Homework 4

625.433

1. (25 pts.) Let X_n be a Markov chain with state space $\{0, 1, 2\}$, and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

and initial distribution $\pi = (0.2, 0.5, 0.3)$. Calculate

(a) $P(X_1 = 2)$

(b) $P(X_2 = 2)$

(c) $P(X_3 = 2|X_0 = 0)$

(d) $P(X_0 = 1|X_1 = 2)$

(e) $P(X_1 = 1, X_3 = 1)$.

2. (25 pts.) Ms. Ella Brum walks back and forth between her home and her office every day. She owns three umbrellas, which are distributed over two umbrella stands (one at home and one at work). When it is not raining, Ms. Brum walks without an umbrella. When it is raining, she takes one umbrella from the stand at the place of her departure, provided there is one available. Suppose that the probability that it is raining at the time of any departure is p. Let X_n denote the number of umbrellas available at the place where Ella arrives afgter walk number n; $n = 1, 2, \dots$, including the one that she possibly brings with her. Calculate the limiting probability that it rains and no umbrella is available.

3. (25 pts.) Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam and Tony. If Helen has the ball, she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball, he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it.

(a) Find the transition probability matrix, and identify which of the states are transient and which are recurrent.

(b) Suppose Dick has the ball at the beginning of the game. What is the probability that Mark will end up with it?

4. (25 pts.) Consider a general chain with state space $S = \{1, 2\}$ and write the transition probability as

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

Use the Markov property to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\left\{P(X_n = 1) - \frac{b}{a+b}\right\}$$

and then conclude

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \left\{P(X_0 = 1) - \frac{b}{a+b}\right\},$$

This shows that if $0 \leq a+b \leq 2$, then $P(X_n = 1)$ converges exponentially fast to its limiting value $\frac{b}{a+b}$,