

THE PERIPLECTIC BRAUER ALGEBRA III: THE DELIGNE CATEGORY

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ABSTRACT. We construct a faithful categorical representation of an infinite Temperley-Lieb algebra on the periplectic analogue of Deligne's category. We use the corresponding combinatorics to classify thick tensor ideals in this periplectic Deligne category. This allows to determine the kernel of the tensor functor going to the module category of the periplectic Lie supergroup, which in turn yields a description of the tensor powers of the natural representation.

INTRODUCTION

This is the third paper in a series studying an analogue of the Brauer algebra which appears in invariant theory for the periplectic Lie superalgebra, see [Mo]. In [Co] the first author studied the cellular properties of the algebras over fields of arbitrary characteristic, leading in particular to a classification of the blocks in characteristic zero. In [CE] we completed this by determining explicitly the Jordan-Hölder decomposition multiplicities of projective and cell modules.

In the current paper we study the periplectic analogue \mathcal{PD} of the *Deligne category* of [De], a strict monoidal supercategory with universal properties, defined in [KT, Se]. We construct a *categorical representation of $\mathrm{TL}_\infty(0)$, the infinite Temperley-Lieb algebra with the circle evaluated at zero*, on \mathcal{PD} . This can be interpreted as a natural analogue of the categorical representation of \mathfrak{sl}_∞ on module categories of symmetric groups or polynomial functors, see [HY, LLT]. Moreover, our approach should be adaptable to construct a categorical representation of $\mathfrak{sl}_{\infty/2} \oplus \mathfrak{sl}_{\infty/2}$ on the ordinary Deligne category $\underline{\mathrm{Rep}}(O_\delta)$ of [De, CH], which relates to [GS].

Our categorical representation of $\mathrm{TL}_\infty(0)$ is a ‘weak categorification’ of a representation in the terminology of [Ma], since there is no known 2-categorical or monoidal notion of categorification of TL_∞ that incorporates the specialisation at 0. We prove that the representation we categorify, which is a representation of $\mathrm{TL}_\infty(0)$ on the bosonic Fock space, is faithful, which might be of use in developing such a notion. The categorical representation of $\mathrm{TL}_\infty(0)$ admits a filtration, where each composition factor corresponds to a cell of the monoidal supercategory \mathcal{PD} . Moreover, we show that the decategorification of the composition factors are isomorphic to representations of $\mathrm{TL}_\infty(0)$ categorified in [BDE+], and that both categorifications are very closely related.

As a consequence of the faithfulness of the representation of $\mathrm{TL}_\infty(0)$ on Fock space, we can realise $\mathrm{TL}_\infty(0)$ as the image of $U(\mathfrak{k})$, under the restriction of the Fock space

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representation of \mathfrak{sl}_∞ to a subalgebra \mathfrak{k} . So far as we know, this is the first realisation of a Temperley-Lieb algebra as the quotient of the universal enveloping algebra of a Lie algebra.

The functor on the Deligne category which lies at the basis of the categorical representation is the tensor product with the generator. Its combinatorics determines explicitly the structure of the tensor product of this generator and an arbitrary indecomposable object. In particular we use this to *classify the thick tensor ideals and cells* in the periplectic Deligne category. The corresponding classification for the Deligne category $\text{Rep}(O_\delta)$ was obtained in [CH]. We use a different approach, compared to [CH], to prove that the combinatorics of this functor is related to the decomposition multiplicities of the periplectic Brauer algebra in [CE]. This approach is much more direct, since it does not rely on liftings of idempotents or classical invariant theory, and can thus be applied in many similar situations (including the one in [CH]).

There exists a tensor functor from the periplectic Deligne category to the category of finite dimensional modules over the periplectic Lie supergroup, see [KT, Se], which is full by results in [DLZ]. Its kernel must be a thick tensor ideal and similarly the pre-image of the class of projective modules is a thick tensor ideal. Our classification of thick tensor ideals allows to determine efficiently those ideals. This thus yields *a classification of the indecomposable direct summands in the tensor powers of the natural representation* for the periplectic Lie supergroup. Furthermore, we determine *which direct summands are projective*. These results are analogues of the corresponding ones for the orthosymplectic Lie supergroups in [CH]. In contrast to [CH], the methods used here do not rely on cohomological tensor functors and instead use simple combinatorial considerations to deduce the classification.

The paper is organised as follows. After recalling some definitions and introducing some notation concerning monoidal supercategories and periplectic Brauer algebras in Section 1, we study the elementary properties of the periplectic Deligne category \mathcal{PD} in Section 2. In Section 3, we study the functor \mathbf{T} on the Deligne category which corresponds to taking the tensor product with the generator. We prove that its action on objects can be described in terms of the decomposition multiplicities of the periplectic Brauer algebra in [CE] and that it decomposes as $\mathbf{T} = \bigoplus_{i \in \mathbb{Z}} \mathbf{T}_i$ according to the eigenvalues of a natural transformation. Section 4 is a purely combinatorial section where we prove uniqueness and existence of a representation of the Temperley-Lieb algebra $\text{TL}_\infty(0)$ on the space of partitions (the Fock space). It then follows that the functors \mathbf{T}_i decategorify to this representation. We also prove that the representation is faithful and establish a filtration. Section 5 contains our main results, the classification of thick tensor ideals in \mathcal{PD} and the description of the higher tensor powers of the natural representation of the periplectic Lie supergroup. Finally, in Section 6 we construct natural transformations related to the functors \mathbf{T}_i in order to improve the above decategorification statements to an actual categorical representation and filtration. Furthermore, we establish a connection between the composition factors of the filtration of our categorical representation and the categorical representations of $\text{TL}_\infty(0)$ in [BDE+].

1. PRELIMINARIES

We set $\mathbb{N} = \{0, 1, 2, \dots\}$. Throughout the paper, \mathbb{k} is an algebraically closed field of characteristic zero. Let $\mathbf{svec}_{\mathbb{k}}$ denote the monoidal category of all \mathbb{Z}_2 -graded \mathbb{k} -vector spaces, with grading preserving morphisms. For elements v of degree $\bar{0}$, resp. $\bar{1}$, in a graded vectorspace, we write $|v| = 0$, resp. $|v| = 1$. For any $r \in \mathbb{Z}_{\geq 1}$, we introduce the sets

$$\mathcal{J}(r) := \{r - 2i \mid 0 \leq i \leq r/2\} \quad \text{and} \quad \mathcal{J}^0(r) := \{r - 2i \mid 0 \leq i < r/2\}.$$

Furthermore, we set $\mathcal{J}(0) = 0 = \mathcal{J}^0(0)$.

1.1. Partitions. We denote the set of partitions of all numbers by \mathbf{Par} . The free \mathbb{Z} -module of \mathbb{Z} -linear combinations of the elements of \mathbf{Par} will be denoted by $\mathbf{Par}_{\mathbb{Z}}$. All matrices that will appear in the paper will have their columns and rows labelled by \mathbf{Par} and have entries in \mathbb{Z} .

We will identify a partition with its Young diagram, using English notation. Each box or node in the diagram has coordinates (i, j) , meaning that the box is in row i and column j . The *content* of a box in position (i, j) in a Young diagram is $j - i$. Any box with content q will be referred to as a q -box. The value $i + j$ will be referred to as the *anticontent* of the box.

By a *rim hook* of λ we mean a removable and connected hook of λ (by a (rim) a -hook for $a \in \mathbb{N}$ we mean a rim hook with a boxes). For any $\lambda \in \mathbf{Par}$, we denote by $\mathcal{R}(\lambda)$ the set of all partitions which can be obtained by removing one box of the Young diagram of λ . In case λ has an addable q -box, we write the partition obtained by adding said box as $\lambda^{(q)}$.

For $k \in \mathbb{N}$, we fix the partition ∂^k of $\frac{1}{2}k(k+1)$, defined as

$$\partial^k := (k, k-1, \dots, 1, 0).$$

The set $\{\partial^k \mid k \in \mathbb{N}\}$ thus consists of all 2-cores, *i.e.* all partitions from which one cannot remove any rim 2-hook. We define the following subsets of \mathbf{Par} :

$$(1.1) \quad \mathbf{Par}^{\geq k} = \{\lambda \mid \partial^k \subseteq \lambda\}, \quad \mathbf{Par}^{\leq k} = \mathbf{Par} \setminus \mathbf{Par}^{\geq (k+1)} \quad \text{and} \quad \mathbf{Par}^k = \mathbf{Par}^{\geq k} \cap \mathbf{Par}^{\leq k}.$$

1.2. Supercategories. We recall some definitions of [BE, Section 1].

1.2.1. Categories, functors and natural transformations. A *supercategory* is defined as a category enriched over $\mathbf{svec}_{\mathbb{k}}$. *Superfunctors* between supercategories are functors enriched in the same way. By definition, supercategories and superfunctors are thus in particular \mathbb{k} -linear.

Consider two supercategories $\mathcal{C}_1, \mathcal{C}_2$ and superfunctors $F, G : \mathcal{C}_1 \rightarrow \mathcal{C}_2$. A *natural transformation of superfunctors* $\xi : F \rightarrow G$ of *parity* p is a family $\{\xi_X : FX \rightarrow GX \mid X \in \text{Ob } \mathcal{C}_1\}$ of morphisms of parity p such that for any homogeneous morphism $\alpha : X \rightarrow Y$ in \mathcal{C}_1 , we have $G(\alpha) \circ \xi_X = (-1)^{p|\alpha|} \xi_Y \circ F(\alpha)$. An even natural transformation of superfunctors is thus just an ordinary natural transformation, where every morphism is even. All functors appearing will be superfunctors, thus all natural transformations appearing are considered as natural transformations of superfunctors.

In the following three paragraphs we recall some standard manipulations of natural transformations. For ease of reading we leave out the categories on which the various functors are defined, as it should be clear from context.

For a functor F and a natural transformation $\xi : G_1 \rightarrow G_2$, we denote by $F(\xi) : F \circ G_1 \rightarrow F \circ G_2$ the natural transformation given by $F(\xi)_X = F(\xi_X)$. The natural transformation $\xi_F : G_1 \circ F \rightarrow G_2 \circ F$ is defined as $(\xi_F)_X = \xi_{FX}$.

For two natural transformations $\xi_1 : F_1 \rightarrow G_1$ and $\xi_2 : F_2 \rightarrow G_2$, we denote the horizontal composition, or *Godement product*, by $\xi_1 \star \xi_2 : F_1 \circ F_2 \rightarrow G_1 \circ G_2$, which is the natural transformation $G_1(\xi_2) \circ (\xi_1)_{F_2} = (\xi_1)_{G_2} \circ F_1(\xi_2)$.

For two natural transformations $\xi_1 : F \rightarrow G$ and $\xi_2 : G \rightarrow H$, we denote the vertical composition by $\xi_2 \circ \xi_1 : F \rightarrow H$, this is the natural transformation defined by $(\xi_2 \circ \xi_1)_X = (\xi_2)_X \circ (\xi_1)_X$.

1.2.2. Kernel of a functor. We say that a functor is *essentially surjective* if any object in the target category is isomorphic to one in the image. The *kernel* of a functor is the full subcategory of the source category of all objects which are sent to zero. A functor is *essentially injective* if it has trivial kernel. A functor is *essentially bijective* if it is both essentially injective and surjective.

1.2.3. Monoidal supercategories. For two supercategories \mathcal{B} and \mathcal{C} , the supercategory $\mathcal{B} \boxtimes \mathcal{C}$ has as objects ordered pairs (X, Y) , with $X \in \text{Ob } \mathcal{B}$ and $Y \in \text{Ob } \mathcal{C}$, and morphism spaces given by

$$\text{Hom}_{\mathcal{B} \boxtimes \mathcal{C}}((X_1, Y_1), (X_2, Y_2)) = \text{Hom}_{\mathcal{B}}(X_1, X_2) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(Y_1, Y_2),$$

with composition defined by the super interchange law

$$(1.2) \quad (f \otimes g) \circ (h \otimes k) = (-1)^{|h||g|} (f \circ h) \otimes (g \circ k).$$

A *strict monoidal supercategory* is a supercategory \mathcal{C} equipped with a superfunctor $\mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ denoted by $- \otimes -$, and a unit object $\mathbb{1}$, such that we have equalities of functors $\mathbb{1} \otimes - = \text{Id} = - \otimes \mathbb{1}$ and $(- \otimes -) \otimes - = - \otimes (- \otimes -)$. When we omit ‘strict’, the three equalities are replaced by even natural isomorphisms.

Remark 1.2.4. As is customary, we use the same notation ‘ \otimes ’ for the functor and for the tensor product of vector spaces. In particular, for $- \otimes - : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ acting on morphisms, we write

$$(- \otimes -)(f \otimes g) = f \otimes g.$$

It should be clear from context whether $f \otimes g$ represents a morphism in $\mathcal{C} \boxtimes \mathcal{C}$ or \mathcal{C} .

1.2.5. For two monoidal supercategories \mathcal{C}_1 and \mathcal{C}_2 , a *monoidal superfunctor* is a superfunctor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ with an even natural isomorphism $c : (F-) \otimes (F-) \rightarrow F \circ (- \otimes -)$ and an even isomorphism $i : \mathbb{1}_{\mathcal{C}_2} \rightarrow F(\mathbb{1}_{\mathcal{C}_1})$ satisfying the usual axioms.

A monoidal supercategory \mathcal{C} is *symmetric* if there exists a braiding B , which is a family of even isomorphisms in \mathcal{C}

$$\{B_{X,Y} : X \otimes Y \rightarrow Y \otimes X \mid X, Y \in \text{Ob } \mathcal{C}\},$$

such that $B_{X',Y'} \circ (f \otimes g) = (-1)^{|f||g|} (g \otimes f) \circ B_{X,Y}$ for any two morphisms $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$. We also have the usual commutative diagrams for $B_{X,Y \otimes Z}$ and $B_{X \otimes Y,Z}$.

1.3. The periplectic Brauer category. The *periplectic Brauer category* \mathcal{A} , was introduced as the category $\mathcal{B}(0, -1)$ in [KT], see also [Se, BE, Co]. It is a small skeletal supercategory with $\text{Ob } \mathcal{A} = \mathbb{N}$. Note that in [KT], contravariant composition of morphisms is used, contrary to [Se, BE, Co]. We thus actually have $\mathcal{A} = \mathcal{B}(0, -1)^{\text{op}}$. This will only manifest itself in equation (5.1).

