Evaluating z-domain transfer functions on the unit circle for Bode and Nyquist plots

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Some people have asked me if there is a faster way to draw z-domain Bode and Nyquist plots by hand. I'm told that you will not be asked to draw these by hand on the exam, but you may be asked to reason about them or to match a transfer function with a plot provided.

Here is my strategy: To evaluate a z-domain transfer function on the unit circle, consider these tricks

1. Recognize familiar forms
2. Check some easy points: ± 1, i, etc.
3. Break down into a product of simpler or familiar components. In log-magnitude and phase, product of components becomes addition (easy!)
4. Look for convenient structure and symmetries in the algebra. $2\cos\theta = e^{i\theta} + e^{-i\theta}$ is handy!
5. Get the poles and zeros, and look for convenient geometric relationships (draw some triangles)
6. If that fails, note that:
   - Phase is the sum/difference of the angles to zeros/poles
   - Magnitude is the product of the distance (inverse distance) to zeros (poles)
   - These can be solved algebraically (but look for shortcuts in the algebra and geometry)

0.0.1 1. Recognize familiar forms

Is the transfer function one you've already seen in the example sheets or past exams? Can it be decomposed into a product of familiar functions? Is it a known function multiplied by additional poles/zeros at zero (e.g. times z^{-1})? Avoiding work by pattern matching is nice (:

0.0.2 2. Evaluate easy points first

The points $z = 1 = e^{0i}$, and $z = -1 = e^{\pi i}$ are easy to compute by hand. These tell you the phase and the magnitude of the DC component and the Nyquist frequency, respectively. The point $z = i = e^{\frac{\pi}{2}i}$ is also quick to evaluate, and may help in sketching the overall curve.

0.0.3 3. Look for familiar pieces

Problems will often have some structure and symmetries that might help reduce a transfer function to simpler, known functions, and more familiar filters, etc. Remember, we're trying to get the phase and the magnitude, which are easy to combine by multiplication. If we know the phase and magnitude for some sub-components, we can estimate the overall phase and magnitude easily.

0.0.4 4. Is there something convenient in the algebra?

In the example sheets, we saw that the relationship $2\cos\theta = e^{i\theta} + e^{-i\theta}$ is quite useful for separating phase and magnitude components! Can you re-arrange the equation, pull out factors of $e^{\theta/2}$, etc, to make something more familiar?

0.0.5 5. Geometric intuition: draw the poles and zeros in the complex plane

Sometimes, a geometric intuition is useful. Recall that the phase, as we move z along the unit circle $z = e^{i\theta}$, is given by the sum of all phase contributions from poles and zeros. Similarly, the magnitude is the product of the distances to the zeros and 1/distances to the poles. Sometimes, nice geometric relationships appear that make it quick to solve for the phase and magnitude. You may be able to solve for certain values by looking at the angles triangles made by drawing vectors between zeros, poles, the origin, and an example point on the unit circle.

0.0.6 Evaluate algebraically

If all else fails, its possible to "roll up our sleeves" and evaulate the transfer function by hand by breaking the transfer function into zeros and poles, and considering the contribution of each. Look for algebra tricks and geometric intuition to accelerate this, along the way, however.

Z-domain transfer functions are a product of simpler poles and zeros For plotting the Bode and Nyquist plots, we evaluate the z-transform along the unit circle $z = e^{i\theta}$. We are interested in getting the phase (ϕ) and magnitude (ρ) of the z-transform evaluated at these points, i.e. for $Y(e^{i\theta}) = \rho e^{i\phi}$ A given transfer function can be decomposed into a product of poles and zeros:

$$Y(z) = \frac{\prod_l (1 - q_l z^{-1})}{\prod_m (1 - p_m z^{-1})}$$

Multiplying complex numbers in polar form is easy Note that multiplying complex numbers in polar coordinates is fairly straightforward

$$\rho_1 e^{i\phi_1} \cdot \rho_2 e^{i\phi_2} = (\rho_1 \rho_2) e^{i(\phi_1 + \phi_2)}$$

It's even easier if we consider log-magnitude $\log(\rho)$, which is used in a typical Bode plot, since the multiplication of the magnitudes becomes addition:

$$e^{\ln \rho_1} e^{i\phi_1} \cdot e^{\ln \rho_2} e^{i\phi_2} = e^{\ln \rho_1 + \ln \rho_2} e^{i(\phi_1 + \phi_2)}$$

So! If we can break a complicated z-transform into its zeros and poles, we can figure out the overall phase by summing up each phase and log-magnitude contribution. If we were drawing a Bode plot, we could draw the plot of each zero and pole separately in the log-magnitude and phase plots, and then sum them up to get the final answer.

