TITAN CODE

REFERENCE MANUAL

by

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DISCLAIMER

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Contents

I. INTRODUCTION

This document describes the equations used in TITAN, a one-dimensional adaptive-grid radiation hydrodynamics code intended for astrophysical calculations. Suggestions about how to use the code are given in the TITAN Users Guide. TITAN follows the basic philosophy of WH80s, the very powerful code written by Karl-Heinz Winkler, except we use the grid equation developed by Dorfi and Drury which is somewhat simpler to implement and use by average users. Any potential user of this code should read carefully the following publications describing WH80s:

(**1**) WH80s: Numerical Radiation Hydrodynamics, K.–H. Winkler and M.L. Norman, in *Astrophysical Radiation Hydrodynamics,* (Dordrecht: Reidel), pp. 71–139, 1986

(**2**) Implicit Adaptive–Grid Radiation Hydrodynamics, K.–H. Winkler M.L. Norman, and D. Mihalas in *Multiple Time Scales,* Computational Techniques, Vol. **2,** ed. J.U. Brackbill and B.I. Cohen, (New York: Academic Press), pp. 145 – 184, 1985

(**3**) Adaptive–Mesh Radiation Hydrodynamics. I. The Radiation Transport Equation in a Completely Adaptive Coordinate System, K.–H. Winkler, M.L. Norman, and D. Mihalas, *J.Q.S.R.T.,* **31,** 473, 1984

(**4**) Adaptive–Mesh Radiation Hydrodynamics. II. The Radiation and Fluid Equations in Relativistic Flows, D. Mihalas, K.–H. Winkler, and M.L. Norman, *J.Q.S.R.T.,* **31,** 479, 1984

In what follows, each of the seven basic equations are discussed in separate sections. In general there are five conservation relations (continuity, gas momentum, radiating fluid energy, radiation energy, and radiation momentum), a mass definition equation, and an adaptive grid equation. The code is written so that it can treat ordinary hydrodynamics without radiation, full radiation hydrodynamics, and time-dependent radiation in a static medium. Further, users can specify an Eulerian grid, a Lagrangean grid, or an adaptive grid. At the beginning of a section, the subsection titled "differential equation" actually contains the original differential equation integrated over a finite volume, and transformed to adaptive coordinates, as described in the references (**1**) and (**3**) above. The advantage of this approach is that the difference equations are strictly conservative (apart from undifferentiated and source/sink terms).

In finite differencing the equations we use a staggered mesh. The medium is assumed to consist of N cells, bounded by $N + 1$ interfaces. The interface $k = 1$ is the leftmost boundary of the domain, and the interface $k = N + 1$ is the rightmost boundary of the domain. In stratified media (e.g. a star) $k = 1$ is the innermost boundary and $k = N + 1$ is the outermost boundary. All thermodynamic variables are cell-centered; thus the gas density ρ_k, temperature T_k, gas internal energy e_k, gas pressure p_k, and radiation pressure P_k are the values of these variables at the center of the cell $(k, k + 1)$. Likewise the radius r_k, mass m_k, velocity u_k, and radiation flux F_k are all centered on interface k. The full nonlinear difference equations are then linearized. The linearized system for the corrections has a block pentadiagonal form. The solution of the system is obtained by the standard Newton-Raphson iteration procedure.

To assist the reader we write all ordinary physical and mathematical variables in *italic type*, and `FORTRAN` quantities in `typewriter type`. In naming `FORTRAN` variables, we also use the elegant mnemonic system devised by Winkler. For example, we denote the matrices E_{-2}, E_{-1}, E_0, E_1, and E_2 of the linearized system as `em2, em1, e00, ep1, and ep2`. For brevity, because several physical terms in one of the difference equations may contribute to a given matrix element, we use the notation

`enn(i,j)`: function(physical variables)

to denote that the quantity on the right–hand side is *added into* the specified matrix element.

II. ADAPTIVE GRID

A. Eulerian Grid

In this case replace the grid equation with

$$dr_k/dt \equiv 0, \qquad (k = 1, \ldots, N+1) \tag{EG1}$$

This freezes all radii: $u_{grid} \equiv 0$, and $u_{rel} \equiv u$. The total mass in the domain is conserved if $u \equiv 0$ at the boundaries; it can change if boundary fluxes are nonzero.

B. Lagrangean Grid

In this case replace the grid equation with

$$dr_k/dt \equiv u_k, \qquad (k = 1, \ldots, N+1) \tag{LG1}$$

Then $u_{grid} \equiv u$, and $u_{rel} \equiv 0$. Therefore the advection term in the equation of continuity is identically zero, which implies that $\rho_k \Delta V_k = \Delta m_k \equiv$ const, $\Rightarrow m_k \equiv$ const. Mass on the computational domain is conserved exactly because the difference form of the continuity equation is exactly conservative, and $u_{rel} \equiv 0$ at the boundaries.

C. Adaptive Grid

1. Basic Grid Equation

(a) References

E.A. Dorfi and L. O'C. Drury, *J. Comp. Phys,* **69,** 175, 1987.
M. Balluch, *Astron. and Astrophys.,* **200,** 58, 1988.
E.A. Dorfi and A. Gautschy, in *Numerical Modeling of Nonlinear Stellar Pulsations,* ed. J.R. Buchler, pp. 289 - 302, 1990.
R.M. Furzeland, J.G. Verwer, and P.A. Zegeling, *J. Comp. Phys,* **89,** 349, 1990.

(b) Equations

$$\nu_2^{n+1} = \nu_1^{n+1} \qquad (k = 2) \qquad \text{(AG1)}$$

$$\frac{\hat{\nu}_k^{n+1}}{R_k^{n+1}} = \frac{\hat{\nu}_{k-1}^{n+1}}{R_{k-1}^{n+1}} \qquad (k = 3, \ldots, N-1) \qquad \text{(AG2)}$$

$$\nu_N^{n+1} = \nu_{N-1}^{n+1} \qquad (k = N) \qquad \text{(AG3)}$$

This system contains $N-1$ equations in $N+1$ unknowns; we thus need two boundary conditions. Equations (AG1) and (AG3) enforce a zero gradient of the grid concentration at the boundaries.

In equation (AG2),

$$\hat{\nu}_k^{n+1} \equiv \tilde{\nu}_k^{n+1} + \left(\frac{\tau}{\Delta t}\right)^{\beta} (\tilde{\nu}_k^{n+1} - \tilde{\nu}_k^{n}) \qquad \text{(AG4)}$$

$$\tilde{\nu}_k^{n+1} \equiv \nu_k^{n+1} - \alpha(\alpha+1)\,(\nu_{k-1}^{n+1} - 2\nu_k^{n+1} + \nu_{k+1}^{n+1}) \qquad \text{(AG5)}$$

and

$$R_k^{n+1} \equiv \left[\, 1 + (\nu_k^{n+1})^2 \sum_{l=1}^{L} W_l\, (S_{kl}^{n+1})^2 \,\right]^{1/2} \qquad \text{(AG6)}$$

In equation (AG6), the sum extends over all physical variables to be resolved by the grid. The W_ls are weights, O(1). The functions ν_k and S_{kl} are chosen according to the type of grid resolution desired. Thus for the abscissa we can choose:

(1) Linear resolution

$$\nu_k \equiv \frac{R_{scale}}{r_{k+1} - r_k} \qquad \text{(AG7)}$$

(2) Logarithmic resolution

$$\nu_k \equiv \frac{1}{2}\left(\frac{r_{k+1} + r_k}{r_{k+1} - r_k}\right) \qquad \text{(AG8)}$$

For the ordinates we can choose:

(1) Linear resolution

$$S_{kl} \equiv \frac{y_{k+1,l} - y_{kl}}{\mathtt{y_{scale,l}}} \tag{AG9}$$

(2) Logarithmic resolution

$$S_{kl} \equiv 2\left(\frac{y_{k+1,l} - y_{kl}}{y_{k+1,l} + y_{kl}}\right) \tag{AG10}$$

(3) Harmonic resolution

$$S_{kl} \equiv \frac{1}{2}\left(\frac{1}{y_{k+1,l}} + \frac{1}{y_{kl}}\right)(y_{k+1,l} - y_{kl}) \tag{AG11}$$

At present, the code uses the following association between the index l and physical variables:

for $l = 1$, $y = m$, cumulative mass
for $l = 2$, $y = \rho$, density
for $l = 3$, $y = T$, temperature
for $l = 4$, $y = E$, radiation energy density
for $l = 5$, $y = p$, gas pressure
for $l = 6$, $y = e$, gas energy density
for $l = 7$, $y = \chi$, opacity
for $l = 8$, $y = q_Q$, artificial viscosity

(c) Interaction of Grid with Boundary Conditions

(1) If the grid is specified to be Eulerian, then Eulerian boundary conditions must be used.

(2) If the grid is specified to be Lagrangean, then Lagrangean boundary conditions must be used.

(3) For an adaptive grid, any combination of Eulerian and Lagrangean boundary conditions at the two boundaries can be used.

2. LINEARIZATION

(a) Eulerian Grid

$$r_k^{n+1} - r_k^n = 0 \qquad (k = 1, \ldots, N+1) \qquad \text{(AG12)}$$

$$\texttt{e00(ir, jr)} = r_k^{n+1} \qquad \text{(AG13)}$$

$$-\texttt{rhs(ir)} = r_k^{n+1} - r_k^n \qquad \text{(AG14)}$$

(b) Lagrangean Grid

$$r_k^{n+1} - r_k^n - u_k^{n+\theta} = 0 \qquad (k = 1, \ldots, N+1) \qquad \text{(AG15)}$$

$$\texttt{e00(ir, jr)} = r_k^{n+1} \qquad \text{(AG16)}$$

$$\texttt{e00(ir, ju)} = -\texttt{unom(k)}\ \theta\ dt \qquad \text{(AG17)}$$

$$-\texttt{rhs(ir)} = r_k^{n+1} - r_k^n - u_k^{n+\theta}\ dt \qquad \text{(AG18)}$$

(c) Adaptive Grid

<u>Derivatives and Matrix Elements from Abscissa Terms</u>

(1) Linear resolution

$$\begin{aligned}\texttt{xnu(k)} &\equiv \nu_k^{n+1}\\ &= R_{scale}\ /\ (r_{k+1}^{n+1} - r_k^{n+1}) \qquad (k = 1, \ldots, N) \qquad \text{(AG19)}\end{aligned}$$

$$\begin{aligned}\texttt{dnudlr00(k)} &\equiv \partial\nu_k^{n+1}/\partial ln r_k^{n+1}\\ &= R_{scale}\ r_k^{n+1}\ /\ (r_{k+1}^{n+1} - r_k^{n+1})^2 \qquad (k = 1, \ldots, N) \qquad \text{(AG20)}\end{aligned}$$

$$\begin{aligned}\texttt{dnudlrp1(k)} &\equiv \partial\nu_k^{n+1}/\partial ln r_{k+1}^{n+1}\\ &= -R_{scale} r_{k+1}^{n+1}\ /\ (r_{k+1}^{n+1} - r_k^{n+1})^2 \qquad (k = 1, \ldots, N) \qquad \text{(AG21)}\end{aligned}$$

(2) Logarithmic resolution

$$\begin{aligned}\texttt{xnu(k)} &\equiv \nu_k^{n+1}\\ &= \tfrac{1}{2}(r_{k+1}^{n+1} + r_k^{n+1})/(r_{k+1}^{n+1} - r_k^{n+1}) \qquad (k = 1, \ldots, N) \qquad \text{(AG22)}\end{aligned}$$

$$\begin{aligned}\texttt{dnudlr00(k)} &\equiv \partial\nu_k^{n+1}/\partial ln r_k^{n+1}\\ &= r_k^{n+1} r_{k+1}^{n+1}\ /\ (r_{k+1}^{n+1} - r_k^{n+1})^2 \qquad (k = 1, \ldots, N) \qquad \text{(AG23)}\end{aligned}$$

$$\begin{aligned}\texttt{dnudlrp1(k)} &\equiv \partial\nu_k^{n+1}/\partial ln r_{k+1}^{n+1}\\ &= -r_k^{n+1} r_{k+1}^{n+1}\ /\ (r_{k+1}^{n+1} - r_k^{n+1})^2 \qquad (k = 1, \ldots, N) \qquad \text{(AG24)}\end{aligned}$$

Then for $k = 2$ we have from (AG1):

```
  em1(ir, jr):                 - dnudlr00(k-1)
  e00(ir, jr):  dnudlr00(k)  - dnudlrp1(k-1)
  ep1(ir, jr):  dnudlrp1(k)
```
(AG25) (AG26) (AG27)

$- \texttt{rhs(ir\quad):}\quad \nu_k^{n+1} - \nu_{k-1}^{n+1}$ (AG28)

For $k = N$, equation (AG3), we use (AG25) - (AG28) with $k = N$.

<u>Derivatives and Matrix Elements from Structure Function</u>

For $k = 3, \ldots, N-1$, we get from (AG2)

$$\sum_m \left[\frac{1}{R_k^{n+1}} \left(\frac{\partial \hat{\nu}_k^{n+1}}{\partial x_m} - \frac{\hat{\nu}_k^{n+1}}{R_k^{n+1}} \frac{\partial R_k^{n+1}}{\partial x_m} \right) - \frac{1}{R_{k-1}^{n+1}} \left(\frac{\partial \hat{\nu}_{k-1}^{n+1}}{\partial x_m} - \frac{\hat{\nu}_{k-1}^{n+1}}{R_{k-1}^{n+1}} \frac{\partial R_{k-1}^{n+1}}{\partial x_m} \right) \right] \delta x_m$$

$$= \frac{\hat{\nu}_{k-1}^{n+1}}{R_{k-1}^{n+1}} - \frac{\hat{\nu}_k^{n+1}}{R_k^{n+1}} \qquad \text{(AG29)}$$

where the sum runs over all physical variables monitored in R. From (AG6) we get

$$\frac{\partial R_k^{n+1}}{\partial x_m} = \frac{\nu_k^{n+1}}{R_k^{n+1}} \left[\frac{\partial \nu_k^{n+1}}{\partial x_m} \sum_l W_l \left(S_{kl}^{n+1} \right)^2 + \nu_k^{n+1} \sum_l W_l S_{kl}^{n+1} \frac{\partial S_{kl}^{n+1}}{\partial x_m} \right] \qquad \text{(AG30)}$$

Now ν and $\hat{\nu}$ depend only on r. Therefore it is convenient to group terms as follows:

$$\sum_m \left\{ \underbrace{\frac{1}{R_k^{n+1}} \left[\frac{\partial \hat{\nu}_k^{n+1}}{\partial x_m} - \frac{\partial \nu_k^{n+1}}{\partial x_m} \frac{\hat{\nu}_k^{n+1} \nu_k^{n+1}}{(R_k^{n+1})^2} \sum_l W_l \; (S_{kl}^{n+1})^2 \right]}_{} \right.$$

A

$$- \underbrace{\frac{1}{R_{k-1}^{n+1}} \left[\frac{\partial \hat{\nu}_{k-1}^{n+1}}{\partial x_m} - \frac{\partial \nu_{k-1}^{n+1}}{\partial x_m} \frac{\hat{\nu}_{k-1}^{n+1} \nu_{k-1}^{n+1}}{(R_{k-1}^{n+1})^2} \sum_l W_l \; (S_{k-1,l}^{n+1})^2 \right]}_{B}$$

$$
\left. \underbrace{-\frac{\hat{\nu}_k^{n+1}(\nu_k^{n+1})^2}{(R_k^{n+1})^3}\sum_l W_l S_{kl}\frac{\partial S_{kl}}{\partial x_m}}_{\mathsf{C}} \quad + \quad \underbrace{\frac{\hat{\nu}_{k-1}^{n+1}(\nu_{k-1}^{n+1})^2}{(R_{k-1}^{n+1})^3}\sum_l W_l S_{k-1,l}\frac{\partial S_{k-1,l}}{\partial x_m}}_{\mathsf{D}} \right\} \delta x_m
$$

$$
= \frac{\hat{\nu}_{k-1}^{n+1}}{R_{k-1}^{n+1}} - \frac{\hat{\nu}_k^{n+1}}{R_k^{n+1}} \qquad \text{(AG31)}
$$

Thus for $k = (2, \ldots, N-1)$ define

$$
\begin{aligned}
\texttt{xnt(k)} \; &\equiv \tilde{\nu}_k^{n+1} \\
&= \texttt{xnu(k)} - \alpha(\alpha+1)[\texttt{xnu(k-1) - 2xnu(k) + xnu(k+1)}] \qquad \text{(AG32)}
\end{aligned}
$$

$$
\texttt{xnc(k)} \; \equiv \hat{\nu}_k^{n+1} = [1 + (\tfrac{\tau}{\Delta t})^\beta] \;\; \texttt{xnt(k)} \; - \; (\tfrac{\tau}{\Delta t})^\beta \;\; \texttt{xnto(k)} \qquad \text{(AG33)}
$$

Then

$$
\texttt{dntdlrm1(k)} \; \equiv \partial\tilde{\nu}_k^{n+1}/\partial ln r_{k-1}^{n+1} = -\alpha(\alpha+1) \;\; \texttt{dnudlr00(k-1)} \qquad \text{(AG34)}
$$

$$
\begin{aligned}
&\texttt{dntdlr00(k)} \; \equiv \partial\tilde{\nu}_k^{n+1}/\partial ln r_k^{n+1} \\
&\quad = -\alpha(\alpha+1) \;\; [\texttt{dnudlrp1(k-1)-2dnudlr00(k)}] \; + \; \texttt{dnudlr00(k)} \qquad \text{(AG35)}
\end{aligned}
$$

$$
\begin{aligned}
&\texttt{dntdlrp1(k)} \; \equiv \partial\tilde{\nu}_k^{n+1}/\partial ln r_{k+1}^{n+1} \\
&\quad = -\alpha(\alpha+1) \;\; [\texttt{-2dnudlrp1(k)+dnudlr00(k+1)}] \; + \; \texttt{dnudlrp1(k)} \qquad \text{(AG36)}
\end{aligned}
$$

$$
\texttt{dntdlrp2(k)} \; \equiv \partial\tilde{\nu}_k^{n+1}/\partial ln r_{k+2}^{n+1} = -\alpha(\alpha+1) \;\; \texttt{dnudlrp1(k+1)} \qquad \text{(AG37)}
$$

$$
\texttt{dncdlrm1(k)} \; \equiv \partial\hat{\nu}_k^{n+1}/\partial ln r_{k-1}^{n+1} = [1 + (\tfrac{\tau}{\Delta t})^\beta] \;\; \texttt{dntdlrm1(k)} \qquad \text{(AG38)}
$$

$$
\texttt{dncdlr00(k)} \; \equiv \partial\hat{\nu}_k^{n+1}/\partial ln r_k^{n+1} = [1 + (\tfrac{\tau}{\Delta t})^\beta] \;\; \texttt{dntdlr00(k)} \qquad \text{(AG39)}
$$

$$
\texttt{dncdlrp1(k)} \; \equiv \partial\hat{\nu}_k^{n+1}/\partial ln r_{k+1}^{n+1} = [1 + (\tfrac{\tau}{\Delta t})^\beta] \;\; \texttt{dntdlrp1(k)} \qquad \text{(AG40)}
$$

$$
\texttt{dncdlrp2(k)} \; \equiv \partial\hat{\nu}_k^{n+1}/\partial ln r_{k+2}^{n+1} = [1 + (\tfrac{\tau}{\Delta t})^\beta] \;\; \texttt{dntdlrp2(k)} \qquad \text{(AG41)}
$$

Further, define:

$$
\texttt{ss(k)} \; \equiv \textstyle\sum_l W_l (S_{kl}^{n+1})^2 \qquad (k = 1, \ldots, N-1) \qquad \text{(AG42)}
$$

and

$$
\texttt{rr(k)} \; \equiv \{1 + [\texttt{xnu(k)}]^2 \texttt{ss(k)}\}^{1/2} \qquad (k = 2, \ldots, N-1) \qquad \text{(AG43)}
$$

Then from terms A and B in equation (AG31) we have

$$\texttt{dabdlrm1(k)} = \texttt{dncdlrm1(k)/rr(k)} \tag{AG44}$$

$$\begin{aligned}\texttt{dabdlr00(k)} = \{ & \texttt{ dncdlr00(k)} \\ & - \texttt{dnudlr00(k)xnc(k)xnu(k)ss(k)}/[\texttt{rr(k)}]^2\}/\texttt{rr(k)}\end{aligned} \tag{AG45}$$

$$\begin{aligned}\texttt{dabdlrp1(k)} = \{ & \texttt{ dncdlrp1(k)} \\ & - \texttt{dnudlrp1(k)xnc(k)xnu(k)ss(k)}/[\texttt{rr(k)}]^2\}/\texttt{rr(k)}\end{aligned} \tag{AG46}$$

$$\texttt{dabdlrp2(k)} = \texttt{dncdlrp2(k)/rr(k)} \tag{AG47}$$

Hence

$$\texttt{em2(ir,jr):}\quad \texttt{dabdlrm2(k)} \tag{AG48}$$

$$\texttt{em1(ir,jr):}\quad \texttt{dabdlrm1(k) - dabdlr00(k-1)} \tag{AG49}$$

$$\texttt{e00(ir,jr):}\quad \texttt{dabdlr00(k) - dabdlrp1(k-1)} \tag{AG50}$$

$$\texttt{ep1(ir,jr):}\quad \texttt{dabdlrp1(k) - dabdlrp2(k-1)} \tag{AG51}$$

$$\texttt{ep2(ir,jr):}\quad \texttt{dabdlrp2(k)} \tag{AG52}$$

Now to calculate terms C and D of equation (AG31), note from equations (AG9) – (AG11) that for linear resolution

$$\frac{\partial S_k}{\partial x} = \left[\left(\frac{\partial y}{\partial x}\right)_{k+1} - \left(\frac{\partial y}{\partial x}\right)_k \right] / \texttt{y}_{\texttt{scale}} \tag{AG53}$$

for logarithmic resolution

$$\frac{\partial S_k}{\partial x} = 4 \left[y_k \left(\frac{\partial y}{\partial x}\right)_{k+1} - y_{k+1} \left(\frac{\partial y}{\partial x}\right)_k \right] / \left(y_k + y_{k+1}\right)^2 \tag{AG54}$$

and for harmonic resolution

$$\frac{\partial S_k}{\partial x} = \frac{1}{2} \left(\frac{y_k}{y_{k+1}} - \frac{y_{k+1}}{y_k} \right) \left[\frac{1}{y_{k+1}} \left(\frac{\partial y}{\partial x}\right)_{k+1} - \frac{1}{y_k} \left(\frac{\partial y}{\partial x}\right)_k \right] \tag{AG55}$$

Here y denotes one of the physical variables monitored in the structure function, and x denotes any physical variable. Now define

$$\texttt{cs(k,l)} \qquad \equiv S_{kl}^{n+1} \tag{AG56}$$

$$\texttt{dcsdlx00(k,l,m)} \equiv \left(\frac{\partial S_{kl}^{n+1}}{\partial ln x_m} \right)_k^{n+1} \tag{AG57}$$

$$\mathtt{dcsdlxp1}(\mathtt{k},\mathtt{l},\mathtt{m}) \equiv \left(\frac{\partial S_{kl}^{n+1}}{\partial ln x_m}\right)_{k+1}^{n+1} \tag{AG58}$$

Then we have:

(1) Linear resolution

$l = 1,\ y = m,$

$$\mathtt{cs(k,\ l)} = (m_{k+1}^{n+1} - m_k^{n+1})/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG59}$$

$$\mathtt{dcsdlx00(k,l,im)} = -m_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG60}$$

$$\mathtt{dcsdlxp1(k,l,im)} = m_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG61}$$

$l = 2,\ y = \rho,$

$$\mathtt{cs(k,\ l)} = (\rho_{k+1}^{n+1} - \rho_k^{n+1})/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG62}$$

$$\mathtt{dcsdlx00(k,l,id)} = -\rho_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG63}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \rho_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG64}$$

$l = 3,\ y = T,$

$$\mathtt{cs(k,\ l)} = (T_{k+1}^{n+1} - T_k^{n+1})/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG65}$$

$$\mathtt{dcsdlx00(k,l,it)} = -T_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG66}$$

$$\mathtt{dcsdlxp1(k,l,it)} = T_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG67}$$

$l = 4,\ y = E,$

$$\mathtt{cs(k,\ l)} = (E_{k+1}^{n+1} - E_k^{n+1})/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG68}$$

$$\mathtt{dcsdlx00(k,l,ie)} = -E_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG69}$$

$$\mathtt{dcsdlxp1(k,l,ie)} = E_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG70}$$

$l = 5,\ y = p,$

$$\mathtt{cs(k,\ l)} = (p_{k+1}^{n+1} - p_k^{n+1})/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG71}$$

$$\mathtt{dcsdlx00(k,l,id)} = -p_k^{n+1}(\partial lnp/\partial ln\rho)_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG72}$$

$$\mathtt{dcsdlxp1(k,l,id)} = p_{k+1}^{n+1}(\partial lnp/\partial ln\rho)_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG73}$$

$$\mathtt{dcsdlx00(k,l,it)} = -p_k^{n+1}(\partial lnp/\partial lnT)_k^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG74}$$

$$\mathtt{dcsdlxp1(k,l,it)} = p_{k+1}^{n+1}(\partial lnp/\partial lnT)_{k+1}^{n+1}/\mathtt{y}_{\mathtt{scale}}(\mathtt{l}) \tag{AG75}$$

$l=6,\ y=e,$

$$\texttt{cs(k, l)} = (e_{k+1}^{n+1} - e_k^{n+1})/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG76}$$

$$\texttt{dcsdlx00(k,l,id)} = -e_k^{n+1}(\partial lne/\partial ln\rho)_k^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG77}$$

$$\texttt{dcsdlxp1(k,l,id)} = e_{k+1}^{n+1}(\partial lne/\partial ln\rho)_{k+1}^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG78}$$

$$\texttt{dcsdlx00(k,l,it)} = -e_k^{n+1}(\partial lne/\partial lnT)_k^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG79}$$

$$\texttt{dcsdlxp1(k,l,it)} = e_{k+1}^{n+1}(\partial lne/\partial lnT)_{k+1}^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG80}$$

$l=7,\ y=\chi,$

$$\texttt{cs(k, l)} = (\chi_{k+1}^{n+1} - \chi_k^{n+1})/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG81}$$

$$\texttt{dcsdlx00(k,l,id)} = -\chi_k^{n+1}(\partial ln\chi/\partial ln\rho)_k^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG82}$$

$$\texttt{dcsdlxp1(k,l,id)} = \chi_{k+1}^{n+1}(\partial ln\chi/\partial ln\rho)_{k+1}^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG83}$$

$$\texttt{dcsdlx00(k,l,it)} = -\chi_k^{n+1}(\partial ln\chi/\partial lnT)_k^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG84}$$

$$\texttt{dcsdlxp1(k,l,it)} = \chi_{k+1}^{n+1}(\partial ln\chi/\partial lnT)_{k+1}^{n+1}/\texttt{y}_{\texttt{scale}}(\texttt{l}) \tag{AG85}$$

$l=8,\ y=q_Q,$

The artificial viscosity indicator is taken to be

$$q_Q \equiv \left(\frac{p_Q}{p+\rho u^2}\right) + q_0 \tag{AG86}$$

$$\texttt{qx(k)} = [\texttt{qf(k)}\ /\ \texttt{pk(k)}] + \texttt{q0} \tag{AG87}$$

where `qf(k)` is given by equation (GM48) and the kinetic pressure `pk(k)` is given by

$$\texttt{pk(k)} \equiv p_k^{n+\theta} + \tfrac{1}{4}\rho_k^{n+\theta}(u_k^{n+\theta} + u_{k+1}^{n+\theta})^2 \tag{AG88}$$

Derivatives of q_Q are somewhat more complicated than terms measuring the behavior of only the basic dependent variables, as above. Thus we have

$$\begin{aligned}\texttt{dpkdld00(k)} &\equiv [\partial \texttt{pk(k)}/\partial ln\rho_k^{n+1}] \\ &= \theta[p_k^{n+1}(\partial lnp/\partial ln\rho)_k^{n+1} + \tfrac{1}{4}\rho_k^{n+1}(u_k^{n+\theta} + u_{k+1}^{n+\theta})^2]\end{aligned} \tag{AG89}$$

$$\texttt{dpkdlu00(k)} \equiv \partial \texttt{pk(k)}/\partial lnu_k^{n+1} = \tfrac{1}{2}\theta\rho_k^{n+\theta}(u_k^{n+\theta} + u_{k+1}^{n+\theta})\texttt{unom(k}\quad\texttt{)} \tag{AG90}$$

$$\mathtt{dpkdlup1(k)} \equiv \partial \mathtt{pk(k)}/\partial ln u_{k+1}^{n+1} = \frac{1}{2}\theta\rho_k^{n+\theta}(u_k^{n+\theta} + u_{k+1}^{n+\theta})\mathtt{unom(k+1)} \quad \text{(AG91)}$$

$$\mathtt{dpkdlt00(k)} \equiv \partial \mathtt{pk(k)}/\partial ln T_k^{n+1} = \theta p_k^{n+1}(\partial ln p/\partial ln T)_k^{n+1} \quad \text{(AG92)}$$

and

$$\mathtt{dqxdlr00(k)} = \mathtt{dqfdlr00(k)/pk(k)} \quad \text{(AG93)}$$

$$\mathtt{dqxdlrp1(k)} = \mathtt{dqfdlrp1(k)/pk(k)} \quad \text{(AG94)}$$

$$\mathtt{dqxdld00(k)} = [\mathtt{dqfdld00(k)\,pk(k)} - \mathtt{qf(k)\,dpkdld00(k)}]/\mathtt{pk(k)}^2 \quad \text{(AG95)}$$

$$\mathtt{dqxdlu00(k)} = [\mathtt{dqfdlu00(k)\,pk(k)} - \mathtt{qf(k)\,dpkdlu00(k)}]/\mathtt{pk(k)}^2 \quad \text{(AG96)}$$

$$\mathtt{dqxdlup1(k)} = [\mathtt{dqfdlup1(k)\,pk(k)} - \mathtt{qf(k)\,dpkdlup1(k)}]/\mathtt{pk(k)}^2 \quad \text{(AG97)}$$

$$\mathtt{dqxdlt00(k)} = [\mathtt{dqfdlt00(k)\,pk(k)} - \mathtt{qf(k)\,dpkdlt00(k)}]/\mathtt{pk(k)}^2 \quad \text{(AG98)}$$

Then for linear resolution we have

$$\mathtt{cs(k,l)} = [\mathtt{qx(k+1)} - \mathtt{qx(k)}]/\mathtt{y_{scale}(l)} \quad \text{(AG99)}$$

$$\mathtt{dcsdlx00(k,l,ir)} = -\mathtt{dqxdlr00(k\quad)}/\mathtt{y_{scale}(l)} \quad \text{(AG100)}$$

$$\mathtt{dcsdlxp1(k,l,ir)} = [\mathtt{dqxdlr00(k+1)} - \mathtt{dqxdlrp1(k)}]/\mathtt{y_{scale}(l)} \quad \text{(AG101)}$$

$$\mathtt{dcsdlxp2(k,l,ir)} = \mathtt{dqxdlrp1(k+1)}/\mathtt{y_{scale}(l)} \quad \text{(AG102)}$$

$$\mathtt{dcsdlx00(k,l,id)} = -\mathtt{dqxdld00(k\quad)}/\mathtt{y_{scale}(l)} \quad \text{(AG103)}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \mathtt{dqxdld00(k+1)}/\mathtt{y_{scale}(l)} \quad \text{(AG104)}$$

$$\mathtt{dcsdlx00(k,l,iu)} = -\mathtt{dqxdlu00(k\quad)}/\mathtt{y_{scale}(l)} \quad \text{(AG105)}$$

$$\mathtt{dcsdlxp1(k,l,iu)} = \ [\mathtt{dqxdlu00(k+1)} - \mathtt{dqxdlup1(k)}]/\mathtt{y_{scale}(l)} \quad \text{(AG106)}$$

$$\mathtt{dcsdlxp2(k,l,iu)} = \ \mathtt{dqxdlrp1(k+1)}/\mathtt{y_{scale}(l)} \quad \text{(AG107)}$$

$$\mathtt{dcsdlx00(k,l,it)} = -\mathtt{dqxdlt00(k\quad)}/\mathtt{y_{scale}(l)} \quad \text{(AG108)}$$

$$\mathtt{dcsdlxp1(k,l,it)} = \ \mathtt{dqxdlt00(k+1)}/\mathtt{y_{scale}(l)} \quad \text{(AG109)}$$

(2) Logarithmic resolution

$l = 1,\ y = m,$

$$\mathtt{cs(k,\ l)} \quad = \quad (m_{k+1}^{n+1} - m_k^{n+1})/(m_k^{n+1} + m_{k+1}^{n+1}) \quad \text{(AG110)}$$

$$\mathtt{dcsdlx00(k,l,im)} \ = -2m_k^{n+1} m_{k+1}^{n+1}/(m_k^{n+1} + m_{k+1}^{n+1})^2 \quad \text{(AG111)}$$

$$\mathtt{dcsdlxp1(k,l,im)} \ = \ 2m_k^{n+1} m_{k+1}^{n+1}/(m_k^{n+1} + m_{k+1}^{n+1})^2 \quad \text{(AG112)}$$

$l = 2,\ y = \rho,$

$$\mathtt{cs(k,\ l)} \quad = \quad (\rho_{k+1}^{n+1} - \rho_k^{n+1})/(\rho_k^{n+1} + \rho_{k+1}^{n+1}) \quad \text{(AG113)}$$

$$\mathtt{dcsdlx00(k,l,id)} \ = -2\rho_k^{n+1} \rho_{k+1}^{n+1}/(\rho_k^{n+1} + \rho_{k+1}^{n+1})^2 \quad \text{(AG114)}$$

$$\mathtt{dcsdlxp1(k,l,id)} \ = \ 2\rho_k^{n+1} \rho_{k+1}^{n+1}/(\rho_k^{n+1} + \rho_{k+1}^{n+1})^2 \quad \text{(AG115)}$$

$l = 3,\ y = T,$

$$\mathtt{cs(k,\ l)} \quad = \quad (T_{k+1}^{n+1} - T_k^{n+1})/(T_k^{n+1} + T_{k+1}^{n+1}) \quad \text{(AG116)}$$

$$\texttt{dcsdlx00(k,l,it)} = -2T_k^{n+1}T_{k+1}^{n+1}/(T_k^{n+1}+T_{k+1}^{n+1})^2 \tag{AG117}$$

$$\texttt{dcsdlxp1(k,l,it)} = 2T_k^{n+1}T_{k+1}^{n+1}/(T_k^{n+1}+T_{k+1}^{n+1})^2 \tag{AG118}$$

$l = 4,\ y = E,$

$$\texttt{cs(k, l)} = (E_{k+1}^{n+1}-E_k^{n+1})/(E_k^{n+1}+E_{k+1}^{n+1}) \tag{AG119}$$

$$\texttt{dcsdlx00(k,l,ie)} = -2E_k^{n+1}E_{k+1}^{n+1}/(E_k^{n+1}+E_{k+1}^{n+1})^2 \tag{AG120}$$

$$\texttt{dcsdlxp1(k,l,ie)} = 2E_k^{n+1}E_{k+1}^{n+1}/(E_k^{n+1}+E_{k+1}^{n+1})^2 \tag{AG121}$$

$l = 5,\ y = p,$

$$\texttt{cs(k, l)} = (p_{k+1}^{n+1}-p_k^{n+1})/(p_k^{n+1}+p_{k+1}^{n+1}) \tag{AG122}$$

$$\texttt{dcsdlx00(k,l,id)} = -2p_k^{n+1}p_{k+1}^{n+1}(\frac{\partial lnp}{\partial ln\rho})_k^{n+1}/(p_k^{n+1}+p_{k+1}^{n+1})^2 \tag{AG123}$$

$$\texttt{dcsdlxp1(k,l,id)} = 2p_k^{n+1}p_{k+1}^{n+1}(\frac{\partial lnp}{\partial ln\rho})_{k+1}^{n+1}/(p_k^{n+1}+p_{k+1}^{n+1})^2 \tag{AG124}$$

$$\texttt{dcsdlx00(k,l,it)} = -2p_k^{n+1}p_{k+1}^{n+1}(\frac{\partial lnp}{\partial lnT})_k^{n+1}/(p_k^{n+1}+p_{k+1}^{n+1})^2 \tag{AG125}$$

$$\texttt{dcsdlxp1(k,l,it)} = 2p_k^{n+1}p_{k+1}^{n+1}(\frac{\partial lnp}{\partial lnT})_{k+1}^{n+1}/(p_k^{n+1}+p_{k+1}^{n+1})^2 \tag{AG126}$$

$l = 6,\ y = e,$

$$\texttt{cs(k, l)} = (e_{k+1}^{n+1} - e_k^{n+1})/(e_k^{n+1} + e_{k+1}^{n+1}) \quad \text{(AG127)}$$

$$\texttt{dcsdlx00(k,l,id)} = -2e_k^{n+1} e_{k+1}^{n+1} (\tfrac{\partial lne}{\partial ln\rho})_k^{n+1} / (e_k^{n+1} + e_{k+1}^{n+1})^2 \quad \text{(AG128)}$$

$$\texttt{dcsdlxp1(k,l,id)} = 2e_k^{n+1} e_{k+1}^{n+1} (\tfrac{\partial lne}{\partial ln\rho})_{k+1}^{n+1} / (e_k^{n+1} + e_{k+1}^{n+1})^2 \quad \text{(AG129)}$$

$$\texttt{dcsdlx00(k,l,it)} = -2e_k^{n+1} e_{k+1}^{n+1} (\tfrac{\partial lne}{\partial lnT})_k^{n+1} / (e_k^{n+1} + e_{k+1}^{n+1})^2 \quad \text{(AG130)}$$

$$\texttt{dcsdlxp1(k,l,it)} = 2e_k^{n+1} e_{k+1}^{n+1} (\tfrac{\partial lne}{\partial lnT})_{k+1}^{n+1} / (e_k^{n+1} + e_{k+1}^{n+1})^2 \quad \text{(AG131)}$$

