拓扑学（模块2）

BIT, Fall 2021

Topological spaces and basis

Definition 0.1 *A **topology** on a set* X *is a collection of subset* $\mathcal{U} \subset \mathcal{P}(\mathrm{X})$ *of* X *s.t.*

(a) $\varnothing \in \mathcal{U}$ *and* $\mathrm{X} \in \mathcal{U}$*,*

(b) for $\mathrm{U}_1, \ldots, \mathrm{U}_n \in \mathcal{U}$ *it holds* $\bigcap_{i=1}^n \mathrm{U}_i \in \mathcal{U}$*, and*

(c) for $\mathrm{U}_i \in \mathcal{U}$ *for* $i \in I$ *it holds* $\bigcup_{i\in I} \mathrm{U}_i \in \mathcal{U}$*.*

The pair $(\mathrm{X},\mathcal{U})$ *is called a **topological space** and the elements of* $\mathcal{U}$ *are called **open subsets**. In general we will omit* $\mathcal{U}$ *in the notation and only write* X *for the topological space.*

Note: The subsets $\mathcal{C} = \{\mathrm{X} \setminus \mathrm{U} \mid \mathrm{U} \in \mathcal{U}\}$ are called **closed subsets** of X. Any definition in the course that is given in terms of open subsets has an equivalent formulation in terms of closed subsets. In general analysis and differential geometry use the definition with respect to open subsets, while algebraic geometry often uses the definitions with respect to closed subsets.

Definition 0.2 *For a set* X *a collection of subsets* $\mathcal{B} \subset \mathcal{P}(\mathrm{X})$ *is called a **basis of a topology** on* X*, if for any finite set* J *and* $U_j \in \mathcal{B}$ *for* $j \in J$ *there exists a set* $L = L(J)$ *and* $V_l \in \mathcal{B}$ *for* $l \in L$ *s.t.*

$$\bigcap_{j\in J} U_j = \bigcup_{l\in L} V_l,$$

i.e. every finite intersection of elements in $\mathcal{B}$ *can be written as a union of elements in* $\mathcal{B}$*.*
The collection $\mathcal{U}(\mathcal{B}) = \{\mathrm{U} \mid \mathrm{U} = \bigcup_{j\in J} \mathrm{B}_j$ *for* J *a set and* $\mathrm{B}_j \in \mathcal{B}\}$ *is the topology on* X ***generated** by* $\mathcal{B}$*.*

Note: As the name suggests, $\mathcal{U}(\mathcal{B})$ forms a topology on X if $\mathcal{B}$ is a basis of a topology on X. This is an easy exercise.

Especially note the extreme cases here. For $J = \varnothing$ in the definition of the basis of a topology, the empty intersection is equal to X by definition, hence $\mathrm{X} \in \mathcal{U}(\mathcal{B})$. While the choice of the empty union in the definition of $\mathcal{U}(\mathcal{B})$ gives by definition that $\varnothing \in \mathcal{U}(\mathcal{B})$.

Example 0.3 *For any set* X*:*

*(a) the **discrete topology*** $\mathcal{U}_{\mathrm{disc}} = \mathcal{P}(\mathrm{X})$*, i.e. every subset of* X *is open, and*

*(b) the **trivial topology*** $\mathcal{U}_{\mathrm{tr}} = \{\varnothing, \mathrm{X}\}$*, i.e.* $\varnothing$ *and* X *are the only open sets.*

Example 0.4 (Euclidean/metric topology) *We usually equip* $\mathbb{R}^n$ *with the **euclidean topology*** $\mathcal{U}_{\mathrm{eu}}$*.*
For $x \in \mathbb{R}^n$ *and* $\varepsilon > 0$ *we denote by* $\mathrm{B}_\varepsilon(x)$ *the open ball centred at* x *with radius* ε*, i.e.*

$$\mathrm{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \varepsilon\}.$$

Then we declare $\mathrm{U} \in \mathcal{U}_{\mathrm{eu}}$ *if for every* $x \in \mathrm{U}$ *there exists* $\varepsilon > 0$ *such that* $\mathrm{B}_\varepsilon(x) \subset \mathrm{U}$*, i.e. for every point in* U *we find a small ball around* x *that is contained in* U*. In other words* $\mathcal{U}_{\mathrm{eu}}$ *is the topology generated by the basis for a topology* $\{\mathrm{B}_\varepsilon(x) \mid x \in \mathbb{R}^n, \varepsilon > 0\}$*.*

The same definition turns any metric space into a topological space.

