These are my answers to the Chapter 2 exercises.

1. Examples of Operations

Which of the following rules are operations on the indicated set? ($\mathbb{Z}$ designates the set of the integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.) For each rule which is not an operation, explain why it is not.

Example $a * b = \dfrac{a+b}{ab}$, on the set $\mathbb{Z}$.

Solution This is not an operation on $\mathbb{Z}$. There are integers a and b such that $(a+b)/ab$ is not an integer. (For example,

$$\frac{2+3}{2\cdot 3} = \frac{5}{6}$$

is not an integer.) Thus, $\mathbb{Z}$ is not closed under $*$.

(1) $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$.
Solution This is not an operation on $\mathbb{Q}$. Here is a counter example: $2 * 1 = \sqrt{|2\cdot 1|} = \sqrt{2}$. The result is in irrational number, and therefore $\mathbb{Q}$ is not closer under $*$.

(2) $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
Solution This is not an operation because it is not closed. A counter example is obtained by choosing $a = 1$ and $b = 0.5$: $1 \cdot 0.5 = \ln 0.5 \approx -0.693 < 0$.

(3) $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.
Solution This is not an operation because it is not uniquely defined. $2 * 2$ is a root of the equation $x^2 - 2^2 2^2 = 0$, where $x = \pm 4$.

(4) Subtraction, on the set $\mathbb{Z}$.
Solution This is an operation. It is defined for all pairs $(a, b)|a, b \in \mathbb{Z}$, it is uniquely defined for each pair, and it is closed.

(5) Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
Solution This is not an operation because it is not closed: $4 * 10 = 4 - 10 = -4 < 0$.

2. Properties of Operations

Each of the following is an operation $*$ on $\mathbb{R}$. Indicate whether or not

(1) it is commutative,
(2) it is associative,
(3) $\mathbb{R}$ has an identity element with respect to $*$,
(4) every $x \in \mathbb{R}$ has an inverse with respect to $*$.

Instructions For (1), computer $x * y$ and $y * x$, and verify whether or not they are equal. For (2), compute $x*(y*z)$ and $(x*y)*z$, and verify whether or not they are equal. For (3), first solve the equation $x*e = x$ for e; if the equation caonnot be solved, there is no identity element. If it *can* be solved, it is still necessary to check that $e * x = x * e = x$ for any $x \in \mathbb{R}$. If it checks, then e is an identity element. For (4), first note that if there is no identity element, there can be no inverses. If there is an identity element e, first solve the equation $x * x' = e$ for x'; if the equation cannot be solved, x does not have an inverse. If it *can* be solved, check to make sure that $x * x' = x' * x = e$. If this checks, x' is the inverse of x.

(1) PROBLEM $x * y = x + 2y + 4$
SOLUTION The solution follows:
(a) The operation is not commutative

$$\begin{aligned} x * y &= x + 2y + 4 \\ y * x &= y + 2x + 4 \end{aligned}$$

(b) The operation is not associative

$$\begin{aligned} x * (y * z) &= x * (y + 2z + 4) \\ &= x + 2(y + 2z + 4) + 4 \\ &= x + 2y + 4z + 12 \\ (x * y) * z &= (x + 2y + 4) * z \\ &= (x + 2y + 4) + 2z + 4 \\ &= x + 2y + 2z + 8 \end{aligned}$$

(c) Check to see if their is an identity element:

$$\begin{aligned} x * e &= x \\ x * e &= x + 2e + 4 \\ x &= x + 2e + 4 \\ 0 &= 2e + 4 \\ 2e &= -4 \\ e &= -2 \end{aligned}$$

Now check to see if $x * e = e * x = x$:

$$\begin{aligned} x * -2 &= x + (2)(-2) + 4 \\ &= x - 4 + 4 \\ &= x \\ -2 * x &= -2 + 2x + 4 \\ &= 2x + 2 \end{aligned}$$

We can see $x * e \neq e * x$ therefore there is no identity element.
(d) No need to check for an inverse. There is no identity.

