Michael Huang
EN.625.633.81
Homework 4

1.

(a)

To find $P(X_1 = 2)$, we can use the property that $\pi^{(t)} = \pi^{(0)}P^t$, and do the simple weighted probability calculation accordingly:

$$P(X_1 = 2) = \sum_{i=0}^{2} P(X_0 = i) \cdot P(X_1 = 2|X_0 = i)$$

$$= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22}$$

$$= 0.2 \cdot 0.6 + 0.5 \cdot 0.2 + 0.3 \cdot 0.2$$

$$= 0.12 + 0.10 + 0.06$$

$$= \boxed{0.28}$$

(b)

To find $P(X_2 = 2)$, we use the same property as before. We therefore need to calculate $P^t = P^2$ in this case to get started:

$$P^2 = \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} (.3)(.3) + (.1)(.4) + (.6)(.1) & (.3)(.1) + (.1)(.4) + (.6)(.7) & (.3)(.6) + (.1)(.2) + (.6)(.2) \\ (.4)(.3) + (.4)(.4) + (.2)(.1) & (.4)(.1) + (.4)(.4) + (.2)(.7) & (.4)(.6) + (.4)(.2) + (.2)(.2) \\ (.1)(.3) + (.7)(.4) + (.2)(.1) & (.1)(.1) + (.7)(.4) + (.2)(.7) & (.1)(.6) + (.7)(.2) + (.2)(.2) \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.09 + 0.04 + 0.06 & 0.03 + 0.04 + 0.42 & 0.18 + 0.02 + 0.12 \\ 0.12 + 0.16 + 0.02 & 0.04 + 0.16 + 0.14 & 0.24 + 0.08 + 0.04 \\ 0.03 + 0.28 + 0.02 & 0.01 + 0.28 + 0.14 & 0.06 + 0.14 + 0.04 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.19 & 0.49 & 0.32 \\ 0.30 & 0.34 & 0.36 \\ 0.33 & 0.43 & 0.24 \end{pmatrix}$$

We can now plug in again:

$$P(X_2 = 2) = \pi^{(0)} \cdot \begin{pmatrix} 0.32 \\ 0.36 \\ 0.24 \end{pmatrix}$$

$$= 0.2 \cdot 0.32 + 0.5 \cdot 0.36 + 0.3 \cdot 0.24$$

$$= 0.064 + 0.18 + 0.072$$

$$= \boxed{0.316}$$

(c)

To find $P(X_3 = 2|X_0 = 0)$, we can use theorem 1 to take this as the $(0, 2)^{th}$ element in P^3. So we first need to find P^3:

$$P^3 = P^2 \cdot P$$

$$P^3 = \begin{pmatrix} 0.19 & 0.49 & 0.32 \\ 0.30 & 0.34 & 0.36 \\ 0.33 & 0.43 & 0.24 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} .19 \cdot .3 + .49 \cdot .4 + .32 \cdot .1 & .19 \cdot .1 + .49 \cdot .4 + .32 \cdot .7 & .19 \cdot .6 + .49 \cdot .2 + .32 \cdot .2 \\ .30 \cdot .3 + .34 \cdot .4 + .36 \cdot .1 & .30 \cdot .1 + .34 \cdot .4 + .36 \cdot .7 & .30 \cdot .6 + .34 \cdot .2 + .36 \cdot .2 \\ .33 \cdot .3 + .43 \cdot .4 + .24 \cdot .1 & .33 \cdot .1 + .43 \cdot .4 + .24 \cdot .7 & .33 \cdot .6 + .43 \cdot .2 + .24 \cdot .2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.057 + 0.196 + 0.032 & 0.019 + 0.196 + 0.224 & 0.114 + 0.098 + 0.064 \\ 0.090 + 0.136 + 0.036 & 0.030 + 0.136 + 0.252 & 0.180 + 0.068 + 0.072 \\ 0.099 + 0.172 + 0.024 & 0.033 + 0.172 + 0.168 & 0.198 + 0.086 + 0.048 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.285 & 0.439 & 0.276 \\ 0.262 & 0.418 & 0.320 \\ 0.295 & 0.373 & 0.332 \end{pmatrix}$$

Taking the corresponding element yields that $\boxed{P(X_3 = 2|X_0 = 0) = 0.276}$.

