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Assignment 2 - Solutions

Exercise 1. Let $B = \mathrm{S}^1$, $C = \mathrm{D}^2$ and denote by $\partial C = \mathrm{S}^1$ the circle contained in C. For a fixed integer $k \geq 1$ we define the map $\phi_k : \partial C \to B$ via $\phi_k(e^{2\pi i s}) = e^{2\pi i k s}$. We then define the following space

$$X_k = (B \coprod C)/_\sim,$$

where $z \sim \varphi_k(z)$ for $z \in \partial C$.

(a) Show that the space X_k is path-connected.

(b) Use the van Kampen theorem to determine $\pi_1(X_k)$.

Hint: The calculation is easier with the choice of a basepoint in $C \setminus \partial C$.

Solutions: Denote by π the quotient map from $B \coprod C$ to X_k. Note that π restricted to $C \setminus \partial C$ is a homeomorphism and π restricted to B is a homeomorphism.

(a) Let $x, y \in X_k$. If $x, y \in \pi(B)$ respectively $x, y \in \pi(C)$ then we can take a path f from $x' \in \pi^{-1}(x)$ and $y' \in \pi^{-1}(y)$ in B respectively C, and $\pi \circ f$ is a path in X_k from x to y. Thus we can assume that $x \in \pi(B)$ and $y \in \pi(C)$. Note that the map ϕ_k is surjective, hence there is $z \in \partial C$ such that $\phi_k(z) = x$ in X_k, hence by the previous discussion y is path connected to z which gets identified with x, hence X_k is path-connected.

(b) Fix $1 > \varepsilon' > \varepsilon > 0$.
Define $U' = \{x \in C \mid |x| > \varepsilon\}$, which is open in C. Set $U = \pi(U')$. Then $\pi^{-1}(U) = U' \coprod B$, which is open, hence U is open in X_k by definition of the quotient topology. Define $V' = \{x \in C \mid |x| < \varepsilon'\}$, which is open in C as well. Set $V = \pi(V')$. Since $V' \subset C \setminus \partial C$ (and π is a homeomorphism when restricted to this subset), V is open in X_k and $V \cap \pi(B) = \varnothing$.

By construction $U \cup V = X_k$ and $U \cap V \cong \{x \in C \mid \varepsilon' > |x| > \varepsilon\}$, since the intersection is contained in $\pi(C \setminus \partial C)$, where π is a homeomorphism. Especially $U \cap V$ is path-connected.

Fix $x_0 \in V \cap U$ and let A be an open ball around $x_0' = \pi^{-1}(x_0)$ that is fully contained in $\{x \in C \mid \varepsilon' > |x| > \varepsilon\}$. Then $\pi(A)$ is open in X_k and contained in both U and V. Since a ball deformation retracts to a point, and U and V are path-connected they satisfy the prerequisites of the van Kampen theorem.

By definition V is contractible, as an open ball. For U, extend the retraction $r : (\mathrm{D}^2 \setminus \{(0,0)\}) \to \mathrm{S}^1$ (from the course) to B by the identity on B to a retraction $r' : (C \setminus \{(0,0)\}) \coprod B \to \partial C \coprod B$. This descents to a retraction on the quotient $\overline{r}$ and so U deformation retracts to $(\partial C \coprod B)/_\sim \cong \mathrm{S}^1$ via $\overline{r}$. Finally $U \cap V$ deformation retract to a circle with radius τ for $\tau = |x_0'|$. Hence $\pi_1(U \cap V, x_0) \cong \mathbb{Z} = \langle [\omega] \rangle$, where we denote by ω a loop around the circle of radius τ based at x_0. Applying the van Kampen theorem we obtain

$$\pi_1(X_k, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0)) / \langle \varphi_{UV}([\omega]) \rangle \cong \pi_1(V, x_0)) / \langle \varphi_{UV}([\omega]) \rangle,$$

where $\varphi_{UV} : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is induced by the inclusion. We omit the term $\varphi_{VU}([\omega])^{-1}$ since $\pi_1(V, x_0) = \{0\}$ by construction, hence the image is trivial anyway. Note that in the quotient

$$\overline{r} \circ \pi(e^{(2\pi i l)/k} x_0') \sim \overline{r}(x_0) \text{ for } 0 \leq l < k.$$

