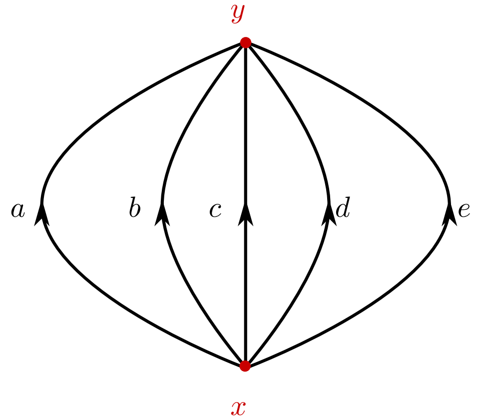
Starting example for homology

We consider an example of a space to introduce the idea of homology. We start with a topological space X_1 and will enlarge it over the course of the example by gluing more and more pieces onto it in higher dimensions.

The base space: As a space X_1 is the quotient of the disjoint union of 5 unit intervals I, where we identify the 0 in all intervals (calling this common point x) and we identify the 1 in all intervals (calling this common point y). We will also denote the embedded intervals into the space as a, b, c, d, and e. For simplicity we will use the same letter for a path that goes from x to y in the respective interval, as seen in the picture



The fundamental group of X_1: For comparison we first calculate the fundamental group of X_1 using the van Kampen theorem, this gives

$$\pi_1(X_1, x) = \langle [a\bar{e}], [b\bar{e}], [c\bar{e}], [d\bar{e}] \rangle \cong *_{i=1}^{4}\mathbb{Z}.$$

We get four generating homotopy classes of loops at x. This group is fairly complicated and it is cumbersome to even explicitly write down an arbitrary loop with respect to the generators. To simplify this, we look at the abelianized version

$$\pi_1(X_1, x)^{\text{ab}} = \pi_1(X_1, x)/[\pi_1(X_1, x), \pi_1(X_1, x)] \cong \mathbb{Z}^4,$$

i.e. the group quotient by its commutator subgroup. Our first aim is to recover this abelian group from X_1 without the need of the fundamental group.

Chain groups: In our example, the space X_1 is glued together from smaller pieces, namely 0-cells (i.e. points) and 1-cells (i.e. unit intervals or equivalently D^1's). Later on we will glue in higher dimensional pieces, namely 2-cells (i.e. copies of D^2) and 3-cells (i.e. copies of D^3). Note that each of these cells is embedded into our space X_1. We define

$$C_n = \mathbb{Z}\{n - \text{cells}\}$$

the free abelian group with basis the n-cells we use in our space. For convenience we will set $C_{-1} = \{0\}$ and we will not make use of cells of dimension higher than three for the example, so $C_n = \{0\}$ for $n \geq 4$ as well. For the space X_1 we thus have

$$C_0 = \mathbb{Z}x \oplus \mathbb{Z}y, \quad C_1 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}d \oplus \mathbb{Z}e, \quad C_2 = \{0\}, \text{ and } \quad C_3 = \{0\}.$$

Cycles: We are interested in so-called cycles in these groups, especially in C_1 for now. For this purpose a cycle is an element of C_1 that comes from a loop in our space under abelianization. We explain this with an example. Consider $2a - b - c$, then this can be rewritten as $(a - b) + (a - c)$. We can view $(a - b)$ as being the image of the loop $[a\bar{b}]$ in the abelianization and similarly $(a - c)$ comes from the loop $[a\bar{c}]$.
Hence for the question: when an element in C_1 is a cycle, i.e. when does it come from a loop at x. We need that it contains as many 1-cells with a positive sign (i.e. path from x to y) as it contains 1-cells with negative signs (i.e. path from y to x). So for a general element $\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e$ it must hold that $\lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0$. This we want to obtain as a kernel of a map from C_1 to C_0.

We see here the motivation to consider C_1, we can decompose loops at x into their building blocks, being made up of the paths $a, \ldots, e$ and their inverses. This is possible since the intervals are simply-connected, hence there is, up to homotopy, a unique path from x to y in each interval.
Chain maps: We define group homomorphisms $\partial_n : C_n \to C_{n-1}$ for $n \geq 0$. In the example X_1 the only non-trivial map is $\partial_1 : C_1 \to C_0$. We need to declare what ∂_1 does on a generator of C_1. A generator is in our case an edge connecting two 0-cells and we have already chosen a path on each edge from x to y that equips our edges with an orientation. We will send an edge to (endpoint) - (starting point). Hence

$$\begin{aligned} \partial_1 : C_1 &\longrightarrow C_0 \\ a, b, c, d, e &\longmapsto y - x. \end{aligned}$$

A simple calculation shows that the kernel of ∂_1 are exactly the elements satisfying $\lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0$, i.e. the cycles in C_1. Note also that in our example X_1, we have that $\partial_n \circ \partial_{n+1} = 0$ or equivalently $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$. This will be true for the later examples as well.

