George Mason University

Frequency Domain Analysis and Design Of Control Systems

Report from laboratory Experiment 3 conducted on 9 February, 2016
As part of ECE 429 Control Systems Lab
Course Instructor: Dr. Daniel M. Lofaro

Michael Kepler
G00804828

16 February, 2016

The Volgenau School of Engineering

Objective

Experiment

The system plant is defined as:

$$G_p(s) = \frac{s+20}{s(s+0.4)(s+5)}$$

and requires the following closed-loop specifications:

$$Steady\ State\ Error: \quad (E_{ss}) \leq 0.01$$

$$Phase\ Margin: \quad (PM) \geq 40\ degrees$$

$$Unity\ Gain\ Crossover\ Frequency\ Range: \quad 1 \leq \omega_{cg} \leq 2\ r/s$$

$$Compensated\ shift\ must\ be\ greater\ than - 180\ for : 0.1 \leq \omega \leq 4\ r/s$$

A lead-lead, lag-lag, or lead-lag compensator must be designed to satisfy the above specifications. The restraints for any given stage of compensation are:

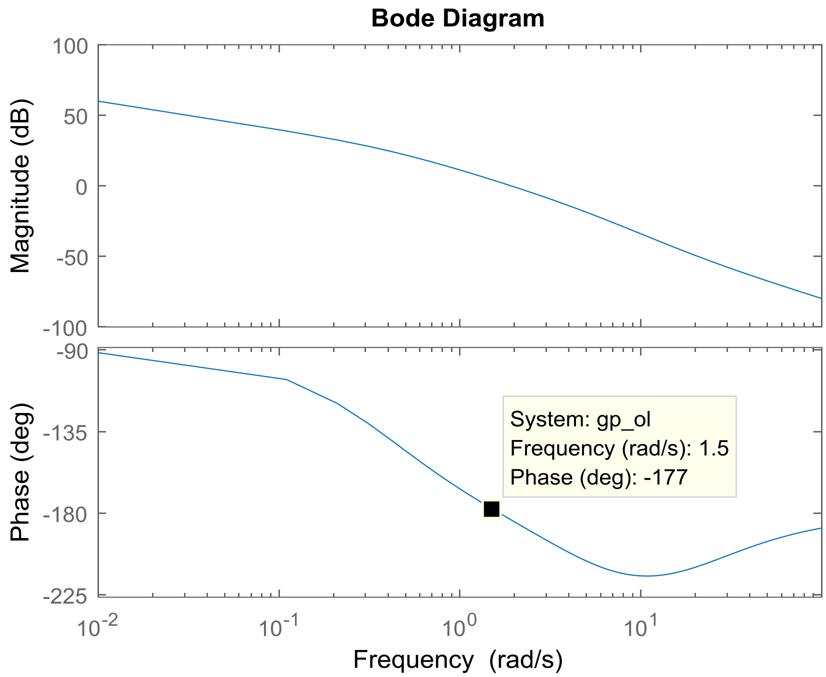
$$\alpha_{lead} = \frac{z_{cd}}{p_{cd}} > 0.05\ and$$

$$\alpha_{lag} = \frac{z_{cg}}{p_{cg}} < 20$$

Obtain the phase at the desired gain crossover frequency ($\omega_{cg} = 1.5$):

```
%Define transfer function
s= tf('s')
k=1;

wgc = 1.5;    %rad/s
pm = 40;      %degrees

%Plant function (gp)
gp_ol = k*(s+20)/(s*(s+0.4)*(s+5));
gp_cl = feedback(gp_ol,1)

figure(1)
bode(gp_ol, 0.01:0.1:100)
[mag, phase] = bode(gp_ol, 1.5)
```

$$\phi = \angle G_p(j\omega_{cg}) = -177.4787$$



Find the necessary phase compensation to achieve a phase margin of 40 degrees (incorporation of a safety factor of 10 degrees):

$$\phi_c = (PM + 10) - (180 + \phi)$$
$$\phi_c = 40 + 10 - (180 - 177.47)$$
$$\phi_c = 47.47$$

$$\boldsymbol{\phi_c > \ 0 \rightarrow Lead\ Compensator}$$
$$\phi_c < \ 0 \rightarrow Lag\ Compensator$$