(2) PROBLEM $x * y = x + 2y - xy$
SOLUTION The solution follows:
(a) The operation is not commutative as we can see from (1) and (2) below:

$$\begin{aligned} x * y &= x + 2y - xy \\ y * x &= y + 2x - yx \\ &= 2x + y - xy \end{aligned}$$

(b) The operation is not associative

$$\begin{aligned}
x * (y * z) &= x * (y + 2z - yz) \\
&= x + 2(y + 2z - yz) - x(y + 2z - yz) \\
&= x + 2y + 4z - 2yz - xy - 2xz + xyz \\
&= x + 2y + 4z - xy - 2xz - 2yz + xyz \\
(x * y) * z &= (x + 2y - xy) * z \\
&= (x + 2y - xy) + 2z - (x + 2y - xy)z \\
&= (x + 2y - xy) + 2z - xz - 2yz - xyz \\
&= x + 2y + 2z - xy - xz - 2yz - xyz
\end{aligned}$$

(c) From the following we can see that $e = 0$.

$$\begin{aligned}
x * e &= x \\
x * e &= x + 2e - xe \\
x &= x + 2e - xe \\
0 &= 2e - xe \\
e(2 - x) &= 0 \\
e &= 0 \quad \text{(when } x \neq 2\text{)}
\end{aligned}$$

We need to check what happens when $x = 0$: we have $0*e = 0+2e-0e$ which reduces to $0 = 2e$ and again we get $e = 0$. Now we must check that $x * e = e * x = x$

$$\begin{aligned}
e * x &= e + 2x - ex \\
&= 0 + 2x - 0 \\
&= 2x \\
&\neq x
\end{aligned}$$

Therefore, there is no identity element.

(d) Since there is no identity element, there are no inverses.

(3) PROBLEM $x * y = |x + y|$

SOLUTION The solution follows:

(a) From the following we can see that the operation is commutative:

$$\begin{aligned}
x * y &= |x + y| \\
y * x &= |y + x| \\
&= |x + y|
\end{aligned}$$

(b) It is not associative. Here is a counter example using $x = 7, y = -13, z = 1$

$$
\begin{aligned}
x * (y * z) &= 7 * |-13 + 1| \\
&= |7 + 12| \\
&= 19 \\
(x * y) * z &= |7 + -13| * 1 \\
&= |-6| * 1 \\
&= |6 + 1| \\
&= 7 \\
&\neq 19
\end{aligned}
$$

(c) The value of 0 works as an identity element for non-negative numbers. However, there is no possible identity element for negative numbers. This is because the result of the absolute value function always returns non-negative values.

(d) Since there is no identity element, there are no inverses.

(4) Problem $x * y = |x - y|$

Solution The solution is as follows:

(a) The operation is commutative:

$$
\begin{aligned}
x * y &= |x - y| \\
&= \begin{cases} x - y, & \text{if } x \geq y; \\ y - x, & \text{otherwise.} \end{cases} \\
y * x &= |y - x| \\
&= \begin{cases} x - y, & \text{if } x \geq y; \\ y - x, & \text{otherwise.} \end{cases}
\end{aligned}
$$

(b) The operation is not associative. A counter example is provided by choosing $x = 1, y = -2, z = 3$.

$$
\begin{aligned}
x * (y * z) &= 1 * |(-2) - 3| \\
&= 1 * 5 \\
&= |1 - 5| \\
&= 4 \\
(x * y) * z &= |1 - (-2)| * 3 \\
&= 3 * 3 \\
&= |3 - 3| \\
&= 0 \\
&\neq 4
\end{aligned}
$$

(c) Again, there can be no identity element, since the result of the operation is always non-negative. Therefore, there can be no number $x * e = x$ for $x < 0$.

(d) Again, since there is no identity element, there can be no inverses. However, $-x$ acts as an inverse for any number assuming that 0 actually was an identity element.

Figure 1. Proof that max function is commutative.

$$\begin{aligned} x * y &= \max(x, y) \\ &= \begin{cases} x, & \text{if } x \geq y; \\ y, & \text{otherwise.} \end{cases} \\ y * x &= \max(y, x) \\ &= \begin{cases} x, & \text{if } x \geq y; \\ y, & \text{otherwise.} \end{cases} \end{aligned}$$

(5) Problem $x * y = xy + 1$
Solution The solution is as follows:
(a) The operation is commutative as seen below.

$$\begin{aligned} x * y &= xy + 1 \\ y * x &= yx + 1 \\ &= xy + 1 \quad \text{(Multiplication is commutative.)} \end{aligned}$$

(b) The operation is not associative as seen below.

