1.17(a) Express the joint characteristic function of $x_1, x_2, ..., x_n$, say,

$$\phi_{x_1,x_2,...,x_n}(\lambda_1, \lambda_2, ...\lambda_n)$$

in terms of $\phi_w(\cdot)$

We will find the characteristic function for the more general case of $x_{t_1}, x_{t_2}, ...x_{t_n}$, which enables a free selection of points rather than constraining them to be consecutive. Later in part(b), this will enable us to prove strict stationary.

$$\begin{aligned} \phi_{x_{t_1},x_{t_2},...x_{t_n}}(\lambda_1, \lambda_2, ...\lambda_n) = \\ E(e^{i\sum_{j=1}^{n} x_{t_j}\lambda_j}) = \\ E(e^{i\sum_{j=1}^{n}(w_{t_j}-w_{t_{j-1}}\theta)\lambda_j}) \end{aligned} \tag{1}$$

The sum in the exponent can be rewritten as follows:

$$\begin{aligned} \sum_{j=1}^{n}(w_{t_j} - w_{t_{j-1}}\theta)\lambda_j = \\ \sum_{j=1}^{n}(w_{t_j}\lambda_j - w_{t_{j-1}}\lambda_j\theta) = \\ \sum_{j=1}^{n} w_{t_j}\lambda_j - \theta\sum_{j=1}^{n} w_{t_{j-1}}\lambda_j = \\ \sum_{j=1}^{n} w_{t_j}\lambda_j - \theta\sum_{j=0}^{n-1} w_{t_j}\lambda_{j+1} = \\ w_{t_n}\lambda_n + \sum_{j=1}^{n-1}(w_{t_j}\lambda_j) - \theta\sum_{j=1}^{n-1}(w_{t_j}\lambda_{j+1}) - \theta w_{t_0}\lambda_1 = \\ w_{t_n}\lambda_n - \theta w_{t_0}\lambda_1 + \sum_{j=1}^{n-1}(w_{t_j}\lambda_j - \theta w_{t_j}\lambda_{j+1}) = \\ w_{t_n}\lambda_n - \theta w_{t_0}\lambda_1 + \sum_{j=1}^{n-1} w_{t_j}(\lambda_j - \theta\lambda_{j+1}) \end{aligned} \tag{2}$$

Replacing this result into the expectation:

$$
\begin{aligned}
E(e^{i\sum_{j=1}^{n}(w_{t_j}-w_{t_{j-1}}\theta)\lambda_j}) &= \\
E(e^{i[w_{t_n}\lambda_n-\theta w_{t_0}\lambda_1+\sum_{j=1}^{n-1}w_{t_j}(\lambda_j-\theta\lambda_{j+1})]} &= \\
E(e^{i[w_{t_n}\lambda_n-\theta w_{t_0}\lambda_1+\sum_{j=1}^{n-1}w_{t_j}(\lambda_j-\theta\lambda_{j+1})]} &= \\
E(e^{i(\lambda_n)w_{t_n}}\cdot e^{i(-\theta\lambda_1)w_{t_0}}\cdot\prod_{j=1}^{n-1}e^{i(\lambda_j-\theta\lambda_{j+1})w_{t_j}}) &= \\
E(e^{i(\lambda_n)w_{t_n}})\cdot E(e^{i(-\theta\lambda_1)w_{t_0}})\cdot\prod_{j=1}^{n-1}E(e^{i(\lambda_j-\theta\lambda_{j+1})w_{t_j}}) &= \\
\phi_{w_{t_n}}(\lambda_n)\cdot\phi_{w_{t_0}}(-\theta\lambda_1)\cdot\prod_{j=1}^{n-1}\phi_{w_{t_j}}(\lambda_j-\theta\lambda_{j+1})
\end{aligned}
\tag{3}
$$

Because the white noise terms are identically distributed, their characteristic functions must also be equal. We can represent the common white noise characteristic function as $\phi_w(\lambda)$, so the above result is

$$
\phi_{x_{t_1},x_{t_2},...x_{t_n}}(\lambda_1,\lambda_2,...\lambda_n)=\phi_w(\lambda_n)\cdot\phi_w(-\theta\lambda_1)\cdot\prod_{j=1}^{n-1}\phi_w(\lambda_j-\theta\lambda_{j+1}) \tag{4}
$$

Letting $t_1=1,t_2=2,...,t_n=n$, the joint characteristic function for $x_1,x_2,...,x_n$ is then

$$
\phi_{x_1,x_2,...,x_n}(\lambda_1,\lambda_2,...\lambda_n)=\phi_w(\lambda_n)\cdot\phi_w(-\theta\lambda_1)\cdot\prod_{j=1}^{n-1}\phi_w(\lambda_j-\theta\lambda_{j+1}) \tag{5}
$$

1.17(b): Deduce from (a) that x_t is strictly stationary.

We must show that for arbitrary h and an arbitrary set of points $t_1,t_2,...,t_n$, the joint distribution of $x_{t_1},x_{t_2},...x_{t_n}$ is the same as $x_{t_1+h},x_{t_2+h},...x_{t_n+h}$. Since characteristic functions are bijective with their underlying distributions, it suffices to show that

$$
\phi_{x_{t_1},x_{t_2},...x_{t_n}}(\lambda_1,\lambda_2,...\lambda_n)=\phi_{x_{t_1+h},x_{t_2+h},...x_{t_n+h}}(\lambda_1,\lambda_2,...\lambda_n)
$$

Eq. (5) gives an expression for the LHS. The RHS is

$$
\begin{aligned}
\phi_{x_{t_1+h},x_{t_2+h},\dots x_{t_n+h}}(\lambda_1,\lambda_2,\dots\lambda_n) &= \\
E(e^{i\sum_{j=1}^{n} x_{t_j+h}\lambda_j}) &= \\
E(e^{i\sum_{j=1}^{n} (w_{t_j+h} - w_{t_{j-1}+h}\theta)\lambda_j}) &
\end{aligned}
\tag{6}
$$

Since only the subscripts on the white noise terms have changed, by following the same calculations in equations (2) and (3), we can derive that the above is

$$
\begin{aligned}
\phi_{w_{t_1+h}}(\lambda_n)\cdot\phi_{w_{t_0+h}}(-\theta\lambda_1)\cdot\prod_{j=1}^{n-1}\phi_{w_{t_j+h}}(\lambda_j-\theta\lambda_{j+1}) &= \\
\phi_w(\lambda_n)\cdot\phi_w(-\theta\lambda_1)\cdot\prod_{j=1}^{n-1}\phi_w(\lambda_j-\theta\lambda_{j+1}) &= \\
\phi_{x_{t_1},x_{t_2},\dots x_{t_n}}(\lambda_1,\lambda_2,\dots\lambda_n) &
\end{aligned}
\tag{7}
$$

Q.E.D.