The lead compensator will be of the form:

$$G_c = \frac{K_c\left(\frac{s}{z_C} + 1\right)}{\left(\frac{s}{P_c} + 1\right)}$$

Find the gain of the lead compensator:

$$K_C = \frac{K_{desired}}{K_p}$$

Where $K_{desired}$ is:

$$K_{desired} = \frac{1}{e_{ss}} = \frac{1}{0.01} = 100$$

And K_p is:

$$K_p = \lim_{s \to 0} G_p(s)$$

$$K_p = \lim_{s\to 0} \frac{s+20}{s(s+0.4)(s+5)} = \frac{20}{0.4*5} = 10$$

Plugging in $K_{desired}$ and K_p, K_c is obtained:

$$K_C = \frac{100}{10} = 10$$

Finding the poles and zeros:

$$z_c = \omega_{gc}\sqrt{\alpha_1}$$
$$p_c = \frac{z_c}{\alpha_1}$$

Where $\alpha_1$is:

$$\alpha_1 = \frac{1-\sin(\phi_c)}{1+\sin(\phi_c)} = 2.0268$$

Plugging in $\alpha_1$to obtain z_c and p_c:

$$z_c = 1.5\sqrt{2.0268} = 0.5836$$
$$p_c = \frac{0.5836}{2.0268} = 3.8551$$

Resulting in the following lead compensator:

$$G_{cLead1} = \frac{10\left(\frac{s}{0.5836}+1\right)}{\frac{s}{3.8551}+1}$$

Thus the compensated system is:

$$G(s) = G_{cLead1}(s)G_p(s)$$

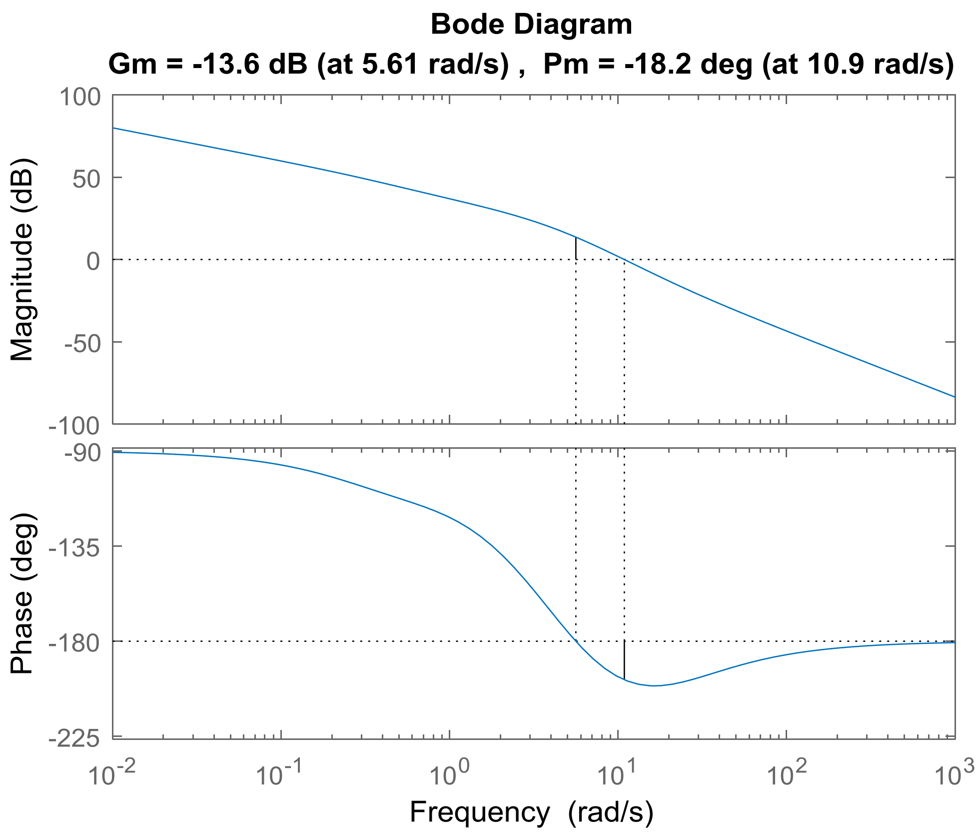
$$G(s) = \frac{10\left(\frac{s}{0.5836}+1\right)}{\frac{s}{3.8551}+1} * \frac{s+20}{s(s+0.4)(s+5)}$$

