Ass Leak

(10 marks)

(a) The Laplace transform of $f(t) = te^{-2t}\, cos3t\, u(t)$ is:

Please choose one:

a) $\dfrac{3s^2+12s+5}{[s^2+4s+13]^2}$

b) $\dfrac{s^2+6s-5}{[s^2+6s+13]^2}$

c) $\dfrac{s^2+4s-5}{[s^2+4s+13]^2}$

d) None of the above

$$f(t) = te^{-2t} \cos(3t)\, u(t)$$

$$F(s) = -\frac{d}{ds}\left(\frac{s+2}{(s+2)^2\ 9}\right)$$

$$= -\frac{s^2+4s+13-(s+2)(2s+4)}{(s^2+4s+13)^2}$$

$$= \frac{2s^2+8s+8-s^2-4s-13}{(s^2+4s+13)^2}$$

$$= \frac{s^2+4s-5}{(s^2+4s+13)^2}$$

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(10 marks)

(b) The inverse Laplace transform of the following function $F(s) = \frac{(3s+3)e^{-3s}}{s^2+3s}$ is:

Please choose one:

a) $[1 + 2e^{-3(t-3)}]u(t-3)$

b) $[1 + 2e^{3(t-3)}]u(t-3)$

c) $[1 + 2e^{-3(t+3)}]u(t+3)$

d) None of the above

$$F(s) = e^{-3s}\left(\frac{3}{s+3} + \frac{3}{s(s+3)}\right)$$

$$= e^{-3s}\left(\frac{3}{s+3} + \frac{A}{s} + \frac{B}{s+3}\right)$$

$$\hookrightarrow A(s+3) + Bs = 3$$

$$\left.\begin{aligned} A+B&=0 \\ 3A&=3 \end{aligned}\right\} \begin{aligned} A&=1 \\ B&=-1 \end{aligned}$$

$$= e^{-3s}\left(\frac{2}{s+3} + \frac{1}{s}\right)$$

$$f(t) = \left[2e^{-3(t-3)} + 1\right]u(t-3)$$ ##

Question 3

(30 marks)

Please scroll to the bottom of page for END of question.

Use fixed-point iteration method to solve

$$f(x) = x^3 - 7x + 2 = 0, \quad x \in [0, 1].$$

Your approximate solution should be correct up to 3 decimal places.

(a) Does the interval $[0, 1]$ contain a root of $f(x) = 0$? (1 mark)

Please choose one:

a) Yes

b) No

(b) Set up a stopping criterion as follows

$$\text{absolute error} = |x_n - x^*| \le \frac{|f(x_n)|}{m} < 0.5 \times 10^{-r}.$$

r = 3 (integer input) (1 mark), m = 4 (integer input) (8 marks),

(c) Show that $f(x) = 0$ can be written as $x = g(x)$ where $g(x) = (x^3 + 2)/c$.
Check whether the fixed-point iteration scheme $x_{n+1} = g(x_n)$ will converge to a root of $f(x) = 0$ in $[0, 1]$. We know that convergence is guaranteed if

$$|g'(x)| \le L < 1 \; for \; all \; x \in [0, 1].$$

c = 7 (integer input) (2 marks)

L = 0.4286 (4 decimals) (8 marks)

Is the convergence of $x_{n+1} = g(x_n)$ guaranteed? (2 marks)

Please choose one:

a) No

b) Yes

(d) Using the $g(x)$ in part (c) and the stopping criterion obtained in part (b), carry out the iteration scheme $x_{n+1} = g(x_n), \; with \; x_0 = 0$. Find an approximate solution of $f(x) = 0, \; x \in [0, 1]$ that is correct to 3 decimal places.

n = 2 (integer input) (3 marks)

x_n = 0.289 (3 decimals) (5 marks)

$f(x) = x^3 - 7x + 2 = 0 \quad , x \in b : [0,1]$

a) $f(0) = 2$, $f(1) = -4$, There is sign change , $\therefore$ Interval Contains solution $\longrightarrow$ Yes

b) for accurate to 3 d.p.