1.3.1. Brauer diagrams. The vector space $\text{Hom}_{\mathcal{A}}(i, j)$ is zero unless $i + j$ is even. Furthermore, the graded vectorspace $\text{Hom}_{\mathcal{A}}(i, j)$ is purely even, resp. purely odd, if $(i - j)/2$ is even, resp. odd. The vector space $\text{Hom}_{\mathcal{A}}(i, j)$ is spanned by (i, j) -Brauer diagrams. These diagrams correspond to all partitions of a set of $i + j$ dots into pairs. They are graphically represented by i dots on a horizontal line and j dots on a second horizontal line, above the first one. The Brauer diagram then consists of $(i + j)/2$ lines, connecting the dots belonging to the same pair. An example of a $(6, 8)$ -Brauer diagram is given below.



The lines in Brauer diagrams which connect the lower and upper horizontal line will be referred to as *propagating lines*.

The composition $d_1 \circ d_2$ of an (i, j) -diagram d_1 and a (k, l) -diagram d_2 is zero unless $i = l$. When $i = l$ we identify the dots on the upper line of d_2 with those on the lower line of d_1 , creating another diagram. If this diagram contains loops, we have $d_1 \circ d_2 = 0$. If it does not contain loops we obtain a (k, j) -Brauer diagram. Then $d_1 \circ d_2$ is equal to that diagram *up to a possible minus sign*. The rules for computing this minus sign were obtained in [KT]. Note that *op. cit.* works with marked Brauer diagrams, whereas we follow the slightly different point of view that the homomorphisms are ordinary diagrams and their composition is to be determined by introducing the marking, see [Co]. The identity morphism of $i \in \text{Ob } \mathcal{A}$ is the diagram with i non-crossing propagating lines, which we will denote by e_i^* .

1.3.2. Strict monoidal supercategory. It is proved in [KT, Theorem 3.2.1], see also [BE, Example 1.5(iii)], that \mathcal{A} is a strict monoidal supercategory. The superfunctor

$$- \otimes - : \mathcal{A} \boxtimes \mathcal{A} \rightarrow \mathcal{A}$$

satisfies $i \otimes j = i + j$ for any $i, j \in \mathbb{N} = \text{Ob } \mathcal{A}$. In particular, $\mathbb{1} = 0 \in \text{Ob } \mathcal{A}$. Now we define the action of $- \otimes -$ on morphisms. For any Brauer diagram d , we have that $d \otimes e_i^*$, resp. $e_i^* \otimes d$, is the Brauer diagram obtained by adding i propagating lines to the right, resp. the left, of d . Now take an (i, j) -Brauer diagram d_1 and a (k, l) -Brauer diagram d_2 . Then we set

$$d_1 \otimes d_2 = (d_1 \otimes e_l^*) \circ (e_i^* \otimes d_2).$$

Thus $d_1 \otimes d_2$ is again a diagram, up to a possible minus sign. The monoidal supercategory \mathcal{A} is symmetric, with braiding morphisms $B_{i,j} : i \otimes j \rightarrow j \otimes i$ given in [KT, Section 3.1].

By [KT, Theorem 3.2.1], the monoidal supercategory \mathcal{A} is generated by four morphisms:

- (1) $I = e_1^*$, the identity morphism of $1 \in \text{Ob } \mathcal{A}$, represented by a straight line;
- (2) X , the crossing in $\text{End}_{\mathcal{A}}(2)$;

- (3) \cup , the unique diagram in $\text{Hom}_{\mathcal{A}}(0, 2)$; and
- (4) \cap , the unique diagram in $\text{Hom}_{\mathcal{A}}(2, 0)$.

1.3.3. *The periplectic Brauer algebra.* The algebras in [Mo] are obtained as the endomorphism algebras in \mathcal{A} . For $r \in \mathbb{N}$, we define the *periplectic Brauer algebra* as

$$A_r := \text{End}_{\mathcal{A}}(r).$$

Note that the A_r are ordinary algebras with trivial \mathbb{Z}_2 -grading, since all elements are even as noted in 1.3.1. The algebra A_r is for instance generated by the elements

$$s_i := I^{\otimes i-1} \otimes X \otimes I^{\otimes r-i-1} \quad \text{and} \quad \epsilon_i := I^{\otimes i-1} \otimes (\cup \circ \cap) \otimes I^{\otimes r-i-1}, \quad \text{for } 1 \leq i < r.$$

The subalgebra generated by $\{s_i\}$ is precisely the symmetric group algebra $\mathbb{K}S_r$. The other relations are given in [Mo, Section 2].

From the monoidal structure on \mathcal{A} we get an embedding of algebras

$$(1.3) \quad A_r \otimes A_s \hookrightarrow A_{r+s}.$$

The embedding of $A_r \otimes A_1 = A_r \otimes \mathbb{K}I$ in A_{r+1} will simply be denoted by $A_r \hookrightarrow A_{r+1}$. Note that $A_0 \hookrightarrow A_1$ is actually an isomorphism.

By [KT, Theorem 4.3.1] or [Co, Theorem 1], the isoclasses of simple modules over A_r , with $r \in \mathbb{N}$, are in one-to-one correspondence with the following subset of \mathbf{Par} :

$$(1.4) \quad \Lambda_r := \{\lambda \vdash j \mid j \in \mathcal{J}^0(r)\}.$$

We denote the projective cover in $A_r\text{-mod}$ of the simple module $L_r(\lambda)$, with $\lambda \in \Lambda_r$, by $P_r(\lambda)$. When $\lambda \in \mathbf{Par} \setminus \Lambda_r$, we set $L_r(\lambda) = P_r(\lambda) = 0$.

1.3.4. *Cell modules.* For $r \in \mathbb{N}$, we set $\mathbf{L}_r := \{\lambda \vdash j \mid j \in \mathcal{J}(r)\}$. For any $\mu \in \mathbf{L}_r$, the cell module $W_r(\mu)$ was introduced in [Co, Section 4]. When $\mu \in \mathbf{Par} \setminus \mathbf{L}_r$, we set $W_r(\mu) = 0$. We use these modules to introduce a matrix \mathbf{c} . For $\lambda, \mu \in \mathbf{Par}$, take an arbitrary $r \in \mathbb{N}$ with $\lambda \in \Lambda_r$ and set

$$\mathbf{c}_{\lambda\mu} := [W_r(\mu) : L_r(\lambda)].$$

The result in [CE, Theorem 1] shows in particular that the definition of \mathbf{c} does not depend on the specific choice of r . Furthermore, we have

$$\mathbf{c}_{\lambda\mu} := \begin{cases} 1 & \text{if } \mu \subseteq \lambda \text{ and } \lambda/\mu \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Here Γ is a set of skew Young diagrams introduced in [CE, Section 3]. In particular, we have $\mathbf{c}_{\lambda\lambda} = 1$. Since $\mathbf{c}_{\lambda\lambda} = 1$ and $\mathbf{c}_{\lambda\mu} = 0$ unless $\mu \subseteq \lambda$, it is possible to construct a matrix \mathbf{c}^{-1} , such that $\mathbf{c}_{\lambda\lambda}^{-1} = 1$, $\mathbf{c}_{\lambda\mu}^{-1} = 0$ unless $\mu \subseteq \lambda$, and

$$\sum_{\mu} \mathbf{c}_{\lambda\mu} \mathbf{c}_{\mu\nu}^{-1} = \delta_{\lambda\nu} \quad \text{and} \quad \sum_{\mu} \mathbf{c}_{\lambda\mu}^{-1} \mathbf{c}_{\mu\nu} = \delta_{\lambda\nu}, \quad \text{for all } \lambda, \nu \in \mathbf{Par}.$$

Note that both summations are actually finite, by the lower diagonal structures of the matrices.

1.3.5. *Primitive idempotents and projective modules.* Take an arbitrary partition λ . We fix for the remainder of the paper a primitive idempotent e_λ in A_j with $j := |\lambda|$, according to the labelling in equation (1.4). Hence we have $P_j(\lambda) \cong A_j e_\lambda$. Examples of the idempotents are e_\emptyset , which is the identity element in $\text{End}_{\mathcal{A}}(0)$, and $e_\square = \text{I}$.

In [Co, Section 3], the algebra

$$C_r := \bigoplus_{i,j \in \mathcal{J}(r)} \text{Hom}_{\mathcal{A}}(i, j),$$

was introduced. By construction, we have $A_j \cong e_j^* C_r e_j^*$ for any $j \in \mathcal{J}(r)$, which allows to interpret e_λ as an idempotent in C_r if $|\lambda| = j \in \mathcal{J}(r)$. By [Co, Lemma 4.6.2], we have

$$(1.5) \quad P_r(\lambda) \cong e_r^* C_r e_\lambda \cong \text{Hom}_{\mathcal{A}}(j, r) e_\lambda, \quad \text{for all } \lambda \vdash j \in \mathcal{J}^0(r).$$

1.3.6. *Restriction and induction.* The embedding $A_r \hookrightarrow A_{r+1}$ of 1.3.3 yields functors

$$\text{Res}_r : A_r\text{-mod} \rightarrow A_{r-1}\text{-mod} \quad \text{and} \quad \text{Ind}_r = A_{r+1} \otimes_{A_r} - : A_r\text{-mod} \rightarrow A_{r+1}\text{-mod}.$$

We introduce the symmetric matrix \mathbf{b} as

$$\mathbf{b}_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu \in \mathcal{R}(\lambda) \text{ or } \lambda \in \mathcal{R}(\mu). \\ 0 & \text{otherwise.} \end{cases}$$

By [Co, Corollary 5.2.4], $\text{Res}_r W_r(\mu)$ (for all $\mu \in \mathbf{Par}$ and $r \in \mathbb{N}$ such that $r - |\mu|$ is even and *strictly* positive) has a filtration with composition factors given by cell modules of A_{r-1} and multiplicities

$$(1.6) \quad (\text{Res}_r W_r(\mu) : W_{r-1}(\lambda)) = \mathbf{b}_{\lambda\mu}, \quad \text{for all } \lambda \in \mathbf{Par}.$$

Note that multiplicities in cell filtrations of arbitrary A_r -modules are actually independent of the chosen filtration, if $r \notin \{2, 4\}$, by [Co, Theorem 4.1.2(3)].

1.3.7. *Jucys-Murphy elements.* The Jucys-Murphy elements for A_r were introduced in [Co, Section 6]. The element $x_r \in A_r$ commutes with the subalgebra A_{r-1} , by [Co, Lemma 6.1.2]. We interpret x_r also as an element of A_s for any $r \geq s$, although $x_r \otimes e_{s-r}^*$ would be more precise. By definition, we have $x_1 = 0$.

We thus have an action of x_r on $\text{Res}_r M$, for an A_r -module M , which commutes with the A_{r-1} -action. In [CE, Section 2], we introduced the notation M_q for the generalised q -eigenspace for x_r . We have $\text{Res}_r M = \bigoplus_{q \in \mathbb{Z}} M_q$ as A_{r-1} -modules. For any $q \in \mathbb{Z}$ and $\lambda, \mu \in \mathbf{Par}$, we set

$$\mathbf{b}_{\lambda\mu}^q = \begin{cases} 1 & \text{if } \mu \text{ is obtained from } \lambda \text{ by adding a } q\text{-box or removing a } q-1\text{-box,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have $\mathbf{b} = \sum_{q \in \mathbb{Z}} \mathbf{b}^q$. By [CE, Proposition 2.12], we can refine (1.6) to

$$(1.7) \quad (W_r(\mu)_q : W_{r-1}(\lambda)) = \mathbf{b}_{\lambda\mu}^q.$$

1.4. **The periplectic Lie superalgebra.** For each $n \in \mathbb{Z}_{n \geq 1}$, the periplectic Lie superalgebra $\mathbf{pe}(n)$ is the subalgebra of the general linear superalgebra $\mathbf{gl}(n|n)$, which preserves an odd bilinear form $\beta : V \times V \rightarrow \mathbb{k}$, see [BDE+, Ch, Co, KT, Mo, Mu], with $V := \mathbb{k}^{n|n}$. Concretely,

$$\mathbf{pe}(n) = \{X \in \mathbf{gl}(n|n) \mid \beta(Xv, w) + (-1)^{|X||v|} \beta(v, Xw) = 0, \quad \text{for all } v, w \in V\}.$$

1.4.1. *The supercategory of integrable modules over $\mathfrak{pe}(n)$.* We consider the category $\mathfrak{pe}(n)$ -smod which has as objects all \mathbb{Z}_2 -graded, finite dimensional, integrable, left $\mathfrak{pe}(n)$ -modules, see [BDE+, Section 2]. The morphism spaces, denoted by $\text{Hom}_{\mathfrak{pe}(n)}(M, N)$ consist of all $\mathfrak{pe}(n)$ -linear morphisms of (ungraded) \mathbb{k} -vector spaces. The morphism spaces are thus naturally \mathbb{Z}_2 -graded vector spaces. The category $\mathfrak{pe}(n)$ -smod is a supercategory. This category is shown to be abelian in [Ch, Section 2.3]. By ‘ $\mathfrak{pe}(n)$ -module’ we will henceforth mean ‘object in $\mathfrak{pe}(n)$ -smod’.

1.4.2. *Monoidal structure.* For $\mathfrak{pe}(n)$ -modules M, N , the tensor product $M \otimes N = M \otimes_{\mathbb{k}} N$ is an object in $\mathfrak{pe}(n)$ -smod, with action given by

$$X(v \otimes w) = Xv \otimes w + (-1)^{|X||v|} v \otimes Xw, \quad \text{for all } X \in \mathfrak{pe}(n).$$

For $f \in \text{Hom}_{\mathfrak{pe}(n)}(M_1, M_2)$ and $g \in \text{Hom}_{\mathfrak{pe}(n)}(N_1, N_2)$, the morphism $f \otimes g$ defined as

$$(f \otimes g)(v \otimes w) = (-1)^{|f||w|} f(v) \otimes g(w)$$

is $\mathfrak{pe}(n)$ -linear. With this rule, we have

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|f||k|} (f \circ h \otimes g \circ k).$$

Comparison with (1.2) shows that $\mathfrak{pe}(n)$ -smod is not a monoidal supercategory for $-\otimes-$, but $(\mathfrak{pe}(n)\text{-smod})^{\text{op}}$ is, with unit object $\mathbb{1} = \mathbb{k}$ the trivial module. Here we take the opposite category in the ordinary, non-super, interpretation.

2. THE PERIPECTIC DELIGNE CATEGORY

2.1. **Construction.** The category \mathcal{PD} , which we will define as the pseudo-abelian envelope of \mathcal{A} , was denoted by $\underline{\text{Rep}} P$ in [Se, Section 4.5] and by $\widehat{\mathcal{B}}(0, -1)$ in [KT, Section 5]. It is the periplectic analogue of the categories $\underline{\text{Rep}} GL_\delta$ and $\underline{\text{Rep}} O_\delta$ introduced by Deligne in [De].