$l = 7,\ y = \chi,$

$$\texttt{cs(k, l)} = (\chi_{k+1}^{n+1} - \chi_k^{n+1})/(\chi_k^{n+1} + \chi_{k+1}^{n+1}) \quad \text{(AG132)}$$

$$\texttt{dcsdlx00(k,l,id)} = -2\chi_k^{n+1} \chi_{k+1}^{n+1} (\tfrac{\partial ln\chi}{\partial ln\rho})_k^{n+1} / (\chi_k^{n+1} + \chi_{k+1}^{n+1})^2 \quad \text{(AG133)}$$

$$\texttt{dcsdlxp1(k,l,id)} = 2\chi_k^{n+1} \chi_{k+1}^{n+1} (\tfrac{\partial ln\chi}{\partial ln\rho})_{k+1}^{n+1} / (\chi_k^{n+1} + \chi_{k+1}^{n+1})^2 \quad \text{(AG134)}$$

$$\texttt{dcsdlx00(k,l,it)} = -2\chi_k^{n+1} \chi_{k+1}^{n+1} (\tfrac{\partial ln\chi}{\partial lnT})_k^{n+1} / (\chi_k^{n+1} + \chi_{k+1}^{n+1})^2 \quad \text{(AG135)}$$

$$\texttt{dcsdlxp1(k,l,it)} = 2\chi_k^{n+1} \chi_{k+1}^{n+1} (\tfrac{\partial ln\chi}{\partial lnT})_{k+1}^{n+1} / (\chi_k^{n+1} + \chi_{k+1}^{n+1})^2 \quad \text{(AG136)}$$

$l = 8,\ y = q_Q,$

$$\mathtt{cs(k,l)} = \frac{\mathtt{qx(k+1) - qx(k)}}{\mathtt{qx(k+1) + qx(k)}} \quad \text{(AG137)}$$

$$\mathtt{dcsdlx00(k,l,ir)} \; = -2\frac{\mathtt{qx(k+1)\,dpxdlr00(k)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG138}$$

$$\mathtt{dcsdlxp1(k,l,ir)} \; =$$

$$2\frac{\mathtt{qx(k)\,dqxdlr00(k+1)-qx(k+1)\,dpxdlrp1(k)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG139}$$

$$\mathtt{dcsdlxp2(k,l,ir)} = \quad 2\frac{\mathtt{qx(k)\,dpxdlr00(k+1)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG140}$$

$$\mathtt{dcsdlx00(k,l,id)} = -2\frac{\mathtt{qx(k+1)\,dpxdld00(k)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG141}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \quad 2\frac{\mathtt{qx(k)\,dpxdld00(k+1)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG142}$$

$$\mathtt{dcsdlx00(k,l,it)} = -2\frac{\mathtt{qx(k+1)\,dpxdlt00(k)}}{[\mathtt{qx(k+1)+qx(k)}]^2} \tag{AG143}$$

$$\texttt{dcsdlxp1(k,l,it)} = \quad 2\frac{\texttt{qx(k)}\,\texttt{dpxdlt00(k+1)}}{[\texttt{qx(k+1)}+\texttt{qx(k)}]^2} \tag{AG144}$$

$$\texttt{dcsdlx00(k,l,iu)} = -2\frac{\texttt{qx(k+1)}\,\texttt{dpxdlu00(k)}}{[\texttt{qx(k+1)}+\texttt{qx(k)}]^2} \tag{AG145}$$

dcsdlxp1(k,l,iu)=

$$2\frac{\texttt{qx(k)}\,\texttt{dqxdlu00(k+1)}-\texttt{qx(k+1)}\,\texttt{dpxdlru1(k)}}{[\texttt{qx(k+1)}+\texttt{qx(k)}]^2} \tag{AG146}$$

$$\texttt{dcsdlxp2(k,l,iu)} = \quad 2\frac{\texttt{qx(k)}\,\texttt{dpxdlu00(k+1)}}{[\texttt{qx(k+1)}+\texttt{qx(k)}]^2} \tag{AG147}$$

(3) Harmonic resolution

$l = 1,\ y = m,$

$$\texttt{cs(k,l)} \qquad = \quad \left(\frac{1}{m_k^{n+1}}+\frac{1}{m_{k+1}^{n+1}}\right)(m_{k+1}^{n+1}-m_k^{n+1}) \tag{AG148}$$

$$\texttt{dcsdlx00(k,l,im)} = -\left(\frac{m_k^{n+1}}{m_{k+1}^{n+1}}+\frac{m_{k+1}^{n+1}}{m_k^{n+1}}\right) \tag{AG149}$$

$$\texttt{dcsdlxp1(k,l,im)} = \left(\frac{m_k^{n+1}}{m_{k+1}^{n+1}} + \frac{m_{k+1}^{n+1}}{m_k^{n+1}}\right) \tag{AG150}$$

$l = 2, y = \rho,$

$$\texttt{cs(k,l)} = \left(\frac{1}{\rho_k^{n+1}} + \frac{1}{\rho_{k+1}^{n+1}}\right)(\rho_{k+1}^{n+1} - \rho_k^{n+1}) \tag{AG151}$$

$$\texttt{dcsdlx00(k,l,id)} = -\left(\frac{\rho_k^{n+1}}{\rho_{k+1}^{n+1}} + \frac{\rho_{k+1}^{n+1}}{\rho_k^{n+1}}\right) \tag{AG152}$$

$$\texttt{dcsdlxp1(k,l,id)} = \left(\frac{\rho_k^{n+1}}{\rho_{k+1}^{n+1}} + \frac{\rho_{k+1}^{n+1}}{\rho_k^{n+1}}\right) \tag{AG153}$$

$l = 3,\ y = T,$

$$\texttt{cs(k,l)} = \left(\frac{1}{T_k^{n+1}} + \frac{1}{T_{k+1}^{n+1}}\right)(T_{k+1}^{n+1} - T_k^{n+1}) \tag{AG154}$$

$$\texttt{dcsdlx00(k,l,it)} = -\left(\frac{T_k^{n+1}}{T_{k+1}^{n+1}} + \frac{T_{k+1}^{n+1}}{T_k^{n+1}}\right) \tag{AG155}$$

$$\texttt{dcsdlxp1(k,l,it)} = \left(\frac{T_k^{n+1}}{T_{k+1}^{n+1}} + \frac{T_{k+1}^{n+1}}{T_k^{n+1}}\right) \tag{AG156}$$

$l = 4,\ y = E,$

$$\texttt{cs(k,l)} = \left(\frac{1}{E_k^{n+1}} + \frac{1}{E_{k+1}^{n+1}}\right)(E_{k+1}^{n+1} - E_k^{n+1}) \tag{AG157}$$

$$\texttt{dcsdlx00(k,l,ie)} = -\left(\frac{E_k^{n+1}}{E_{k+1}^{n+1}} + \frac{E_{k+1}^{n+1}}{E_k^{n+1}}\right) \tag{AG158}$$

$$\texttt{dcsdlxp1(k,l,ie)} = \left(\frac{E_k^{n+1}}{E_{k+1}^{n+1}} + \frac{E_{k+1}^{n+1}}{E_k^{n+1}}\right) \tag{AG159}$$

$l = 5,\ y = p,$

$$\texttt{cs(k,l)} = (p_{k+1}^{n+1} - p_k^{n+1})\left(\frac{1}{p_k^{n+1}} + \frac{1}{p_{k+1}^{n+1}}\right) \tag{AG160}$$

$$\texttt{dcsdlx00(k,l,id)} = -\left(\frac{\partial lnp}{\partial ln\rho}\right)_k^{n+1}\left(\frac{p_k^{n+1}}{p_{k+1}^{n+1}} + \frac{p_{k+1}^{n+1}}{p_k^{n+1}}\right) \tag{AG161}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \left(\frac{\partial ln p}{\partial ln \rho}\right)_{k+1}^{n+1} \left(\frac{p_k^{n+1}}{p_{k+1}^{n+1}} + \frac{p_{k+1}^{n+1}}{p_k^{n+1}}\right) \tag{AG162}$$

$$\mathtt{dcsdlx00(k,l,it)} = -\left(\frac{\partial ln p}{\partial ln T}\right)_{k}^{n+1} \left(\frac{p_k^{n+1}}{p_{k+1}^{n+1}} + \frac{p_{k+1}^{n+1}}{p_k^{n+1}}\right) \tag{AG163}$$

$$\mathtt{dcsdlxp1(k,l,it)} = \left(\frac{\partial ln p}{\partial ln T}\right)_{k+1}^{n+1} \left(\frac{p_k^{n+1}}{p_{k+1}^{n+1}} + \frac{p_{k+1}^{n+1}}{p_k^{n+1}}\right) \tag{AG164}$$

$l = 6,\ y = e,$

$$\mathtt{cs(k,l)} = (e_{k+1}^{n+1} - e_k^{n+1}) \left(\frac{1}{e_k^{n+1}} + \frac{1}{e_{k+1}^{n+1}}\right) \tag{AG165}$$

$$\mathtt{dcsdlx00(k,l,id)} = -\left(\frac{\partial ln e}{\partial ln \rho}\right)_{k}^{n+1} \left(\frac{e_k^{n+1}}{e_{k+1}^{n+1}} + \frac{e_{k+1}^{n+1}}{e_k^{n+1}}\right) \tag{AG166}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \left(\frac{\partial ln e}{\partial ln \rho}\right)_{k+1}^{n+1} \left(\frac{e_k^{n+1}}{e_{k+1}^{n+1}} + \frac{e_{k+1}^{n+1}}{e_k^{n+1}}\right) \tag{AG167}$$

$$\mathtt{dcsdlx00(k,l,it)} = -\left(\frac{\partial lne}{\partial lnT}\right)_k^{n+1}\left(\frac{e_k^{n+1}}{e_{k+1}^{n+1}} + \frac{e_{k+1}^{n+1}}{e_k^{n+1}}\right) \tag{AG168}$$

$$\mathtt{dcsdlxp1(k,l,it)} = \left(\frac{\partial lne}{\partial lnT}\right)_{k+1}^{n+1}\left(\frac{e_k^{n+1}}{e_{k+1}^{n+1}} + \frac{e_{k+1}^{n+1}}{e_k^{n+1}}\right) \tag{AG169}$$

$l = 7,\ y = \chi,$

$$\mathtt{cs(k,l)} = (\chi_{k+1}^{n+1} - \chi_k^{n+1})\left(\frac{1}{\chi_k^{n+1}} + \frac{1}{\chi_{k+1}^{n+1}}\right) \tag{AG170}$$

$$\mathtt{dcsdlx00(k,l,id)} = -\left(\frac{\partial ln\chi}{\partial ln\rho}\right)_k^{n+1}\left(\frac{\chi_k^{n+1}}{\chi_{k+1}^{n+1}} + \frac{\chi_{k+1}^{n+1}}{\chi_k^{n+1}}\right) \tag{AG171}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \left(\frac{\partial ln\chi}{\partial ln\rho}\right)_{k+1}^{n+1}\left(\frac{\chi_k^{n+1}}{\chi_{k+1}^{n+1}} + \frac{\chi_{k+1}^{n+1}}{\chi_k^{n+1}}\right) \tag{AG172}$$

$$\mathtt{dcsdlx00(k,l,it)} = -\left(\frac{\partial ln\chi}{\partial lnT}\right)_k^{n+1}\left(\frac{\chi_k^{n+1}}{\chi_{k+1}^{n+1}} + \frac{\chi_{k+1}^{n+1}}{\chi_k^{n+1}}\right) \tag{AG173}$$

$$\mathtt{dcsdlxp1(k,l,it)} = \left(\frac{\partial ln\chi}{\partial lnT}\right)^{n+1}_{k+1} \left(\frac{\chi^{n+1}_{k}}{\chi^{n+1}_{k+1}} + \frac{\chi^{n+1}_{k+1}}{\chi^{n+1}_{k}}\right) \tag{AG174}$$

$l = 8,\ y = q_Q,$

$$\mathtt{cs(k,l)} = \frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} - \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}} \tag{AG175}$$

$$\mathtt{dcsdlx00(k,l,ir)} = -\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlr00(k)}}{\mathtt{qx(k)}} \tag{AG176}$$

$\mathtt{dcsdlxp1(k,l,ir)}$ =

$$\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \left[\frac{\mathtt{dqxdlr00(k+1)}}{\mathtt{qx(k+1)}} - \frac{\mathtt{dqxdlrp1(k)}}{\mathtt{qx(k)}}\right] \tag{AG177}$$

$$\mathtt{dcsdlxp2(k,l,ir)} = \left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlrp1(k+1)}}{\mathtt{qx(k+1)}} \tag{AG178}$$

$$\mathtt{dcsdlx00(k,l,id)} = -\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdld00(k)}}{\mathtt{qx(k)}} \tag{AG179}$$

$$\mathtt{dcsdlxp1(k,l,id)} = \left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdld00(k+1)}}{\mathtt{qx(k+1)}} \quad \text{(AG180)}$$

$$\mathtt{dcsdlx00(k,l,iu)} = -\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlu00(k)}}{\mathtt{qx(k)}} \quad \text{(AG181)}$$

$$\mathtt{dcsdlxp1(k,l,iu)} =$$

$$\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \left[\frac{\mathtt{dqxdlu00(k+1)}}{\mathtt{qx(k+1)}} - \frac{\mathtt{dqxdlup1(k)}}{\mathtt{qx(k)}}\right] \quad \text{(AG182)}$$

$$\mathtt{dcsdlxp2(k,l,iu)} = \left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlup1(k+1)}}{\mathtt{qx(k+1)}} \quad \text{(AG183)}$$

$$\mathtt{dcsdlx00(k,l,it)} = -\left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlt00(k)}}{\mathtt{qx(k)}} \quad \text{(AG184)}$$

$$\mathtt{dcsdlxp1(k,l,it)} = \left[\frac{\mathtt{qx(k+1)}}{\mathtt{qx(k)}} + \frac{\mathtt{qx(k)}}{\mathtt{qx(k+1)}}\right] \frac{\mathtt{dqxdlt00(k+1)}}{\mathtt{qx(k+1)}} \quad \text{(AG185)}$$

Finally, we can assemble these results to obtain the derivatives of terms C and D in equation (AG31). Thus for $(k = 1, \ldots, N-1)$ define

$$\texttt{dssdlx00(k,m)} \equiv 2 \sum_{l} W_l \; \texttt{cs(k,l)} \; \texttt{dcsdlx00(k,l,m)} \tag{AG186}$$

$$\texttt{dssdlxp1(k,m)} \equiv 2 \sum_{l} W_l \; \texttt{cs(k,l)} \; \texttt{dcsdlxp1(k,l,m)} \tag{AG187}$$

$$\texttt{dssdlxp2(k,m)} \equiv 2 \sum_{l} W_l \; \texttt{cs(k,l)} \; \texttt{dcsdlxp2(k,l,m)} \tag{AG188}$$

and, for $k = 2, \ldots, N-1$,

$$\texttt{dcddlx00(k,m)} \equiv \tfrac{1}{2} \; \texttt{xnc(k)} \; [\texttt{xnu(k)}^2/\texttt{rr(k)}^3] \; \texttt{dssdlx00(k,m)} \tag{AG189}$$

$$\texttt{dcddlxp1(k,m)} \equiv \tfrac{1}{2} \; \texttt{xnc(k)} \; [\texttt{xnu(k)}^2/\texttt{rr(k)}^3] \; \texttt{dssdlxp1(k,m)} \tag{AG190}$$

$$\texttt{dcddlxp2(k,m)} \equiv \tfrac{1}{2} \; \texttt{xnc(k)} \; [\texttt{xnu(k)}^2/\texttt{rr(k)}^3] \; \texttt{dssdlxp2(k,m)} \tag{AG191}$$

Then, for $k = 3, \ldots, N-1$,

`em1(ir,jr):  dcddlx00(k-1,jr)` (AG192)

`e00(ir,jr):  dcddlxp1(k-1,jr) – dcddlx00(k,jr)` (AG193)

`ep1(ir,jr):  dcddlxp2(k-1,jr) – dcddlxp1(k,jr)` (AG194)

`ep2(ir,jr):                   – dcddlxp2(k,jr)` (AG195)

`em1(ir,jm):  dcddlx00(k-1,jm)` (AG196)

`e00(ir,jm):  dcddlxp1(k-1,jm) – dcddlx00(k,jm)` (AG197)

`ep1(ir,jm):  dcddlxp2(k-1,jm) – dcddlxp1(k,jm)` (AG198)

$$\texttt{ep2(ir,jm):} \qquad\qquad\qquad\qquad - \texttt{dcddlxp2(k,jm)} \tag{AG199}$$

$$\texttt{em1(ir,jd):} \quad \texttt{dcddlx00(k-1,jd)} \tag{AG200}$$

$$\texttt{e00(ir,jd):} \quad \texttt{dcddlxp1(k-1,jd)} - \texttt{dcddlx00(k,jd)} \tag{AG201}$$

$$\texttt{ep1(ir,jd):} \quad \texttt{dcddlxp2(k-1,jd)} - \texttt{dcddlxp1(k,jd)} \tag{AG202}$$

$$\texttt{ep2(ir,jd):} \qquad\qquad\qquad\qquad - \texttt{dcddlxp2(k,jd)} \tag{AG203}$$

$$\texttt{em1(ir,ju):} \quad \texttt{dcddlx00(k-1,ju)} \tag{AG204}$$

$$\texttt{e00(ir,ju):} \quad \texttt{dcddlxp1(k-1,ju)} - \texttt{dcddlx00(k,ju)} \tag{AG205}$$

$$\texttt{ep1(ir,ju):} \quad \texttt{dcddlxp2(k-1,ju)} - \texttt{dcddlxp1(k,ju)} \tag{AG206}$$

$$\texttt{ep2(ir,ju):} \qquad\qquad\qquad\qquad - \texttt{dcddlxp2(k,ju)} \tag{AG207}$$

$$\texttt{em1(ir,jt):} \quad \texttt{dcddlx00(k-1,jt)} \tag{AG208}$$

$$\texttt{e00(ir,jt):} \quad \texttt{dcddlxp1(k-1,jt)} - \texttt{dcddlx00(k,jt)} \tag{AG209}$$

$$\texttt{ep1(ir,jt):} \quad \texttt{dcddlxp2(k-1,jt)} - \texttt{dcddlxp1(k,jt)} \tag{AG210}$$

$$\texttt{ep2(ir,jt):} \qquad\qquad\qquad\qquad - \texttt{dcddlxp2(k,jt)} \tag{AG211}$$

$$\texttt{em1(ir,je):} \quad \texttt{dcddlx00(k-1,je)} \tag{AG212}$$

$$\texttt{e00(ir,je):}\quad \texttt{dcddlxp1(k-1,je)} - \texttt{dcddlx00(k,je)} \tag{AG213}$$

$$\texttt{ep1(ir,je):}\quad \texttt{dcddlxp2(k-1,je)} - \texttt{dcddlxp1(k,je)} \tag{AG214}$$

$$\texttt{ep2(ir,je):}\quad - \texttt{dcddlxp2(k,je)} \tag{AG215}$$

III. CUMULATIVE MASS

A. Differential Equation

$$\Delta m = \rho \Delta V \tag{M1}$$

B. Difference Equation

For $(k = 2, \ldots, N+1)$,

$$m_k^{n+1} - m_{k-1}^{n+1} - \rho_{k-1}^{n+1}[(r_k^{n+1})^{\mu+1} - (r_{k-1}^{n+1})^{\mu+1}]/(\mu+1) = 0 \tag{M2}$$

where $\mu = 0$, 1, or 2.

Equation (M2) provides N relations connecting masses at N+1 gridpoints. Thus we will require boundary condition. Also notice that because it is a definition and not a differential equation in time, equation (M2) uses variables only at the advanced time level, t^{n+1}.

C. Linearization

For $(k = 1, \ldots, N+1)$ define

$$\texttt{rmup1o(k)} \equiv (r_k^{n})^{\mu+1} \tag{M3}$$

$$\texttt{rmup1 (k)} \equiv (r_k^{n+\theta})^{\mu+1} \tag{M4}$$

$$\texttt{rmup1n(k)} \equiv (r_k^{n+1})^{\mu+1} \tag{M5}$$

And for $(k = 1, \ldots, N)$ define

$$\texttt{dvolo(k)} \equiv \Delta V_k^{n} = (\texttt{rmup1o(k+1)} - \texttt{rmup1o(k)})/(\mu+1) \tag{M6}$$

$$\texttt{dvol (k)} \equiv \Delta V_k^{n+\theta} = (\texttt{rmup1 (k+1)} - \texttt{rmup1 (k)})/(\mu+1) \tag{M7}$$

$$\texttt{dvoln(k)} \equiv \Delta V_k^{n+1} = (\texttt{rmup1n(k+1)} - \texttt{rmup1n(k)})/(\mu+1) \tag{M8}$$

Then

$$\texttt{em1(im,jm)}: -m_{k-1}^{n+1} \tag{M9}$$

$$\texttt{e00(im,jm)}: \ m_k^{n+1} \tag{M10}$$

$$\texttt{em1(im,jr)}: \ \rho_{k-1}^{n+1}\texttt{rmup1n(k-1)} \tag{M11}$$

$$\texttt{e00(im,jr)}: -\rho_{k-1}^{n+1}\texttt{rmup1n(k)} \tag{M12}$$

$$\texttt{em1(im,jd)}: -\rho_{k-1}^{n+1}\texttt{dvoln(k-1)} \tag{M13}$$

and

$$-\texttt{rhs(im\ \ \ \ ):}\ m_k^{n+1} - m_{k-1}^{n+1} - \rho_{k-1}^{n+1}\texttt{dvoln(k-1)} \tag{M14}$$

IV. CONTINUITY

A. Differential Equation

$$\frac{d(\rho\Delta V)}{dt} + \Delta(\rho r^{\mu} u_{rel}) = \Delta(r^{\mu}\sigma_{\rho}\frac{\Delta\rho}{\Delta r}) \tag{C1}$$

B. Difference Equation

For $(k = 2, \ldots, N-1)$,

$$\frac{\rho_k^{n+1}[(r_{k+1}^{n+1})^{\mu+1} - (r_k^{n+1})^{\mu+1}] - \rho_k^{n}[(r_{k+1}^{n})^{\mu+1} - (r_k^{n})^{\mu+1}]}{(\mu+1)\ dt}$$

$$+\ (r_{k+1}^{n+\theta})^{\mu}\left[u_{k+1}^{n+\theta} - \left(\frac{r_{k+1}^{n+1} - r_{k+1}^{n}}{dt}\right)\right]\overline{\rho}_{k+1}$$

$$-\ (r_{k}^{n+\theta})^{\mu}\left[u_{k}^{n+\theta} - \left(\frac{r_{k}^{n+1} - r_{k}^{n}}{dt}\right)\right]\overline{\rho}_{k}$$

$$-2\sigma_{\rho}\left[(r_{k+1}^{n+\theta})^{\mu}\left(\frac{\rho_{k+1}^{n+\theta} - \rho_{k}^{n+\theta}}{r_{k+2}^{n+\theta} - r_{k}^{n+\theta}}\right) - (r_{k}^{n+\theta})^{\mu}\left(\frac{\rho_{k}^{n+\theta} - \rho_{k-1}^{n+\theta}}{r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}}\right)\right] = 0 \tag{C2}$$

where $\mu = 0,\ 1,\ \text{or}\ 2$.

Equation (C2) provides $N-2$ relations connecting the densities at N grid-points. We thus require two boundary conditions. In equation (C2) we have written time-centered values of the physical variables as:

$$x^{n+\theta} \equiv \theta x^{n+1} + (1-\theta)x^{n} \tag{C3}$$

Values for the density advected at cell interfaces are defined as

$$\overline{\rho}_k \equiv (0.5+s_k)[\theta(\rho_{k-1}^{n+1}+0.5D\rho_{k-1}^{n+1})+(1-\theta)(\rho_{k-1}^{n}+0.5D\rho_{k-1}^{n})]$$
$$+(0.5-s_k)[\theta(\rho_{k}^{n+1}-0.5D\rho_{k}^{n+1})+(1-\theta)(\rho_{k}^{n}\quad-0.5D\rho_{k}^{n}\quad)] \tag{C4}$$

The switch s_k

$$s_k \equiv \texttt{cvmgp}[0.5, -0.5, u_{rel,k}] \tag{C5}$$

chooses the upstream direction, and the slope $D\rho_k^{n+1}$ inside the cell is given by

$$D\rho_k^{n+1} \equiv \texttt{cvmgp}\left[\frac{C\Delta\rho_{k-1}^{n+1}\Delta\rho_k^{n+1}}{\Delta\rho_{k-1}^{n+1}+\Delta\rho_k^{n+1}},\ 0,\ \Delta\rho_k^{n+1}\right] \tag{C6}$$

which yields monotonized van Leer advection if $C=2$, and donor cell advection if $C=0$. Here

$$\Delta\rho_{k+1}^{n+1} \equiv \rho_{k+1}^{n+1}-\rho_k^{n+1} \tag{C7}$$

C. Linearization

For $(k=1,\ldots,N+1)$ define

$\texttt{rmuo(k)} \equiv (r_k^n)^\mu$ (C8)

$\texttt{rmu (k)} \equiv (r_k^{n+\theta})^\mu$ (C9)

$\texttt{rmun(k)} \equiv (r_k^{n+1})^\mu$ (C10)

$\texttt{urel(k)} \equiv u_k^{n+\theta}-(r_k^{n+1}-r_k^n)\,/\,dt$ (C11)

$a_{s,k}^{n+\theta} \equiv \sqrt{\gamma p_k^{n+\theta}/\rho_k^{n+\theta}}$ (C12)

Calculating matrix elements we find:

(1) Time Derivative

$\texttt{e00(id,jr)}$: $-\rho_k^{n+1}\texttt{rmup1n(k)}\,/\,dt$ (C13)

$\texttt{ep1(id,jr)}$: $\rho_k^{n+1}\texttt{rmup1n(k+1)}\,/\,dt$ (C14)

$\texttt{e00(id,jd)}$: $\rho_k^{n+1}\texttt{dvoln (k)}\,/\,dt$ (C15)

$$\mathtt{e00(id,jr):rmu(k}\qquad)\overline{\rho}_k \quad \left[\quad \left(\frac{r_k^{n+1}}{dt}\right) - \mu\theta\left(\frac{r_k^{n+1}}{r_k^{n+\theta}}\right)\mathtt{urel(k}\qquad)\right] \quad \text{(C16)}$$

$$\mathtt{e00(id,jr):rmu(k+1)}\overline{\rho}_{k+1}\left[-\left(\frac{r_{k+1}^{n+1}}{dt}\right) + \mu\theta\left(\frac{r_{k+1}^{n+1}}{r_{k+1}^{n+\theta}}\right)\mathtt{urel(k+1)}\right] \quad \text{(C17)}$$

$$\mathtt{e00(id,ju):}-\theta\ \mathtt{rmu(k}\quad)\ \overline{\rho}_k\quad \mathtt{unom(k}\quad) \qquad \text{(C18)}$$

$$\mathtt{ep1(id,ju):}\ \ \theta\ \mathtt{rmu(k+1)}\ \overline{\rho}_{k+1}\ \mathtt{unom(k+1)} \qquad \text{(C19)}$$

The quantities needed to calculate the derivatives of the advection term with respect to density are generated in a separate subroutine, named `advectc` to indicate that it treats the advection of cell-centered variables such as ρ, e, and $e + (E/\rho)$. For generality, and to avoid repetition, we shall denote the advected quantity as q. The inputs required by the subroutine are, for $(k = -1, \ldots, N+2)$,

$$
\begin{aligned}
q \quad &\equiv\ q_k^{n+1} &&= \text{advected quantity at advanced time}\\
qo \quad &\equiv\ q_k^{n} &&= \text{advected quantity at old time}\\
qso \quad &\equiv Dq_k^{n} &&= \text{monotonized slope at old time}\\
\mathtt{flow(k)} &\equiv \mathtt{urel(k)} &&= \text{direction of flow at interface } k
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathtt{s(k)} &&\equiv s_k = \mathtt{cvmgp}[0.5, -0.5, \mathtt{flow(k)}] && (k = 1, \ldots, N+1) \quad \text{(C20)}\\
&\mathtt{dq(k)} &&\equiv \Delta q_k^{n+1} = q_{k+1}^{n+1} - q_k^{n+1} && (k = 0, \ldots, N+1) \quad \text{(C21)}\\
&\mathtt{denom(k)} &&\equiv \mathtt{cvmgm}[\,\mathtt{dq(k-1)} + \mathtt{dq(k)} - \epsilon, &&\\
& && \qquad\qquad \mathtt{dq(k-1)} + \mathtt{dq(k)} + \epsilon, &&\\
& && \qquad\qquad \mathtt{dq(k-1)} + \mathtt{dq(k)}\,] && (k = 1, \ldots, N+1) \quad \text{(C22)}
\end{aligned}
$$

$$\mathtt{qr(k)} \equiv R_k^{n+1} = C_{adv}\mathtt{dq(k-1)}\,\mathtt{dq(k)}\,/\,\mathtt{denom(k)}\,(k = 1, \ldots, N+1) \quad \text{(C23)}$$

$$\mathtt{qsn(k)} \equiv Dq_k^{n+1} = \mathtt{cvmgp[qr(k),0,dq(k-1)dq(k)]}(k=0,\ldots,N+1) \quad (C24)$$

and

$$\mathtt{qb(k)} \equiv \overline{q}_k^{n+1} =$$

$$(0.5+s_k)\{\theta[\mathtt{q(k-1)} + 0.5\mathtt{qsn(k-1)}] + (1-\theta)[\mathtt{qo(k-1)} + 0.5\mathtt{qso(k-1)}]\}$$
$$+(0.5-s_k)\{\theta[\mathtt{q(k\quad)} + 0.5\mathtt{qs(k\quad)}] + (1-\theta)[\mathtt{qo(k\quad)} + 0.5\mathtt{qso(k\quad)}]\}$$
$$(k=1,\ldots,N+1) \quad (C25)$$

Hence

$$\mathtt{dqrdlqm1(k)} \equiv \partial R_k^{n+1}/\partial ln q_{k-1}^{n+1} = -C_{adv} q_{k-1}^{n+1}[\mathtt{dq(k)/denom(k)}]^2 \quad (C26)$$

$$\mathtt{dqrdlq00(k)} \equiv \partial R_k^{n+1}/\partial ln q_k^{n+1} =$$
$$-C_{adv} q_k^{n+1}[\mathtt{dq(k)-dq(k-1)}]/\mathtt{denom(k)} \quad (C27)$$

$$\mathtt{dqrdlqp1(k)} \equiv \partial R_k^{n+1}/\partial ln q_{k+1}^{n+1} = +C_{adv} q_{k+1}^{n+1}[\mathtt{dq(k-1)/denom(k)}]^2 \quad (C28)$$

and

$$\mathtt{dqsdlqm1(k)} \equiv \partial(Dq_k^{n+1})/\partial ln q_{k-1}^{n+1} =$$
$$\mathtt{cvmgp[dqrdlqm1(k),0,dq(k-1)dq(k)]} \quad (C29)$$

$$\mathtt{dqsdlq00(k)} \equiv \partial(Dq_k^{n+1})/\partial ln q_k^{n+1} =$$
$$\mathtt{cvmgp[dqrdlq00(k),0,dq(k-1)dq(k)]} \quad (C30)$$

$$\mathtt{dqsdlqp1(k)} \equiv \partial(Dq_k^{n+1})/\partial ln q_{k+1}^{n+1} =$$
$$\mathtt{cvmgp[dqrdlqp1(k),0,dq(k-1)dq(k)]} \quad (C31)$$

Then

$$\delta\overline{q}_k^{n+1} = \theta[(0.5+s_k)(\delta q_{k-1}^{n+1} + 0.5\delta Dq_{k-1}^{n+1}) + (0.5-s_k)(\delta q_k^{n+1} - 0.5\delta Dq_k^{n+1})] \quad (C32)$$

Therefore

$$\mathtt{dqbdlqm2(k)} \equiv \partial(\delta\overline{q}_k^{n+1})/\partial ln q_{k-2}^{n+1} = \tfrac{1}{2}\theta(0.5+s_k)\mathtt{dqsdlqm1(k-1)} \quad (C33)$$

$$\texttt{dqbdlqm1(k)} \equiv \partial(\delta\overline{q}_k^{n+1})/\partial ln q_{k-1}^{n+1} =$$

$$\theta\{(0.5+s_k)[\texttt{q(k}-\texttt{1)} + \tfrac{1}{2}\texttt{dqsdlq00(k}-\texttt{1)}] - \tfrac{1}{2}(0.5-s_k)\texttt{dqsdlqm1(k)}\} \quad (C34)$$

$$\texttt{dqbdlq00(k)} \equiv \partial(\delta\overline{q}_k^{n+1})/\partial ln q_k^{n+1} =$$

$$\theta\{\tfrac{1}{2}(0.5+s_k)\texttt{dqsdlqp1(k}-\texttt{1)} + (0.5-s_k)[\texttt{q(k)} - \tfrac{1}{2}\texttt{dqsdlq00(k)}]\} \quad (C35)$$

$$\texttt{dqbdlqp1(k)} \equiv \partial(\delta\overline{q}_k^{n+1})/\partial ln q_{k+1}^{n+1} = -\tfrac{1}{2}\theta(0.5-s_k)\texttt{dqsdlqp1(k)} \quad (C36)$$

For the continuity equation, the advected quantity $q_k \equiv \rho_k$, hence $\partial/\partial q \equiv \partial/\partial\rho$. Thus

```
em2(id,jd):                        − rmu(k) urel(k) dqbdlqm2(k)   (C37)
em1(id,jd):  rmu(k+1) urel(k+1) dqbdlqm2(k+1)
                                   − rmu(k) urel(k) dqbdlqm1(k)   (C38)
e00(id,jd):  rmu(k+1) urel(k+1) dqbdlqm1(k+1)
                                   − rmu(k) urel(k) dqbdlq00(k)   (C39)
ep1(id,jd):  rmu(k+1) urel(k+1) dqbdlq00(k+1)
                                   − rmu(k) urel(k) dqbdlqp1(k)   (C40)
ep2(id,jd):  rmu(k+1) urel(k+1) dqbdlqp1(k+1)                     (C41)
```

(3) Diffusion

The quantities needed to calculate the derivatives of the diffusion term are generated in a separate subroutine, named `diffuse`. It treats the diffusion of both the density ρ in the continuity equation, and the gas energy density e in the radiating fluid energy equation. For generality, and to avoid repetition, we shall denote the quantity diffused as q. The inputs required by the subroutine are, for $(k = 1, \ldots, N+1)$:

			Mass diffusion	Energy diffusion
`qd (k)` $\equiv$	$q_k^{n+\theta}$ =	quantity diffused at $t^{n+\theta}$ =	$\rho_k^{n+\theta}$	$e_k^{n+\theta}$
`qdn(k)` $\equiv$	q_k^{n+1} =	quantity diffused at t^{n+1} =	ρ_k^{n+1}	e_k^{n+1}
	σ =	diffusion coefficient =	σ_ρ	σ_e
	`zet` =	density exponent =	0	1

Then the diffusion flux $\mathtt{df(k)}$ and its derivatives for $(k = 1, \ldots, N+1)$ are $\mathtt{df(k)} \equiv Q_k^{n+\theta} =$

$$2^{(1-\mathtt{zet})}\,\sigma\,\mathtt{rmu(k)}(\rho_k^{n+\theta} + \rho_{k-1}^{n+\theta})^{\mathtt{zet}}(\mathtt{qd(k)}-\mathtt{qd(k-1)})/(r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}) \quad \text{(C42)}$$

$$\mathtt{ddfdlrm1(k)} \equiv \partial Q_k^{n+\theta}/\partial ln r_{k-1}^{n+1} = \theta\,\mathtt{df(k)}\,r_{k-1}^{n+1}/(r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}) \quad \text{(C43)}$$

$$\mathtt{ddfdlr00(k)} \equiv \partial Q_k^{n+\theta}/\partial ln r_k^{n+1} = \mu\theta\,\mathtt{df(k)}(r_k^{n+1}/r_k^{n+\theta}) \quad \text{(C44)}$$

$$\mathtt{ddfdlrp1(k)} \equiv \partial Q_k^{n+\theta}/\partial ln r_{k+1}^{n+1} = -\theta\,\mathtt{df(k)}\,r_{k+1}^{n+1}/(r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}) \quad \text{(C45)}$$

$$\mathtt{ddfdlqm1(k)} \equiv \partial Q_k^{n+\theta}/\partial ln q_{k-1}^{n+1} =$$

$$-2^{(1-\mathtt{zet})}\,\sigma\theta\;\mathtt{qdn(k-1)}\,(\rho_k^{n+\theta} + \rho_{k-1}^{n+\theta})^{\mathtt{zet}}\,(r_k^{n+\theta})^{\mu}/(r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}) \quad \text{(C46)}$$

$$\mathtt{ddfdlq00(k)} \equiv \partial Q_k^{n+\theta}/\partial ln q_k^{n+1} =$$

$$2^{(1-\mathtt{zet})}\,\sigma\theta\;\mathtt{qdn(k\quad)}\,(\rho_k^{n+\theta} + \rho_{k-1}^{n+\theta})^{\mathtt{zet}}\,(r_k^{n+\theta})^{\mu}/(r_{k+1}^{n+\theta} - r_{k-1}^{n+\theta}) \quad \text{(C47)}$$

$$\mathtt{ddfdldm1(k)} \equiv \partial Q_k^{n+\theta}/\partial ln \rho_{k-1}^{n+1} = \mathtt{zet}\theta\,\mathtt{df(k)}\,\rho_{k-1}^{n+1}/(\rho_k^{n+\theta} + \rho_{k-1}^{n+\theta}) \quad \text{(C48)}$$

$$\mathtt{ddfdld00(k)} \equiv \partial Q_k^{n+\theta}/\partial ln \rho_k^{n+1} = \mathtt{zet}\theta\,\mathtt{df(k)}\,\rho_k^{n+1}/(\rho_k^{n+\theta} + \rho_{k-1}^{n+\theta}) \quad \text{(C49)}$$

Thus

$$\mathtt{em1(id,jr)}: \quad \mathtt{ddfdlrm1(k)} \quad \text{(C50)}$$

$$\mathtt{e00(id,jr)}: \quad \mathtt{ddfdlr00(k)} - \mathtt{ddfdlrm1(k+1)} \quad \text{(C51)}$$

$$\mathtt{ep1(id,jr)}: \quad \mathtt{ddfdlrp1(k)} - \mathtt{ddfdlr00(k+1)} \quad \text{(C52)}$$

$$\mathtt{ep2(id,jr)}: \quad - \mathtt{ddfdlrp1(k+1)} \quad \text{(C53)}$$

$$\texttt{em1(id,jd):}\quad \texttt{ddfdlqm1(k)} \tag{C54}$$

$$\texttt{e00(id,jd):}\quad \texttt{ddfdlq00(k)} \;-\; \texttt{ddfdlqm1(k+1)} \tag{C55}$$

$$\texttt{ep1(id,jd):}\qquad\qquad\qquad -\; \texttt{ddfdlq00(k+1)} \tag{C56}$$

Notice that in equations (C55) and (C56) we have made explicit use of the facts that for mass diffusion, $\texttt{zet} \equiv 0$, and $(\partial ln q/\partial ln \rho) \equiv 1$. Finally,

$$\begin{aligned} -\texttt{rhs(id)} = {} & [\,\rho_k^{n+1}\texttt{dvoln(k)} - \rho_k^n\,\texttt{dvolo(k)}]\;/\;dt \\ & + \texttt{rmu}(\texttt{k}+1)\,\texttt{urel}(\texttt{k}+1)\,\overline{\rho}_{k+1} - \texttt{rmu}(\texttt{k})\,\texttt{urel}(\texttt{k})\,\overline{\rho}_k \\ & - \texttt{df}(\texttt{k}+1) + \texttt{df}(\texttt{k}) \end{aligned} \tag{C57}$$

V. GAS MOMENTUM

A. Differential Equation

$$\frac{d}{dt}[u(\rho\Delta V)] - \Delta\left(\frac{dm}{dt}u\right) + r^\mu \Delta p + \left(\frac{2-\mu}{2}g + \frac{2\pi\mu G m}{r^\mu}\right)(\rho\Delta V)$$

$$-\phi_Q \Delta V - \frac{\chi_F}{c}F(\rho\Delta V) = 0 \quad \text{(GM1)}$$

Here $\mu = 0$, 1, or 2, and

$$\phi_Q \equiv \frac{4}{3}r^{-(\mu/2)}\Delta\left[\rho\mu_Q r^{(3\mu/2)}\left(\frac{\Delta u}{\Delta r} - \frac{\mu}{2} < \frac{u}{r} >\right)\right]/\Delta V, \quad \text{(GM2)}$$

where

$$\mu_Q \equiv C_1 \ell a_s - min[\, C_2 \ell^2 \Delta(r^\mu u)/\Delta V,\ 0\,] \quad \text{(GM3)}$$

and $\ell \equiv \ell_0 + \ell_1 r$. ϕ_Q is interface centered, and μ_Q is cell centered.