Using MATLAB to obtain information about the system:

```
%Define transfer function
s= tf('s')
k=1;

%Plant function (gp)
gp_ol = k*(s+20)/(s*(s+0.4)*(s+5));
```

```
%Lead Compensator
gclead1=10*(s/0.5836 + 1)/(s/3.8551+1)

%Compensated System 1
sys_comp1_ol = gp_ol * gclead1

wgc = 1.5;    %rad/s
[gain, phase] = bode(sys_comp1_ol,wgc)

margin(sys_comp1_ol
```

This is the Bode Diagram of the compensated system:



The phase at the crossover frequency ($\omega_{gc} = 1.5\ rad/s$), is:

$$\phi = \angle G_{comp1}(j\omega_{cg}) = -129.9988$$

Yielding a satisfactory phase margin of:

$$PM = 180 - 129.9988 = 50.0012$$

The gain of the system at the crossover frequency ($\omega_{gc} = 1.5\ rad/s$), is:

$$Gain = |G_{cLead1}(j\omega_{gc})||G_p(j\omega_{gc})| = 42.4069$$

Thus, without affecting the phase at that frequency, a compensator must be designed so that the magnitude is unity at the desired crossover frequency, ($\omega_{gc} = 1.5\ rad/s$):

Defining α as:

$$\alpha_{comp2} \triangleq Gain\ to\ be\ reduced$$

A compensator needs to be designed to achieve unity gain at the desired crossover frequency:

$$G_c = \frac{\frac{s}{z_c} + 1}{\frac{s}{p_c} + 1}$$

But given the constraint on any stage of a lag compensator:

$$\alpha_{lag} = \frac{z_{cg}}{p_{cg}} < 20$$

A 2-stage lag compensator will be required since

$$\alpha_{stage} = \sqrt[n]{\alpha_{lag_{total}}}$$

Where n is the number of stages.

Resulting in the following α_{stage} :

$$\alpha_{stage} = \sqrt[2]{42.4069} = 6.5115$$

Let the zero of the compensator be 50x closer to the origin than ω_{gc} :

$$z_c = \frac{\omega_{gc}}{10} = \frac{1.5}{50} = 0.03$$

Then using z_c and α_{stage}, the pole, p_c, can be calculated:

$$p_c = \frac{z_c}{\alpha_{stage}} = \frac{0.15}{6.5115} = 0.0046$$

Resulting in the following two-stage compensator:

$$G_{cLag} = \left(\frac{\frac{s}{0.03}+1}{\frac{s}{0.0046}+1}\right)^2$$

Given the compensated system of:

