II) Easy examples:

1) $M = \mathbb{C}$ the trivial repr. of $\mathcal{U}(\mathfrak{gl}_n)$.

$\mathcal{M}$ = category of f. dim'l $\mathbb{C}$-v. spaces.

$\mathcal{E}_i, \mathcal{F}_i$ are all the zero functor.

$$[\mathcal{M}] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C} \simeq M$$

$[\mathcal{E}_i] = [\mathcal{F}_i] = 0 \quad \forall i$ is the correct action

and all isomorphisms of functors are satisfied.

2) $M = \mathbb{C}^n$ the natural (or defining) repr. of $\mathcal{U}(\mathfrak{gl}_n)$.

weight spaces of M are $M_\mu = \begin{cases} \mathbb{C} & \text{if } \mu = \tau_i \text{ for some } i \\ \{0\} & \text{otherwise.} \end{cases}$

Set $\mathcal{M}_{\tau_i}$ = f. dim'l $\mathbb{C}$-v. spaces

$\mathcal{M}_\mu = \{0\}$ if $\mu \neq \tau_i \ \forall i$.

$$\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_{\tau_i}$$

$$\mathcal{E}_i|_{\mathcal{M}_{\tau_{i+1}}} : \mathbb{C} \longmapsto \underset{\mathcal{M}_{\tau_i}}{\underset{\cap}{\mathbb{C}}} \qquad \mathcal{E}_i|_{\mathcal{M}_{\tau_j}} = 0 \quad \text{if } j \neq i+1$$

$$\mathcal{F}_i|_{\mathcal{M}_{\tau_i}} : \mathbb{C} \longmapsto \underset{\mathcal{M}_{\tau_{i+1}}}{\underset{\cap}{\mathbb{C}}} \qquad \mathcal{F}_i|_{\mathcal{M}_{\tau_j}} = 0 \quad \text{if } j \neq i.$$

Let $M = \langle v_1, \ldots, v_n \rangle_{\mathbb{C}}$ s.t. $M_{\tau_i} = \langle v_i \rangle_{\mathbb{C}}$

the standard basis of $\mathbb{C}^n$.

$$\Rightarrow \quad [M_{\tau_i}] \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} M_{\tau_i}$$
$$[\mathbb{C}] \longmapsto v_i$$

<u>Check:</u> The $[\mathcal{E}_i], [\mathcal{F}_i]$ give exactly the action of elementary matrices

$$[\mathcal{E}_i] \longleftrightarrow E_{i,i+1}$$
$$[\mathcal{F}_i] \longleftrightarrow E_{i+1,i}$$

<u>Check:</u> Isomorphisms of functors. Most of the compositions in part c) are zero in this case (e.g. all Serre relations are zero on both sides thus trivial)

$$\mathcal{E}_i \circ \mathcal{F}_i |_{M_{\tau_i}} : \begin{array}{ccccc} \mathbb{C} & \xmapsto{\mathcal{F}_i} & \mathbb{C} & \xmapsto{\mathcal{E}_i} & \mathbb{C} \\ & & \cap & & \cap \\ & & M_{\tau_{i+1}} & & M_{\tau_i} \end{array}$$

$$\Rightarrow \quad \mathcal{E}_i \circ \mathcal{F}_i |_{M_{\tau_i}} \simeq \mathrm{Id}_{M_{\tau_i}}^{\oplus 1}$$

(since $\mu_i - \mu_{i+1} = 1$ for $\mu = \tau_i$)

and similar for other cases.

<u>Remark:</u> After this it gets too complicated to just use vector spaces anymore.

Even if one tries to categorify

$$\boxed{M = S^2\mathbb{C}^2 \quad \text{as a } gl_2\text{-module}}$$

one cannot find a categorification of the form:

$$\mathcal{M} = \mathcal{M}_{2\tau_1} \oplus \mathcal{M}_{\tau_1-\tau_2} \oplus \mathcal{M}_{-2\tau_2}$$

where $\mathcal{M}_{2\tau_1}$ = f. dim'l v. spaces

$\mathcal{M}_{\tau_1-\tau_2}$ = —— " ——

$\mathcal{M}_{-2\tau_2}$ = —— " ——

even though all weight spaces of M are 1-dim'l.

<u>Exercise:</u> One can get a categorification if one puts

$$\mathcal{M}_{\tau_1-\tau_2} = \mathbb{C}[x]/(x^2)\text{-mod}^{\text{f. dim'l.}}$$

and defines more interesting functors $\mathcal{E}_1$ and $\mathcal{F}_1$