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Homework 2

1.

(a)

(b)

(c)

(d)

2.

(a)

(b)

3.

Find $P(X^2 < Y < X)$ where $f(x, y) = 2x$ for $0 \leq x, y \leq 1$ [1].

We first establish the bounds – given that we need $X^2 < Y < X$, it is obvious that we must have $0 < x < 1$. Given these two bounds, we can now establish the double integral for representing the joint pdf and evaluate:

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^x 2x dy dx$$

$$P(X^2 < Y < X) = \int_0^1 2x|_{x^2}^x dx$$

$$P(X^2 < Y < X) = \int_0^1 2x^2 - 2x^3 dx$$

$$P(X^2 < Y < X) = \frac{2x^3}{3} - \frac{2x^4}{4}|_0^1$$

$$P(X^2 < Y < X) = (\frac{2}{3} - \frac{2}{4}) - (0 - 0)$$

$$\boxed{P(X^2 < Y < X) = \frac{1}{6}}$$

4.

(a)

Let's begin by calculating the marginal distributions from the table.

$$P(X = 1) = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$$

$$P(X = 2) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$$

$$P(X = 3) = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$$

$$P(Y = 2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$$

$$P(Y = 3) = \frac{1}{6} + 0 + \frac{1}{6} = \frac{1}{3}$$

$$P(Y = 4) = 0 + \frac{1}{3} + 0 = \frac{1}{3}$$

Now, we can simply check for independence. We know that X and Y are independent if $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$. Most values fulfill the condition, but we find that for $Y = 3$ and $X = 1$:

$$P(X = 1, Y = 3) = P(X = 1) \cdot P(Y = 3)$$
$$\frac{1}{6} = \frac{1}{4} \cdot \frac{1}{3}$$

$$\frac{1}{6} \neq \frac{1}{12}$$

So X and Y are not independent, and therefore dependent.

(b)

We can do this by simply recreating the table, except multiply the marginals together. This ensures that they are independent, taking $X = U$ and $Y = V$:

	$U = 1$	$U = 2$	$U = 3$
$V = 2$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$
$V = 3$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$
$V = 4$	$\frac{1}{4} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{4} \cdot \frac{1}{3}$

	$U = 1$	$U = 2$	$U = 3$
$V = 2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$V = 3$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$V = 4$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

5.

6.

We are given the distribution of the average of the independent measurements as $\bar{X} \sim N\left(\mu, \frac{25}{n}\right)$. We apply the Central Limit Theorem after standardizing the probability expression to $N(0, 1)$ with $Z = \frac{\bar{X}-\mu}{\sigma}$:

$$P(|\bar{X} - \mu| < 1)$$
$$P(\frac{-1}{\sqrt{\frac{25}{n}}} < Z < \frac{1}{\sqrt{\frac{25}{n}}})$$
$$P(-\frac{\sqrt{n}}{5} < Z < \frac{\sqrt{n}}{5})$$

Since we are looking for tails on either side, we look for 2 symmetrical regions that give us the the remaining $1 - 0.95 = 0.05$:

$$2\phi(\frac{\sqrt{n}}{5}) = 0.05$$

$$\phi(\frac{\sqrt{n}}{5}) = 0.025$$
$$\frac{\sqrt{n}}{5} = z_{0.025}$$

We now do a lookup for the corresponding z-value, and substitute:

$$\frac{\sqrt{n}}{5} = 1.96$$
$$\sqrt{n} = 9.8$$
$$n = 96.04$$

We round up as we need a whole number, which gives us the final answer that we need the number of measurements $\boxed{n \geq 97}$.

7.

(a)

(b)

8.

We are given the exponential distribution with probability density $f(x|\lambda) = \lambda e^{-\lambda x}$. We first take the proper expression for the likelihood function corresponding to the n observed values of X:

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

which we can easily simplify by multiplying to get

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

We then proceed to take the log likelihood:

$$l(\lambda) = \ln(\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i})$$
$$l(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

We then try to maximize this, so find a critical point by taking the derivative and setting equal to 0:

$$0 = \frac{dl}{d\lambda}$$

$$0 = \frac{d}{d\lambda}(n\ln(\lambda) - \lambda\sum_{i=1}^{n} x_i)$$

$$0 = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = \frac{n}{\lambda}$$

$$\lambda\sum_{i=1}^{n} x_i = n$$

$$\lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

We can also ensure that this is a maximum by taking the derivative again:

$$\frac{d^2l}{d\lambda^2} = \frac{d}{d\lambda}(\frac{n}{\lambda} - \sum_{i=1}^{n} x_i)$$

$$\frac{d^2l}{d\lambda^2} = -\frac{n}{\lambda^2} - 0 = -\frac{n}{\lambda^2}$$

The original probability density must be positive, so we know that $\lambda > 0$ always. We see that this value of $\frac{d^2l}{d\lambda^2} < 0$ for all $\lambda > 0$, which means that this is indeed a maximum. We can now finally conclude that $\hat{\lambda}_{\text{MLE}} = \frac{n}{\sum_{i=1}^{n} X_i} = \boxed{\frac{1}{\bar{X}}}$