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Homework 4

1.

(a)

To find $P(X_1 = 2)$, we can use the property that $\pi^{(t)} = \pi^{(0)}P^t$, and do the simple weighted probability calculation accordingly:

$$\begin{aligned}
P(X_1 = 2) &= \sum_{i=0}^{2} P(X_0 = i) \cdot P(X_1 = 2 | X_0 = i) \\
&= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} \\
&= 0.2 \cdot 0.6 + 0.5 \cdot 0.2 + 0.3 \cdot 0.2 \\
&= 0.12 + 0.10 + 0.06 \\
&= \boxed{0.28}
\end{aligned}$$

(b)

To find $P(X_2 = 2)$, we use the same property as before. We therefore need to calculate $P^t = P^2$ in this case to get started:

$$P^2 = \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

$$P^2 = \left(\begin{matrix} (.3)(.3) + (.1)(.4) + (.6)(.1) & (.3)(.1) + (.1)(.4) + (.6)(.7) & (.3)(.6) + (.1)(.2) + (.6)(.2 \\ (.4)(.3) + (.4)(.4) + (.2)(.1) & (.4)(.1) + (.4)(.4) + (.2)(.7) & (.4)(.6) + (.4)(.2) + (.2)(.2 \\ (.1)(.3) + (.7)(.4) + (.2)(.1) & (.1)(.1) + (.7)(.4) + (.2)(.7) & (.1)(.6) + (.7)(.2) + (.2)(.2 \end{matrix} \right.$$

$$P^2 = \begin{pmatrix} 0.09 + 0.04 + 0.06 & 0.03 + 0.04 + 0.42 & 0.18 + 0.02 + 0.12 \\ 0.12 + 0.16 + 0.02 & 0.04 + 0.16 + 0.14 & 0.24 + 0.08 + 0.04 \\ 0.03 + 0.28 + 0.02 & 0.01 + 0.28 + 0.14 & 0.06 + 0.14 + 0.04 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.19 & 0.49 & 0.32 \\ 0.30 & 0.34 & 0.36 \\ 0.33 & 0.43 & 0.24 \end{pmatrix}$$

We can now plug in again:

$$P(X_2 = 2) = \pi^{(0)} \cdot \begin{pmatrix} 0.32 \\ 0.36 \\ 0.24 \end{pmatrix}$$

$$= 0.2 \cdot 0.32 + 0.5 \cdot 0.36 + 0.3 \cdot 0.24$$

$$= 0.064 + 0.18 + 0.072$$

$$= \boxed{0.316}$$

(c)

To find $P(X_3 = 2|X_0 = 0)$, we can use theorem 1 to take this as the $(0,2)^{th}$ element in P^3. So we first need to find P^3:

$$P^3 = P^2 \cdot P$$

$$P^3 = \begin{pmatrix} 0.19 & 0.49 & 0.32 \\ 0.30 & 0.34 & 0.36 \\ 0.33 & 0.43 & 0.24 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} .19 \cdot .3 + .49 \cdot .4 + .32 \cdot .1 & .19 \cdot .1 + .49 \cdot .4 + .32 \cdot .7 & .19 \cdot .6 + .49 \cdot .2 + .32 \cdot .2 \\ .30 \cdot .3 + .34 \cdot .4 + .36 \cdot .1 & .30 \cdot .1 + .34 \cdot .4 + .36 \cdot .7 & .30 \cdot .6 + .34 \cdot .2 + .36 \cdot .2 \\ .33 \cdot .3 + .43 \cdot .4 + .24 \cdot .1 & .33 \cdot .1 + .43 \cdot .4 + .24 \cdot .7 & .33 \cdot .6 + .43 \cdot .2 + .24 \cdot .2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.057 + 0.196 + 0.032 & 0.019 + 0.196 + 0.224 & 0.114 + 0.098 + 0.064 \\ 0.090 + 0.136 + 0.036 & 0.030 + 0.136 + 0.252 & 0.180 + 0.068 + 0.072 \\ 0.099 + 0.172 + 0.024 & 0.033 + 0.172 + 0.168 & 0.198 + 0.086 + 0.048 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.285 & 0.439 & 0.276 \\ 0.262 & 0.418 & 0.320 \\ 0.295 & 0.373 & 0.332 \end{pmatrix}$$

Taking the corresponding element yields that $\boxed{P(X_3 = 2|X_0 = 0) = 0.276}$.

