Example 1.50: The wedge sum of spaces

Let $\{(X_\alpha, x_\alpha)\}_\alpha$ be a family of pointed spaces and let $X = \bigvee_\alpha X_\alpha$ with basepoint x_0.
We want to apply the van Kampen theorem to X. First we need the covering by open subsets.
Let $U_\alpha \subset X_\alpha$ be an open neighbourhood of x_α in X_α.
Then the subspaces $A_\alpha = X_\alpha \vee \bigvee_{\beta\neq\alpha} U_\beta$ form an open cover of X and they each contain x_0.
To apply the van Kampen theorem we have to look at the prerequisites:

- The A_α are a covering by open subsets each containing x_0.
- Each A_α needs to be path-connected. To achieve this we assume: X_α **is path-connected and** U_α **is a path-connected neighbourhood of** x_α.

To obtain that φ from the van-Kampen theorem is surjective we need that $A_\alpha \cap A_\beta$ is path-connected for $\alpha \neq \beta$. By construction

$$A_\alpha \cap A_\beta = \bigvee_\gamma U_\gamma \text{ for } \alpha \neq \beta,$$

which, by the new assumption, is path-connected. Hence φ is surjective. By construction triple intersection are also path-connected, hence we know that the kernel of φ is of the form

$$N = \left\langle g\varphi_{\alpha\beta}(\omega)\varphi_{\beta\alpha}(\omega)^{-1}g^{-1} \mid \alpha,\beta \text{ and } \omega \in \pi_1(A_\alpha \cap A_\beta, x_0), g \in \pi_1(X, x_0)\right\rangle.$$

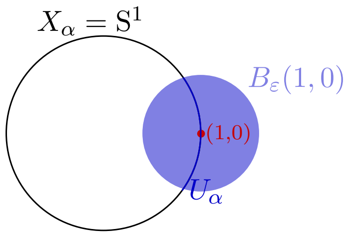
To get any information out of the theorem we need to have a situation where we know how $\pi_1(A_\alpha \cap A_\beta, x_0)$ looks like. The easiest general statement is the following:

- We assume: U_α **deformation retracts to** $\{x_\alpha\}$.

With this extra assumption each $A_\alpha \cap A_\beta$, for $\alpha \neq \beta$, deformation retracts to a single point and is thus contractible and $\pi_1(A_\alpha \cap A_\beta, x_0)$ is trivial. Furthermore we obtain that $A_\alpha \simeq X_\alpha$ since A_α is a wedge sum of X_α with contractible spaces.
Thus under the two extra assumption we made above, i.e. each X_α is path-connected and each U_α deformation retracts to the basepoint, we obtain

$$\pi_1(X) \cong *_\alpha \pi_1(X_\alpha).$$

We can leave out the basepoint, since under our assumptions X will be automatically path-connected as well.
If we apply this to the situation $X = \mathrm{S}^1 \vee \mathrm{S}^1$ with basepoint $(1,0)$ in each circle. We see that each S^1 is path-connected and for U_α we can intersect the circle with a small open ball of radius $\varepsilon < 2$, U_α is then an open arc on the circle as seen in the picture



Hence we get $\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1) \cong \mathbb{Z} * \mathbb{Z}$.

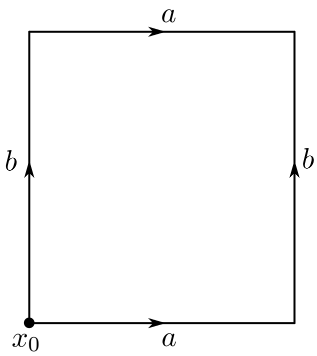
Example 1.51: Surfaces of genus g

(a) We start with the example of $X = \mathrm{S}^2$ with basepoint $x_0 = (1, 0, 0)$. We define $A_1 = X \cap (\mathbb{R}^2 \times (-\infty, 1/2))$ and $A_2 = X \cap (\mathbb{R}^2 \times (-1/2, \infty))$, i.e. we intersect the sphere with two open half-spaces in $\mathbb{R}^3$. Then A_1 and A_2 are path-connected and contain x_0. It is easy to see that $A_1 \cong B^2 \cong A_2$ are homeomorphic to an open disc in $\mathbb{R}^2$. To obtain that the map φ from the van Kampen theorem is now surjective, we need to look at $A_1 \cap A_2$, which is equal to $X \cap (\mathbb{R}^2 \times (-1/2, 1/2))$. This is obviously homeomorphic to $\mathrm{S}^1 \times (-1/2, 1/2)$, which is path-connected. Hence we can apply the van Kampen theorem and obtain