$$\begin{aligned} x * (y * z) &= x * (yz + 1) \\ &= x(yz + 1) + 1 \\ &= x + xyz + 1 \\ (x * y) * z &= (xy + 1) * z \\ &= (xy + 1)z + 1 \\ &= z + xyz + 1 \\ &\neq x + xyz + 1 \end{aligned}$$

(c) We can see below that their is no constant value defined for the identity element. Our equations find a formula for e which depens on x. This is not a constant value.

$$\begin{aligned} x * e &= x \\ x * e &= xe + 1 \\ x &= xe + 1 \\ x - 1 &= xe \\ xe &= x - 1 \\ e &= \frac{x - 1}{x} \end{aligned}$$

(d) There is no identity element, so there are no inverses.

(6) Problem $x * y = \max(x, y)$, the larger of the two numbers x and y.
Solution The solution is as follows.
(a) The operation is commutative. This can be seen in figure 1 on page 5.

FIGURE 2. Proof that the max function is associative.

$$
\begin{aligned}
x * (y * z) &= x * \max(y, z) \\
&= \max(x, \max(y, z)) \\
&= \begin{cases} \max(x, y), \text{if } y >= z; \\ \max(x, z), \text{otherwise.} \end{cases} \\
&= \begin{cases} x, & \text{if } x \geq y \wedge x \geq z; \\ y, & \text{if } x < y \wedge y \geq z; \\ z, & \text{otherwise.} \end{cases} \\
(x * y) * z &= \max(x, y) * z \\
&= \max(\max(x, y), z) \\
&= \begin{cases} \max(x, z), \text{if } x >= y; \\ \max(y, z), \text{otherwise.} \end{cases} \\
&= \begin{cases} x, & \text{if } x >= y \wedge x >= z; \\ y, & \text{if } x < y \wedge y >= z; \\ z, & \text{otherwise.} \end{cases}
\end{aligned}
$$

FIGURE 3. Proof that the equation of problem 2.7 (B7 in the text) is commutative.

$$
\begin{aligned}
x * y &= \frac{xy}{x + y + 1} \\
y * x &= \frac{yx}{y + x + 1} \\
&= \frac{xy}{x + y + 1} && (+, \cdot \text{ are commutative})
\end{aligned}
$$

(b) The operation is associative. The proof can be seen in figure 2 on page 6.
(c) There is no identity element. We will prove this by contradiction. Assume that there is some identity element e. Then by definition $x * e = x$ forall x. Let us choose a value $m = e - 1$. Then we have $m * e = (e - 1) * e = \max(e - 1, e) = e \neq m$. Therefore, e is not an identity element. $\square$.
(d) Since there is no identity element, there are no inverses.

(7) PROBLEM $x * y = \dfrac{xy}{x + y + 1}$ on the set of positive real numbers.
SOLUTION The solution is as follows
(a) The operation is commutative. The proof is shown in figure 3 on page 6.

(b) The operation is not associative. This can be demonstrated with the values $x = 2, y = 3, z = 4$.

$$\begin{aligned}
2 * (3 * 4) &= 2 * \frac{3 \cdot 4}{3 + 4 + 1} \\
&= 2 * (3/2) \\
&= \frac{2 \cdot (3/2)}{2 + (3/2) + 1} \\
&= \frac{6/2}{9/2} \\
&= \frac{6}{9} \\
&= \frac{2}{3} \\
(2 * 3) * 4 &= \frac{2 \cdot 3}{2 + 3 + 1} * 4 \\
&= \frac{6}{7} * 4 \\
&= \frac{(6/7) \cdot 4}{(3/7) + 4 + 1} \\
&= \frac{24/7}{38/7} \\
&= \frac{24}{38} \\
&= \frac{12}{19}
\end{aligned}$$

(c) There is no identity element. As a matter of fact, the equation $x * e = x$ can only be solved when $x = 0$ or $x = -1$, as shown below.

$$\begin{aligned}
\frac{ex}{e + x + 1} &= x \\
ex &= x^2 + ex + x \\
x^2 + ex + x &= ex \\
x^2 + x &= 0 \\
x(x + 1) &= 0 \\
x &= 0, -1
\end{aligned}$$

(d) Since there is no identity, there can be no inverses.

3. Operations on a Two-Element Set

Let A be the two-element set $A = \{a, b\}$.