B. Difference Equation

For $(k = 2, \ldots, N)$,

$$\left[u_k^{n+1}(\rho_{k-1}^{n+1}\Delta V_{k-1}^{n+1} + \rho_k^{n+1}\Delta V_k^{n+1}) - u_k^n(\rho_{k-1}^n \Delta V_{k-1}^n + \rho_k^n \Delta V_k^n)\right]/2dt$$

$$-[(m_k^{n+1} - m_k^n) + (m_{k+1}^{n+1} - m_{k+1}^n)]\overline{u}_k \quad /2dt$$
$$+[(m_{k-1}^{n+1} - m_{k-1}^n) + (m_k^{n+1} - m_k^n)]\overline{u}_{k-1}/2dt$$

$$+(r_k^{n+\theta})^\mu (p_k^{n+\theta} - p_{k-1}^{n+\theta})$$

$$+\left[\frac{2-\mu}{2}g + \frac{2\pi\mu G m_k^{n+\theta}}{(r_k^{n+\theta})^\mu} - \frac{<\chi>_k^{n+\theta}\ F_k^{n+\theta}}{c}\right]\frac{\rho_{k-1}^{n+\theta}\Delta V_{k-1}^{n+\theta} + \rho_k^{n+\theta}\Delta V_k^{n+\theta}}{2}$$

$$-\frac{4\rho_k^{n+\theta}(\mu_Q)_k^{n+\theta}}{3(r_k^{n+\theta})^{\mu/2}}\left(\frac{r_k^{n+\theta} + r_{k+1}^{n+\theta}}{2}\right)^{3\mu/2}\left[\frac{u_{k+1}^{n+\theta} - u_k^{n+\theta}}{r_{k+1}^{n+\theta} - r_k^{n+\theta}} - \frac{\mu}{4}\left(\frac{u_{k+1}^{n+\theta}}{r_{k+1}^{n+\theta}} + \frac{u_k^{n+\theta}}{r_k^{n+\theta}}\right)\right]$$

$$+\frac{4\rho_{k-1}^{n+\theta}(\mu_Q)_{k-1}^{n+\theta}}{3(r_k^{n+\theta})^{\mu/2}}\left(\frac{r_{k-1}^{n+\theta}+r_k^{n+\theta}}{2}\right)^{3\mu/2}\left[\frac{u_k^{n+\theta}-u_{k-1}^{n+\theta}}{r_k^{n+\theta}-r_{k-1}^{n+\theta}}-\frac{\mu}{4}\left(\frac{u_k^{n+\theta}}{r_k^{n+\theta}}+\frac{u_{k-1}^{n+\theta}}{r_{k-1}^{n+\theta}}\right)\right]$$

$$=0 \tag{GM4}$$

Equation (GM4) provides $N-1$ relations connecting the velocities at $N+1$ gridpoints. We thus require two boundary conditions. In equation (GM4)

$$(\mu_Q)_k^{n+\theta}=$$

$$C_1\ell_k(a_s)_k^{n+\theta}-C_2\ell_k^2\min\left\{\frac{(\mu+1)[(r_{k+1}^{n+\theta})^\mu u_{k+1}^{n+\theta}-(r_k^{n+\theta})^\mu u_k^{n+\theta}]}{[(r_{k+1}^{n+\theta})^{\mu+1}-(r_k^{n+\theta})^{\mu+1}]},0\right\} \tag{GM5}$$

where

$$\ell_k\equiv\ell_0+\ell_1\tfrac{1}{2}(r_k^{n+\theta}+r_{k+1}^{n+\theta}) \tag{GM6}$$

Values for the velocity advected at cell centers are defined as

$$\begin{aligned}\overline{u}_k\equiv\ &(0.5+s_k)[\theta(u_k^{n+1}+0.5Du_k^{n+1})+(1-\theta)(u_k^n\quad+0.5Du_k^n\quad)]\\ &+(0.5-s_k)[\theta(u_{k+1}^{n+1}-0.5Du_{k+1}^{n+1})+(1-\theta)(u_{k+1}^n-0.5Du_{k+1}^n)]\end{aligned} \tag{GM7}$$

The switch s_k

$$s_k\equiv\texttt{cvmgp}\left[0.5,-0.5,-\left(\tfrac{dm_k}{dt}+\tfrac{dm_{k+1}}{dt}\right)\right] \tag{GM8}$$

chooses the upstream direction, and the slope Du_k^{n+1} inside the cell is given by

$$Du_k^{n+1}\equiv\texttt{cvmgp}\left[\frac{C\Delta u_{k-1}^{n+1}\Delta u_k^{n+1}}{\Delta u_{k-1}^{n+1}+\Delta u_k^{n+1}},\ 0,\ \Delta u_k^{n+1}\right] \tag{GM9}$$

which yields monotonized van Leer advection if $C=2$, and donor cell ad–vection if $C=0$. Here

$$\Delta u_{k+1}^{n+1}\equiv u_{k+1}^{n+1}-u_k^{n+1} \tag{GM10}$$

The opacity at an interface $<\chi>_k^{n+\theta}$ is defined to be

$$\frac{1}{<\chi>_k^{n+\theta}} \equiv \frac{1}{2}\left(\frac{1}{\chi_{k-1}^{n+\theta}} + \frac{1}{\chi_k^{n+\theta}}\right) \tag{GM11}$$

C. Linearization

Calculating matrix elements we find:

(1) Time Derivative

$$\texttt{em1(iu,jr)}: -u_k^{n+1}\rho_{k-1}^{n+1}\ \texttt{rmup1n(k}-\texttt{1)}/2dt \tag{GM12}$$
$$\texttt{e00(iu,jr)}:\ u_k^{n+1}(\rho_{k-1}^{n+1} - \rho_k^{n+1})\ \texttt{rmup1n(k)}/2dt \tag{GM13}$$
$$\texttt{ep1(iu,jr)}:\ u_k^{n+1}\rho_k^{n+1}\ \texttt{rmup1n(k+1)}/2dt \tag{GM14}$$
$$\texttt{em1(iu,jd)}:\ u_k^{n+1}\rho_{k-1}^{n+1}\ \texttt{dvoln(k}-\texttt{1)}/2dt \tag{GM15}$$
$$\texttt{e00(iu,jd)}:\ u_k^{n+1}\rho_k^{n+1}\ \texttt{dvoln(k\ \ \ )}/2dt \tag{GM16}$$
$$\texttt{e00(iu,ju)}:\texttt{unom(k)}[\rho_{k-1}^{n+1}\ \texttt{dvoln(k}-\texttt{1)}\ +\rho_k^{n+1}\ \texttt{dvoln(k)}]/2dt \tag{GM17}$$

(2) Advection

The quantities needed to calculate the derivatives of the advection term with respect to density are generated in subroutine `advecti`, which treats the advection of interface centered variables such as u and F/ρ. As we did for cell centered quantities, denote the advected quantity as q. The inputs required by the subroutine are, for $(k = 0, \ldots, N+2)$,

$$\begin{array}{lll} \texttt{q(k)} & \equiv\ q_k^{n+1} & = u_k^{n+1} \\ \texttt{qo(k)} & \equiv\ q_k^{n} & = u_k^{n} \\ \texttt{qso(k)} & \equiv Dq_k^{n} & = \text{monotonized slope at old time} \end{array}$$

and

$$\texttt{flow(k)} \equiv -\ [\texttt{dmdt(k)+dmdt(k+1)}] \tag{GM18}$$

where

$$\texttt{dmdt(k)} \equiv \quad (m_k^{n+1} - m_k^n)\,/\,dt \tag{GM19}$$

Then, proceeding as in (C20) - (C36), and using the fact that $q_k \equiv u_k$, we get

`em1(iu,jm):` $\quad m_{k-1}^{n+1}\overline{u}_{k-1}/2\,dt$ (GM20)

`e00(iu,jm):` $\quad m_{k}^{n+1}(\overline{u}_{k-1}-\overline{u}_k)/2\,dt$ (GM21)

`ep1(iu,jm):` $-m_{k+1}^{n+1}\overline{u}_{k}/2\,dt$ (GM22)

```
em2(iu,ju):unom(k-2)[dmdt(k-1)+dmdt(k)]dqbdqm1(k-1)/2   (GM23)
em1(iu,ju):unom(k-1){[dmdt(k-1)+dmdt(k)]dqbdq00(k-1)
                    -[dmdt(k+1)+dmdt(k)]dqbdqm1(k  )}/2 (GM24)
e00(iu,ju):unom(k  ){[dmdt(k-1)+dmdt(k)]dqbdqp1(k-1)
                    -[dmdt(k+1)+dmdt(k)]dqbdq00(k  )}/2 (GM25)
ep1(iu,ju):unom(k+1){[dmdt(k-1)+dmdt(k)]dqbdqp2(k-1)
                    -[dmdt(k+1)+dmdt(k)]dqbdqp1(k  )}/2 (GM26)
ep2(iu,ju):-unom(k+2)[dmdt(k+1)+dmdt(k)]dqbdqp2(k  )/2  (GM27)
```

(3) Pressure Gradient

`e00(iu,jr):` $\mu\theta\,\mathtt{rmu(k)}(r_k^{n+1}/r_k^{n+\theta})(p_k^{n+\theta}-p_{k-1}^{n+\theta})$ (GM28)

`em1(iu,jd):` $-\theta\,\mathtt{rmu(k)}\,p_{k-1}^{n+1}(\partial lnp/\partial ln\rho)_{k-1}^{n+1}$ (GM29)

`e00(iu,jd):` $\theta\,\mathtt{rmu(k)}\,p_{k}^{n+1}(\partial lnp/\partial ln\rho)_{k}^{n+1}$ (GM30)

`em1(iu,jt):` $-\theta\,\mathtt{rmu(k)}\,p_{k-1}^{n+1}(\partial lnp/\partial lnT)_{k-1}^{n+1}$ (GM31)

`e00(iu,jt):` $\theta\,\mathtt{rmu(k)}\,p_{k}^{n+1}(\partial lnp/\partial lnT)_{k}^{n+1}$ (GM32)

(4) Gravity

For brevity, define

$$G_1 \equiv \left[\left(1-\frac{\mu}{2}\right)g+\frac{\mu}{2}\frac{4\pi G m_k^{n+\theta}}{(r_k^{n+\theta})^\mu}\right] \quad \text{and} \quad G_2 \equiv \frac{\mu\pi G m_k^{n+\theta}}{(r_k^{n+\theta})^\mu}$$

Then

$$\texttt{em1(iu,jr)}: -\tfrac{1}{2}\theta G_1\ \rho_{k-1}^{n+\theta} \qquad \texttt{rmup1(k-1)} \qquad (r_{k-1}^{n+1}/r_{k-1}^{n+\theta}) \tag{GM33}$$

$$\texttt{ep1(iu,jr)}: \ \tfrac{1}{2}\theta G_1\ \rho_{k}^{n+\theta} \qquad \texttt{rmup1(k+1)} \qquad (r_{k+1}^{n+1}/r_{k+1}^{n+\theta}) \tag{GM34}$$

$$\begin{aligned}\texttt{e00(iu,jr)}: \ & \tfrac{1}{2}\theta G_1(\rho_{k-1}^{n+\theta}-\rho_{k}^{n+\theta})\,\texttt{rmup1(k)} \qquad (r_{k}^{n+1}/r_{k}^{n+\theta}) \\ & -\mu\theta G_2[\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)}+\rho_{k}^{n+\theta}\texttt{dvol(k)}](r_{k}^{n+1}/r_{k}^{n+\theta})\end{aligned} \tag{GM35}$$

$$\texttt{e00(iu,jm)}: \ \theta G_2[\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)}+\rho_{k}^{n+\theta}\texttt{dvol(k)}](m_{k}^{n+1}/m_{k}^{n+\theta}) \tag{GM36}$$

$$\texttt{em1(iu,jr)}: \ \tfrac{1}{2}\theta G_1\ \rho_{k-1}^{n+1}\texttt{dvol(k-1)} \tag{GM37}$$

$$\texttt{e00(iu,jr)}: \ \tfrac{1}{2}\theta G_1\ \rho_{k}^{n+1}\texttt{dvol(k)} \tag{GM38}$$

(5) Radiation Force

These matrix elements are the negative of those given for the radiation momentum equation, (RM45) – (RM52), with the index `if` replaced by the index `iu`.

(6) Viscous Force

The quantities needed to calculate the viscous force and its derivatives are generated in a separate subroutine, named `viscous`. The viscous momentum deposition rate (i.e. viscous force) ϕ_Q is an interface centered quantity needed only within the computational domain, that is at $(k = 2, \ldots, N)$. Thus we need the radius r, velocity u, and some auxiliary vectors defined below at $(k = 1, \ldots, N)$, and we apply the algorithm for `qf(k)`$= (\phi_Q)_k^{n+\theta}$ at $(k = 2, \ldots, N)$.

Define:

$$f0\ (r_0, r_+) \equiv \quad [\tfrac{1}{2}(r_0 + r_+)]^{3\mu/2} \tag{GM39}$$

$$f0r(r_0, r_+) \equiv \tfrac{3\mu}{4}[\tfrac{1}{2}(r_0 + r_+)]^{(3\mu/2)-1} \tag{GM40}$$

$$f1\ (r_0 \quad) \equiv \qquad r_0^{-\mu/2} \tag{GM41}$$

$$f1r(r_0 \quad) \equiv -\frac{1}{2}\mu r_0^{-(\mu/2)-1} \tag{GM42}$$

and

$$\mathtt{r3(k)} \quad \equiv \quad f0\ \ (r_k^{n+\theta}, r_{k+1}^{n+\theta}) \tag{GM43}$$

$$\mathtt{dr3dlr00(k)} \equiv \theta r_k^{n+1} f0r(r_k^{n+\theta}, r_{k+1}^{n+\theta}) \tag{GM44}$$

$$\mathtt{dr3dlrp1(k)} \equiv \theta r_{k+1}^{n+1} f0r(r_k^{n+\theta}, r_{k+1}^{n+\theta}) \tag{GM45}$$

$$\mathtt{r1(k)} \quad \equiv \quad f1\ \ (r_k^{n+\theta}) \tag{GM46}$$

$$\mathtt{dr1dlr00(k)} \equiv \theta r_k^{n+1} f1r(r_k^{n+\theta}) \tag{GM47}$$

Then the viscous force is

$$\mathtt{qf(k)\ =}\ -\frac{4}{3}\rho_k^{n+\theta}\ \mathtt{qm(k)\ dudr(k)} \tag{GM48}$$

where `qm(k)` is given by (FE81) and `dudr(k)` is given by FE(63). Then

$$\mathtt{dqfdlr00(k)} \equiv \partial \mathtt{qf}/\partial ln r_k^{n+1} = -\frac{4}{3}\rho_k^{n+\theta}\mathtt{[dqmdlr00(k)dudr(k)+qm(k)durdlr00(k)]} \tag{GM49}$$

$$\mathtt{dqfdlrp1(k)} \equiv \partial \mathtt{qf}/\partial ln r_{k+1}^{n+1} = -\frac{4}{3}\rho_k^{n+\theta}\mathtt{[dqmdlrp1(k)dudr(k)+qm(k)durdlrp1(k)]} \tag{GM50}$$

$$\mathtt{dqfdlu00(k)} \equiv \partial \mathtt{qf}/\partial ln u_k^{n+1} = -\frac{4}{3}\rho_k^{n+\theta}\mathtt{[dqmdlu00(k)dudr(k)+qm(k)durdlu00(k)]} \tag{GM51}$$

$$\mathtt{dqfdlup1(k)} \equiv \partial \mathtt{qf}/\partial ln u_{k+1}^{n+1} = -\frac{4}{3}\rho_k^{n+\theta}\mathtt{[dqmdlup1(k)dudr(k)+qm(k)durdlup1(k)]} \tag{GM52}$$

$$\texttt{dqfdlt00(k)} \equiv \partial\texttt{qf}/\partial lnT_k^{n+1} = -\frac{4}{3}\rho_k^{n+\theta}\ \texttt{dqmdlt00(k)dudr(k)} \quad \text{(GM53)}$$

$$\texttt{dqfdld00(k)} \equiv \partial\texttt{qf}/\partial ln\rho_k^{n+1} =$$

$$-\frac{4}{3}\rho_k^{n+\theta}\texttt{dqmdld00(k)dudr(k)} - \frac{4}{3}\theta\rho_k^{n+1}\texttt{qm(k)dudr(k)} \quad \text{(GM54)}$$

Thus we find

```
em1(iu,jr):

      -r1(k)[dr3dlr00(k-1)qf(k-1)+r3(k-1)dqfdlr00(k-1)]
```
(GM55)

```
e00(iu,jr):  dr1dlr00(k)[r3(k  )qf(k  )-r3(k-1)qf(k-1)]

      +r1(k)[dr3dlr00(k  )qf(k  )-dr3dlrp1(k-1)qf(k-1)]

      +r1(k)[r3(k  )dqfdrl00(k  )-r3(k-1)dqfdlrp1(k-1)]
```
(GM56)

```
ep1(iu,jr):

       r1(k)[dr3dlrp1(k  )qf(k  )+r3(k  )dqfdlrp1(k  )]
```
(GM57)

```
em1(iu,ju):-r1(k)r3(k-1)dqfdlu00(k-1)
```
(GM58)

```
e00(iu,ju): r1(k)[r3(k  )dqfdlu00(k  )

                           -r3(k-1)dqfdlup1(k-1)]
```
(GM59)

```
ep1(iu,ju): r1(k)r3(k  )dqfdlup1(k  )
```
(GM60)

$$\texttt{em1(iu,jt):-r1(k)r3(k-1)dqfdlt00(k-1)} \tag{GM61}$$

$$\texttt{e00(iu,jt): r1(k)r3(k\ \ )dqfdlt00(k\ \ )} \tag{GM62}$$

$$\texttt{em1(iu,jd):-r1(k)r3(k-1)dqfdld00(k-1)} \tag{GM63}$$

$$\texttt{e00(iu,jd): r1(k)r3(k\ \ )dqfdld00(k\ \ )} \tag{GM64}$$

(7) Right Hand Side

$$-\texttt{rhs(iu)}=$$

$$\{u_k^{n+1}[\rho_{k-1}^{n+1}\texttt{dvoln(k-1)}+\rho_k^{n+1}\texttt{dvoln(k)}]$$

$$-u_k^n[\rho_{k-1}^n\texttt{dvolo(k-1)}+\rho_k^n\texttt{dvolo(k)}]\}/2dt$$

$$-\frac{1}{2}\{[\texttt{dmdt(k)+dmdt(k+1)}]\overline{u}_k-[\texttt{dmdt(k-1)-dmdt(k)}]\overline{u}_{k-1}\}$$

$$+\frac{1}{2}[(1-\frac{\mu}{2})g+\frac{2\pi\mu G m_k^{n+\theta}}{(r_k^{n+\theta})^\mu}-\frac{<\chi>_k^{n+\theta}F_k^{n+\theta}}{c}][\rho_{k-1}^{n+\theta}\texttt{dvolo(k-1)}+\rho_k^{n+\theta}\texttt{dvolo(k)}]$$

$$+\texttt{rmu(k)}(p_k^{n+\theta}-p_{k-1}^{n+\theta})+\texttt{r1(k)[r3(k)qf(k)}-\texttt{r3(k-1)qf(k-1)]} \tag{GM65}$$

VI. RADIATING FLUID ENERGY

A. Differential Equation

$$\frac{d}{dt}\left[\left(e+\frac{E}{\rho}\right)\rho\Delta V\right]-\Delta\left[\frac{dm}{dt}\left(e+\frac{E}{\rho}\right)-r^\mu F\right]$$

$$+(p+P)\Delta(r^\mu u)+(E-3P)\frac{u}{r}\Delta V=\Delta\left(r^\mu\sigma_e\frac{\Delta e}{\Delta r}\right)+\epsilon_Q\Delta V \qquad \text{(FE1)}$$

where $\mu = 0$ or 2, and

$$\epsilon_Q \equiv \frac{4}{3}\mu_Q\,\rho\left(\frac{\Delta u}{\Delta r}-\frac{\mu}{2}<\frac{u}{r}>\right)^2 \qquad \text{(FE2)}$$

Note that ϵ_Q is cell centered, as is μ_Q.

B. Difference Equation

For $(k = 2, \ldots, N-1)$,

$$\left[(\rho_k^{n+1}e_k^{n+1}+E_k^{n+1})\Delta V_k^{n+1}-(\rho_k^n e_k^n+E_k^n)\Delta V_k^n\right]/dt$$

$$-[(m_{k+1}^{n+1}-m_{k+1}^n)(\overline{e+\tfrac{E}{\rho}})_{k+1}-(m_k^{n+1}-m_k^n)(\overline{e+\tfrac{E}{\rho}})_k]/dt$$

$$+(r_{k+1}^{n+\theta})^\mu F_{k+1}^{n+\theta}-(r_k^{n+\theta})^\mu F_k^{n+\theta}$$

$$+(p_k^{n+\theta}+f_k^{n+\theta}E_k^{n+\theta})[(r_{k+1}^{n+\theta})^\mu u_{k+1}^{n+\theta}-(r_k^{n+\theta})^\mu u_k^{n+\theta}]$$

$$+\frac{\mu}{4}(1-3f_k^{n+\theta})E_k^{n+\theta}\left(\frac{u_{k+1}^{n+\theta}}{r_{k+1}^{n+\theta}}+\frac{u_k^{n+\theta}}{r_k^{n+\theta}}\right)\Delta V_k^{n+\theta}$$

$$-\sigma_e\left[(r_{k+1}^{n+\theta})^\mu(\rho_k^{n+\theta}+\rho_{k+1}^{n+\theta})\left(\frac{e_{k+1}^{n+\theta}-e_k^{n+\theta}}{r_{k+2}^{n+\theta}-r_k^{n+\theta}}\right)\right.$$

$$\left.-(r_k^{n+\theta})^\mu(\rho_{k-1}^{n+\theta}+\rho_k^{n+\theta})\left(\frac{e_k^{n+\theta}-e_{k-1}^{n+\theta}}{r_{k+1}^{n+\theta}-r_{k-1}^{n+\theta}}\right)\right]$$

$$-\frac{4}{3}(\mu_Q)_k^{n+\theta}\rho_k^{n+\theta}\left[\frac{u_{k+1}^{n+\theta}-u_k^{n+\theta}}{r_{k+1}^{n+\theta}-r_k^{n+\theta}}-\frac{\mu}{2}\left(\frac{u_{k+1}^{n+\theta}}{r_{k+1}^{n+\theta}}+\frac{u_k^{n+\theta}}{r_k^{n+\theta}}\right)\right]^2\Delta V_k^{n+\theta}$$

$$=0 \tag{FE3}$$

Equation (FE3) provides $N-2$ relations connecting the the energy density of the radiating fluid at N gridpoints. We thus require two boundary conditions.

C. Linearization

Calculating matrix elements we find:

(1) Time Derivative

e00(it,jr): $-(\rho_k^{n+1}e_k^{n+1}+E_k^{n+1})$ rmup1n(k)$/dt$ (FE4)

ep1(it,jr): $(\rho_k^{n+1}e_k^{n+1}+E_k^{n+1})$ rmup1n(k+1)$/dt$ (FE5)

e00(it,jd): $\rho_k^{n+1}e_k^{n+1}[1+(\partial lne/\partial ln\rho)_k^{n+1}]$ dvoln(k)$/dt$ (FE6)

e00(it,jt): $\rho_k^{n+1}e_k^{n+1} \quad (\partial lne/\partial lnT)_k^{n+1}$ dvoln(k)$/dt$ (FE7)

e00(it,je): E_k^{n+1}dvoln(k)$/dt$ (FE8)

(2) Advection

The quantities needed to calculate the derivatives of the advection term are generated in the subroutine `advectc`. As before, denote the advected quantity as q. The inputs required by the subroutine for $(k=0,\ldots,N+2)$ are:

q (k)	=	q_k^{n+1}	$\equiv (e+\frac{E}{\rho})_k^{n+1}$
qo (k)	=	q_k^n	$\equiv (e+\frac{E}{\rho})_k^n$
qso (k)	=	Dq_k^n	= monotonized slope at old time
flow(k)	=	−dmdt(k)	direction of flow at interface k

Then, proceeding as in (C20) - (C36), and using the facts that

$$\left(\frac{\partial lnq}{\partial lnE}\right)_k^{n+1} = \left(\frac{E}{\rho}\right)_k^{n+1} / \left(e + \frac{E}{\rho}\right)_k^{n+1} = \texttt{dlqdle(k)} \qquad \text{(FE9)}$$

$$\left(\frac{\partial lnq}{\partial lnT}\right)_k^{n+1} = \left(e\frac{\partial lne}{\partial lnT}\right)_k^{n+1} / \left(e + \frac{E}{\rho}\right)_k^{n+1} = \texttt{dlqdlt(k)} \qquad \text{(FE10)}$$

$$\left(\frac{\partial lnq}{\partial ln\rho}\right)_k^{n+1} = \left(e\frac{\partial lne}{\partial ln\rho} - \frac{E}{\rho}\right)_k^{n+1} / \left(e + \frac{E}{\rho}\right)_k^{n+1} = \texttt{dlqdld(k)} \qquad \text{(FE11)}$$

we get

$\texttt{dqbdlem2(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnE_{k-2}^{n+1} = \texttt{dqbdlqm2(k) dlqdle(k-2)}$ (FE12)

$\texttt{dqbdlem1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnE_{k-1}^{n+1} = \texttt{dqbdlqm1(k) dlqdle(k-1)}$ (FE13)

$\texttt{dqbdle00(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnE_{k}^{n+1} = \texttt{dqbdlq00(k) dlqdle(k)}$ (FE14)

$\texttt{dqbdlep1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnE_{k+1}^{n+1} = \texttt{dqbdlqp1(k) dlqdle(k+1)}$ (FE15)

$\texttt{dqbdltm2(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnT_{k-2}^{n+1} = \texttt{dqbdlqm2(k) dlqdlt(k-2)}$ (FE16)

$\texttt{dqbdltm1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnT_{k-1}^{n+1} = \texttt{dqbdlqm1(k) dlqdlt(k-1)}$ (FE17)

$\texttt{dqbdlt00(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnT_{k}^{n+1} = \texttt{dqbdlq00(k) dlqdlt(k)}$ (FE18)

$\texttt{dqbdltp1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnT_{k+1}^{n+1} = \texttt{dqbdlqp1(k) dlqdlt(k+1)}$ (FE19)

$\texttt{dqbdldm2(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k-2}^{n+1} = \texttt{dqbdlqm2(k) dlqdld(k-2)}$ (FE20)

$\texttt{dqbdldm1(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k-1}^{n+1} = \texttt{dqbdlqm1(k) dlqdld(k-1)}$ (FE21)

$\texttt{dqbdld00(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k}^{n+1} = \texttt{dqbdlq00(k) dlqdld(k)}$ (FE22)

$\texttt{dqbdldp1(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k+1}^{n+1} = \texttt{dqbdlqp1(k) dlqdld(k+1)}$ (FE23)

Thus

$\texttt{e00(it,jm):}\ \ \overline{q}_k \quad m_k^{n+1}/dt$ (FE24)

$\texttt{ep1(it,jm):} -\overline{q}_{k+1} m_{k+1}^{n+1}/dt$ (FE25)

$\texttt{em2(it,jd):dmdt(k)dqbdldm2(k)}$ (FE26)

$\texttt{em1(it,jd):dmdt(k)dqbdldm1(k)} - \texttt{dmdt(k+1)dqbdldm2(k+1)}$ (FE27)

$$\texttt{e00(it,jd):dmdt(k)dqbdld00(k)} - \texttt{dmdt(k+1)dqbdldm1(k+1)} \tag{FE28}$$

$$\texttt{ep1(it,jd):dmdt(k)dqbdldp1(k)} - \texttt{dmdt(k+1)dqbdld00(k+1)} \tag{FE29}$$

$$\texttt{ep2(it,jd):} \qquad - \texttt{dmdt(k+1)dqbdldp1(k+1)} \tag{FE30}$$

$$\texttt{em2(it,jt):dmdt(k)dqbdltm2(k)} \tag{FE31}$$

$$\texttt{em1(it,jt):dmdt(k)dqbdltm1(k)} - \texttt{dmdt(k+1)dqbdltm2(k+1)} \tag{FE32}$$

$$\texttt{e00(it,jt):dmdt(k)dqbdlt00(k)} - \texttt{dmdt(k+1)dqbdltm1(k+1)} \tag{FE33}$$

$$\texttt{ep1(it,jt):dmdt(k)dqbdltp1(k)} - \texttt{dmdt(k+1)dqbdlt00(k+1)} \tag{FE34}$$

$$\texttt{ep2(it,jt):} \qquad - \texttt{dmdt(k+1)dqbdltp1(k+1)} \tag{FE35}$$

$$\texttt{em2(it,je):dmdt(k)dqbdlem2(k)} \tag{FE36}$$

$$\texttt{em1(it,je):dmdt(k)dqbdlem1(k)} - \texttt{dmdt(k+1)dqbdetm2(k+1)} \tag{FE37}$$

$$\texttt{e00(it,je):dmdt(k)dqbdle00(k)} - \texttt{dmdt(k+1)dqbdlem1(k+1)} \tag{FE38}$$

$$\texttt{ep1(it,je):dmdt(k)dqbdlep1(k)} - \texttt{dmdt(k+1)dqbdle00(k+1)} \tag{FE39}$$

$$\texttt{ep2(it,je):} \qquad - \texttt{dmdt(k+1)dqbdlep1(k+1)} \tag{FE40}$$

(3) Flux Divergence

These matrix elements are the same as those given for the radiation energy equation, (RE6) – (RE9), with the index `ie` replaced by the index `it`.

(4) Work

$$\texttt{e00(it,jr):} -\theta\mu(p_k^{n+\theta} + f_k^{n+\theta} E_k^{n+\theta})(r_k^{n+1}/r_k^{n+\theta})\,\texttt{rmu(k\ \ \ )}\,u_k^{n+\theta} \tag{FE41}$$

$$\texttt{ep1(it,jr):}\ \ \theta\mu(p_k^{n+\theta} + f_k^{n+\theta} E_k^{n+\theta})(r_{k+1}^{n+1}/r_{k+1}^{n+\theta})\,\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} \tag{FE42}$$

$$\texttt{e00(it,jd):}\ \ \theta p_k^{n+1}(\partial lnp/\partial ln\rho)_k^{n+1}[\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} - \texttt{rmu(k)}\,u_k^{n+\theta}] \tag{FE43}$$

$$\texttt{e00(it,ju):} -\theta(p_k^{n+\theta} + f_k^{n+\theta} E_k^{n+\theta})\,\texttt{rmu(k\ \ \ )unom(k\ \ \ )} \tag{FE44}$$

$$\texttt{ep1(it,ju):}\ \ \theta(p_k^{n+\theta} + f_k^{n+\theta} E_k^{n+\theta})\,\texttt{rmu(k+1)unom(k+1)} \tag{FE45}$$

$$\texttt{e00(it,jt):}\ \ \theta p_k^{n+1}(\partial lnp/\partial lnT)_k^{n+1}[\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} - \texttt{rmu(k)}\,u_k^{n+\theta}] \tag{FE46}$$

$$\texttt{e00(it,je):}\ \ \theta f_k^{n+\theta} E_k^{n+1}[\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} - \texttt{rmu(k)}\,u_k^{n+\theta}] \tag{FE47}$$

(5) Anisotropy

These matrix elements are the same as those given for the radiation energy equation, (RE15) – (RE19), with the index `ie` replaced by the index `it`.

(6) Diffusion

The quantities needed to calculate the derivatives of the energy diffusion term are generated in subroutine `diffuse` in the same way as the mass diffusion in the continuity equation (*q.v.*). In this case the inputs required are `qd(k)` $= e_k^{n+\theta}$ and `qdn(k)` $= e_k^{n+1}$, for $(k = 1, \ldots, N+1)$, and also $\sigma = \sigma_e$, and $\zeta = 0$. Then, as in the continuity equation,

`em1(it,jr):`	`ddfdlrm1(k)`		(FE48)
`e00(it,jr):`	`ddfdlr00(k)`	$-$ `ddfdlrm1(k+1)`	(FE49)
`ep1(it,jr):`	`ddfdlrp1(k)`	$-$ `ddfdlr00(k+1)`	(FE50)
`ep2(it,jr):`		$-$ `ddfdlrp1(k+1)`	(FE51)
`em1(it,jd):`	`ddfdlqm1(k)`		(FE52)
`e00(it,jd):`	`ddfdlq00(k)`	$-$ `ddfdlqm1(k+1)`	(FE53)
`ep1(it,jd):`		$-$ `ddfdlq00(k+1)`	(FE54)
`em1(it,jd):`	`ddfdltm1(k)`		(FE55)
`e00(it,jd):`	`ddfdlt00(k)`	$-$ `ddfdltm1(k+1)`	(FE56)
`ep1(it,jd):`		$-$ `ddfdlt00(k+1)`	(FE57)

Here we have defined

$$\texttt{ddfdltm1(k)} \equiv \partial Q_k^{n+\theta} / \partial lnT_{k-1}^{n+1} = \texttt{ddfdlqm1(k)} (\partial lne / \partial lnT)_{k-1}^{n+1} \qquad \text{(FE58)}$$

$$\texttt{ddfdlt00(k)} \equiv \partial Q_k^{n+\theta} / \partial lnT_k^{n+1} = \texttt{ddfdlq00(k)} (\partial lne / \partial lnT)_k^{n+1} \qquad \text{(FE59)}$$

(7) Viscous Heating

The quantities needed to calculate the viscous energy dissipation rate `qe(k)` $= (\epsilon_Q)_k$ and its derivatives are generated in subroutine `viscous`. It is a cell-centered quantity needed only within the computational domain. Thus at $(k = 1, \ldots, N+1)$ we need the radius r, velocity u, and some auxiliary vectors defined below, and we apply the algorithm only at $(k = 2, \ldots, N-1)$.