$$\frac{|f(x_n)|}{m} < 0.5 \times 10^{-3} \longrightarrow r = 3$$

$$\hookrightarrow m = \min_{x \in b} |f'(x_n)|$$
$$= \min_{x \in b} |3x^2 - 7|$$
$$= |3(1)^2 - 7|$$
$$= 4$$

c) $g(x_n) = x_{n+1} = \frac{x^3 + 2}{7} \longrightarrow c = 7$

for convergence

$\hookrightarrow |g'(x_n)| \leq L < 1$

$\hookrightarrow L = \max_{x \in b} |g'(x)|$

$$= \max_{x \in b} \left| \frac{3}{7} x^2 \right|$$
$$= \frac{3}{7}(1)^2$$
$$= \frac{3}{7} \approx 0.4286 \text{ [4dp]}$$

$\therefore \quad n = 2, \; f(x_n) = 0.289 \; [3dp]$

Question 4

(30 marks)

Solve the following initial value problem using the method of Laplace Transform

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0, \qquad y(0) = 4, \quad y'(0) = 2.$$

Let $Y(s)$ be the Laplace transform of $y(t)$.

Please scroll to the bottom of page for END of question.

(a) Find $Y(s)$. We write, in simplest fraction form, (12 marks)

$$Y(s) = \frac{c_1 s^2 + c_2 s + c_3}{c_4 s^2 + c_5 s + c_6},$$

where c_1, c_2, ..., c_6 are nonnegative integers. Give the answers of c_1, c_2, ..., c_6 (nonnegative integers).

c_1 = 0

c_2 = 4

c_3 = 22

c_4 = 1

c_5 = 5

c_6 = 4

(b) Do a partial fraction expansion for $Y(s)$. We get

$$Y(s) = \frac{a_1}{s+1} - \frac{a_2}{a_3 s + a_4},$$

where a_1, ..., a_4 are nonnegative integers. Give the answers of a_1, ..., a_4 (nonnegative integers). (12 marks)

a_1 = 6

a_2 = 2

a_3 = 1

a_4 = 4

(c) Find $y(t)$. We write

$$y(t) = \left(b_1 e^{-t} + b_2 e^{b_3 t}\right) u(t),$$

where b_1, b_2, b_3 are integers. Give the answers of b_1, b_2, b_3 (integers). (6 marks)

b_1 = 6

b_2 = -2

b_3 = -4

$$s\left(sY(s) - y(0)\right) - y'(0) + 5\left(sY(s) - y(0)\right) + 4Y(s) = 0$$

$$Y(s)\left(s^2 + 5s + 4\right) - 4s - 2 - 20 = 0$$

a) $$Y(s) = \frac{4s+22}{s^2+5s+4}$$

$C_1 = 0$ $\quad C_4 = 1$

$C_2 = 4$ $\quad C_5 = 5$

$C_3 = 22$ $\quad C_6 = 4$

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b) $$Y(s) = \frac{A}{s+1} + \frac{B}{s+4}$$

$a_1 = 6$

$a_2 = 2$

$a_3 = 1$

$a_4 = 4$ #

$\hookrightarrow A(s+4) + B(s+1) = 4s + 22$

$$\left.\begin{aligned} A + B &= 4 \\ 4A + B &= 22 \end{aligned}\right\} \quad \begin{aligned} A &= 6 \\ B &= -2 \end{aligned}$$

$$= \frac{6}{s+1} - \frac{2}{s+4}$$

c) $$y(t) = \left[6e^{-t} - 2e^{-4t}\right]u(t)$$

$b_1 = 6$

$b_2 = -2$

$b_3 = -4$

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Question 2

(20 marks)

Consider the initial value problem (IVP)

$$y'' + x^2y' + 2xy = 3,$$

$$y(2) = 3, \qquad y'(2) = 0.$$

Use the Improved Euler Method with step size $h = 0.1$ to find approximate values of $y(2.2)$ and $y'(2.2)$.

Please scroll to the bottom of page for END of question.