(d)

To find $P(X_0 = 1|X_1 = 2)$, we can first decompose this expression into parts that we can calculate more easily using Bayes' theorem:

$$P(X_0 = 1|X_1 = 2) = \frac{P(X_1 = 2|X_0 = 1) \cdot P(X_0 = 1)}{P(X_1 = 2)}$$

We can first substitute using P, directly taking the $(1, 2)^{th}$ element:

$$= \frac{0.2 \cdot P(X_0 = 1)}{P(X_1 = 2)}$$

Next, we can substitute using $\pi^{(0)}$:

$$= \frac{0.2 \cdot 0.5}{P(X_1 = 2)}$$

Finally, we can substitute using what we found in part (a):

$$= \frac{0.10}{0.28} = \frac{10}{28} = \boxed{\frac{5}{14}}$$

(e)

To find $P(X_1 = 1, X_3 = 1)$, we can first break this down by calculating the probabilities of the full chain, taking all states into account. In other words, we can set up the expression such that

$$P(X_1 = 1, X_3 = 1) = \sum_{i=0}^{2} P(X_0 = i) \cdot P(X_1 = 1|X_0 = i) \cdot P(X_3 = 1|X_1 = 1)$$

$$= P(X_3 = 1|X_1 = 1) \sum_{i=0}^{2} \pi_i \cdot P_{i,1}$$

Given the properties of the Markov chain, we can just treat this as a normal 2-step transition and therefore use P^2 for this. So we can equate this to $P^2_{1,1} \cdot P(X_1 = 1)$. Plug this in and evaluate these terms, given what we've already previously calculated:

$$= P^2_{1,1} \cdot P(X_1 = 1)$$

$$= 0.34 \cdot (\pi_0 \cdot P_{0,1} + \pi_1 \cdot P_{1,1} + \pi_2 \cdot P_{2,1})$$

$$= 0.34 \cdot (0.2 \cdot 0.1 + 0.5 \cdot 0.4 + 0.3 \cdot 0.7)$$

$$= 0.34 \cdot (0.02 + 0.2 + 0.21)$$

$$= 0.34 \cdot 0.43$$

$$= \boxed{0.1462}$$

2.

Let's begin by constructing the applicable transition matrix. We are given the following information:

- From state 0 (no umbrellas): No matter what whether it rains or doesn't, there's no umbrellas to take, so we always arrive to state 3, with the remaining 3 umbrellas on the other side.

- From any other nonzero state i (1-3 umbrellas): If it rains with probability p, then take one umbrella across – the other side has $3 - i$ umbrellas, and then we bring one more, so we will reach state $3 - i + 1 = 4 - i$ with probability p. Conversely, it doesn't rain with probability $1 - p$, so she leaves the umbrella, so we go to state $3 - i$ with probability $1 - p$.

Visualizing this all as a matrix:

	0	1	2	3
0	0	0	0	1
1	0	0	$1 - p$	p
2	0	$1 - p$	p	0
3	$1 - p$	p	0	0

Using this, we can find the limiting distribution π_i by solving $\pi = \pi P$. Let's set up the equations that represent this:

$$\pi_0 = (1-p)\cdot\pi_3$$
$$\pi_1 = (1-p)\cdot\pi_2 + p\cdot\pi_3$$
$$\pi_2 = (1-p)\cdot\pi_1 + p\cdot\pi_2$$
$$\pi_3 = 1\cdot\pi_0 + p\cdot\pi_1$$

Substituting π_0 into π_3:

$$\pi_3 = 1\cdot\pi_0 + p\cdot\pi_1$$
$$\pi_3 = (1-p)\cdot\pi_3 + p\cdot\pi_1$$
$$\pi_3 = \pi_3 - p\cdot\pi_3 + p\cdot\pi_1$$
$$p\cdot\pi_3 = p\cdot\pi_1$$
$$\pi_3 = \pi_1$$

And plugging this into the equation for π_1:

$$\pi_1 = (1-p)\cdot\pi_2 + p\cdot\pi_1$$
$$(1-p)\cdot\pi_1 = (1-p)\cdot\pi_2$$
$$\pi_1 = \pi_2$$

By definition, we know that the total probability of these probabilities must add up to 1, i.e.

