Variational Inference Univariate Normal

Univariate Normal Model

$$
\begin{aligned}
y_i &\sim N(\mu, \sigma^2) \\
\mu &\sim N(\mu_0, \sigma_0^2) \\
\sigma^2 &\sim \text{inv-gamma}(\alpha_0, \beta_0)
\end{aligned}
$$

The posterior distribution we are interested in is given by

$$
p(\mu, \sigma^2 | y_{1:N}) \propto \prod_{i=1}^{N} p(y_i | \mu, \sigma^2) p(\mu) p(\sigma^2).
$$

Mean-Field Variational Family

Assume the family of approximate densities to $p(\mu, \sigma^2)$ has the form,

$$
q(\mu, \sigma^2) = q_\mu(\mu) q_{\sigma^2}(\sigma^2)
$$

Then the optimal densities are

$$
q_\mu^*(\mu) \propto \exp\Big\{ E_{\sigma^2} \log \prod_{i=1}^{N} p(y_i | \mu, \sigma^2) p(\mu) \Big\}.
$$

Which is,

$$
\begin{aligned}
q_\mu^*(\mu) &\sim N(\mu_{q_\mu(\mu)}, \text{Var}_\mu(\mu)) \\
\text{Var}_{q_\mu(\mu)} &= \frac{N}{E_{\sigma^2}(\sigma^2)} + \frac{1}{\sigma_0^2} \\
\mu_{q_\mu(\mu)} &= \text{Var}_\mu(\mu) \Big(\frac{\sum y_i}{E_{\sigma^2}(\sigma^2)} + \frac{\mu_0}{\sigma_0^2} \Big).
\end{aligned}
$$

And

$$
q_{\sigma^2}^*(\sigma^2) \propto \exp\Big\{ E_\mu \log \prod_{i=1}^{N} p(y_i | \mu, \sigma^2) p(\sigma^2) \Big\}.
$$

Which is,

$$
\begin{aligned}
q_{\sigma^2}^*(\sigma^2) &\sim \text{inv-gamma}\Big(A_{q(\sigma^2)}, B_{q(\sigma^2)} \Big) \\
A_{q(\sigma^2)} &= \alpha_0 + \frac{N}{2} \\
B_{q(\sigma^2)} &= \beta_0 + \frac{1}{2} \sum_{i=1}^{N} E_\mu (y_i - \mu)^2 = \beta_0 + \frac{1}{2} \Big(\sum_{i=1}^{N} (y_i - E_\mu(\mu))^2 - N \text{Var}_{q(\mu)}(\mu) \Big)
\end{aligned}
$$