Categorical notions

Definition 0.5 *For a two topological spaces* $(\mathrm{X},\mathcal{U})$ *and* $(\mathrm{Y},\mathcal{U}')$*, a map* $f:(\mathrm{X},\mathcal{U})\to(\mathrm{Y},\mathcal{U}')$ *is called* ***continuous*** *if* $f^{-1}(\mathrm{U})\in\mathcal{U}$ *for* $U\in\mathcal{U}'$*.*

Note: For any topological space X the identity map is obviously continuous and similarly the composition of continuous maps is continuous. Hence one can form the category of topological spaces **Top** consisting of topological spaces and continuous maps between them.

Definition 0.6 *Let* $(\mathrm{X}_j,\mathcal{U}_j)_{j\in J}$ *be a family of topological spaces. We denote by* $\coprod_j \mathrm{X}_j$ *the disjoint union. The* ***disjoint union topology*** $\mathcal{U}_{\coprod}$ *is given by* $\mathrm{U}\in\mathcal{U}_{\coprod}$ *if and only if* $\mathrm{U}\cap\mathrm{X}_j\in\mathcal{U}_j$ *for all* $j\in J$*.*

Remarks on the disjoint union topology

(a) Let $\iota_i:\mathrm{X}_i\to\coprod_j\mathrm{X}_j$ be the inclusion, then $\mathrm{U}\in\mathcal{U}_{\coprod}$ if and only if $\iota_i^{-1}(\mathrm{U})\in\mathcal{U}_i$ for all $i\in J$.

(b) One can reformulate (a) by saying that the disjoint union topology is the **finest** (= **strongest=largest**) topology on $\coprod_i\mathrm{X}_i$ such that all ι_i are continuous. Here **finest** means that it has more open sets than other topology for which this is true.

(c) The disjoint union $\coprod$ with $\mathcal{U}_{\coprod}$ is the coproduct in the category **Top**.

Definition 0.7 *Let* $(\mathrm{X}_j,\mathcal{U}_j)_{j\in J}$ *be a family of topological spaces. We denote by* $\prod_j\mathrm{X}_j$ *the cartesian product and by* $\pi_i:\prod_j\mathrm{X}_j\to\mathrm{X}_i$ *the projection maps. The basis for the* ***product topology*** $\mathcal{U}_{\prod}$ *is* $\{\pi_i^{-1}(\mathrm{U})|i\in J,\mathrm{U}\in\mathcal{U}_i\}$*.*

Remarks on the product topology

(a) The product topology is the **coarsest** (= **weakest=smallest**) topology on $\prod_j\mathrm{X}_j$ such that all π_i are continuous. This means that it has the least open subsets such that the projection maps are continuous.

(b) The product $\prod$ with $\mathcal{U}_{\prod}$ is the product in the category **Top**.

Definition 0.8 *For* $\mathrm{A}\subset\mathrm{X}$ *in a topological space* $(\mathrm{X},\mathcal{U})$*, we define the* ***subspace topology*** *on* A *as* $\mathcal{U}_{\mathrm{A}}=\{\mathrm{A}\cap\mathrm{U}|\mathrm{U}\in\mathcal{U}\}$*.*

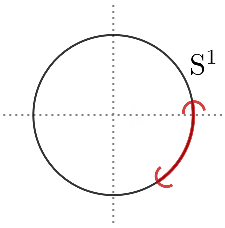
Remarks on the subspace topology
The subspace topology is the **coarsest** topology, s.t. the inclusion $\iota_{\mathrm{A}}:\mathrm{A}\to\mathrm{X}$ is continuous.