Hence $\overline{r} \circ \omega$ is a loop at $\overline{r}(x_0)$ that passes $\overline{r}(x_0)$ a total of $k+1$ times. Hence generates the subgroup $k\mathbb{Z}$ inside $\pi_1(U, x_0) \cong \pi_1(\mathrm{S}^1) \cong \mathbb{Z}$. Hence we obtain $\pi_1(X_k, x_0) \cong \mathbb{Z}/k\mathbb{Z}$.

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Exercise 2. Let $X \subset \mathbb{R}^n$ and assume that $X = \bigcup_{i=1}^m X_i$ where each X_i is an open convex subset of $\mathbb{R}^n$. Assume that any triple intersection $X_i \cap X_j \cap X_k \neq \varnothing$ for $1 \leq i, j, k \leq m$. Show that X is simply-connected.

Solutions: We do this inductively on the number of convex subsets used. Let $Y_k = \bigcup_{i=1}^k X_i$ for $1 \leq k \leq m$. By assumption $Y_1 = X_1$ is open and convex in $\mathbb{R}^n$, hence Y_1 is simply-connected. Assume now that we already know that Y_{k-1} is simply connected.

It holds $Y_k = Y_{k-1} \cup X_k$ with $Y_{k-1} \cap X_k \neq \varnothing$ by assumption. Thus we can choose a point in the intersection that is path-connected to any point in Y_{k-1} by assumption and path-connected to any point in X_k since X_k is convex. Hence Y_k is path-connected.

To calculate the fundamental group of Y_k we use the open subsets Y_{k-1} and X_k. We need to check that the intersection $Z_k = Y_{k-1} \cap X_k$ is path-connected. Let $x, y \in Z_k$, then there exists $1 \leq i, j < k$ such that $x \in X_i$ and $y \in X_j$. Since $X_i \cap X_j \cap X_k$ is non-empty we can choose a point in the triple intersection that is connected to x (since X_i is convex) and connected to y (since X_j is convex). Hence x and y are path-connected and so Z_k is path-connected. Thus we can apply the van Kampen theorem to the open subsets Y_{k-1} and X_k with an arbitrary basepoint $x_0 \in Z_k$. Then

$$\pi_1(X_k, x_0) \cong (\pi_1(Y_{k-1}, x_0) * \pi_1(X_k, x_0))/N,$$

where N is the subgroup given in the van Kampen theorem. But by assumption Y_{k-1} is simply-connected and X_k is convex, hence also simply-connected. Thus $\pi_1(X_k, x_0) = \{0\}$ and so X_k is simply-connected. By induction the claim then follows.

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Exercise 3. Let X be a space with $X = U \cup V$ for U, V, and $U \cap V$ all open, non-empty and path-connected.

(a) Show that X is path-connected.

(b) Assume that $V \cap U$ is simply-connected and show that $\pi(X) \cong \pi_1(U) * \pi_1(V)$.

Solutions: **(a)** Since $U \cap V \neq \varnothing$ we can choose $x_0 \in U \cap V$. Then by assumption x_0 is path-connected to any point in U and to any point in V, since they are both path-connected. Hence x_0 is path-connected to any point in X and so X is path-connected.

(b) We already know that X is path-connected. Assume now that also $V \cap U$ is simply-connected and let x_0 be as in part (a). We can apply the van Kampen theorem to the open subsets U and V, since they are both open, path-connected, contain x_0 and $V \cap U$ is path-connected. Hence by the van Kampen theorem

$$\pi_1(X, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0))/N,$$

where N is given as in the van Kampen theorem. But by assumption $\pi_1(U \cap V, x_0) = \{0\}$, hence the maps induced from the embedding of $U \cap V$ into U respectively V are both trivial and thus $N = \{0\}$. Hence the claim follows.