Take $f1$ to be a discrete representation of $\frac{du}{dr} - \frac{\mu}{2} < \frac{u}{r} >$. Then define

$$f1 \quad (r_0, r_+, u_0, u_+) = \frac{u_+ - u_0}{r_+ - r_0} - \frac{\mu}{4}(\frac{u_+}{r_+} + \frac{u_0}{r_0}) \tag{FE60}$$

$$f1u0(r_0, r_+, u_0, u_+) \equiv \frac{\partial f_1}{\partial u_0} = -\frac{1}{r_+ - r_0} - \frac{\mu}{4}\frac{1}{r_0} \tag{FE61}$$

$$f1up(r_0, r_+, u_0, u_+) \equiv \frac{\partial f_1}{\partial u_+} = \quad \frac{1}{r_+ - r_0} - \frac{\mu}{4}\frac{1}{r_+} \tag{FE62}$$

$$f1r0(r_0, r_+, u_0, u_+) \equiv \frac{\partial f_1}{\partial r_0} = \quad \frac{(u_+ - u_0)}{(r_+ - r_0)^2} + \frac{\mu}{4}\frac{u_0}{r_0^2} \tag{FE63}$$

$$f1rp(r_0, r_+, u_0, u_+) \equiv \frac{\partial f_1}{\partial r_+} = -\frac{(u_+ - u_0)}{(r_+ - r_0)^2} + \frac{\mu}{4}\frac{u_+}{r_+^2} \tag{FE64}$$

Then

$$\texttt{dudr(k)} \equiv f1(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{FE65}$$

$$\texttt{dudlr00(k)} \equiv \frac{\partial f_1}{\partial ln r_0} = \theta r_k^{n+1} f1r0(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{FE66}$$

$$\texttt{dudlrp1(k)} \equiv \frac{\partial f_1}{\partial ln r_+} = \theta r_{k+1}^{n+1} f1rp(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{FE67}$$

$$\texttt{dudlu00(k)} \equiv \frac{\partial f_1}{\partial ln u_0} = \theta\, \texttt{unom}(k \quad) f1u0(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{FE68}$$

$$\texttt{dudlup1(k)} \equiv \frac{\partial f_1}{\partial ln u_+} = \theta\, \texttt{unom}(k+1) f1up(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{FE69}$$

The dissipation length is

$$\texttt{ql(k)} \equiv \ell_0 + \tfrac{1}{2}(r_k^{n+\theta} + r_{k+1}^{n+\theta})\,\ell_1 \quad \text{(FE70)}$$

$$\texttt{dqldr00(k)} \equiv (\partial ql/\partial ln r_k^{n+1}) = \tfrac{1}{2}\theta r_k^{n+1}\, l_1 \quad \text{(FE71)}$$

$$\texttt{dqldrp1(k)} \equiv (\partial ql/\partial ln r_{k+1}^{n+1}) = \tfrac{1}{2}\theta r_{k+1}^{n+1}\, l_1 \quad \text{(FE72)}$$

The velocity divergence is

$$\texttt{div(k)} \equiv \texttt{[rmu(k+1)}u_{k+1}^{n+\theta} - \texttt{rmu(k)}u_k^{n+\theta}\texttt{] / dvol(k)} \quad \text{(FE73)}$$

$$\texttt{ddivdr00(k)} \equiv \partial\texttt{div(k)}/\partial r_k^{n+1} =$$

$$[\mu\,\texttt{rmum1(k)}u_k^{n+\theta} - \texttt{rmu(k)div(k)}]/\texttt{dvol(k)} \quad \text{(FE74)}$$

$$\texttt{ddivdrp1(k)} \equiv \partial\texttt{div(k)}/\partial r_{k+1}^{n+1} =$$

$$-[\mu\,\texttt{rmum1(k+1)}u_{k+1}^{n+\theta} - \texttt{rmu(k+1)div(k)}]/\texttt{dvol(k)} \quad \text{(FE75)}$$

$$\texttt{ddivdu00(k)} \equiv \partial\texttt{div(k)}/\partial u_k^{n+1} = -\texttt{rmu(k)/dvol(k)} \quad \text{(FE76)}$$

$$\texttt{ddivdup1(k)} \equiv \partial\texttt{div(k)}/\partial u_{k+1}^{n+1} = \texttt{rmu(k+1)/dvol(k)} \quad \text{(FE77)}$$

Define

$$\texttt{qv(k)} \equiv \min[\texttt{div(k)}, 0] \quad \text{(FE78)}$$

Then

$$\texttt{dqvdlr00(k) = cvmgm[}\theta r_k^{n+1}\texttt{ddivdr00(k), 0, div(k)]} \quad \text{(FE79)}$$

$$\texttt{dqvdlrp1(k) = cvmgm[}\theta r_{k+1}^{n+1}\texttt{ddivdrp1(k), 0, div(k)]} \quad \text{(FE80)}$$

$$\texttt{dqvdlu00(k) = cvmgm[}\theta\, unom\texttt{(k)ddivdu00(k), 0, div(k)]} \quad \text{(FE81)}$$

$$\texttt{dqvdlup1(k)} = \texttt{cvmgm}[\theta\, unom\texttt{(k+1)ddivdru1(k), 0, div(k)}] \quad \text{(FE82)}$$

The coefficient of viscosity is

$$\texttt{qm(k)} \equiv (\mu_Q)_k^{n+\theta} = C_1\texttt{ql(k)}(a_s)_k^{n+\theta} - C_2[\texttt{ql(k)}]^2\texttt{qv(k)} \quad \text{(FE83)}$$

$$\texttt{dqmdlr00(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln r_k^{n+1} =$$

$$\texttt{dqldr00(k)}\{C_1 a_{s,k}^{n+\theta} - C_2[\texttt{2ql(k)qv(k)+ql(k)}^2\texttt{dqvdlr00(k)}]\} \quad \text{(FE84)}$$

$$\texttt{dqmdlrp1(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln r_{k+1}^{n+1} =$$

$$\texttt{dqldrp1(k)}\{C_1 a_{s,k}^{n+\theta} - C_2[\texttt{2ql(k)qv(k)+ql(k)}^2\texttt{dqvdlrp1(k)}]\} \quad \text{(FE85)}$$

$$\texttt{dqmdld00(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln \rho_k^{n+1} =$$

$$\tfrac{1}{2}\theta C_1\texttt{ql(k)}\, a_{s,k}^{n+\theta}\left[(p_k^{n+1}/p_k^{n+\theta})\left(\frac{\partial lnp}{\partial lnT}\right)_k^{n+1} - (\rho_k^{n+1}/\rho_k^{n+\theta})\right] \quad \text{(FE86)}$$

$$\texttt{dqmdlt00(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln T_k^{n+1} =$$

$$\tfrac{1}{2}\theta C_1\texttt{ql(k)}\, a_{s,k}^{n+\theta}(p_k^{n+1}/p_k^{n+\theta})\left(\frac{\partial lnp}{\partial lnT}\right)_k^{n+1} \quad \text{(FE87)}$$

$$\texttt{dqmdlu00(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln u_k^{n+1} = -C_2\texttt{ql(k)}^2\texttt{dqvdlu00(k)} \quad \text{(FE88)}$$

$$\texttt{dqmdlup1(k)} \equiv \partial(\mu_Q)_k^{n+\theta}/\partial ln u_{k+1}^{n+1} = -C_2\texttt{ql(k)}^2\texttt{dqvdlup1(k)} \quad \text{(FE89)}$$

The rate of viscous energy dissipation is (cf. FE2 and GM48)

$$\mathtt{qe(k)} = -\tfrac{4}{3}\rho_k^{n+\theta}\mathtt{qm(k)}\,[\mathtt{dudr(k)}]^2\mathtt{dvol(k)} \equiv \mathtt{qf(k)dudr(k)dvol(k)} \quad \text{(FE90)}$$

Then

$$\mathtt{e00(it,jr):dqfdlr00(k)dudr(k)dvol(k)\ +\ qf(k)durlu00(k)dvol(k)}$$

$$-\theta\,\mathtt{qf(k)dudr(k)rmu(k\quad)}r_k^{n+1} \quad \text{(FE91)}$$

$$\mathtt{ep1(it,jr):dqfdlrp1(k)dudr(k)dvol(k)\ +\ qf(k)durlup1(k)dvol(k)}$$

$$-\theta\,\mathtt{qf(k)dudr(k)rmu(k+1)}r_{k+1}^{n+1} \quad \text{(FE92)}$$

e00(it,ju):

$$\mathtt{dqfdlu00(k)dudr(k)dvol(k)\ +\ qf(k)durlu00(k)dvol(k)} \quad \text{(FE93)}$$

ep1(it,ju):

$$\mathtt{dqfdlup1(k)dudr(k)dvol(k)\ +\ qf(k)durlup1(k)dvol(k)} \quad \text{(FE94)}$$

$$\mathtt{e00(it,jt):dqfdlt00(k)dudr(k)dvol(k)} \quad \text{(FE95)}$$

$$\mathtt{e00(it,jd):dqfdld00(k)dudr(k)dvol(k)} \quad \text{(FE96)}$$

(8) Right Hand Side

$$
\begin{aligned}
&- \texttt{rhs(it)}= \\
&\quad \left[(\rho e + E)_k^{n+1} \texttt{dvoln(k)} - (\rho e + E)_k^{n} \texttt{dvolo(k)} \right] / dt \\
&- \texttt{dmdt(k+1)} \overline{(e + \tfrac{E}{\rho})}_{k+1} \ + \ \texttt{dmdt(k)} \overline{(e + \tfrac{E}{\rho})}_{k} \\
&+ \texttt{rmu(k+1)}\, F_{k+1}^{n+\theta} \ - \ \texttt{rmu(k)} F_k^{n+\theta} \\
&+ (p + fE)_k^{n+\theta} [\texttt{rmu(k+1)}\, u_{k+1}^{n+\theta} - \texttt{rmu(k)}\, u_k^{n+\theta}] \\
&+ \tfrac{\mu}{4} (1 - 3 f_k^{n+\theta}) E_k^{n+\theta} \left[(u_{k+1}^{n+\theta} / r_{k+1}^{n+\theta}) + (u_k^{n+\theta} / r_k^{n+\theta}) \right] \texttt{dvol(k)} \\
&+ \texttt{df(k)} - \texttt{df(k+1)} + \texttt{qe(k)}
\end{aligned}
\tag{FE97}
$$

VII. RADIATION ENERGY

A. Differential Equation

$$\frac{d}{dt}\left[\left(\frac{E}{\rho}\right)\rho\Delta V\right]-\Delta\left[\frac{dm}{dt}\left(\frac{E}{\rho}\right)-r^\mu F\right]+P\Delta(r^\mu u)+(E-3P)\frac{u}{r}\Delta V$$
$$= (4\pi\kappa_P B - c\kappa_E E)\rho\Delta V \quad \text{(RE1)}$$

where $\mu = 0$ or 2.

B. Difference Equation

For $(k = 2, \ldots, N-1)$,

$$(E_k^{n+1}\Delta V_k^{n+1} - E_k^n\Delta V_k^n)/dt$$

$$-[(m_{k+1}^{n+1} - m_{k+1}^n)(\tfrac{\overline{E}}{\rho})_{k+1} - (m_k^{n+1} - m_k^n)(\tfrac{\overline{E}}{\rho})_k]/dt$$

$$+(r_{k+1}^{n+\theta})^\mu F_{k+1}^{n+\theta} - (r_k^{n+\theta})^\mu F_k^{n+\theta}$$

$$+f_k^{n+\theta}E_k^{n+\theta}[(r_{k+1}^{n+\theta})^\mu u_{k+1}^{n+\theta} - (r_k^{n+\theta})^\mu u_k^{n+\theta}]$$

$$+\frac{\mu}{4}(1-3f_k^{n+\theta})E_k^{n+\theta}\left(\frac{u_{k+1}^{n+\theta}}{r_{k+1}^{n+\theta}} + \frac{u_k^{n+\theta}}{r_k^{n+\theta}}\right)\Delta V_k^{n+\theta}$$

$$+[c(\kappa_E)_k^{n+\theta}E_k^{n+\theta} - 4\pi(\kappa_P)_k^{n+\theta}B_k^{n+\theta}]\rho_k^{n+\theta}\Delta V_k^{n+\theta} = 0 \quad \text{(RE2)}$$

Equation (RE2) provides $N-2$ relations connecting the radiation energy density at N gridpoints. We thus require two boundary conditions.

C. Linearization

Calculating matrix elements we find:

(1) Time Derivative

$$\texttt{e00(ie,jr):} -E_k^{n+1}\,\texttt{rmup1n(k\ \ )}/dt \tag{RE3}$$

$$\texttt{ep1(ie,jr):}\ \ E_k^{n+1}\,\texttt{rmup1n(k+1)}/dt \tag{RE4}$$

$$\texttt{e00(ie,je):}\ \ E_k^{n+1}\texttt{dvoln (k\ \ )}/dt \tag{RE5}$$

(2) Flux Divergence

$$\texttt{e00(ie,jr):} -\theta\,\mu F_k^{n+\theta}(r_k^{n+1}/r_k^{n+\theta})\ \texttt{rmu(k\ \ )} \tag{RE6}$$

$$\texttt{ep1(ie,jr):}\ \ \theta\,\mu F_{k+1}^{n+\theta}(r_{k+1}^{n+1}/r_{k+1}^{n+\theta})\ \texttt{rmu(k+1)} \tag{RE7}$$

$$\texttt{e00(ie,jf):} -\theta\,\texttt{frnom(k\ \ )rmu(k\ \ )} \tag{RE8}$$

$$\texttt{ep1(ie,jf):}\ \ \theta\,\texttt{frnom(k+1)rmu(k+1)} \tag{RE9}$$

where

$$\texttt{frnom(k)} \equiv\ (1-\tfrac{\mu}{2})\,\sigma_R T_{eff}^4 + \tfrac{\mu}{2}[\,L\,/\,4\pi\,(r_k^{n+1})^2\,] \tag{RE10}$$

(3) Work

$$\texttt{e00(ie,jr):} -\theta\mu f_k^{n+\theta}E_k^{n+\theta}(r_k^{n+1}/r_k^{n+\theta})\,\texttt{rmu(k\ \ )}\,u_k^{n+\theta} \tag{RE11}$$

$$\texttt{ep1(ie,jr):}\ \ \theta\mu f_k^{n+\theta}E_k^{n+\theta}(r_{k+1}^{n+1}/r_{k+1}^{n+\theta})\,\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} \tag{RE12}$$

$$\texttt{e00(ie,ju):} -\theta f_k^{n+\theta}E_k^{n+\theta}\,\texttt{rmu(k\ \ )unom(k\ \ )} \tag{RE13}$$

$$\texttt{ep1(ie,ju):}\ \ \theta f_k^{n+\theta}E_k^{n+\theta}\,\texttt{rmu(k+1)unom(k+1)} \tag{RE14}$$

$$\texttt{e00(ie,je):}\ \ \theta f_k^{n+\theta}E_k^{n+1}\,[\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} - \texttt{rmu(k)}\,u_k^{n+\theta}] \tag{RE15}$$

(4) Anisotropy

Define

$$f1\ \ (r_0, r_p, u_0, u_p) \equiv\ \ \tfrac{1}{2}(\tfrac{u_0}{r_0} + \tfrac{u_p}{r_p}) \tag{RE16}$$

$$f1r0(r_0, r_p, u_0, u_p) \equiv -\tfrac{1}{2}\tfrac{u_0}{r_0^2} \tag{RE17a}$$

$$f1rp(r_0, r_p, u_0, u_p) \equiv -\tfrac{1}{2}\tfrac{u_p}{r_p^2} \tag{RE17b}$$

$$f1up(r_0, r_p, u_0, u_p) \equiv \quad \frac{1}{2}\frac{1}{r_0} \tag{RE18a}$$

$$f1up(r_0, r_p, u_0, u_p) \equiv \quad \frac{1}{2}\frac{1}{r_p} \tag{RE18b}$$

and

$$\texttt{aur(k)} \qquad \equiv f1(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{RE19}$$

$$\texttt{dardlr00(k)} \equiv \theta\, r_k^{n+1} f1r0(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{RE20}$$

$$\texttt{dardlrp1(k)} \equiv \theta\, r_{k+1}^{n+1} f1rp(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{RE21}$$

$$\texttt{dardlu00(k)} \equiv \theta\, \texttt{unom(k\ \ \ )}\, f1u0(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{RE22}$$

$$\texttt{dardlup1(k)} \equiv \theta\, \texttt{unom(k+1)}\, f1up(r_k^{n+\theta}, r_{k+1}^{n+\theta}, u_k^{n+\theta}, u_{k+1}^{n+\theta}) \tag{RE23}$$

Then

$$\texttt{e00(ie,jr)}: \tfrac{1}{2}\mu \;\; (1 - 3f_k^{n+\theta})\, E_k^{n+\theta}[-\theta\, r_k^{n+1} \texttt{rmu(k\ \ \ )}\,\texttt{aur(k)} + \texttt{dvol(k)}\,\texttt{dardlu00(k)}] \tag{RE24}$$

$$\texttt{ep1(ie,jr)}: \tfrac{1}{2}\mu \;\; (1 - 3f_k^{n+\theta})\, E_k^{n+\theta}[\;\; \theta\, r_{k+1}^{n+1} \texttt{rmu(k+1)}\,\texttt{aur(k)} + \texttt{dvol(k)}\,\texttt{dardlup1(k)}] \tag{RE25}$$

$$\texttt{e00(ie,ju)}: \tfrac{1}{2}\mu \;\; (1 - 3f_k^{n+\theta})\, E_k^{n+\theta} \texttt{dvol(k)}\,\texttt{dardlu00(k)} \tag{RE26}$$

$$\texttt{ep1(ie,ju)}: \tfrac{1}{2}\mu \;\; (1 - 3f_k^{n+\theta})\, E_k^{n+\theta} \texttt{dvol(k)}\,\texttt{dardlup1(k)} \tag{RE27}$$

$$\texttt{e00(ie,je)}: \tfrac{1}{2}\mu\theta\, (1 - 3f_k^{n+\theta})\, E_k^{n+1} \texttt{dvol(k)}\,\texttt{aur(k)} \tag{RE28}$$

(5) Sources and Sinks

`e00(ie,jr)`:

$$-\theta[c(\kappa_E)_k^{n+\theta} E_k^{n+\theta} - 4\pi(\kappa_P)_k^{n+\theta} B_k^{n+\theta}]\rho_k^{n+\theta}(r_k^{n+1}/r_k^{n+\theta})\, \texttt{rmup1(k\ \ \ )} \tag{RE29}$$

`ep1(ie,jr)`:

$$\theta[c(\kappa_E)_k^{n+\theta} E_k^{n+\theta} - 4\pi(\kappa_P)_k^{n+\theta} B_k^{n+\theta}]\rho_k^{n+\theta}(r_{k+1}^{n+1}/r_{k+1}^{n+\theta})\, \texttt{rmup1(k+1)} \tag{RE30}$$

$$\texttt{e00(ie,jd)}:\theta[c(\kappa_E)_k^{n+\theta}E_k^{n+\theta}-4\pi(\kappa_P)_k^{n+\theta}B_k^{n+\theta}]\rho_k^{n+1}\,\texttt{dvol(k)}$$
$$+\theta\;\;c(\kappa_E)_k^{n+1}(\partial ln\kappa_E/\partial ln\rho)_k^{n+1}E_k^{n+\theta}\quad\rho_k^{n+\theta}\,\texttt{dvol(k)}$$
$$-\theta\,4\pi(\kappa_P)_k^{n+1}(\partial ln\kappa_P/\partial ln\rho)_k^{n+1}B_k^{n+\theta}\quad\rho_k^{n+\theta}\,\texttt{dvol(k)} \tag{RE31}$$

$$\texttt{e00(ie,jt)}:\theta c(\kappa_E)_k^{n+1}(\partial ln\kappa_E/\partial lnT)_k^{n+1}E_k^{n+\theta}\rho_k^{n+\theta}\,\texttt{dvol(k)}$$
$$-\theta\,4\pi(\kappa_P)_k^{n+1}(\partial ln\kappa_P/\partial lnT)_k^{n+1}B_k^{n+\theta}\rho_k^{n+\theta}\,\texttt{dvol(k)}$$
$$-\theta\,4\pi(\kappa_P)_k^{n+\theta}(\partial B/\partial lnT)_k^{n+1}\qquad\rho_k^{n+\theta}\texttt{dvol(k)} \tag{RE32}$$

$$\texttt{e00(ie,je)}:\theta c(\kappa_E)_k^{n+\theta}E_k^{n+1}\rho_k^{n+\theta}\;\texttt{dvol(k)} \tag{RE33}$$

(6) Advection

The quantities needed to calculate the derivatives of the advection term are generated in subroutine `advectc`. As before, denote the advected quantity as q. The inputs required by the subroutine for $(k = 0, \ldots, N+2)$ are: `q(k)` $\equiv (E/\rho)_k^{n+1}$, `qo(k)` $\equiv (E/\rho)_k^n$,`qso(k)` $\equiv Dq_k^n$, and `flow(k) = -dmdt(k).`

Note that

$$(\partial lnq/\partial lnE)_k^{n+1} \equiv +1 \tag{RE34}$$

and

$$(\partial lnq/\partial ln\rho)_k^{n+1} \equiv -1 \tag{RE35}$$

Therefore

$$\partial(\delta\overline{q}_k)/\partial lnE_k^{n+1} = \texttt{dqbdle00(k)} \equiv \;\texttt{dqbdlq00(k)} \tag{RE36}$$

and

$$\partial(\delta\overline{q}_k)/\partial ln\rho_k^{n+1} = \texttt{dqbdld00(k)} \equiv -\texttt{dqbdlq00(k)} \tag{RE37}$$

and so on for $k-2$, $k-1$, $k+1$. Thus

$$\texttt{e00(ie,jm)}:\;\;\overline{q}_k\quad m_k^{n+1}/dt \tag{RE38}$$

$$\texttt{ep1(ie,jm)}:-\overline{q}_{k+1}m_{k+1}^{n+1}/dt \tag{RE39}$$

$$\texttt{em2(ie,jd):dmdt(k)dqbdldm2(k)} \tag{RE40}$$

$$\texttt{em1(ie,jd):dmdt(k)dqbdldm1(k)}-\texttt{dmdt(k+1)dqbdldm2(k+1)} \tag{RE41}$$

$$\texttt{e00(ie,jd):dmdt(k)dqbdld00(k)}-\texttt{dmdt(k+1)dqbdldm1(k+1)} \tag{RE42}$$

$$\texttt{ep1(ie,jd):dmdt(k)dqbdldp1(k)}-\texttt{dmdt(k+1)dqbdld00(k+1)} \tag{RE43}$$

$$\texttt{ep2(ie,jd):}\qquad\qquad -\texttt{dmdt(k+1)dqbdldp1(k+1)} \tag{RE44}$$

$$\texttt{em2(ie,je):dmdt(k)dqbdlem2(k)} \tag{RE45}$$

$$\texttt{em1(ie,je):dmdt(k)dqbdlem1(k)}-\texttt{dmdt(k+1)dqbdetm2(k+1)} \tag{RE46}$$

$$\texttt{e00(ie,je):dmdt(k)dqbdle00(k)}-\texttt{dmdt(k+1)dqbdlem1(k+1)} \tag{RE47}$$

$$\texttt{ep1(ie,je):dmdt(k)dqbdlep1(k)}-\texttt{dmdt(k+1)dqbdle00(k+1)} \tag{RE48}$$

$$\texttt{ep2(ie,je):} \qquad -\texttt{dmdt(k+1)dqbdlep1(k+1)} \tag{RE49}$$

(7) Right Hand Side

$$
\begin{aligned}
&- \ \texttt{rhs(ie)} \ = \\
&\quad [E_k^{n+1}\texttt{dvoln}(\texttt{k}) - E_k^n \texttt{dvolo}(\texttt{k})] \,/\, dt \\
&- \ \texttt{dmdt(k+1)}\overline{(E/\rho)}_{k+1} \ + \ \texttt{dmdt(k)}\overline{(E/\rho)}_k \\
&+ \ \texttt{rmu(k+1)}\,F_{k+1}^{n+\theta} \ - \ \texttt{rmu(k)}F_k^{n+\theta} \\
&+ f_k^{n+\theta} E_k^{n+\theta} [\texttt{rmu(k+1)}\,u_{k+1}^{n+\theta} - \texttt{rmu(k)}\,u_k^{n+\theta}] \\
&+ \ \tfrac{\mu}{4}(1 - 3f_k^{n+\theta})E_k^{n+\theta} \left[\left(u_{k+1}^{n+\theta}/r_{k+1}^{n+\theta}\right) + \left(u_k^{n+\theta}/r_k^{n+\theta}\right)\right]\texttt{dvol(k)} \\
&+ \ [c(\kappa_E)_k^{n+\theta} E_k^{n+\theta} - 4\pi(\kappa_P)_k^{n+\theta} B_k^{n+\theta}]\rho_k^{n+\theta}\texttt{dvol(k)}
\end{aligned} \tag{RE50}
$$

VIII. RADIATION MOMENTUM

A. Differential Equation

$$\frac{d}{dt}\left[\left(\frac{F}{c^2\rho}\right)\rho\Delta V\right]-\Delta\left[\frac{dm}{dt}\left(\frac{F}{c^2\rho}\right)\right]+r^\mu\left(\Delta P+\frac{F\Delta u}{c^2}\right)+\frac{(3P-E)}{r}\Delta V$$

$$=-\frac{1}{c}\chi_F F\rho\Delta V \quad \text{(RM1)}$$

where $\mu = 0$ or 2.

B. Difference Equation

For $(k = 2, \ldots, N)$,

$$\left[\left(\frac{F_k^{n+1}}{\rho_k^{n+1}}\right)(\rho_{k-1}^{n+1}\Delta V_{k-1}^{n+1}+\rho_k^{n+1}\Delta V_k^{n+1})\right.$$

$$\left.-\left(\frac{F_k^n}{\rho_k^n}\right)(\rho_{k-1}^n\Delta V_{k-1}^n+\rho_k^n\Delta V_k^n)\right]/2c^2dt$$

$$-\left\{\left[(m_k^{n+1}-m_k^n)+(m_{k+1}^{n+1}-m_{k+1}^n)\right]\left(\frac{F}{\rho}\right)_k\right.$$

$$\left.-\left[(m_{k-1}^{n+1}-m_{k-1}^n)+(m_k^{n+1}-m_k^n)\right]\left(\frac{F}{\rho}\right)_{k-1}\right\}/2c^2dt$$

$$+\left(r_k^{n+\theta}\right)^\mu\left[f_k^{n+\theta}E_k^{n+\theta}-f_{k-1}^{n+\theta}E_{k-1}^{n+\theta}+F_k^{n+\theta}(u_{k+1}^{n+\theta}-u_{k-1}^{n+\theta})/2c^2\right]$$

$$+\mu\left[(3f_{k-1}^{n+\theta}-1)E_{k-1}^{n+\theta}\Delta V_{k-1}^{n+\theta}+(3f_k^{n+\theta}-1)E_k^{n+\theta}\Delta V_k^{n+\theta}\right]/(4r_k^{n+\theta})$$

$$+<\chi>_k^{n+\theta}\ F_k^{n+\theta}\left(\rho_{k-1}^{n+\theta}\Delta V_{k-1}^{n+\theta}+\rho_k^{n+\theta}\Delta V_k^{n+\theta}\right)/2c=0 \quad \text{(RM2)}$$

Equation (RM2) provides $N-1$ relations connecting the radiation momentum density at $N+1$ interfaces. We thus require two boundary conditions.

The opacity at an interface $<\chi>_k^{n+\theta}$ is defined to be

$$\frac{1}{<\chi>_k^{n+\theta}} \equiv \frac{1}{2}\left(\frac{1}{\chi_{k-1}^{n+\theta}} + \frac{1}{\chi_k^{n+\theta}}\right) \tag{RM3}$$

C. Linearization

Calculating matrix elements we find:

(1) Time Derivative

$$\texttt{em1(if,jr):} \quad -(F/\rho)_k^{n+1}\ \rho_{k-1}^{n+1} \qquad \texttt{rmup1n(k-1)}/2c^2dt \tag{RM4}$$

$$\texttt{e00(if,jr):} \quad (F/\rho)_k^{n+1}(\rho_{k-1}^{n+1} - \rho_k^{n+1})\,\texttt{rmup1n(k\ \ )}/2c^2dt \tag{RM5}$$

$$\texttt{ep1(if,jr):} \quad (F/\rho)_k^{n+1} \qquad \rho_k^{n+1}\ \texttt{rmup1n(k+1)}/2c^2dt \tag{RM6}$$

$$\texttt{em1(if,jd):} \quad (F/\rho)_k^{n+1}\rho_{k-1}^{n+1}\,\texttt{dvoln(k-1)}/2c^2dt \tag{RM7}$$

$$\texttt{e00(if,jd):} \quad -(F/\rho)_k^{n+1}\rho_k^{n+1}\,\texttt{dvoln(k-1)}/2c^2dt \tag{RM8}$$

$$\texttt{e00(if,jf):} \quad \left(\frac{\texttt{frnom(k)}}{\rho_k^{n+1}}\right)[\rho_{k-1}^{n+1}\texttt{dvoln(k-1)} + \rho_k^{n+1}\texttt{dvoln(k)}]/2c^2dt \tag{RM9}$$

(2) Advection

The quantities needed to calculate the derivatives of the advection term are generated in subroutine `advecti`. As before, denote the advected quantity as q. The inputs required by the subroutine for $(k = 0, \ldots, N+2)$ are:

$$\begin{array}{llll}
\texttt{q\ \ \ (k)} & = \ \ q_k^{n+1} & \equiv (F/\rho)_k^{n+1} \\
\texttt{qo\ \ (k)} & = \ \ q_k^{n} & \equiv (F/\rho)_k^{n} \\
\texttt{qso\ (k)} & = Dq_k^{n} & = \text{monotonized slope at old time} \\
\texttt{flow(k)} & = -\texttt{dmdt(k)} & \ \ \text{direction of flow at interface k}
\end{array}$$

Then, proceeding as in (C20) - (C36), and using the facts that

$$\partial q_k^{n+1}/\partial F_k^{n+1} = \quad 1/\rho_k^{n+1} \tag{RM10}$$

and

$$\partial q_k^{n+1}/\partial \rho_k^{n+1} = -q_k^{n+1}/\rho_k^{n+1} \tag{RM11}$$

we have

$$\texttt{dqbdlfm1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnF_{k-1}^{n+1} = \texttt{dqbdqm1(k)frnom(k-1)}/\rho_{k-1}^{n+1} \tag{RM12}$$

$$\texttt{dqbdlf00(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnF_{k}^{n+1} = \texttt{dqbdq00(k)frnom(k)}/\rho_{k}^{n+1} \tag{RM13}$$

$$\texttt{dqbdlfp1(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnF_{k+1}^{n+1} = \texttt{dqbdqp1(k)frnom(k+1)}/\rho_{k+1}^{n+1} \tag{RM14}$$

$$\texttt{dqbdlfp2(k)} \equiv \partial(\delta\overline{q}_k)/\partial lnF_{k+2}^{n+1} = \texttt{dqbdqp2(k)frnom(k+2)}/\rho_{k+2}^{n+1} \tag{RM15}$$

and

$$\texttt{dqbdldm1(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k-1}^{n+1} = -\texttt{dqbdqm1(k)}\,q_{k-1}^{n+1} \tag{RM16}$$

$$\texttt{dqbdld00(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k}^{n+1} = -\texttt{dqbdq00(k)}\,q_{k}^{n+1} \tag{RM17}$$

$$\texttt{dqbdldp1(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k+1}^{n+1} = -\texttt{dqbdqp1(k)}\,q_{k+1}^{n+1} \tag{RM18}$$

$$\texttt{dqbdldp2(k)} \equiv \partial(\delta\overline{q}_k)/\partial ln\rho_{k+2}^{n+1} = -\texttt{dqbdqp2(k)}\,q_{k+2}^{n+1} \tag{RM19}$$

Hence

$$\texttt{em1(if,jm):} \quad m_{k-1}^{n+1}\ \overline{q}_{k-1} \qquad /2c^2dt \tag{RM20}$$

$$\texttt{e00(if,jm):} \quad m_{k}^{n+1}(\overline{q}_{k-1} - \overline{q}_k)/2c^2dt \tag{RM21}$$

$$\texttt{ep1(if,jm):} \quad -m_{k+1}^{n+1}\ \overline{q}_{k} \qquad /2c^2dt \tag{RM22}$$

$$\texttt{em2(if,jd): [dmdt(k-1)+dmdt(k)]dqbdldm1(k-1)}\ /2c^2 \tag{RM23}$$

$$\texttt{em1(if,jd):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdld00(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdldm1(k)}\}/2c^2 \tag{RM24}$$

$$\texttt{e00(if,jd):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdldp1(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdld00(k)}\}/2c^2 \qquad \text{(RM25)}$$

$$\texttt{ep1(if,jd):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdldp2(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdldp1(k)}\}/2c^2 \qquad \text{(RM26)}$$

$$\texttt{ep2(if,jd):-[dmdt(k)+dmdt(k+1)]dqbdldp2(k) }/2c^2 \qquad \text{(RM27)}$$

and

$$\texttt{em2(if,jf): [dmdt(k-1)+dmdt(k)]dqbdlfm1(k-1) /}2c^2 \qquad \text{(RM28)}$$

$$\texttt{em1(if,jf):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdlf00(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdlfm1(k)}\}/2c^2 \qquad \text{(RM29)}$$

$$\texttt{e00(if,jf):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdldf1(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdlf00(k)}\}/2c^2 \qquad \text{(RM30)}$$

$$\texttt{ep1(if,jf):}\{\texttt{[dmdt(k-1)+dmdt(k)]dqbdldf2(k-1)}$$

$$\texttt{-[dmdt(k)+dmdt(k+1)]dqbdldf1(k)}\}/2c^2 \qquad \text{(RM31)}$$

$$\texttt{ep2(if,jf):} -[\texttt{dmdt(k }) + \texttt{dmdt(k+1)}]\,\texttt{dqbdlfp2(k }) \;/2c^2 \qquad \text{(RM32)}$$

(3) Pressure Gradient

$$\texttt{e00(if,jr):}\;\; \theta\mu\left(\frac{r_k^{n+1}}{r_k^{n+\theta}}\right)\;\texttt{rmu(k)}[f_k^{n+\theta}E_k^{n+\theta} - f_{k-1}^{n+\theta}E_{k-1}^{n+\theta}$$

$$+F_k^{n+\theta}(u_{k+1}^{n+\theta} - u_{k-1}^{n+\theta})/2c^2] \qquad \text{(RM33)}$$

$$\texttt{em1(if,ju):} -\theta F_k^{n+\theta} \quad \texttt{rmu(k)}\,\texttt{unom(k-1)}/2c^2 \qquad \text{(RM34)}$$

$$\texttt{ep1(if,ju):}\;\; \theta F_k^{n+\theta} \quad \texttt{rmu(k)}\,\texttt{unom(k+1)}/2c^2 \qquad \text{(RM35)}$$

$$\texttt{em1(if,je):} -\theta f_{k-1}^{n+\theta}E_{k-1}^{n+1}\texttt{rmu(k)} \qquad \text{(RM36)}$$

$$\texttt{e00(if,je):}\;\; \theta f_k^{n+\theta}E_k^{n+1}\texttt{rmu(k)} \qquad \text{(RM37)}$$

$$\texttt{e00(if,jf):}\;\; \theta\,\texttt{frnom(k)}\;\texttt{rmu(k)}\,(u_{k+1}^{n+\theta} - u_{k-1}^{n+\theta})/2c^2 \qquad \text{(RM38)}$$

(4) Isotropy

$$\texttt{em1(if,jr):} -\tfrac{1}{4}\mu\theta\;\;(3f_{k-1}^{n+\theta} - 1)E_{k-1}^{n+1}\texttt{rmu(k-1)}(r_k^{n+1}/r_k^{n+\theta}) \qquad \text{(RM39)}$$

$$\texttt{ep1(if,jr):}\;\; \tfrac{1}{4}\mu\theta\;\;(3f_k^{n+\theta} - 1)E_k^{n+1}\texttt{rmu(k+1)}(r_{k+1}^{n+1}/r_k^{n+\theta}) \qquad \text{(RM40)}$$

$$\texttt{e00(if,jr):}\ \ \frac{1}{4}\mu\theta\{\ [(3f_{k-1}^{n+\theta}-1)E_{k-1}^{n+\theta}-(3f_k^{n+\theta}-1)E_k^{n+\theta}]\ \texttt{rmum1(k)}$$

$$-[(3f_{k-1}^{n+\theta}-1)E_{k-1}^{n+\theta}\texttt{dvol(k-1)}$$

$$+\ (3f_k^{n+\theta}-1)E_k^{n+\theta}\texttt{dvol(k\ \ \ )}]/r_k^2\}r_k^{n+1} \qquad \text{(RM41)}$$

$$\texttt{em1(if,je):}\ \ \frac{1}{4}\mu\theta\ \ (3f_{k-1}^{n+\theta}-1)E_{k-1}^{n+1}\texttt{dvol(k-1)}/r_k^{n+\theta} \qquad \text{(RM42)}$$

$$\texttt{e00(if,je):}\ \ \frac{1}{4}\mu\theta\ \ (3f_k^{n+\theta}-1)E_k^{n+1}\texttt{dvol(k\ \ \ )}/r_k^{n+\theta} \qquad \text{(RM43)}$$

