Quantitative Test Questions

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January 2025

1 Introduction

Advanced Algebra Questions

1. Prove that for a commutative ring R, the set of nilpotent elements forms an ideal. Is this ideal always finitely generated? Provide a counterexample if not.
2. Let G be a finite group of order n. Show that if n is divisible by 3, then G has an element of order 3.
3. Solve for $x \in Q$ in the following equation:
$$x^3 - 3x^2 + 2x - 1 = 0.$$
4. Let $f(x) = x^5 - x + 1$. Prove that $f(x)$ is irreducible over Q.
5. Show that the ring $Z[\sqrt{-5}]$ is not a unique factorization domain (UFD).
6. Determine the Galois group of the polynomial $x^4 - 4x^2 + 4$ over Q.
7. Let R be a commutative ring with unity, and I, J be ideals of R. Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
8. Prove that every finite integral domain is a field.
9. If G is a finite cyclic group of order n, show that the number of generators of G is $\phi(n)$, where ϕ is the Euler totient function.
10. Let R be a principal ideal domain (PID). Prove that every submodule of a free R-module is free.

Linear Algebra Questions

1. Let $A \in R^{n\times n}$. Prove that if A is invertible, then all its eigenvalues are non-zero.

2. Suppose $A \in C^{n \times n}$ is a Hermitian matrix. Prove that all eigenvalues of A are real.
3. Find the Jordan canonical form of the matrix:

$$A = 54201 - 1003.$$

4. Let V be a finite-dimensional vector space over C. Show that every linear operator on V has at least one eigenvalue.
5. If $A \in R^{n \times n}$ is a skew-symmetric matrix, prove that $A^T = -A$ and that all eigenvalues of A are purely imaginary.
6. Prove that the determinant of a block diagonal matrix is the product of the determinants of its diagonal blocks.
7. Compute the singular value decomposition (SVD) of the matrix:

$$A = 300400050.$$

8. Let $A, B \in C^{n \times n}$. Prove that if $AB = BA$, then A and B are simultaneously triangularizable.
9. Let $A \in R^{n \times n}$ be a positive definite matrix. Show that there exists a unique $L \in R^{n \times n}$ such that $A = LL^T$ and L is lower triangular with positive diagonal entries (Cholesky decomposition).
10. Given $A \in C^{n \times n}$, prove that A is normal if and only if there exists a unitary matrix U such that $U^* AU$ is diagonal.

Differential Calculus Questions

1. Let $f(x, y) = x^2 + xy + y^2$. Find and classify the critical points of f.
2. Prove that if $f : R^n \to R$ is differentiable and $\nabla f = 0$, then f is constant on any connected open subset of R^n.
3. Find the Taylor series expansion of $f(x, y) = e^{xy}$ up to the second-order terms at $(0, 0)$.
4. Let $f(x, y, z) = x^2 + y^2 + z^2 - xyz$. Find the equation of the tangent plane at the point $(1, 1, 1)$.
5. Use the implicit function theorem to prove that the equation $x^3 + y^3 - 3xy + z = 0$ defines z as a differentiable function of x and y near $(1, 1, -2)$.
6. Show that the function $f(x, y) = |x| + |y|$ is not differentiable at $(0, 0)$.
7. Find the Jacobian determinant of the transformation $u = x^2 - y^2, v = 2xy$.

8. Let $f(x) = x^x$. Compute $\frac{d^2}{dx^2} f(x)$.
9. Prove that if $f : R \to R$ is twice differentiable and $f''(x) > 0$ for all x, then f is strictly convex.
10. Solve the optimization problem: Maximize $f(x, y) = xy$ subject to $x^2 + y^2 = 1$.

