Sequential Games

ECON 420: Game Theory

Spring 2018

Centipede game

1. 2 players play for 10 rounds
2. Each round another unit of payoff is added to the pot (starting with one unit)
3. Players alternate turns, choose to either:
   - Stop, and collect the entire pot for themselves
   - Continue, one is added to the pot and next player chooses

Sequential games

- Games where there is a strict order of play
- Games where players take turns moving are sequential
- Real-world games are generally combinations of sequential and simultaneous games

Game trees

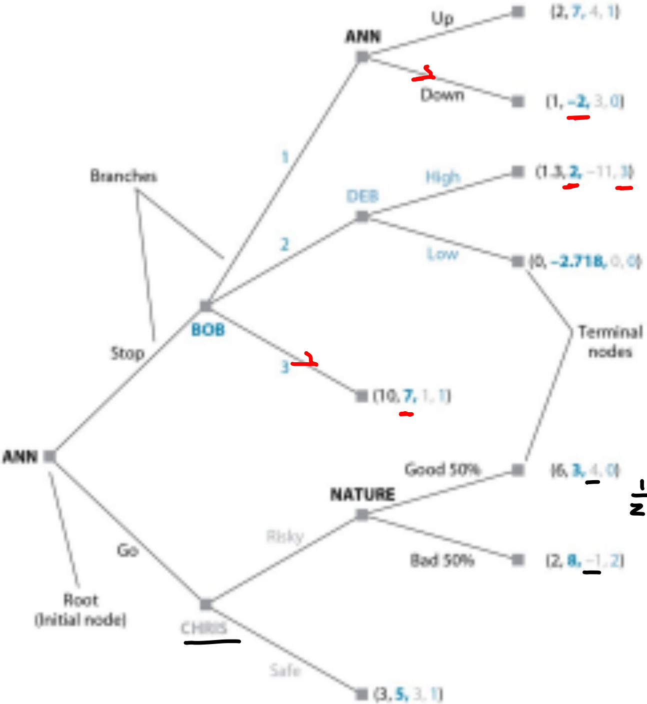
- We will visualize games using **game trees**
- Representing a game as a tree is known as the "extensive form" of a game
- The tree shows all components of a game: players, actions and strategies, payoffs

Nodes and branches

- *Nodes* are points on the tree where choices are made
  - The first node is called the *root node*
  - The last nodes (without branches) are *terminal nodes*
- *Branches* show the actions available for the player to choose among at any node
- A node (and its branches) represent a "turn" for a player
- Payoffs are listed at the terminal nodes
  - Each player in the game gets a payoff at each node
  - Remember: Higher numbers are always better

External uncertainty

- With external uncertainty, we introduce nature as a "player"
- Nature gets its own node, branches are possible outcomes
- Players calculate expected payoffs across the possible outcomes of nature's "choice"



$\frac{1}{2} \cdot 4 + \frac{1}{2}(-1)$

$2 - \frac{1}{2}$

$= 1\frac{1}{2}$

Moves vs strategy

- A choice of action at a node is called a *move*
- A strategy is a *complete plan of action*
  - A set of moves that will be performed if a certain situation arises
  - Strategies are collections of statements like "if X then Y" for *any possible* X

(1) choose stop, then up (stop, up)
(2) choose stop, then down (stop, down)
(3) go, up
(4) go, down

Example

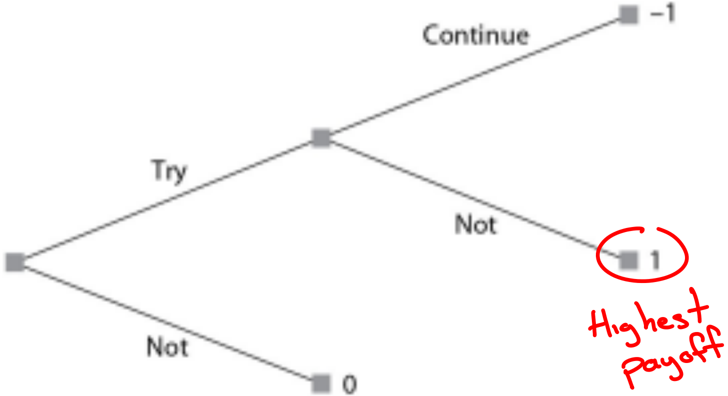
- How many strategies does Ann have?
- What are they?

Strategies

- Strategies must include actions at *each node where a player can move*
- This includes the nodes that won't be reached if a player chooses a particular set of actions
- This is because hypothetical moves might help determine which moves should be chosen at earlier nodes
- Choices early in a game are affected by *expectations* about what will happen later in the game

Finding equilibria in game trees

- Consider one person's decision tree (is this a game?)
- The player (Carmen) is considering whether or not to start smoking
- Carmen first decides whether to start, then decides whether to continue
- What should Carmen do?


Continue
−1
Try
Not
1
Highest payoff
Not
0

A decision tree as a game

- Previous decision tree ignore that Carmen may become addicted if she starts smoking
  - Once addicted, quitting becomes worse (payoffs are lower)
- Carmen knows she may become addicted and that her payoffs might change if she starts smoking
- We can think of this as a game where the players are Carmen today and Carmen in the future (after the initial decision is made)
- Today's Carmen and future Carmen have different payoffs

Today's carmen: N

Future Carmen: C

Pruning

- Starting at the end, we can "prune" the branches that we know will not be chosen
- When there is one action remaining at the final nodes, this means that the "final" decision moves back to the previous node (rollback)
- Starting at the end and moving backward by pruning allows today's Carmen to choose the best option for herself
- When all players use rollback analysis, the result of the game is called a *rollback equilibrium*