In all cases above, if we talk about a disjoint union, product or a subspace of one or more topological spaces, we always imply that we use the corresponding disjoint union, product, or subspace topology.
This is easy for our basic example as the product topology on $\mathbb{R}^n$ agrees with the euclidean topology on $\mathbb{R}^n$.

Example 0.9 (The circle) *The* ***circle*** $\mathrm{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ *will be the most important example of a connected topological space in the course. Depending on the situation we can also view* S^1 *as being complex numbers with absolute value one.*

We will always consider the ***subspace topology*** *on* S^1 *from* $\mathbb{R}^2$*. Since a basis of the topology of* $\mathbb{R}^2$ *is given by open balls* $B_\varepsilon(x)$*, the basis of the topology of* S^1 *is given by open arcs on the circle, which are the intersection of an open ball with the circle.*

In pictures



For notations we have the following subsets of $\mathbb{R}^2$

(a) the closed disc $\mathrm{D}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$*, and*

(b) the open disc $\mathrm{B}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$*,*

and $\mathrm{D}^2 = \mathrm{S}^1 \coprod \mathrm{B}^2$*.*

Example 0.10 (Higher spheres) *An* n***-sphere*** $\mathrm{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$*, for* $|\cdot|$ *the standard norm on* $\mathbb{R}^{n+1}$*, is the obvious generalization of the circle. Again it will be equipped with the subspace topology from* $\mathbb{R}^{n+1}$ *and we have the corresponding subsets of* $\mathbb{R}^{n+1}$

(a) the closed ball $\mathrm{D}^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| \le 1\}$*, and*

(b) the open ball $\mathrm{B}^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$*,*

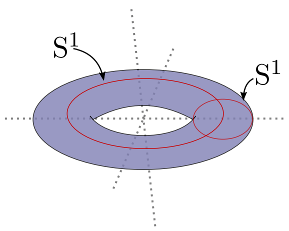
and $\mathrm{D}^{n+1} = \mathrm{S}^n \coprod \mathrm{B}^{n+1}$*.*

Note: The 0-sphere $\mathrm{S}^0 = \{\pm 1\}$ is the only sphere that is not connected.

Example 0.11 (The torus) *The* ***torus*** $\mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ *is equipped with the product topology. There is an obvious embedding into* $\mathbb{R}^4$*, by embedding each circle in* $\mathbb{R}^2$*. More useful is the inclusion into* $\mathbb{R}^3$ *via the map* $\varphi : \mathrm{S}^1 \times \mathrm{S}^1 \to \mathbb{R}^3$ *given by*

$$\varphi((x_1,y_1),(x_2,y_2)) = ((2+x_1)x_2, y_1, (2+x_1)y_2).$$

The image of φ *is a hollow ring with major radius 2 and minor radius 1. In pictures*



Where the smaller S^1 *is the image of the first copy of* S^1 *with the choice* $(1,0)$ *in the second copy. While the bigger* S^1 *is the image of the second copy of* S^1 *with the choice* $(0,1)$ *in the first copy.*

More generally, $\mathrm{T}^n = \prod_{i=1}^n \mathrm{S}^1$ *is the* n***-dimensional torus****. It can also be viewed as being embedded into* $\mathbb{R}^{2n}$ *or* $\mathbb{R}^{n+1}$ *similar to* T^2*.*

Special maps

Definition 0.12 *A continuous map $f : \mathrm{X} \to \mathrm{Y}$ is called a **homeomorphism** if there exists a continuous map $g : \mathrm{Y} \to \mathrm{X}$ such that $f \circ g = \mathrm{id}_{\mathrm{Y}}$ and $g \circ f = \mathrm{id}_{\mathrm{X}}$. In this case we say that X is **homeomorphic** to Y and write $\mathrm{X} \cong \mathrm{Y}$.*

Remarks on homeomorphisms

(a) The map g is called the **inverse** and is unique as it is also the inverse as a map of sets.