Integral Calculus Questions

1. Evaluate the improper integral:
$$\int_0^\infty \frac{x}{e^x - 1}\, dx.$$
2. Show that the integral
$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$
3. Compute the surface area of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane.
4. Solve the double integral:
$$\int_0^1 \int_0^x e^{x^2+y^2}\, dy\, dx.$$
5. Prove that for any $n \in N$,
$$\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2},$$
where $n!!$ denotes the double factorial.
6. Evaluate the line integral:
$$\int_C (x^2 + y^2)\, ds,$$
where C is the unit circle $x^2 + y^2 = 1$.
7. Show that the volume of the solid obtained by revolving the curve $y = \sqrt{x}$ about the x-axis for $0 \le x \le 1$ is $\frac{\pi}{2}$.
8. Use Fubini's theorem to evaluate:
$$\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy.$$
9. Find the Fourier transform of $f(x) = e^{-|x|}$.
10. Prove that
$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Differential Equations Questions

1. Solve the partial differential equation:

$$u_t + u_x = 0, \quad u(x,0) = e^{-x^2}.$$

2. Solve the boundary value problem:

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

3. Verify that $y = x^2 + Cx^{-1}$ is a general solution of the differential equation:

$$x^2 y'' - 3xy' + 4y = 0.$$

4. Use the method of Frobenius to find a series solution near $x = 0$ for:

$$x^2 y'' + xy' - y = 0.$$

5. Solve the nonlinear first-order differential equation:

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

6. Prove that the solution of the heat equation:

$$u_t = \alpha^2 u_{xx}, \quad u(x,0) = \sin(\pi x),$$

satisfies the boundary conditions $u(0,t) = u(1,t) = 0$.

7. Solve the system of differential equations:

$$\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = 4x + 3y.$$

8. Solve the Laplace equation:

$$\nabla^2 u = 0, \quad u(x,0) = x, \quad u(0,y) = y.$$

9. Show that $y = Ce^{\lambda x}$ is a solution of the eigenvalue problem:

$$y'' + \lambda y = 0, \quad y(0) = y(1) = 0.$$

10. Use the method of separation of variables to solve:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Probability Questions

1. Prove that if $X \sim N(0,1)$, then $E[X^2] = 1$.
2. Let X and Y be independent random variables. Prove that $Var(X+Y) = Var(X) + Var(Y)$.
3. Find the maximum likelihood estimator for λ given a sample $X_1, X_2, \ldots, X_n$ from a Poisson distribution with mean λ.
4. A fair coin is flipped 10 times. What is the probability of getting exactly 5 heads?
5. Compute the moment-generating function of an exponential random variable with rate parameter λ.
6. If $X \sim U(0,1)$, find the probability density function of $Y = -\ln(X)$.
7. Show that the covariance of X and Y, defined as $Cov(X,Y) = E[XY] - E[X]E[Y]$, is symmetric.
8. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with mean μ and variance σ^2. Prove the central limit theorem.
9. Prove that for any random variable X, $Var(X) \geq 0$.
10. Compute the expected value of the sum of two independent uniform random variables:
$$X, Y \sim U(0,1).$$

Discrete Mathematics Questions

1. Prove that the sum of the degrees of all vertices in a graph is twice the number of edges.
2. Show that the number of subsets of a set with n elements is 2^n.
3. Let G be a bipartite graph. Prove that G has no odd-length cycles.
4. Prove that the recurrence relation $T(n) = 2T(n/2) + n$ has a solution $T(n) = O(n \log n)$.
5. Find the chromatic number of the complete graph K_n.
6. Use mathematical induction to prove that:
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$
7. Prove that every planar graph satisfies $V - E + F = 2$, where V, E, and F are the number of vertices, edges, and faces, respectively.

8. Determine whether the following logical formula is satisfiable:

$$(P \lor Q) \land (\neg P \lor R) \land (\neg Q \lor \neg R).$$

9. Show that the number of ways to distribute n identical objects into r distinct boxes is $n + r - 1r - 1$.

10. Prove that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$, where F_k is the k-th Fibonacci number.

Statistics Questions

1. A random sample of size n is drawn from a population with mean μ and variance σ^2. Derive the maximum likelihood estimator for μ.

2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Show that the sample mean $\bar{X}$ is an unbiased estimator of μ.

3. Prove that the variance of the sample mean $\bar{X}$ is σ^2/n.

4. Compute the 95% confidence interval for the mean μ of a normal distribution with known variance σ^2.

5. Test the hypothesis $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ for a normal population with known variance σ^2 at a significance level α.