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Exercise 4.

(a) Let $p : \widetilde{X} \to X$ be a covering space. Let $A \subset X$ be a subspace and set $\widetilde{A} = p^{-1}(A)$. Show that the restriction of p to $\widetilde{A}$ is a covering space.

(b) Let $p_1 : \widetilde{X}_1 \to X_1$ and $p_2 : \widetilde{X}_2 \to X_2$ be covering spaces. Show that the product $\widetilde{X}_1 \times \widetilde{X}_2$ can be made into a covering space for $X_1 \times X_2$.

(c) Let $X = \mathrm{S}^2 \cup D$, where $D = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$, i.e. the 2-sphere together with a line segment connecting two anti-podal points. Construct a simply-connected covering space $\widetilde{X}_1$ of X and a non simply-connected covering space $\widetilde{X}_2 \neq X$ of X.

Solutions: **(a)** Let $x \in A$. Since $\widetilde{X}$ is a covering space, there exists an open neighbourhood U of x such that $p^{-1}(U) = \coprod \widetilde{U}_\alpha$ is a disjoint union of open subsets and p restricted to each $p : \widetilde{U}_\alpha \to U$ is a homeomorphism. Set $V = U \cap A$, then

$$p^{-1}(V) = p^{-1}(A) \cap p^{-1}(U) = \widetilde{A} \cap \coprod_\alpha \widetilde{U}_\alpha = \coprod_\beta (\widetilde{A} \cap \widetilde{U}_\beta),$$

where the final disjoint union only runs over those β such that $\widetilde{V}_\beta = \widetilde{A} \cap \widetilde{U}_\beta \neq \varnothing$. By definition $\widetilde{V}_\beta$ is open in $\widetilde{A}$ and by construction $p : \widetilde{V}_\beta \to V$ is bijective and continuous. Since an open subset in $\widetilde{V}_\beta$ is an open subset of $\widetilde{U}_\beta$ intersected with $\widetilde{A}$ it also follows immediately that the image of any open subset is open, thus the inverse map is also continuous and so p is a homeomorphism and $\widetilde{A}$ is a covering space of A.

(b) Let $(x_1, x_2) \in X_1 \times X_2$. Let U be an open neighbourhood of x_1 in X_1 such that $p_1^{-1}(U) = \coprod_{\alpha \in J_1} \widetilde{U}_\alpha$ satisfies the covering space property and similarly let V be an open neighbourhood of x_2 in X_2 such that $p_2^{-1}(V) = \coprod_{\beta \in J_2} \widetilde{V}_\beta$ satisfies the covering space property. Now define $p : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ be defined via $p(\widetilde{x}, \widetilde{y}) = (p_1(\widetilde{x}), p_2(\widetilde{y}))$. By definition of the product topology, p is continuous and $U \times V$ is an open subset of $X_1 \times X_2$. Furthermore

$$p^{-1}(U \times V) = \coprod_{\alpha \in J_1, \beta \in J_2} \widetilde{U}_\alpha \times \widetilde{V}_\beta,$$

with $\widetilde{U}_\alpha \times \widetilde{V}_\beta$ being open in $\widetilde{X}_1 \times \widetilde{X}_2$ by definition of the product topology. By construction $p : \widetilde{U}_\alpha \times \widetilde{V}_\beta \to U \times V$ is a homeomorphism in each component, hence a homeomorphism. Thus $\widetilde{X}_1 \times \widetilde{X}_2$ is a covering space of $X_1 \times X_2$.