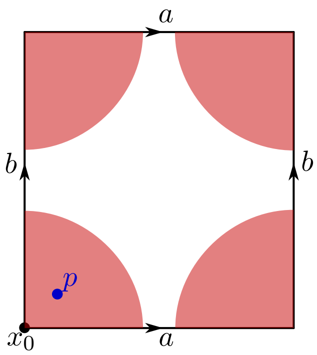
$$\pi_1(\mathrm{S}^2) = \pi_1(B^2) * \pi_1(B^2) = \{0\},$$

we do not have to worry about the quotient, since the free product is already trivial.

(b) We now increase to genus 1, which is the torus $X = \mathrm{T}^2$. We indicate the choices in the picture below as usual for the torus where we identify edges according to their orientation

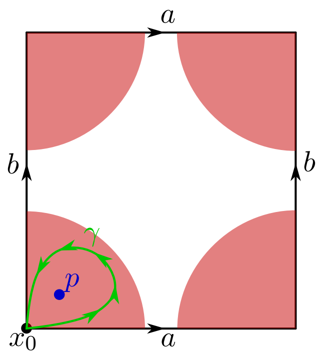


As seen in the picture, we realize the torus again as a quotient of $\mathrm{I} \times \mathrm{I}$. We have the basepoint x_0 as the corners that all get identified in the quotient. For A_1 we take four open balls of radius $0 < \varepsilon < 1/2$, one at each corner of $\mathrm{I} \times \mathrm{I}$ and intersect them with $\mathrm{I} \times \mathrm{I}$ (outlined as the red area in the picture below). The image of the intersections (shown in red in the picture) in the quotient is then A_1. For A_2 we take T^2 and remove a point p inside A_1 and the interior of $\mathrm{I} \times \mathrm{I}$ as indicated by the blue point in the picture below



In the quotient, A_1 is just four quarter discs glued together, hence it is homeomorphic to an open ball in $\mathbb{R}^2$ and thus contractible. While for A_2, we note that $\mathrm{I} \times \mathrm{I} \setminus \{p\}$ is homotopic to the boundary of $\mathrm{I} \times \mathrm{I}$. This is done in the same way as showing that the punctured disc is homotopic to a circle. Then we see that in the quotient T^2 the boundary of $\mathrm{I} \times \mathrm{I}$ becomes $\mathrm{S}^1 \times \mathrm{S}^1$. Hence A_2 is homotopic to $\mathrm{S}^1 \vee \mathrm{S}^1$.

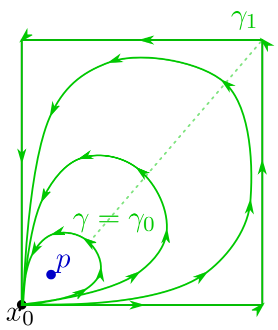
Finally note that the intersection $A_1 \cap A_2$ is homeomorphic to a punctured ball B^2, hence homotopic to S^1. The generator $[\gamma]$ of $\pi_1(A_1 \cap A_2, x_0)$ is indictated in the following picture as the homotopy class of the path γ that goes from x_0 and goes "around" p inside A_1



Since A_1, A_2, and $A_1 \cap A_2$ are all path-connected we can apply the van Kampen theorem and obtain

$$\pi_1(X, x_0) \cong \pi_1(A_1, x_0) * \pi_1(A_2, x_0)/N \cong \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)/\left\langle [\gamma'] \right\rangle,$$