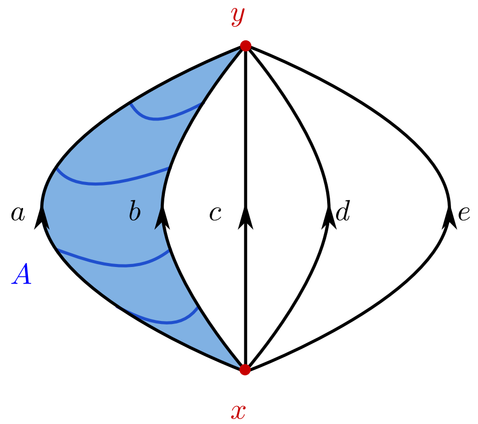
Homology groups: We can now define the homology groups of our space as

$$H_n(X) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

Example X_1: In our example we thus see that

$$\begin{aligned} H_0(X_1) &= C_0/\mathrm{Im}(\partial_1) = C_0/\langle y - x \rangle \cong \mathbb{Z} \\ H_1(X_1) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) \\ &= \mathrm{Ker}(\partial_1) = \{\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e \mid \lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0\} \cong \mathbb{Z}^4 \\ H_n(X_1) &= \{0\} \text{ for } n \geq 2. \end{aligned}$$

Example X_2: We now enlarge our space by gluing in a 2-cell. Denote the 2-cell by A, its boundary is a circle S^1, take two antipodal points on the circle, identify these with x respectively y, and identify the two half-circles with a respectively b. In pictures this looks as follows



Then we now have $C_2 = \mathbb{Z}A$ and the map $\partial_2 : C_2 \to C_1$ is defined via $\partial_2(A) = a - b$. For the map consider the boundary of A to have an orientation (let us take clock-wise in the picture) and compare it with the orientation of a and b, then we see that a is in the correct orientation, while b has the opposite orientation. Hence we send A to its boundary decomposed into the 1-cells, but with a sign, depending on whether they have the correct orientation or not. We now obtain the following homology groups:

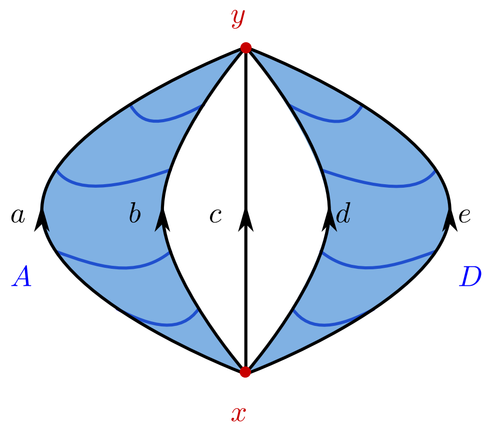
$$\begin{aligned}
H_0(X_2) &= C_0/\mathrm{Im}(\partial_1) = C_0/\langle y - x\rangle \cong \mathbb{Z} \\
H_1(X_2) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) \\
&= \{\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e \mid \lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0\}/\langle a - b\rangle \cong \mathbb{Z}^3 \\
H_2(X_2) &= \{0\} \text{ since } \partial_2 \text{ is injective} \\
H_n(X_2) &= \{0\} \text{ for } n \geq 3.
\end{aligned}$$

The only thing that really changed is that the image of ∂_2 is now $\langle a - b\rangle$. The meaning for this is that in X_2 the loop $a\bar{b}$ is nullhomotopic, which it was not in X_1.
To give some "informal" interpretation of the homology groups:

- That H_0 has rank 1 for both X_1 and X_2 comes from the fact that both spaces have one path-connected component.
- That H_1 changed from $\mathbb{Z}^4$ (for X_1) to $\mathbb{Z}^3$ (for X_2) can be interpreted as saying that X_1 has 4 holes in dimension 2, i.e. is missing the interiors of circles S^1, where the circles are given by the four loops $a\bar{b}$, $b\bar{c}$, $c\bar{d}$, and $d\bar{e}$. But In X_2, one of these circles is now "filled up" by the new 2-cell, hence H_1 has decreased in rank by 1.
- Since we have not yet created an S^2 inside our space, there is no hole in dimension 3, hence H_2 is trivial in both example spaces.

This easy interpretation only works because our spaces have such a nice form, but one can already see that homology groups carry a lot of topological information.