6. Derive the least-squares estimators for the coefficients in a simple linear regression model $Y = \beta_0 + \beta_1 X + \epsilon$.

7. Show that the correlation coefficient r satisfies $-1 \leq r \leq 1$.

8. Let $X \sim N(\mu, \sigma^2)$. Prove that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, where S^2 is the sample variance.

9. Use the central limit theorem to approximate the probability:

$$P\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \leq z\right),$$

where X_i are independent and identically distributed random variables.

10. Perform a chi-square test for goodness of fit for a multinomial distribution.

Numerical Methods Questions

1. Derive the Newton-Raphson formula for finding roots of a nonlinear equation $f(x) = 0$.

2. Apply the bisection method to solve $x^3 - x - 2 = 0$ in the interval $[1, 2]$ up to 4 iterations.

3. Prove the convergence of the fixed-point iteration method under the condition $|g'(x)| < 1$ for all x in the interval of interest.

4. Use Lagrange interpolation to find a polynomial that passes through the points $(1, 1), (2, 4), (3, 9)$.

5. Derive the trapezoidal rule for approximating $\int_a^b f(x)\,dx$.

6. Analyze the error term in Simpson's rule for numerical integration.

7. Solve the linear system $Ax = b$ using Gaussian elimination for the matrix:

$$A = 21 - 1 - 3 - 12 - 212, \quad b = 8 - 11 - 3.$$

8. Perform power iteration to find the dominant eigenvalue of the matrix:

$$A = 4123.$$

9. Use Euler's method to approximate the solution of the differential equation $\frac{dy}{dx} = x + y$, $y(0) = 1$, for $x \in [0, 1]$ with step size $h = 0.1$.

10. Prove that the Jacobi method converges for a diagonally dominant matrix.

Python Programming Questions

1. What is the output of the following Python code?

```
print(2 + 3 * 5)
```

2. How do you define a function in Python? Provide an example function that returns the square of a number.

```
def square(x):

return x²
```

2. What does the `lambda` keyword do in Python? Provide an example.

```
f = lambda x:  x²print(f(4))    Output will be 16
```

2. What is the difference between a list and a tuple in Python?

```
list_example = [1, 2, 3]
tuple_example = (1, 2, 3)
```

Lists are mutable, while tuples are immutable.

3. How do you handle exceptions in Python? Write an example using `try` and `except`.

```
try:
x = 1 / 0
except ZeroDivisionError:
print("Cannot divide by zero!")
```

4. What is a dictionary in Python? Provide an example of a dictionary.

```
my_dict = { "name":  "Alice", "age":  25 }
print(my_dict["name"])    Output will be "Alice"
```

5. What is the difference between `del` and `pop` in Python?
6. How do you add an item to the end of a list in Python?

```
list_example.append(4)
```

7. What is list slicing in Python? Provide an example.

```
my_list = [1, 2, 3, 4, 5]
print(my_list[1:4])    Output will be [2, 3, 4]
```

8. How do you remove an item from a list in Python by its value?

```
my_list.remove(3)
```

9. What is the output of the following code?

```
print("Hello" + " " + "World")
```

10. What are the different types of loops in Python?

11. How can you iterate over a dictionary in Python?

```
for key, value in my_dict.items():
print(key, value)
```

12. What is a class in Python? Provide an example of a simple class.

```
class Person:
def __init__(self, name, age):
self.name = name
self.age = age
```

13. How do you call a method of a class in Python?

```
person = Person("Alice", 30)
print(person.name)
```

14. What is inheritance in Python? Provide an example.

```
class Student(Person):
def __init__(self, name, age, grade):
super().__init__(name, age)
self.grade = grade
```

15. What is a generator in Python?

```
def my_generator():
yield 1
yield 2
yield 3
```

16. How do you create a Python set? Provide an example.

```
my_set = {1, 2, 3, 4, 5}
```

17. What is the difference between `is` and `==` in Python?

18. How do you convert a string to an integer in Python?

```
my_int = int("123")
print(my_int)    Output will be 123
```