Homework 2

625.433

1. (15 points). Let X and Y have joint density f given by

$$f(x, y) = cxy \quad 0 \le y \le x, \quad 0 \le x \le 1.$$

   (a) Determine the normalization constant c.
   (b) Determine $P(X + 2Y \le 1)$.
   (c) Find $E(X|Y = y)$.
   (d) Find $E(X)$.

2. (15 points). Let $\mathbf{X}$ be $N(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu}^T = (2, -3, 1)$ and

$$\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

   (a) Find the distribution of $3X_1 - 2X_2 + X_3$.
   (b) Find a 2×1 vector $\mathbf{a}$ such that X_2 and $X_2 - \mathbf{a}^T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ are independent.

3. (10 points). Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with pdf

$$f(x, y) = 2x \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

4. (15 points). The random pair (X, Y) has the distribution

		X		
		1	2	3
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
Y	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

   (a) Show that X and Y are dependent.
   (b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

5. (15 points). Suppose that $X_1, X_2, \ldots, X_{20}$ are independent random variables with density functions $f(x) = 2x \quad 0 \le x \le 1$. Let $S = X_1 + X_2 + \cdots + X_{20}$. Use the central limit theorem to approximate $P(S \le 10)$.

6. (10 points). Suppose that a measurement has mean μ and variance $\sigma^2 = 25$. Let $\overline{X}$ be the average of n such independent measurements. How large should n be so that $P\left(|\overline{X} - \mu| < 1\right) = 0.95$?

7. (10 points). The file `RPrograms_Week2.R` contains an R program entitled 'limitTheorem'. This function generates 10000 samples of size n from a Poisson distribution with $\lambda = 4.2$ and then calculates the fraction of sample averages that are within ϵ of $\mu = 4.2$.

   (a) Use this function to illustrate the law of large numbers and the central limit theorem. In other words, show that the smaller ϵ becomes, the smaller the fraction becomes.
   (b) Change the function a little to illustrate the central limit theorem.

8. (10 points). Suppose $X_1, X_2, \ldots, X_n$ are an i.i.d. sample from the exponential distribution with probability density $f(x|\lambda) = \lambda \exp\{-x \cdot \lambda\}$.. What is the maximum likelihood estimator for λ?