2.1.1. The periplectic Brauer category \mathcal{A} is \mathbb{k} -linear, so in particular pre-additive. It thus admits a unique (up to equivalence) additive envelope. We define such a supercategory $\overline{\mathcal{A}}$ which has as objects finite (non-ordered) formal sums of objects in \mathcal{A} , denoted by $\bigoplus_{\alpha \in S} r_\alpha$ for certain $r_\alpha \in \mathbb{N} = \text{Ob } \mathcal{A}$, and morphisms interpreted as matrices of coefficients. By construction, $\overline{\mathcal{A}}$ is still skeletal. The category $\overline{\mathcal{A}}$ inherits a structure of a symmetric strict monoidal supercategory from its subcategory \mathcal{A} , with $-\otimes-$ extended as a bi-additive functor.

2.1.2. The additive category $\overline{\mathcal{A}}$ admits a unique (up to equivalence) Karoubi envelope. We define such a supercategory \mathcal{PD} as the idempotent completion of $\overline{\mathcal{A}}$. Concretely, the objects in \mathcal{PD} are pairs (X, e) with $X \in \text{Ob } \overline{\mathcal{A}}$ and e an idempotent in $\text{End}_{\overline{\mathcal{A}}}(X)$. Any object is thus of the form

$$(2.1) \quad \left(\bigoplus_{\alpha \in S} r_\alpha, \sum_{\alpha \in S} e^{(\alpha)} \right),$$

with each $e^{(\alpha)}$ an idempotent in A_{r_α} . Morphism superspaces in \mathcal{PD} are given by

$$(2.2) \quad \text{Hom}_{\mathcal{PD}}((X, e), (Y, f)) = \{ \alpha \in \text{Hom}_{\overline{\mathcal{A}}}(X, Y) \mid \alpha = f \circ \alpha \circ e \} = f \text{Hom}_{\overline{\mathcal{A}}}(X, Y) e.$$

Now \mathcal{PD} is no longer skeletal. However, since \mathcal{PD} is karoubian, additive and \mathbb{k} -linear with finite dimensional endomorphism algebras, it is Krull-Schmidt. It also inherits naturally from its subcategory $\overline{\mathcal{A}}$ the structure of a symmetric monoidal supercategory, with $- \otimes -$ a bi-additive functor. For $i, j \in \mathbb{N} = \text{Ob } \mathcal{A} \subset \text{Ob } \overline{\mathcal{A}}$ and idempotents $e \in A_i$ and $f \in A_j$, we have for instance

$$(2.3) \quad (j, f) \otimes (i, e) = (j + i, f \otimes e),$$

with $f \otimes e$ interpreted as an element in A_{j+i} as in (1.3). We will follow the convention to denote coproducts \amalg in \mathcal{PD} (now only defined up to isomorphism) as direct sums \bigoplus .

Remark 2.1.3. Since, $\text{char}(\mathbb{k}) = 0$, the category $\bigoplus_{i \in \mathbb{N}} \mathbb{k}\mathbb{S}_i\text{-mod}$ is a pseudo-abelian envelope of the \mathbb{k} -linear category \mathcal{C} with objects \mathbb{N} , $\text{Hom}_{\mathcal{C}}(i, j) = 0$ if $i \neq j$ and $\text{End}_{\mathcal{C}}(i) = \mathbb{k}\mathbb{S}_i$. It is in this spirit that our categorical representation is an analogue of the one for \mathfrak{sl}_{∞} in [LLT, HY].

2.2. Classification of indecomposable objects. For any $\lambda \in \text{Par}$, we set

$$R(\lambda) := (|\lambda|, e_{\lambda}) \in \mathcal{PD},$$

with e_{λ} the primitive idempotent in $A_{|\lambda|}$ of 1.3.5. In particular, $R(\emptyset) = \mathbb{1}$ and $R(\square) = (1, \mathbb{I})$.

Theorem 2.2.1. *The assignment $\lambda \mapsto R(\lambda)$ gives a bijection between Par and the set of isomorphism classes of non-zero indecomposable objects in \mathcal{PD} .*

Proof. Let X be an arbitrary non-zero indecomposable object in \mathcal{PD} . Clearly X is isomorphic to (r, e) for some $r \in \mathbb{N} = \text{Ob } \mathcal{A} \subset \text{Ob } \overline{\mathcal{A}}$ and e a primitive idempotent in $A_r = \text{End}_{\overline{\mathcal{A}}}(r)$. If $r = 0$, then $X = R(\emptyset)$, so we can assume $r > 0$. Let μ be the partition of $r - 2i \in \mathcal{J}^0(r)$ such that $A_r e \cong P_r(\mu)$. We will show in two steps that $R(\mu) \cong X$.

By [Co, 4.2.1], there exist $a \in \text{Hom}_{\mathcal{A}}(r - 2i, r)$ and $b \in \text{Hom}_{\mathcal{A}}(r, r - 2i)$ such that $ba = e_{r-2i}^*$. Consequently, $\bar{e}_{\mu} := ae_{\mu}b$ is an idempotent in A_r . We define, using (2.2),

$$\begin{aligned} x &:= e_{\mu}b = e_{\mu}b\bar{e}_{\mu} \in e_{\mu}\text{Hom}_{\mathcal{A}}(r, r - 2i)\bar{e}_{\mu} = \text{Hom}_{\mathcal{PD}}((r, \bar{e}_{\mu}), R(\mu)) \quad \text{and} \\ y &:= ae_{\mu} = \bar{e}_{\mu}ae_{\mu} \in \bar{e}_{\mu}\text{Hom}_{\mathcal{A}}(r - 2i, r)e_{\mu} = \text{Hom}_{\mathcal{PD}}(R(\mu), (r, \bar{e}_{\mu})). \end{aligned}$$

Since $xy = e_{\mu}$ and $yx = \bar{e}_{\mu}$, the identity morphisms of $R(\mu)$ and (r, \bar{e}_{μ}) , we have $R(\mu) \cong (r, \bar{e}_{\mu})$.

By equation (1.5) and the properties of a and b , we have isomorphisms of left A_r -modules:

$$A_r e \cong e_r^* C_r e_{\mu} \cong A_r \bar{e}_{\mu}$$

This means that there exist $\alpha \in e A_r \bar{e}_{\mu}$ and $\beta \in \bar{e}_{\mu} A_r e$, corresponding to the mutual inverses in

$$e\text{Hom}_{\mathcal{A}}(r, r)\bar{e}_{\mu} = \text{Hom}_{\mathcal{PD}}((r, \bar{e}_{\mu}), (r, e)) \quad \text{and} \quad \bar{e}_{\mu}\text{Hom}_{\mathcal{A}}(r, r)e = \text{Hom}_{\mathcal{PD}}((r, e), (r, \bar{e}_{\mu})).$$

Hence $(r, e) \cong (r, \bar{e}_{\mu}) \cong R(\mu)$, so we find that any indecomposable object in \mathcal{PD} is isomorphic to some $R(\lambda)$.

Now assume that for $\lambda \neq \mu$ we have $R(\mu) \cong R(\lambda)$. The corresponding isomorphism which must exist in $e_{\mu}\text{Hom}_{\mathcal{A}}(t, s)e_{\lambda}$ with $t = |\lambda|$ and $s = |\mu|$ implies that $t - s$ is even and that $C_r e_{\lambda} \cong C_r e_{\mu}$ in $C_r\text{-mod}$, for r such that $s, t \in \mathcal{J}(r)$. This is contradicted by [Co, Section 3]. \square

Remark 2.2.2. The proof of Theorem 2.2.1 implies that for an arbitrary primitive idempotent $e \in A_r$, we have $R(\lambda) \cong (r, e)$ if and only if $A_r e \cong P_r(\lambda)$.

2.3. The split Grothendieck group. We let $[\mathcal{PD}]_{\oplus}$ denote the split Grothendieck group of the small additive category \mathcal{PD} , see [Ma, Section 1.2]. Concretely, $[\mathcal{PD}]_{\oplus}$ is the free abelian group with elements the isomorphism classes $[X]$ of objects X in \mathcal{PD} , modulo the relations $[X] = [Y] + [Z]$, whenever $X = Y \oplus Z$. As an immediate consequence of Theorem 2.2.1, we thus find the following.

Corollary 2.3.1. *The map $\Psi : \text{Par}_{\mathbb{Z}} \rightarrow [\mathcal{PD}]_{\oplus}$ determined by $\lambda \mapsto [R(\lambda)]$ is a \mathbb{Z} -module isomorphism.*

In the terminology of [Ma, Section 1.3], (\mathcal{PD}, Ψ) is a \mathbb{Z} -categorification of $\text{Par}_{\mathbb{Z}}$.

3. TENSOR PRODUCT WITH THE GENERATOR

In this section, we study the functor \mathbf{T} , the additive endo-superfunctor of \mathcal{PD} given by

$$\mathbf{T}(-) := - \otimes R(\square).$$

For idempotents $e \in A_r$, $f \in A_s$ and $a \in f\text{Hom}_{\mathcal{A}}(r, s)e = \text{Hom}_{\mathcal{PD}}((r, e), (s, f))$, we thus have

$$(3.1) \quad \mathbf{T}(r, e) = (r + 1, e \otimes \mathbf{I}) \quad \text{and} \quad \mathbf{T}(a) = a \otimes \mathbf{I},$$

by definition and equation (2.3).

3.1. The combinatorics of \mathbf{T} . We use the Krull-Schmidt category \mathcal{PD} to define a matrix \mathbf{a} .

Definition 3.1.1. For all $\nu, \mu \in \text{Par}$, we define $\mathbf{a}_{\nu, \mu} \in \mathbb{N}$, by

$$\mathbf{T}(R(\nu)) = R(\nu) \otimes R(\square) \cong \bigoplus_{\kappa} R(\kappa)^{\oplus \mathbf{a}_{\nu, \kappa}}.$$

Recall the matrices \mathbf{b} and \mathbf{c} introduced in Section 1.3.

Theorem 3.1.2. *We have $\mathbf{a} = \mathbf{c} \mathbf{b} \mathbf{c}^{-1}$. Concretely, for all $\nu, \mu \in \text{Par}$, we have*

$$(3.2) \quad \mathbf{a}_{\nu, \kappa} = \sum_{\lambda \subseteq \nu} \sum_{\mu \supseteq \kappa} \mathbf{c}_{\nu, \lambda} \mathbf{b}_{\lambda, \mu} \mathbf{c}_{\mu, \kappa}^{-1}.$$

Proof. Take $r = |\nu|$. Equation (3.1) and Remark 2.2.2 imply that $\mathbf{a}_{\nu, \kappa}$ is the number of times the projective A_{r+1} -module $P_{r+1}(\kappa)$ appears as a direct summand of

$$A_{r+1}(e_{\nu} \otimes I) \cong \text{Ind}_r P_r(\nu).$$

Consequently

$$(3.3) \quad \mathbf{a}_{\nu, \kappa} = \dim \text{Hom}_{A_{r+1}}(\text{Ind}_r P_r(\nu), L_{r+1}(\kappa)) = [\text{Res}_{r+1} L_{r+1}(\kappa) : L_r(\nu)].$$

In particular, we thus find

$$\sum_{\kappa} \mathbf{a}_{\nu, \kappa} \mathbf{c}_{\kappa, \lambda} = \sum_{\kappa} [W_{r+1}(\lambda) : L_{r+1}(\kappa)] [\text{Res}_{r+1} L_{r+1}(\kappa) : L_r(\nu)] = [\text{Res}_{r+1} W_{r+1}(\lambda) : L_r(\nu)].$$

On the other hand, by equation (1.6), we have

$$\sum_{\mu} c_{\nu\mu} b_{\mu\lambda} = \sum_{\mu} [\text{Res}_{r+1} W_{r+1}(\lambda) : W_r(\mu)] [W_r(\mu) : L_r(\nu)] = [\text{Res}_{r+1} W_{r+1}(\lambda) : L_r(\nu)].$$

This shows that $\mathbf{a} = \mathbf{c} \mathbf{b} \mathbf{c}^{-1}$, where we can restrict the summation by 1.3.4. \square

Remark 3.1.3. Equation (3.3) shows the explicit connection between \mathbf{T} on \mathcal{PD} and Res between the Brauer algebras. This explains the similarities between translation functors for the periplectic Lie superalgebra [BDE+, Corollary 4.4.6] and the restriction functors [CE, Proposition 2.3.1].

3.2. The natural transformation ξ . For an object $X = (r, e)$ in \mathcal{PD} , we define

$$\xi_X \in \text{End}_{\mathcal{PD}}(X \otimes R(\square)) = (e \otimes \mathbf{I}) A_{r+1} (e \otimes \mathbf{I}), \quad \text{as}$$

$$\xi_X = (e \otimes \mathbf{I}) x_{r+1} (e \otimes \mathbf{I}) = (e \otimes \mathbf{I}) x_{r+1} = x_{r+1} (e \otimes \mathbf{I}),$$

with $x_{r+1} \in A_{r+1}$ the Jucys-Murphy element. The different identities for ξ_X are equal since x_{r+1} commutes with elements of A_r . We can easily extend this to arbitrary objects X in \mathcal{PD} , see (2.1).

Proposition 3.2.1. *The family of morphisms $\{\xi_X \mid X \in \text{Ob } \mathcal{PD}\}$ yields an even natural transformation of the superfunctor \mathbf{T} on \mathcal{PD} .*

Proof. Consider objects $X = (r, e)$, $Y = (s, f)$ and a morphism

$$a \in \text{Hom}_{\mathcal{PD}}(X, Y) = f \text{Hom}_{\mathcal{A}}(r, s) e.$$

We claim that $\mathbf{T}(a) \circ \xi_X = \xi_Y \circ \mathbf{T}(a)$. Indeed, by (3.1), the left-hand, resp. right-hand side becomes

$$(a \otimes \mathbf{I})(e \otimes \mathbf{I}) x_{r+1} = (a \otimes \mathbf{I}) x_{r+1}, \quad \text{resp.} \quad x_{s+1} (f \otimes \mathbf{I})(a \otimes \mathbf{I}) = x_{s+1} (a \otimes \mathbf{I}).$$

The claim then follows from the subsequent Lemma 3.2.2. \square

Lemma 3.2.2. *For arbitrary $a \in \text{Hom}_{\mathcal{A}}(s, r)$, we have $(a \otimes \mathbf{I}) x_{r+1} = x_{s+1} (a \otimes \mathbf{I})$.*

Proof. The case $r = s$ is precisely the fact that x_{r+1} commutes with A_r , see 1.3.7. This means that it suffices to prove that, for $r \geq 2$,

$$(\cup \otimes \mathbf{I}^{\otimes r-1}) x_{r-1} = x_{r+1} (\cup \otimes \mathbf{I}^{\otimes r-1}) \quad \text{and} \quad (\cap \otimes \mathbf{I}^{\otimes r-1}) x_{r+1} = x_{r-1} (\cap \otimes \mathbf{I}^{\otimes r-1}).$$

These easy calculations are left to the reader. \square

3.3. The functors \mathbf{T}_q . We introduce some elements of A_r . On any A_r -module, $x_r \in A_r$ only attains integer eigenvalues, see [Co, Section 6.2]. If $r > 0$, we can thus construct mutually orthogonal idempotents $\gamma_i^{(r)} \in A_r$, for $i \in \mathbb{Z}$, which are in the subalgebra generated by x_r , such that

$$(3.4) \quad 1_{A_r} = e_r^* = \sum_{i \in \mathbb{Z}} \gamma_i^{(r)}, \quad \text{and} \quad (x_r - i)^p \gamma_i^{(r)} = 0, \text{ for some } p \in \mathbb{N}.$$

Since we keep track of r in the notation, we can with slight abuse of notation also write $\gamma_j^{(r)}$ for $\gamma_j^{(r)} \otimes e_{s-r}^* \in A_s$. By construction, $\gamma_i^{(r)}$ commutes with any element of $A_{r-1} \subset A_r$. We also set $\gamma_i^{(0)} = \delta_{i0} \in \mathbb{k} = A_1$.

