Algebraic Geometry

Beijing Institute of Technology, 2019

Exercises for chapter 3: Affine algebraic varieties

We use the notations from the course, i.e. $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ its algebraic closure, and so on.

Exercise 1. In the following we will always look at the closure in the Zariski topology of the variety. Show the following statements

(a) Let $\mathbb{C}^*_{\text{fin}} \subset \mathbb{C}^*$ be the elements of finite multiplicative order in $\mathbb{C}^*$. Show that the closure of $\mathbb{C}^*_{\text{fin}}$ is $\mathbb{C}^*$.

(b) Let $O_2(\mathbb{C}) = \{A \in Gl_2(\mathbb{C}) \mid A^T = A^{-1}\} \subset Gl_2(\mathbb{C})$ be the complex orthogonal matrices. Show that $O_2(\mathbb{C})$ is closed in $Gl_2(\mathbb{C})$.

(c) Let $U_2(\mathbb{C}) = \{A \in Gl_2(\mathbb{C}) \mid \overline{A}^T = A^{-1}\} \subset Gl_2(\mathbb{C})$ be the complex hermitian matrices, where $\overline{A}^T$ is the complex conjugate of the transpose of A. Show that the closure of $U_2(\mathbb{C})$ is $Gl_2(\mathbb{C})$.

Exercise 2. Assume $\mathbb{K} = \overline{\mathbb{K}}$ and $(X, \mathbb{K}[X])$ is an affine algebraic variety over $\mathbb{K}$.

(a) Show that $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in $(X \times X, \mathbb{K}[X] \otimes \mathbb{K}[X])$.

(b) Assume that X is irreducible and let $f \in \mathbb{K}(X)$. Show that $\mathrm{Pol}(f)$, the subset of poles of f in X, is an affine variety.

(c) Let $(Y, \mathbb{K}[Y])$ be an affine algebraic variety and $\Phi : X \to Y$ a morphism of affine algebraic varieties. Show that if Φ^* is surjective then $\mathrm{Im}(\Phi)$ is a closed subset of Y.

Exercise 3. Assume $\mathbb{K} = \overline{\mathbb{K}}$. Show the following

(a) $\{A \in M_n(\mathbb{K}) \mid \mathrm{rank}(A) \le r\}$ for $0 \le r \le n$ is a closed subvariety of $M_n(\mathbb{K})$,

(b) $\{A \in M_n(\mathbb{K}) \mid A \text{ nilpotent}\}$ is a closed and irreducible subvariety of $M_n(\mathbb{K})$,

(c) $\{A \in M_n(\mathbb{K}) \mid A \text{ diagonizable}\}$ is an irreducible subset of $M_n(\mathbb{K})$.