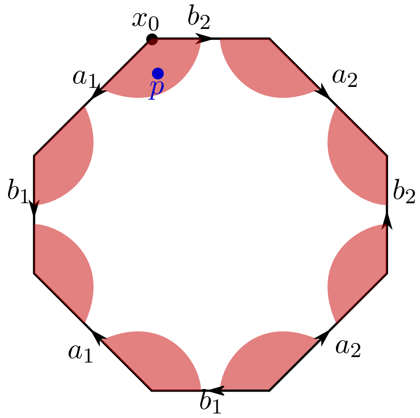
where the second isomorphism comes from the fact that A_1 is contractible, A_2 is homotopic to $\mathrm{S}^1 \vee \mathrm{S}^1$, and γ' is the image of the generating loop pictured above in $\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)$.
If we deformation retract $\mathrm{I} \times \mathrm{I} \setminus \{p\}$ to the boundary, we have a homotopy F between the identity of $\mathrm{I} \times \mathrm{I} \setminus \{p\}$ and the retraction to the boundary. Looking at $\gamma_t = F(-, t) \circ \gamma$ for $t \in \mathrm{I}$ we get a family of paths that look as follows



where $\gamma_0 = \gamma$ and γ_1 is the image of γ under the deformation retraction on the boundary and for t moving from 0 to 1 we get a family of paths that gets closer and closer to the boundary until at $t = 1$ the path is contained in the boundary. Hence in the quotient, we see that the deformation retraction maps $[\gamma]$ to the element $[\gamma'] = aba^{-1}b^{-1}$ if we denote the generating elements of $\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)$ as a and b. Thus we get

$$\pi_1(X, x_0) \cong \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)/\langle [\gamma] \rangle = \mathbb{Z} * \mathbb{Z}/\left\langle aba^{-1}b^{-1} \right\rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

(c) If we look at even higher genus, one can deduce the fundamental group for the general case of $X = M_g$, for $g \geq 1$. For this we use the construction of M_g as a quotient defined in the course. The proof is nearly the same as in the torus case. We give the picture here for M_2



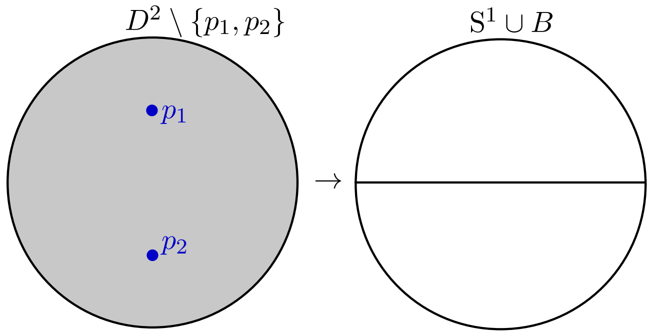
Again the red part being A_1, while A_2 is $M_2 \setminus \{p\}$. For general M_g we always get that A_1 is homeomorphic to an open ball and hence contractible, while A_2 is homotopic to the boundary in the quotient, which is $\bigvee_{i=1}^{2g} \mathrm{S}^1$. Finally the subgroup of N is generated by $a_1 b_1 a_1^{-1} b_2^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Note that the polygon for M_g has $4g$ edges. Then start at any corner and choose an orientation to go around the polygon. The occurring the edges are, in order, $a_1, b_1, a_1^{-1}, b_2^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$, where the inverse just indicates that the edge has on orientation opposite to the one chosen. This can be seen above for the cases $g = 1$ and $g = 2$.

Example 1.52: Removing a "small" subspace from $\mathbb{R}^3$

The next example we look at is a bit more involved and shows that one sometimes has to first pass to a space that is homotopic to the original to use the van Kampen theorem.
For this let $A = \{(x/2, y/2, 0) \mid (x, y) \in \mathbb{S}^1\}$ be a circle in the (x, y)-plane of $\mathbb{R}^3$. Then we look at the space $X = \mathbb{R}^3 \setminus A$. There is no obvious candidates for subsets to use for the van Kampen theorem, hence we first look for a homotopic space.