(b) Homeomorphisms are by definition the isomorphisms of the category **Top**.

Note for Example 0.11: If we restrict the map φ from Example 0.11 to its image it becomes bijective. One can then check that the inverse map of sets is also continuous and so φ restricted to its image is a homeomorphism.

Definition 0.13 *Let X be a topological space and $\mathrm{A} \subset \mathrm{X}$ a subspace. A **retraction** of X onto A is a continuous map $r : \mathrm{X} \to \mathrm{X}$ such that $r(\mathrm{X}) \subset \mathrm{A}$ and $r(a) = a$ for $a \in \mathrm{A}$.*

Remarks on retractions

(a) Depending on the situations, one can view a retraction from X to A also as a map from X to A by definition.

(b) Retractions will play a similar role to projection maps in algebra, for example from a vector space to a quotient space.

Example 0.14 (The "standard" retraction) *We set $\mathrm{X} = \mathbb{R}^2 \backslash \{(0,0)\}$, i.e. 2-dimensional real space minus the origin, and $\mathrm{A} = \mathrm{S}^1$.*
Then $r : \mathrm{X} \to \mathrm{X}$ given by $r(x,y) = (\sqrt{x^2+y^2})^{-1}(x,y)$ is continuous with image in A and the identity when restricted to A. Hence r is a retraction from X onto A.

Note that one cannot extend this map to all of $\mathbb{R}^2$. One can check this by lengthy calculations, but it will automatically follow from our studies later in the course.

Remarks on isomorphisms in topology and algebra
In algebra one usually considers isomorphisms (of groups, rings, vector spaces, etc.) as the only notion of "similarity" between objects.

We will see that for our purposes the spaces $\mathrm{X} = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathrm{A} = \mathrm{S}^1$ give the same results for all of our constructions later on, even though they are not homeomorphic. From the viewpoint of algebraic topology, X is "similar" to A, since one can shrink X down to A in a "good" way. We will need to formalize what we mean by this.

Definition 0.15 *Let* X, Y *be topological spaces,* $\mathrm{A} \subset \mathrm{X}$, *and* $\mathrm{I} = [0,1] \subset \mathbb{R}$.

(a) A continuous map $F : \mathrm{X} \times \mathrm{I} \to \mathrm{Y}$ *is called a* ***homotopy****. In this case we call* $f_0(x) = F(x,0)$ *and* $f_1(x) = F(x,1)$ ***homotopic*** *and write* $f_0 \simeq f_1$.

(b) A homotopy $F : \mathrm{X} \times \mathrm{I} \to \mathrm{Y}$ *is called* ***relative to*** A *if* $F(a,t)$ *is independent of* t *for all* $a \in \mathrm{A}$.

(c) A homotopy $F : \mathrm{X} \times \mathrm{I} \to \mathrm{X}$ *relative to* A *is called a* ***deformation retraction*** *of* X *onto* A *if* $f_0 = \mathrm{id}_\mathrm{X}$ *and* f_1 *is a retraction of* X *onto* A. *In this case* A *is called a* ***deformation retract*** *of* X.

(d) A continuous map $f : \mathrm{X} \to \mathrm{Y}$ *is called a* ***homotopy equivalence*** *if there exists a continuous map* $g : \mathrm{Y} \to \mathrm{X}$, *s.t.* $f \circ g \simeq \mathrm{id}_\mathrm{Y}$ *and* $g \circ f \simeq \mathrm{id}_\mathrm{X}$. *The map* g *is called a* ***homotopy inverse*** *(it is not unique). We write* $\mathrm{X} \simeq \mathrm{Y}$ *and call them* ***homotopy equivalent*** *(or of the same* ***homotopy type****).*

Lemma 0.16 *If* $F : \mathrm{X} \times \mathrm{I} \to \mathrm{X}$ *is a deformation retract from* X *onto* A, *then the retraction* f_1 *induces a homotopy equivalence between* X *and* A.

