Cobb-Douglas production functions

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

$$0 < \alpha < 1$$

Code: K[t]^α L[t]^(1-α)

Decreasing MPK:

$$MPK = \frac{\partial F(K_t, L_t)}{\partial K_t}$$

$$= \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

$$= \alpha \frac{L_t^{1-\alpha}}{K_t^{1-\alpha}}$$

$$= \alpha \left(\frac{L_t}{K_t}\right)^{1-\alpha}$$

Question: What happens to MPK as K_t increases?

MPK goes down.

Returns to scale:

$$F(zK_t, zL_t) = (zK_t)^{\alpha}(zL_t)^{1-\alpha}$$

$$= z^{\alpha} K_t^{\alpha} z^{1-\alpha} L_t^{1-\alpha}$$

$$= \underbrace{z^{\alpha} z^{1-\alpha}}_{z} \cdot \underbrace{K_t^{\alpha} L_t^{1-\alpha}}_{F(K_t, L_t)}$$

$\longrightarrow$ Constant RTS

Note: Book says $F(K, L) = A K^{\alpha} L^{1-\alpha}$

We'll ignore A for now ($A = 1$)

Let's set $z = 1/L_t$

$$F(zK_t, zL_t) = F(K_t/L_t, 1)$$

$$\frac{1}{L_t} F(K_t, L_t) = \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$Y_t = F(K_t, L_t)$$

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^{\alpha}$$

Define: $k_t = \dfrac{K_t}{L_t}$

$$y_t = \frac{Y_t}{L_t}$$

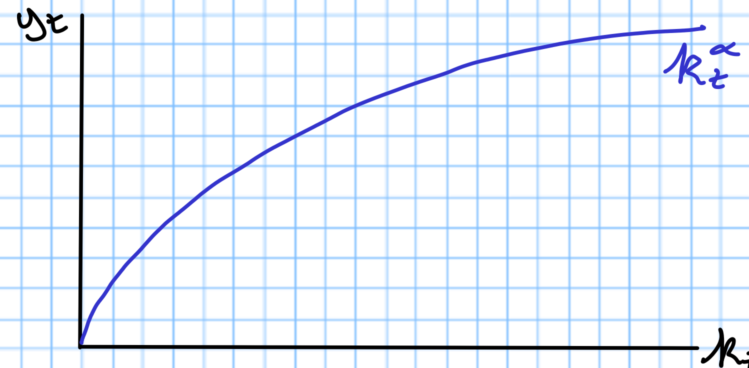
Then $y_t = k_t^{\alpha}$

→ Lower-case variable will mean "per worker"

$k_t = \dfrac{K_t}{L_t}$ capital per worker

$y_t = \dfrac{Y_t}{L_t}$ income per worker

$y_t = k_t^{\alpha}$ is the per-worker production function



Solow model

(AKA: Solow-Swan, neoclassical growth, exogenous growth)

Capital law of Motion

$$K_{t+1} = K_t + I_t - D_t$$

capital next year = capital this year + new capital (investments) − depreciation

Assume output is either consumed or saved

- Constant savings rate γ

$$I_t = \gamma Y_t$$

(consumption $(1-\gamma)Y_t$)

MPC

- Constant depreciation rate δ

$$D_t = \delta K_t$$

- Constant population

$$L_t = \bar{L}$$

CLM:

$$K_{t+1} = K_t + \gamma Y_t - \delta K_t$$

Now divide both sides by $\bar{L}$

$$\frac{K_{t+1}}{\bar{L}} = \frac{K_t}{\bar{L}} + \gamma \frac{Y_t}{\bar{L}} - \delta \frac{K_t}{\bar{L}}$$

