# 3.13

Time Series Analysis by Shumway and Stoffer 4th ed.

Michael Jagdharry

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Let

$$\epsilon_t = x_t - \sum_{i=1}^{h-1} a_i x_{t-i}$$

$$\delta_{t-h} = x_{t-h} - \sum_{j=1}^{h-1} b_j x_{t-j}$$

be two residuals where $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ minimize $E(\epsilon_t^2)$ and $E(\delta_{t-h}^2)$ respectively.

If $x_t^h$ is the best linear predictor of $x_t$ given $\{x_{t-1}, x_{t-2}, \cdots, x_{t-h}\}$, that is,

$$x_t^h = \alpha_{h1}x_{t-1} + \alpha_{h2}x_{t-2} + \cdots + \alpha_{hh}x_{t-h}$$

Show that

$$\phi_{hh} = \frac{E(\epsilon_t \delta_{t-h})}{\sqrt{E(\epsilon_t^2)E(\delta_{t-h}^2)}} = \frac{\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}} = \alpha_{hh}$$

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Before proceeding, let's establish the following vector definitions:

$$\begin{aligned}
\gamma_{h-1} &= [\gamma(1)\ \gamma(2)\ \cdots\ \gamma(h-1)]^T \\
\rho_{h-1} &= [\rho(1)\ \rho(2)\ \cdots\ \rho(h-1)]^T \\
\tilde{\rho}_{h-1} &= [\rho(h-1)\ \rho(h-2)\ \cdots\ \rho(1)]^T \\
\phi &= [\phi_{h1}\ \phi_{h2}\ \cdots\ \phi_{h2}]^T \\
\alpha &= [\alpha_{h1}\ \alpha_{h2}\ \cdots\ \alpha_{hh}]^T \\
\alpha_1 &= [\alpha_{h1}\ \alpha_{h2}\ \cdots\ \alpha_{h(h-1)}]^T \\
a &= [a_1\ a_2\ \cdots\ a_{h-1}]^T \\
b &= [b_1\ b_2\ \cdots\ b_{h-1}]^T \\
x &= [x_{t-1}\ x_{t-2}\ \cdots\ x_{t-h+1}]^T
\end{aligned}$$

From the results of (3.53) and (3.54), we see the coefficient of $x_{t+i}$ in (3.54) is the coefficient of $x_{t+h-i}$ in (3.53). Applying these results here, the coefficient of $x_{t-i}$ in $\epsilon_t$ is the coefficient of $x_{t-h+i} = x_{t-(h-i)}$ in $\delta_{t-h}$, This means that $a_i = b_{h-i}$, i.e.

$$b = [b_1\ b_2\ \cdots\ b_n]^T = [a_{h-1}\ a_{h-2}\ \cdots\ a_1]^T = \tilde{a}$$

With the knowledge that for any two column vectors c and k, $c^T k = \sum c_i k_i$, we can express $E(\epsilon_t \delta_{t-h})$ as

$$\begin{aligned}
E(\epsilon_t \delta_{t-h}) &= \\
E[(x_t - \sum_{i=1}^{h-1} a_i x_{t-i})(x_{t-h} - \sum_{j=1}^{h-1} b_j x_{t-j})] &= \\
E[(x_t - a^T x)(x_{t-h} - \sum_{j=1}^{h-1} a_{h-j} x_{t-j})] &= \\
E[(x_t - a^T x)(x_{t-h} - \tilde{a}^T x)] &= \\
E[(x_t x_{t-h} - x_{t-h} a^T x - x_t \tilde{a}^T x + a^T x x^T \tilde{a}] &= \\
E(x_t x_{t-h}) - a^T E(x_{t-h} x) - \tilde{a}^T E(x_t x) + a^T E(x x^T)\tilde{a}] &= \\
\gamma(h) - a^T \tilde{\gamma}_{h-1} - \tilde{a}^T \gamma_{h-1} + a^T \Gamma_{h-1} \tilde{a}
\end{aligned}$$

In this problem, the vectors $a = \alpha$, and since the coefficients $\alpha_{hi}$ are such that they produce the BLP for $x_t^h$, we can apply the prediction equations (3.63) and (3.64) to them, for $n = h-1$. In other words, $a = \alpha = \phi = \Gamma_{h-1}^{-1}\gamma_{h-1}$ in (3.64), and $\Gamma_{h-1}\tilde{a} = \tilde{\gamma}_{h-1}$. Furthermore, we can show the following relationship between the variance-covariance matrix $\Gamma$ and the correlation matrix $R$:

$$\begin{aligned}
\phi_n &= \Gamma_n^{-1}\gamma_n \\
\Gamma_n \phi_n &= \gamma_n \\
\gamma(0) R_n \phi_n &= \gamma(0)\rho_n \\
R_n \phi_n &= \rho_n \\
\phi_n &= R^{-1}\rho_n
\end{aligned}$$