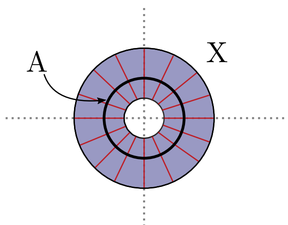
Proof: Let $i : \mathrm{A} \to \mathrm{X}$ be the inclusion and $r : \mathrm{X} \to \mathrm{A}$ be the map f_1 with domain restricted to A. Then $r \circ i = \mathrm{id}_\mathrm{A}$, since f_1 is a retraction, and $i \circ r = f_1 \simeq \mathrm{id}_\mathrm{X}$, since F is a deformation retract.

Example 0.17 *Let* $\mathrm{X} = \mathbb{R}^2 \backslash \{(0,0)\}$ *and* $r : \mathrm{X} \to \mathrm{X}$ *as in Example 0.14. We set* $F : \mathrm{X} \times \mathrm{I} \to \mathrm{X}$ *as*

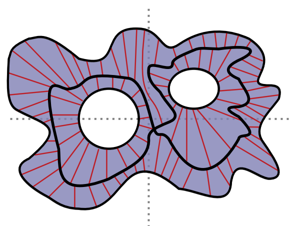
$$F((x,y),t) = \frac{(1-t)\sqrt{x^2+y^2}+t}{\sqrt{x^2+y^2}}(x,y).$$

This is continuous with $F((x,y),0) = \mathrm{id}_\mathrm{X}$ *and* $F((x,y),1) = r$, *hence a retraction. Thus* F *is a deformation retraction of* X *onto* S^1.

When spaces have an obvious embedding into 2-dimensional space and an obvious deformation retract we will illustrate it as a family of lines that shows how a point in X moves towards a point in A depending on t.



Here X is an annulus and $\mathrm{A} = \mathrm{S}^1$. The deformation retract is given by moving a point along a red line towards the intersection of the line with A. Hence it is exactly the deformation retraction as given in Example 0.17.



In this case there is no common sense methods to either parametrize the surrounding space or the subspace A, thus this is not an acceptable example for an illustration of a deformation retraction.

Quotient spaces

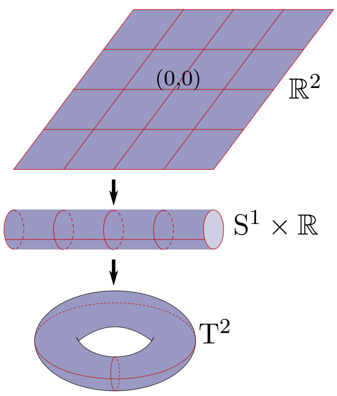
Definition 0.18 *Let* $(\mathrm{X},\mathcal{U})$ *be a topological space and* $\sim$ *an equivalence relation on the set* X. *Let* $\pi : \mathrm{X} \to \mathrm{X}/_\sim$ *be the quotient map with* $\pi(x) = [x]$ *(the equivalence class of* x*). Then we define the* ***quotient topology*** $\mathcal{U}_\sim$ *via* $\mathrm{U} \in \mathcal{U}_\sim$ *if and only if* $\pi^{-1}(\mathrm{U}) \in \mathcal{U}$.

Facts on the quotient topology
The quotient topology is the **finest** topology such that π is continuous.