(d)

To find $P(X_0 = 1|X_1 = 2)$, we can first decompose this expression into parts that we can calculate more easily using Bayes' theorem:

$$P(X_0 = 1|X_1 = 2) = \frac{P(X_1 = 2|X_0 = 1) \cdot P(X_0 = 1)}{P(X_1 = 2)}$$

We can first substitute using P, directly taking the $(1, 2)^{th}$ element:

$$= \frac{0.2 \cdot P(X_0 = 1)}{P(X_1 = 2)}$$

Next, we can substitute using $\pi^{(0)}$:

$$= \frac{0.2 \cdot 0.5}{P(X_1 = 2)}$$

Finally, we can substitute using what we found in part (a):

$$= \frac{0.10}{0.28} = \frac{10}{28} = \boxed{\frac{5}{14}}$$

(e)

To find $P(X_1 = 1, X_3 = 1)$, we can first break this down by calculating the probabilities of the full chain, taking all states into account. In other words, we can set up the expression such that

$$P(X_1 = 1, X_3 = 1) = \sum_{i=0}^{2} P(X_0 = i) \cdot P(X_1 = 1|X_0 = i) \cdot P(X_3 = 1|X_1 = 1)$$

$$= P(X_3 = 1|X_1 = 1) \sum_{i=0}^{2} \pi_i \cdot P_{i,1}$$

Given the properties of the Markov chain, we can just treat this as a normal 2-step transition and therefore use P^2 for this. So we can equate this to $P^2_{1,1} \cdot P(X_1 = 1)$. Plug this in and evaluate these terms, given what we've already previously calculated:

$$= P^2_{1,1} \cdot P(X_1 = 1)$$

$$= 0.34 \cdot (\pi_0 \cdot P_{0,1} + \pi_1 \cdot P_{1,1} + \pi_2 \cdot P_{2,1})$$

$$= 0.34 \cdot (0.2 \cdot 0.1 + 0.5 \cdot 0.4 + 0.3 \cdot 0.7)$$

$$= 0.34 \cdot (0.02 + 0.2 + 0.21)$$

$$= 0.34 \cdot 0.43$$

$$= \boxed{0.1462}$$

2.

3.

(a)

(b)

4.

We essentially aim to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\{P(X_n = 1) - \frac{b}{a+b}\}$$

As directed by the problem, we start with expanding $P(X_{n+1} = 1)$ based on the condition of the chain at n:

$$P(X_{n+1} = 1) = P(X_{n+1} = 1 | X_n = 1) \cdot P(X_n = 1) + P(X_{n+1} = 1 | X_n = 2) \cdot P(X_n = 2)$$

And we now plug in using the given P:

$$= (1-a) \cdot P(X_n = 1) + b \cdot P(X_n = 2)$$

We can also plug in $P(X_n = 2) = 1 - P(X_n = 1)$:

$$= (1-a) \cdot P(X_n = 1) + b \cdot (1 - P(X_n = 1))$$

$$= (1-a) \cdot P(X_n = 1) + b - b \cdot P(X_n = 1)$$

$$= (1-a-b) \cdot P(X_n = 1) + b$$

Subtract $\frac{b}{a+b}$ from both sides:

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot P(X_n = 1) + b - \frac{b}{a+b}$$

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot P(X_n = 1) + \frac{b(a+b) - b}{a+b}$$

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot P(X_n = 1) + \frac{ab + b^2 - b}{a+b}$$
$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot P(X_n = 1) + \frac{ab + b^2 - b}{a+b}$$
$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot P(X_n = 1) - \frac{(1-a-b) \cdot b}{a+b}$$
$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \cdot \{P(X_n = 1) - \frac{b}{a+b}\}$$

as we expected to find in the first step. Let's continue by trying to find something to fit the exponent of the $(1-a-b)$ term. We can more generally look at the terms as multiplying by a common factor of $(1-a-b)$ by substituting the $P(X_{n+1} = 1) - \frac{b}{a+b}$ value. In other words, by replacing with some arbitrary $x_n = P(X_n = 1) - \frac{b}{a+b}$, we can simplify the expression to be

$$x_{n+1} = (1-a-b) \cdot x_n$$

which yields the appropriate exponential form. In other words, we multiply by $(1-a-b)$ each time we get the next term, yielding $(1-a-b)^n$ term relative to x_0, i.e.

$$x_{n+1} = (1-a-b)^{n+1} \cdot x_0$$

or equivalently

$$x_n = (1-a-b)^n \cdot x_0$$

and substituting back:

$$P(X_n = 1) - \frac{b}{a+b} = (1-a-b)^n \cdot \{P(X_0 = 1) - \frac{b}{a+b}\}$$
$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \cdot \{P(X_0 = 1) - \frac{b}{a+b}\}$$

as we exactly sought to find. Evaluating the last statement, given $0 \leq a + b \leq 2$, we can find that $1-(a+b)$ has range $[-1, 1]$, and given the exponential of n approaching infinity, the second term reaches 0, so $P(X_n = 1)$ does indeed converge to be just the first term of $\frac{b}{a+b}$.