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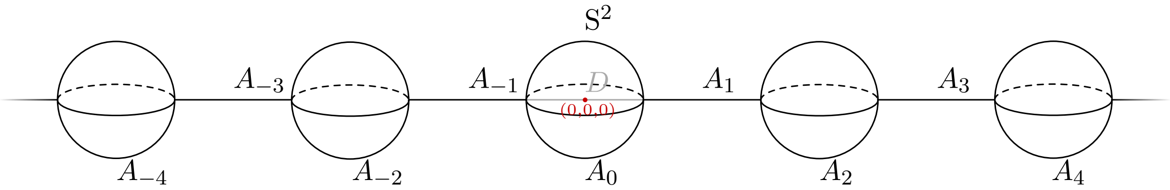
(c) We will first construct $\widetilde{X}_1$ and then use a quotient of $\widetilde{X}_1$ to construct $\widetilde{X}_2$.

Definition of the space $\widetilde{X}_1$: We first construct $\widetilde{X}_1$ as a subspace of $\mathbb{R}^3$. For $k \in \mathbb{Z}$, let $\Theta_k : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\Theta_k(x, y, z) = (x+k, y, z)$, i.e. the translation by k in the first coordinate. Each Θ_k is obviously a homeomorphism with inverse Θ_{-k}. Now define $A_k = \Theta_{2k}(\mathrm{S}^2)$, for k even, and $A_k = \Theta_{2k}(D)$, for k odd, and set $\widetilde{X}_1 = \bigcup_{k\in\mathbb{Z}} A_k \subset \mathbb{R}^3$. In a picture this looks as follows



with $X = \mathrm{S}^2 \cup D$ in the center and the translated spheres and the translations of D to the left and right.

$\widetilde{X}_1$ is a covering space: For the covering map p, let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\sigma(x, y, z) = (-x, y, z)$. Note that $\sigma : D \to D$ is a homeomorphism. Then p is defined on A_k, for k even, as Θ_{-2k}, which by construction maps A_k homeomorphically onto S^2. On A_k, for k odd, p is defined as $\sigma \circ \Theta_{-2k}$, i.e. first translate onto D and then apply the reflection, again this is a homeomorphism from A_k onto D. To check that this map is well-defined, we need to check intersection points, i.e. points of the form $(2k + 1, 0, 0) \in A_k \cap A_{k+1}$. Then we see that for k even

$$\Theta_{-2k}(2k + 1, 0, 0) = (1, 0, 0) = \sigma \circ \Theta_{-2(k+1)}(2k + 1, 0, 0),$$

and similar for k odd. Hence p is well-defined. For $x \in X$, let $U_\varepsilon = X \cap B_\varepsilon(x)$ be the intersection of X with a 3-dimensional ball of radius $\varepsilon > 0$. If x is contained in either $D \setminus \mathrm{S}^2$ or $\mathrm{S}^2 \setminus D$ one can choose ε such that U_ε is contained in the same subset and then the preimage under p is just an infinite number of translated copies of U_ε in case of $\mathrm{S}^2 \setminus D$ (or reflected and then translated in case of $D \setminus \mathrm{S}^2$), which obviously satisfies the covering space property. In case x is either $(1, 0, 0)$ or $(-1, 0, 0)$, one can choose for example $\varepsilon = 1/2$ and also sees immediately that the preimage are disjoint copies of subsets of $\mathbb{R}^3$ that are homeomorphically mapped onto U_ε. Hence $\widetilde{X}_1$ is a covering space.

$\widetilde{X}_1$ is simpy-connected: To see that $\widetilde{X}$ is simply-connected, we first note that it is obviously path-connected. Now fix a loop γ_0 at $(0, 1, 0)$ in $\widetilde{X}_1$. Since I is compact, the image of I under γ is also compact. Since $\widetilde{X}_1$ is a closed subset of $\mathbb{R}^3$, the image is compact in $\mathbb{R}^3$. Hence it is bound and closed. Thus there exists $n, m > 0$ such that