(5) Radiation Force

For $i = k-1,\ k$

$$\frac{\partial <\chi>_k^{n+\theta}}{\partial ln q_i^{n+1}} = \frac{\theta}{2} <\chi>_k^{n+\theta}\left(\frac{<\chi>_k^{n+\theta}}{\chi_i^{n+\theta}}\right)\left(\frac{\chi_i^{n+1}}{\chi_i^{n+\theta}}\right)\left(\frac{\partial ln\chi}{\partial lnq}\right)_i^{n+1} \qquad \text{(RM44)}$$

where q denotes any physical quantity. Then

$$\texttt{em1(if,jr):}-\theta <\chi>_k^{n+\theta}\ F_k^{n+\theta}\rho_{k-1}^{n+\theta}\texttt{rmu(k-1)}r_{k-1}^{n+1}/2c \qquad \text{(RM45)}$$

$$\texttt{ep1(if,jr):}\ \ \theta <\chi>_k^{n+\theta}\ F_k^{n+\theta}\rho_k^{n+\theta}\texttt{rmu(k+1)}r_{k+1}^{n+1}/2c \qquad \text{(RM46)}$$

$$\texttt{e00(if,jr):}\ \ \theta <\chi>_k^{n+\theta}\ F_k^{n+\theta}(\rho_{k-1}^{n+\theta}-\rho_k^{n+\theta})\texttt{rmu(k)}r_k^{n+1}/2c \qquad \text{(RM47)}$$

$$\texttt{em1(if,jd):}\;\; \theta < \chi >_k^{n+\theta}\; F_k^{n+\theta} \left\{ \rho_{k-1}^{n+1}\;\; \texttt{dvol(k-1)} \right.$$
$$\left. +\frac{1}{2} \left(\frac{<\chi>_k^{n+\theta}}{\chi_{k-1}^{n+\theta}} \right) \left(\frac{\chi_{k-1}^{n+1}}{\chi_{k-1}^{n+\theta}} \right) \left(\frac{\partial ln\chi}{\partial ln\rho} \right)_{k-1}^{n+1} [\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)} + \rho_k^{n+\theta}\texttt{dvol(k)}] \right\} /2c \tag{RM48}$$

$$\texttt{e00(if,jd):}\;\; \theta < \chi >_k^{n+\theta}\; F_k^{n+\theta} \left\{ \rho_k^{n+1}\;\; \texttt{dvol(k)} \right.$$
$$\left. +\frac{1}{2} \left(\frac{<\chi>_k^{n+\theta}}{\chi_k^{n+\theta}} \right) \left(\frac{\chi_k^{n+1}}{\chi_k^{n+\theta}} \right) \left(\frac{\partial ln\chi}{\partial ln\rho} \right)_k^{n+1} [\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)} + \rho_k^{n+\theta}\texttt{dvol(k)}] \right\} /2c \tag{RM49}$$

$$\texttt{em1(if,jt):}\;\; \theta < \chi >_k^{n+\theta}\; F_k^{n+\theta}$$
$$\times \left(\frac{<\chi>_k^{n+\theta}}{\chi_{k-1}^{n+\theta}} \right) \left(\frac{\chi_{k-1}^{n+1}}{\chi_{k-1}^{n+\theta}} \right) \left(\frac{\partial ln\chi}{\partial lnT} \right)_{k-1}^{n+1} [\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)}+\rho_k^{n+\theta}\texttt{dvol(k)}]/\texttt{4c} \tag{RM50}$$

$$\texttt{e00(if,jt):}\;\; \theta < \chi >_k^{n+\theta}\; F_k^{n+\theta}$$
$$\times \left(\frac{<\chi>_k^{n+\theta}}{\chi_k^{n+\theta}} \right) \left(\frac{\chi_k^{n+1}}{\chi_k^{n+\theta}} \right) \left(\frac{\partial ln\chi}{\partial lnT} \right)_k^{n+1} [\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)}+\rho_k^{n+\theta}\texttt{dvol(k)}]/\texttt{4c} \tag{RM51}$$

$$\texttt{e00(if,jf):}\;\; \theta < \chi >_k^{n+\theta}\texttt{frnom(k)}[\rho_{k-1}^{n+\theta}\texttt{dvol(k-1)}+\rho_k^{n+\theta}\texttt{dvol(k)}]/2c \tag{RM52}$$

(6) Right Hand Side

$$- \mathtt{rhs(if)} =$$

$$\left\{ (F/\rho)_k^{n+1} \left[\rho_{k-1}^{n+1} \mathtt{dvoln(k-1)} + \rho_k^{n+1} \mathtt{dvoln(k)} \right] \right.$$

$$\left. - (F/\rho)_k^n \left[\rho_{k-1}^n \mathtt{dvolo(k-1)} + \rho_k^n \mathtt{dvolo(k)} \right] \right\} / 2c^2 dt$$

$$- \left\{ [\mathtt{dmdt(k)} + \mathtt{dmdt(k+1)}] \left(\overline{F/\rho} \right)_k \right.$$

$$\left. - [\mathtt{dmdt(k)} + \mathtt{dmdt(k+1)}] \left(\overline{F/\rho} \right)_{k-1} \right\} / 2c^2 dt$$

$$+ \mathtt{rmu(k)} \left[f_k^{n+\theta} E_k^{n+\theta} - f_{k-1}^{n+\theta} E_{k-1}^{n+\theta} + F_k^{n+\theta} (u_{k+1}^{n+\theta} - u_{k-1}^{n+\theta}) / 2c^2 \right]$$

$$+ \mu \left[(3 f_{k-1}^{n+\theta} - 1) E_{k-1}^{n+\theta} \mathtt{dvol(k-1)} + (3 f_k^{n+\theta} - 1) E_k^{n+\theta} \mathtt{dvol(k)} \right] / 4 r_k^{n+\theta}$$

$$+ < \chi >_k^{n+\theta} \; F_k^{n+\theta} \left[\rho_{k-1}^{n+\theta} \mathtt{dvol(k-1)} + \; \rho_k^{n+\theta} \mathtt{dvol(k)} \right] / 2c = 0 \qquad \text{(RM53)}$$

IX. RADIATION DIFFUSION

For both nonequilibrium and equilibrium diffusion we set all Eddington factors $f_k \equiv \frac{1}{3}$, and set the surface flux Eddington factor $g_N \equiv \frac{1}{2}$ (consistent with isotropic radiation). Further we drop the time derivative and velocity-dependent terms.

A. Nonequilibrium Diffusion (Flux-Limited)

Replace (RM2) with the flux-limited diffusion diffusion equation

$$F_k^{n+1} - \tfrac{c}{3}(r_k^{n+1})^\mu (E_{k-1}^{n+1} - E_k^{n+1})/D_k^{n+1} = 0 \tag{RD1}$$

where

$$\begin{aligned} D_k^{n+1} \equiv < \chi >_k^{n+1} \; & \tfrac{1}{2}[\rho_{k-1}^{n+1}\texttt{dvoln(k-1)} + \rho_k^{n+1}\texttt{dvoln(k)}] \\ & + \tfrac{2\lambda}{3}(r_k^{n+1})^\mu |E_{k-1}^{n+1} - E_k^{n+1}|/(E_{k-1}^{n+1} + E_k^{n+1}) \end{aligned} \tag{RD2}$$

and

$$\frac{1}{< \chi >_k^{n+1}} \equiv \frac{1}{2}\left(\frac{1}{\chi_{k-1}^{n+1}} + \frac{1}{\chi_k^{n+1}}\right) \tag{RD3}$$

For convenience write (RD2) as

$$D_k^{n+1} = \alpha_k^{n+1} + \tfrac{2}{3}\lambda\beta_k^{n+1} \tag{RD4}$$

For no flux-limiting set $\lambda = 0$, and for flux limiting set $\lambda = 1$. Boundary conditions (BC22) - (BC25), and (BC35) - (BC36) remain unchanged.

B. Equilibrium Diffusion

To implement equilibrium diffusion, in addition to all replacements in equation (RD2) described above, replace (RE2) with

$$E_k^{n+1} = a_R(T_k^{n+1})^4 \tag{RD5}$$

C. Linearization

(1) Nonequilibrium Diffusion

Define

$$\texttt{chdvol(k)} \equiv \alpha_k^{n+1} = \frac{1}{2} < \chi >_k^{n+1} [\rho_{k-1}^{n+1}\texttt{dvoln(k-1)} + \rho_k^{n+1}\texttt{dvoln(k)}] \quad \text{(RD6)}$$

$$\texttt{chdvolrm(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln r_{k-1}^{n+1}} = -\frac{1}{2} < \chi >_k^{n+1} \texttt{rmup1n(k-1)} \rho_{k-1}^{n+1} \quad \text{(RD7)}$$

$$\texttt{chdvolr0(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln r_k^{n+1}} = \frac{1}{2} < \chi >_k^{n+1} \texttt{rmup1n(k)} (\rho_{k-1}^{n+1} - \rho_k^{n+1}) \quad \text{(RD8)}$$

$$\texttt{chdvolrp(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln r_{k+1}^{n+1}} = \frac{1}{2} < \chi >_k^{n+1} \texttt{rmup1n(k+1)} \rho_k^{n+1} \quad \text{(RD9)}$$

$$\texttt{chdvoldm(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln \rho_{k-1}^{n+1}} =$$

$$\frac{1}{2} < \chi >_k^{n+1} \texttt{dvoln(k)} \rho_{k-1}^{n+1} + \frac{1}{2} \texttt{chdvol(k)} \frac{<\chi>_k^{n+1}}{\chi_{k-1}^{n+1}} \left(\frac{\partial ln \chi}{\partial ln \rho} \right)_{k-1}^{n+1} \quad \text{(RD10)}$$

$$\texttt{chdvold0(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln \rho_k^{n+1}} =$$

$$\frac{1}{2} < \chi >_k^{n+1} \texttt{dvoln(k)} \rho_k^{n+1} + \frac{1}{2} \texttt{chdvol(k)} \frac{<\chi>_k^{n+1}}{\chi_k^{n+1}} \left(\frac{\partial ln \chi}{\partial ln \rho} \right)_k^{n+1} \quad \text{(RD11)}$$

$$\texttt{chdvoltm(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial ln T_{k-1}^{n+1}} = \frac{1}{2} \texttt{chdvol(k)} \frac{<\chi>_k^{n+1}}{\chi_{k-1}^{n+1}} \left(\frac{\partial ln \chi}{\partial ln T} \right)_{k-1}^{n+1} \quad \text{(RD12)}$$

$$\texttt{chdvolt0(k)} \equiv \frac{\partial \alpha_k^{n+1}}{\partial lnT_k^{n+1}} = \frac{1}{2}\texttt{chdvol(k)}\frac{<\chi>_k^{n+1}}{\chi_k^{n+1}}\left(\frac{\partial ln\chi}{\partial lnT}\right)_k^{n+1} \quad \text{(RD13)}$$

Further define

$$\texttt{elim(k)} = \frac{2}{3}\lambda\beta_k^{n+1} = \frac{2}{3}\lambda\,\texttt{rmun(k)}\,|E_{k-1}^{n+1} - E_k^{n+1}|/(E_{k-1}^{n+1} + E_k^{n+1}) \quad \text{(RD14)}$$

$$\texttt{elimr0(k)} = \frac{2}{3}\lambda\partial\beta_k^{n+1}/\partial r_k^{n+1} = \mu\ \texttt{elim(k)} \quad \text{(RD15)}$$

$$\begin{aligned}\texttt{elimem(k)} &= \frac{2}{3}\lambda\partial\beta_k^{n+1}/\partial E_{k-1}^{n+1}\\ &= -\frac{[\texttt{elim(k)}E_{k-1}^{n+1} + \frac{2}{3}\lambda\,\texttt{rmun(k)}\mathrm{sgn}(E_{k-1}^{n+1} - E_k^{n+1})]}{(E_{k-1}^{n+1} + E_k^{n+1})}\end{aligned} \quad \text{(RD16)}$$

$$\begin{aligned}\texttt{elime0(k)} &= \frac{2}{3}\lambda\partial\beta_k^{n+1}/\partial E_k^{n+1}\\ &= -\frac{[\texttt{elim(k)}E_k^{n+1} + \frac{2}{3}\lambda\,\texttt{rmun(k)}\mathrm{sgn}(E_{k-1}^{n+1} - E_k^{n+1})]}{(E_{k-1}^{n+1} + E_k^{n+1})}\end{aligned} \quad \text{(RD17)}$$

and

$$\texttt{fd(k)} \equiv F_k^{n+1}D_k^{n+1} = (c/3)\ \texttt{rmun(k)}(E_{k-1}^{n+1} - E_k^{n+1}) \quad \text{(RD18)}$$

$$\texttt{dfddem(k)} \equiv \partial\texttt{fd(k)}/\partial lnE_{k-1}^{n+1} = (c/3)\ \texttt{rmun(k)}\,E_{k-1}^{n+1} \quad \text{(RD19)}$$

$$\texttt{dfdde0(k)} \equiv \partial\texttt{fd(k)}/\partial lnE_k^{n+1} = -(c/3)\ \texttt{rmun(k)}\,E_k^{n+1} \quad \text{(RD20)}$$

$$\texttt{dfddr0(k)} \equiv \partial\texttt{fd(k)}/\partial ln r_k^{n+1} \;=\; \mu\ \texttt{fd(k)} \qquad \text{(RD21)}$$

$$\texttt{denom(k)} \equiv 1/D_k^{n+1} = 1/(\texttt{chdvol(k)}+\texttt{elim(k)}+10^{-30}) \qquad \text{(RD22)}$$

Then

$$\texttt{dif(k)} \quad = F_k^{n+1} \quad = \texttt{denom(k)fd(k)} \qquad \text{(RD23)}$$

$$\texttt{ddifdrm(k)} = \partial F_k^{n+1}/\partial ln r_{k-1}^{n+1} = -\texttt{denom(k)dif(k)chdvolrm(k)} \qquad \text{(RD24)}$$

$$\texttt{ddifdr0(k)} = \partial F_k^{n+1}/\partial ln r_k^{n+1}$$

$$= \texttt{denom(k)}\{\texttt{dfddr0(k)}-\texttt{dif(k)}[\texttt{chdvolr0(k)}+\texttt{elimr0(k)}]\} \qquad \text{(RD25)}$$

$$\texttt{ddifdrp(k)} = \partial F_k^{n+1}/\partial ln r_{k+1}^{n+1} = -\texttt{denom(k)dif(k)chdvolrp(k)} \qquad \text{(RD26)}$$

$$\texttt{ddifddm(k)} = \partial F_k^{n+1}/\partial ln \rho_{k-1}^{n+1} = -\texttt{denom(k)dif(k)chdvoldm(k)} \qquad \text{(RD27)}$$

$$\texttt{ddifdd0(k)} = \partial F_k^{n+1}/\partial ln \rho_k^{n+1} = -\texttt{denom(k)dif(k)chdvold0(k)} \qquad \text{(RD28)}$$

$$\texttt{ddifdtm(k)} = \frac{\partial F_k^{n+1}}{\partial ln T_{k-1}^{n+1}} \quad = -\texttt{denom(k)dif(k)chdvoltm(k)} \qquad \text{(RD29)}$$

$$\texttt{ddifdt0(k)} = \frac{\partial F_k^{n+1}}{\partial ln T_k^{n+1}} \quad = -\texttt{denom(k)dif(k)chdvolt0(k)} \qquad \text{(RD30)}$$

$$\texttt{ddifdem(k)} = \frac{\partial F_k^{n+1}}{\partial ln E_{k-1}^{n+1}} = \texttt{denom(k)}[\texttt{dfddem(k)}-\texttt{dif(k)elimem(k)}] \qquad \text{(RD31)}$$

$$\texttt{ddifde0(k)} = \frac{\partial F_k^{n+1}}{\partial ln E_k^{n+1}} = \texttt{denom(k)[dfdde0(k)} - \texttt{dif(k)elime0(k)]} \quad \text{(RD32)}$$

Then

$$\texttt{em1(if,jr):} - \texttt{ddifdrm(k)} \quad \text{(RD33)}$$

$$\texttt{e00(if,jr):} - \texttt{ddifdr0(k)} \quad \text{(RD34)}$$

$$\texttt{ep1(if,jr):} - \texttt{ddifdrp(k)} \quad \text{(RD35)}$$

$$\texttt{em1(if,jd):} - \texttt{ddifddm(k)} \quad \text{(RD36)}$$

$$\texttt{e00(if,jd):} - \texttt{ddifdd0(k)} \quad \text{(RD37)}$$

$$\texttt{em1(if,jt):} - \texttt{ddifdtm(k)} \quad \text{(RD38)}$$

$$\texttt{e00(if,jt):} - \texttt{ddifdt0(k)} \quad \text{(RD39)}$$

$$\texttt{em1(if,je):} - \texttt{ddifdem(k)} \quad \text{(RD40)}$$

$$\texttt{e00(if,je):} - \texttt{ddifde0(k)} \quad \text{(RD41)}$$

$$\texttt{e00(if,jf): frnom(k)} \quad \text{(RD42)}$$

Finally

$$- \texttt{rhs(if)} = F_k^{n+1} - \texttt{dif(k)} \tag{RD43}$$

(2) Equilibrium Diffusion

$$\texttt{e00(if,jt)}: -4a_R (T_k^{n+1})^4 \tag{RD44}$$

$$\texttt{e00(if,je)}: \ E_k^{n+1} \tag{RD45}$$

$$- \texttt{rhs(ie)} = E_k^{n+1} - 4a_R (T_k^{n+1})^4 \tag{RD46}$$

X. EDDINGTON FACTORS

A. Plane Geometry

(1) Solution of the Transfer Equation

The planar transfer equation is

$$\mu \frac{\partial I}{\partial r} = \rho(\eta - \chi I) \tag{EF1}$$

or

$$\mu \frac{\partial I}{\partial \tau} = I - S \tag{EF2}$$

where

$$S \equiv \eta/\chi = [(\rho\chi - n_e\sigma_e)B + n_e\sigma_e(cE/4\pi)]/\rho\chi \tag{EF3}$$

and

$$d\tau = -\rho\chi dr \tag{EF4}$$

Discretizing, we have

$$\Delta\tau_k \equiv -\rho_k\chi_k(r_{k+1} - r_k) = \tau_k - \tau_{k+1} \tag{EF5}$$

and

$$\begin{aligned} S_k(\tau) &= [S_k - \tfrac{1}{2}(dS/d\tau)_k\Delta\tau_k] + (dS/d\tau)_k(\tau - \tau_{k+1}) \\ &\equiv a_k + b_k(\tau - \tau_{k+1}) \qquad (\tau_{k+1} \le \tau \le \tau_k) \end{aligned} \tag{EF6}$$

where (dS/dτ) is monotonized as described in section X.C below. Then for an outgoing ray,

$$I^+_{j,k+1} = a_k(1 - e^{-\Delta\tau_k/\mu_j}) + b_k\mu_j\{1 - [1 + (\Delta\tau_k/\mu_j)]e^{-\Delta\tau_k/\mu_j}\} + I^+_{jk}e^{-\Delta\tau_k/\mu_j} \tag{EF7}$$

For incoming rays,

$$\begin{aligned} I^-_{j,k+1} = I^-_{j,k+1}&e^{-\Delta\tau_k/|\mu_j|} \\ &+ a_k(1 - e^{-\Delta\tau_k/|\mu_j|}) + b_k|\mu_j|[e^{-\Delta\tau_k/|\mu_j|} + (\Delta\tau_k/|\mu_j|) - 1] \end{aligned} \tag{EF8}$$

In this formulation, the intensities are naturally centered at the interfaces k and $k+1$. However, in order to compute Eddington factors, we actually need intensities at cell centers. Thus in practice we perform the integration in two steps, from the first interface to the cell center, then from the cell center to the next interface. This procedure greatly increases accuracy, at negligible cost. Therefore we augment the interfacial radial shells with a second set of shells through cell centers: $r_{k+\frac{1}{2}} \equiv \frac{1}{2}(r_k + r_{k+1})$. We index the combined sets of shells with a new index $\ell \equiv 2k - 1$, with $(k = 1,\ \frac{3}{2},\ 2,\ \frac{5}{2}, \ldots,\ N+1)$.

(2) Boundary Conditions

At the inner boundary $(k = 1)$ take

$$I^+_{j,1} \equiv I^+_L \tag{EF9}$$

for `lribc` $= 1$, and

$$I^+_{j,1} \equiv \frac{1}{4\pi}(cE_1 + \tfrac{3}{2}F_1\Delta\tau_1) + \frac{3}{4\pi}F_1\mu_j \tag{EF10}$$

for `lribc` $= 3$.

At the outer boundary $(k = N + 1)$ take

$$I^-_{j,N+1} \equiv I^-_R \tag{EF11}$$

for `lrobc` $= 1$, and

$$I^-_{j,N+1} \equiv I^+_{j,N+1} \tag{EF12}$$

for `lrobc` $= 2$. For *both* `lribc` $= 2$ *and* `lrobc` $= 2$, one must use a globally consistent solution, described in section X.B.3 below.

(3) Two Reflecting Boundaries

The physical problem is to find the ougoing intensity I^+_{jk} $(k = 1, \ldots, N+1)$ and the incoming intensity I^-_{jk} $(k = N+1, \ldots, 1)$ subject to reflecting boundary conditions at both the inner and outer boundaries. Because of the reflections, the intensities traveling in both directions are coupled.

From equations (EF7), (EF10), and reflecting boundary conditions we get a linear system of the following form: Starting with the jth outward traveling ray, the system is bidiagonal with elements on the principal diagonal and the first lower diagonal, $(k = 1, \ldots, N+1)$, representing the coupling of $I^+_{j,k+1}$

to I^+_{jk}, and a column vector on the right hand side containing the contributions of the source terms in cell $(k, k+1)$. We then extend the system from $k = N + 2$ to $k = 2N + 2$ to determine the inward depth variation of the incoming intensity $I^-_{jk'}$ where $k' \equiv 2N + 3 - k$. This part of the system is again bidiagonal, with the principal and the first lower diagonal. At the outer boundary $(k = N + 2)$ we equate $I^-_{j,N+1}$ to $I^+_{j,N+1}$. Likewise, at the inner boundary $(k = 1)$ we equate I^+_{j1} to I^-_{j1}, which introduces one additional element in the last column of the first row.

This system is easy to generate and solve. Let d_{jk} denote an element on the main diagonal, c_{jk} an element on the lower diagonal, e_{jk} an element in the rightmost column, and s_{jk} an element of the source vector on the right hand side. Then

$k = 1$

$$d_{j1} = 1, e_{j1} = -1, s_{j1} = 0 \tag{EF13}$$

$k = 2, \ldots, N + 1$

$$c_{jk} = -e^{-\Delta\tau_k/\mu_j}, d_{jk} = 1, e_{jk} = 0, s_{jk} = \text{sources on rhs of (EF7).} \tag{EF14}$$

$k = N + 2$

$$c_{j,N+2} = -1, d_{j,N+2} = 1, e_{j,N+2} = 0, s_{j,N+2k} = 0 \tag{EF15}$$

$k = 2N + 3 - k'$, where $k' = N, \ldots, 1$

$$c_{jk} = -e^{-\Delta\tau_k/|\mu_j|}, d_{jk} = 1, e_{jk} = 0, s_{jk} = \text{sources on rhs of (EF10).} \tag{EF16}$$

To solve the system, carry out the forward elimination:

$k = 2, \ldots, 2N + 1$

$$e_{jk} = -c_{jk}e_{j,k-1} \tag{EF17}$$
$$s_{jk} = s_{jk} - c_{jk}s_{j,k-1} \tag{EF18}$$

$k = 2N + 2$

$$d_{jk} = 1 - c_{jk}e_{j,k-1} \tag{EF19}$$
$$s_{jk} = s_{jk} - c_{jk}e_{j,k-1} \tag{EF20}$$
$$I^-_{j1} = s_{j1}/d_{j1} = I^+_{j1} \tag{EF21}$$

Given this globally consistent value for I^+_{j1}, we can now go back to (EF5) - (EF12) for boundary and cell center intensities.

B. Spherical Geometry

For each radial shell $(k, k+1)$ we know the cell-center values $S_{k+\frac{1}{2}}$ and can determine the monotonized derivative $(dS/d\tau)_{k+\frac{1}{2}}$, see section X.C. From this information we can calculate values for S_L and S_R at the left and right interfaces. Thus intensities are naturally centered at the interfaces k and $k+1$. However, in order to compute Eddington factors, we actually need the intensities at cell centers. Thus as in the planar case we perform the integration in two steps, from the first interface to the cell center, then from the cell center to the next interface. Therefore we augment the interfacial radial shells with a second set of shells through cell centers: $r_{k+\frac{1}{2}} \equiv \frac{1}{2}(r_k + r_{k+1})$. We index the combined sets of shells with a new index $\ell = 2k-1$, with $(k = 1,\ \frac{3}{2},\ 2,\ \frac{5}{2}, \ldots,\ N+1)$.

First compute all interface values of the source function:

$$S_{L\ell} = S_{k+\frac{1}{2}} + \frac{1}{2}\Delta\tau_k \left(\frac{dS}{d\tau}\right)_{k+\frac{1}{2}} \qquad (= S_{Lk}) \tag{EF22}$$

$$S_{R\ell} = S_{k+\frac{1}{2}} \tag{EF23}$$

$$S_{L,\ell+1} = S_{k+\frac{1}{2}} \tag{EF24}$$

$$S_{R,\ell+1} = S_{k+\frac{1}{2}} - \frac{1}{2}\Delta\tau_k \left(\frac{dS}{d\tau}\right)_{k+\frac{1}{2}} \qquad (= S_{Rk}) \tag{EF25}$$

$$\Delta\tau_k = \tau_k - \tau_{k+1} \tag{EF26}$$

Then using the augmented set of shells, define the impact parameters of a parallel set of rays tangent to the spherical shells as

$$p_j = (j-1)r_1/\mathtt{ncore} \qquad (1 \le j \le \mathtt{ncore}) \tag{EF27}$$

and

$$p_j = r_{j-\mathtt{ncore}} \qquad (\mathtt{ncore}+1 \le j \le \mathtt{nray}) \tag{EF28}$$

where

$$\mathtt{nray} = \mathtt{ncore} + 2N + 1 \tag{EF29}$$

Here `nray` is the total number of rays, and `ncore` is the number of rays penetrating *inside* the first (innermost) radial shell. In general, on shell ℓ we have *core rays* for $1 \le j \le$ `ncore`, and *envelope rays* for $\mathtt{ncore} + 1 \le j \le \mathtt{ncore} + \ell$.

For incoming rays we have

$$x_{j\ell} \quad \equiv \left(r_\ell^2 \quad - p_j^2\right)^{\frac{1}{2}} \tag{EF30}$$

$$x_{j,\ell+1} \equiv \left(r_{\ell+1}^2 - p_j^2\right)^{\frac{1}{2}} \tag{EF31}$$

and

$$\Delta\tau_{j\ell} = \rho_k \chi_k (x_{j\ell} - x_{j,\ell+1}) \tag{EF32}$$

where $k = (\ell+1)/2$ using FORTRAN integer arithmetic rules. For the interval $(\ell, \ell+1)$ take

$$S_{j\ell}(t) = S_{L\ell} + \left(\frac{S_{R\ell} - S_{L\ell}}{\Delta\tau_{j\ell}}\right) t = a_\ell + b_{j\ell} t \tag{EF33}$$

where $t \equiv (\tau_\ell - \tau)$, $\quad 0 \le t \le \Delta\tau_{j\ell}$. Then

$$I_{j,\ell}^- = I_{j,\ell+1}^- e^{-\Delta\tau_{j\ell}} + a_\ell(1 - e^{-\Delta\tau_{j\ell}}) + b_{j\ell}[1 - (1 + \Delta\tau_{j\ell})e^{-\Delta\tau_{j\ell}}] \tag{EF34}$$

and

$$\mu_{j\ell} = \left(1 - p_j^2/r_\ell^2\right)^{\frac{1}{2}}, \qquad 0 \le \mu_{jl} \le 1 \tag{EF35}$$

The outer boundary condition at $\ell = \ell_{max}$ is

$$I_j^- = 0, \qquad (j = 1, ..., \mathtt{nray}) \tag{EF36}$$

For outgoing rays in the interval $(\ell - 1, \ell)$ take

$$S_{j,\ell-1}(t) = S_{R,\ell-1} + \left(\frac{S_{L,\ell-1} - S_{R,\ell-1}}{\Delta\tau_{j,\ell-1}}\right) t = a_{\ell-1} + b_{j,\ell-1} t \tag{EF37}$$

where $t \equiv (\tau - \tau_\ell)$, $\quad 0 \le t \le \Delta\tau_{j,\ell-1}$. Then

$$\begin{aligned} I_{j\ell}^+ = {} & e^{-\Delta\tau_{j,\ell-1}} I_{j,\ell-1}^+ \\ & + a_{\ell-1}(1 - e^{-\Delta\tau_{j,\ell-1}}) + b_{j,\ell-1}[1 - (1 + \Delta\tau_{j,\ell-1})e^{-\Delta\tau_{j,\ell-1}}] \end{aligned} \tag{EF38}$$

At the inner boundary $(k = 1)$ take

$$I_1^+ \equiv I_L^+ \tag{EF39}$$

for `lribc` = 1, and for `lribc` = 3 take

$$I_1^+ \equiv \tfrac{1}{4\pi}(cE_1 + \tfrac{3}{2}F_1\Delta\tau_1) + \tfrac{3}{4\pi}F_1\mu \tag{EF40}$$

where $\Delta\tau_1$ is the optical thickness of the innermost shell $(k,\, k+1) = (1,\, 2)$. The $\Delta\tau$ term in equation (EF40) should be very small for a star: $\mathrm{O}(T_{eff}^4/T_1^4)$ where T_{eff} is the effective temperature of the star, and T_1 is the temperature at the inner boundary.

C. Monotonic Interpolation of Slopes

Because the variation in cell size on an adaptive grid can be extreme, the physical accuracy of the integration along rays can be improved by using monotonized slopes of the source function on the ray. Thus define

$$\Delta x_k \equiv \tfrac{1}{2}(x_{k+2} - x_k) \tag{EF41}$$

$$\Delta q_k \equiv \quad q_{k+1} - q_k \tag{EF42}$$

Then take

$$\left(\frac{dx}{dq}\right)_k = \frac{1}{2}\left(\frac{\Delta x_{k-1}}{\Delta q_{k-1}} + \frac{\Delta x_k}{\Delta q_k}\right) \tag{EF43}$$

or

$$\overline{\left(\frac{dq}{dx}\right)}_k = \frac{2\Delta q_{k-1}\Delta q_k}{\Delta x_{k-1}\Delta q_k + \Delta x_k \Delta q_{k-1}} \tag{EF44}$$

which reduces to van Leer's formula for equal step sizes. If we use (EF44) when $\Delta q_{k-1}\Delta q_k > 0$, and $\overline{(dq/dx)} = 0$ otherwise, we get monotonic interpolation. For example, at interface $k+1$

$$q_{I,k+1} = q_k + \frac{1}{2}\Delta x_k \left(\frac{dq}{dx}\right)_k = q_k + \frac{\Delta q_k}{\left(\frac{\Delta q_k}{\Delta q_{k-1}}\frac{\Delta x_{k-1}}{\Delta x_k} + 1\right)} \tag{EF45}$$

The term in the denominator lies on the range $(1, \infty)$. Negative values of the factor $(\Delta q_k / \Delta q_{k-1})$ are excluded because the filter sets $(dq/dx)_k = 0$ when $(\Delta q_{k-1} \Delta q_k) \le 0$. Therefore the interface value can never exceed $q_k + \Delta q_k$.

D. Angle Quadrature

On the interval $\mu_j \le \mu \le \mu_{j+1}$ represent the angular variation of the specific intensity as

$$I(\mu) = I_j + \frac{(I_{j+1} - I_j)}{(\mu_{j+1} - \mu_j)} (\mu - \mu_j) \equiv \alpha_j + \beta_j(\mu - \mu_j) \tag{EF46}$$

Then

$$\int_{\mu_j}^{\mu_{j+1}} [(\alpha_j - \beta_j \mu_j) + \beta_j \mu] \mu^n \, d\mu$$

$$= (\alpha_j - \beta_j \mu_j) \frac{(\mu_{j+1}^{n+1} - \mu_j^{n+1})}{(n+1)} + \beta_j \frac{(\mu_{j+1}^{n+2} - \mu_j^{n+2})}{(n+2)}$$

$$= \frac{(I_j \mu_{j+1} - I_{j+1} \mu_j)}{(n+1)} \frac{(\mu_{j+1}^{n+1} - \mu_j^{n+1})}{(\mu_{j+1} - \mu_j)} + \frac{(I_j \mu_{j+1} - I_{j+1} \mu_j)}{(n+2)} \frac{(\mu_{j+1}^{n+2} - \mu_j^{n+2})}{(\mu_{j+1} - \mu_j)} \tag{EF47}$$

Hence

$$\mathcal{I}_{j0} \equiv \int_{\mu_j}^{\mu_{j+1}} I(\mu) \quad d\mu = \frac{1}{2}(I_j + I_{j+1})(\mu_{j+1} - \mu_j) \tag{EF48}$$

$$\mathcal{I}_{j1} \equiv \int_{\mu_j}^{\mu_{j+1}} I(\mu)\mu \;\; d\mu = \frac{1}{2}(I_j \mu_{j+1} - I_{j+1} \mu_j)(\mu_j + \mu_{j+1})$$

$$+\frac{1}{3}(I_{j+1} - I_j)(\mu_j^2 + \mu_j \mu_{j+1} + \mu_{j+1}^2) \tag{EF49}$$

$$\mathcal{I}_{j2} \equiv \int_{\mu_j}^{\mu_{j+1}} I(\mu)\mu^2 d\mu = \frac{1}{3}(I_j\mu_{j+1} - I_{j+1}\mu_j)(\mu_j^2 + \mu_j\mu_{j+1} + \mu_{j+1}^2)$$

$$+\frac{1}{4}(I_{j+1} - I_j)[(\mu_j + \mu_{j+1})(\mu_j^2 + \mu_{j+1}^2)] \tag{EF50}$$

Thus the Eddington factor at the center of cell $(k, k+1)$ is

$$\texttt{fedd(k)} = f_k = \sum_{j=1}^{J} \mathcal{I}_{\ell j0} / \sum_{\ell j=1}^{J} \mathcal{I}_{\ell j2} \tag{EF51}$$

where $\ell = 2k$.

For planar geometry the code uses a fixed set of angles $\{\mu_j\}$, $(1 \leq j \leq J+1)$ which may be changed by the user. For spherical geometry the code uses the set of $\{\mu_j\}$, $(1 \leq j \leq \ell + \texttt{ncore})$, induced on each shell by the tangent rays that penetrate that shell.

XI. TOTAL ENERGY

A. Differential Equation

Summing the gas mechanical energy equation, the radiating fluid energy equation, and the radiation momentum equation we get

$$\frac{d}{dt}\left[\left(e+\frac{E}{\rho}+\frac{1}{2}u^2+\left[\left(1-\frac{\mu}{2}\right)gr-\frac{\mu}{2}\frac{4\pi Gm}{r}\right]\right)\Delta\xi\right]$$

$$-\Delta\left[\frac{dm}{dt}\left(e+\frac{E}{\rho}+\frac{1}{2}u^2+\left[\left(1-\frac{\mu}{2}\right)gr-\frac{\mu}{2}\frac{4\pi Gm}{r}\right]\right)\right]$$

$$+\Delta\left[r^\mu\, u\,(p+P)+r^\mu\, F\right]-\Delta\left(r^\mu\,\sigma_e\rho\frac{\Delta e}{\Delta r}\right)-(\epsilon_Q+u\phi_Q)\Delta V=0 \qquad \text{(TE1)}$$

where $\mu = 0$ or 2.