(1) Problem Write the tables of all 16 operations on A. (Use the format explained on page 20.) Label these operations O_1 to O_{16}.
Solution The tables are shown in table 1 on page 8.
(2) Problem Identify which of the operations O_1 to O_{16} are commutative.

TABLE 1. The sixteen different operations for a two element set.

O_{16}

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	a

O_1

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	b

O_2

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	a

O_3

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	b

O_4

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	a

O_5

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	b

O_6

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	a

O_7

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	b

O_8

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	a

O_9

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	b

O_{10}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	a

O_{11}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	b

O_{12}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	a

O_{13}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	b

O_{14}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	a

O_{15}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	b

SOLUTION This can be solved very easily by looking at the second and third entries in each table to see if $a*b = b*a$. The commutative entries are $O_1, O_6, O_7, O_8, O_9, O_{14}, O_{15}, O_{16}$. commutative.

(3) PROBLEM Identify which operations, among O_1 to O_{16}, are associative.
SOLUTION In general there are eight cases to check. These cases are shown in table 2 on page 9.

Let's consider how many cases there are when $*$ is commutative. It will be shown that only two cases need to be checked: 2 and 4. All the others are trivially true by commutativity, or else they are true if 2 or 4 is true. See table 3 on page 9 to see the list of proofs.

Now we need to start checking all the cases.

$\mathbf{O_1}$: This operation is commutative. Cases 2 and 4 are true, so this operation is associative. See figure 4

$\mathbf{O_2}$: This operation is not commutative. Therefore there are eight cases to check. Rather than check them all, we'll use boolean algebra. We assume a is the value `false` and b is the value `true`. The operation is equivalent

TABLE 2. The eight cases of associativity.

$a * (a * a) = (a * a) * a$	1
$a * (a * b) = (a * a) * b$	2
$a * (b * a) = (a * b) * a$	3
$a * (b * b) = (a * b) * b$	4
$b * (a * a) = (b * a) * a$	5
$b * (a * b) = (b * a) * b$	6
$b * (b * a) = (b * b) * a$	7
$b * (b * b) = (b * b) * b$	8

TABLE 3. Equations proving only 2 cases must be checked when operation is commutative

$$a * (a * a) = (a * a) * a \qquad \text{(Commutativity)}$$

$$a * (a * b) = (a * a) * b \qquad \text{(Must be checked)}$$

$$\begin{aligned} a * (b * a) &= a * (a * b) && \text{(Commutativity)} \\ &= (a * b) * a && \text{(Commutativity)} \end{aligned}$$

$$a * (b * b) = (a * b) * b \qquad \text{(Must be checked)}$$

$$\begin{aligned} b * (a * a) &= (a * a) * b && \text{(Commutativity)} \\ &= a * (a * b) && \text{(By case 2)} \\ &= a * (b * a) && \text{(Commutativity)} \\ &= (b * a) * a && \text{(Commutativity)} \end{aligned}$$

$$\begin{aligned} b * (a * b) &= b * (b * a) && \text{(Commutativity)} \\ &= (b * a) * b && \text{(Commutativity)} \end{aligned}$$

$$\begin{aligned} b * (b * a) &= b * (a * b) && \text{(Commutativity)} \\ &= (a * b) * b && \text{(Commutativity)} \\ &= a * (b * b) && \text{(By case 4)} \\ &= (b * b) * a && \text{(Commutativity)} \end{aligned}$$

$$b * (b * b) = (b * b) * b \qquad \text{(Commutativity)}$$

FIGURE 4. The proofs for O_1.

$$\begin{aligned}
a * (a * b) &= a * a && \text{(Left hand side of case 2.)}\\
&= a\\
(a * a) * b &= a * b && \text{(Right hand side of case 2.)}\\
&= a && \text{(Case 2 is true.)}\\
a * (b * b) &= a * b && \text{(Left hand side of case 4.)}\\
&= a\\
(a * b) * b &= a * b && \text{(Right hand side of case 4.)}\\
&= a && \text{(Case 4 is true.)}
\end{aligned}$$

FIGURE 5. The 3 variable boolean equations for O_2.

$$\begin{aligned}
x * (y * z) &= x * (y \wedge \neg z)\\
&= x \wedge \neg(y \wedge \neg z)\\
&= x \wedge (\neg y \vee z)\\
(x * y) * z &= (x \wedge \neg y) * z\\
&= (x \wedge \neg y) \wedge \neg z\\
&= x \wedge \neg y \wedge \neg z
\end{aligned}$$

to the boolean equation $x \wedge \neg y$. We calculate the appropriate equations for $x * (y * z)$ and $(x * y) * z$ in figure 5.