$$\gamma(I) \subset A_{m,n} = \bigcup_{-m\le k\le n} A_k \subset \mathbb{R}^3.$$

Hence a finite union of translated spheres and line segments. If $m = n = 0$ we are done since then γ is a loop inside S^2, which is simply-connected. Thus we can assume that either $n > 0$ or $m > 0$. Without loss of generality we assume that $n > 0$ (the case $m > 0$ uses the same type or arguments). Let $0 < s < 1$ be the minimal value such that $\gamma(s) \in A_n$ and let $s < t < 1$ be the maximal value such that $\gamma([s, t]) \subset A_n$ but $\gamma([s, t]) \not\subset A_{n-1}$. If no such value exists then $\gamma(I) \subset A_{m,n-1}$. Otherwise we use that A_n is simply-connected and replace γ by a homotopic loop γ' that is equal to γ outside of $[s, t]$ and $\gamma'([s, t]) \subset \{(2n - 1, 0, 0)\}$, i.e. constant with image the intersection point of A_n and A_{n-1}. Since I is compact we can

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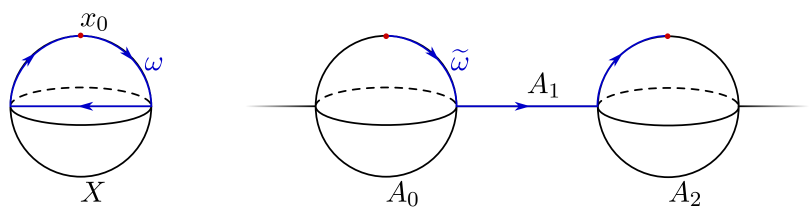
apply this procedure finitely many times until we arrive at a homotopic loop that is fully contained in $A_{m,n-1}$. Repeating this step n times we arrive at a loop that is homotopic to γ and contained in $A_{m,0}$. Now we use the analogous arguments on the negative side until we arrive at a homotopic loop contained in $A_{0,0} = \mathrm{S}^2$, which is nullhomotopic. Thus $\widetilde{X}_1$ is simply-connected.

Definition of the space $\widetilde{X}_2$: For the construction of $\widetilde{X}_2$ we use a quotient of $\widetilde{X}_1$. Choose an integer $k > 0$. Note that by construction $\Theta_{4k} : \widetilde{X}_1 \to \widetilde{X}_1$ is a homeomorphism from $\widetilde{X}_1$ to itself. We set $Y_k = \widetilde{X}_1/_{\sim}$ where the equivalence relation is defined by $x \sim y$ iff $\Theta_{4rk}(x) = y$ for some $r \in \mathbb{Z}$. The map p descents to a map $p_k : Y_k \to X$, since it is constant on equivalence classes by construction. It is also obviously a covering space, instead of infinitely many copies of the open subsets described above, one now has k copies. Note that $p_1 : Y_1 \to X$ is a homeomorphism.

Showing that $\widetilde{X}_2 \ncong X$ for $k > 1$: Assume $k > 1$. To see that Y_k is not homeomorphic to X for $k > 1$ we first note from the lecture that Example 1.52 showed that $\pi_1(X, (0, 1, 0)) \cong \mathbb{Z}$ (it was a step in the example), the generator being the homotopy class of a loop ω at $(0, 1, 0)$ such that $\omega^{-1}(x, 0, 0)$ has cardinality 1 for every $-1 \leq x \leq 1$ (i.e. it passed through D exactly once). Instead of giving a formula for the loop we picture it below to the left



If $(p_k)_*$ would be surjective then ω has to lift to a loop in Y_k. We can write down a lift $\widetilde{\omega}$ as shown in the picture above to the right as a path connecting the equivalence class of the point $(0, 1, 0)$ to the equivalence class of $(4, 1, 0)$. This lift is unique by the lifting property. In case of $k > 1$ these two equivalence classes are distinct, hence ω does not lift to a loop. Since anything homotopic to ω lifts to a path with the same start and end point (since the homotopy lifts as well) also no other loop in the homotopy class lifts to a loop in Y_k. Hence Y_k is not homeomorphic to X and for $k > 1$ and so is a choice for $\widetilde{X}_2$.