STAT302: Midterm #1 Revision

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1 Number of possible pins

I got both parts of this problem wrong due to some mis-steps while approaching them using permutations (and eliminating duplicates) instead of combinations.

1.1 Two 0s in the pin

The solution is $\binom{4}{2} \times 9^2 = 486$. The solution given is for 5-spaces in the pin, but the solution is generally $\binom{n}{2} \times 9^{n-2}$ where n is the number of spaces.

The solution I gave on WebWork was as follows

$$\frac{9 \times 4!}{2! \times 2!} + \frac{9 \times 8 \times 4!}{2!} = 918$$

I was (overly) concerned about the ordering of the digits, so I opted to use permutations, and wanted to ensure I eliminated duplicates. There were two cases here, a string like (a) "00XX" and a string like (b) "00XY", leading to the two terms. We always have $4!$ ways of arranging 4 characters. In (a), we eliminate two sets of duplicates and only have 9 choices for X (and only 1 choice for the other X). Likewise in (b), we only have to cancel one set of duplicates and have 9×8 choices for X and Y.

It becomes clear how "close" (or far) I was if we just re-write my solution in terms of $\binom{n}{k}$.

$$\begin{aligned}
\frac{1 \times 1 \times 9 \times 1 \times 4!}{2! \times 2!} + \frac{1 \times 1 \times 9 \times 8 \times 4!}{2!} &= \frac{9 \times 8 \times 4!}{2! \times 2!} + \frac{9 \times 8 \times 4! \times 2!}{2! \times 2!} \\
&= \frac{9 \times 8 \times 4! + 9 \times 8 \times 4! \times 2!}{2! \times 2!} \\
&= \frac{9 \times 4! \times (1 + 8 \times 2!)}{2! \times 2!} \\
&= \frac{9 \times 4! \times 17}{2! \times 2!} \\
&= \frac{4!}{2! \times 2!} \times 9 \times 17 \\
&= \binom{4}{2} \times 9 \times 17 \\
&= 918
\end{aligned} \tag{1}$$

I considered solutions like $\frac{9 \times 9 \times 4!}{2!2!}$ but the magic extra $2!$ on the bottom for "some other duplicate" (other than "00") made me believe it was incorrect so I scratched it out. The proposed solution I gave was, after some squinting, actually correct everywhere in except one term. I was aware that I need to order 4 things, to eliminate duplicate in the zeroes, and to eliminate "some other duplicate" in the other letters, but overly complicated my approach. To correctly do this, I should have added a second $2!$ to the bottom of the second term to account for duplicates there as well. Then, it would've been completely correct, as

$$\frac{9 \times 4!}{2! \times 2!} + \frac{9 \times 8 \times 4!}{2! \times 2!} = 54 + 432 = 486$$

as desired.

1.2 1 and 2 in the pin

(The argument here is practically identical, so feel free to skim a bit.) The solution is $2\binom{4}{2} \times 8^2 = 768$. The solution given is for 5-spaces in the pin, but the solution is generally $2\binom{n}{2} \times 8^{n-2}$ where n is the number of spaces.

The solution I gave on WebWork was as follows

$$\frac{1 \times 1 \times 8 \times 1 \times 4!}{2!} + \frac{1 \times 1 \times 8 \times 7 \times 4!}{1} = 1440$$

I was (overly) concerned about the order of the digits and wanted ensure I was handling cases where there were duplicates in the string. In the first case, I imagined we had a string like "12XX" where X was some digit other than 0 (and the "X"s are of course equal). There are $4!$ ways to re-arrange these characters and we need to divide by $2!$ to account for re-arranging XX. In the second case, I imagined we had a string like "12XY". This explains the 8×7 part of the term, since we have 8 choices for X and then just 7 for Y. The same logic follows from before, but this time we have no duplicates (I assumed).

It becomes clear how "close" I was if we just re-write my term in terms of $\binom{n}{k}$.

$$\begin{aligned} \frac{8 \times 4!}{2!} + \frac{8 \times 7 \times 4!}{1} &= \frac{8 \times 4! \times 2!}{2! \times 2!} + \frac{8 \times 7 \times 4! \times 2! \times 2!}{2! \times 2!} \\ &= \frac{8 \times 4! \times 2! + 8 \times 7 \times 4! \times 2! \times 2!}{2! \times 2!} \\ &= \frac{8 \times 4! \times 2! \times (1 + 7 \times 2!)}{2! \times 2!} \\ &= \frac{8 \times 4! \times 2! \times 15}{2! \times 2!} \\ &= \frac{4!}{2! \times 2!} \times 8 \times 15 \times 2 \\ &= \binom{4}{2} \times 8 \times 15 \times 2 \\ &= 1440 \end{aligned} \tag{2}$$

I considered solutions like $\frac{8 \times 8 \times 4!}{2!2!}$, thinking the second $2!$ could somehow magically erase all the "XX" cases above (which is correct) but scratched them out as I was concerned about eliminating too many cases... alas. The proposed solution I gave was (after some squinting) correct everywhere except for one term, due to over-counting certain cases. Similar to the first part, the term again missed was a 2!, and again in this case, it should be placed in the denominator in the second term. We'd see

$$\frac{8 \times 4!}{2!} + \frac{8 \times 7 \times 4!}{2!} = 96 + 672 = 768$$

as desired.

The moral? In both questions, I tried to "split" the question into two cases, when you can't neatly do that in these counting questions and thus *underestimated* the number of duplicates I had. A missing $2!$ in both parts would've earned me full points.

(I've summarized my work shown here, but I'm happy to walk through the submission I uploaded to Canvas as well and "prove" that this was my thinking via the scratch-work. Apologies for the long write-up, but this one took me a minute to see!)