Lecture 09: Matrix Equations

Multiplying a Matrix by a Vector

given an $\underset{\text{rows}}{m} \times \underset{\text{cols}}{n}$ matrix A

with colums $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$

and if $x \in \mathbb{R}^m$ then

a_1 represents a colwn

the product of A and x

$$A\vec{x} = \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}}$$

$$= x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$$

Example

Multiply

colums # = entries #

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}}_{\text{colums \#}} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}}_{\text{entries \#}}$$

$$= 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$ answer

Three Forms

Linear Equation

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1$$

Vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Given a matrix equation

$$A\vec{x} = \vec{b}$$

this equation has the same solutions as the subsystem of linear equations whose aug matrix is

$$[a_1 \ a_2 \ \dots \ a_n \ b]$$

or

$$[A \ \vec{b}]$$

Existence of Solutions

- $Ax = b$ has a solution if b is a linear combination of the columns of A
- We have considered the question: is $\vec{b}$ is Span$\{A\}$
- Is $Ax = b$ consistent

Theorom

Let A be a $\overset{\text{rows}}{m} \times \overset{\text{cols}}{n}$ matrix
The following statements are logically equivalent

1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = b$ has a solution

2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A

3. The columns of A span $\mathbb{R}^m$

4. The matrix A has a pivot in every row

if you prove one is true,
all are true
if one is false,
all are false

The Row-Vector Rule

$$\overset{A}{\begin{bmatrix} 4 & -1 & 0 & 1 \\ 2 & 5 & -2 & 0 \\ 0 & 3 & -4 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}}$$

$$= \begin{bmatrix} 4 \cdot 2 + -1 \cdot -1 + 0 \cdot 0 + 1 \cdot 5 \\ 2 \cdot 2 + 5 \cdot -1 + -2 \cdot 0 + 0 \cdot 5 \\ 0 \cdot 2 + 3 \cdot -1 + -4 \cdot 0 + 1 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 1 + 0 + 5 \\ 4 + -5 + -2 + 0 \\ 0 + -3 + -4 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ -3 \\ -2 \end{bmatrix} \, ?$$

did something wrong

$$\begin{bmatrix} 14 \\ -1 \\ 2 \end{bmatrix}$$

Algebraic Properties

A is a $m \times n$ matrix

$\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^m$

c is a scalar

① $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ distributive

② $A(c\vec{u}) = c(A\vec{u})$

① Proof

$A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$A(\vec{u}+\vec{v}) = A\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$

$= (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2$
$(u_3+v_3)\vec{a}_3$

$= u_1\vec{a}_1 + v_1\vec{a}_1 + u_2\vec{a}_2 + v_2 a^2$
$+ u_3\vec{a}_3 + v_3\vec{a}_3$

$= A\vec{u} + A\vec{v}$