

1.

(a)

(b)

$$\begin{aligned}
 & P(A \cup B \cup C) && \text{Given} \\
 & P((A \cup B) \cup C) && \text{Assoc. prop. of union} \\
 & P(A \cup B) + P(C) - P((A \cup B) \cap C) && \text{Def. of union} \\
 & P(A) + P(B) - P(A \cap B) + P(C) - P((A \cup B) \cap C) && \text{Def. of union} \\
 & P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C)) && \text{Distr. prop. of intersection} \\
 & P(A) + P(B) - P(A \cap B) + P(C) \\
 & -P(A \cap C) - P(B \cap C) + P(A \cap B \cap A \cap C) && \text{Assoc. prop. of union} \\
 & P(A) + P(B) - P(A \cap B) + P(C) \\
 & -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) && \text{Prop. of intersection}
 \end{aligned}$$

which ultimately simplifies to

$$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

as we exactly sought to prove.

2.

We are given $p = 0.05$. Given the independent trials and the fact that we want to find the minimum number of throws n such that the chance of having at least one hit in those n throws is greater than 0.5. As calculating the probability of at least one hit being greater than 0.5 is messy as we need to compute for all terms from 1 to n , let's instead take the complement of this, which is easier to calculate. We set this up as

$$P(\text{one hit in } n) = 1 - P(\text{no hits in } n) = 1 - 0.95^n$$

We can then solve accordingly:

$$\begin{aligned}
 1 - 0.95^n &\geq 0.5 && \text{Given} \\
 -0.95^n &\geq -0.5 && \text{Arithmetic} \\
 0.95^n &\leq 0.5 && \text{Arithmetic} \\
 n \cdot \ln(0.95) &\leq \ln(0.5) && \text{Property of logarithm} \\
 n &\geq \frac{\ln(0.5)}{\ln(0.95)} && \text{As } \ln(0.95) \text{ is negative} \\
 n &\geq 13.513407334 && \text{Arithmetic}
 \end{aligned}$$

As we need an integer number of throws, we find the minimum number of throws to be **[14]**.

3.

As the outcomes of each player shooting is deemed to be independent, we can model the PMF for the result for the number of attempts by breaking it into two scenarios: Player A makes the first shot, or Player B makes the first shot. Since Player A goes first, we can say that the make is on an odd number of attempts, and Player B making it would be on an even number of attempts. We therefore break this down into

$$P(N) = P(N = 2k - 1) + P(N = 2k)$$

where N is the number of attempts. Given that they make with probability p_1 and p_2 , we can break each scenario down using the complements of these probabilities:

$$P(N = 2k - 1) = ((1 - p_1) \cdot (1 - p_2))^{k-1} \cdot p_1$$

and

$$P(N = 2k) = ((1 - p_1) \cdot (1 - p_2))^{k-1} \cdot (1 - p_1) \cdot p_2$$

where $k \geq 1$. So putting this together, we have the PMF for the make being on the N th shot being

$$p_N(k) = ((1 - p_1) \cdot (1 - p_2))^{k-1} \cdot p_1 + ((1 - p_1) \cdot (1 - p_2))^{k-1} \cdot (1 - p_1) \cdot p_2$$

for $k = 1, 2, 3, \dots$

4.

We can use the uniform distribution here, i.e. we can model the point at which we cut with $X \sim \text{Unif}(0, 1)$. We want to find $P(\text{Longer Piece} > 2 \cdot \text{Shorter Piece})$, which we can divide into the following scenarios:

Scenario 1: We cut above the halfway point, so $X > \frac{1}{2}$:

$$X > 2(1 - X)$$

$$X > 2 - 2X$$

$$3X > 2$$

$$X > \frac{2}{3}$$

Scenario 2: We cut below the halfway point, so $X < \frac{1}{2}$:

$$1 - X > 2X$$

$$1 - X > 2X$$

$$1 > 3X$$

$$X < \frac{1}{3}$$

Putting this together, we can find the total probability of the scenario to be

$$P(X \geq \frac{2}{3}) + P(X \leq \frac{1}{3})$$

which we very intuitively interpret using the Uniform distribution to be

$$\frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

5.

We are given the binomial r.v. np , i.e. $X \sim \text{Binom}(n, p)$. We can first define that we have n trials in the binomial form and take the expected value by setting up a Bernoulli r.v. with equivalent success rate p . We can then sum the total number of successes into X by breaking it down into X_i for each i th trial, i.e.

$$X = X_1 + X_2 + X_3 + \dots + X_n$$

with success being 1 and failure being 0. By definition, we have a bunch of Bernoulli r.v.s that we have now strung together into n trials, which is exactly what a binomial r.v. is. We take the expected value of X :

$$E[X] = E[X_1 + X_2 + X_3 + \dots + X_n]$$

And use the property of expectation of r.v.s being the expectation of its components, i.e. linearity of expectation:

$$E[X] = E[X_1] + E[X_2] + E[X_3] + \dots + E[X_n]$$

And using the definition of the expected value of a Bernoulli variable, we can find that

$$E[X] = p + p + p + \dots + p = np$$

as we sought to find.

