

1.

(a)

We can find c by starting with the fact that the total integral of the density must equal 1, as it represents a probability, i.e.:

$$\int_0^1 \int_0^x cxy dy dx = 1$$

And we can integrate and solve from here:

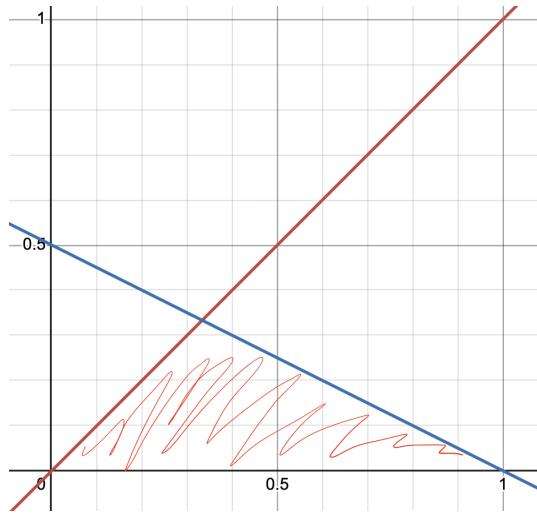
$$\begin{aligned} \int_0^1 cx \cdot \frac{y^2}{2} \Big|_0^x dx &= 1 \\ \int_0^1 cx \cdot \left(\frac{x^2}{2} - 0\right) dx &= 1 \\ \int_0^1 \frac{cx^3}{2} dx &= 1 \\ \frac{c}{2} \cdot \frac{x^4}{4} \Big|_0^1 &= 1 \\ \frac{c}{2} \cdot \left(\frac{1}{4} - 0\right) &= 1 \\ \frac{c}{8} &= 1 \\ \boxed{c = 8} \end{aligned}$$

(b)

To determine $P(X + 2Y \leq 1)$, we intersect the line of $y = -\frac{x}{2} + \frac{1}{2}$ with $y = x$ from the original problem statement:

$$\begin{aligned} x &= -\frac{x}{2} + \frac{1}{2} \\ \frac{3x}{2} &= \frac{1}{2} \\ x &= \frac{1}{3} \end{aligned}$$

So we have regions from $0 \leq x \leq \frac{1}{3}$ and $\frac{1}{3} \leq x \leq 1$. For the first region, we see that



$y = x$ is the binding equation, while we see that for the second region, $y = \frac{-x}{2} + \frac{1}{2}$ is the binding equation. Let's evaluate accordingly. Region 1:

$$\begin{aligned}
& \int_0^{\frac{1}{3}} \int_0^x 8xy dy dx \\
&= \int_0^{\frac{1}{3}} 8x \cdot \frac{y^2}{2} \Big|_0^x dx \\
&= \int_0^{\frac{1}{3}} 8x \cdot \left(\frac{x^2}{2} - 0\right) dx = \int_0^{\frac{1}{3}} 4x^3 dx \\
&= x^4 \Big|_0^{\frac{1}{3}} \\
&= \frac{1}{81}
\end{aligned}$$

Region 2:

$$\begin{aligned}
& \int_{\frac{1}{3}}^1 \int_0^{\frac{-x+1}{2}} 8xy dy dx \\
&= \int_{\frac{1}{3}}^1 8x \cdot \frac{y^2}{2} \Big|_0^{\frac{-x+1}{2}} dx \\
&= \int_{\frac{1}{3}}^1 8x \cdot \frac{(-x+1)^2}{4 \cdot 2} dx = \int_{\frac{1}{3}}^1 8x \cdot \frac{x^2 + 1 - 2x}{8} dx = \int_{\frac{1}{3}}^1 x^3 + x - 2x^2 dx \\
&= \frac{x^4}{4} + \frac{x^2}{2} - \frac{2x^3}{3} \Big|_{\frac{1}{3}}^1 \\
&= \left(\frac{1}{4} + \frac{1}{2} - \frac{2}{3}\right) - \left(\frac{1}{324} + \frac{1}{18} - \frac{2}{81}\right) = \frac{1}{12} - \frac{11}{324}
\end{aligned}$$

$$= \frac{27}{324} - \frac{11}{324} = \frac{16}{324} = \frac{4}{81}$$

So in total, we have $\frac{1}{81} + \frac{4}{81} = \boxed{\frac{5}{81}}$

(c)