Example X_2': We now glue another 2-cell into X_2, this time between the edges d and e, called D. In pictures this looks as follows

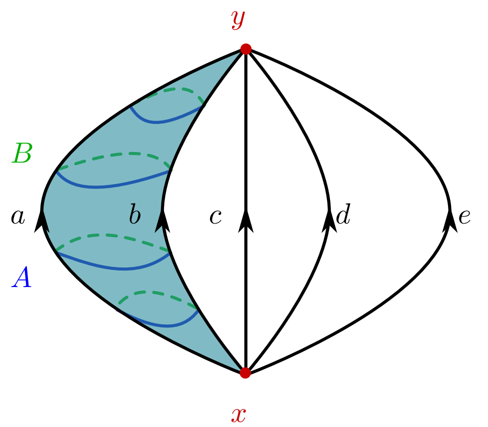


Then we need to make the following changes $C_2 = \mathbb{Z}A \oplus \mathbb{Z}D$ and the map $\partial_2 : C_2 \to C_1$ is defined via $\partial_2(A) = a - b$ and $\partial_2(D) = d - e$. Then we obtain the homology groups

$$\begin{aligned}
H_0(X_2') &= C_0/\mathrm{Im}(\partial_1) = C_0/\langle y - x\rangle \cong \mathbb{Z}\\
H_1(X_2') &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2)\\
&= \{\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e \mid \lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0\}/\langle a - b, d - e\rangle \cong \mathbb{Z}^2\\
H_2(X_2') &= \{0\} \text{ since } \partial_2 \text{ is still injective}\\
H_n(X_2') &= \{0\} \text{ for } n \geq 3.
\end{aligned}$$

We see that we "filled up" another hole in dimension 2 and H_1 has gotten smaller again. Next we want to see what would have happened if instead of gluing the second 2-cell between d and e, we glue it again between a and b.

Example X_2'': In contrast to example X_2' we now glue another 2-cell into X_2 between the edges a and b, called B. In pictures this looks as follows

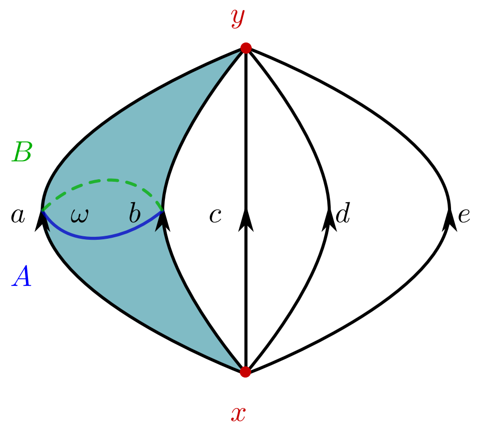


Then we need to make the following changes $C_2 = \mathbb{Z}A \oplus \mathbb{Z}B$ and the map $\partial_2 : C_2 \to C_1$ is defined via $\partial_2(A) = a - b$ and $\partial_2(B) = a - b$. Hence we choose the same orientation on B as for A. Then we obtain the homology groups

$$\begin{aligned}
H_0(X_2'') &= C_0/\mathrm{Im}(\partial_1) = C_0/\langle y-x\rangle \cong \mathbb{Z} \\
H_1(X_2'') &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) \\
&= \{\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e \mid \lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0\}/\langle a-b\rangle \cong \mathbb{Z}^3 \\
H_2(X_2'') &= \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3) \\
&= \langle A-B\rangle \cong \mathbb{Z} \\
H_n(X_2'') &= \{0\} \text{ for } n \geq 3.
\end{aligned}$$

Since the image of ∂_2 is the same as for X_2, H_1 agrees with X_2, but now ∂_2 is not injective anymore and we get a non-trivial H_2. This fits with the interpretation as we have now created a 2-sphere inside our space by gluing A and B back to back into the same circle inside X_1. Since the sphere is missing its interior, we have a hole in dimension 3.

Example X_3: For our final example we take X_2'' and fill the 2-sphere with a 3-cells, i.e. for a D^3 choose an equator on the boundary, identify that with the circle formed by x, y, a, and b. Finally identify the two half spheres on opposite sides of the equator with A respectively B. In the pictures we cannot really visualize this



Of course now $C_3 = \mathbb{Z}\omega$, where ω is the new 3-cell and we define $\partial_3 : C_3 \to C_2$ is defined via $\partial_2(\omega) = A - B$. Then we obtain the homology groups

$$\begin{aligned}
H_0(X_3) &= C_0/\mathrm{Im}(\partial_1) = C_0/\langle y-x\rangle \cong \mathbb{Z} \\
H_1(X_3) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) \\
&= \{\lambda_a a + \lambda_b b + \lambda_c c + \lambda_d d + \lambda_e e \mid \lambda_a + \lambda_b + \lambda_c + \lambda_d + \lambda_e = 0\}/\langle a-b\rangle \cong \mathbb{Z}^3 \\
H_2(X_3) &= \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3) \\
&= \langle A-B\rangle / \langle A-B\rangle \cong \{0\} \\
H_3(X_3) &= \{0\} \text{ since } \partial_3 \text{ is injective} \\
H_n(X_3) &= \{0\} \text{ for } n \geq 4.
\end{aligned}$$

We "filled up" the hole in dimension 3, hence H_2 is now trivial again. Since we don't have any 3-spheres in our space, there are no holes in dimension 4 and hence H_3 is still trivial.