Step 1: First note that $A \subset B^3$, especially it is fully contained inside D^3. Thus we can apply the deformation retraction from $\mathbb{R}^3 \setminus B^3$ onto S^2 to our space $\mathbb{R}^3 \setminus A$ by extending it with the identity on D^3. Then $\mathbb{R}^3 \setminus A$ deformation retracts to $D^3 \setminus A$.

Step 2: Now we claim that $D^3 \setminus A$ deformation retracts to $\mathrm{S}^2 \cup B$ where $B = \{(0, 0, z) \mid z \in [-1, 1]\}$. For this we look at the intersection of $D^3 \setminus A$ with the (x, z)-plane. This gives a 2-disc with two points removed, i.e. the following picture to the left



where the two points are the two intersections of A with the (x, z)-plane. Note that if we cut this picture along the z-axis, we really look at two punctured discs that both deformation retract to a circle, hence the intersection with the (x, y)-plane deformation retracts to the picture above on the right, i.e. a circle union with B. Fix this deformation retract and apply it to all rotations of the (x, y)-plane around the z-axis and we get a deformation retraction of $D^3 \setminus A$ onto $\mathrm{S}^2 \cup B$.

Step 3: For the final step, consider $C^+ = \{(x, y, z) \mid (x, y, z) \in \mathrm{S}^2, x \geq 0\}$ and $C^- = \{(x, y, z) \mid (x, y, z) \in \mathrm{S}^2, x \leq 0\}$, hence the half spheres above and below the (y, z)-plane. In particular $\mathrm{S}^2 = (C^+ \coprod C^-)/_\sim$ where the equivalence relation is given by identifying the points with x coordinate 0 that lie in both C^+ and C^-. Note that C^+ is homeomorphic to a disc D^2, hence deformation retracts to a point. Thus we obtain a homotopy equivalence between $\mathrm{S}^2 \cup B$ and $(C^- \cup B)/_\sim$ where the equivalence relation is given by identifying all points in C^- with x coordinate 0. It is fairly obvious that $(C^- \cup B)/_\sim \cong \mathrm{S}^2 \vee \mathrm{S}^1$, where the S^2 comes from identifying all the boundary points of C^-, while the S^1 comes from identifying the two points in B that also lie in C^-.

Hence in total we obtain that $\pi_1(X) \cong \pi_1(\mathrm{S}^2 \vee \mathrm{S}^1) \cong \mathbb{Z}$. Where the final isomorphism comes from our application of the van Kampen theorem in Example 1.50.

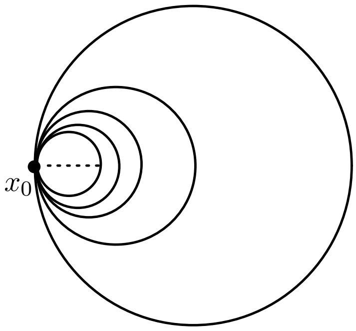
Remark: This type of example can be modified to instead of one circle, removing two circles from $\mathbb{R}^3$. With very similar steps one obtains $\mathbb{Z} * \mathbb{Z}$ if the two circles are not linked and $\mathbb{Z} \times \mathbb{Z}$ if the two circles are linked.

Example 1.53: An infinite union of circles

The following is a non-example for the van Kampen theorem. We take

$$X = \bigcup_{n\geq 1} \left(\left(\frac{1}{n}, 0 \right) + \frac{1}{n} \mathrm{S}^1 \right)$$

the union of infinitely many circles that all intersect in the origin of $\mathbb{R}^2$. In a picture this looks as follows



We use the subspace topology of $\mathbb{R}^2$ and choose as a base point $x_0 = (0, 0)$. If we want to apply the van Kampen theorem we have to find subsets A_α that are open and each contain x_0. But an open subset of X will always contain $B_\varepsilon(0, 0) \cap X$ for sufficiently small ε by definition of the subspace topology. But for any ε, $B_\varepsilon(0, 0) \cap X$ will contain $(\frac{1}{n}, 0) + \frac{1}{n}\mathrm{S}^1$ for sufficiently big n. Hence any choice of A_α will have a fundamental group that is at least as difficult to compute as the one for X itself since it contains infinitely many of the circles used to define X in the first place.