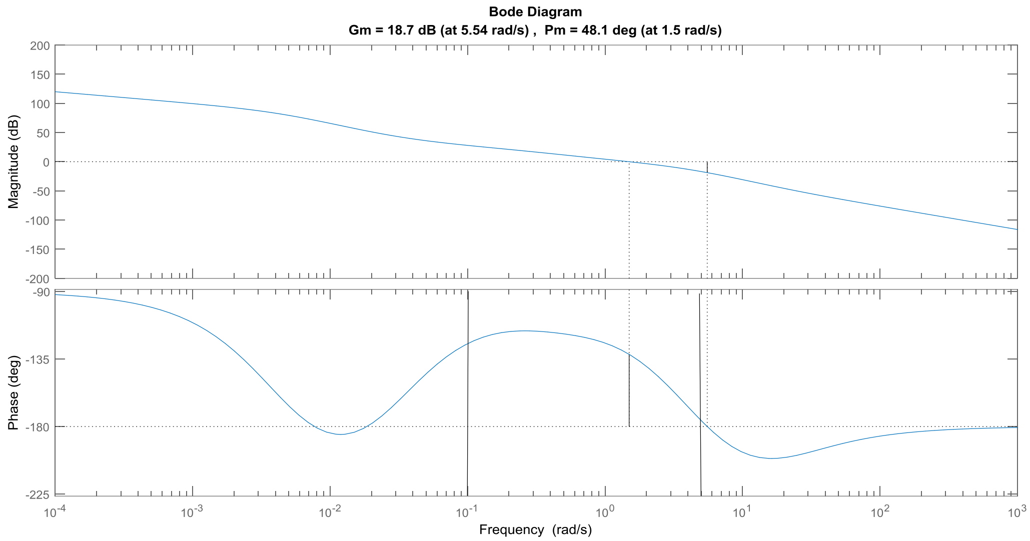
$$G(s) = G_{cLag}(s)G_{cLead1}(s)G_p(s)$$

$$G(s) = \left(\frac{\frac{s}{0.03}+1}{\frac{s}{0.0046}+1}\right)^2 \left(\frac{10\left(\frac{s}{0.5836}+1\right)}{\frac{s}{3.8551}+1}\right)\left(\frac{s+20}{s(s+0.4)(s+5)}\right)$$

Using MATLAB to obtain information about the system:

```
%Define transfer function
s= tf('s')
k=1;

%Plant function (gp)
gp_ol = k*(s+20)/(s*(s+0.4)*(s+5));

%Lead Compensator
gclead1= 10*(s/0.5836 + 1)/(s/3.8551+1)

%Lag Compensator
gclag = ((s/0.03 + 1)/(s/0.0046+1))^2
%Compensated System 1
sys_comp1_ol = gp_ol * gclead1 *gclag

wgc = 1.5;   %rad/s
[gain, phase] = bode(sys_comp1_ol,wgc)

margin(sys_comp1_ol)
```

This is the Bode Diagram of the compensated system:



The phase at the crossover frequency ($\omega_{gc} = 1.5\ rad/s$), is:

$$\phi = \angle G_{comp1}(j\omega_{cg}) = -131.9389$$

Yielding a satisfactory phase margin (> 40) of:

$$PM = 180 - 131.9389 = 48.0611$$

As well as, achieving a unity gain at the desired crossover frequency ($\omega_{gc} = 1.5\ rad/s$):

$$Gain = |G_{cLead1}(j\omega_{gc})||G_{cLag}(j\omega_{gc})||G_p(j\omega_{gc})| = 0.9974 \approx 1$$

Also note that the phase margin is less negative than -180 for all frequencies between 0.1 r/s and 4 r/s, as outlined on the Bode diagram.

The steady state-error of the system can be approximated by:

$$E_{ss} = \frac{1}{K_V}$$

Where K_V is:

$$K_V = \lim_{s \to 0} s\ G_{cLead1}(s) G_{cLag}(s) G_p(s)$$

$$K_v = \lim_{s \to 0} s \left(\frac{\frac{s}{0.03} + 1}{\frac{s}{0.0046} + 1}\right)^2 \left(\frac{10\left(\frac{s}{0.5836} + 1\right)}{\frac{s}{3.8551} + 1}\right)\left(\frac{s + 20}{s(s + 0.4)(s + 5)}\right)$$

