Income Shares

Percentage of total income that is received by various productive factors

Price of labor: $w = MPL$

$$MPL = \frac{\partial F(K_t, L_t)}{\partial L_t}$$

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

$$w = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}$$

$$r = \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

Total income received by capital: $r K_t$

Capital share of total income:

$$\frac{r K_t}{Y_t} = \frac{\alpha K_t^{\alpha-1} L_t^{1-\alpha} \cdot K_t}{K_t^{\alpha} L_t^{1-\alpha}}$$

$$= \alpha \frac{\cancel{K_t^{\alpha} L_t^{1-\alpha}}}{\cancel{K_t^{\alpha} L_t^{1-\alpha}}}$$

"capital share" $= \alpha$

labor share $= 1-\alpha$

Data: $\alpha \approx 1/3$

Development accounting

$$Y_t = A K_t^\alpha (h L_t)^{1-\alpha}$$

A: productivity

output = productivity × factors

Per-worker terms:

$$\frac{Y_t}{L_t} = \frac{A K_t^\alpha (h L_t)^{1-\alpha}}{L_t}$$

$$y_t = \frac{A K_t^\alpha h^{1-\alpha} L_t^{1-\alpha}}{L_t}$$

$$y_t = A K_t^\alpha h^{1-\alpha} L_t^{-\alpha}$$

$$y_t = \frac{A K_t^\alpha h^{1-\alpha}}{L^\alpha} = A k_t^\alpha h^{1-\alpha}$$

Let's compare 2 countries:

$$\frac{y_{it}}{y_{jt}} = \frac{A_i k_{it}^\alpha h_i^{1-\alpha}}{A_j k_{jt}^\alpha h_j^{1-\alpha}}$$

$$\underbrace{\frac{y_{it}}{y_{jt}}}_{\text{observed}} = \underbrace{\frac{A_i}{A_j}}_{\text{unobserved}} \underbrace{\frac{k_{it}^\alpha h_i^{1-\alpha}}{k_{jt}^\alpha h_j^{1-\alpha}}}_{\text{observed}}$$

$$\frac{A_i}{A_j} = \frac{y_{it}/y_{jt}}{(k_{it}^\alpha h_i^{1-\alpha})/(k_{jt}^\alpha h_j^{1-\alpha})}$$

Example

	y_t	k_t	h
i	24	27	8
j	1	1	1

$$\frac{A_i}{A_j} = \frac{24}{27^{1/3} \cdot 8^{2/3}}$$

$$= \frac{24}{3\left(8^{1/3}\right)^2}$$

$$= \frac{24}{3 \cdot 4}$$

$$= 2$$

Growth accounting

$$Y_t = A_t K_t^{\alpha} (h_t L_t)^{1-\alpha}$$

Notation:

- use "^" to denote growth rates
- Example: $\hat{X}$ is the growth rate of X_t

  $\hat{Y}$ is growth of Y_t

$$\hat{X} = \frac{X_{t+1} - X_t}{X_t}$$

Production function in per-worker terms:

$$\frac{Y_t}{L_t} = \frac{A_t K_t^{\alpha} (h_t L_t)^{1-\alpha}}{L_t}$$

$$y_t = A_t K_t^{\alpha} h_t^{1-\alpha} L_t^{1-\alpha} L_t^{-1}$$

$$= A_t K_t^{\alpha} h_t^{1-\alpha} L_t^{-\alpha}$$

$$= \frac{A_t K_t^{\alpha} h_t^{1-\alpha}}{L_t^{\alpha}}$$

$$= A_t \left(\frac{K_t}{L_t}\right)^{\alpha} h_t^{1-\alpha}$$

$$= A_t k_t^{\alpha} h_t^{1-\alpha}$$

$$y_{t+1} = A_{t+1} k_{t+1}^{\alpha} h_{t+1}^{1-\alpha}$$

Goal: calculate $\hat{y}$ as a function of everything else $(\hat{A}, \hat{k}, \hat{L}, \hat{h})$

$$\hat{y} = \frac{y_{t+1} - y_t}{y_t}$$

$$\hat{y} = \frac{y_{t+1}}{y_t} - \frac{y_t}{y_t}$$

$$\hat{y} = \frac{y_{t+1}}{y_t} - 1$$

$$\hat{y} = \frac{A_{t+1} k_{t+1}^{\alpha} h_{t+1}^{1-\alpha}}{A_t k_t^{\alpha} h_t^{1-\alpha}} - 1$$

$$\hat{y} = \left(\frac{A_{t+1}}{A_t}\right)\left(\frac{k_{t+1}}{k_t}\right)^{\alpha}\left(\frac{h_{t+1}}{h_t}\right)^{1-\alpha} - 1$$

$$\hat{A} = \frac{A_{t+1} - A_t}{A_t}$$

$$\hat{A} = \frac{A_{t+1}}{A_t} - 1$$

$$\underline{\hat{A}+1} = \frac{A_{t+1}}{A_t}$$

$$\hat{y} = (\hat{A}+1)(\hat{k}+1)^{\alpha}(\hat{h}+1)^{1-\alpha} - 1$$

$$(\hat{y}+1) = (\hat{A}+1)(\hat{k}+1)^{\alpha}(\hat{h}+1)^{1-\alpha}$$

Take logs of both sides:

Rules: $\ln(X \cdot Y) = \ln(X) + \ln(Y)$

$\ln(X^a) = a\ln(X)$

$$\ln(\hat{y}+1) = \ln\left[(\hat{A}+1)(\hat{k}+1)^{\alpha}(\hat{h}+1)^{1-\alpha}\right]$$

$$\ln(\hat{y}+1) = \ln(\hat{A}+1) + \ln\left[(\hat{k}+1)^{\alpha}\right] + \ln\left[(\hat{h}+1)^{1-\alpha}\right]$$

$$\ln(\hat{y}+1) = \ln(\hat{A}+1) + \alpha\ln(\hat{k}+1) + (1-\alpha)\ln(\hat{h}+1)$$

<u>Taylor's theorem</u>

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

1st order Taylor approximation of $f(x) = \ln(1+x)$ around $a = 0$

$$\ln(x+1) \approx \ln(0+1) + \frac{\frac{d}{dx}\ln(0+1)}{1!}(x)$$

$$\ln(x+1) \approx 0 + \frac{\frac{1}{1}}{1!} \cdot x$$

$$\ln(x+1) \approx x$$

whenever x is "small"

$$\boxed{\hat{y} = \hat{A} + \alpha \hat{k} + (1-\alpha)\hat{h}}$$

We observe everything in this equation except $\hat{A}$

$$\hat{A} = \hat{y} - \alpha \hat{k} - (1-\alpha)\hat{h}$$

$\hat{A}$ (the growth rate of productivity) is the proportion of income growth ($\hat{y}$) that is not explained by the growth rate of factors

$\hat{A}$ is the "Solow residual"