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$
$$(1-p)\cdot\pi_3 + \pi_3 + \pi_1 + \pi_3 = 1$$
$$(1-p)\cdot\pi_3 + \pi_3 + \pi_3 + \pi_3 = 1$$
$$(4-p)\cdot\pi_3 = 1$$
$$\pi_3 = \frac{1}{4-p}$$

This also tells us that

$$\pi_0 = 1 - \pi_1 - \pi_2 - \pi_3$$

$$\pi_0 = 1 - \frac{1}{4-p} - \frac{1}{4-p} - \frac{1}{4-p}$$
$$\pi_0 = 1 - \frac{3}{4-p}$$
$$\pi_0 = \frac{4-p-3}{4-p}$$
$$\pi_0 = \frac{1-p}{4-p}$$

Our end goal was to find the limiting probability that it rains and no umbrella is available. By definition, the probability that it rains is p, and the limiting probability that no umbrella is available is $P(X_n = 0) = \pi_0$. This is equivalent to

$$p \cdot \pi_0 = p \cdot \frac{1-p}{4-p} = \boxed{\frac{p(1-p)}{4-p}}$$

3.

(a)

Restating the information from the problem to calculate the transition matrix:

- Dick is equally likely to throw it to Helen, Mark, Sam, and Tony. In other words, we have probability $\frac{1}{4}$ to each of them.
- Helen is equally likely to throw to Dick, Joni, Sam, and Tony. In other words, we have probability $\frac{1}{4}$ to each of them.
- Sam is equally likely to throw to Dick, Helen, Mark, and Tony. In other words, we have probability $\frac{1}{4}$ to each of them.
- Joni and Tony throw only to each other. So each have probability 1 to throw to each other.
- Mark runs away with the ball. So we have probability 1 to "throw to himself."

Let's create the matrix based on this information:

	Dick	Helen	Sam	Joni	Tony	Mark
Dick	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Helen	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
Sam	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
Joni	0	0	0	0	1	0
Tony	0	0	0	1	0	0
Mark	0	0	0	0	0	1

Looking at the matrix, we can see that Dick, Helen, and Sam are transient states/people since there's a chance that the ball is thrown to Joni, Tony, and Mark, who will never return the ball to any of the group of transient people.

Similarly, we can see that Joni, Tony, and Mark are recurrent states/people – Joni and Tony will only go back and forth with each other, and Mark will only go to himself.

(b)

We aim to find the probability that Mark will end up with the ball given that Dick has the ball at the beginning of the game. We can start by understanding the recurrent states:
- Mark will always get the ball if he has the ball, so $P(M|M) = 1$
- Tony and Joni go back and forth with each other, so $P(M|T) = 0$ and $P(M|J) = 0$
We aren't concerned with these cases as much as the other people, which we actually have to solve for. Starting with Dick's case:

$$P(M|D) = \frac{1}{4} \cdot P(M|H) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4} \cdot P(M|M) + \frac{1}{4} \cdot P(M|T)$$

$$P(M|D) = \frac{1}{4} \cdot P(M|H) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 0$$

$$P(M|D) = \frac{1}{4} \cdot P(M|H) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4}$$

For Helen:

$$P(M|H) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4} \cdot P(M|J) + \frac{1}{4} \cdot P(M|T)$$

$$P(M|H) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0$$

$$P(M|H) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|S)$$

Finally, for Sam:

$$P(M|S) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|H) + \frac{1}{4} \cdot P(M|T) + \frac{1}{4} \cdot P(M|M)$$

$$P(M|S) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|H) + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1$$

$$P(M|S) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|H) + \frac{1}{4}$$

Let's get to solving – let's substitute $P(M|H)$ into the equation for $P(M|D)$:

$$P(M|D) = \frac{1}{4} \cdot (\frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|S)) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4}$$

$$P(M|D) = \frac{1}{16} \cdot P(M|D) + \frac{1}{16} \cdot P(M|S) + \frac{1}{4} \cdot P(M|S) + \frac{1}{4}$$

$$P(M|D) = \frac{1}{16} \cdot P(M|D) + \frac{5}{16} \cdot P(M|S) + \frac{1}{4}$$

$$\frac{15}{16} \cdot P(M|D) = \frac{5}{16} \cdot P(M|S) + \frac{1}{4}$$

$$15 \cdot P(M|D) = 5 \cdot P(M|S) + 4$$

Let's now substitute $P(M|H)$ into the equation for $P(M|S)$:

$$P(M|S) = \frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot (\frac{1}{4} \cdot P(M|D) + \frac{1}{4} \cdot P(M|S)) + \frac{1}{4}$$