Example 0.19 (The torus as a quotient) *Define the equivalence relation* $(x, y) \sim (u, v)$ *iff* $(x - u, y - v) \in \mathbb{Z}^2$ *on* $\mathbb{R}^2$. *Then* $\mathrm{T}^2 \cong \mathbb{R}^2/_\sim$ *with the homeomorphism given by*

$$\varphi : \mathbb{R}^2/_\sim \longrightarrow \mathrm{S}^1 \times \mathrm{S}^1, \quad [(x, y)] \longmapsto ((\cos(2\pi x),\sin(2\pi x)),(\cos(2\pi y),\sin(2\pi y))).$$

In pictures this looks as follows



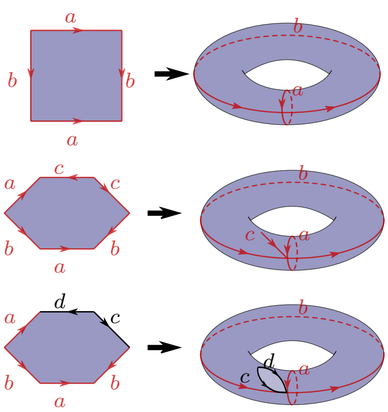
In the top picture we have a part of $\mathbb{R}^2$ with the points that have either coordinate being an integer highlighted.

In the middle picture we have identified points along the second coordinate resulting in an infinite cylinder, i.e. $\mathrm{S}^1 \times \mathbb{R}$.

In the bottom picture we have then also identified along the first coordinate. The two red circles are the image of the highlighted points from $\mathbb{R}^2$ under the quotient map.

As one can see it would be easier to simply start with $\mathrm{I} \times \mathrm{I}$ instead of $\mathbb{R}^2$ and identify opposing edges. The idea of taking polygons and gluing together edges is a very useful way to obtain many important surfaces in topology and analysis.

Example 0.20 (Glueing polygons) *To the left we have different examples of polygons with labelled edges and an orientation. In each case we identify in the quotient space of the polygon edges with the same label along the orientation, i.e. the starting points of the edges with each other, the end points with each other and in a continuous bijective way the points in between.*



In the top picture we get the torus in the same way as in our previous example.

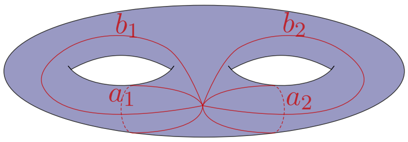
In the middle one we also get the torus, but it will have a distinguished line segment on it, the image of the edge with label c.

In the bottom picture the result is a torus, that is missing a small open ball on it. The boundary of the ball is the images of the two edges that were not identified with any other edge.

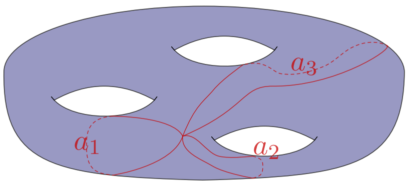
Example 0.21 *Denote by* X_i*, for* $0 \leq i \leq g-1$ *and* $g \in \mathbb{N}$*, the space we obtained in the last example, i.e. the torus with a missing open ball and denote by* c_i *and* d_i *the edges as before. The two distinguished circles on* X_i *we denote by* a_i *and* b_i*, i.e. the images of the edges with label* a *respectively* b*. Then we can identify the edge* c_i *in* X_i *with the edge* d_{i+1} *in* X_{i+1} *where we look at the indices modulo* g*.*

In case $g = 1$ *we just obtain the torus again. This was our example of the torus with a line segment on it. But as a space this is still just the torus.*

If we view the torus as a 2*-sphere with a "hole" in the middle, i.e. a "donut", then in case of arbitrary* g *we obtain a* 2*-sphere with* g *holes. This is denoted by* M_g *and called an* ***orientable surface of genus*** g*. One additionally declares* $\mathrm{M}_0 = \mathrm{S}^2$*.*



The surface M_2



The surface M_3

See the picture to the left for the $g = 2$ *and* $g = 3$ *case.*

In the $g = 2$ *case we illustrate the images of all four circles meeting at a single point. These circles are the images of the edges labelled* a *and* b *in the two original hexagons.*

While in the $g = 3$ *case we only illustrate the circles* a_1*,* a_2 *and* a_3 *that go "through" precisely one hole in the chosen illustration. We omit the circles that go "around" a single hole. Note that any two of the six circles only intersect in the single distinguished point.*

The orientable surfaces of genus g play an important role in differential geometry and analysis. As one can see here, they have a very nice topology, since open subsets are just subsets that have an open preimage in the polygon which is by itself just a subset of $\mathbb{R}^2$.