3.3.1. *Example.* We have $x_2^2 = 1$ and consequently $\gamma_1^{(2)} = \frac{1}{2}(1 + x_2)$, $\gamma_{-1}^{(2)} = \frac{1}{2}(1 - x_2)$ and $\gamma_i^{(2)} = 0$ if $i \notin \{1, -1\}$.

For an idempotent $e \in A_r$, we set

$$e[j] = \gamma_j^{(r+1)}(e \otimes \mathbf{I}) = (e \otimes \mathbf{I})\gamma_j^{(r+1)}.$$

Definition 3.3.2. For any $j \in \mathbb{Z}$, the additive functor \mathbf{T}_j on \mathcal{PD} is defined as

$$\mathbf{T}_j(r, e) = (r + 1, e[j]), \quad \text{for all } r \in \mathbb{N} \text{ and } e \text{ an idempotent in } A_r, \text{ and}$$

$$\mathbf{T}_j(a) = f[j](a \otimes \mathbf{I})e[j] = \gamma_j^{(s+1)}(a \otimes \mathbf{I})\gamma_j^{(r+1)}, \quad \text{for all } a \in \text{Hom}_{\mathcal{PD}}((r, e), (s, f)).$$

By construction, we have $\mathbf{T} = \bigoplus_{j \in \mathbb{Z}} \mathbf{T}_j$. Following Definition 3.1.1, for each $q \in \mathbb{Z}$, we define a matrix \mathbf{a}^q by

$$\mathbf{T}_q(R(\nu)) = \bigoplus_{\kappa} R(\kappa)^{\oplus \mathbf{a}_{\nu\kappa}^q}.$$

Proposition 3.3.3. *For each $q \in \mathbb{Z}$, we have $\mathbf{a}^q = \mathbf{c} \mathbf{b}^q \mathbf{c}^{-1}$.*

Proof. This is an analogue of the proof of Theorem 3.1.2. Consider $\nu \in \mathbf{Par}$ with $r = |\nu|$. We have

$$a_{\nu\kappa} = \dim_{\mathbb{K}}(e_{\nu} \otimes \mathbf{I})L_{r+1}(\kappa) = \dim_{\mathbb{K}} e_{\nu} \text{Res}_{r+1} L_{r+1}(\kappa).$$

Correspondingly, we find

$$a_{\nu\kappa}^q = \dim_{\mathbb{K}} e_{\nu}[q]L_{r+1}(\kappa).$$

Since $e_{\nu}[q](x_{r+1} - q)^k = 0$ for some $k \in \mathbb{N}$, we find

$$(3.5) \quad a_{\nu\kappa}^q = \dim_{\mathbb{K}} e_{\nu} L_{r+1}(\kappa)_q = \dim_{\mathbb{K}} \text{Hom}_{A_r}(A_r e_{\nu}, L_{r+1}(\kappa)_q) = [L_{r+1}(\kappa)_q : L_r(\nu)].$$

This and equation (1.7) imply that

$$(\mathbf{a}^q \mathbf{c})_{\nu\lambda} = [W_{r+1}(\lambda)_q : L_r(\nu)] = (\mathbf{c} \mathbf{b}^q)_{\nu\lambda},$$

which concludes the proof. \square

Lemma 3.3.4. *Let ν be a partition with κ obtained from ν by removing a q -box.*

- (i) *If κ has a removable $q - 1$ -box, then $\mathbf{a}_{\nu\kappa}^{q-1} = 1$.*
- (ii) *If κ has a removable $q + 1$ -box, then $\mathbf{a}_{\nu\kappa}^{q+1} = 1$.*

Proof. Part (i) is [CE, Lemma 2.2.3], by equation (3.5).

For part (ii), we claim that ν does not admit an addable $q + 1$ -box. Indeed, in order for κ to have a removable $q + 1$ -box, there must be a $q + 1$ -box above the q -box in ν , such that there is no $q + 2$ -box to the right of the $q + 1$ -box. Part (ii) then follows from [CE, Lemma 2.2.1], by equation (3.5). \square

An alternative way to prove Lemma 3.3.4 is to use the results in Section 4.

Lemma 3.3.5. *If λ admits an addable q -box, then $\mathbf{a}_{\lambda\lambda^{(q)}}^q = 1$.*

Proof. By Proposition 3.3.3, we have

$$\mathbf{a}_{\lambda\lambda^{(q)}}^q = \sum_{\mu \subseteq \lambda} \sum_{\nu \supseteq \lambda^{(q)}} c_{\lambda\mu} \mathbf{b}_{\mu\nu}^q c_{\nu\lambda^{(q)}}^{-1}.$$

The summation thus goes over $\mu, \nu \in \mathbf{Par}$ with $\mu \subseteq \lambda \subsetneq \lambda^{(q)} \subseteq \nu$. On the other hand $\mathbf{b}_{\mu\nu}^q = 0$ unless μ and ν differ by precisely one box. Hence we have

$$\mathbf{a}_{\lambda\lambda^{(q)}}^q = \mathbf{c}_{\lambda\lambda} \mathbf{b}_{\lambda\lambda^{(q)}}^q \mathbf{c}_{\lambda^{(q)}\lambda^{(q)}}^{-1} = 1,$$

which concludes the proof. \square

Corollary 3.3.6. *If λ admits an addable q -box, then $(\mathbf{c} \mathbf{b}^q)_{\lambda\nu} \geq \mathbf{c}_{\lambda^{(q)}\nu}$ for all $\nu \in \mathbf{Par}$.*

Proof. By Lemma 3.3.5, we have $\mathbf{a}_{\lambda\eta}^q \geq \delta_{\eta\lambda^{(q)}}$, for all $\eta \in \mathbf{Par}$, where positivity of the entries of \mathbf{a}^q follows by Definition 3.3.2. Since the entries of \mathbf{c} are also positive, we thus find

$$(\mathbf{c} \mathbf{b}^q)_{\lambda\nu} = (\mathbf{a}^q \mathbf{c})_{\lambda\nu} \geq \mathbf{c}_{\lambda^{(q)}\nu},$$

where the first equation is Proposition 3.3.3. \square

4. THE FOCK SPACE REPRESENTATION OF THE INFINITE TEMPERLEY-LIEB ALGEBRA

Consider the \mathbb{Z} -algebra with generators $\{T_i \mid i \in \mathbb{Z}\}$ and relations (with $|i - j| > 1$)

$$T_i^2 = 0, \quad T_i T_j = T_j T_i \quad \text{and} \quad T_i T_{i\pm 1} T_i = T_i.$$

This is the infinite Temperley-Lieb algebra over \mathbb{Z} for parameter zero, $\mathrm{TL}_\infty(0)$. In this section, we will consider two representations of $\mathrm{TL}_\infty(0)$ on $\mathbf{Par}_\mathbb{Z}$, related by an automorphism of $\mathbf{Par}_\mathbb{Z}$. Due to its close connection with the Fock space representation of \mathfrak{sl}_∞ , we will refer to one as the Fock space representation of $\mathrm{TL}_\infty(0)$. The twisted version is the one that will describe the combinatorics of the periplectic Deligne category and will be referred to as Ξ .

4.1. The representation Ξ . By Propositions 4.2.3, 4.3.2 and 4.5.1, we have the following theorem.

Theorem 4.1.1. *There exists a unique representation*

$$\Xi : \mathrm{TL}_\infty(0) \rightarrow \mathrm{End}_\mathbb{Z}(\mathbf{Par}_\mathbb{Z})$$

which satisfies for all $q \in \mathbb{Z}$:

- $\Xi(T_q)(\emptyset) = \delta_{q0} \square;$
- $\Xi(T_q)(\lambda) = \lambda^{(q)}$ for any $\lambda \in \mathbf{Par}$ which admits an addable q -box.

Moreover, the representation Ξ is faithful.

The following theorem is an immediate consequence of the realisation of Ξ in Proposition 4.3.2.

Theorem 4.1.2. *The \mathbb{Z} -module isomorphism $\Psi : \mathbf{Par}_\mathbb{Z} \rightarrow [\mathcal{PD}]_\oplus$ in Corollary 2.3.1 satisfies $[\mathbf{T}_j] \circ \Psi = \Psi \circ \Xi(T_j)$. Hence, the functors \mathbf{T}_j satisfy the properties (with $|i - j| > 1$)*

$$[\mathbf{T}_i]^2 = 0, \quad [\mathbf{T}_i][\mathbf{T}_j] = [\mathbf{T}_j][\mathbf{T}_i] \quad \text{and} \quad [\mathbf{T}_i][\mathbf{T}_{i\pm 1}][\mathbf{T}_i] = [\mathbf{T}_i].$$

This means that $(\mathcal{PD}, \Psi, \{\mathbf{T}_i \mid i \in \mathbb{Z}\})$ is a naïve \mathbb{Z} -categorification of the $\mathrm{TL}_\infty(0)$ -representation Ξ , in the terminology of [Ma, Section 2.2]. We will improve this statement in Section 6.

In the following, we will usually write $T_q(\lambda)$ instead of $\Xi(T_q)(\lambda)$.

4.2. Uniqueness of the representation. The combinatorial arguments in this subsection are inspired by the results and proofs in [BDE+, Section 7.2].

4.2.1. For arbitrary $\lambda \in \text{Par}$ and $q \in \mathbb{Z}$, there are 5 mutually exclusive possibilities:

- (a) λ admits an addable q -box;
- (b) λ has a removable q -box;
- (c) λ has no boxes with content in $\{q-1, q, q+1\}$ (and $\lambda \neq \emptyset$ when $q = 0$);
- (d) there is a box right of the (existing) rim q -box of λ , but not below;
- (e) there is a box below the (existing) rim q -box of λ , but not to its right.

We draw the $q-1$, q and $q+1$ boxes on the rim of λ in the ‘generic’ cases (meaning assuming that all three contents appear in λ) corresponding to (a), (b), (d) and (e):

$$(a) : \begin{array}{|c|c|} \hline q & \\ \hline \square & \\ \hline \end{array}, \quad (b) : \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline q \\ \hline \end{array}, \quad (d) : \begin{array}{|c|c|c|} \hline & q & \\ \hline \end{array}, \quad (e) : \begin{array}{|c|} \hline \square \\ \hline q \\ \hline \square \\ \hline \end{array}.$$

If it is clear from context which q is referred to, we will simply say that λ is of type (a), (b), etc.

4.2.2. We also introduce some terminology for (rim) hooks. A hook is called *balanced* if its height (the number of rows it has boxes in) is the same as its width (the number of columns it has boxes in). A rim hook of λ such that the minimal, resp. maximal, content of its boxes is q is called a *rim hook starting at q* , resp. a *rim hook ending at q* .

In case (d) there will always be a rim hook starting at q and one starting at $q+1$, in case (e) there will always be a rim hooks ending at q and $q-1$.

Proposition 4.2.3. *Assume that a representation of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$ satisfies, for any $q \in \mathbb{Z}$ and $\lambda \in \text{Par}$:*

- (I) $T_q(\emptyset) = \delta_{q0} \square$;
- (II) $T_q(\lambda) = \lambda^{(q)}$ if λ is of type (a).

Then we have the following:

- (III) If λ is of type (b) or (c), $T_q(\lambda) = 0$;
- (IV) If λ is of type (d), $T_q(\lambda)$ is the partition obtained by removing the minimal balanced rim hook starting at $q+1$, if that exists, otherwise $T_q(\lambda) = 0$;
- (V) If λ is of type (e), $T_q(\lambda)$ is the partition obtained by removing the minimal balanced rim hook ending at $q-1$, if that exists, otherwise $T_q(\lambda) = 0$.

In particular, there is at most one representation of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$ satisfying (I) and (II).

We prove this in four lemmata and denote by Ω an arbitrary representation of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$.

Lemma 4.2.4. *Assume that Ω satisfies (I) and (II), then it satisfies (III).*

Proof. Assume first that λ has a removable q -box (type (b)). Then (II) implies that $\lambda = T_q(\mu)$ for some $\mu \in \mathcal{R}(\lambda)$. Hence we find $T_q(\lambda) = T_q^2(\mu) = 0$.

Now assume that λ is of type (c). Then $\lambda = T_{p_1} T_{p_2} \cdots T_{p_k}(\emptyset)$, with $k = |\lambda|$ and each $p_i \notin \{q-1, q, q+1\}$ by (II). The Temperley-Lieb relations thus imply that

$$T_q(\lambda) = T_{p_1} T_{p_2} \cdots T_{p_k} T_q(\emptyset).$$

As we can clearly assume that $q \neq 0$, this must be zero by (I), which concludes the proof. \square

If λ is of type (d) for q , we let $t \in \mathbb{N}$ denote the maximal number such that there is a box in λ with content $q + t + 1$ on the row of the rim q -box. We then specify that λ is of type $(d, [r, t])$, with $r = |\lambda|$.

Lemma 4.2.5. *Assume that Ω satisfies (II), then it satisfies condition (IV) for all λ of types $(d, [r, 0])$ and $(d, [0, t])$.*

Proof. If λ is of type $(d, [r, 0])$, then λ has a removable $q + 1$ -box, so by (II) we have $\lambda = T_{q+1}(\mu)$, with μ obtained by removing the rim $q + 1$ -box. Furthermore, μ has a removable q -box, so

$$\lambda = T_{q+1}T_q(\nu)$$

with ν obtained from λ by removing the q and $q + 1$ boxes on its rim. We thus find

$$T_q(\lambda) = T_qT_{q+1}T_q(\nu) = T_q(\nu) = \mu.$$

In conclusion, $T_q(\lambda)$ is the partition obtained from λ by removing the rim $q + 1$ -box. Clearly, the rim $q + 1$ -box is the minimal balanced rim hook starting at $q + 1$.