$$k_{t+1} = k_t + \gamma y_t - \delta k_t$$

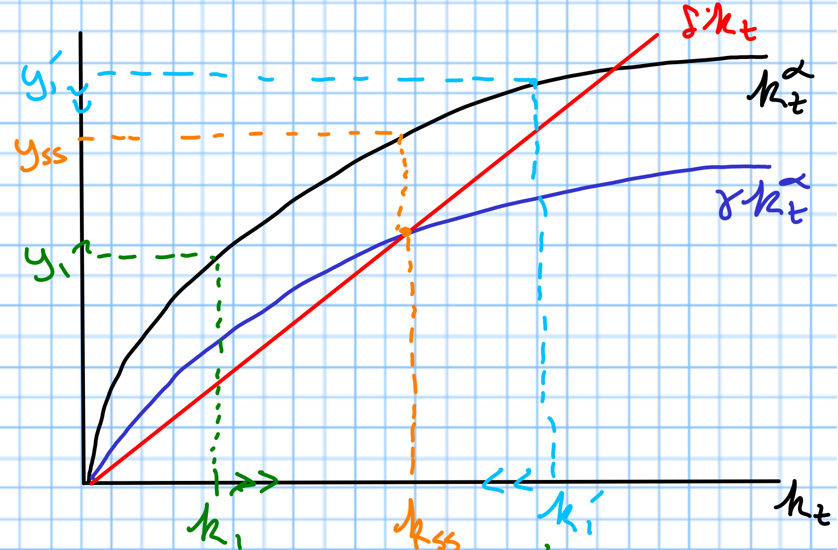
$$k_{t+1} - k_t = \gamma y_t - \delta k_t$$

$$\Delta k_t = \gamma y_t - \delta k_t$$

change in k from t to $t+1$

Recall: $y_t = k_t^\alpha$

$$\Delta k_t = \gamma k_t^\alpha - \delta k_t$$



when $k_t = k_1$ $\quad \gamma k_1^\alpha > \delta k_1$

$\Delta k_1 > 0$

when $k_t = k_1'$ $\quad \gamma k_1^\alpha < \delta k_1$

$\Delta k_1' < 0$

When $k_t = k_{ss}$ $\gamma k_t^{\alpha} = \delta k_t$

$\Delta k_t = 0$

k_{ss} is the steady state level of capital per worker

When $k_t = k_{ss}$, $\Delta k_t = 0$

$$0 = \gamma k_t^{\alpha} - \delta k_t$$

$$\frac{\gamma k_t^{\alpha}}{k_t^{\alpha}} = \frac{\delta k_t}{k_t^{\alpha}}$$

$$\gamma = \delta k_t^{1-\alpha}$$

$$k_t^{1-\alpha} = \frac{\gamma}{\delta}$$

$$\boxed{k_{ss} = \left(\frac{\gamma}{\delta}\right)^{1/1-\alpha}}$$

$$y_{ss} = k_{ss}^{\alpha}$$

$$= \left[\left(\frac{\gamma}{\delta}\right)^{1/1-\alpha}\right]^{\alpha}$$

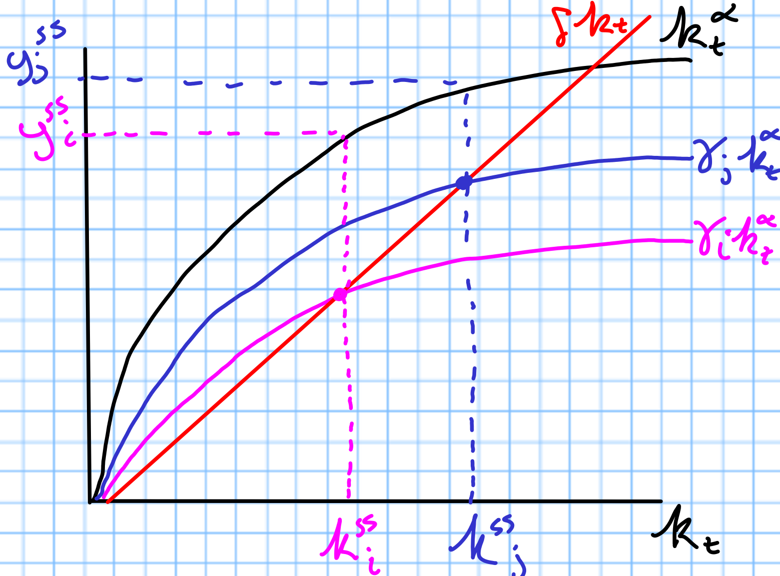
$$\boxed{y_{ss} = \left(\frac{\gamma}{\delta}\right)^{\alpha/1-\alpha}}$$