For variance, we know that each X_i is independent by definition of the binomial variable, and that $\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2$. We know that for each trial, $E[X_i^2] = E[X_i]$ since the value is just 0 or 1. Therefore, we know that $\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i] - (E[X_i])^2 = p - p^2$. Putting this together:

$$\text{Var}[X] = \text{Var}[X_1 + X_2 + X_3 + \dots + X_n]$$

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \dots + \text{Var}(X_n)$$

$$\text{Var}(X) = p - p^2 + p - p^2 + p - p^2 + \dots + p - p^2$$

$$\text{Var}(X) = n(p - p^2) = np(1 - p)$$

exactly as we sought to find as well.

6.

We are given an experiment with $X_1, X_2, \dots, X_n \sim \text{Unif}[0, 1]$ with all X_i being independent. To find the pdf of $X_{(1)} = \min(X_1, \dots, X_n)$

We can approach this by finding the cdf and taking the derivative of the cdf. By definition, we can find the cdf of the uniform distribution to be

$$F_X(x) = P(X \leq x) = P(X_{(1)} \leq x)$$

To make this easier, we instead take the complement and continue deriving:

$$\begin{aligned} P(X_{(1)} \leq x) &= 1 - P(X_{(1)} > x) && \text{Complement} \\ &= 1 - P(X_1 > x \cap X_2 > x \cap \dots \cap X_n > x) && \text{Definition of } X_{(1)} \\ &= 1 - P(X_1 > x) \cap P(X_2 > x) \cap \dots \cap P(X_n > x) && \text{Property of set intersection} \\ &= 1 - (1-x) \cdot (1-x) \cdot \dots \cdot (1-x) && \text{Definition of uniform distr.} \\ &= 1 - (1-x)^n && \text{Property of set intersection} \end{aligned}$$

Now, to find the pdf, we differentiate:

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{d}{dx}(1 - (1-x)^n) && \text{Definition of pdf} \\ &= \frac{d}{dx}(1) - \frac{d}{dx}(1-x)^n && \text{Differentiation} \\ &= 0 - \frac{d}{dx}(1-x)^n && \text{Differentiation} \\ &= -nu^{n-1} \cdot \frac{du}{dx} = -n(1-x)^{n-1} \cdot (0-1) && \text{Chain rule, with } u = 1-x \\ &= \boxed{n(1-x)^{n-1}} && \text{Arithmetic} \end{aligned}$$

for $0 \leq x \leq 1$, and 0 otherwise.

7.

(a)

We are given $Y \sim e^X$, with $X \sim N(0, 1)$. We'll use the methodology from the transformation section. From this, we know that we can determine that the pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

We know that $Y \sim g(X) \sim e^X$, so

$$g(y) = e^y$$

$$\begin{aligned} g^{-1}(y) &= \ln(y) \\ \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} \ln(y) = \frac{1}{y} \end{aligned}$$

Now we can substitute accordingly. We know as well by definition that for $X \sim N(0, 1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Putting this all together:

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| \\ f_Y(y) &= f_X(\ln(y)) \left| \frac{1}{y} \right| \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln(y)^2}{2}} \left| \frac{1}{y} \right| \\ f_Y(y) &= \boxed{\frac{1}{y} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln(y)^2}{2}}} \end{aligned}$$

for $y > 0$, as by definition $Y \sim e^X > 0$ always, and 0 otherwise.

(b)

We want to find $E[Y] = E[e^X]$. We know that in general, $E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy$. Given the pdf we found in part (a), let's start by plugging in:

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y \cdot \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln(y)^2}{2}} dy \\ E[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\ln(y)^2}{2}} dy \end{aligned}$$

$$E[Y] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\ln(y)^2}{2}} dy$$

We can perform integration by substitution. Using $x = \ln(y)$ (and subsequently, $y = e^x$ and $dy = e^x dx$):

$$\begin{aligned} E[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^x dx \\ E[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x - \frac{x^2}{2}} dx \\ E[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x - \frac{x^2}{2}} dx \end{aligned}$$

We want to make the exponent of the e term more interpretable, in order to get something with the form of the normal pdf, as we know that generally for $Z \sim N(\mu, \sigma^2)$, $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$. We complete the square accordingly to get to this form:

$$x - \frac{x^2}{2} = -\frac{1}{2}(x^2 - 2x)$$

Set up with $(x - 1)^2 = x^2 + 1 - 2x$:

$$\begin{aligned} x^2 - 2x + 1 &= (x - 1)^2 \\ x^2 - 2x &= (x - 1)^2 - 1 \\ -\frac{1}{2}(x^2 - 2x) &= -\frac{1}{2}((x - 1)^2 - 1) = \frac{1}{2} - \frac{(x - 1)^2}{2} \end{aligned}$$

So substituting back:

$$\begin{aligned} E[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2} - \frac{(x-1)^2}{2}} dx \\ E[Y] &= e^{\frac{1}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{2}} dx \end{aligned}$$

And we use the fact that we know the form of the normal pdf to find it is a pdf with $\mu = 1$, $\sigma = 1$. The integral of this kind of distribution is simply 1, so:

$$E[Y] = e^{\frac{1}{2}} \cdot 1 = \boxed{e^{\frac{1}{2}}}$$