The case $(d, [0, t])$ is empty, since $\lambda = \emptyset$ is never of type (d). \square

Lemma 4.2.6. *Assume that Ω satisfies (II), (III) in general and (IV) for all partitions of type $(d, [r', -])$ with $r' < r$, then it satisfies condition (IV) for λ of type $(d, [r, 1])$.*

Proof. By assumption on λ and (II) we have $\lambda = T_{q+2}(\mu)$, with μ obtained from λ by removing the rim $q + 2$ -box. Hence we have

$$T_q(\lambda) = T_qT_{q+2}(\mu) = T_{q+2}T_q(\mu).$$

Furthermore μ is of type $(d, [r - 1, 0])$, so (IV) holds true which means $T_q(\mu) = \nu$, with ν obtained by removing the rim $q + 1$ -box in μ . Hence, we have

$$T_q(\lambda) = T_{q+2}(\nu).$$

We review the two possibilities for ν .

If the rim q -box of λ was on the highest row, then ν contains no box with content in $\{q + 1, q + 2, q + 3\}$, so $T_q(\lambda) = T_{q+2}(\nu) = 0$ by (III). In this case, λ has no balanced rim hook starting at $q + 1$, so (IV) is indeed satisfied for λ .

If there is a row above the rim q -box, then ν is clearly again of type (d), now for $q + 2$. Furthermore, $|\nu| < |\lambda| = r$ so ν satisfies (IV). Moreover, we have a clear one-to-one correspondence between the rim hooks of ν starting at $q + 3$ and the rim hooks of λ starting at $q + 1$, by adding the rim $q + 1$ and $q + 2$ -boxes in λ to the former hook. This correspondence preserves the notion of balancedness. Hence we find that λ satisfies (IV). \square

Lemma 4.2.7. *Assume that Ω satisfies (II) in general and (IV) for all partitions of type $(d, [r', -])$ with $r' < r$, then it satisfies condition (IV) for λ of type $(d, [r, t])$ with $t > 1$.*

Proof. We have $\lambda = T_{q+t+1}(\mu)$, with μ obtained by removing the rim $q + t + 1$ -box from λ . We thus have

$$T_q(\lambda) = T_qT_{q+t+1}(\mu) = T_{q+t+1}T_q(\mu),$$

where now μ is of type $(d, [r-1, t-1])$ for q , and thus satisfies (IV). Hence, by assumption, $T_q(\mu)$ is obtained from μ by removing the minimal balanced rim hook starting at $q+1$, if it exists and zero otherwise. There is an obvious one-to-one correspondence between the rim hooks starting at $q+1$ for λ and μ , corresponding to ‘moving’ the $q+t+1$ -box of the hook. This correspondence thus preserves the notion of balancedness.

If λ does not have a balanced rim hook starting at q , we thus find $T_q(\lambda) = T_{q+t+1}T_q(\mu) = 0$ since μ satisfies (IV). If λ does have a balanced rim hook starting at q , then $T_q(\mu)$ is obtained from μ by removing its minimal rim balanced rim hook starting at q . By construction $T_q(\mu)$ then allows an addable $q+t+1$ -box and $T_q(\lambda) = T_{q+t+1}T_q(\mu)$ is obtained by adding this box by (II). Hence also in this case, λ satisfies indeed (IV). \square

Proof of Proposition 4.2.3. By Lemma 4.2.4, (III) is satisfied. By Lemma 4.2.5, (IV) is satisfied for all partitions of types $(d, [r, 0])$ and $(d, [0, t])$. Lemmata 4.2.6 and 4.2.7 then allow to prove (IV) in general by induction on r .

The proof of (V) is completely symmetrical to that of (IV). \square

4.3. Existence of the representation. First we construct the Fock space representation.

Lemma 4.3.1. *We have a representation Ξ' of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$, determined by*

$$T_q(\lambda) = \sum_{\mu} b_{\lambda\mu}^q \mu, \quad \text{for all } \lambda \in \text{Par}.$$

Proof. This is an easy combinatorial exercise, see also the proof of [CE, Proposition 2.3.1]. \square

We can identify the matrix c with an automorphism of $\text{Par}_\mathbb{Z}$, defined by

$$(4.1) \quad \lambda \mapsto \sum_{\mu} c_{\lambda\mu} \mu.$$

Note that the above summation is finite, by 1.3.4. We twist the representation in Lemma 4.3.1 by this automorphism and use Proposition 3.3.3.

Proposition 4.3.2. *The representation of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$ defined by*

$$T_q(\nu) = \sum_{\lambda, \mu, \kappa} c_{\nu\lambda} b_{\lambda\mu}^q c_{\mu\kappa}^{-1} \kappa = \sum_{\kappa} a_{\nu\kappa}^q \kappa,$$

satisfies $T_q(\emptyset) = \delta_{q0} \square$ and $T_q(\lambda) = \lambda^{(q)}$ if λ has an addable q -box.

Proof. Using the elementary properties of c in 1.3.4 and the definition of b^q , we find

$$T_q(\emptyset) = \sum_{\mu, \kappa} b_{\emptyset\mu}^q c_{\mu\kappa}^{-1} \kappa = \delta_{q,0} \sum_{\kappa} c_{\square\kappa}^{-1} \kappa = \delta_{q,0} \square.$$

To prove the second relation, we need to show that, for any fixed λ with an addable q -box, we have

$$(4.2) \quad \sum_{\mu} c_{\lambda\mu} b_{\mu\nu}^q = c_{\lambda^{(q)}\nu}, \quad \text{for all } \nu \in \text{Par}.$$

By Corollary 3.3.6, we have $(cb^q)_{\lambda\nu} \geq c_{\lambda^{(q)}\nu}$, so we focus on the inequality in the other direction.

We first reformulate (4.2) combinatorially. We will assume the reader is familiar with the set Γ_0 of connected hooks and the set Γ of skew Young diagrams introduced in [CE, Section 3.3], which describe the matrix \mathbf{c} . Let $\mathcal{S}_1(\lambda)$ denote the *multiset* of partitions ν obtained by the following procedure, first take a partition $\mu \subseteq \lambda$ such that $\lambda/\mu \in \Gamma$, then either add a q -box to μ or remove a $(q-1)$ -box from μ to obtain the partition ν . This multiset is linked to the left-hand side of (4.2). Concretely, each $\nu \in \mathbf{Par}$ appears $(\mathbf{cb}^q)_{\lambda\nu}$ times in $\mathcal{S}_1(\lambda)$. Let $\mathcal{S}_2(\lambda)$ denote the set of partitions $\nu \subseteq \lambda^{(q)}$ such that $\lambda^{(q)}/\nu \in \Gamma$. This describes the right-hand side of (4.2). First we will show that each element in $\mathcal{S}_1(\lambda)$ is also an element in $\mathcal{S}_2(\lambda)$ and then secondly that $\mathcal{S}_1(\lambda)$ is actually a set. In conclusion, we have $\mathcal{S}_1(\lambda) \subseteq \mathcal{S}_2(\lambda)$ and hence $(\mathbf{cb}^q)_{\lambda\nu} \leq \mathbf{c}_{\lambda^{(q)}\nu}$, which thus implies the proposition.

We start with the following observation, which follows from immediate application of the properties of Γ . Let μ be a partition such that $\lambda/\mu = \gamma \in \Gamma$, with decomposition $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r$, such that each γ_i is a disjoint union of connected rim hooks belonging to Γ_0 in the partition $\lambda \setminus (\gamma_1 \sqcup \dots \sqcup \gamma_{i-1})$. Under the assumption that λ has an addable q -box, there is a k , $1 \leq k \leq r$, such that $\gamma_1, \dots, \gamma_{k-1}$ all contain a q , $(q-1)$ and $(q+1)$ -box, while γ_k does not contain a q -box, and $\gamma_{k+1}, \dots, \gamma_r$ contain no boxes with any of the three contents.

Each $\gamma_1, \dots, \gamma_{k-1}$ will thus contain a shape of the form

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & \\ \hline \end{array},$$

with a being a q -box. By swapping a for the q -box below b and to the right of c we thus obtain $\gamma'_1, \dots, \gamma'_{k-1} \in \Gamma_0$, such that the skew Young diagram $\gamma'_1 \sqcup \dots \sqcup \gamma'_{k-1} \in \Gamma$ is removable from $\lambda^{(q)}$.

From now on, to avoid additional notation, we denote by a , b and c the boxes, in the same configuration as above with a being a q -box, that are on the rim of the partition $\lambda \setminus (\gamma_1 \sqcup \dots \sqcup \gamma_{k-1})$. Furthermore denote by d the q -box directly below b and to the right of c . By construction we know that γ_k does not contain a , but may contain any of the other two boxes, and that d is directly adjacent to γ'_{k-1} (it was the box in γ_{k-1} that was swapped for another box to obtain γ'_{k-1}). We treat the three possible cases one by one.

Case 1: γ_k contains neither b nor c . In this case, we can always add a q -box to μ (the box d) and sometimes it is possible to remove a $q-1$ -box from μ (the box c).

(i) If $\nu \in \mathcal{S}_1(\lambda)$ is obtained by adding the box d to μ , we set $\gamma'_j = \gamma_j$ for $j \geq k$ and obtain $\nu = \lambda^{(q)} \setminus (\gamma'_1 \sqcup \dots \sqcup \gamma'_r)$, so $\nu \in \mathcal{S}_2(\lambda)$.

(ii) If $\nu \in \mathcal{S}_1(\lambda)$ is obtained by removing the $(q-1)$ -box c from μ , we define γ'_k as the union of γ_k and the boxes c and d . By construction γ'_k is either an element of Γ_0 or the disjoint union of two elements of Γ_0 . Furthermore, we set $\gamma'_j = \gamma_j$ for $j > k$ and we have $\nu = \lambda^{(q)} \setminus (\gamma'_1 \sqcup \dots \sqcup \gamma'_r) \in \mathcal{S}_2(\lambda)$.

Case 2: γ_k contains c but not b . In this case it is obvious that one cannot add the q -box d to μ and one can also not remove the $(q-1)$ -box directly to the left of a from μ . Thus this case will not produce any elements in $\mathcal{S}_1(\lambda)$.

Case 3: γ_k contains b but not c . As in Case 2, it is not possible to add the q -box d , but it can be possible to remove the $(q-1)$ -box c . In case that this is possible we add the two boxes c and d to γ_k as in Case 1 above to obtain γ'_k and set $\gamma'_j = \gamma_j$ for $j > k$. Thus $\nu = \lambda^{(q)} \setminus (\gamma'_1 \sqcup \dots \sqcup \gamma'_r)$.

In this way we have realised every element of $\mathcal{S}_1(\lambda)$ as an element of $\mathcal{S}_2(\lambda)$.

Now we prove that $\mathcal{S}_1(\lambda)$ is in fact a set, by showing that each element of $\mathcal{S}_2(\lambda)$ can only be created in at most one of the above ways from the construction in the definition of $\mathcal{S}_1(\lambda)$. For this, note that in the different cases we obtain the following:

- *Case 1(i):* for $p \in \{q-1, q, q+1\}$, the skew diagram $\lambda^{(q)}/\nu$ contains $k-1$ p -boxes.
- *Case 1(ii):* for $p \in \{q-1, q\}$, the skew diagram $\lambda^{(q)}/\nu$ contains k p -boxes and $k-1$ $q+1$ -boxes.
- *Case 3:* for $p \in \{q-1, q, q+1\}$, the skew diagram $\lambda^{(q)}/\nu$ contains k p -boxes.

Clearly there is no overlap between 1(ii) and the other cases. To distinguish elements obtained from Case 1(i) and 3, we look at the unique hook α in Γ_0 , in the covering (see [CE, 3.3]) of $\lambda^{(q)}/\nu \in \Gamma$, which contains the $q-1$ -box with minimal anticontent. In case 1(i), we have $\alpha \subset \gamma'_{k-1}$ and the fact that the connected hooks in γ_{k-1} must satisfy the D-condition in [CE, Definition 3.3.4] shows that γ_{k-1} and also α contains a $q-2$ -box. In Case 3, we have $\alpha \subset \gamma'_k = \gamma_k \sqcup \{c, d\}$. Since the box c was not contained in $\lambda/\mu \supset \gamma_k$, neither was the $q-2$ -box left of c . Hence α does not contain that $q-2$ -box. The $q-2$ -box below c belongs to γ'_{k-1} , so also not to α . In conclusion, α is different for cases 1(i) and (3). A fixed element of $\mathcal{S}_2(\lambda)$ can thus only be identified in at most one way with an element of $\mathcal{S}_1(\lambda)$. \square

4.4. A filtration of Ξ . Recall the set $\text{Par}^{\geq k}$ and Par^k from (1.1)

Proposition 4.4.1. *The representation Ξ of $\text{TL}_\infty(0)$ on $\text{Par}_\mathbb{Z}$ restricts to subrepresentations $\Xi^{\geq k}$ on $\text{Par}_\mathbb{Z}^{\geq k}$ for each $k \in \mathbb{N}$.*

We denote the composition factors of the above filtration by

$$(4.3) \quad \Xi^k : \text{TL}_\infty(0) \rightarrow \text{End}_\mathbb{Z}(\text{Par}_\mathbb{Z}^k).$$

The proposition follows immediately from the following lemma.

Lemma 4.4.2. *If $\lambda \in \text{Par}^{\geq k}$, for some $k \in \mathbb{N}$, and $\mathbf{a}_{\nu\kappa}^q \neq 0$, then $\kappa \in \text{Par}^{\geq k}$.*

Proof. By Propositions 4.3.2 and 4.2.3, κ is obtained from λ either by adding a q -box (in which case $\partial^k \subseteq \lambda \subset \kappa$) or by removing a rim hook as described in 4.2.3(IV) or (V). We restrict to the case (IV) for simplicity. The rim hooks which are removed are balanced and minimal with that property. This means that the minimal anticontent of a box in the hook is attained by the $q+1$ -box, since otherwise one could construct a smaller balanced rim hook which ends the box before the first one with strictly smaller anticontent.

Assume first that the rim q -box of λ is inside ∂^k . It is then necessarily a removable box in ∂^k (in other words a box with maximal anticontent in ∂^k). As the rim $q+1$ -box has anticontent one higher than the q -box, the above observation on the anticontent shows that no boxes in the rim hook are contained in ∂^k . If the rim q -box already is not contained in ∂^k then the q -box already has higher content than the ones in ∂^k and the same reasoning thus allows to conclude that no boxes in the rim hook are contained in ∂^k . \square

4.5. Faithfulness of Ξ .

Proposition 4.5.1. *The representations Ξ and Ξ' are faithful.*

Before we get to the proof we need some preparatory results.