Comparing the initial and final equations here,

$$\Gamma_n^{-1}\gamma_n = R^{-1}\rho_n$$

Continuing from where we left off,

$$\begin{aligned}
\gamma(h) - a^T \tilde{\gamma}_{h-1} - \tilde{a}^T \gamma_{h-1} + a^T \Gamma_{h-1}\tilde{a} &= \\
\gamma(h) - \tilde{a}^T \gamma_{h-1} - a^T \tilde{\gamma}_{h-1} + a^T \tilde{\gamma}_{h-1} &= \\
\gamma(h) - \tilde{a}^T \gamma_{h-1} &= \\
\gamma(h) - (\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1})^T \gamma_{h-1} &= \\
\gamma(h) - \tilde{\gamma}_{h-1}^T \Gamma_{h-1}^{-1} \gamma_{h-1} &= \\
\gamma(0)\rho(h) - \gamma(0)\tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1} &= \\
\gamma(0)[\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}] &= \\
E(\epsilon_t \delta_{t-h})
\end{aligned}$$

We can compute $E(\epsilon_t^2)$ and $E(\delta_{t-h}^2)$ as follows:

$$E(\epsilon_t^2) =$$
$$E[(x_t - a^T x)^2] =$$
$$E(x_t^2 - 2a^T x_t x + a^T x x^T a) =$$
$$E(x_t^2) - 2a^T E(x_t x) + a^T E(x x^T) a =$$
$$E(x_t^2) - 2a^T E(x_t x) + a^T E(x x^T) a =$$
$$\gamma(0) - 2a^T \gamma_{h-1} + a^T \Gamma_{h-1} a =$$
$$\gamma(0) - 2a^T \gamma_{h-1} + a^T \gamma_{h-1} =$$
$$\gamma(0) - a^T \gamma_{h-1}$$

$$E(\delta_{t-h}^2) =$$
$$E[(x_t - \tilde{a}^T x)^2]$$

Replacing a with $\tilde{a}$ and proceeding with the same steps as above, we obtain

$$\gamma(0) - \tilde{a}^T \tilde{\gamma}_{h-1}$$

To show that $E(\epsilon_t^2) = E(\delta_{t-h}^2)$, we need only show that $\tilde{a}^T \tilde{\gamma}_{h-1} = a^T \gamma_{h-1}$. This is true for the following:

$$\tilde{a}^T \tilde{\gamma}_{h-1} =$$
$$a_{h-1}\gamma(h-1) + a_{h-2}\gamma(h-2) + \cdots + a_2\gamma(2) + a_1\gamma(1)$$

$$a^T \gamma_{h-1} =$$
$$a_1\gamma(1) + a_2\gamma(2) + \cdots + a_{h-2}\gamma(h-2) + a_{h-1}\gamma(h-1)$$

But these are exactly the same sums, with one's summands in the reverse order of the other. Hence,

$$\tilde{a}^T \tilde{\gamma}_{h-1} = a^T \gamma_{h-1}$$

It follows

$$E(\epsilon_t^2) = E(\delta_{t-h}^2)$$
$$\frac{E(\epsilon_t \delta_{t-h})}{\sqrt{E(\epsilon_t^2) E(\delta_{t-h}^2)}} = \frac{E(\epsilon_t \delta_{t-h})}{\sqrt{E(\epsilon_t^2)^2}} = \frac{E(\epsilon_t \delta_{t-h})}{E(\epsilon_t^2)} =$$

$$\frac{\gamma(0)[\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}]}{\gamma(0) - \tilde{a}^T \tilde{\gamma}_{h-1}} =$$

$$\frac{\gamma(0)[\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}]}{\gamma(0) - \tilde{\gamma}_{h-1}^T \Gamma_{h-1}^{-1} \tilde{\gamma}_{h-1}} =$$

$$\frac{\gamma(0)[\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}]}{\gamma(0) - \gamma(0) \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}} =$$

$$\frac{\gamma(0)[\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}]}{\gamma(0)[1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}]} =$$

$$\frac{\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}}$$

as desired.

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To show the rightmost equality

$$\frac{\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}} = \alpha_{hh}$$

Write the matrix equation $R_n \alpha_n = \rho_n$ in partitioned form as:

$$\begin{bmatrix} R_{h-1} & \tilde{\rho}_{h-1} \\ \tilde{\rho}_{h-1}^T & \rho(0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_{hh} \end{bmatrix} = \begin{bmatrix} \rho_{h-1} \\ \rho(h) \end{bmatrix}$$

The first implied equation is

$$\begin{aligned} R_{h-1}\alpha_1 + \tilde{\rho}_{h-1}\alpha_{hh} &= \rho_{h-1} \\ R_{h-1}\alpha_1 &= \rho_{h-1} - \tilde{\rho}_{h-1}\alpha_{hh} \\ \alpha_1 &= R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\alpha_{hh}) \end{aligned}$$

Substituting into the second implied equation yields

$$\begin{aligned} \tilde{\rho}_{h-1}^T \alpha_1 + \rho(0)\alpha_{hh} &= \rho(h) \\ \tilde{\rho}_{h-1}^T R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\alpha_{hh}) + \alpha_{hh} &= \rho(h) \\ \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\rho_{h-1} - \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\tilde{\rho}_{h-1}\alpha_{hh} + \alpha_{hh} &= \rho(h) \\ \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\rho_{h-1} + \alpha_{hh}(1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\tilde{\rho}_{h-1}) &= \rho(h) \\ \alpha_{hh}(1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\tilde{\rho}_{h-1}) &= \rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1}\rho_{h-1} \\ \alpha_{hh} &= \frac{\rho(h) - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}_{h-1}^T R_{h-1}^{-1} \tilde{\rho}_{h-1}} \end{aligned}$$

as desired.