$$P(M|S) = \frac{1}{4} \cdot P(M|D) + \frac{1}{16} \cdot P(M|D) + \frac{1}{16} \cdot P(M|S) + \frac{1}{4}$$

$$\frac{15}{16} \cdot P(M|S) = \frac{5}{16} \cdot P(M|D) + \frac{1}{4}$$

$$15 \cdot P(M|S) = 5 \cdot P(M|D) + 4$$

$$15 \cdot P(M|S) - 4 = 5 \cdot P(M|D)$$

$$P(M|D) = 3 \cdot P(M|S) - \frac{4}{5}$$

We can now plug this into the first equation and solve:

$$15 \cdot P(M|D) = 5 \cdot P(M|S) + 4$$

$$15 \cdot (3 \cdot P(M|S) - \frac{4}{5}) = 5 \cdot P(M|S) + 4$$

$$45 \cdot P(M|S) - 12 = 5 \cdot P(M|S) + 4$$

$$40 \cdot P(M|S) = 16$$

$$P(M|S) = \frac{16}{40} = \frac{2}{5}$$

And subsequently,

$$P(M|D) = 3 \cdot P(M|S) - \frac{4}{5}$$

$$P(M|D) = 3 \cdot \frac{2}{5} - \frac{4}{5}$$

$$P(M|D) = \frac{6}{5} - \frac{4}{5} = \boxed{\frac{2}{5}}$$

4.

We essentially aim to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1 - a - b)\{P(X_n = 1) - \frac{b}{a+b}\}$$

As directed by the problem, we start with expanding $P(X_{n+1} = 1)$ based on the condition of the chain at n:

$$P(X_{n+1} = 1) = P(X_{n+1} = 1|X_n = 1) \cdot P(X_n = 1) + P(X_{n+1} = 1|X_n = 2) \cdot P(X_n = 2)$$

And we now plug in using the given P:

$$= (1 - a) \cdot P(X_n = 1) + b \cdot P(X_n = 2)$$

We can also plug in $P(X_n = 2) = 1 - P(X_n = 1)$:

$$= (1 - a) \cdot P(X_n = 1) + b \cdot (1 - P(X_n = 1))$$

$$= (1 - a) \cdot P(X_n = 1) + b - b \cdot P(X_n = 1)$$

$$= (1 - a - b) \cdot P(X_n = 1) + b$$

Subtract $\frac{b}{a+b}$ from both sides:

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot P(X_n=1)+b-\frac{b}{a+b}$$
$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot P(X_n=1)+\frac{b(a+b)-b}{a+b}$$
$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot P(X_n=1)+\frac{ab+b^2-b}{a+b}$$
$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot P(X_n=1)+\frac{ab+b^2-b}{a+b}$$
$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot P(X_n=1)-\frac{(1-a-b)\cdot b}{a+b}$$
$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\cdot \{P(X_n=1)-\frac{b}{a+b}\}$$

as we expected to find in the first step. Let's continue by trying to find something to fit the exponent of the $(1-a-b)$ term. We can more generally look at the terms as multiplying by a common factor of $(1-a-b)$ by substituting the $P(X_{n+1}=1)-\frac{b}{a+b}$ value. In other words, by replacing with some arbitrary $x_n = P(X_n=1)-\frac{b}{a+b}$, we can simplify the expression to be

$$x_{n+1}=(1-a-b)\cdot x_n$$

which yields the appropriate exponential form. In other words, we multiply by $(1-a-b)$ each time we get the next term, yielding $(1-a-b)^n$ term relative to x_0, i.e.

$$x_{n+1}=(1-a-b)^{n+1}\cdot x_0$$

or equivalently

$$x_n=(1-a-b)^n\cdot x_0$$

and substituting back:

$$P(X_n=1)-\frac{b}{a+b}=(1-a-b)^n\cdot \{P(X_0=1)-\frac{b}{a+b}\}$$
$$P(X_n=1)=\frac{b}{a+b}+(1-a-b)^n\cdot \{P(X_0=1)-\frac{b}{a+b}\}$$

as we exactly sought to find. Evaluating the last statement, given $0\le a+b\le 2$, we can find that $1-(a+b)$ has range $[-1,1]$, and given the exponential of n approaching infinity, the second term reaches 0, so $P(X_n = 1)$ does indeed converge to be just the first term of $\frac{b}{a+b}$.