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Homework 3

1.

(a)

Given mean $\mu = 2$ with the exponential distribution, we can directly plug in:

$$P(X > (s-t)) = e^{-\lambda \cdot (s-t)}$$
$$P(X > 2) = e^{-\frac{1}{\mu} \cdot 2}$$
$$P(X > 2) = e^{-\frac{1}{2} \cdot 2}$$
$$P(X > 2) = e^{-1} = \boxed{0.36787944117}$$

(b)

We set up the equivalent conditional property expression:

$$P(X > 5|X > 3) = \frac{P(X > 5 \cap X > 3)}{P(X > 3)}$$

Logically, we know that if $P(X > 5)$, $P(X > 3)$ for sure, so we can remove the latter part, i.e. $P(X > 5 \cap X > 3) = P(X > 5)$. Plugging back in:

$$P(X > 5|X > 3) = \frac{P(X > 5)}{P(X > 3)}$$

And now using the exponential pdf:

$$P(X > 5|X > 3) = \frac{e^{-5\lambda}}{e^{-3\lambda}}$$
$$P(X > 5|X > 3) = \frac{e^{-5 \cdot \frac{1}{2}}}{e^{-3 \cdot \frac{1}{2}}}$$
$$P(X > 5|X > 3) = \frac{e^{\frac{-5}{2}}}{e^{\frac{-3}{2}}}$$
$$P(X > 5|X > 3) = e^{-1} = \boxed{0.36787944117}$$

2.

Given $X \sim \text{Exp}(\lambda)$, we aim to prove that $P(X \geq a+t|X \geq a) = P(X \geq t)$:

$$
\begin{aligned}
P(X \geq a+t|X \geq a) &= \frac{P(X \geq a+t \cap X \geq a)}{P(X \geq a)} && \text{Conditional probability} \\
&= \frac{P(X \geq a+t)}{P(X \geq a)} && \text{As above, } X \geq a+t \text{ implies } X \geq a \\
&= \frac{e^{-\lambda \cdot (a+t)}}{e^{-\lambda a}} && \text{Definition of exponential distribution} \\
&= \frac{e^{-\lambda a} e^{-\lambda t}}{e^{-\lambda a}} \\
&= e^{-\lambda t} \\
&= P(X \geq t)
\end{aligned}
$$

as we sought to show.

3.

4.

5.

(a)

We are given that $N(5) = 2$, and Poisson process $\lambda = 10$. As the arrival times are uniformly distributed, $N(t) = n$ in time $[0, s]$ follows the Binomial with $p = \frac{s}{t}$. We therefore can directly use the formula to calculate $P(k = 2)$:

$$P(N(s) = l|N(t) = k) = \binom{k}{l}(\frac{s}{t})^l(1 - \frac{s}{t})^{k-l}$$

$$P(N(s) = 2|N(5) = 2) = \binom{2}{2}(\frac{2}{5})^2(1 - \frac{2}{5})^{2-2}$$

$$P(N(s) = 2|N(5) = 2) = \binom{2}{2}(\frac{2}{5})^2(\frac{3}{5})^0$$

$$P(N(s) = 2|N(5) = 2) = 1 \cdot \frac{4}{25} \cdot 1 = \boxed{\frac{4}{25}}$$

(b)

The probability that at least one customer arrived in the first two minutes is easier found by taking its complement, i.e. 1 - the probability that no customers arrived in the first two minutes:

$$P(N(s) = l|N(t) = k) = \binom{k}{l}(\frac{s}{t})^l(1 - \frac{s}{t})^{k-l}$$

$$P(N(2) = 0|N(5) = 2) = \binom{2}{0}(\frac{2}{5})^0(1 - \frac{2}{5})^{2-0}$$

$$P(N(2) = 0|N(5) = 2) = 1 \cdot 1 \cdot (\frac{3}{5})^2$$

$$P(N(2) = 0|N(5) = 2) = \frac{9}{25}$$

Taking the compliment, we see that

$$P(N(2) \geq 1|N(5) = 2) = 1 - \frac{9}{25} = \boxed{\frac{16}{25}}$$

6.

(a)

We are given Poisson $N(t)$ with $\lambda = 2$. To find $P(N(2) = 1, N(3) = 4, N(5) = 5)$, we can break this up into independent periods with how many events occur within each period, and then multiply. Let's begin:
1 event from $[0, 1]$:

$$P[N(s) = n] = \frac{e^{-\lambda s} \cdot (\lambda s)^n}{n!}$$

$$P[N(2) = 1] = \frac{e^{-2 \cdot 2} \cdot (2 \cdot 2)^1}{1!}$$

$$P[N(2) = 1] = \frac{e^{-4} \cdot 4^1}{1} = 4e^{-4}$$

4-1 = 3 events from $[2, 3]$:

$$P[N(3 - 2) = 3] = \frac{e^{-2 \cdot (3-2)} \cdot (2 \cdot (3 - 2))^3}{3!}$$

$$P[N(1) = 3] = \frac{e^{-2} \cdot 2^3}{6} = \frac{4e^{-2}}{3}$$

5-4 $= 1$ event from $[3, 5]$:

$$P[N(5-3) = 1] = \frac{e^{-2\cdot(5-3)} \cdot (2 \cdot (5-3))^1}{1!}$$
$$P[N(2) = 1] = \frac{e^{-2\cdot 2} \cdot (2 \cdot 2)^1}{1} = 4e^{-4}$$

So finally,

$$P(N(2) = 1, N(3) = 4, N(5) = 5) = P(N(2) = 1) \cdot P(N(3) - N(2) = 3) \cdot P(N(5) - N(3) = 1)$$
$$= 4e^{-4} \cdot \frac{4e^{-2}}{3} \cdot 4e^{-4} = \frac{4 \cdot 4 \cdot 4}{3} e^{-4-2-4}$$
$$= \frac{64}{3} e^{-10} = \boxed{0.00096853183}$$

(b)

To find $P(N(4) = 3|N(2) = 1, N(3) = 2)$, we can first remove the $N(2) = 1$ part because we know that $N(3) = 2$ is the only relevant part given the independent time increments. Therefore, we just need to find $P(N(4) = 3|N(3) = 2)$:

$$P(N(4) = 3|N(3) = 2) = P[N(4-3) = 3-2]$$
$$P[N(s) = n] = \frac{e^{-\lambda s} \cdot (\lambda s)^n}{n!}$$
$$P[N(1) = 1] = \frac{e^{-2\cdot 1} \cdot (2 \cdot 1)^1}{1!}$$
$$P[N(1) = 1] = \frac{e^{-2} \cdot 2^1}{1}$$
$$P[N(1) = 1] = 2e^{-2} = \boxed{0.27067056647}$$

(c)

$$
\begin{aligned}
E(N(4)|N(2) = 2) && \text{Given} \\
E(N(4) - N(2) + N(2)|N(2) = 2) && N(4) \text{ in independent increments} \\
E[N(4) - N(2)|N(2) = 2] + E[N(2)|N(2) = 2] && \text{Linearity of expectation} \\
E[N(4) - N(2)|N(2) = 2] + 2 && \text{Definition of expectation} \\
E[N(4) - N(2)] + 2 && \text{Independent increments} \\
E[N(4 - 2)] + 2 && \text{Definition of Poisson process} \\
E[N(2)] + 2 && \\
\lambda \cdot 2 + 2 && \text{Definition of Poisson process} \\
2 \cdot 2 + 2 && \\
4 + 2 = \boxed{6} &&
\end{aligned}
$$

7.

8.

9.

(a)

(b)