Now we need to find a difference. So we'll calculate a triple of values where the first formula is true, but the second formula is not. We'll do this with an equation of the form $f \wedge \neg s$ where f is the first equation and s is the second equation.

$$\begin{aligned}
f \wedge s &= x \wedge (\neg y \vee z) \wedge \neg(x \wedge \neg y \wedge \neg z))\\
&= (x \wedge \neg y \vee x \wedge z) \wedge (\neg x \vee y \vee z)\\
&= (x \wedge y \wedge z) \vee (x \wedge \neg y \wedge z) \vee (x \wedge z)\\
&= (x \wedge z) \wedge (y \vee \neg y \vee \top)\\
&= x \wedge z
\end{aligned}$$

So the two formulas differ when $x = b, z = b$. Let's check.

$$\begin{aligned}
b * (a * b) &= b * a\\
&= b\\
(b * a) * b &= b * b\\
&= a\\
&\neq b
\end{aligned}$$

$\mathbf{O_3}$: This operation is not commutative. We'll solve it the same way we solve O_2. This operation is equivalent to the equation $x * y = x$.

$$\begin{aligned} x * (y * z) &= x * y \\ &= x \\ (x * y) * z &= x * z \\ &= x \end{aligned}$$

Since the equations are identical we conclude that O_3 is associative.

$\mathbf{O_4}$: This operation is not commutative. It is equivalent to the equatioin $x * y = \neg x \wedge y$.

$$\begin{aligned} x * (y * z) &= x * (\neg y \wedge z) \\ &= \neg x \wedge \neg y \wedge z \\ (x * y) * z &= (\neg x \wedge y) * z \\ &= \neg(\neg x \wedge y) \wedge z \end{aligned}$$

The first formula is more restrictive. Let's find a set of values where the second formula is true but the first is not: $s \wedge \neg f$.

$$\begin{aligned} s \wedge \neg f &= (\neg(\neg x \wedge y) \wedge z) \wedge \neg(\neg x \wedge \neg y \wedge z) \\ &= (x \vee \neg y) \wedge z \wedge (x \vee y \vee \neg z) \\ &= (x \vee \neg y) \wedge ((x \wedge z) \vee (y \wedge z) \vee \perp) \\ &= (x \wedge z) \vee (x \wedge y \wedge z) \vee (\neg y \wedge x \wedge z) \vee (\neg y \wedge y \wedge z) \\ &= (x \wedge z) \wedge (y \vee \neg y) \\ &= x \wedge z \end{aligned}$$

When $x = b, z = b$ we find a difference, therefore O_4 is not associative.

$$\begin{aligned} b * (b * b) &= b * a \\ &= a \\ (b * b) * b &= a * b \\ &= b \\ &\neq a \end{aligned}$$

$\mathbf{O_5}$: This is not commutative either. It is equivalent to $x * y = y$. It is associative. We can see this since both formulas evaluate to the same thing.

$$\begin{aligned} x * (y * z) &= x * z \\ &= z \\ (x * y) * z &= y * z \\ &= z \end{aligned}$$

$\mathbf{O_6}$: This operation is commutative. We just have to check cases 2 and 4 from above. Below we see both cases are true, therefore this operation is associative.

$$\begin{aligned}
a * (a * b) &= a * b && \text{(Left hand side of case 2.)} \\
&= b \\
(a * a) * b &= a * b && \text{(Right hand side of case 2.)} \\
&= b && \text{(Case 2 is true.)} \\
a * (b * b) &= a * a && \text{(Left hand side of case 4.)} \\
&= a \\
(a * b) * b &= b * b && \text{(Right hand side of case 4.)} \\
&= a && \text{(Case 4 is true.)}
\end{aligned}$$

O_7: This operation is commutative. Cases 2 and 4 are true, therefore this operation is associative.