4.5.2. For every sequence of integers $\underline{i} = (i_1, \dots, i_r)$, we define the element $T_{\underline{i}} = T_{i_1} \cdots T_{i_r}$ in $\text{TL}_\infty(0)$. We also denote by $\ell(\underline{i}) = r$ the length of \underline{i} . We multiply sequences of integers by concatenation and, for $a \leq b$, we write $[a, b]$ for the sequence $(a, a+1, \dots, b)$. Sequences

$$(4.4) \quad \underline{w} = [a_1, b_1] \cdot [a_2, b_2] \cdots [a_r, b_r],$$

for some $r \geq 0$ such that $a_1 > a_2 > \dots > a_r$ and $b_1 > b_2 > \dots > b_r$, will be called *fully commutative sequences*. We denote the set of such sequences by **fcs**. The segments $[a_j, b_j]$ will be called the *intervals* of \underline{w} .

4.5.3. The algebra $\text{TL}_\infty(0)$ admits a *basis* of the form $\{T_{\underline{w}} \mid \underline{w} \in \text{fcs}\}$. By [Fa, Proposition 1], a basis of $\text{TL}_\infty(0)$ is given by products of generators corresponding to fully commutative elements in \mathbb{S}_∞ , see [Fa, Section 1] for a definition of fully commutative elements. Furthermore, by [St, Corollary 5.8], fully commutative elements have a normal form given by the elements in **fcs**. Such an expression for a fully commutative element in \mathbb{S}_∞ is unique, see [St, Section 1.3].

4.5.4. Consider the Fock space representation Ξ' of $\text{TL}_\infty(0)$. For $\underline{w} \in \text{fcs}$ and $\lambda \in \text{Par}$, we denote by $\langle T_{\underline{w}}(\lambda) \rangle_m$, the part of the summation in $T_{\underline{w}}(\lambda)$ of partitions of size $|\lambda| - \ell(\underline{w})$.

Lemma 4.5.5. *Consider arbitrary $\underline{w}, \underline{v}$ in **fcs**.*

- (i) *For an arbitrary $\lambda \in \text{Par}$, we have either $\langle T_{\underline{w}}(\lambda) \rangle_m = 0$ or $\langle T_{\underline{w}}(\lambda) \rangle_m$ is a partition.*
- (ii) *There exists $\lambda \in \text{Par}$, such that $\langle T_{\underline{w}}(\lambda) \rangle_m \neq 0$.*
- (iii) *If $\ell(\underline{w}) = \ell(\underline{v})$ and $\langle T_{\underline{w}}(\lambda) \rangle_m = \langle T_{\underline{v}}(\lambda) \rangle_m \neq 0$ for some $\lambda \in \text{Par}$, then $\underline{w} = \underline{v}$.*

Proof. We will assume \underline{w} is of the form (4.4) for the entire proof. Consider first the interval $[a_r, b_r]$. Then $T_{[a_r, b_r]}$ can remove $b_r - a_r + 1$ boxes in $\lambda \in \text{Par}$ if and only if there is an $i \in \mathbb{Z}_{\geq 1}$ such that

$$\lambda_i - i = b_r - 1 \quad \text{and} \quad \lambda_i - \lambda_{i+1} \geq b_r - a_r + 1.$$

In this case, the unique partition $\bar{\lambda}$ of size $|\lambda| - (b_r - a_r + 1)$ in the summation $T_{[a_r, b_r]}(\lambda)$ is obtained from λ by removing $b_r - a_r + 1$ boxes in row i . We can use the above argument on $T_{[a_{r-1}, b_{r-1}]}(\bar{\lambda})$. Moreover, since $b_{r-1} > b_r$, it follows that the row from which boxes are removed in this step is strictly above the previous one.

It follows that the unique partition of $|\lambda| - \ell(\underline{w})$ which can appear in $T_{\underline{w}}\lambda$ is obtained by removing $b_j - a_j + 1$ boxes in the unique row k for which $\lambda_k = b_j + j - 1$. This already proves part (i). Furthermore, since the number of boxes which are removed in each row reflects the lengths of the intervals of \underline{w} and the rows in which they are removed determines the values b_j , we obtain part (iii).

Now we prove part (ii). Take $p \in \mathbb{N}$ such that $p \geq 2 - a_r - r$. We define $\lambda \in \text{Par}$ of length $p + r$, by setting

$$\begin{cases} \lambda_l &= p + 1 + b_1 - 1, & \text{for } 1 \leq l \leq p, \\ \lambda_{p+i} &= p + i + b_i - 1, & \text{for } 1 \leq i \leq r. \end{cases}$$

That this is a partition follows from $\underline{w} \in \text{fcs}$ and the definition of p . Clearly, by acting with $T_{[a_r, b_r]}$ we can remove $b_r - a_r + 1$ boxes in row $p + r$. As such we obtain a partition $\bar{\lambda}$ with

$$\bar{\lambda}_{p+r-1} = \lambda_{p+r-1} = p + r + b_{r-1} - 2 \quad \text{and} \quad \bar{\lambda}_{p+r} = p + r + a_r - 2.$$

In particular

$$\bar{\lambda}_{p+r-1} - \bar{\lambda}_{p+r} = b_{r-1} - a_r \geq b_{r-1} - a_{r-1} + 1.$$

Hence, $\langle T_{[a_{r-1}, b_{r-1}]}(\bar{\lambda}) \rangle_m$ will again be non-zero and we can proceed iteratively. \square

Proof of Proposition 4.5.1. Since the representation Ξ in Proposition 4.3.2 is obtained from Ξ' in Lemma 4.3.1 by applying an automorphism we will only prove faithfulness of the latter.

Fix an arbitrary element x in $\text{TL}_\infty(0)$, written as $\sum_{k=1}^m r_k T_{\underline{w}^k}$, with $r_k \in R$ and the $\underline{w}^k \in \text{fcs}$ distinct. Assume that $\underline{w} := \underline{w}^1$ has maximal $\ell(\underline{w})$. By Lemma 4.5.5(ii), there exists $\lambda \in \text{Par}$ such that the summation $T_{\underline{w}}\lambda$ contains a partition of size $|\lambda| - \ell(\underline{w})$, say ν , with coefficient 1. If $\ell(\underline{w}^j) < \ell(\underline{w})$, then clearly $T_{\underline{w}^j}\lambda$ will not contain ν , since all appearing partitions will be of strictly bigger size. Furthermore, if $\ell(\underline{w}^j) = \ell(\underline{w})$, Lemma 4.5.5(i) and (iii) imply that $T_{\underline{w}^j}\lambda$ does not contain ν either. This proves that $x(\lambda) \neq 0$. \square

4.6. The Temperley-Lieb algebra as an enveloping algebra. Consider the \mathbb{Z} -Lie algebra \mathfrak{sl}_∞ with standard Chevalley generators $\{e_i, f_i \mid i \in \mathbb{Z}\}$. The Fock space representation Φ of $\mathfrak{sl}(\infty)$ on $\text{Par}_\mathbb{Z}$, see e.g. [HY, Section 2.3], is clearly such that

$$\Phi(e_i + f_{i-1}) = \Xi'(T_i), \quad \text{for all } i \in \mathbb{Z}.$$

Let \mathfrak{k} denote the Lie subalgebra of \mathfrak{sl}_∞ generated by $\{e_i + f_{i-1} \mid i \in \mathbb{Z}\}$. By construction, we have

$$\Phi(U(\mathfrak{k})) = \Xi'(\text{TL}_\infty(0)),$$

as subalgebras of $\text{End}_\mathbb{Z}(\text{Par}_\mathbb{Z})$. By Proposition 4.5.1, we thus have

$$\text{TL}_\infty(0) \cong U(\mathfrak{k})/K$$

with K the kernel of $\Phi|_{U(\mathfrak{k})}$. So far as we know this is the first realisation of a Temperley-Lieb algebra as the enveloping algebra of a Lie algebra in a representation.

5. MAIN THEOREMS

5.1. Thick tensor ideals and cells in the periplectic Deligne category.

5.1.1. Thick tensor ideals. A thick tensor ideal in a Krull-Schmidt monoidal (super)category \mathcal{C} is a full subcategory \mathcal{I} which is

- *an ideal*: $X \otimes Y \in \mathcal{I}$, whenever $X \in \mathcal{I}$ or $Y \in \mathcal{I}$;
- *thick*: if $Z \in \mathcal{I}$ satisfies $Z \cong X \oplus Y$, then $X, Y \in \mathcal{I}$.

For \mathcal{C} and \mathcal{I} as above, the monoidal supercategory \mathcal{C}/\mathcal{I} is defined as the quotient category of \mathcal{C} with respect to all morphisms which factor through objects in \mathcal{I} .

Remark 5.1.2. The first condition simplifies for symmetric monoidal supercategories, such as \mathcal{PD} . The second condition implies in particular that \mathcal{I} is *strictly full*. Sometimes it is imposed that a thick tensor ideal \mathcal{I} must also be an *additive* subcategory. As all thick tensor ideals in \mathcal{PD} , using the above definition, will be generated by one indecomposable object, they are obviously additive.

Let \mathcal{I}_k denote the thick tensor ideal in \mathcal{PD} generated by $R(\partial^k)$. Concretely, \mathcal{I}_k is the strictly full additive subcategory which contains all direct summands of $R(\partial^k) \otimes R(\nu)$ for all $\nu \in \text{Par}$.

Theorem 5.1.3. *The set $\{\mathcal{I}_k \mid k \in \mathbb{N}\}$ yields a complete set of thick tensor ideals in \mathcal{PD} . The indecomposable objects in \mathcal{I}_k are (up to isomorphism) given by $\{R(\lambda) \mid \partial^k \subseteq \lambda\}$. We thus have one chain of ideals*

$$\mathcal{PD} = \mathcal{I}_0 \supsetneq \mathcal{I}_1 \supsetneq \cdots \supsetneq \mathcal{I}_k \supsetneq \mathcal{I}_{k+1} \supsetneq \cdots.$$

Proof. Proposition 4.3.2 implies that $\mathbf{T}(R(\nu)) = R(\nu) \otimes R(\square) = \bigoplus_{\kappa} R(\kappa)^{\oplus a_{\nu\kappa}}$ contains any $R(\kappa)$, with κ obtained by adding a box to ν . Consequently, \mathcal{I}_k contains $R(\lambda)$ for all partitions λ which contain ∂^k . On the other hand, Lemma 4.4.2 implies that $R(\lambda) \in \mathcal{I}_k$ requires $\partial^k \subseteq \lambda$.

It thus suffices to show that there are no more thick tensor ideals. Let \mathcal{I} be such an ideal and ∂^k the largest 2-core which is contained in all λ with $R(\lambda) \in \mathcal{I}$. Let ν be a partition with $R(\nu) \in \mathcal{I}$, with $\partial^{k+1} \not\subseteq \nu$ and which has minimal $|\nu|$ under those two restrictions. Assume first that $\nu \neq \partial^k$. Then ν must contain a removable rim 2-hook and by Lemma 3.3.4 there exists $\kappa \subsetneq \nu$ such that $R(\kappa)$ is a direct summand of $\mathbf{T}(R(\nu))$. This violates the minimality of $|\nu|$, so $\nu = \partial^k$. By the above paragraph we then find $\mathcal{I} = \mathcal{I}_k$. \square

5.1.4. Two-sided cells. Following [MM, Section 3], we have the notions of left, right and two-sided cells on a monoidal supercategory. As we work with symmetric categories, these three notions coincide. The quasi-order \preceq on the set of isomorphism classes of indecomposable objects in a Krull-Schmidt symmetric monoidal (super)category \mathcal{C} is determined by

$$[X] \preceq [Y], \quad \text{if there exists } Z \in \text{Ob } \mathcal{C} \text{ such that } Y \text{ is a direct summand of } X \otimes Z.$$

There is a corresponding equivalence relation, defined as $[X] \sim [Y]$ if we have both $[X] \preceq [Y]$ and $[Y] \preceq [X]$. We denote the equivalence class of $[X]$ under \sim by $[[X]]$.

For each $c = [[X]]$, we consider the additive strictly full subcategory \mathcal{C}_c generated by the indecomposable objects $Y \in \text{Ob } \mathcal{C}$ with $[Y] \geq [X]$. This is the thick tensor ideal in \mathcal{C} generated by X . Furthermore, we have the additive strictly full subcategory $\overline{\mathcal{C}}_c$ of \mathcal{C}_c corresponding to the indecomposable objects $Y \in \text{Ob } \mathcal{C}$ with $[Y] \not\sim [X]$. The *cells* of \mathcal{C} are the quotient categories $\mathcal{C}_c / \overline{\mathcal{C}}_c$. We call a cell *maximal* if it corresponds to indecomposable objects which are maximal in the quasi-order. A maximal cell hence corresponds to a subcategory of \mathcal{C} .

Recall the subsets of Par in (1.1). Clearly, for \mathcal{PD} , the quasi-order \preceq is total:

$$[R(\lambda)] \preceq [R(\mu)] \quad \text{if and only if } k \leq l, \text{ with } \lambda \in \text{Par}^k \text{ and } \mu \in \text{Par}^l.$$

Corollary 5.1.5. *The set $\{\mathcal{I}_k / \mathcal{I}_{k+1} \mid k \in \mathbb{N}\}$ yields a complete set of cells in \mathcal{PD} .*

We clearly have $[\mathcal{I}_k / \mathcal{I}_{k+1}]_{\oplus} \cong \text{Par}_{\mathbb{Z}}^k$.

5.2. The kernel of the universal tensor functor. By [KT, Section 5] (see also [Se, Section 4.5]), for any $n \in \mathbb{Z}_{\geq 1}$, we have an additive monoidal superfunctor

$$(5.1) \quad F_n : \mathcal{PD} \rightarrow \mathfrak{pe}(n)\text{-smod}^{\text{op}},$$

where $i \in \mathbb{N} \subset \text{Ob } \mathcal{PD}$ gets mapped to $V^{\otimes i}$ and $F_n(\cup) \in \text{Hom}_{\mathfrak{pe}(n)}(V^{\otimes 2}, \mathbb{k})$ is given by $F_n(\cup)(v \otimes w) = \beta(v, w)$, with β the defining bilinear form in Section 1.4. In particular, F_n induces the algebra morphisms

$$\phi_n^r : A_r \rightarrow \text{End}_{\mathfrak{pe}(n)}(V^{\otimes r})^{\text{op}},$$

first introduced in [Mo, Proposition 2.4].

Theorem 5.2.1. *The additive monoidal superfunctor (5.1) is full and its kernel is given by \mathcal{I}_{n+1} .*

We start with two preparatory lemmata.