Example 0.22 *A more general example is the* ***mapping cylinder****. Let* X *and* Y *be topological spaces and* $f : \mathrm{X} \to \mathrm{Y}$ *a continuous map.*

The mapping cylinder M_f *is the space*

$$\mathrm{M}_f = ((\mathrm{X} \times \mathrm{I}) \coprod \mathrm{Y})/_\sim,$$

where $(x, 1) \sim f(x)$ *for* $x \in \mathrm{X}$*, i.e. if we view* $\mathrm{X} \times \mathrm{I}$ *as a cylinder over the space* X*, then* M_f *glues* Y *to one end of that cylinder.*

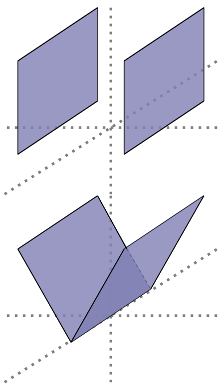
This is especially useful because this construction comes automatically with a deformation retraction of M_f *onto* Y.

Note that the map π *restricted to* Y *is injective, so we can view* Y *as a subspace of* M_f.

For the homotopy define $F : \mathrm{M}_f \times \mathrm{I} \to \mathrm{M}_f$ *via* $F([y], t) = [y]$ *for any* $y \in \mathrm{Y}$ *and for* $(x, s) \in \mathrm{X} \times \mathrm{I}$ *set* $F([(x, s)], t) = [(x, s + t(1 - s))]$*. Then* $F([(x, 1)], t) = [(x, 1)] = [f(x)] = F([f(x)], t)$ *for* $x \in \mathrm{X}$ *and so* F *is a well-defined continuous map.*

It holds $f_0 = \mathrm{id}_{\mathrm{M}_f}$*. Furthermore* f_1 *is a retraction of* M_f *onto* Y*, since* $f_1([(x, s)]) = [(x, s + (1 - s))] = [(x, 1)] = [f(x)] \in \mathrm{Y} \subset \mathrm{M}_f$ *and* $f_1([y], t) = [y]$*. Hence* $\mathrm{M}_f \simeq \mathrm{Y}$ *in this case.*

Morally one takes the space $\mathrm{X} \times \mathrm{I}$ *and presses it down onto* Y *along the interval* I*. As an example this looks as follows*



Here $\mathrm{X} = \mathrm{I} \coprod \mathrm{I}$*. Then the picture to the left realizes* $\mathrm{X} \times \mathrm{I}$ *inside* $\mathbb{R}^3$*.*

For $\mathrm{Y} = \mathrm{I}$ *we choose the map* f *to be the identity on both copies of* I *in* X*. Then the mapping cylinder* M_f *can be realized inside* $\mathbb{R}^3$ *as shown to the left.*

Example 0.23 *There are generalisations and variations of the mapping cylinder construction that appear very often in topology and analysis.*

The ***double mapping cylinder*** *is constructed by considering topological spaces* X*,* Y_0 *and* Y_1 *and two continuous maps* $g_0 : \mathrm{X} \to \mathrm{Y}_0$ *and* $g_1 : \mathrm{X} \to \mathrm{Y}_1$*. Then the double mapping cylinder is the quotient space* $\mathrm{M}_{g_0, g_1} = (\mathrm{Y}_0 \coprod (\mathrm{X} \times \mathrm{I}) \coprod \mathrm{Y}_1)/_\sim$ *where* $(x, 0) \sim g_0(x)$ *and* $(x, 1) \sim g_1(x)$ *for all* $x \in \mathrm{X}$*. Now each “end” of* $\mathrm{X} \times \mathrm{I}$ *is glued to another space.*

