1. Is this true or false:

$$(n \in \mathcal{N})(n \in \mathcal{N})(3m + 5m = 12)$$

Answer: This is false
Proof: By cases
Case 1: $m = 1, n = 1$: $\quad 3 \times 1 + 5 \times 1 = 8$
Case 2: $m = 2, n = 1$: $\quad 3 \times 2 + 5 \times 1 = 11$
Case 3: $m = 1, n = 2$: $\quad 3 \times 1 + 5 \times 2 = 13$
For all larger values of m or n, $(3m + 5n) > 12$.
So there are no values of m, n giving 12. This concludes the proof.

2. Is the sum of five consecutive integers always divisible by 5.
Answer: This is true.
Proof: Any 5 consecutive integers can be written as

$n - 2, n - 1, n, n + 1, n + 2$ where $n \in \mathcal{Z}$

The sum of these 5 integers is

$$\begin{aligned} &n - 2 + n - 1 + n + n + 1 + n + 2 \\ =&5n + 0 \end{aligned}$$

which is obviously divisible by 5. QED.

3. Is $(n^2 + n + 1)$ always odd for an integer n.
Answer: This is true
Proof:
We can rewrite $(n^2 + n + 1) = n(n + 1) + 1$
Either n is even, or $(n + 1)$ is even.
Therefore $n(n + 1)$ is even.
Therefore $n(n + 1) + 1$ is odd.
This concludes the proof.

4. Prove that every odd natural number is one of $4n + 1$ or $4n + 3$

Proof:
By the division theorem, all natural numbers can be written as one of

$$4n, 4n + 1, 4n + 2, 4n + 3$$

where $n \in \mathcal{Z}$
Both $4n$ and $4n + 2 = 2(2n + 1)$ are even, and $4n + 1$ and $4n + 3$ are odd.
Since this covers all natural numbers, all the odd numbers must be one of $4n + 1$ or $4n + 3$. QED

5. Prove that for any integer n, at least one of $n, n + 2, n + 4$ is divisible by 3.
Proof: By the division theorem, any integer n can be written

$$n = 3q + r$$

where $q \in \mathcal{Z}, r \in \mathcal{Z}$ and $0 \leq r < 3$
Case 1: $r = 0$ then $n = 3q$ and is divisible by 3

Case 2: $r = 1$ then

$$\begin{aligned} n + 2 &= 3q + r + 2 \\ &= 3q + 1 + 2 \\ &= 3q + 3 = 3(q + 1) \end{aligned}$$

which is divisible by 3

Case 3: $r = 2$ then

$$\begin{aligned} n + 4 &= 3q + r + 4 \\ &= 3q + 2 + 4 \\ &= 3q + 6 = 3(q + 2) \end{aligned}$$

which is divisible by 3

Hence for all possible values of r, one of $n, n + 2, n + 4$ is divisible by 3. QED.

6. Prove that the only prime triple is 3,5,7.
Proof: A prime triple is of form $n, n + 2, n + 4$ where all are prime numbers.
We just showed (Q5) that one of $n, n+2, n+4$ must be divisible by 3. Since a prime number is only divisible by itself and 1, all values of the triplet for $n > 3$ must be composite (divisible by 3 and another number). Therefore there are no other prime triplets than 3,5,7. QED.

7. Prove that for any natural number n

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$$

Proof: By induction

We need to show that

$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$$

For $n = 1$, the left hand side $= 2^1 = 2$ and the right hand side is

$$2^{n+1} - 2 = 2^2 - 2 = 4 - 2 = 2$$

So the expression is valid for $n = 1$
Then for $n + 1$,

$$\sum_{k=1}^{n+1} 2^k = \sum_{k=1}^{n} 2^k + 2^{n+1} \quad \text{separating out last term}$$

By induction hypothesis, assuming the expression is valid for n, this is

$$\begin{aligned} \sum_{k=1}^{n+1} 2^k &= 2^{n+1} - 2 + 2^{n+1} \\ &= 2 \times 2^{n+1} - 2 \\ &= 2^{n+2} - 2 \end{aligned}$$

which is the value of the expression for n+1 as required.
Hence, the theorem is proved by mathematical induction.

8. Theorem: If $\{a_n\}_{n=1}^{\infty}$ tends to limit L then for $M > 0$ the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.
Proof:

Let $\epsilon > 0$ be given. If $\{a_n\}$ tends to limit L, we can choose N large enough that

$$|a_N - L| < \frac{\epsilon}{M}$$

Hence, multiplying by M

$$M|a_N - L| < \epsilon$$
$$|Ma_N - ML| < \epsilon$$

By the definition of a limit, this proves the theorem.

9. Give an example of a family of intervals $A_n = 1, 2, ...$ such that $A_{n+1} \subset A_n$ for all n and intersection $= \emptyset$

Answer: The family $A_n = (0, 1/n]$ fulfils these criteria.
Proof:
a) Proof that $A_{n+1} \subset A_n$
Since $(n+1) > n$, it follows that $1/(n+1) < 1/n$
Hence the interval $(0, 1/n]$ includes the interval $(0, 1/(n+1)]$ and thus $A_{n+1} \subset A_n$

b) Proof that intersection is $\emptyset$
Proof by contradiction. If the intersection is not $\emptyset$ then there must be at least one member. Let x be the minimal such member.

By the Archimedean property, there is a $m \in \mathcal{N}$ such that $m > 1/x$ and thus x will not be included in the intervals A_n for $n \geq m$
This is a contradiction, and thus there are no members and the intersection is the empty set.
Thus A_n fulfils the requirements. QED.

10. Give an example of a family of intervals $A_n = 1, 2, ...$ such that $A_{n+1} \subset A_n$ for all n and the intersection has one real member.

Answer: The family $A_n = [0, 1/n]$ fulfils these criteria.
Proof:
This is the same family as Q9, but with the addition of zero to the lower limit.
The proof that $A_{n+1} \subset A_n$ is as Q9.

Proof that the intersection has one real member:
As shown in Q9 , the intersections of intervals excluding 0 has no members. There is no value of n for which $1/n = 0$ thus the interval $[0, 1/n]$ will always include 0 which is the single member of the intersection.
Thus A_n fulfils the requirements. QED.