To find $E(X|Y = y)$, we can start with calculating the conditional density $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. We have $f(x,y)$, so we need to find $f_Y(y)$. We can derive this from $f(x,y)$:

$$\begin{aligned} f_Y(y) &= \int_y^1 8xy dx \\ f_Y(y) &= 8y \cdot \frac{x^2}{2} \Big|_y^1 \\ f_Y(y) &= 8y \cdot \left(\frac{1}{2} - \frac{y^2}{2}\right) \\ f_Y(y) &= 4y - 4y^3 \end{aligned}$$

And plug in accordingly:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ f_{X|Y}(x|y) &= \frac{8xy}{4y - 4y^3} \\ f_{X|Y}(x|y) &= \frac{2x \cdot 4y}{4y(1 - y^2)} = \frac{2x}{1 - y^2} \end{aligned}$$

We can now finally find the expectation:

$$\begin{aligned} E(X|Y = y) &= \int_y^1 x \cdot f_{X|Y}(x|y) dx \\ E(X|Y = y) &= \int_y^1 x \cdot \frac{2x}{1 - y^2} dx \\ E(X|Y = y) &= \int_y^1 \frac{2x^2}{1 - y^2} dx \\ E(X|Y = y) &= \frac{2}{1 - y^2} \cdot \frac{x^3}{3} \Big|_y^1 \\ E(X|Y = y) &= \frac{2}{1 - y^2} \cdot \frac{1 - y^3}{3} \\ E(X|Y = y) &= \frac{2}{(1 + y)(1 - y)} \cdot \frac{(1 - y)(y^2 + y + 1)}{3} = \boxed{\frac{2(y^2 + y + 1)}{3(y + 1)}} \end{aligned}$$

(d)

To do this, we simply need to find $f_X(x)$:

$$\begin{aligned} f_X(x) &= \int_0^x 8xy dy \\ f_X(x) &= 8x \cdot \frac{y^2}{2} \Big|_0^x \\ f_X(x) &= 8x \cdot \left(\frac{x^2}{2} - 0\right) = 4x^3 \end{aligned}$$

and thusly find $E(X)$:

$$\begin{aligned} E(X) &= \int_0^1 x \cdot f_X(x) dx \\ E(X) &= \int_0^1 x \cdot 4x^3 dx = \int_0^1 4x^4 dx \\ E(X) &= \frac{4x^5}{5} \Big|_0^1 \\ E(X) &= \frac{4}{5} - 0 = \boxed{\frac{4}{5}} \end{aligned}$$

2.

(a)

(b)

3.

Find $P(X^2 < Y < X)$ where $f(x, y) = 2x$ for $0 \leq x, y \leq 1$ [1].

We first establish the bounds – given that we need $X^2 < Y < X$, it is obvious that we must have $0 < x < 1$. Given these two bounds, we can now establish the double integral for representing the joint pdf and evaluate:

$$\begin{aligned} P(X^2 < Y < X) &= \int_0^1 \int_{x^2}^x 2x dy dx \\ P(X^2 < Y < X) &= \int_0^1 2x \Big|_{x^2}^x dx \end{aligned}$$

$$\begin{aligned}
P(X^2 < Y < X) &= \int_0^1 2x^2 - 2x^3 dx \\
P(X^2 < Y < X) &= \frac{2x^3}{3} - \frac{2x^4}{4} \Big|_0^1 \\
P(X^2 < Y < X) &= \left(\frac{2}{3} - \frac{2}{4}\right) - (0 - 0) \\
P(X^2 < Y < X) &= \boxed{\frac{1}{6}}
\end{aligned}$$

4.

(a)

Let's begin by calculating the marginal distributions from the table.

$$P(X = 1) = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$$

$$P(X = 2) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$$

$$P(X = 3) = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$$

$$P(Y = 2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$$

$$P(Y = 3) = \frac{1}{6} + 0 + \frac{1}{6} = \frac{1}{3}$$

$$P(Y = 4) = 0 + \frac{1}{3} + 0 = \frac{1}{3}$$

Now, we can simply check for independence. We know that X and Y are independent if $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$. Most values fulfill the condition, but we find that for $Y = 3$ and $X = 1$:

$$P(X = 1, Y = 3) = P(X = 1) \cdot P(Y = 3)$$

$$\frac{1}{6} = \frac{1}{4} \cdot \frac{1}{3}$$

$$\frac{1}{6} \neq \frac{1}{12}$$

So X and Y are not independent, and therefore dependent.

(b)

We can do this by simply recreating the table, except multiply the marginals together. This ensures that they are independent, taking $X = U$ and $Y = V$:

	$U = 1$	$U = 2$	$U = 3$
$V = 2$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$
$V = 3$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$
$V = 4$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$

	$U = 1$	$U = 2$	$U = 3$
$V = 2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$V = 3$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$V = 4$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

5.