Lemma 5.2.2. *For $\lambda \in \text{Par}$, we have $F_n(R(\lambda)) = 0$ if and only if $\phi_n^r(e) = 0$ for an arbitrary $r \in \mathbb{N}$ with $|\lambda| \in \mathcal{J}^0(r)$ and $e \in A_r$ an idempotent corresponding to $L_r(\lambda)$.*

Proof. By Remark 2.2.2, we have $R(\lambda) \cong (r, e)$ in \mathcal{PD} . Furthermore, by definition of ϕ_n^r , we have $F_n((r, e)) = \text{im } \phi_n^r(e)$. This concludes the proof. \square

Lemma 5.2.3. *For any partition λ with $\lambda_{n+1} > n$, we have $F_n(R(\lambda)) = 0$.*

Proof. By Lemma 5.2.2, it suffices to prove that $\phi_n^r(e_\lambda) = 0$, with $r = |\lambda|$.

When we restrict the action ϕ_n^r from A_r to the subalgebra $\mathbb{k}\mathbb{S}_r$ (see 1.3.3), the image commutes with the $\mathfrak{gl}(n|n)$ action on $V^{\otimes r}$, see [BR, Theorem 4.14]. Hence, we have a commuting diagram:

$$\begin{array}{ccc} A_r & \xrightarrow{\phi_n^r} & \text{End}_{\mathfrak{pe}(n)}(V^{\otimes r})^{\text{op}} \\ \uparrow & & \uparrow \\ \mathbb{k}\mathbb{S}_r & \longrightarrow & \text{End}_{\mathfrak{gl}(n|n)}(V^{\otimes r})^{\text{op}}. \end{array}$$

Now, let $f_\lambda \in \mathbb{k}\mathbb{S}_r$ be a primitive idempotent corresponding to the (simple) Specht module for λ . By the choice of the labelling of simple modules over A_r , e_λ appears (up to conjugation) in the decomposition of f_λ into primitive idempotents in A_r , see e.g. [Co, Corollary 4.3.3].

The hook condition in [BR, Theorem 3.20] and the above commuting diagram imply that $\phi_n^r(f_\lambda) = 0$ if $\lambda_{n+1} > n$, so in particular $\phi_n^r(e_\lambda) = 0$. \square

Proof of Theorem 5.2.1. By additivity it suffices to show that the restriction $\mathcal{A} \rightarrow \mathfrak{pe}(n)\text{-smod}$ is full. By [KT, Section 5.3], the surjectivity of

$$\text{Hom}_{\mathcal{A}}(i, j) \rightarrow \text{Hom}_{\mathfrak{pe}(n)}(V^{\otimes j}, V^{\otimes i}),$$

for any $i, j \in \mathbb{N}$, is equivalent to surjectivity of

$$\text{Hom}_{\mathcal{A}}(0, i + j) \rightarrow \text{Hom}_{\mathfrak{pe}(n)}(V^{\otimes(j+i)}, \mathbb{k}).$$

The latter is precisely [DLZ, Section 4.9].

As F_n is an additive monoidal superfunctor, its kernel \mathcal{K}_n is a thick tensor ideal. By Theorem 5.1.3, we thus have $\mathcal{K}_n = \mathcal{I}_k$ for some $k \in \mathbb{N}$.

By [Co, Corollary 8.2.7], for $\lambda \vdash r$ with $\lambda_{n+1} = 0$, we have $\phi_n^r(e_\lambda) \neq 0$. By Lemma 5.2.2, we find that in particular $R(\partial^n) \notin \mathcal{K}_n$, which implies $\mathcal{K}_n \neq \mathcal{I}_k$ when $k \leq n$.

By Lemma 5.2.3, we have $R(\lambda) \in \mathcal{K}_n$ for $\lambda = (n+1, \dots, n+1)$, the partition of $(n+1)^2$ of length $n+1$. As $\partial^k \not\leq \lambda$ for $k > n+1$, we find $\mathcal{K}_n \neq \mathcal{I}_k$ when $k > n+1$. This concludes the proof. \square

5.3. Tensor powers of the natural representation of $\mathfrak{pe}(n)$. The results in the previous subsection allow to classify the indecomposable summands in the $\mathfrak{pe}(n)$ -module $V^{\otimes r}$ up to isomorphism. In this subsection we further determine when the direct summands are projective.

Theorem 5.3.1. *The assignment*

$$\lambda \mapsto R_n(\lambda) := F_n(R(\lambda)),$$

is a bijection between $\text{Par}^{\leq n}$ and the set of isomorphism classes of indecomposable summands in $\bigoplus_{r \in \mathbb{N}} V^{\otimes r}$. $R_n(\lambda)$ appears in $V^{\otimes r}$ if $|\lambda| \in \mathcal{J}^0(r)$. The module $R_n(\lambda)$ is projective if and only if $\lambda \in \text{Par}^n$.

We denote the full subcategory of projective modules in $\mathfrak{pe}(n)$ -smod by $\mathfrak{pe}(n)$ -proj.

Theorem 5.3.2. *The subcategory $\mathfrak{pe}(n)$ -proj is the unique maximal cell in $\mathfrak{pe}(n)$ -smod. The functor F_n restricts to an essentially surjective functor $\mathcal{I}_n \rightarrow \mathfrak{pe}(n)$ -proj with kernel \mathcal{I}_{n+1} . Hence, there exists an additive tensor functor*

$$\mathcal{I}_n / \mathcal{I}_{n+1} \rightarrow \mathfrak{pe}(n)\text{-proj},$$

which is essentially bijective and full.

Now we prove these two theorems. There is a duality $*$ on $\mathfrak{pe}(n)$ -smod, see [BDE+, Section 2]. Furthermore, the right adjoint of $- \otimes M$ is $- \otimes M^*$, for a module M , see e.g. [BDE+, Section 4.4]. This implies that $M \otimes N$ is projective as soon as either M or N , so in particular that $\mathfrak{pe}(n)$ -proj is a thick tensor ideal.

Lemma 5.3.3. *Let Q_1, Q_2 be arbitrary indecomposable projective modules in $\mathfrak{pe}(n)$ -smod. Then Q_1 is a direct summand of $Q_2 \otimes V^{\otimes k}$ for some $k \in \mathbb{N}$.*

Proof. It is well-known that injective and projective modules coincide in $\mathfrak{pe}(n)$ -smod, see e.g. [BDE+, Ch]. In particular, the duality $*$ maps projective modules to projective modules. We then find that there exists $j, i \in \mathbb{N}$ such that Q_1 is a direct summand of $V^{\otimes j}$ and Q_2^* is a direct summand in $V^{\otimes i}$, by [Co, Lemma 8.3.2]. Then we have a composition of epimorphisms

$$Q_2 \otimes V^{\otimes i+j} \twoheadrightarrow Q_2 \otimes Q_2^* \otimes Q_1 \twoheadrightarrow Q_1.$$

Since Q_1 is projective, the corresponding epimorphism splits, which concludes the proof. \square

Corollary 5.3.4. *The full subcategory $\mathfrak{pe}(n)$ -proj is the unique maximal cell in $\mathfrak{pe}(n)$ -smod.*

Proof. Lemma 5.3.3 implies that all indecomposable projective modules are equivalent for the relation of 5.1.4. This implies that the thick tensor ideal is in fact a cell. Moreover, since $V^{\otimes k}$ contains a projective direct summand for $k \gg 0$ ([Co, Lemma 8.3.2]), $M \otimes V^{\otimes k}$ for any module M contains a projective direct summand, showing that $\mathfrak{pe}(n)$ -proj is the unique maximal cell. \square

Proof of Theorem 5.3.1. The classification of indecomposable summands is an immediate consequence of Theorem 5.2.1 and Lemma 5.2.2.

The projective modules form a thick tensor ideal in $\mathfrak{pe}(n)$ -smod. This implies that the corresponding pre-image under the tensor superfunctor F_n also forms a thick tensor ideal

\mathcal{J} in \mathcal{PD} . By Theorem 5.2.1, we have $\mathcal{J}_{n+1} \subseteq \mathcal{J}$. Since each projective module appears as direct summands of $V^{\otimes j}$ for some $j \in \mathbb{N}$ ([Co, Lemma 8.3.2]), we even have $\mathcal{J}_{n+1} \subsetneq \mathcal{J}$. By Theorem 5.1.3, we thus have $\mathcal{J} = \mathcal{J}_k$ for some $k \leq n$. Because F_n is full (Theorem 5.2.1), Corollary 5.3.4 implies that $\mathcal{J}_k/\mathcal{J}_{n+1}$ must be a cell, so $k = n$. \square

Proof of Theorem 5.3.2. This follows from Corollary 5.3.4 and Theorems 5.2.1 and 5.3.1. \square

6. CATEGORIFICATION OF THE REPRESENTATION Ξ

6.1. Categorical action of $\mathrm{TL}_\infty(0)$ on \mathcal{PD} . In this section we upgrade the naïve categorification $(\mathcal{PD}, \Psi, \{\mathbf{T}_i \mid i \in \mathbb{Z}\})$ from Theorem 4.1.2 to a ‘weak categorification’ of the $\mathrm{TL}_\infty(0)$ -representation Ξ , in the sense of [Ma, Definition 2.7].

Theorem 6.1.1. *We have natural isomorphisms of functors, for all $i, j \in \mathbb{Z}$ with $|i - j| > 1$,*

$$\mathbf{T}_i^2 \cong 0, \quad \mathbf{T}_i \mathbf{T}_j \cong \mathbf{T}_j \mathbf{T}_i \quad \text{and} \quad \mathbf{T}_i \mathbf{T}_{i \pm 1} \mathbf{T}_i \cong \mathbf{T}_i.$$

The second natural isomorphism is even, the third one odd.

Remark 6.1.2. Theorem 6.1.1 implies also that we get a weak categorification of the $\mathrm{TL}_\infty(0)$ -representation Ξ^k in (4.3) on the cell $\mathcal{J}_k/\mathcal{J}_{k+1}$.

The rest of this subsection is devoted to the proof.

6.1.3. We define two families of odd morphisms. For $X = (r, e) \in \mathcal{PD}$, we set, using (2.2),

$$\begin{aligned} \varepsilon_X : \mathbf{T}\mathbf{T}(X) &\rightarrow X, & \varepsilon_X &= (e \otimes \cap) \in e\mathrm{Hom}_{\mathcal{A}}(r+2, r)(e \otimes \mathbf{I} \otimes \mathbf{I}), \quad \text{and} \\ \eta_X : X &\rightarrow \mathbf{T}\mathbf{T}(X), & \eta_X &= (e \otimes \cup) \in (e \otimes \mathbf{I} \otimes \mathbf{I})\mathrm{Hom}_{\mathcal{A}}(r, r+2)e. \end{aligned}$$

These extend easily to arbitrary objects $X \in \mathrm{Ob} \mathcal{PD}$, see (2.1).

Lemma 6.1.4. *The families $\{\varepsilon_X\}$ and $\{\eta_X\}$ define odd natural transformations $\varepsilon : \mathbf{T}\mathbf{T} \rightarrow \mathrm{Id}$ and $\eta : \mathrm{Id} \rightarrow \mathbf{T}\mathbf{T}$.*

Proof. We prove the claim for ε , the case η is proved identically. Take idempotents $e \in A_r$ and $f \in A_s$ and set $X = (r, e)$ and $Y = (s, f)$. Consider $a \in f\mathrm{Hom}_{\mathcal{A}}(r, s)e = \mathrm{Hom}_{\mathcal{PD}}((r, e), (s, f))$. By definition, we need to show that

$$a \circ (e \otimes \cap) = (-1)^{|a|} (f \otimes \cap) \circ (a \otimes \mathbf{I} \otimes \mathbf{I}).$$

This equation holds true by equation (1.2) and $ae = a = fa$. \square

Lemma 6.1.5. *We have equalities of natural transformations*

$$\mathbf{T}(\varepsilon) \circ \eta_{\mathbf{T}} = 1_{\mathbf{T}} \quad \text{and} \quad \varepsilon_{\mathbf{T}} \circ \mathbf{T}(\eta) = -1_{\mathbf{T}}.$$

Proof. By [KT, Theorem 3.2.1], we have

$$(\cap \otimes \mathbf{I}) \circ (\mathbf{I} \otimes \cup) = \mathbf{I} \quad \text{and} \quad (\mathbf{I} \otimes \cap) \circ (\cup \otimes \mathbf{I}) = -\mathbf{I}.$$

Set $X = (r, e) \in \mathrm{Ob} \mathcal{PD}$. By 6.1.3, equation (3.1) and the above formula, we have

$$\mathbf{T}(\varepsilon_X) \circ \eta_{\mathbf{T}X} = (e \otimes \cap \otimes \mathbf{I}) \circ (e \otimes \mathbf{I} \otimes \cup) = (e \otimes \mathbf{I}) = 1_{\mathbf{T}X}.$$

The second relation follows identically. \square

6.1.6. We introduce natural transformations $\iota^i : \mathbf{T}_i \rightarrow \mathbf{T}$ and $\pi^i : \mathbf{T} \rightarrow \mathbf{T}_i$. For $X = (r, e)$, the morphism ι_X^i , resp. π_X^i , is to be identified with

$$(e \otimes \mathbf{I})\gamma_i^{(r+1)} = \gamma_i^{(r+1)}(e \otimes \mathbf{I})\gamma_i^{(r+1)} = \gamma_i^{(r+1)}(e \otimes \mathbf{I}),$$

which can be interpreted inside $\text{Hom}_{\mathcal{PD}}(\mathbf{T}_i X, \mathbf{T} X)$, resp. $\text{Hom}_{\mathcal{PD}}(\mathbf{T} X, \mathbf{T}_i X)$, as in (2.2). Furthermore, $(\iota^i \circ \pi^i)_X \in \text{Hom}_{\mathcal{PD}}(\mathbf{T} X, \mathbf{T} X)$ and $1_{\mathbf{T}_i X} = (\pi^i \circ \iota^i)_X \in \text{Hom}_{\mathcal{PD}}(\mathbf{T}_i X, \mathbf{T}_i X)$ can also be interpreted as the above element. All this extends to arbitrary $X \in \text{Ob } \mathcal{PD}$ in (2.1).

