_*} \widetilde{H}_{n-1}(X,A) \to \cdots ,$$

where $\widetilde{H}_n(X,A)$ is the homology of the complex $\widetilde{C}_\bullet(X,A)$. Note that $\widetilde{C}_n(X,A) = C_n(X,A)$ for **all** $n \in \mathbb{Z}$, even $n = -1$.

**(b)** We could also look at more "relative" situations, i.e. we choose $B \subset A \subset X$. Then we can consider:

$$C^1_\bullet = C_\bullet(A,B),\ C^2_\bullet = C_\bullet(X,B),\ \text{and } C^3_\bullet = C_\bullet(X,A).$$

For the chain map $f$, consider the chain map $C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(B)$, this induces a chainmap $f : C^1_\bullet \to C^2_\bullet$ by applying the homomorphism theorem for abelian groups. For the chain map $g$ consider the natural quotient map $C_\bullet(X) \to C_\bullet(X)/C_\bullet(A)$ and apply the homomorphism theorem for abelian groups to get an induced map $g : C^2_\bullet \to C^3_\bullet$. We can then apply the long exact sequence of homology theorem, which gives us

$$\begin{aligned} \cdots \to H_n(A,B) \xrightarrow{f_*} H_n(X,B) \xrightarrow{g_*} H_n(X,A) \xrightarrow{\partial} \\ \xrightarrow{\partial} H_{n-1}(A,B) \xrightarrow{f_*} H_{n-1}(X,B) \xrightarrow{g_*} H_{n-1}(X,A) \to \cdots . \end{aligned}$$

So we get a long exact sequence of terms only involving versions of relative homology.

**(c)** We briefly mentioned at the end of §2.1 that many homology theories are defined by constructing a complex and then taking homology of that complex.
In every one of these homology theories there are situations to get complexes $C^1_\bullet$, $C^2_\bullet$, and $C^3_\bullet$ with chain maps $f : C^1_\bullet \to C^2_\bullet$ and $g : C^2_\bullet \to C^3_\bullet$ such that they satisfy the assumption of the long exact sequence of homology theorem. Hence the theorem constructs exact complexes of homology for every single one of these homology theories. Sometimes even mixing different types of homology theories.