The point here is that X is not homeomorphic to $\bigvee_{i=1}^{\infty} \mathrm{S}^1$ because we used the subspace topology of $\mathbb{R}^2$. There is a bijection between the two, but not a homeomorphisms. Thus, although we can apply the van Kampen theorem to $\bigvee_{i=1}^{\infty} \mathrm{S}^1$ by using Example 1.50, we cannot apply it to X.

Example 1.54: Graphs as topological spaces

For the final example we look at a connected graph X seen as a topological space. For this let E be the set of edges of X, V the set of vertices of X, and for an edge $e \in E$ we denote by $s(e)$ and $t(e)$ the starting and ending vertex of the edge. For each $e \in E$ we fix $\mathrm{I}_e = \mathrm{I}$ a copy of the unit interval and for each $v \in V$ we fix a one pointed space $\{x_v\}$. Then we give X the structure of a topological space as follows

$$X = \left(\coprod_{e \in E} \mathrm{I}_e \right) \coprod \left(\coprod_{v \in V} \{x_v\} \right) /\sim$$

where the equivalence relation is given by $0 \sim x_{s(e)}$ and $1 \sim x_{t(e)}$ for $0, 1 \in \mathrm{I}_e$. Thus we glue together the intervals according to the start and endpoint of the edges. For simplicity we assume that X has finitely many edges, but this assumption can be relaxed.
Now we fix $T \subset E$ a maximal tree, i.e. a connected subgraph of X that does not contain any cycles and is maximal with this property. Note that such a choice is not unique. Thus $T \cup e$ for $e \in E \setminus T$ will contain a unique cycle. We note a few basic statements that one checks easily (especially with the assumption that X is a finite graph)

- A collection of edges $E' \subset E$ defined a subspace $X_{E'}$.
- If the collection of edges is a tree T, then X_T is contractible.
- If the collection of edges E' contains a unique cycle, then $X_{E'}$ is deformation retracts to S^1.

We fix $x_0 \in X_T$ an arbitrary basepoint. For each $\alpha \in E \setminus T$ we define an open subset as the image of

$$\left(\coprod_{e \in T \cup \{\alpha\}} \mathrm{I}_e \right) \coprod \left(\coprod_{e \in E \setminus T, e \neq \alpha} \mathrm{I}_e \setminus \left\{ \frac{1}{2} \right\} \right).$$

in the quotient, i.e. for each edge in $T \cup \{\alpha\}$ we take the whole interval and for all other intervals we take the interval minus the point $\frac{1}{2}$. Then A_α is path-connected and deformation retracts to the subspace $X_{T \cup \{\alpha\}}$, which in turn deformation retracts to S^1. Furthermore, for $\alpha \neq \beta$, we see that $A_\alpha \cap A_\beta$ is path-connected and deformation retracts to X_T, which in turn deformation retracts to a point. The same also holds for any triple intersections.
Hence in total we can apply the van Kampen theorem and obtain

$$\pi_1(X) \cong *_{\alpha \notin T} \pi_1(A_\alpha) \cong *_{\alpha \notin T} \mathbb{Z}.$$

So we get one copy of $\mathbb{Z}$ for each edge outside of T. This can be applied to any graph that possess a maximal tree even if it is not finite.

Remark: Note that an immediate corollary of this example is that for a finite graph any maximal tree has the same cardinality of edges. Since for our choice T, we obtain that $\pi_1(X)^{\mathrm{ab}} \cong \mathbb{Z}^{|E \setminus T|}$, where $\pi_1(X)^{\mathrm{ab}}$ is $\pi_1(X)$ modulo its commutator subgroup. Since the fundamental group is independent of our choice of T, the power $|E \setminus T|$ needs to be independent of T.