A special case of this is the ***mapping cone****, in this case* $\mathrm{Y}_0 = \mathrm{pt}$ *is just a point and the map* g_0 *is the map that send all of* X *to the point. Then we write* C_{g_1} *for the mapping cone and can simplify the construction* $\mathrm{C}_{g_1} = ((\mathrm{X} \times \mathrm{I}) \coprod \mathrm{Y}_1)/_\sim$ *where* $(x, 0) \sim (x', 0)$ *and* $(x, 1) \sim g_1(x)$ *for all* $x, x' \in \mathrm{X}$*. Thus we glue* Y_1 *to the bottom as in the mapping cylinder, but we identify the whole upper boundary of the cylinder.*

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Final remarks

In the study of fundamental groups we will need the notion of a **pointed topological space**, this is a pair (X, x_0) with X a topological space and $x_0 \in \mathrm{X}$. The point x_0 is usually called the **basepoint**.

By looking only at continuous maps $f : (\mathrm{X}, x_0) \to (\mathrm{Y}, y_0)$ such that $f(x_0) = y_0$ one obtains the category of pointed topological spaces $\mathbf{Top}_\bullet$.

While the product of two pointed topological spaces (X, x_0) and (Y, y_0) is naturally again a pointed topological space $(\mathrm{X} \times \mathrm{Y}, (x_0, y_0))$. The disjoint union is not.

Definition 0.24 *Let (X, x_0) and (Y, y_0) be pointed topological spaces, then the **wedge sum** $\mathrm{X} \vee \mathrm{Y}$ is defined as $\mathrm{X} \coprod \mathrm{Y}/_\sim$, where the only non-trivial equivalence relation is $x_0 \sim y_0$.*

*More generally, for a family $(\mathrm{X}_i, x_i)_{i \in I}$ for some set I, the **wedge sum** $\bigvee_i \mathrm{X}_i = \coprod_i \mathrm{X}_i/_\sim$ is given by the only non-trivial equivalence relations begin $x_i \sim x_j$ for all $i, j \in I$.*

In both cases the class of the distinguished points makes the wedge sum into a pointed topological space.

Facts on the wedge sum
The wedge sum is the coproduct in the category $\mathbf{Top}_\bullet$.

Some conventions for the rest of the course

- We call a topological space X simply a space.
- We assume that all given maps are continuous.
- For $f : (\mathrm{X}, x_0) \to (\mathrm{Y}, y_0)$ a map of pointed spaces we assume $f(x_0) = y_0$.
- For a homotopy $F : \mathrm{X} \times \mathrm{I} \to \mathrm{Y}$ we define f_t via $f_t(x) = F(x, t)$. And similar for other uppercase letters for the homotopy and lowercase letters for the family of continuous maps, e.g. a homotopy G and the family of maps g_t.

Lemma 0.25 *Let $F : \mathrm{X} \times \mathrm{I} \to \mathrm{Y}$ be a homotopy and $g : \mathrm{Y} \to \mathrm{Z}$ a map. Then $f_t \simeq f_{t'}$ for any $t, t' \in \mathrm{I}$ and $g \circ F$ is a homotopy from $g \circ f_0$ to $g \circ f_1$.*

Some notations
If a space X is homotopic to a single point, we call X **contractible**.

If $f \simeq g$ are homotopic maps such that g is a constant map, i.e. the image is a single point, then we say that f is **nullhomotopic**.

Remark
One has to be very careful with the notations for the relation between spaces, while $\mathrm{X} \cong \mathrm{Y}$ says that the spaces are **homeomorphic**, i.e. there exists a homeomorphism between X and Y, $\mathrm{X} \simeq \mathrm{Y}$ says that the two spaces are **homotopy equivalent**. By definition

$$\mathrm{X} \cong \mathrm{Y} \Longrightarrow \mathrm{X} \simeq \mathrm{Y},$$

but the converse is usually not true. Being homeomorphic is a much stronger assumption than being homotopy equivalent.