B. Spacetime Integral

Integrate over the entire spatial domain and over all times since the start of the calculation to get

$$\mathcal{E}^{n+1}+\mathcal{S}^{n+1}+\mathcal{W}^{n+1}+\mathcal{L}^{n+1}-\mathcal{Q}^{n+1}=\mathcal{E}^1\equiv \text{Const} \qquad \text{(TE2)}$$

where the *total energy* is

$$\mathcal{E}^{n+1}\equiv\sum_{k=1}^{N}\left\{e_k^{n+1}+\frac{E_k^{n+1}}{\rho_k^{n+1}}+\frac{1}{4}\left[(u_k^{n+1})^2+(u_{k+1}^{n+1})^2\right]\right.$$

$$+\left(1-\frac{\mu}{2}\right)\frac{g}{2}\left[(r_k^{n+1}+r_{k+1}^{n+1})-(r_k^1+r_{k+1}^1)\right]$$

$$\left.-\frac{\mu}{2}2\pi G\left[\left(\frac{m_k^{n+1}}{r_k^{n+1}}+\frac{m_{k+1}^{n+1}}{r_{k+1}^{n+1}}\right)-\left(\frac{m_k^1}{r_k^1}+\frac{m_{k+1}^1}{r_{k+1}^1}\right)\right]\right\}\Delta\xi_k^{n+1} \qquad \text{(TE3)}$$

the net fluid energy *loss by transport* through the boundary surfaces is

$$\mathcal{S}^{n+1} \equiv \sum_{\nu=1}^{n} \Bigg\{ (r_{N+1}^{\nu+\frac{1}{2}})^{\mu}\, \Phi_R^0 \Bigg[e_N^{\nu+\frac{1}{2}} + \frac{E_N^{\nu+\frac{1}{2}}}{\rho_N^{\nu+\frac{1}{2}}} + \frac{1}{2}(u_{N+1}^{\nu+\frac{1}{2}})^2$$

$$+ \left(1 - \frac{\mu}{2}\right) \frac{g}{2}(r_{N+1}^{\nu+\frac{1}{2}} - r_{N+1}^{1}) - \frac{\mu}{2} 4\pi G \left(\frac{m_{N+1}^{\nu+\frac{1}{2}}}{r_{N+1}^{\nu+\frac{1}{2}}} - \frac{m_{N+1}^{1}}{r_{N+1}^{1}} \right) \Bigg]$$

$$- (r_1^{\nu+\frac{1}{2}})^{\mu}\, \Phi_L^0 \Bigg[e_1^{\nu+\frac{1}{2}} + \frac{E_1^{\nu+\frac{1}{2}}}{\rho_1^{\nu+\frac{1}{2}}} + \frac{1}{2}(u_1^{\nu+\frac{1}{2}})^2$$

$$+ \left(1 - \frac{\mu}{2}\right) \frac{g}{2}(r_1^{\nu+\frac{1}{2}} - r_1^{1}) - \frac{\mu}{2} 4\pi G \left(\frac{m_1^{\nu+\frac{1}{2}}}{r_1^{\nu+\frac{1}{2}}} - \frac{m_1^{1}}{r_1^{1}} \right) \Bigg] \Bigg\} \Delta t^{\nu+\frac{1}{2}} \tag{TE4}$$

the *work* done by the fluid at the boundary surfaces is

$$\mathcal{W}^{n+1} \equiv \sum_{\nu=1}^{n} \Bigg[(r_{N+1}^{\nu+\frac{1}{2}})^{\mu}\, u_{N+1}^{\nu+\frac{1}{2}} (p_N^{\nu+\frac{1}{2}} + P_N^{\nu+\frac{1}{2}})$$

$$- (r_1^{\nu+\frac{1}{2}})^{\mu}\, u_1^{\nu+\frac{1}{2}} (p_1^{\nu+\frac{1}{2}} + P_1^{\nu+\frac{1}{2}}) \Bigg] \Delta t^{\nu+\frac{1}{2}} \tag{TE5}$$

and the *luminosity* lost through the boundary surfaces is

$$\mathcal{L}^{n+1} \equiv \sum_{\nu=1}^{n} \left[(r_{N+1}^{\nu+\frac{1}{2}})^{\mu}\, F_{N+1}^{\nu+\frac{1}{2}} - (r_1^{\nu+\frac{1}{2}})^{\mu}\, F_1^{\nu+\frac{1}{2}} \right] \Delta t^{\nu+\frac{1}{2}} \tag{TE6}$$

To calculate the *viscous energy dissipation*, take the viscous pressure Q to be

$$Q = \frac{4}{3}\rho\mu_Q\left(\frac{\partial u}{\partial r} - \frac{\mu}{2}\frac{u}{r}\right) \tag{TE7}$$

where $\mu = 0$ or 2. Then one can show that

$$u\phi_Q \Delta V = \left(\frac{u}{r^{\mu/2}}\right) \Delta\left(r^{3\mu/2} Q\right) \tag{TE8}$$

and

$$\epsilon_Q \Delta V = r^{3\mu/2} Q \Delta\left(\frac{u}{r^{\mu/2}}\right) \tag{TE9}$$

Therefore

$$(u\phi_Q + \epsilon_Q)\,\Delta V = \Delta\left(r^\mu u Q\right) \tag{TE10}$$

Summing over all cells we get

$$\mathcal{Q}^{n+1} \equiv \sum_{\nu=1}^{n}\left[\left(r^\mu u Q\right)_{N+1}^{\nu+\frac{1}{2}} - \left(r^\mu u Q\right)_{1}^{\nu+\frac{1}{2}}\right]\Delta t^{\nu+\frac{1}{2}} \tag{TE11}$$

Diffusion terms are ignored because in the absence of information outside the boundaries, we can only assume that gradients, hence the diffusion fluxes, are zero.

XII. BOUNDARY CONDITIONS

A. Eulerian

(1) Generic BCs

Assume an *imposed flow* from *outside* the computational domain. Then for $k = 1$ or $k = N + 1$ we have:

$$\text{Adaptive Grid:} \quad \dot{r}_k \equiv 0, \quad \Rightarrow \quad u_{grid,k} \equiv 0, \quad \text{and} \quad u_{rel,k} \equiv u_k \tag{BC1}$$

Continuity:	$(\rho u_{rel})_k$	$\equiv$	Φ_k^0	$\mathrm{g\,cm^{-2}s^{-1}}$	$=$	$(\rho u)_k$	(BC2)
Gas Momentum:	$(\rho u u_{rel})_k$	$\equiv$	Φ_k^1	$\mathrm{g\,cm^{-1}s^{-2}}$	$=$	$(\rho u^2)_k$	(BC3)
Gas Energy:	$(\rho e u_{rel})_k$	$\equiv$	Φ_k^2	$\mathrm{erg\,cm^{-2}s^{-1}}$	$=$	$(\rho e u)_k$	(BC4)

The last equality holds only at an Eulerian boundary where by definition $u_{rel} \equiv u$. In principle all the quantities $\rho_k, u_k, e_k, u_{rel,k}$, and $\Phi_k^{0,1,2}$ can be functions of time; but in practice the current code assumes that they are constant.

At the inner (left) boundary ($k = L$) we have

$$\Phi_L^0 \geq 0, \qquad \Phi_L^1 \geq 0, \qquad \Phi_L^2 \geq 0, \tag{BC5}$$

and at the outer (right) boundary ($k = R$) we have

$$\Phi_R^0 \leq 0, \qquad \Phi_R^1 \geq 0, \qquad \Phi_R^2 \leq 0. \tag{BC6}$$

For nonzero fluxes the values of the physical variables at the boundaries are given by

$\rho_L \equiv$	$(\Phi_L^0)^2/\Phi_L^1$	(BC7)	$\rho_R \equiv$	$(\Phi_R^0)^2/\Phi_R^1$	(BC8)
$u_L \equiv u_1 \equiv$	$\Phi_L^1 \ / \Phi_L^0$	(BC9)	$u_R \equiv u_{N+1} \equiv$	$\Phi_R^1 \ / \Phi_R^0$	(BC10)
$e_L \equiv$	$\Phi_L^2 \ / \Phi_L^0$	(BC11)	$e_R \equiv$	$\Phi_R^2 \ / \Phi_R^0$	(BC12)

(2) Inner Boundary ($k = 1$)

(a) Adaptive Grid

$$r_1^{n+1} - r_1^n = 0 \tag{BC13}$$

(b) Mass Definition

$$m_1^{n+1} = m_1^n - (r_1^{n+\theta})^\mu \, \Phi_L^0 \, dt \qquad \text{(BC14)}$$

Linearization of equations (BC13) and (BC14) is trivial.

(c) Continuity

$$\frac{\rho_1^{n+1}\Delta V_1^{n+1} - \rho_1^n \Delta V_1^n}{dt} + \ (r_2^{n+\theta})^\mu \left[u_2^{n+\theta} - \left(\frac{r_2^{n+1} - r_2^n}{dt} \right) \right] \overline{\rho}_2 - (r_1^{n+\theta})^\mu \Phi_L^0$$

$$-2\sigma_\rho \left[(r_2^{n+\theta})^\mu \left(\frac{\rho_2^{n+\theta} - \rho_1^{n+\theta}}{r_3^{n+\theta} - r_1^{n+\theta}} \right) - (r_1^{n+\theta})^\mu \left(\frac{\rho_1^{n+\theta} - \rho_0^{n+\theta}}{r_2^{n+\theta} - r_0^{n+\theta}} \right) \right] = 0 \qquad \text{(BC15)}$$

Linearization of equation (BC15) is identical to the liniearization of equation (C2) except for the term in Φ_L^0.

(d) Gas Momentum

$$u_1^{n+1} = u_L \qquad \text{(BC16)}$$

Linearization of equation (BC16) is trivial.

(e) Radiating Fluid Energy

$$\left[(\rho_1^{n+1} e_1^{n+1} + E_1^{n+1}) \Delta V_1^{n+1} - (\rho_1^n e_1^n + E_1^n) \Delta V_1^n \right] / dt$$

$$- \left(\overline{e + \tfrac{E}{\rho}} \right)_2 (dm/dt)_2^{n+\theta} - \left(r_1^{n+\theta} \right)^\mu \{ \Phi_L^2 + [u_1^{n+\theta} - (dr/dt)_1^{n+\theta}] \, E_1^{n+\theta} \}$$

$$+ \ (r_2^{n+\theta})^\mu F_2^{n+\theta} - (r_1^{n+\theta})^\mu F_1^{n+\theta}$$

$$+ \ (p_1^{n+\theta} + f_1^{n+\theta} E_1^{n+\theta}) [(r_2^{n+\theta})^\mu u_2^{n+\theta} - (r_1^{n+\theta})^\mu u_1^{n+\theta}]$$

$$+ \ \frac{\mu}{4} (1 - 3 f_1^{n+\theta}) E_1^{n+\theta} \left[\left(\frac{u_2^{n+\theta}}{r_2^{n+\theta}} \right) + \left(\frac{u_1^{n+\theta}}{r_1^{n+\theta}} \right) \right] \Delta V_1^{n+\theta}$$

$$
-\frac{4}{3}(\mu_Q)_1^{n+\theta}\rho_1^{n+\theta}\left[\frac{u_2^{n+\theta}-u_1^{n+\theta}}{r_2^{n+\theta}-r_1^{n+\theta}}-\frac{\mu}{4}\left(\frac{u_2^{n+\theta}}{r_2^{n+\theta}}+\frac{u_1^{n+\theta}}{r_1^{n+\theta}}\right)\right]^2\Delta V_1^{n+\theta}
$$

$$
-\sigma_e\left[(r_2^{n+\theta})^\mu(\rho_1^{n+\theta}+\rho_2^{n+\theta})\left(\frac{e_2^{n+\theta}-e_1^{n+\theta}}{r_3^{n+\theta}-r_1^{n+\theta}}\right)\right.
$$

$$
\left.-(r_1^{n+\theta})^\mu(\rho_0^{n+\theta}+\rho_1^{n+\theta})\left(\frac{e_1^{n+\theta}-e_0^{n+\theta}}{r_2^{n+\theta}-r_0^{n+\theta}}\right)\right]
$$

$$
=0 \tag{BC17}
$$

Note that for an Eulerian boundary, the term in $(\dot{r})_1^{n+\theta}$ is identically zero; we retain it here so the same formula can be used for a Lagrangean boundary. Linearization of equation (BC17) is identical to the linearization of (FE3) except for the advection term at the inner (left) boundary.

(f) Radiation Energy

$$
\begin{aligned}
&(E_1^{n+1}\Delta V_1^{n+1}-E_1^n\Delta V_k^1)/dt \\
&-\left(\frac{E}{\rho}\right)_2(dm/dt)_2^{n+\theta}-\left(r_1^{n+\theta}\right)^\mu\left[u_1^{n+\theta}-(dr/dt)_1^{n+\theta}\right]E_1^{n+\theta} \\
&+(r_2^{n+\theta})^\mu F_2^{n+\theta}-(r_1^{n+\theta})^\mu F_1^{n+\theta}+f_1^{n+\theta}E_1^{n+\theta}\left[(r_2^{n+\theta})^\mu u_2^{n+\theta}-(r_1^{n+\theta})^\mu u_1^{n+\theta}\right] \\
&+\tfrac{\mu}{4}(1-3f_1^{n+\theta})E_1^{n+\theta}\left[\left(u_2^{n+\theta}/r_2^{n+\theta}\right)+\left(u_1^{n+\theta}/r_1^{n+\theta}\right)\right]\Delta V_1^{n+\theta} \\
&=\left[c(\kappa_E)_1^{n+\theta}E_1^{n+\theta}-4\pi(\kappa_P)_1^{n+\theta}B_1^{n+\theta}\right]\rho_1^{n+\theta}\Delta V_1^{n+\theta}
\end{aligned} \tag{BC18}
$$

Note that for an Eulerian boundary, the term in $(\dot{r})_1^{n+\theta}$ is identically zero; we retain it here so the same formula can be used for a Lagrangean boundary. Linearization of equation (BC18) is identical to the linearization of (RE2) except for the advection term at the inner (left) boundary *(g) Radiation Momentum*

`(1) Optically Transmitting Boundary`

At the inner boundary, assume an incident outward (rightward) propa- gating external radiation field $I_L^+(\mu)$, taken to be isotropic. At this boundary, the inward (leftward) propagating radiation is simply the internal radiation field, i.e. $I^-(\mu) = I_{internal}(-\mu)$. Then

$$cE_1 = 2\pi \int_0^1 (I^+ + I^-) d\mu = 2\pi I_L^+ + 2\pi \int_0^1 I^- d\mu \qquad \text{(BC19)}$$

and

$$F_1 = 2\pi \int_0^1 (I^+ + I^-) \mu d\mu = \pi I_L^+ - 2\pi \int_0^1 I^- \mu d\mu \qquad \text{(BC20)}$$

Define the surface Eddington factor

$$g_1 \equiv \int_0^1 I^- \mu d\mu / \int_0^1 I^- d\mu \qquad \text{(BC21)}$$

which is obtained from the formal solution. Then

$$F_1 = (2g_1 + 1)\pi I_L^+ - c g_1 E_1 \qquad \text{(BC22)}$$

which eliminates F_1 in terms of E_1, thus closing the system. Linearization of equation (BC22) is trivial.

`(2) Optically Reflecting Boundary`

$$F_1 \equiv 0 \qquad \text{(BC23)}$$

Linearization of equation (BC23) is trivial.

`(3) Imposed Net Flux (diffusion limit)`

$$F_1 = L_1(t)/4\pi r_1^2 \qquad \text{(spherical medium)} \qquad \text{(BC24)}$$

$$= \sigma[T_{eff}(t)]^4 \qquad \text{(planar medium)} \qquad \text{(BC25)}$$

In principle we can have $F_1 = f(t)$, but at present the code assumes that F_1 is constant. Linearization of equation (BC23) is trivial.

(3) Outer Boundary $(k = N + 1)$

(3A) Boundary Fluxes Specified

(a) Adaptive Grid

$$r_{N+1}^{n+1} - r_{N+1}^{n} \equiv 0 \tag{BC26}$$

Linearization of equation (BC26) is trivial.

(b) Mass Definition

The boundary condition is given by equation (M2) written for $k = N+1$. As a consistency check, sum the mass definition equation over the entire domain, to obtain

$$m_{N+1}^{n+1} - m_1^{n+1} = \sum_{k=1}^{N} \rho_k^{n+1} \Delta V_k^{n+1} \tag{BC27}$$

Now sum the continuity equation over the entire domain, obtaining

$$\frac{d}{dt}\left(\sum_{k=1}^{N} \rho_k \Delta V_k\right) + (r_{N+1})^{\mu}\,\Phi_R^0 \; - \; (r_1)^{\mu}\,\Phi_L^0 = 0 \tag{BC28}$$

which, recalling that both the positions of the Eulerian boundaries and the boundary fluxes are constant in time, yields

$$(m_{N+1}^{n+1} - m_1^{n+1}) - (m_{N+1}^{n} - m_1^{n}) = [\,(r_1)^{\mu}\,\Phi_L^0 \; - \; (r_{N+1})^{\mu}\,\Phi_R^0\,]\,dt \tag{BC29}$$

Then from (BC14) we get

$$m_{N+1}^{n+1} = m_{N+1}^{n} - (r_{N+1})^{\mu}\,\Phi_R^0\,dt \tag{BC30}$$

as expected from first principles.

(c) Continuity ($k = N$)

$$\begin{aligned} &\frac{\rho_N^{n+1}\Delta V_N^{n+1} - \rho_N^{n}\Delta V_N^{n}}{dt} \\ &+ (r_{N+1}^{n+\theta})^{\mu}\;\Phi_R^0 \; - \; (r_N^{n+\theta})^{\mu}\left[u_N^{n+\theta} - \left(\frac{r_N^{n+1} - r_N^{n}}{dt}\right)\right]\overline{\rho}_N \\ &-2\sigma_\rho\left[(r_{N+1}^{n+\theta})^{\mu}\left(\frac{\rho_{N+1}^{n+\theta} - \rho_N^{n+\theta}}{r_{N+2}^{n+\theta} - r_N^{n+\theta}}\right) - (r_N^{n+\theta})^{\mu}\left(\frac{\rho_N^{n+\theta} - \rho_{N-1}^{n+\theta}}{r_{N+1}^{n+\theta} - r_{N-1}^{n+\theta}}\right)\right] = 0 \end{aligned} \tag{BC31}$$

Linearization of equation (BC31) is identical to the linearization of equation (C2) except for the term in Φ_R^0.

(d) Gas Momentum $(k = N+1)$

$$u_{N+1}^{n+1} = u_R \tag{BC32}$$

Linearization of equation (BC32) is trivial.

(e) Radiating Fluid Energy $(k = N)$

$$\begin{aligned}
&\left[(\rho_N^{n+1} e_N^{n+1} + E_N^{n+1})\Delta V_N^{n+1} - (\rho_N^n e_N^n + E_N^n)\Delta V_N^n\right]/dt \\
&+ \left(r_{N+1}^{n+\theta}\right)^\mu \{\Phi_R^2 + [u_{N+1}^{n+\theta} - (dr/dt)_{N+1}^{n+\theta}]\, E_N^{n+\theta}\} + \left(\overline{e + \tfrac{E}{\rho}}\right)_N (dm/dt)_N^{n+\theta} \\
&+ (r_{N+1}^{n+\theta})^\mu F_{N+1}^{n+\theta} - (r_N^{n+\theta})^\mu F_N^{n+\theta} \\
&+ (p_N^{n+\theta} + f_N^{n+\theta} E_N^{n+\theta})[(r_{N+1}^{n+\theta})^\mu u_{N+1}^{n+\theta} - (r_N^{n+\theta})^\mu u_N^{n+\theta}] \\
&+ \frac{\mu}{4}(1 - 3f_N^{n+\theta}) E_N^{n+\theta} \left[\left(\frac{u_{N+1}^{n+\theta}}{r_{N+1}^{n+\theta}}\right) + \left(\frac{u_N^{n+\theta}}{r_N^{n+\theta}}\right)\right] \Delta V_N^{n+\theta} \\
&- \sigma_e \left[(r_{N+1}^{n+\theta})^\mu (\rho_N^{n+\theta} + \rho_{N+1}^{n+\theta}) \left(\frac{e_{N+1}^{n+\theta} - e_N^{n+\theta}}{r_{N+2}^{n+\theta} - r_N^{n+\theta}}\right)\right. \\
&\qquad\qquad \left. - (r_N^{n+\theta})^\mu (\rho_{N-1}^{n+\theta} + \rho_N^{n+\theta}) \left(\frac{e_N^{n+\theta} - e_{N-1}^{n+\theta}}{r_{N+1}^{n+\theta} - r_{N-1}^{n+\theta}}\right)\right] \\
&- \frac{4}{3}(\mu_Q)_N^{n+\theta} \rho_N^{n+\theta} \left[\frac{u_{N+1}^{n+\theta} - u_N^{n+\theta}}{r_{N+1}^{n+\theta} - r_N^{n+\theta}} - \frac{\mu}{4}\left(\frac{u_{N+1}^{n+\theta}}{r_{N+1}^{n+\theta}} + \frac{u_N^{n+\theta}}{r_N^{n+\theta}}\right)\right]^2 \Delta V_N^{n+\theta} = 0
\end{aligned} \tag{BC33}$$

Note that for an Eulerian boundary, the term in $(\dot{r})_{N+1}^{n+\theta}$ is identically zero; we retain it here so the same formula can be used for a Lagrangean boundary. Linearization of equation (BC33) is identical to the linearization of (FE3) except for the advection term at the outer (right) boundary.

(f) Radiation Energy $(k = N)$

$$
\begin{aligned}
&(E_N^{n+1}\Delta V_N^{n+1} - E_N^n \Delta V_N^n)/dt \\
&+ \left(r_{N+1}^{n+\theta}\right)^\mu [u_{N+1}^{n+\theta} - (dr/dt)_{N+1}^{n+\theta}]\, E_N^{n+\theta} + \left(\frac{\overline{E}}{\rho}\right)_N (dm/dt)_N^{n+\theta} \\
&+ (r_{N+1}^{n+\theta})^\mu F_{N+1}^{n+\theta} - (r_N^{n+\theta})^\mu F_N^{n+\theta} + f_N^{n+\theta} E_N^{n+\theta}[(r_{N+1}^{n+\theta})^\mu u_{N+1}^{n+\theta} - (r_N^{n+\theta})^\mu u_N^{n+\theta}] \\
&+ \tfrac{\mu}{4}(1 - 3f_N^{n+\theta})E_N^{n+\theta}\left[\left(u_{N+1}^{n+\theta}/r_{N+1}^{n+\theta}\right) + \left(u_N^{n+\theta}/r_N^{n+\theta}\right)\right] \Delta V_N^{n+\theta} \\
&+ [c(\kappa_E)_N^{n+\theta} E_N^{n+\theta} - 4\pi(\kappa_P)_N^{n+\theta} B_N^{n+\theta}]\rho_N^{n+\theta} \Delta V_N^{n+\theta} = 0
\end{aligned}
\tag{BC34}
$$

Note that for an Eulerian boundary, the term in $(\dot{r})_{N+1}^{n+\theta}$ is identically zero; we retain it here so the same formula can be used for a Lagrangean boundary. Linearization of equation (BC34) is identical to the linearization of (RE2) except for the advection term at the outer (right) boundary.

(g) Radiation Momentum $(k = N + 1)$

`(1) Optically Transmitting Boundary`

At the outer boundary, assume an incident inward (leftward) propagating external radiation field $I_R^-(\mu)$, taken to be isotropic. At this boundary, the outward (rightward) propagating radiation is simply the internal radiation field, i.e. $I^+(\mu) = I_{internal}(\mu)$. Then by an analysis similar to that leading to equation (BC22) we find

$$F_{N+1} = cg_{N+1}E_{N+1} - (2g_{N+1} + 1)\pi I_R^- \tag{BC35}$$

`(2) Optically Reflecting Boundary`

$$F_{N+1} \equiv 0 \tag{BC36}$$

`(3) Imposed Net Flux`

$$r_{N+1}^\mu F_{N+1} = r_N^\mu F_N \tag{BC37}$$

Linearization of equations (BC35) – (BC37) is trivial.

(3B) Transmitting Outer Boundary

(a) Adaptive Grid ($k = N + 1$)

The boundary condition is given by equation (BC26).

(b) Mass Definition

The boundary condition is given by equation (M2) written for $k = N + 1$.

(c) Continuity ($k = N$)

The boundary condition is the same as equation (BC31) with

$$(r_{N+1}^{n+\theta})^{\mu}\ \Phi_R^0 \quad \rightarrow \quad (r_{N+1}^{n+\theta})^{\mu}\left[u_{N+1}^{n+\theta} - \left(r_{N+1}^{n+1} - r_{N+1}^{n}\right)/dt\right]\overline{\rho}_{N+1} \qquad \text{(BC38)}$$

(d) Gas Momentum

Replace the momentum equation with an equation for the advection of fluid velocity along the outgoing characteristic C^{+}:

$$\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x} = 0 \qquad \Rightarrow \qquad \frac{Du}{Dt} + a\frac{\partial u}{\partial x} = 0 \qquad \text{(BC39)}$$

The above equation neglects terms $O(\lambda/R)$, where λ is the wavelength of a disturbance, and R is the radius of curvature of the boundary surface. Rewriting equation (BC39) on the adaptive grid we get:

$$\frac{d}{dt}[u(\rho\Delta V)] - \Delta(\frac{dm}{dt}u) + r^{\mu}\rho a\Delta u = 0 \qquad \text{(BC40)}$$

Or, in finite difference form

$$\begin{aligned} &[(u_N^{n+1} + u_{N+1}^{n+1})\rho_N^{n+1}\Delta V_N^{n+1} - (u_N^n + u_{N+1}^n)\rho_N^n\Delta V_N^n]\ /\ 2dt \\ -\ &[(m_{N+1}^{n+1} - m_{N+1}^{n})u_{N+1}^{n+\theta} - (m_N^{n+1} - m_N^n)u_N^{n+\theta}]\ /\ dt \\ +&\tfrac{1}{2}[(r_N^{n+\theta})^{\mu} + (r_{N+1}^{n+\theta})^{\mu}]\rho_N^{n+\theta}a_N^{n+\theta}(u_{N+1}^{n+\theta} - u_N^{n+\theta}) = 0 \qquad \text{(BC41)} \end{aligned}$$

Linearization of equation (BC41) is straightforward. From the time derivative term we get

$$\texttt{em1(iu,jr):}\ -\ (u_N^{n+1}+u_N^{n+1})\,\rho_N^{n+1}\,\texttt{rmup1n(N\ \ \ )}/2dt \tag{BC42}$$

$$\texttt{e00(iu,jr):}\quad (u_N^{n+1}+u_N^{n+1})\,\rho_{N+1}^{n+1}\texttt{rmup1n(N+1)}/2dt \tag{BC43}$$

$$\texttt{em1(iu,jd):}\quad (u_N^{n+1}+u_N^{n+1})\,\rho_N^{n+1}\,\texttt{dvoln (N\ \ \ )}/2dt \tag{BC44}$$

$$\texttt{em1(iu,ju):}\quad (\texttt{unom(N)\ \ \ \ })\rho_N^{n+1}\,\texttt{dvoln (N\ \ \ )}/2dt \tag{BC45}$$

$$\texttt{e00(iu,ju):}\quad (\texttt{unom(N+1)\ \ })\rho_N^{n+1}\,\texttt{dvoln (N\ \ \ )}/2dt \tag{BC46}$$

From the advection term we get

$$\texttt{em1(iu,jm):}\quad m_N^{n+1}\,u_N^{n+\theta}\ \ /\ \ dt \tag{BC47}$$

$$\texttt{e00(iu,jm):}\ -\ m_{N+1}^{n+1}u_{N+1}^{n+\theta}/\ \ dt \tag{BC48}$$

$$\texttt{em1(iu,ju):}\quad \theta\texttt{dmdt(N\ \ \ ) unom(N\ \ )} \tag{BC49}$$

$$\texttt{e00(iu,ju):}\ -\ \theta\texttt{dmdt(N+1) unom(N+1)} \tag{BC50}$$

From the term in sound speed times velocity gradient we get

$$\texttt{em1(iu,jr):}\ \ \tfrac{1}{2}\theta\mu r_N^{n+1}\,\texttt{rmum1(N\ \ \ \ )}\rho_N^{n+\theta}a_N^{n+\theta}(u_{N+1}^{n+\theta}-u_N^{n+\theta}) \tag{BC51}$$

$$\texttt{e00(iu,jr):}\ \ \tfrac{1}{2}\theta\mu r_{N+1}^{n+1}\texttt{rmum1(N+1)}\rho_N^{n+\theta}a_N^{n+\theta}(u_{N+1}^{n+\theta}-u_N^{n+\theta}) \tag{BC52}$$

$$\texttt{em1(iu,ju):}-\tfrac{1}{2}\theta[\texttt{rmu(N)+rmu(N+1)}]\rho_N^{n+\theta}a_N^{n+\theta}\texttt{unom(N\ \ \ )} \tag{BC53}$$

$$\texttt{e00(iu,ju):}\ \ \tfrac{1}{2}\theta[\texttt{rmu(N)+rmu(N+1)}]\rho_N^{n+\theta}a_N^{n+\theta}\texttt{unom(N+1)} \tag{BC54}$$

$$\texttt{em1(iu,jt):}\ \ \tfrac{1}{4}\theta[\texttt{rmu(N)+rmu(N+1)}]\rho_N^{n+\theta}a_N^{n+\theta}(u_{N+1}^{n+\theta}-u_N^{n+\theta}) \times\left(\frac{p_N^{n+1}}{p_N^{n+\theta}}\right)\left(\frac{\partial lnp}{\partial lnT}\right)_N^{n+1} \tag{BC55}$$

$$\texttt{em1(iu,jd):}\ \ \tfrac{1}{4}\theta[\texttt{rmu(N)+rmu(N+1)}]\rho_N^{n+\theta}a_N^{n+\theta}(u_{N+1}^{n+\theta}-u_N^{n+\theta}) \times\left[\left(\frac{\rho_N^{n+1}}{\rho_N^{n+\theta}}\right)+\left(\frac{p_N^{n+1}}{p_N^{n+\theta}}\right)\left(\frac{\partial lnp}{\partial ln\rho}\right)_N^{n+1}\right] \tag{BC56}$$

(e) Radiating Fluid Energy

The boundary condition is the same as equation (BC33) with

$$\left(r_{N+1}^{n+\theta}\right)^{\mu}\left[\Phi_R^2 + (u_{rel})_{N+1}^{n+\theta} E_N^{n+\theta}\right] \rightarrow -\left(\overline{e + \frac{E}{\rho}}\right)_{N+1} (dm/dt)_{N+1}^{n+\theta} \quad \text{(BC57)}$$

(f) Radiation Energy

The boundary condition is the same as equation (BC34) with

$$\left(r_{N+1}^{n+\theta}\right)^{\mu}(u_{rel})_{N+1}^{n+\theta} E_N^{n+\theta} \quad \rightarrow \quad -\left(\overline{\frac{E}{\rho}}\right)_{N+1} (dm/dt)_{N+1}^{n+\theta} \quad \text{(BC58)}$$

(g) Radiation Momentum

The boundary condition is given by equations (BC35) – (BC37).

B. Lagrangean

(1) Generic BCs

Because there can be no flow across a Lagrangean boundary, set

$$\Phi_L^0 = \Phi_L^1 = \Phi_L^2 = \Phi_R^0 = \Phi_R^1 = \Phi_R^2 \equiv 0 \quad \text{(BC59)}$$

Then the main differences between the Lagrangean and Eulerian cases are:

(a) Adaptive Grid

$$\dot{r}_k \equiv u_k \Rightarrow (u_{rel})_k \equiv 0 \qquad (k = 1,\ N+1) \quad \text{(BC60)}$$

(c) Continuity

$$\dot{m}_k \equiv 0 \qquad (k = 1,\ N+1) \quad \text{(BC61)}$$

(d) Gas Momentum

We can either prescribe a driven motion of the boundary surface (i.e. as a “piston”):

$$u_k \quad \equiv \quad U_k \qquad\qquad (k = 1 \text{ or } L, \text{ and } k = N+1 \text{ or } R) \quad \text{(BC62)}$$

or we can prescribe an external pressure outside the computational domain which acts on a boundary surface:

$$(p_{external})_k \quad \equiv \quad \Pi_k \qquad\qquad (k = 1 \text{ or } L, \text{ and } k = N+1 \text{ or } R) \quad \text{(BC63)}$$

In principle we can have $U_k = f(t)$ or $\Pi_k = f(t)$, but at present the code assumes that U_k and Π_k are constant.

(2) INNER BOUNDARY ($k = 1$)

(a) Adaptive Grid

$$r_1^{n+1} = r_1^n + u_1^{n+\theta} dt \qquad \text{(BC64)}$$

(b) Mass Definition

$$m_1^{n+1} - m_1^n \equiv 0 \qquad \text{(BC65)}$$

(c) Continuity

The boundary condition is given by equation (BC15).

(d) Gas Momentum

$$u_1^{n+1} \equiv U_1(t^{n+1}) \qquad \text{(BC66)}$$

(e) Radiating Fluid Energy

The boundary condition is given by equation (BC17).

(f) Radiation Energy

The boundary condition is given by equation (BC18).

(g) Radiation Momentum

The boundary condition is given by equations (BC22) – (BC25).

(3) OUTER BOUNDARY ($k = N + 1$)

(a) Adaptive Grid

$$r_{N+1}^{n+1} = r_{N+1}^n + u_{N+1}^{n+\theta} dt \qquad \text{(BC67)}$$

(b) Mass Definition

$$m_{N+1}^{n+1} - m_{N+1}^n \equiv 0 \qquad \text{(BC68)}$$

(c) Continuity

The boundary condition is given by equation (BC31).

(d) Gas Momentum

(1) Piston

$$u_{N+1}^{n+1} \equiv U_{N+1}(t^{n+1}) \tag{BC69}$$

(2) Specified external pressure

There are two variants of this boundary condition commonly used in the literature. The first of these is by Christy (cf. *Rev. Mod. Phys.*, **36**,555, 1964), who assumes that $p \equiv p_{ext}$ at the boundary surface r_{N+1}, and that there is no material outside of r_{N+1}. In the paper cited, Christy chooses $p_{ext} = 0$. A sketch of the geometry and indexing near the outer boundary surface is shown below.

$\leftarrow$	$\Delta\xi_N$	$\rightarrow$	VOID	
	ρ_N		$\rho = 0$	
	p_N		p_{ext}	
	•		•	
r_N	$r_{N+\frac{1}{2}}$	r_{N+1}	$r_{N+\frac{3}{2}}$	r_{N+2}
m_N		m_{N+1}		m_{N+2}
u_N		u_{N+1}		u_{N+2}

The Lagrangean equation of motion is

$$\frac{Du}{Dt} = -\left(\frac{2-\mu}{2}\right) g - \frac{2\pi\mu G m}{r^{\mu}} - r^2 \frac{\partial p}{\partial m} + \frac{\phi_Q}{\rho} - \frac{\chi_F F}{c} \tag{BC70}$$

Integrating over the zone $\left(r_{N+\frac{1}{2}}, r_{N+\frac{3}{2}}\right)$,recalling that $\left(r_N, r_{N+\frac{3}{2}}\right)$ is void, and converting to adaptive grid form, we get

$$\frac{d}{dt}\left(u_{N+1}\frac{\Delta\xi_N}{2}\right) - (\overline{u}_{N+1}\dot{m}_{N+\frac{3}{2}} - \overline{u}_N\dot{m}_{N+\frac{1}{2}}) + (r_{N+1})^{\mu}(p_{ext} - p_N)$$

$$+\left[\left(\frac{2-\mu}{2}\right)g + \frac{2\pi\mu Gm}{r^{\mu}} - \frac{\chi_F F}{c}\right]\frac{\Delta\xi_N}{2} - \left(\frac{\phi_Q}{\rho}\right)_{N+1}\frac{\Delta\xi_N}{2} = 0 \quad \text{(BC71)}$$

But $\dot{m}_{N+\frac{3}{2}} \equiv 0$ (because the space outside the boundary is void), and $\dot{m}_{N+\frac{1}{2}} = \frac{1}{2}(\dot{m}_N + \dot{m}_{N+1}) = \frac{1}{2}\dot{m}_N$ because the boundary is Lagrangean. Thus

$$\frac{d}{dt}(u_{N+1}\Delta\xi_N) + \overline{u}_N\dot{m}_N + 2(r_{N+1})^{\mu}(p_{ext} - p_N)$$

$$+\left[\left(\frac{2-\mu}{2}\right)g + \frac{2\pi\mu Gm}{r^{\mu}} - \frac{\chi_F F}{c}\right]\Delta\xi_N - \left(\frac{\phi_Q}{\rho}\right)_{N+1}\Delta\xi_N = 0 \quad \text{(BC72)}$$

In finite difference form

$$\left[u_{N+1}^{n+1}\rho_N^{n+1}\Delta V_N^{n+1} - u_{N+1}^{n}\rho_N^{n}\Delta V_N^{n}\right] \ / \ dt + \ (m_N^{n+1} - m_N^n)\overline{u}_N \ / \ dt$$

$$+2(r_{N+1}^{n+\theta})^{\mu}(p_{ext} - p_N^{n+\theta}) + \left[\frac{2-\mu}{2}g + \frac{2\pi\mu Gm_{N+1}^{n+\theta}}{(r_{N+1}^{n+\theta})^{\mu}} - \frac{\chi_N^{n+\theta}F_{N+1}^{n+\theta}}{c}\right]\rho_N^{n+\theta}\Delta V_N^{n+\theta}$$

$$+\frac{4\rho_N^{n+\theta}(\mu_Q)_N^{n+\theta}}{3(r_{N+1}^{n+\theta})^{\mu/2}}\left[\frac{1}{2}\left(r_N^{n+\theta} + r_{N+1}^{n+\theta}\right)\right]^{3\mu/2}\left[\frac{u_{N+1}^{n+\theta} - u_N^{n+\theta}}{r_{N+1}^{n+\theta} - r_N^{n+\theta}} - \frac{\mu}{4}\left(\frac{u_N^{n+\theta}}{r_N^{n+\theta}} + \frac{u_{N+1}^{n+\theta}}{r_{N+1}^{n+\theta}}\right)\right]$$

$$= 0 \quad \text{(BC73)}$$

To avoid the appearence of unknown quantities in ϕ_Q we assume the velocity divergence is zero outside the computational domain.