$$\begin{aligned}
a * (a * b) &= a * b && \text{(Left hand side of case 2.)} \\
&= b \\
(a * a) * b &= a * b && \text{(Right hand side of case 2.)} \\
&= b && \text{(Case 2 is true.)} \\
a * (b * b) &= a * b && \text{(Left hand side of case 4.)} \\
&= b \\
(a * b) * b &= b * b && \text{(Right hand side of case 4.)} \\
&= b && \text{(Case 4 is true.)}
\end{aligned}$$

O_8: This operation is commutative. Case 2 is false, therefore this operation is not associative.

$$\begin{aligned}
a * (a * b) &= a * a && \text{(Left hand side of case 2.)} \\
&= b \\
(a * a) * b &= b * b && \text{(Right hand side of case 2.)} \\
&= a && \text{(Case 2 is false.)}
\end{aligned}$$

O_9: This operation is commutative. Cases 2 and 4 are true, therefore this operation is associative.

$$\begin{aligned}
a * (a * b) &= a * a \\
&= b \\
(a * a) * b &= b * b \\
&= b && \text{(Case 2 is true.)} \\
a * (b * b) &= a * b \\
&= a \\
(a * b) * b &= a * b \\
&= a && \text{(Case 4 is true.)}
\end{aligned}$$

O_{10}: This operation is not commutative. This operation corresponds to the boolean equation $x * y = \neg y$. See figure 6 for proof that this operation is not associative.

FIGURE 6. Proof that O_{10} is not associative.

$$\begin{aligned} x * y &= \neg y && \text{(Boolean equivalent equation for } O_{10}.) \\ x * (y * z) &= x * \neg z \\ &= z \\ (x * y) * z &= \neg y * z \\ &= \neg z && \text{(Doesn't equal } z.). \end{aligned}$$

FIGURE 7. Derivation of the boolean equations $x * (y * z)$ and $(x * y) * z$ for O_{11}.

$$\begin{aligned} x * y &= x \vee (\neg z \wedge \neg y) \\ x * (y * z) &= x * (y \vee (\neg y \wedge \neg z)) \\ &= x \vee (\neg x \wedge \neg(y \vee (\neg y \wedge \neg z))) \\ &= x \vee (\neg x \wedge (\neg y \wedge \neg(\neg y \wedge \neg z))) \\ &= x \vee (\neg x \wedge (\neg y \wedge (y \vee z))) \\ &= x \vee (\neg x \wedge \neg y \wedge y) \vee (\neg x \wedge \neg y \wedge z) \\ &= x \vee (\neg x \wedge \neg y \wedge z) \\ (x * y) * z &= (x \vee (\neg x \wedge \neg y) * z \\ &= (x \vee (\neg x \wedge \neg y) \vee (z \wedge (x \vee (\neg x \wedge \neg y))) \\ &= x \vee (\neg x \wedge \neg y) \vee (x \wedge z) \vee (\neg x \wedge \neg y \wedge z) \\ &= x \vee (\neg x \wedge \neg y) \end{aligned}$$

$\mathbf{O_{11}}$: This operation is not commutative. The equivalent binary equation is $x*y = x \vee \neg x \wedge \neg y$. See the derivation of the boolean equivalent equations for $x*(y*z)$ and $(x*y)*z$ in figure 7 to see that they equal $x \vee \neg * x \wedge \neg y \wedge z$ and $x \vee \neg x \wedge \neg y$ respectively.

From the equations in figure 7 you can see that one of the equations is true for $x = a, y = a, z = a$ and the other one is false for the same set of values. So this is our counter example that O_{11} is not associative. See figure 8 for the derivation.

$\mathbf{O_{12}}$: This operation is not commutative. Its boolean equivalent equation is $x * y = \neg x$. This operation is not associative The two equations evaluate to different values. See figure 9.

FIGURE 8. Derivation of $x = a, y = a, z = a$ to show O_{11} is not associative. One evaluates to a and the other b.

$$\begin{aligned} a * (a * a) &= a * b \\ &= a \\ (a * a) * a &= b * a \\ &= b \end{aligned}$$

FIGURE 9. Evaluation of binary equivalent equations for O_{12} to show that it is not associative.

$$\begin{aligned} x * y &= \neg x \\ x * (y * z) &= x * \neg z \\ &= \neg x \\ (x * y) * z &= \neg x * z \\ &= x \end{aligned}$$