(a) Rewrite the IVP into first order differential equations and write down the Improved Euler Method explicitly. (3 marks)

Let $y_1 = y$ and $y_2 = y'$. It follows that

$$y_1' = f_1(x, y_1, y_2),$$

$$y_2' = f_2(x, y_1, y_2).$$

(i) What is the first function $f_1(x, y_1, y_2)$?

Please choose one:

a) $f_1(x, y_1, y_2) = \dfrac{3 - y_3 - x^2y_2}{2x}$

b) $f_1(x, y_1, y_2) = y'$

c) $f_1(x, y_1, y_2) = y_2$

d) $f_1(x, y_1, y_2) = \dfrac{3 - y'' - x^2y'}{2x}$

(ii) What is the second function $f_2(x, y_1, y_2)$?

Please choose one:

a) $f_2(x, y_1, y_2) = 3 - x^2y' - 2xy$

b) $f_2(x, y_1, y_2) = 3 - x^2y_2 - 2xy_1$

c) $f_2(x, y_1, y_2) = y''$

d) $f_2(x, y_1, y_2) = y_3$

(b) Fill in the blanks in the following table. (Give your answers in 8 decimal places. Give exact value if the answer is less than 8 decimal places.) (15 marks)

n	x_n	$y_{1,n}$	$y_{2,n}$	a_1	a_2	b_1	b_2
0	x_0	$y_{1,0}$	$y_{2,0}$	$a_{1,0}$	$a_{2,0}$	$b_{1,0}$	$b_{2,0}$
1	x_1	$y_{1,1}$	$y_{2,1}$	$a_{1,1}$	$a_{2,1}$	$b_{1,1}$	$b_{2,1}$
2	x_2	$y_{1,2}$	$y_{2,2}$	-	-	-	-

x_0 = 2.0

x_1 = 2.1

x_2 = 2.2

$y_{1,0}$ = 3

$y_{2,0}$ = 0

$a_{1,0}$ = 0

$a_{2,0}$ = -0.9

$b_{1,0}$ = -0.09

$b_{2,0}$ = -0.5631

$y_{1,1}$ = 2.955

$y_{2,1}$ = -0.73155

$a_{1,1}$ = -0.073155

$a_{2,1}$ = -0.61848645

$b_{1,1}$ = -0.13500365

$b_{2,1}$ = -0.31459416

$y_{1,2}$ = 2.85092068

$y_{2,2}$ = -1.19809030

(c) From the above calculations, obtain approximate values of $y(2.2)$ and $y'(2.2)$. (Give your answers in 4 decimal places.) (2 marks)

$y(2.2) \approx$ 2.8509

$y'(2.2) \approx$ -1.1981

a) i) $y_1' = f_1(x, y_1, y_2) = y_2$ #

ii) $y_2' = f_2(x, y_1, y_2) = 3 - x^2 y_2 - 2x y_1$ #

b) $\tilde{y}_{n+1} = \tilde{y}_n + \frac{1}{2}(\tilde{a}_n + \tilde{b}_n)$

$a_{1,n} = 0.1\, y_{2,n}$ $\qquad$ $a_{2,n} = 0.1(3 - x_n^2 y_{2,n} - 2x_n y_{1,n})$

$b_{1,n} = 0.1(y_{2,n} + a_{2,n})$ $\qquad$ $b_{2,n} = 0.1(3 - x_{n+1}^2(y_{2,n} + a_{2,n})$
$- 2x_{n+1}(y_{1,n} + a_{1,n}$

n	x_n	$y_{1,n}$	$y_{2,n}$	$a_{1,n}$	$a_{2,n}$	$b_{1,n}$	$b_{2,n}$
0	2	3	0	0	-0.9	-0.09	-0.5631
1	2.1	2.955	-0.73155	-0.073155	-0.61848645	-0.185008645 $\approx$ -0.18500865	-0.314594 $\approx$ -0.314
2	2.2	2.850920678 $\approx$ 2.85092068	-1.198090304 $\approx$ -1.19809030				

c) $f(2.2) \approx 2.8509$ [4dp] #

$f'(2.2) \approx -1.1981$ [4dp] #