STAT302: Midterm #1

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1 Number of possible pins

I got both parts of this problem wrong due to some mis-steps while approaching them using (primarily) permutations (and eliminating duplicates) instead of combinations. Each is show separately below.

1.1 Two 0s in the pin

The solution is $\binom{4}{2} \times 9^2 = 486$. The solution given is for 5-spaces in the pin, but the solution is generally $\binom{n}{2} \times 9^{n-2}$ where n is the number of spaces.

The solution I gave on WebWork was as follows

$$\frac{9 \times 4!}{2! \times 2!} + \frac{9 \times 8 \times 4!}{2!}$$

I was (overly) concerned about the ordering of the digits, so I opted to use permutations, and wanted to ensure I eliminated duplicates. There were two cases here, a string like (a) "00XX" and a string like (b) "00XY", leading to the two terms. We always have $4!$ ways of arranging 4 characters. In (a), we eliminate two sets of duplicates and only have 9 choices for X. Likewise in (b), we only have to cancel one set of duplicates and have 9×8 choices for X and Y.

It becomes clear how "close" (or far) I was if we just re-write my solution in terms of $\binom{n}{k}$.

$$\begin{aligned}
\frac{1 \times 1 \times 9 \times 1 \times 4!}{2! \times 2!} + \frac{1 \times 1 \times 9 \times 8 \times 4!}{2!} &= \frac{9 \times 8 \times 4!}{2! \times 2!} + \frac{9 \times 8 \times 4! \times 2!}{2! \times 2!} \\
&= \frac{9 \times 8 \times 4! + 9 \times 8 \times 4! \times 2!}{2! \times 2!} \\
&= \frac{9 \times 4! \times (1 + 8 \times 2!)}{2! \times 2!} \\
&= \frac{9 \times 4! \times 17}{2! \times 2!} \\
&= \frac{4!}{2! \times 2!} \times 9 \times 17 \\
&= \binom{4}{2} \times 9 \times 17 \\
&= 918
\end{aligned} \tag{1}$$

I considered solutions like $\frac{9 \times 9 \times 4!}{2!2!}$ but the magic extra 2! on the bottom for "some other duplicate" (other than "00") made me believe it was incorrect so I scratched it out. The proposed solution I gave was, after some squinting, actually correct everywhere in except one term. (TODO: Why over-counting?)

1.2 1 and 2 in the pin

The solution is $2\binom{4}{2} \times 8^2 = 768$. The solution given is for 5-spaces in the pin, but the solution is generally $2\binom{n}{2} \times 8^{n-2}$ where n is the number of spaces.

The solution I gave on WebWork was as follows

$$\frac{1 \times 1 \times 8 \times 1 \times 4!}{2!} + \frac{1 \times 1 \times 8 \times 7 \times 4!}{1}$$

I was (overly) concerned about the order of the digits and wanted ensure I was handling cases where there were duplicates in the string. In the first case, I imagined we had a string like 12XX where X is some digit other than 0 (and the "X"s are of course equal). There are $4!$ ways to re-arrange these characters and we need to divide by $2!$ to account for "flipping" XX. In the second case, I imagined we had a string like 12XY. This explains the 8×7 part of the term, since we have 8 choices for X and then just 7 for Y. The same logic follows from before, but this time we have no duplicates.

It becomes clear how "close" I was if we just re-write my term in terms of $\binom{n}{k}$.

$$\begin{aligned}
\frac{8 \times 4!}{2!} + \frac{8 \times 7 \times 4!}{1} &= \frac{8 \times 4! \times 2!}{2! \times 2!} + \frac{8 \times 7 \times 4! \times 2! \times 2!}{2! \times 2!} \\
&= \frac{8 \times 4! \times 2! + 8 \times 7 \times 4! \times 2! \times 2!}{2! \times 2!} \\
&= \frac{8 \times 4! \times 2! \times (1 + 7 \times 2!)}{2! \times 2!} \\
&= \frac{8 \times 4! \times 2! \times 15}{2! \times 2!} \\
&= \frac{4!}{2! \times 2!} \times 8 \times 15 \times 2 \\
&= \binom{4}{2} \times 8 \times 15 \times 2 \\
&= 1440
\end{aligned} \tag{2}$$

I considered solutions like $\frac{8 \times 8 \times 4!}{2!2!}$, thinking the second $2!$ could somehow magically erase all the "XX" cases above (which is correct) but scratched them out as I was concerned about eliminating too many cases... alas. The proposed solution I gave was (after some squinting) correct everywhere except for one term, due to over-counting certain cases. (TODO: Why over-counting?)