Examples

- Suppose there are two countries i and j

$\gamma_i < \gamma_j$

Country i saves more than j



$\gamma_j > \gamma_i \rightarrow k_j^{ss} > k_i^{ss}$

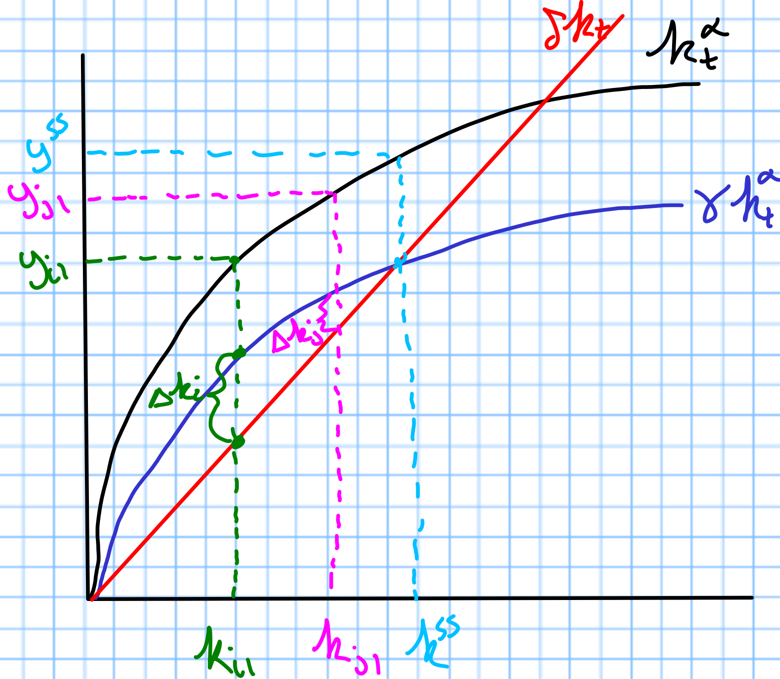
$\rightarrow y_j^{ss} > y_i^{ss}$

- All else equal, country with higher savings rate has higher steady state

Example: Suppose i and j are the same in every way, except $y_{j1} > y_{i1}$

y_{j1}: j's income in period 1

$\rightarrow$ Country i starts out with less income per worker than j



$$\Delta k_{it} = \gamma k_{it}^\alpha - \delta k_{it}$$

$$\Delta k_i > \Delta k_j$$

All else equal, a country that starts further away from the steady state will grow faster

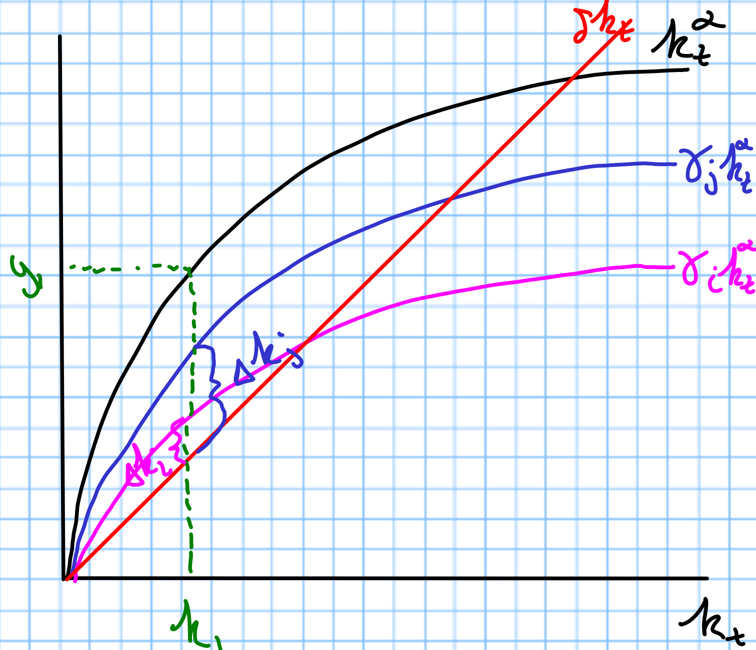
"catch-up effect"

→ poor countries tend to "catch up" to rich countries

<u>Example:</u>

Suppose $y_i = y_j$ (less than steady state)