Castor (thesis, 1966), Spangenberg (thesis, 1975), and Stellingwerf (thesis, 1975) extended Christy's approach by taking into account the mass of the outer envelope and atmosphere of the star. They assume that the masses of exterior zones are given by $\Delta\xi_{k+1} = \Delta\xi_k/\omega$, $k \geq N$, $\omega \geq 1$. Then the total mass to be associated with r_{N+1} is $\frac{1}{2}(\frac{\omega+1}{\omega-1})\Delta\xi_N$. Because the grid is adaptive, $\Delta\xi_N$ can change, but the mass outside the boundary remains fixed (Lagrangean boundary). Therefore write $\Delta m = \frac{1}{2}\Delta\xi_N + \Delta\xi_0/(\omega-1)$ where $\Delta\xi_0 = (\Delta\xi_N)_{initial}$ = constant for the mass associated with r_{N+1}. Then

$$\Delta m = \left[1 + \frac{2}{\omega - 1}\left(\frac{\Delta\xi_0}{\Delta\xi_N}\right)\right]\frac{\Delta\xi_N}{2} \tag{BC74}$$

For simplicity we shall set $(\Delta\xi_0/\Delta\xi_N) \equiv 1$. Then we can write $\Delta m = \Delta\xi_N/2\mathcal{R}$, where

$$\mathcal{R} \equiv (\tfrac{\omega-1}{\omega+1}) \tag{BC75}$$

Thus the equation of motion (BC72) becomes

$$\frac{d}{dt}(u_{N+1}\Delta\xi_N) + \mathcal{R}\left[\overline{u}_N\dot{m}_N + 2(r_{N+1})^{\mu}(p_{ext} - p_N)\right]$$

$$+ \left[\left(\frac{2-\mu}{2}\right)g + \frac{2\pi\mu G m}{r^{\mu}} - \frac{\chi_F F}{c}\right]\Delta\xi_N - \left(\frac{\phi_Q}{\rho}\right)_{N+1}\Delta\xi_N = 0 \tag{BC76}$$

and similarly for (BC73). Christy's formulae are recovered as $\omega \rightarrow \infty$ and $\mathcal{R} \rightarrow 1$.

Linearization of equations (BC73) and (BC76) yields

(α) Time Derivative

$$\texttt{em1(iu,jr):}\quad -u^{n+1}_{N+1}\rho^{n+1}_N\texttt{rmup1n(N)} / dt \tag{BC77}$$

$$\texttt{e00(iu,jr):}\quad u^{n+1}_{N+1}\rho^{n+1}_N\texttt{rmup1n(N+1)} / dt \tag{BC78}$$

$$\texttt{em1(iu,jd):}\quad u^{n+1}_{N+1}\rho^{n+1}_N\texttt{dvoln (N)} / dt \tag{BC79}$$

$$\texttt{em1(iu,ju):}\; \texttt{unom(N+1)}\rho^{n+1}_N\texttt{dvoln (N)} / dt \tag{BC80}$$

(β) Advection

$$\texttt{em1(iu,jm)}: \mathcal{R}\, m_N^{n+1} \overline{u}_N \,/\, dt \tag{BC81}$$
$$\texttt{em2(iu,ju)}: \mathcal{R}\, \texttt{dmdt(N) dqbdqm1(N) unom(N-1)} \tag{BC82}$$
$$\texttt{em1(iu,ju)}: \mathcal{R}\, \texttt{dmdt(N) dqbdq00(N) unom(N)} \tag{BC83}$$
$$\texttt{e00(iu,ju)}: \mathcal{R}\, \texttt{dmdt(N) dqbdqp1(N) unom(N+1)} \tag{BC84}$$
$$\texttt{ep1(iu,ju)}: \mathcal{R}\, \texttt{dmdt(N) dqbdqp2(N) unom(N+2)} \tag{BC85}$$

(γ) Pressure Gradient

$$\texttt{e00(iu,ju)}: \quad 2\mathcal{R}\, \theta \texttt{rmu(N+1)} \mu (r_{N+1}^{n+1}/r_{N+1}^{n+\theta})(p_{ext} - p_N^{n+\theta}) \tag{BC86}$$
$$\texttt{em1(iu,jd)}: -2\mathcal{R}\, \theta \texttt{rmu(N+1)} p_N^{n+1} (\partial p/\partial \rho)_N^{n+1} \tag{BC87}$$
$$\texttt{em1(iu,jt)}: -2\mathcal{R}\, \theta \texttt{rmu(N+1)} p_N^{n+1} (\partial p/\partial T)_N^{n+1} \tag{BC88}$$

(δ) Gravity

$$\texttt{em1(iu,jr)}: -\theta \left[(2-\mu)\frac{g}{2} + \frac{2\pi\mu G m_{N+1}^{n+\theta}}{\texttt{rmu(N + 1)}} \right] \texttt{rmu(N)}\, \rho_N^{n+1} r_N^{n+1} \tag{BC89}$$

$$\begin{aligned} \texttt{e00(iu,jr)}: \quad & \theta \left[(2-\mu)\frac{g}{2} + \frac{2\pi\mu G m_{N+1}^{n+\theta}}{\texttt{rmu(N + 1)}} \right] \texttt{rmu(N + 1)}\, \rho_N^{n+1} r_{N+1}^{n+1} \\ & -\theta \frac{2\pi\mu^2 G m_{N+1}^{n+\theta}}{\texttt{rmup1(N + 1)}} \texttt{dvol(N)}\, \rho_N^{n+\theta} r_{N+1}^{n+1} \end{aligned} \tag{BC90}$$

$$\texttt{e00(iu,jm)}: \quad 2\pi\mu\theta G m_{N+1}^{n+1} \rho_N^{n+\theta} \texttt{dvol(N)} \,/\, \texttt{rmu(N + 1)} \tag{BC91}$$

$$\texttt{em1(iu,jd)}: \theta \left[(2-\mu)\frac{g}{2} + \frac{2\pi\mu G m_{N+1}^{n+\theta}}{\texttt{rmu(N + 1)}} \right] \rho_N^{n+1} \texttt{dvol(N)} \tag{BC92}$$

(ϵ) Radiation Force

$$\texttt{em1(iu,jr)}: \quad \tfrac{\theta}{c} \chi_N^{n+\theta} F_{N+1}^{n+\theta} \rho_N^{n+\theta} \texttt{rmu(N)} r_N^{n+1} \tag{BC93}$$
$$\texttt{e00(iu,jr)}: -\tfrac{\theta}{c} \chi_N^{n+\theta} F_{N+1}^{n+\theta} \rho_N^{n+\theta} \texttt{rmu(N+1)} r_{N+1}^{n+1} \tag{BC94}$$
$$\texttt{em1(iu,jd)}: -\tfrac{\theta}{c} \chi_N^{n+\theta} F_{N+1}^{n+\theta} \rho_N^{n+1} \texttt{dvol(N)} \left[1 + \tfrac{\rho_N^{n+\theta}}{\rho_N^{n+1}} \tfrac{\chi_N^{n+1}}{\chi_N^{n+\theta}} \left(\tfrac{\partial ln\chi}{\partial \ln T} \right)_N^{n+1} \right] \tag{BC95}$$
$$\texttt{em1(iu,jt)}: -\tfrac{\theta}{c} \chi_N^{n+\theta} F_{N+1}^{n+\theta} \rho_N^{n+\theta} \texttt{dvol(N)} \left(\tfrac{\partial ln\chi}{\partial \ln T} \right)_N^{n+1} \tag{BC96}$$
$$\texttt{em1(iu,jf)}: -\tfrac{\theta}{c} \chi_N^{n+\theta} \texttt{frnom(N+1)} \rho_N^{n+\theta} \texttt{dvol(N)} \tag{BC97}$$

(ζ) Viscous Force

$$\texttt{em1(iu,jr)}: \texttt{-r1(N+1)[dr3dlr00(N)qf(N)+r3(N)dqfdlr00(N)]} \quad \text{(BC98)}$$

$$\texttt{e00(iu,jr)}: \texttt{-r1(N+1)[dr3dlrp1(N)qf(N)+r3(N)dqfdlrp1(N)]}$$
$$\texttt{-dr1dlr00(N+1)r3(N)qf(N)} \quad \text{(BC99)}$$

$$\texttt{em1(iu,ju)}: \texttt{-r1(N+1)r3(N)dqfdlu00(N)} \quad \text{(BC100)}$$

$$\texttt{e00(iu,ju)}: \texttt{-r1(N+1)r3(N)dqfdlup1(N)} \quad \text{(BC101)}$$

$$\texttt{em1(iu,jd)}: \texttt{-r1(N+1)r3(N)dqfdld00(N)} \quad \text{(BC102)}$$

$$\texttt{em1(iu,jt)}: \texttt{-r1(N+1)r3(N)dqfdlt00(N)} \quad \text{(BC103)}$$

(η) Right Hand Side

$$- \ \texttt{rhs(iu)} =$$
$$[u_{N+1}^{n+1}\,\rho_N^{n+1}\,\texttt{dvoln(N)} - u_{N+1}^{n}\,\rho_N^{n}\,\texttt{dvolo(N)}]\ /\ dt$$
$$+\ \mathcal{R}\,\texttt{dmdt(N)}\,\overline{u}_N + 2\,\mathcal{R}\,\texttt{rmu(N+1)}\,(p_{ext} - p_N^{n+\theta})$$
$$+\left[\left(\frac{2-\mu}{2}\right)g + \frac{2\pi\mu G m_{N+1}^{n+\theta}}{(r_{N+1}^{n+\theta})^{\mu}} - \frac{\chi_N^{n+\theta}F_{N+1}^{n+\theta}}{c}\right]\rho_N^{n+\theta}\texttt{dvol(N)}$$
$$-\ \texttt{r1(N+1)}\,\texttt{r3(N)}\,\texttt{qf(N)} \quad \text{(BC104)}$$

(3) Transmitting Boundary

The boundary condition is given by equation (BC41).

(e) Radiating Fluid Energy

The boundary condition is given by equation (BC33).

(f) Radiation Energy

The boundary condition is given by equation (BC34).

(g) Radiation Momentum

The boundary condition is given by equation (BC35) or (BC36).

XIII. PHANTOM ZONES

Because the advection and diffusion algorithms have a five-point stencil, we need to specify the physical variables in "phantom zones" outside of the computational domain. We briefly examine these algorithms at the boundaries to determine how many points are needed outside the domain.

1. Advection of zone-centered quantities

The advected zone-centered quantities of interest are $q = \rho$, E/ρ, and $(e + E/\rho)$. Inside the computational domain $(k = 2, \dots, N-1)$ we need $\overline{q}_2, \dots, \overline{q}_{N-1}$, which implies we need q_0 and q_{N+1}. At the inner boundary $(k = 1)$ we need only $\overline{q}_2$, hence no new information about q. At the outer boundary $(k = N)$ we need $\overline{q}_N$ or (transmitting boundary) $\overline{q}_{N+1}$. Thus we need to run the advection algorithm from $k = 2$ to $k = N + 1$, which means we need to set r_0 and r_{N+2}, and q_0 and q_{N+2}.

2. Advection of interface-centered quantities

The advected interface-centered quantities of interest are $q = u$ and F/ρ. Inside the computational domain we need $\overline{q}_1, \dots, \overline{q}_N$, which implies we need q_0 and q_{N+1}. At the inner boundary $(k = 1)$ and the outer boundary $(k = N + 1)$ q is set by boundary conditions, and no additional information is required. Thus we need run the advection algorithm from $k = 1$ to $k = N$, which means we need to set r_0 and r_{N+2}, and q_0 and q_{N+2}.

3. Zone-centered viscous energy dissipation: ϵ_Q

Inside the computational domain $(k = 2, \dots, N-1)$ we need $r_2, \dots, r_N$, and $u_2, \dots, u_N$. At the inner boundary $(k = 1)$ we need r_1 and u_1, and at the outer boundary $(k = N - 1)$ we need r_{N+1} and u_{N+1}. Thus we need to run the dissipation algorithm from $k = 1, \dots, N$, and compute ϵ_Q for $k = 1, \dots, N$ from quantities known on the grid; no phantom zones are required.

4. Interface-centered viscous momentum deposition: ϕ_Q

ϕ_Q is needed only inside the computational domain $(k = 2, \dots, N)$. Thus we need only $r_1, \dots, r_{N+1}$; $u_1, \dots, u_{N+1}$; $qh_1, \dots, qh_{N+1}$; $qm_1, \dots, qm_{N+1}$; and $qu_1, \dots, qu_{N+1}$. We run the algorithm from $k = 2$ to $k = N$.

5. Diffusion

We allow for artificial diffusion of $q = \rho$ and e. Inside the computational domain ($k = 2, \ldots , N-1$) we need $r_1, \ldots , r_{N+1}$, and $q_1, \ldots , q_N$. At the inner boundary ($k = 1$) we need r_0 and q_0; at the outer boundary ($k = N$) we need r_{N+2} and q_{N+1}. Thus we need to run the diffusion algorithm from $k = 1$ to $k = N$, and set r_0, r_{N+2}, and q_0, q_{N+1}.

A. Inner Boundary

(1) Eulerian with zero flux: ($\Phi_L^{0,1,2} \equiv 0$)

Here we must demand symmetry with respect to the boundary ($k = 1$); then

$$r_0 = r_1 - (r_2 - r_1) \quad \text{(PZ1)} \qquad \delta r_0 \equiv 0 \quad \text{(PZ2)}$$
$$m_0 = 2m_1 - m_2 \quad \text{(PZ3)} \qquad \delta m_0 = -\delta m_2 \quad \text{(PZ4)}$$
$$\rho_0 = \rho_1 \quad \text{(PZ5)} \qquad \delta\rho_0 = \delta\rho_1 \quad \text{(PZ6)}$$
$$u_0 = -u_2 \quad \text{(PZ7)} \qquad \delta u_0 = -\delta u_2 \quad \text{(PZ8)}$$
$$e_0 = e_1 \quad \text{(PZ9)} \qquad \delta e_0 = \delta e_1 \quad \text{(PZ10)}$$

(2) Eulerian with nonzero flux: ($\Phi_L^{0,1,2} > 0$)

We choose an arbitrary Δr_L, so that

$$r_0 = r_1 - \Delta r_L \quad \text{(PZ11)} \qquad \delta r_0 \equiv 0 \quad \text{(PZ12)}$$
$$m_0 = m_1 - \rho_L(r_1^{\mu+1} - r_0^{\mu+1})/(\mu+1) \quad \text{(PZ13)} \qquad \delta m_0 = \delta m_1 \quad \text{(PZ14)}$$
$$\rho_0 = \rho_L \quad \text{(PZ15)} \qquad \delta\rho_0 = 0 \quad \text{(PZ16)}$$
$$u_0 = u_L \quad \text{(PZ17)} \qquad \delta u_0 = 0 \quad \text{(PZ18)}$$
$$e_0 = e_L \quad \text{(PZ19)} \qquad \delta e_0 = 0 \quad \text{(PZ20)}$$

(3) Lagrangean

In cell zero demand $\Delta m_L =$ constant, and that all thermodynamic properties are the same as in cell one. Then

$$r_0^{\mu+1} = r_1^{\mu+1} - (\mu+1)\Delta m_L/\rho_0 \quad \text{(PZ21)} \qquad \delta r_0 = (\tfrac{r_1}{r_0})^\mu \delta r_1 + \tfrac{\Delta m_L}{r_0^\mu \rho_0^2}\delta\rho_0 \quad \text{(PZ22)}$$
$$m_0 = m_1 - \Delta m_L \quad \text{(PZ23)} \qquad \delta m_0 = 0 \quad \text{(PZ24)}$$
$$\rho_0 = \rho_1 \quad \text{(PZ25)} \qquad \delta\rho_0 = \delta\rho_1 \quad \text{(PZ26)}$$
$$u_0 = u_1 \quad \text{(PZ27)} \qquad \delta u_0 = \delta u_1 \quad \text{(PZ28)}$$
$$e_0 = e_1 \quad \text{(PZ29)} \qquad \delta e_0 = \delta e_1 \quad \text{(PZ30)}$$

(4) Radiation

$$E_0 = E_1 \quad \text{(PZ31)} \qquad \delta E_0 = \delta E_1 \quad \text{(PZ32)}$$

(a) Optically transmitting:

$$F_0 = F_1 \quad \text{(PZ33)} \qquad \delta F_0 = \delta F_1 \quad \text{(PZ34)}$$

(b) Optically reflecting:

$$F_0 = -F_2 \quad \text{(PZ35)} \qquad \delta F_0 = -\delta F_2 \quad \text{(PZ36)}$$

(c) Imposed flux:

$$F_0 = (\tfrac{r_1}{r_0})^\mu F_1 \quad \text{(PZ37)} \qquad \delta F_0 = (\tfrac{r_1}{r_0})^\mu \delta F_1 + \mu(\tfrac{\delta r_1}{r_1} - \tfrac{\delta r_0}{r_0})(\tfrac{r_1}{r_0})^\mu F_1 \quad \text{(PZ38)}$$

B. Outer Boundary

(1) Eulerian with zero flux: ($\Phi_R^{0,1,2} \equiv 0$)

Again, demand symmetry with respect to the boundary ($k = N + 1$); then

$$\begin{aligned}
r_{N+2} &= r_{N+1} + (r_{N+1} - r_N) && \text{(PZ39)} & \delta r_{N+2} &\equiv 0 && \text{(PZ40)}\\
m_{N+2} &= 2m_{N+1} - m_N && \text{(PZ41)} & \delta m_{N+2} &= -\delta m_N && \text{(PZ42)}\\
\rho_{N+1} &= \rho_N && \text{(PZ43)} & \delta \rho_{N+1} &= \delta \rho_N && \text{(PZ44)}\\
u_{N+2} &= -u_N && \text{(PZ45)} & \delta u_{N+2} &= -\delta u_N && \text{(PZ46)}\\
e_{N+1} &= e_N && \text{(PZ47)} & \delta e_{N+1} &= \delta e_N && \text{(PZ48)}
\end{aligned}$$

(2) Eulerian with nonzero flux: ($\Phi_R^{0,1,2} > 0$)

We choose an arbitrary Δr_R, so that

$$\begin{aligned}
r_{N+2} &= r_{N+1} + \Delta r_R && \text{(PZ49)} & \delta r_{N+2} &\equiv 0 && \text{(PZ50)}\\
m_{N+2} &= m_{N+1} + \frac{\rho_R(r_{N+2}^{\mu+1} - r_{N+1}^{\mu+1})}{(\mu+1)} && \text{(PZ51)} & \delta m_{N+2} &= \delta m_{N+1} && \text{(PZ52)}\\
\rho_{N+1} &= \rho_R && \text{(PZ53)} & \delta \rho_{N+1} &= 0 && \text{(PZ54)}\\
u_{N+2} &= u_R && \text{(PZ55)} & \delta u_{N+2} &= 0 && \text{(PZ56)}\\
e_{N+1} &= e_R && \text{(PZ57)} & \delta e_{N+1} &= 0 && \text{(PZ58)}
\end{aligned}$$

(3) Transmitting Eulerian

$$r_{N+2} = r_{N+1} + \Delta r_R \quad \text{(PZ59)} \qquad \delta r_{N+2} \equiv 0 \quad \text{(PZ60)}$$

$$m_{N+2} = m_{N+1} + \rho_{N+1} \Delta V_{N+1} \quad \text{(PZ61)}$$

$$\delta m_{N+2} = \delta \rho_{N+1} \Delta V_{N+1} + \delta m_{N+1} \quad \text{(PZ62)}$$

$$\rho_{N+1} = \rho_N \qquad \text{(PZ63)} \qquad \delta\rho_{N+1} = \delta\rho_N \qquad \text{(PZ64)}$$
$$u_{N+2} = u_{N+1} \qquad \text{(PZ65)} \qquad \delta u_{N+2} = \delta u_{N+1} \qquad \text{(PZ66)}$$
$$e_{N+1} = e_N \qquad \text{(PZ67)} \qquad \delta e_{N+1} = \delta e_N \qquad \text{(PZ68)}$$

(4) Lagrangean or Transmitting Lagrangean

In cell $N+1$ demand $\Delta m_R =$ constant, and that all thermodynamic properties are the same as in cell N. Then

$$r_{N+2}^{\mu+1} = r_{N+1}^{\mu+1} - \frac{(\mu+1)\Delta m_R}{\rho_{N+1}} \qquad \text{(PZ69)}$$
$$\delta r_{N+2} = \left(\frac{r_{N+1}}{r_{N+2}}\right)^{\mu} \delta r_{N+1} - \frac{\Delta m_R}{r_{N+2}^{\mu}\rho_{N+1}^2}\delta\rho_{N+1} \qquad \text{(PZ70)}$$
$$m_{N+2} = m_{N+1} - \Delta m_R \qquad \text{(PZ71)} \qquad \delta m_{N+2} = 0 \qquad \text{(PZ72)}$$
$$\rho_{N+1} = \rho_N \qquad \text{(PZ73)} \qquad \delta\rho_{N+1} = \delta\rho_N \qquad \text{(PZ74)}$$
$$u_{N+2} = u_{N+1} \qquad \text{(PZ75)} \qquad \delta u_{N+2} = \delta u_{N+1} \qquad \text{(PZ76)}$$
$$e_{N+1} = e_N \qquad \text{(PZ77)} \qquad \delta e_{N+1} = \delta e_N \qquad \text{(PZ78)}$$

(5) Radiation

(a) Optically transmitting:

$$E_{N+1} = E_N \qquad \text{(PZ79)} \qquad \delta E_{N+1} = \delta E_N \qquad \text{(PZ80)}$$
$$F_{N+2} = F_{N+1} \qquad \text{(PZ81)} \qquad \delta F_{N+2} = \delta F_{N+1} \qquad \text{(PZ82)}$$

(b) Optically reflecting:

$$E_{N+1} = E_N \qquad \text{(PZ83)} \qquad \delta E_{N+1} = \delta E_N \qquad \text{(PZ84)}$$
$$F_{N+2} = -F_N \qquad \text{(PZ85)} \qquad \delta F_{N+2} = -\delta F_N \qquad \text{(PZ86)}$$

(c) Imposed flux:

$$E_{N+1} = E_N \qquad \text{(PZ87)} \qquad \delta E_{N+1} = \delta E_N \qquad \text{(PZ88)}$$
$$F_{N+2} = \left(\frac{r_{N+1}}{r_{N+2}}\right)^{\mu} F_{N+1} \qquad \text{(PZ89)}$$
$$\delta F_{N+2} = \left(\frac{r_{N+1}}{r_{N+2}}\right)^{\mu}\delta F_{N+1} + \mu\left(\frac{\delta r_{N+1}}{r_{N+1}} - \frac{\delta r_{N+2}}{r_{N+2}}\right)\left(\frac{r_{N+1}}{r_{N+2}}\right)^{\mu} F_{N+1} \qquad \text{(PZ90)}$$

XIV. LOGICAL SWITCHES

A. Geometry

`lgeom = 0`	Planar geometry $\Rightarrow \mu = 0$	
`lgeom = 1`	Cylindrical geometry $\Rightarrow \mu = 1$, hydro only	
`lgeom = 2`	Spherical geometry $\Rightarrow \mu = 2$	

B. Gravity

`lgrav = 0` No gravity. Set $G = 0$ for spherical geometry, and $g = 0$ for planar geometry.

`lgrav = 1` Gravity. Set $4\pi G$ for spherical geometry, and g for planar geometry.

C. Adaptive Grid

`lgrid = 1`	Adaptive grid
`lgrid = 2`	Eulerian grid
`lgrid = 3`	Lagrangean grid

If the grid is adaptive, then
`ladx = 1, 2, 3`
for linear, logarithmic, and harmonic resolution, respectively.
Likewise, for structure function number `l, (l = 1, ...,mad)`,
`lady(l) = 1, 2, 3`
for linear, logarithmic, and harmonic resolution respectively.

D. EOS

`leos = 1`	Tables
`leos = 2`	Gamma law
`leos = 3`	Stellingwerf formula

E. Opacity

`lopac = 1`	Tables
`lopac = 2`	Stellingwerf fit
`lopac = 3`	Constant opacity
`lopac = 4`	Thompson electron scattering opacity

F. Radiation

`lrad = 0`	No radiation
`lrad = 1`	Radiation

G. Transfer

`ltran = 1`	Full transport equation
`ltran = 2`	Nonequilibrium diffusion
`ltran = 3`	Equilibrium diffusion
`lam = 0`	No flux limiting in diffusion equation
`lam = 1`	Flux limiting in diffusion equation

H. Hydrodynamics

`lhydr = 0`	No hydro
`lhydr = 1`	Hydro

I. Boundary Conditions

`leibc = 1`	Eulerian (inner|left) BC with zero hydro flux
`leibc = 2`	Eulerian (inner|left) BC with nonzero hydro flux
`leobc = 1`	Eulerian (outer|right) BC with zero hydro flux
`leobc = 2`	Eulerian (outer|right) BC with nonzero hydro flux
`leobc = 3`	Eulerian transmitting (outer|right) boundary
`llibc = 1`	Lagrangean (inner|left) BC, driven piston

`llobc = 1`	Lagrangean (outer|right) BC, driven piston
`llobc = 2`	Lagrangean (outer|right) BC, specified pressure
`llobc = 3`	Lagrangean (outer|right) BC, transmitting boundary
`lribc = 1`	Optically transmitting (inner|left) boundary
`lribc = 2`	Optically reflecting left boundary (planar) or Milne inner BC for hollow core (spherical)
`lribc = 3`	Imposed net flux at (inner|left boundary
`lrobc = 1`	Optically transmitting (outer|right) boundary
`lrobc = 2`	Optically reflecting right boundary (planar only)

Note: in planar geometry, if only one optically reflecting boundary is set, it *must* be at the right boundary.

J. Control

`linit = 1`	Initial model
`lboos = 1`	Perform boost iteration; not set by user

K. Numerics

`lcray = 0`	Computer is non-CRAY
`lcray = 1`	Use CAL coded routines for CRAY
`lband = 0`	Solve block pentadiagonal system
`lband = 1`	Solve banded system

XV. INPUT PARAMETERS

A. Hydrodynamics

Time-centering parameter: θ

Pseudoviscous length scale: $\ell = \ell_0 + \ell_1 r$

Pseudoviscosity coefficients: C_1, C_2

Floor on pseudoviscous pressure ratio: q_0

Order of advection: C_{adv}

Overflow protection in advection switch: ϵ_{adv}

Artificial mass-diffusion coefficient: σ_ρ

Artificial energy-diffusion coefficient: σ_e

B. Constitutive Relations

Ideal gas polytropic exponent: γ

Ideal gas mean molecular weight: μ_m

Constant total opacity $[cm^{-1}]$: $(\rho\chi)_0$

Ratio of Planck to flux mean opacity: ζ

Chemical abundances: X, Y, Z

C. Adaptive Grid

Adaptive grid spatial scale: R_{scale}

Adaptive grid ordinate scale: $(F_{scale})_l, l = 1, \ldots,$ `mad`

Adaptive grid spatial diffusion coefficient: α

Adaptive grid time constant: τ

Adaptive grid time delay exponent: β

Adaptive grid weights: $W_m, W_\rho, W_T, W_E, W_p, W_e, W_\kappa, W_q$

D. Boundary Conditions

(1) Eulerian

Mass flux at (left|right) boundary: $(\Phi_L^0 \mid \Phi_R^0)$

Momentum flux at (left|right) boundary: $(\Phi_L^1 \mid \Phi_R^1)$

Energy flux at (left|right) boundary: $(\Phi_L^2 \mid \Phi_R^2)$

Displacement of phantom zone boundary from (left|right) flow boundary:
$(\Delta r_L \mid \Delta r_R)$

Note: The relevant hydrodynamic fluxes *must* be zero for:
`leibc = 1, leobc = 1 or 3, llibc = 1, and llobc > 0.`

(2) Lagrangean

Mass of phantom zone at (left|right) boundary: $(\Delta m_L \mid \Delta m_R)$

Driven piston velocity at (left|right) boundary: $(U_L(t) \mid U_R(t))$

Imposed external pressure at outer boundary: $\Pi_R(t)$

Ratio of mass zones in exterior atmosphere: ω

Note: currently the code assumes that U_L, U_R, and Π_R, if specified, are constant in time.

(3) Radiation Field and Surface Gravity

(Outgoing|incoming) specific intensity incident on (inner|outer) boundary: $(I_L^+(t) \mid I_R^-(t))$

Luminosity incident on inner boundary (spherical geometry): $L_1(t)$

Effective temperature of radiation field incident on lower boundary (planar geometry): $T_{eff}(t)$

Surface gravity in planar geometry: g

Note: currently the code assumes that I_L^+, I_R^-, L_1, and T_{eff}, if specified, are constant in time.

E. Array Dimensions

`neqn` = actual number of equations used at each grid point
`ngrs` = starting index for fluid cells
`ngre` = ending index for fluid cells

`meqn` = maximum number of equations allowed at each grid point
`mgr` = maximum number of grid points (including phantom zones)
`mad` = maximum number of structure functions in adaptive grid equation
`madv` = maximum number of advection scratch vectors
`mdfz` = maximum number of diffusion scratch vectors

Note: `ngrs` *must* be ≥ 4, and `ngre` *must* be $\leq$ `mgr` $- 4$

F. Iteration Control

`dtol` = maximum fractional change allowed in physical variables
`ctol` = maximum fractional change allowed in cell size
`conv` = convergence criterion for Newton-Raphson iteration
`niter` = maximum number of Newton-Raphson iterations allowed
`ntry` = maximum number of tries for convergence using reduced timesteps

G. Integration Control

`jdump`	=	dump number of current dump to dump file
`jsteps`	=	dump number of first model in current run
`jstepe`	=	dump number of last model in current run
`ndump`	=	number of time steps between dumps of models
`stol`	=	maximum fractional change allowed between timesteps
`nback`	=	maximum number of reintegrations with reduced timestep if the maximum error in an integration step is ≥ 2 `stol`
`next`	=	maximum number of tries to preserve a monotonic mesh in extrapolation to the new time level

Note: the record number of dump number `jdump` in the dumpfile is

`irec` = `jdump + 2`.

XVI. SOLUTION

A. Scaling of Variables

In the Newton-Raphson iteration procedure, take as unknowns the dimensionless fractional changes $\delta r/r, \delta m/m, \delta \rho/\rho, \delta T/T$, and $\delta E/E$. For u and F, which can pass through zero, use $\delta u/u_{nom}$, and $\delta F/F_{nom}$. Both u_{nom} and F_{nom} are constant under linearization, even though they may be a function of depth; they are merely scale factors. Examples of physically reasonable choices for these quantities are $u_{nom} = a_{sound}$, and $F_{nom} = L/4\pi r^2$ or σT_{eff}^4 for spherical and planar geometry respectively. In all test problems (as described in the TITAN Code User's Guide) we use $u_{nom} = |u|$ and $F_{nom} = |F|$ instead. With this scaling, all coefficients in a given equation will have the correct relative numerical size, that is, no arbitrary factors will be introduced because the different physical variables are measured in different units.

If we use systems solvers with pivoting, it is prudent to scale each equation by a numerical factor designed to eliminate arbitrary scale factors (units) between rows. For example, choose

$$(\text{scale factor})_{\mathrm{i}} = 1.0 \; / \; \max_{\mathrm{j}} \; [\text{abs} \; (\mathrm{a_{ij}}) \;] \tag{S1}$$

This scaling procedure assures that the largest element in each row is unity. Other procedures may also be used. If the system is solved with a nonpivoting solver (our usual choice), row scaling is irrelevant.

B. Block Pentadiagonal System

In general, the system to be solved is of the form

$$E_{-2k}x_{k-2} + E_{-1k}x_{k-1} + E_{0k}x_k + E_{1k}x_{k+1} + E_{2k}x_{k+2} = e_k \tag{S2}$$

For $k = 1$,

$$E_{01}x_1 + E_{11}x_2 + E_{21}x_3 = e_1 \tag{S3}$$

Therefore

$$x_1 = a_1x_2 + b_1x_3 + c_1 \tag{S4}$$

where

$$a_1 = -E_{01}^{-1}E_{11}, \quad b_1 = -E_{01}^{-1}E_{21}, \quad c_1 = -E_{01}^{-1}e_1 \tag{S5}$$

For $k = 2$,

$$E_{-12}x_1 + E_{02}x_2 + E_{12}x_3 + E_{22}x_4 = e_2 \tag{S6}$$

Substituting from equation (S4), we have

$$(E_{02} + E_{-12}a_1)x_2 + (E_{12} + E_{-12}b_1)x_3 + E_{22}x_4 = e_2 - E_{-12}c_1 \tag{S7}$$

Therefore

$$x_2 = a_2x_3 + b_2x_4 + c_2 \tag{S8}$$

where

$$a_2 = -(E_{02} + E_{-12}a_1)^{-1}(E_{12} + E_{-12}b_1) \tag{S9}$$
$$b_2 = -(E_{02} + E_{-12}a_1)^{-1}E_{22} \tag{S10}$$
$$c_2 = \quad (E_{02} + E_{-12}a_1)^{-1}(e_2 - E_{-12}c_1) \tag{S11}$$

For $k = 3$,

$$E_{-23}x_1 + E_{13}x_2 + E_{03}x_3 + E_{13}x_4 + E_{23}x_5 = e_3 \tag{S12}$$

Substituting from equation (S4), we have

$$(E_{-13} + E_{-23}a_1)x_2 + (E_{03} + E_{-23}b_1)x_3 + E_{13}x_4 + E_{23}x_5 = e_3 - E_{-23}c_1 \tag{S13}$$

Then substituting from equation (S8), we have

$$[(E_{-13} + E_{-23}a_1)a_2 + E_{03} + E_{-23}b_1]x_3 + [(E_{-13} + E_{-23}a_1)b_2 + E_{13}]x_4 + E_{23}x_5 = e_3 - E_{-23}c_1 - (E_{-13} + E_{-23}a_1)c_2 \tag{S14}$$

Therefore

$$x_3 = a_3x_4 + b_3x_5 + c_3 \tag{S15}$$

where

$$a_3 = -[E_{03} + E_{-13}a_2 + E_{-23}(a_1a_2 + b_1)]^{-1}(E_{13} + E_{-13}b_2 + E_{-23}a_1b_2) \tag{S16}$$
$$b_3 = -[E_{03} + E_{-13}a_2 + E_{-23}(a_1a_2 + b_1)]^{-1}E_{23} \tag{S17}$$
$$c_3 = \quad [E_{03} + E_{-13}a_2 + E_{-23}(a_1a_2 + b_1)]^{-1}[e_3 - E_{-23}(c_1 + a_1c_2) - E_{-13}c_2] \tag{S18}$$

The general recursion rule can now be written by inspection:

$$x_k = a_kx_{k+1} + b_kx_{k+2} + c_k \tag{S19}$$

where

$$a_k \equiv -d_k^{-1}(E_{1k} + E_{-1k}b_{k-1} + E_{-2k}a_{k-2}b_{k-1}) \tag{S20}$$
$$b_k \equiv -d_k^{-1}E_{2k} \tag{S21}$$
$$c_k \equiv \quad d_k^{-1}[e_k - E_{-2k}(c_{k-2} + a_{k-2}c_{k-1}) - E_{-1k}c_{k-1}] \tag{S22}$$

and

$$d_k \equiv [E_{-2k}(b_{k-2} + a_{k-2}a_{k-1}) - E_{-1k}a_{k-1} + E_{0k}] \quad \text{(S23)}$$

One can generate a_k, b_k, c_k, and d_k with one scratch matrix and one scratch vector. To solve the system we use the standard standard BLAS and LINPACK routines `sgemv`, `sgemm`, etc. We solve the system with nonpivoting versions of `sgefa` and `sgesl`. On CRAY machines we use special hand-coded CAL versions of these routines (CALMATH).

C. Band System

An alternative way to view the system is as a band matrix whose band-with is determined by the number of equations in each pentadiagonal block matrix. Define:

I	= dimension of (square) block matrices
i	= row index in block matrix
j	= column index in block matrix
k	= depth index of block matrix in grand system
K	= total number of depth levels
l	= position index (-2, -1, 0, 1, 2) of matrix at a given level
M	= number of principal diagonal in band system (M = 3I)
m'	= diagonal number in band system
n	= column number in band system

The relationship between indices of an element of the pentadiagonal system and its indices in the band system is:

$$m' = j - i + lI + M \qquad 1 \le m' \le 2M - 1 \quad \text{(S24)}$$

and

$$n = j + (k + l - 1)I \qquad 1 \le n \le KI \quad \text{(S25)}$$

For simplicity we always generate the system in pentadiagonal form. Then if the system is to be solved as a band matrix we map all elements into band format. To create the band system in correct form for LINPACK `sgefa` and `sgesl` we have the mapping $(i, j, k, l) \to (m, n)$ where

$$m = i - j - lI + 2M - 1 \tag{S26}$$

and

$$n = j + (k + l - 1)I \tag{S27}$$

To create the band system in correct form for T. Jordan's CAL coded `bglsdc` we have the mapping $(i, j, k, l) \rightarrow (m, n)$ where

$$m = i - j - lI + M \tag{S28}$$

and

$$n = j + (k + l - 1)I \tag{S29}$$

D. Timing

The timings, in seconds, given below are for the solution of the system for 300 depth points and seven equations on a CRAY X-MP.