Lemma 6.1.7. *We have equalities of natural transformations, for all $i \in \mathbb{Z}$,*

$$\begin{aligned} \varepsilon \circ \iota_{\mathbf{T}}^i &= \varepsilon \circ (\iota^i \star (\iota^{i+1} \circ \pi^{i+1})), & \text{for } \mathbf{T}_i \mathbf{T} &\rightarrow \text{Id}, \\ \varepsilon \circ \mathbf{T}(\iota^i) &= \varepsilon \circ ((\iota^{i-1} \circ \pi^{i-1}) \star \iota^i), & \text{for } \mathbf{T} \mathbf{T}_i &\rightarrow \text{Id}, \\ \mathbf{T}(\pi^i) \circ \eta &= ((\iota^{i+1} \circ \pi^{i+1}) \star \pi^i) \circ \eta, & \text{for } \text{Id} &\rightarrow \mathbf{T} \mathbf{T}_i, \\ \pi_{\mathbf{T}}^i \circ \eta &= (\pi^i \star (\iota^{i-1} \circ \pi^{i-1})) \circ \eta, & \text{for } \text{Id} &\rightarrow \mathbf{T}_i \mathbf{T}. \end{aligned}$$

Proof. By [Co, Lemma 6.3.1(1)], for $k \in \mathbb{N}$, we have

$$\begin{aligned} (\mathbf{I}^{\otimes k} \otimes \cap) \circ x_{k+1} &= (\mathbf{I}^{\otimes k} \otimes \cap) \circ (x_{k+2} + \mathbf{I}^{\otimes k+2}) \quad \text{and} \\ x_{k+2} \circ (\mathbf{I}^{\otimes k} \otimes \cup) &= (x_{k+1} + \mathbf{I}^{\otimes k+2}) \circ (\mathbf{I}^{\otimes k} \otimes \cup). \end{aligned}$$

From the first equation we find

$$(6.1) \quad (\mathbf{I}^{\otimes k} \otimes \cap) \gamma_i^{(k+2)} \gamma_j^{(k+1)} = 0, \quad \text{unless } j = i + 1.$$

Equation (3.4) then further implies

$$(6.2) \quad (\mathbf{I}^{\otimes k} \otimes \cap) \gamma_i^{(k+2)} = (\mathbf{I}^{\otimes k} \otimes \cap) \gamma_i^{(k+2)} \gamma_{i+1}^{(k+1)} = (\mathbf{I}^{\otimes k} \otimes \cap) \gamma_{i+1}^{(k+1)}.$$

These equations, and their analogues for \cup can be used to prove the proposed equalities. We do this explicitly for the first one.

For $X = (r, e)$, we calculate, using equation (3.1) and the definitions in 6.1.3 and 6.1.6, that

$$(\varepsilon \circ \iota_{\mathbf{T}}^i)_X = (e \otimes \cap) \circ (e \otimes \mathbf{I} \otimes \mathbf{I}) \gamma_i^{(r+2)} = (e \otimes \cap) \gamma_i^{(r+2)}$$

and

$$\begin{aligned} (\varepsilon \circ (\iota^i \star (\iota^{i+1} \circ \pi^{i+1})))_X &= \varepsilon_X \circ \iota_{\mathbf{T} X}^i \circ \mathbf{T}_i(\iota_X^{i+1} \circ \pi_X^{i+1}) \\ &= (e \otimes \cap) \circ (e \otimes \mathbf{I} \otimes \mathbf{I}) \gamma_i^{(r+2)} \circ ((e \otimes \mathbf{I}) \gamma_{i+1}^{(r+1)} \otimes \mathbf{I}) \gamma_i^{(r+2)} \\ &= (e \otimes \cap) \gamma_i^{(r+2)} \gamma_{i+1}^{(r+1)}. \end{aligned}$$

By (6.2), these two morphisms are the same indeed. \square

The following two lemmata were inspired by [ES, Lemmata 2.7 and 2.8].

Lemma 6.1.8. *For $\psi_r := s_{r-1}(x_{r-1} - x_r) + 1 \in A_r$, with $r \geq 2$, we have*

- (i) $x_r \psi_r = \psi_r x_{r-1} - \epsilon_{r-1}$,
- (ii) $x_{r-1} \psi_r = \psi_r x_r - \epsilon_{r-1}$,
- (iii) $\psi_r^2 = 1 - (x_{r-1} - x_r)^2$,
- (iv) $\psi_r \circ (a \otimes \mathbf{I} \otimes \mathbf{I}) = (a \otimes \mathbf{I} \otimes \mathbf{I}) \circ \psi_s$, for any $a \in \text{Hom}_{\mathcal{A}}(s-2, r-2)$.

Proof. Parts (i)-(iii) are direct applications of the commutation relations in [Co, Lemma 6.3.1]. Part (iv) follows from the fact that s_{r-1} is equal to $I^{\otimes r-2} \otimes X$ and Lemma 3.2.2. \square

Now we define a family of even morphisms. For each $X = (r, e) \in \text{Ob } \mathcal{PD}$, we set

$$\varphi_X : \mathbf{TT}(X) \rightarrow \mathbf{TT}(X), \quad \varphi_X = \psi_{r+2}(e \otimes I \otimes I) = (e \otimes I \otimes I)\psi_{r+2}.$$

Again we extend to arbitrary objects in \mathcal{PD} and we obtain a natural transformation $\varphi : \mathbf{TT} \rightarrow \mathbf{TT}$ by Lemma 6.1.8(iv).

Lemma 6.1.9. *Take $i, j \in \mathbb{Z}$, such that $|i - j| > 1$, then*

- (i) $\varphi \circ (\iota^i \star \iota^j) = (\iota^j \star \iota^i) \circ (\pi^j \star \pi^i) \circ \varphi \circ (\iota^i \star \iota^j)$ as natural transformations $\mathbf{T}_i \mathbf{T}_j \rightarrow \mathbf{TT}$;
- (ii) $(\pi^i \star \pi^j) \circ \varphi \circ \varphi \circ (\iota^i \star \iota^j)$ is a natural isomorphism of $\mathbf{T}_i \mathbf{T}_j$.

Proof. We start by proving part (i). For $X = (r, e)$, we have

$$(\varphi \circ (\iota^i \star \iota^j))_X = (e \otimes I \otimes I)\psi_{r+2}\gamma_i^{(r+2)}\gamma_j^{(r+1)}.$$

By Lemma 6.1.8(i), we have

$$(x_{r+2}-j)(e \otimes I \otimes I)\psi_{r+2}\gamma_i^{(r+2)}\gamma_j^{(r+1)} = (e \otimes I \otimes I) \left(\psi_{r+2}(x_{r+1}-j)\gamma_i^{(r+2)}\gamma_j^{(r+1)} - \epsilon_{r+1}\gamma_i^{(r+2)}\gamma_j^{(r+1)} \right).$$

As we assume that $j \neq i + 1$, the last term vanishes by equation (6.1). Multiplying $(\varphi \circ (\iota^i \star \iota^j))_X$ with $(x_{r+2}-j)^p$ from the left for the appropriate $p \in \mathbb{N}$ will thus yield zero, meaning

$$(e \otimes I \otimes I)\psi_{r+2}\gamma_i^{(r+2)}\gamma_j^{(r+1)} = \gamma_j^{(r+2)}(e \otimes I \otimes I)\psi_{r+2}\gamma_i^{(r+2)}\gamma_j^{(r+1)}.$$

The corresponding reasoning for $(x_{r+1}-i)$ concludes the proof of part (i).

Now we consider part (ii). By Lemma 6.1.8(iii), for $X = (r, e)$, we have

$$((\pi^i \star \pi^j) \circ \varphi \circ \varphi \circ (\iota^i \star \iota^j))_X = (e \otimes I \otimes I)(1 - (x_{r+1} - x_{r+2})^2)\gamma_i^{(r+2)}\gamma_j^{(r+1)}.$$

For any $c \in \mathbb{k}$, we can expand

$$1 - (x_{r+1} - x_{r+2})^2 = (1 - c^2) - (x_{r+1} - x_{r+2} - c)^2 - 2c(x_{r+1} - x_{r+2} - c)$$

If we set $c = j - i$, then this allows to write the above morphism as the sum of $(1 - c^2)1_{\mathbf{T}_i \mathbf{T}_j X}$ and a nilpotent one. Since $c^2 \neq 1$, this means that the morphism is an isomorphism of $\mathbf{T}_i \mathbf{T}_j X$. \square

Proof of Theorem 6.1.1. The relation $\mathbf{T}_i^2 \cong 0$ follows immediately from Theorem 4.1.2.

Now assume that $|i - j| > 1$. The composition

$$(\pi^i \star \pi^j) \circ \varphi \circ (\iota^j \star \iota^i) \circ (\pi^j \star \pi^i) \circ \varphi \circ (\iota^i \star \iota^j)$$

corresponding to

$$\mathbf{T}_i \mathbf{T}_j \rightarrow \mathbf{TT} \rightarrow \mathbf{TT} \rightarrow \mathbf{T}_j \mathbf{T}_i \rightarrow \mathbf{TT} \rightarrow \mathbf{TT} \rightarrow \mathbf{T}_i \mathbf{T}_j$$

is an isomorphism, by Lemma 6.1.9. We hence have even natural transformations $\alpha : \mathbf{T}_i \mathbf{T}_j \rightarrow \mathbf{T}_j \mathbf{T}_i$ and $\beta : \mathbf{T}_j \mathbf{T}_i \rightarrow \mathbf{T}_i \mathbf{T}_j$ such that $\beta \circ \alpha$ is an isomorphism. Since $\mathbf{T}_i \mathbf{T}_j X \cong \mathbf{T}_j \mathbf{T}_i X$ for all $X \in \text{Ob } \mathcal{PD}$, see Theorem 4.1.2, this means that both α and β must be isomorphisms.

Now we consider the natural transformation

$$\pi^i \circ \mathbf{T}(\varepsilon) \circ ((\iota^i \circ \pi^i) \star (\iota^{i-1} \circ \pi^{i-1}) \star (\iota^i \circ \pi^i)) \circ \eta_{\mathbf{T}} \circ \iota^i,$$

corresponding to

$$\mathbf{T}_i \rightarrow \mathbf{T} \rightarrow \mathbf{TTT} \rightarrow \mathbf{T}_i \mathbf{T}_{i-1} \mathbf{T}_i \rightarrow \mathbf{TTT} \rightarrow \mathbf{T} \rightarrow \mathbf{T}_i.$$

Using the standard interchange laws $\eta_{\mathbf{T}} \circ \iota^i = \mathbf{TT}(\iota^i) \circ \eta_{\mathbf{T}_i}$ and $\pi^i \circ \mathbf{T}(\varepsilon) = \mathbf{T}_i(\varepsilon) \circ \pi_{\mathbf{TT}}^i$, subsequently Lemma 6.1.7, again the interchange laws, and finally Lemma 6.1.5, shows that the composition above is equal to $1_{\mathbf{T}_i} = \pi^i \circ \iota^i$. In particular, we find odd natural transformations $\alpha : \mathbf{T}_i \rightarrow \mathbf{T}_i \mathbf{T}_{i-1} \mathbf{T}_i$ and $\beta : \mathbf{T}_i \mathbf{T}_{i-1} \mathbf{T}_i \rightarrow \mathbf{T}_i$ such that $\beta \circ \alpha = 1_{\mathbf{T}_i}$. As $\mathbf{T}_i(X) \cong \mathbf{T}_i \mathbf{T}_{i-1} \mathbf{T}_i(X)$ for all $X \in \text{Ob } \mathcal{PD}$, see Theorem 4.1.2, it follows that α and β are isomorphisms. The relation for $i + 1$ follows similarly. \square

6.2. Relation with other categorical representations. A categorical representation of $\text{TL}_\infty(0)$ on $\mathbf{pe}(n)$ -smod was constructed in [BDE+, Theorem 4.5.1]. That result served as inspiration for the statement in Theorem 6.1.1. Both categorical representations are actually intimately connected, despite the fact that one is on an abelian and one on an additive category. We briefly explore the relation in this section.

6.2.1. We have the functor F_n between \mathcal{PD} and $\mathbf{pe}(n)$ -smod in (5.1). It follows almost by definition that F_n in (5.1) intertwines \mathbf{T} and the translation functor $\Theta' = - \otimes V$ in [BDE+, equation (12)]. Moreover, it then follows easily from [Co, 8.5.3] that the natural transformation ξ of \mathbf{T} is intertwined with the natural transformation Ω of Θ' in [BDE+, Definition 4.1.3]. It thus follows indeed that F_n links the functors \mathbf{T}_i in Theorem 6.1.1 with the functors Θ_i in [BDE+, Theorem 4.5.1]. Hence, the decategorification of F_n is a morphism of $\text{TL}_\infty(0)$ -modules. Since F_n has a kernel, this is not a monomorphism. Moreover, F_n is not essentially surjective and more importantly it is not clear whether the induced morphism

$$[\mathcal{PD}]_\oplus \rightarrow [\mathbf{pe}(n)\text{-smod}]$$

from the split Grothendieck group of \mathcal{PD} to the Grothendieck group of $\mathbf{pe}(n)$ -smod is surjective.

6.2.2. Since $\mathbf{pe}(n)$ -smod has infinite global dimension, the canonical group monomorphism

$$[\mathbf{pe}(n)\text{-proj}]_\oplus \hookrightarrow [\mathbf{pe}(n)\text{-smod}]$$

will not be an isomorphism. In fact, by the results in Section 5.3, the functor Θ' restricts to the full additive subcategory $\mathbf{pe}(n)$ -proj and this subcategory constitutes the socle of the categorical $\text{TL}_\infty(0)$ -representation on $\mathbf{pe}(n)$ -smod in [BDE+, Theorem 4.5.1].

By Theorem 5.3.2, we have an essentially bijective (and full) additive functor from $\mathcal{I}_n/\mathcal{I}_{n+1}$ to $\mathbf{pe}(n)$ -proj, so in particular

$$[\mathcal{I}_n/\mathcal{I}_{n+1}]_\oplus \cong [\mathbf{pe}(n)\text{-proj}]_\oplus.$$

By construction and 6.2.1, this is an isomorphism between the $\text{TL}_\infty(0)$ -representation Ξ^n in (4.3) and the decategorification of the socle of the categorical representation of [BDE+, Theorem 4.5.1] on $\mathbf{pe}(n)$ -smod.

In conclusion, our categorical representation Ξ of $\text{TL}_\infty(0)$ on \mathcal{PD} admits a filtration, labelled by $n \in \mathbb{N}$, where each composition factor ‘corresponds to’ a categorical representation of $\text{TL}_\infty(0)$ on the category of projective modules over $\mathbf{pe}(n)$ introduced in [BDE+].

6.2.3. Contrary to the representation Ξ , the decategorification of [BDE+, Theorem 4.5.1] for a fixed $\mathfrak{pe}(n)$ is not a faithful representation of $\mathrm{TL}_\infty(0)$. Indeed, using the combinatorics of [BDE+, Section 5.2], it follows that a generic functor of the form

$$\Theta_{i_1} \Theta_{i_2} \cdots \Theta_{i_p}$$

with $p > n$ will send any (thick) Kac module to zero. Since the functor is exact this automatically implies that it maps any module to zero. In particular $T_{i_1} T_{i_2} \cdots T_{i_p}$ will generically act as zero on the decategorified representation of $\mathrm{TL}_\infty(0)$.

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