$$k_v = \frac{(10)(20)}{(0.4)(5)} = 100$$

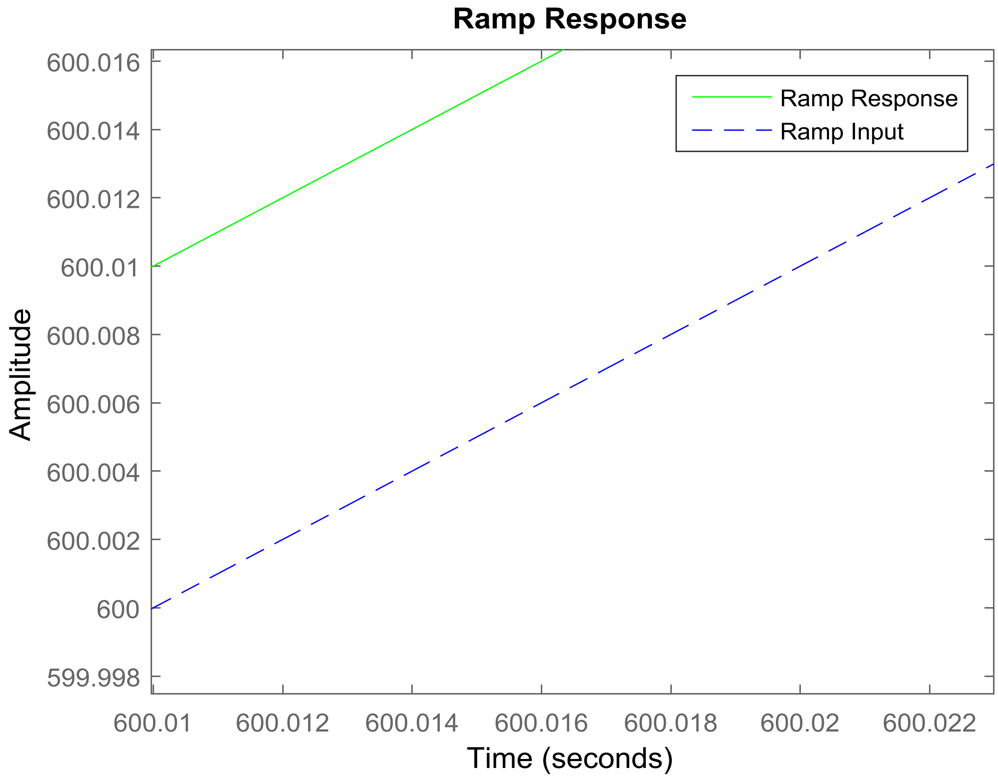
Plugging in K_V, we obtain an approximation for E_{ss}:

$$E_{ss} = \frac{1}{100} = 0.01$$

Which satisfies the overall system requirement of $E_{ss} \leq 0.01$.

The following is the MATLAB simulation used to verify the steady-state error for a ramp input:

```
%Define transfer function
s= tf('s')
k=1;

%Plant function (gp)
gp_ol = k*(s+20)/(s*(s+0.4)*(s+5));

%Lead Compensator
gclead1= 10*(s/0.5836 + 1)/(s/3.8551+1)

%Lag Compensator
gclag = ((s/0.03 + 1)/(s/0.0046+1))^2

%Compensated System 1
sys_comp1_ol = gp_ol * gclead1 *gclag
sys_comp1_cl = feedback(sys_comp1_ol,1)

t=0:0.001:2000;
plot(t,t,'g')
hold on
step(sys_comp1_cl/s,t,'--b')
title('Ramp Response')
legend('Ramp Response', 'Ramp Input')
```



Apparent from the above ramp response, the system meets the steady-state error requirements.

Conclusion

The design of the system was done primarily with respect to the frequency domain. The use of Bode diagrams facilitates designing for crossover frequency, phase margin, and gain margin. In summary a lead compensator was initially used to achieve the proper phase margin. After viewing the Bode diagram of the compensated system, it was apparent that the gain needed to be reduced. Since there were constraints on the compensators, a two-stage lag compensator needed to be implemented. Ultimately all the system requirements met:

1. **Steady-state error <= 0.01**: Steady-state error = 0.01
2. **PM > 40**: PM = 48.0611
3. **1 r/s =< Crossover frequency =< 2 r/s**: ~1.5 r/s
4. **Phase shift > -180 for (0.1r/s<w<4r/s):** Verified from the Bode diagram
5. $\boldsymbol{\alpha_{lead} > 0.05}$: $\alpha_{lead} = 0.1513$
6. $\boldsymbol{\alpha_{lag} < 20}$: $\alpha_{lag} = 6.5115$