But $\gamma_i < \gamma_j$



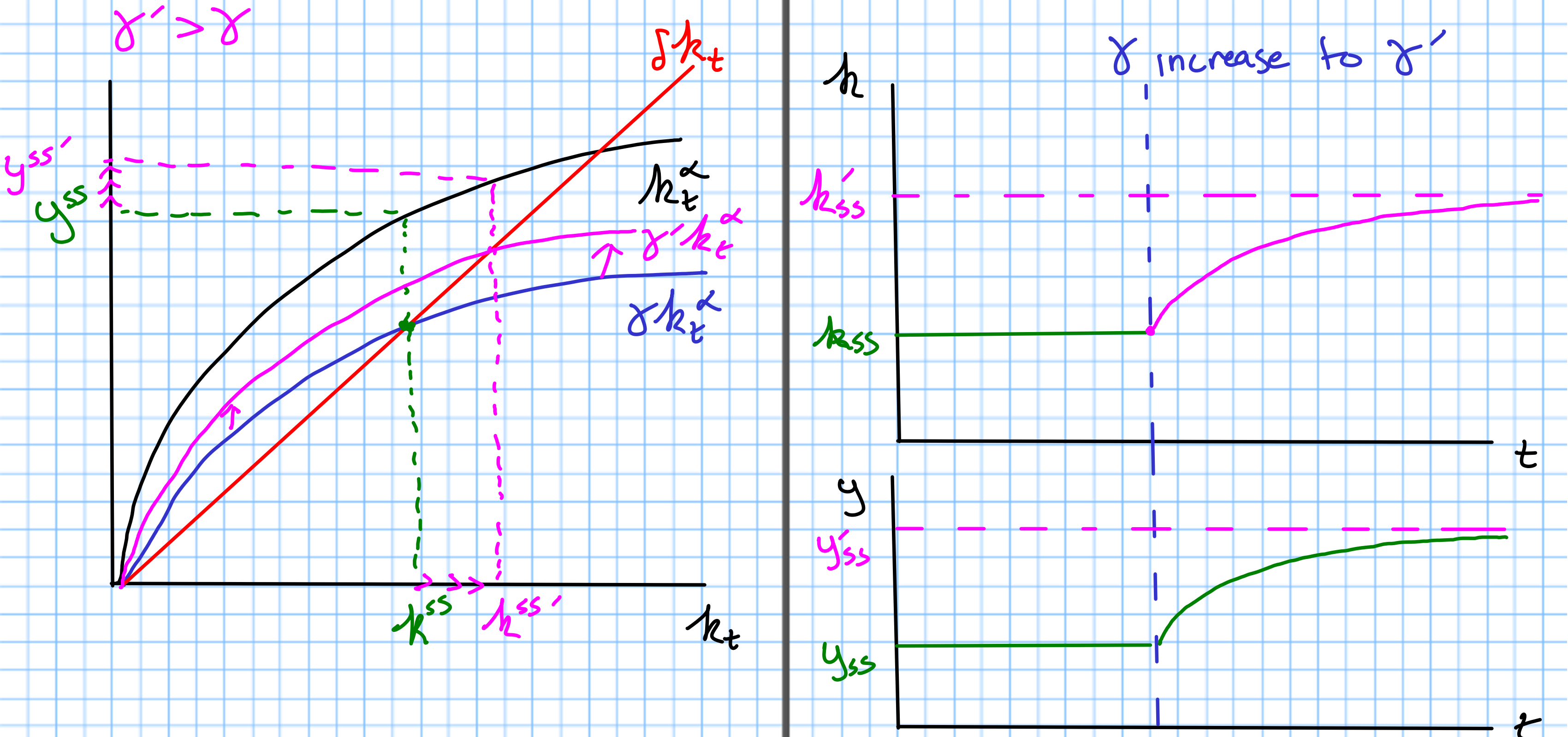
$\Delta k_i < \Delta k_j$

→ All else equal, country with higher savings rate grows faster (in the short run)

→ j has a higher steady state, and is starting further away

Example: (like the HW)

Suppose a country is at its steady state, then it increases γ.



γ' > γ
δk_t
k_t^α
γ'k_t^α
γk_t^α
y^ss'
y^ss
k^ss
k^ss'
k_t
γ increase to γ'
k
k'_ss
k_ss
t
y
y'_ss
y_ss
t