Homework 3

625.433

1. (10 pts.) Suppose that the time to repair a machine is exponentially distributed random variable with mean 2. (a) What is the probability the repair takes more than 2 hours. (b) What is the probability that the repair takes more than 5 hours given that it takes more than three hours.

2. (10 pts.) Assume $X \sim \text{Exp}(\lambda)$. Prove that for any value $a > 0$,

$$P(X \geq a + t | X \geq a) = P(X \geq t).$$

   This is referred to as the memoryless property of the exponential distribution. Given X hasn't happened after time a, the probability that X is t units away does not change at all.

3. (10 pts.) Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes. (a) What is the expected total time for Carol to complete her business? (b) What is the expected total time until the last of the three customers leaves? Hint: use the memoryless property of the exponential.

4. (10 pts.) A machine has two critically important parts and is subject to three different types of shocks. Shocks of type I occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let U and V be the failure times of the two parts. (a) Find $P(U > s, V > t)$. (b) Find the distribution of U and the distribution of V. (c) Are U and V independent?

5. (10 pts.) Customers arrive at a bank according to a Poisson process with $\lambda = 10$ per hour. Given that two customers arrived in the first five minutes, what is the probability that

   (a) Both arrived in the first two minutes.

   (b) At least one arrived in the first two minutes.

6. (15 pts.) Let $N(t)$ be a Poisson process with rate $\lambda = 2$. Find

   (a) $P(N(2) = 1, N(3) = 4, N(5) = 5)$

   (b) $P(N(4) = 3 | N(2) = 1, N(3) = 2)$

   (c) $E(N(4) | N(2) = 2)$

7. (15 pts.) In the program `PoissonProcess.R`, you can simulate a Poisson process from scratch. It allows you to do the following *nmbSim* times (*nmbSim* is the total number of simulations): simulate *tot* exponentials with parameter λ and count the fraction of times n customers arrive before time s. As you've learned in this module, the number of occurrences before time s is distributed as a Poisson random variable. Run the program I've written for you, and use the function `dpois` in `R` to illustrate that the number occurrences before time s does, in fact, follow a Poisson random variable.

8. (10 pts.) Illustrate (with the `R` program I wrote `differentArrivalProbs.R`) Theorem 1 and Corollary 1. That is, show how the accuracy of the Poisson approximation varies with the maximum of the arrival probabilities.

9. (10 pts.) Consider the context of Example 4 in the notes.

   (a) What is the distribution of the number of arrivals in a day?

   (b) What is the probability that no one arrives before noon?