Therefore, it is enough to consider the behavior of a single zero or pole along the circle $z = e^{i\theta}$. If we can predict the phase and log-magnitude of this, we can handle any general z-domain transfer function.

Poles are just 1/zero Inverting complex numbers is simple in polar coordinates:

$$\frac{1}{\rho e^{i\phi}} = \frac{1}{\rho}e^{-i\phi}$$

Since a pole is, conceptually, the inverse of a zero, we can consider just the case of finding the Bode or Nyquist diagram for zeros. In the case of poles, we can consider the $(1 - pz^{-1})$ term as if it were a zero, then invert it (as above) to get its contribution to the overall plot.

So! We've reduced the problem to quickly determining the Bode/Nyquist diagrams for a single term, $(1 - qz^{-1})$.

Evaluating $1 - qz^{-1}$ along the unit circle $z = e^{i\phi}$ Is there a simple way to evaluate $1 - qz^{-1}$ along the unit circle $z = e^{i\phi}$? For me, I find it simpler to re-write the expression $1 - qz^{-1}$ as $(z - q)z^{-1}$.

$$1 - qe^{-i\theta} = (e^{i\theta} - q) \cdot e^{i\theta}$$

The term $z^{-1} = e^{-i\theta}$ is already in polar form, so we just need to find a polar form for $z - q$. This is a difference between two complex numbers.

The distance between two complex numbers The magnitude of $z - q$ is just the distance between $z = e^{i\theta}$ and $q = re^{i\psi}$. It might be easier to solve this by examining the geometry of the pole-zero plot. If that fails, it can be calculated by hand. The usual Cartesian distance formula may be useful. Distance can also be calculated from the polar representation:

$$\begin{aligned}
z &= x_z + iy_z \\
q &= x_q + iy_q \\
|z - q|^2 &= (x_z - x_q)^2 + (y_z - y_q)^2 \\
&= x_z^2 + x_q^2 + y_z^2 + y_q^2 - 2x_z x_q - 2y_z y_q \\
&= |z|^2 + |q|^2 - 2(x_z x_q + y_z y_q) \\
&= 1 + r^2 - 2r[\cos\theta\cos\psi + \sin\theta\sin\psi] \\
&= 1 + r^2 - 2r\cos(\theta - \psi)
\end{aligned}$$

Take the square root of the above gives $|z - q|$.

In general, the squared distance between two complex numbers, in polar coordinates, can be remembered as: the sum of the squared magnitudes, minus twice the product of their magnitudes with the cosine of their phase difference (i.e. twice their dot-product, if these were 2D vectors). It might just be easier to work through the Cartesian form though!

The angle between two complex numbers The phase of $z - q$ is the angle of a line drawn from q to z. The fastest way to do this is to first check if there is some geometric intuition, and see if the angle can be found via trigonometry. Some points like $q = 1$ or $q = i$ might have convenient relationships that bypass difficult math. Also look for convenient structure in the algebra, which might be faster than applying memorized formulas. If all else fails, you can solve directly for the phase:

$$\angle(z-q) = -i \log\left(\frac{z-q}{|z-q|}\right) + 2\pi n, \quad n \in \mathbb{Z}$$

1 In summary:

Try to use the big picture and pattern matching to be as lazy as possible. See if the problem can be reduced to simpler and familiar forms, and see if there are some convenient algebraic tricks or geometric intuition to solve things quickly. If you dive deep into algebra immediately, you might miss some short-cuts. That said, it's always possible to reason about a transfer function in terms of the contributions to the phase and magnitude from each zero and pole, and you can solve for these using some relatively simple formulae.