Pentadiagonal	
SCILIB: `sgefa` + `sgesl` + `sgemm`	0.12
Jordan `sgefa` + `sgesl` + `sgeslm`	0.096

Band	
SCILIB: `sgbfa` + `sgbsl`	0.062
Jordan `bglsdc` + `bglssl`	0.030

From the above data we see that solving the banded system is factor of 2 to 3 faster than solving the pentadiagonal system, and that on a CRAY machine using Jordan's CAL-coded routines leads to a factor of 1.5 to 2 speedup relative to SCILIB.

XVII. CONTROL

A. Convergence Control

Define

$$\mathtt{dmax}_l \;\equiv\; \max_{i,j}\left|\frac{\delta x_{ik}^l}{x_{ik}^l}\right|$$

the actual maximum fractional change in all physical variables in iteration number l, and

$$\mathtt{cmax}_l \;\equiv\; \max_{k}\left|\frac{\delta r_{k+1}^l - \delta r_k^l}{r_{k+1}^l - r_k^l}\right|$$

the actual maximum fractional change in cell size in iteration number l. Further, define

$$\mathtt{s1} \equiv \min(1,\ \mathtt{dtol}/\mathtt{dmax}_l)$$

$$\mathtt{s2} \equiv \min(1,\ \mathtt{ctol}/\mathtt{cmax}_l)$$

and

$$\mathtt{sc}_l \equiv \min(\mathtt{s1},\ \mathtt{s2})$$

where `dtol` is the maximum allowed fractional change in all physical variables and `ctol` is the maximum allowed fractional change in cell size. Then update all variables with the formula

$$x_{ik}^l = \left(1 + \mathtt{sc}_l \frac{\delta x_{ik}^l}{x_{ik}^l}\right) x_{ik}^l$$

Convergence is reached when `dmax` $\leq$ `conv`.

B. Timestep Control

Timestep control is done using the algorithm discussed in detail in section V.D of

K.-H. Winkler and M.L. Norman, in *Astrophysical Radiation Hydrodynamics,* (Dordrecht: D. Reidel), pp. 71 - 139, 1986.

In addition, define

$$\mathtt{smax} \;\equiv\; \max_{i,k} \left| \frac{x_{ik}^{n+1} - x_{ik}^{n}}{x_{ik}^{n}} \right|$$

the actual maximum fractional change in all physical variables between the two levels of a completed integration step. Then if `smax` is found to be ≥ 2 `stol`, where `stol` is a prechosen allowed maximum, cut the timestep, and do the integration over, even though convergence was attained at t^{n+1}.

XVIII. GLOSSARY

common/agrid/

`alph`	α	spatial diffusion coefficient
`cs(k,l)`	S_{kl}^{n+1}	
`dabdlrm1(k)`	`dncdlrm1(k)`$/R_k^{n+1}$	
`dabdlr00(k)`	`dncdlr00(k)`$/R_k^{n+1}$ -`dnudlr00(k)`$\hat{\nu}_k^{n+1}\nu_k^{n+1}$`ss(k)`$/(R_k^{n+1})^3$	
`dabdlrp1(k)`	`dncdlrp1(k)`$/R_k^{n+1}$ -`dnudlrp1(k)`$\hat{\nu}_k^{n+1}\nu_k^{n+1}$`ss(k)`$/(R_k^{n+1})^3$	
`dabdlrp2(k)`	`dncdlrp2(k)`$/R_k^{n+1}$	
`dcddlx00(k,m)`	$\frac{1}{2}\hat{\nu}_k^{n+1}[(\nu_k^{n+1})^2/(R_k^{n+1})^3]$`dssdlx00(k,m)`	
`dcddlxp1(k,m)`	$\frac{1}{2}\hat{\nu}_k^{n+1}[(\nu_k^{n+1})^2/(R_k^{n+1})^3]$`dssdlxp1(k,m)`	
`dcddlxp2(k,m)`	$\frac{1}{2}\hat{\nu}_k^{n+1}[(\nu_k^{n+1})^2/(R_k^{n+1})^3]$`dssdlxp2(k,m)`	
`dcsdlx00(k,l,m)`	$(\partial S_{kl}^{n+1}/\partial lnx_m)_k^{n+1}$	
`dcsdlxp1(k,l,m)`	$(\partial S_{kl}^{n+1}/\partial lnx_m)_{k+1}^{n+1}$	
`dcsdlxp2(k,l,m)`	$(\partial S_{kl}^{n+1}/\partial lnx_m)_{k+2}^{n+1}$	
`dncdlrm1(k)`	$\partial\hat{\nu}_k^{n+1}/\partial lnr_{k-1}^{n+1}$	
`dncdlr00(k)`	$\partial\hat{\nu}_k^{n+1}/\partial lnr_{k}^{n+1}$	
`dncdlrp1(k)`	$\partial\hat{\nu}_k^{n+1}/\partial lnr_{k+1}^{n+1}$	
`dncdlrp2(k)`	$\partial\hat{\nu}_k^{n+1}/\partial lnr_{k+2}^{n+1}$	
`dntdlrm1(k)`	$\partial\tilde{\nu}_k^{n+1}/\partial lnr_{k-1}^{n+1}$	
`dntdlr00(k)`	$\partial\tilde{\nu}_k^{n+1}/\partial lnr_{k}^{n+1}$	
`dntdlrp1(k)`	$\partial\tilde{\nu}_k^{n+1}/\partial lnr_{k+1}^{n+1}$	
`dntdlrp2(k)`	$\partial\tilde{\nu}_k^{n+1}/\partial lnr_{k+2}^{n+1}$	
`dnudlr00(k)`	$\partial\nu_k^{n+1}/\partial lnr_{k}^{n+1}$	
`dnudlrp1(k)`	$\partial\nu_k^{n+1}/\partial lnr_{k+1}^{n+1}$	
`dssdlx00(k,m)`	$2\sum_l W_l$`cs(k,l)dcsdlx00(k,l,m)`	
`dssdlxp1(k,m)`	$2\sum_l W_l$`cs(k,l)dcsdlxp1(k,l,m)`	
`dssdlxp2(k,m)`	$2\sum_l W_l$`cs(k,l)dcsdlxp2(k,l,m)`	
`ibet`	β	time delay exponent
`rr(k)`	$R_k^{n+1} \equiv \{1+[\mathrm{xnu(k)}]^2\mathrm{ss(k)}\}^{1/2}$	
`ss(k)`	$\sum_l W_l(S_{kl}^{n+1})^2$	
`tau`	τ	time constant
`wt(l)`	W_l	weights
`xnc(k)`	$\hat{\nu}_k^{n+1}$	

`xnt(k)`	$\tilde{\nu}_k^{n+1}$	
`xnto(k)`	$\tilde{\nu}_k^{n}$	
`xnu(k)`	ν_k^{n+1}	grid concentration at t^{n+1}
`xnuo(k)`	ν_k^{n+1}	grid concentration at t^{n}
`xscale`	R_{scale}	spatial scale
`yscl(l)`	$F_{scale,l}$	ordinate scale

common/avuor/

`aur(k)`	$\left\langle \frac{u}{r} \right\rangle_k^{n+1}$
`dardlr00(k)`	$\partial < u/r >_k^{n+1} / \partial lnr_k^{n+1}$
`dardlrp1(k)`	$\partial < u/r >_k^{n+1} / \partial lnr_{k+1}^{n+1}$
`dardlu00(k)`	$\partial < u/r >_k^{n+1} / \partial lnu_k^{n+1}$
`dardlup1(k)`	$\partial < u/r >_k^{n+1} / \partial lnu_{k+1}^{n+1}$
`dudr(k)`	$[\frac{du}{dr} - \frac{\mu}{2} \left\langle \frac{u}{r} \right\rangle]_k^{n+1}$
`durdlr00(k)`	$\partial(du/dr)_k^{n+1}/\partial lnr_k^{n+1}$
`durdlrp1(k)`	$\partial(du/dr)_k^{n+1}/\partial lnr_{k+1}^{n+1}$
`durdlu00(k)`	$\partial(du/dr)_k^{n+1}/\partial lnu_k^{n+1}$
`durdlup1(k)`	$\partial(du/dr)_k^{n+1}/\partial lnu_{k+1}^{n+1}$

common/bc/

`delml`	Δm_L	mass of phantom zone at left (inner) Lagrangean boundary
`delmr`	Δm_R	mass of phantom zone at right (outer) Lagrangean boundary
`delrl`	Δr_L	radial size of phantom zone at left (inner) Eulerian boundary
`delrr`	Δr_R	radial size of phantom zone at right (outer) Eulerian boundary
`dl`	ρ_L	density at left (inner) boundary
`dr`	ρ_R	density at right (outer) boundary
`egasl`	e_L	gas energy density at left (inner) boundary
`egasr`	e_R	gas energy density at right (outer) boundary
`omega`	$\frac{\Delta\xi_k}{\Delta\xi_{k+1}}$	ratio of masses in successive zones in extended atmosphere
`pextr`	p_{ext}	imposed external pressure at Lagrangean boundary
`phil0`	Φ_L^0	imposed mass flux at left (inner) boundary
`phil1`	Φ_L^1	imposed momentum flux at left (inner) boundary
`phil2`	Φ_L^2	imposed energy flux at left (inner) boundary
`phir0`	Φ_R^0	imposed mass flux at right (outer) boundary
`phir1`	Φ_R^1	imposed momentum flux at right (outer) boundary
`phir2`	Φ_R^2	imposed energy flux at right (outer) boundary
`tl`	T_L	material temperature at left (inner) boundary
`tr`	T_R	material temperature at right (outer) boundary
`uextl`	$u_{ext,L}$	imposed piston velocity at left (inner) Lagrangean boundary
`uextr`	$u_{ext,R}$	imposed piston velocity at right (outer) Lagrangean boundary
`ul`	u_L	flow velocity at left (inner) boundary
`ur`	u_R	flow velocity at right (outer) boundary

common/const/

`car`	a_R	radiation constant
`cc`	c	speed of light
`cgas`	k/m_H	molar gas constant
`cgrav`	G	Newtonian gravitation constant
`ck`	k	Boltzmann's constant
`clsol`	$L_\odot$	solar luminosity
`cm0`	m_0	atomic mass unit
`cmsol`	$M_\odot$	solar mass
`cpi`	π	
`crsol`	$R_\odot$	solar radius
`csige`	σ_e	electron scattering cross-section
`csigr`	σ_R	Stefan-Boltzmann constant

common/diffus/

`ddfdldm1(k)`	$\partial Q_k^{n+\theta}/\partial ln\rho_{k-1}^{n+1}$	
`ddfdld00(k)`	$\partial Q_k^{n+\theta}/\partial ln\rho_k^{n+1}$	
`ddfdlqm1(k)`	$\partial Q_k^{n+\theta}/\partial lnq_{k-1}^{n+1}$	
`ddfdlq00(k)`	$\partial Q_k^{n+\theta}/\partial lnq_k^{n+1}$	
`ddfdlrm1(k)`	$\partial Q_k^{n+\theta}/\partial lnr_{k-1}^{n+1}$	
`ddfdlr00(k)`	$\partial Q_k^{n+\theta}/\partial lnr_k^{n+1}$	
`ddfdlrp1(k)`	$\partial Q_k^{n+\theta}/\partial lnr_{k+1}^{n+1}$	
`ddfdltm1(k)`	$\partial Q_k^{n+\theta}/\partial lnT_{k-1}^{n+1}$	
`ddfdlt00(k)`	$\partial Q_k^{n+\theta}/\partial lnT_k^{n+1}$	
`df(k)`	$Q_k^{n+\theta}$	diffusion flux, see equation (C42)
`qd(k)`	$q_k^{n+\theta}$	quantity diffused at $t^{n+\theta}$
`qdn(k)`	q_k^{n+1}	quantity diffused at t^{n+1}
`sig`	σ	diffusion coefficient
`zet`	`0.` or `1.`	density exponent, see equation (C42)

common /dot/

`dmdt(k)`	$(dm/dt)_k^{n+\theta}$	mass flux across interface k at $t^{n+\theta}$
`dmdto(k)`	$(dm/dt)_k^{n-1+\theta}$	mass flux across interface k at $t^{n-1+\theta}$
`drdt(k)`	$(dr/dt)_k^{n+\theta}$	velocity of interface k at $t^{n+\theta}$
`drdto(k)`	$(dr/dt)_k^{n-1+\theta}$	velocity of interface k at $t^{n-1+\theta}$
`urel(k)`	$[u-(dr/dt)]_k^{n+\theta}$	velocity of fluid relative to grid at $t^{n+\theta}$
`urelo(k)`	$[u-(dr/dt)]_k^{n-1+\theta}$	velocity of fluid relative to grid at $t^{n-1+\theta}$

common/edding/

`ang(mang)`	μ_j	angle cosines in planar geometry
`asf(l)`	a_l	constant in piecewise linear source function on ray
`bsf(l)`	b_{jl}	monotonized slope in piecewise linear source function on ray
`cjk(j)`	c_{jk}	elements of system for two reflecting boundaries
`dsf(k)`	$\Delta S_k \equiv S_k - S_{k+1}$	change in S between radial shells
`dtau(k)`	$\Delta\tau_k \equiv \tau_k - \tau_{k+1}$	optical depth increment between radial shells
`ejk(j)`	e_{jk}	elements of system for two reflecting boundaries
`epsedf`	ϵ_{edf}	monotonization overflow protection switch
`nang`		number of angles in planar geometry
`ncor`		number of rays penetrating innermost shell
`p(j)`	p_j	impact parameters
`rp(j)`	r_l	augmented radial grid
`sf(k)`	S_k	source function on radial grid
`sjk(j)`	s_{jk}	rhs of system for two reflecting boundaries
`sl(l)`	S_{Ll}	source function at left interface of cell in augmented grid
`sr(l)`	S_{Rl}	source function at right interface of cell in augmented grid
`xang(j)`	μ_j	angle cosines on spherical shell
`xi0 (j)`		intensity at cell center along a ray integration
`xim(j)`		intensity at previous interface along a ray integration
`xip(j)`		intensity at next interface along a ray integration
`xmom0(l)`	$\int_{-1}^{1} I_l(\mu) d\mu$	zeroth angular moment of intensity
`xmom2(l)`	$\int_{-1}^{1} I_l(\mu) \mu^2 d\mu$	second angular moment of intensity

common/energy/

`r0(k)`		interface radii at t = 0
`tee`	$\mathcal{E}^{n+1}$	total fluid energy inside domain
`tel`	$\mathcal{L}^{n+1}$	luminosity through boundaries at t^{n+1}
`telo`	$\mathcal{L}^{n}$	luminosity through boundaries at t^{n}
`teq`	$\mathcal{Q}^{n+1}$	viscous energy dissipation at t^{n+1}
`teqo`	$\mathcal{Q}^{n}$	viscous energy dissipation at t^{n}
`tes`	$\mathcal{S}^{n+1}$	fluid energy transport at boundary at t^{n+1}
`teso`	$\mathcal{S}^{n}$	fluid energy transport at boundary at t^{n}
`tetot`	$\mathcal{E}^{n+1} + \mathcal{S}^{n+1} + \mathcal{W}^{n+1}$ $+\mathcal{L}^{n+1} - \mathcal{Q}^{n+1}$	total fluid energy
`tew`	$\mathcal{W}^{n+1}$	work done by fluid at t^{n+1}
`tewo`	$\mathcal{W}^{n}$	work done by fluid at t^{n}
`tescr(k)`		scratch vector
`xm0(k)`		interface masses at t = 0

common/eostab/

`dlegdldn(k)`	$(\partial lne_g/\partial ln\rho)_k^{n+1}$	
`dlegdltn(k)`	$(\partial lne_g/\partial lnT)_k^{n+1}$	
`dlpedldn(k)`	$(\partial lnp_e/\partial ln\rho)_k^{n+1}$	
`dlpedltn(k)`	$(\partial lnp_e/\partial lnT)_k^{n+1}$	
`dlpgdldn(k)`	$(\partial lnp_g/\partial ln\rho)_k^{n+1}$	
`dlpgdltn(k)`	$(\partial lnp_g/\partial lnT)_k^{n+1}$	
`feg(mxe,mye)`	lne_g	log of gas energy density
`fpe(mxe,mye)`	lnp_e	log of electron pressure
`fpg(mxe,mye)`	lnp_g	log of gas pressure
`fegx(mxe,mye)`	$\partial lne_g/\partial ln\rho$	
`fpex(mxe,mye)`	$\partial lnp_e/\partial ln\rho$	
`fpgx(mxe,mye)`	$\partial lnp_g/\partial ln\rho$	
`fegxy(mxe,mye)`	$\partial^2 lne_g/\partial ln\rho\partial lnT$	
`fpexy(mxe,mye)`	$\partial^2 lnp_e/\partial ln\rho\partial lnT$	
`fpgxy(mxe,mye)`	$\partial^2 lnp_g/\partial ln\rho\partial lnT$	
`fegy(mxe,mye)`	$\partial lne_g/\partial lnT$	
`fpey(mxe,mye)`	$\partial lnp_e/\partial lnT$	
`fpgy(mxe,mye)`	$\partial lnp_g/\partial lnT$	
`gam`	γ	ideal gas adiabatic index
`gmu`	μ_m	ideal gas mean molecular weight
`xabun`	X	hydrogen mass abundance
`yabun`	Y	helium mass abundance
`zabun`	Z	"metals" mass abundance

common/geom/

`mu`	μ	exponent of r in metric (integer)
`mum1`	$\mu-1$	
`mup1`	$\mu+1$	
`xmu`	μ	exponent of r in metric (floating)
`xmum1`	$\mu-1$	
`xmup1`	$\mu+1$	

common/hydro/

`cadv`	C_{adv}	order of advection
`cq1`	C_1	coefficient of linear pseudoviscosity
`cq2`	C_2	coefficient of quadratic pseudoviscosity
`cqvis`	$\frac{4}{3}$	coefficient of artificial stress tensor
`epsadv`	ϵ_{adv}	advection overflow protection switch
`q0`	q_0	floor on pseudoviscous pressure ratio

`ql0`	q_0	fixed absolute length for pseudoviscosity (planar)
`ql1`	q_1	fixed relative length for pseudoviscosity (spherical)
`sigd`	σ_ρ	artificial mass-diffusion coefficient
`sige`	σ_e	artificial energy-diffusion coefficient
`thet`	θ	time-centering parameter

common/index/

`id`	row index of continuity equation
`ie`	row index of radiation energy equation
`if`	row index of radiation momentum equation
`im`	row index of mass equation
`ir`	row index of radius equation
`it`	row index of radiating fluid energy equation
`iu`	row index of gas momentum equation
`jd`	column index of continuity equation
`je`	column index of radiation energy equation
`jf`	column index of radiation momentum equation
`jm`	column index of mass equation
`jr`	column index of radius equation
`jt`	column index of radiating fluid energy equation
`ju`	column index of gas momentum equation
`jb`	dummy index used in grid equation

common /integrat/

`dtime`	timestep
`jback`	number of current reintegration with reduced timestep
`jext`	number of current try to preserve monotonic grid
`jstep`	serial number of current model
`jstepe`	dump number of last model of current run
`jsteps`	dump number of first model of current run
`nback`	maximum number of reintegrations with reduced timestep
`next`	maximum number of tries to preserve monotonic grid
`smax`	global maximum change in physical variables
`stol`	maximum fractional change allowed between timesteps
`sx(mgr, meqn)`	change in physical variables at all depths
`tfac`	multiplier used in setting next timestep
`timen`	t^{n+1}, time at end of timestep
`timeo`	t^n, time at beginning of timestep

common/io/

`idoc`	unit number of parameter documentation file
`idump`	unit number of dump file
`ieos`	unit number of eos file
`ihist`	unit number of history file
`iin`	unit number of input file
`iinit`	unit number of initial model file
`iopac`	unit number of opacity file
`iout`	unit number of output file
`itty`	unit number of terminal
`jdump`	number of current dump
`jhist`	number of current history record
`ldump`	length of dump file
`lhist`	length of history file
`ndump`	number of time steps between dumps of models
`nout`	number of time steps between outputs of models

common/iterate/

`cmax`	global maximum fractional change in cell size
`conv`	Newton-Raphson convergence criterion
`ctol`	maximum allowed fractional change in cell size
`cx(mgr)`	actual fractional changes in cell size
`dmax`	global maximum fractional change in physical variables
`dtol`	maximum allowed fractional change in physical variables
`dx(meqn)`	actual fractional changes in physical variables
`iter`	number of current Newton-Raphson iteration
`jtry`	number of current try using reduced timesteps
`kx(meqn)`	depth index of maximum change in each physical variable
`niter`	maximum number of Newton-Raphson iterations
`ntry`	maximum number of tries for convergence using reduced timesteps

common/logic/

`ladx`	specify type of x-coordinate for adaptive grid
`lady(mad)`	specify type of y-coordinate for adaptive grid
`lam`	select flux limiting in radiation diffusion equation
`lband`	specify whether system is band or pentadiagonal matrix
`lboos`	= 1,perform boost iteration; set internally
`lcray`	specify whether computer is cray machine or not
`leibc`	select Eulerian inner BC
`leobc`	select Eulerian outer BC
`leos`	select eos, formula or table
`lgeom`	select planar or spherical geometry
`lgrav`	specify whether gravity is present
`lgrid`	select type of grid
`lhydr`	specify whether hydrodynamics is treated
`linit`	flag initial model
`llibc`	select Lagrangean inner BC
`llobc`	select Lagrangean outer BC
`lopac`	select opacity, formula or table
`lprob`	specify nature of problem to be solved
`lrad`	specify whether radiation is treated
`lribc`	select inner radiation BC
`lrobc`	select outer radiation BC
`ltran`	select radiation transport or diffusion (equib or nonequib)

common/matrix/

`em2(meqn, meqn, mgr)`	E_{-2k}	linearized block pentadiagonal system at $k-2$
`em1(meqn, meqn, mgr)`	E_{-1k}	linearized block pentadiagonal system at $k-1$
`e00(meqn, meqn, mgr)`	E_{0k}	linearized block pentadiagonal system at k
`ep1(meqn, meqn, mgr)`	E_{1k}	linearized block pentadiagonal system at $k+1$
`ep2(meqn, meqn, mgr)`	E_{2k}	linearized block pentadiagonal system at $k+2$
`rhs(meqn,       mgr)`		right hand side of block pentadiagonal system
`bm(3*mpd-2, meqn*mgr)`		linearized band matrix system
`br(meqn*mgr)`		right hand side of band matrix system
`tt(meqn, meqn, 3)`		scratch space for solution
`v(meqn)`		scratch vector
`ipv(meqn*mgr)`		pivot vector

common/mid/ Note: all variables are evaluated at time $t^{n+\theta}$

`as(k)`	$(a_S)_k^{n+\theta}$	sound speed
`avchi(k)`	$< \chi_F >_k^{n+\theta}$	flux-weighted opacity at interface
`chif(k)`	$(\chi_F)_k^{n+\theta}$	flux-mean opacity
`d(k)`	$\rho_k^{n+\theta}$	material density
`ds(k)`	$D\rho_k^{n+\theta}$	monotonized slope of material density
`dvol(k)`	$dV_k^{n+\theta}$	volume element
`eg(k)`	$e_k^{n+\theta}$	material energy density per gram
`egr(k)`	$(e+\frac{E}{\rho})_k^{n+\theta}$	total energy density per gram
`egrs(k)`	$D(e+\frac{E}{\rho})_k^{n+\theta}$	monotonized slope of total energy density
`egs(k)`	$De_k^{n+\theta}$	monotonized slope of material energy density
`er(k)`	$E_k^{n+\theta}$	radiation energy density per cm^3
`ers(k)`	$DE_k^{n+\theta}$	monotonized slope of radiation density
`fedd(k)`	$f_k^{n+\theta}$	Eddington factor
`fr(k)`	$F_k^{n+\theta}$	radiation flux
`frnom(k)`	$(F_{nom})_k^{n+\theta}$	nominal radiation flux
`frs(k)`	$DF_k^{n+\theta}$	monotonized slope of radiation flux
`geddl`	g_L	Eddington factor at left (inner) boundary
`geddr`	g_R	Eddington factor at right (outer) boundary
`pg(k)`	$p_k^{n+\theta}$	gas pressure
`plf(k)`	$B_k^{n+\theta}$	Planck function
`rmum1(k)`	$(r_k^{n+\theta})^{\mu-1}$	radius to power $\mu-1$
`rmu(k)`	$(r_k^{n+\theta})^{\mu}$	radius to power μ
`rmup1(k)`	$(r_k^{n+\theta})^{\mu+1}$	radius to power $\mu+1$
`r(k)`	$r_k^{n+\theta}$	radius
`t(k)`	$T_k^{n+\theta}$	temperature
`u(k)`	$u_k^{n+\theta}$	velocity
`unom(k)`	$(u_{nom})_k^{n+\theta}$	nominal velocity
`us(k)`	$Du_k^{n+\theta}$	monotonized slope of velocity
`xke(k)`	$(\kappa_E)_k^{n+\theta}$	energy weighted opacity per gram
`xkp(k)`	$(\kappa_P)_k^{n+\theta}$	Planck mean opacity per gram
`xm(k)`	$m_k^{n+\theta}$	interior mass
`xne(k)`	$(N_e)_k^{n+\theta}$	electron density per cm^3

common/new/ Note: all variables are evaluated at advanced time t^{n+1}

`asn(k)`	$(a_S)_k^{n+1}$	sound speed
`avchin(k)`	$< \chi_F >_k^{n+1}$	flux-weighted opacity at interface
`chifn(k)`	$(\chi_F)_k^{n+1}$	flux-weighted opacity
`dn(k)`	ρ_k^{n+1}	material density
`dsn(k)`	$D\rho_k^{n+1}$	monotonized slope of material density
`dvoln(k)`	dV_k^{n+1}	volume element
`egn(k)`	e_k^{n+1}	material energy density per gram
`egrn(k)`	$(e + \frac{E}{\rho})_k^{n+1}$	total energy density per gram
`egrsn(k)`	$D(e + \frac{E}{\rho})_k^{n+1}$	monotonized slope of total energy density
`egsn(k)`	De_k^{n+1}	monotonized slope of material energy density
`ern(k)`	E_k^{n+1}	radiation energy density per cm^3
`ersn(k)`	DE_k^{n+1}	monotonized slope of radiation density
`frn(k)`	F_k^{n+1}	radiation flux
`frsn(k)`	DF_k^{n+1}	monotonized slope of radiation flux
`pgn(k)`	p_k^{n+1}	gas pressure
`plfn(k)`	B_k^{n+1}	Planck function
`rmum1n(k)`	$(r_k^{n+1})^{\mu-1}$	radius to power $\mu - 1$
`rmun(k)`	$(r_k^{n+1})^{\mu}$	radius to power μ
`rmup1n(k)`	$(r_k^{n+1})^{\mu+1}$	radius to power $\mu + 1$
`rn(k)`	r_k^{n+1}	radius
`tn(k)`	T_k^{n+1}	temperature
`un(k)`	u_k^{n+1}	velocity
`usn(k)`	Du_k^{n+1}	monotonized slope of velocity
`xken(k)`	$(\kappa_E)_k^{n+1}$	energy weighted opacity per gram
`xkpn(k)`	$(\kappa_P)_k^{n+1}$	Planck mean opacity per gram
`xmn(k)`	m_k^{n+1}	interior mass
`xnen(k)`	$(N_e)_k^{n+1}$	electron density per cm^3

common/old/ Note: all variables are evaluated at old time t^n

`aso(k)`	$(a_S)_k^n$	sound speed
`avchio(k)`	$< \chi_F >_k^n$	flux-weighted opacity at interface
`chifo(k)`	$(\chi_F)_k^n$	flux-weighted opacity
`do(k)`	ρ_k^n	material density
`dso(k)`	$D\rho_k^n$	monotonized slope of material density
`dvolo(k)`	dV_k^n	volume element
`ego(k)`	e_k^n	material energy density per gram
`egro(k)`	$(e + \frac{E}{\rho})_k^n$	total energy density per gram
`egrso(k)`	$D(e + \frac{E}{\rho})_k^n$	monotonized slope of total energy density
`egso(k)`	De_k^n	monotonized slope of material energy density
`ero(k)`	E_k^n	radiation energy density per cm^3
`erso(k)`	DE_k^n	monotonized slope of radiation density
`fro(k)`	F_k^n	radiation flux
`frso(k)`	DF_k^n	monotonized slope of radiation flux
`pgo(k)`	p_k^n	gas pressure
`plfo(k)`	B_k^n	Planck function
`rmum1o(k)`	$(r_k^n)^{\mu-1}$	radius to power $\mu - 1$
`rmuo(k)`	$(r_k^n)^{\mu}$	radius to power μ
`rmup1o(k)`	$(r_k^n)^{\mu+1}$	radius to power $\mu + 1$
`ro(k)`	r_k^n	radius
`to(k)`	T_k^n	temperature
`uo(k)`	u_k^n	velocity
`uso(k)`	Du_k^n	monotonized slope of velocity
`xkeo(k)`	$(\kappa_E)_k^n$	energy weighted opacity per gram
`xkpo(k)`	$(\kappa_P)_k^n$	Planck mean opacity per gram
`xmo(k)`	m_k^n	interior mass
`xneo(k)`	$(N_e)_k^n$	electron density per cm^3

common/opactab/

`dlcfdldn(k)`	$(\partial ln\chi_F/\partial ln\rho)_k^{n+1}$
`dlcfdltn(k)`	$(\partial ln\chi_F/\partial lnT)_k^{n+1}$
`dlkedldn(k)`	$(\partial ln\kappa_E/\partial ln\rho)_k^{n+1}$
`dlkedltn(k)`	$(\partial ln\kappa_E/\partial lnT)_k^{n+1}$
`dlkpdldn(k)`	$(\partial ln\kappa_P/\partial ln\rho)_k^{n+1}$
`dlkpdltn(k)`	$(\partial ln\kappa_P/\partial lnT)_k^{n+1}$
`dplfdltn(k)`	$(\partial B/\partial lnT)_k^{n+1}$

`fcf(mxo,myo)`	$ln\chi_F$	log of flux-mean opacity (N.B. code uses Rosseland mean)
`fcfx(mxo,myo)`	$\partial ln\chi_F/\partial ln\rho$	
`fcfxy(mxo,myo)`	$\partial^2 ln\chi_F/\partial ln\rho\partial lnT$	
`fcfy(mxo,myo)`	$\partial ln\chi_F/\partial lnT$	
`fke(mxo,myo)`	$ln\kappa_E$	log of energy-mean opacity (N.B. code uses Planck mean)
`fkex(mxo,myo)`	$\partial ln\kappa_E/\partial ln\rho$	
`fkexy(mxo,myo)`	$\partial^2 ln\kappa_E/\partial ln\rho\partial lnT$	
`fkey(mxo,myo)`	$\partial ln\kappa_E/\partial lnT$	

common/rad/

`ximr`	I_R^-	incoming intensity at outer (right) boundary
`xipl`	I_L^+	outgoing intensity at inner (left) boundary
`xlam`	$\frac{2}{3}\lambda$	coefficient in flux-limited diffusion theory

common/r3o2/

`r1(k)`	$(r_k^{n+\theta})^{-\mu/2}$
`r3(k)`	$[\frac{1}{2}(r_k^{n+\theta}+r_{k+1}^{n+\theta})]^{3\mu/2}$
`dr1dlr00(k)`	$\partial \mathrm{r1}/\partial ln r_k^{n+1}$
`dr3dlr00(k)`	$\partial \mathrm{r3}/\partial ln r_k^{n+1}$
`dr3dlrp1(k)`	$\partial \mathrm{r3}/\partial ln r_{k+1}^{n+1}$

common/rdif/

`ddifdrm(k)`	$\partial F_k^{n+1}/\partial ln r_{k-1}^{n+1}$	
`ddifdr0(k)`	$\partial F_k^{n+1}/\partial ln r_k^{n+1}$	
`ddifdrp(k)`	$\partial F_k^{n+1}/\partial ln r_{k+1}^{n+1}$	
`ddifdem(k)`	$\partial F_k^{n+1}/\partial ln E_{k-1}^{n+1}$	
`ddifde0(k)`	$\partial F_k^{n+1}/\partial ln E_k^{n+1}$	
`ddifddm(k)`	$\partial F_k^{n+1}/\partial ln\rho_{k-1}^{n+1}$	
`ddifdd0(k)`	$\partial F_k^{n+1}/\partial ln\rho_k^{n+1}$	
`ddifdtm(k)`	$\partial F_k^{n+1}/\partial lnT_{k-1}^{n+1}$	
`ddifdt0(k)`	$\partial F_k^{n+1}/\partial lnT_k^{n+1}$	
`dif(k)`	F_k^{n+1}	radiation diffusion flux see equations (RD1) – (RD3)

common/star/

`chif0`	$(\rho\chi)_0$	opacity parameter for stellar envelope initialization
`g`	g	stellar surface gravity
`header`		identifying information
`ratio`	ζ	opacity multiplier for stellar envelope initialization
`rho0`	ρ_0	density parameter for stellar envelope initialization
`teff`	T_{eff}	stellar effective temperature
`tmax`	t_{max}	time at which integration terminates
`xindef`		preset arithmetic indefinite
`xlum`	L	stellar luminosity
`xmass`	M	stellar mass
`xrad`	R	stellar radius

common/viscos1/

div(k)	$\frac{\texttt{rmu(k+1)}u_{k+1}^{n+\theta}-\texttt{rmu(k)}u_k^{n+\theta}}{\texttt{dvol(k)}}$	velocity divergence
dqmdldm1(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial ln\rho_{k-1}^{n+1}$	
dqmdld00(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial ln\rho_k^{n+1}$	
dqmdlrm1(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnr_{k-1}^{n+1}$	
dqmdlr00(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnr_k^{n+1}$	
dqmdlrp1(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnr_{k+1}^{n+1}$	
dqmdltm1(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnT_{k-1}^{n+1}$	
dqmdlt00(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnT_k^{n+1}$	
dqmdlu00(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnu_k^{n+1}$	
dqmdlup1(k)	$\partial(\mu_Q)_k^{n+\theta}/\partial lnu_{k+1}^{n+1}$	
dqxdld00(k)	$\partial\mathrm{qx}/\partial\rho_k^{n+1}$	
dqxdldp1(k)	$\partial\mathrm{qx}/\partial\rho_{k+1}^{n+1}$	
dqxdlr00(k)	$\partial\mathrm{qx}/\partial r_k^{n+1}$	
dqxdlrp1(k)	$\partial\mathrm{qx}/\partial r_{k+1}^{n+1}$	
dqxdlt00(k)	$\partial\mathrm{qx}/\partial T_k^{n+1}$	
dqxdlu00(k)	$\partial\mathrm{qx}/\partial u_k^{n+1}$	
dqxdlup1(k)	$\partial\mathrm{qx}/\partial u_{k+1}^{n+1}$	
qe(k)	$-\frac{4}{3}\rho_k^{n+\theta}\texttt{qm(k)}[\texttt{dudr(k)}]^2\texttt{dvol(k)}$	rate of viscous energy dissipation
ql(k)	$\ell_0+\frac{1}{2}(r_k^{n+\theta}+r_{k+1}^{n+\theta})\ell_1$	dissipation length
qm(k)	$(\mu_Q)_k^{n+\theta}$	coefficient of viscosity
qx(k)	`[qf(k)pk(k)] + q0`	artificial viscosity indicator

common/viscos2/

`dpkdld00(k)`	$\partial \mathrm{pk}/\partial ln\rho_k^{n+1}$	
`dpkdlt00(k)`	$\partial \mathrm{pk}/\partial lnT_k^{n+1}$	
`dpkdlu00(k)`	$\partial \mathrm{pk}/\partial lnu_k^{n+1}$	
`dpkdlup1(k)`	$\partial \mathrm{pk}/\partial lnu_{k+1}^{n+1}$	
`dqfdld00(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial ln\rho_k^{n+1}$	
`dqfdlr00(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial lnr_k^{n+1}$	
`dqfdlrp1(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial lnr_{k+1}^{n+1}$	
`dqfdlt00(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial lnT_k^{n+1}$	
`dqfdlu00(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial lnu_k^{n+1}$	
`dqfdlup1(k)`	$\partial(\phi_Q)_k^{n+\theta}/\partial lnu_{k+1}^{n+1}$	
`dqvdldm1(k)`	$\partial \mathrm{qv}/\partial ln\rho_{k-1}^{n+1}$	
`dqvdld00(k)`	$\partial \mathrm{qv}/\partial ln\rho_k^{n+1}$	
`dqvdlrm1(k)`	$\partial \mathrm{qv}/\partial lnr_{k-1}^{n+1}$	
`dqvdlr00(k)`	$\partial \mathrm{qv}/\partial lnr_k^{n+1}$	
`dqvdlrp1(k)`	$\partial \mathrm{qv}/\partial lnr_{k+1}^{n+1}$	
`dqvdltm1(k)`	$\partial \mathrm{qv}/\partial lnT_{k-1}^{n+1}$	
`dqvdlt00(k)`	$\partial \mathrm{qv}/\partial lnT_k^{n+1}$	
`dqvdlu00(k)`	$\partial \mathrm{qv}/\partial lnu_k^{n+1}$	
`dqvdlup1(k)`	$\partial \mathrm{qv}/\partial lnu_{k+1}^{n+1}$	
`pk(k)`	$p_k^{n+1} + \frac{1}{4}\rho_k^{n+1}(u_k^{n+1} + u_{k+1}^{n+1})^2$	kinetic pressure
`qf(k)`	$(\phi_Q)_k^{n+\theta}$	viscous force
`qv(k)`	min[`div(k), 0`]	