6.

We are given the distribution of the average of the independent measurements as $\bar{X} \sim N(\mu, \frac{25}{n})$. We apply the Central Limit Theorem after standardizing the probability expression to $N(0, 1)$ with $Z = \frac{\bar{X} - \mu}{\sigma}$:

$$P(|\bar{X} - \mu| < 1)$$

$$P\left(\frac{-1}{\sqrt{\frac{25}{n}}} < Z < \frac{1}{\sqrt{\frac{25}{n}}}\right)$$

$$P\left(-\frac{\sqrt{n}}{5} < Z < \frac{\sqrt{n}}{5}\right)$$

Since we are looking for tails on either side, we look for 2 symmetrical regions that give us the the remaining $1 - 0.95 = 0.05$:

$$2\phi\left(\frac{\sqrt{n}}{5}\right) = 0.05$$

$$\phi\left(\frac{\sqrt{n}}{5}\right) = 0.025$$

$$\frac{\sqrt{n}}{5} = z_{0.025}$$

We now do a lookup for the corresponding z -value, and substitute:

$$\frac{\sqrt{n}}{5} = 1.96$$

$$\sqrt{n} = 9.8$$

$$n = 96.04$$

We round up as we need a whole number, which gives us the final answer that we need the number of measurements $n \geq 97$.

7.

(a)

We run the code a few times, reducing n and keeping ϵ constant:

```
> limitTheorem(100, 0.5)
[1] 0.9874
> limitTheorem(50, 0.5)
[1] 0.928
> limitTheorem(20, 0.5)
[1] 0.7522
> limitTheorem(5, 0.5)
[1] 0.4088
```

Here, we see that law of large numbers applies with higher n , with a higher fraction of numbers landing within the $\epsilon = 0.5$ of the mean $\mu = 4.2$.

We now run reducing ϵ and keeping n constant:

```
> limitTheorem(50, 1)
[1] 0.9992
> limitTheorem(50, 0.5)
[1] 0.9179
> limitTheorem(50, 0.1)
[1] 0.2635
```

We see that the fraction reduces as we expect it to.

(b)

While we already showed this previously, something we can do to further show this is to further standardize the averages, and we see that the fraction further decreases and the histogram becomes more normalized:

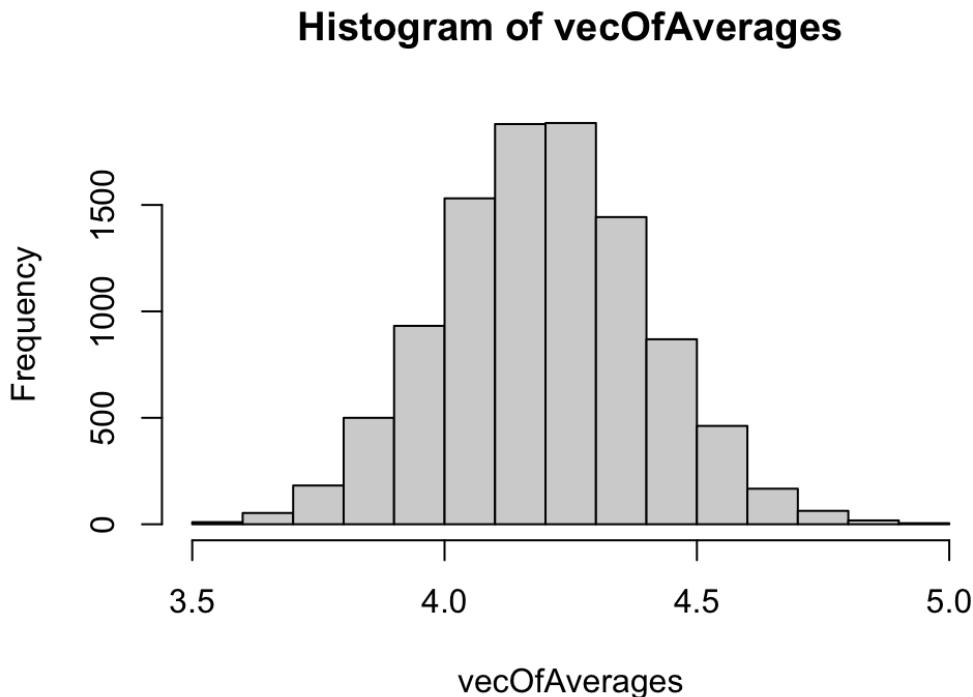
```
> limitTheorem = function(n,epsilon) {
+
+   vecOfAverages = c(rep(0,10000))
+   for (i in 1:10000) {
+     smpl = rpois(n,4.2)
+     avg = mean(smpl);
+     vecOfAverages[i] = avg;
+   }
+   standardizedAverages = (vecOfAverages - 4.2) / sqrt(4.2/n)
+   hist(standardizedAverages);
+   vecOfDifferences = standardizedAverages - 4.2*c(rep(1,10000));
+   nmbCloseToTruth = length(vecOfDifferences[abs(vecOfDifferences) <= epsilon])
+   print(nmbCloseToTruth/10000);
```

```

+ }
>
> limitTheorem(100, 0.5)
[1] 1e-04

```

Before: After:



8.

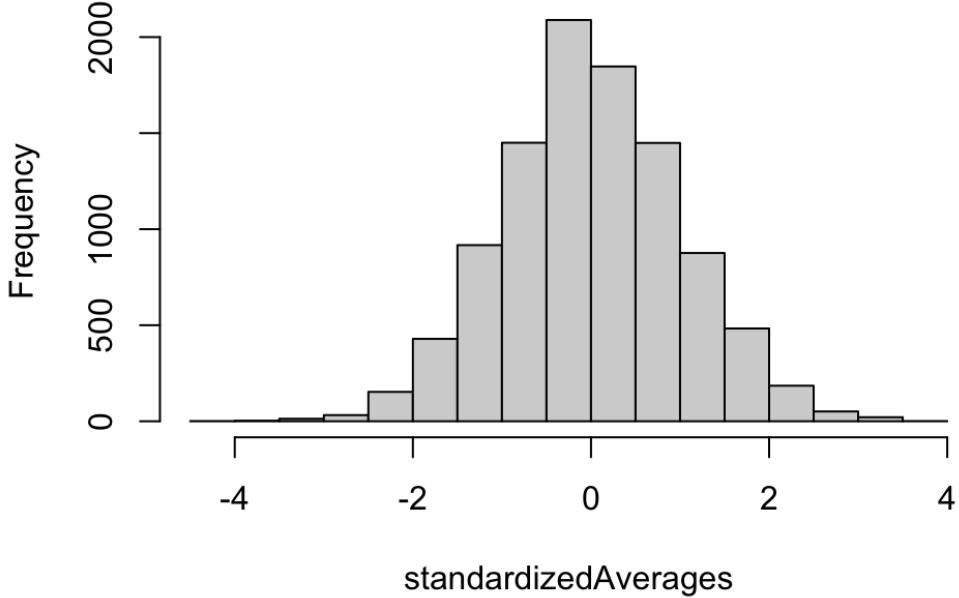
We are given the exponential distribution with probability density $f(x|\lambda) = \lambda e^{-\lambda x}$. We first take the proper expression for the likelihood function corresponding to the n observed values of X :

$$L(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i}$$

which we can easily simplify by multiplying to get

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

Histogram of standardizedAverages



We then proceed to take the log likelihood:

$$l(\lambda) = \ln(\lambda^n e^{-\lambda \sum_{i=1}^n x_i})$$

$$l(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

We then try to maximize this, so find a critical point by taking the derivative and setting equal to 0:

$$0 = \frac{dl}{d\lambda}$$

$$0 = \frac{d}{d\lambda} (n \ln(\lambda) - \lambda \sum_{i=1}^n x_i)$$

$$0 = \frac{n}{\lambda} - \sum_{i=1}^n x_i$$

$$\sum_{i=1}^n x_i = \frac{n}{\lambda}$$

$$\lambda \sum_{i=1}^n x_i = n$$

$$\lambda = \frac{n}{\sum_{i=1}^n x_i}$$

We can also ensure that this is a maximum by taking the derivative again:

$$\begin{aligned}\frac{d^2l}{d\lambda^2} &= \frac{d}{d\lambda}\left(\frac{n}{\lambda} - \sum_{i=1}^n x_i\right) \\ \frac{d^2l}{d\lambda^2} &= -\frac{n}{\lambda^2} - 0 = -\frac{n}{\lambda^2}\end{aligned}$$

The original probability density must be positive, so we know that $\lambda > 0$ always. We see that this value of $\frac{d^2l}{d\lambda^2} < 0$ for all $\lambda > 0$, which means that this is indeed a maximum. We can now finally conclude that $\hat{\lambda}_{\text{MLE}} = \frac{n}{\sum_{i=1}^n X_i} = \boxed{\frac{1}{\bar{X}}}$