Smoking game

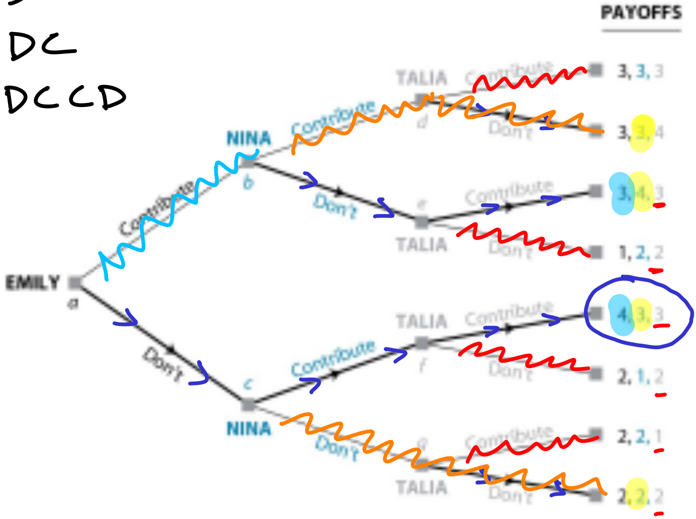
- What are the rollback equilibrium strategies?
- Can either player do better by changing their strategies?

Equilibrium strategies

Emily: D

Nina: DC

Talia: DCCD



Emily's strategies: 1 C
2 D

Three-player game

- How many strategies does each player have?
- What is the rollback equilibrium?
- What are the rollback equilibrium strategies?

Nina: 1 Contribute always
2 C if Emily C, D if Emily D
3 D if Emily C, C if Emily D
4 Always D

Nina: (1) (C,C)

(2) C at b, D at C or (C,D)

choice at c

choice at b

(3) (D,C)

(4) (D,D)

Talia's strategies in the form (d,e,f,g)

(1) (D,C,C,D)

(2) (D,D,D,D)

(3) (C,C,C,C)

(4) (C,D,C,D)

(5) (D,C,D,C)

(6) (D,D,D,C)

(7) (D,D,C,C)

(8) (D,C,C,C)

(9) (C,D,D,C)

(10) (C,C,C,D)

(11) (C,C,D,D)

(12) (C,D,D,D)

(13) (D,D,C,D)

(14) CCDC

(15)

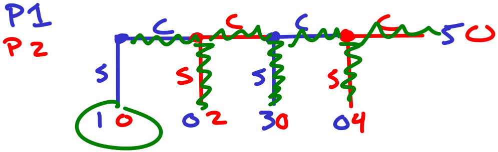
⋮

use a system

CCCC
CCCD
CCDC
CCDD
CDCC
CDCD
CDDC
CDDD
DCCC
DCCD
DCDC
DCDD
DDCC
DDCD
DDDC
DDDD

Example: Ultimatum game

- Player 1:
  - Choose how to split 10 units so that both players get at least one unit
- Player 2:
  - Choose to either:
    1. Accept the split (you get what player 1 chooses for you)
    2. Reject the split (neither player gets anything)



Example: Centipede game (5 round version)

- What does the game tree look like?
- What are the strategies for each player?
- What is the rollback equilibrium outcome? 1,0
- What are the rollback equilibrium strategies? $\underbrace{\text{P1: SS, P2: SS}}$
- Is this the outcome we observe in practice?

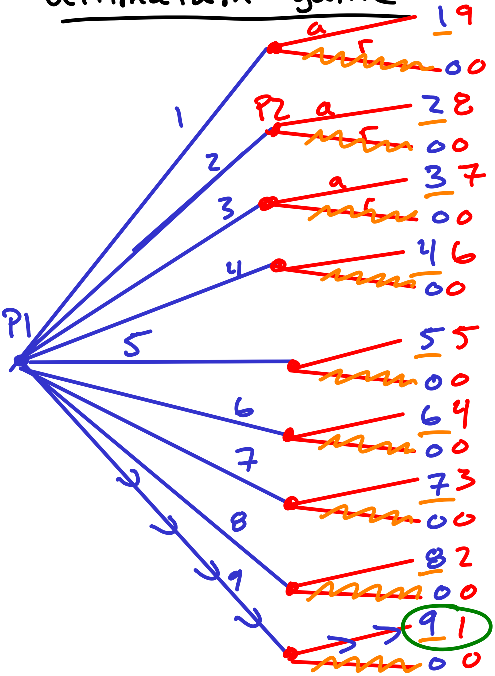
unique roll-back equilibrium

strategies:

P1: C, C
C, S
S, C
S, S

P2: C C
C S
S C
S S

Ultimatum game



rollback equilibrium

Equilibrium strategies

P1: 9

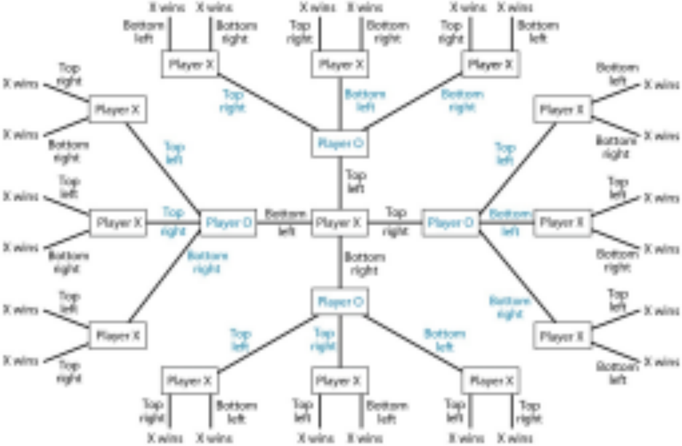
P2: (a,a,a,a,...

Suppose P2 chooses:

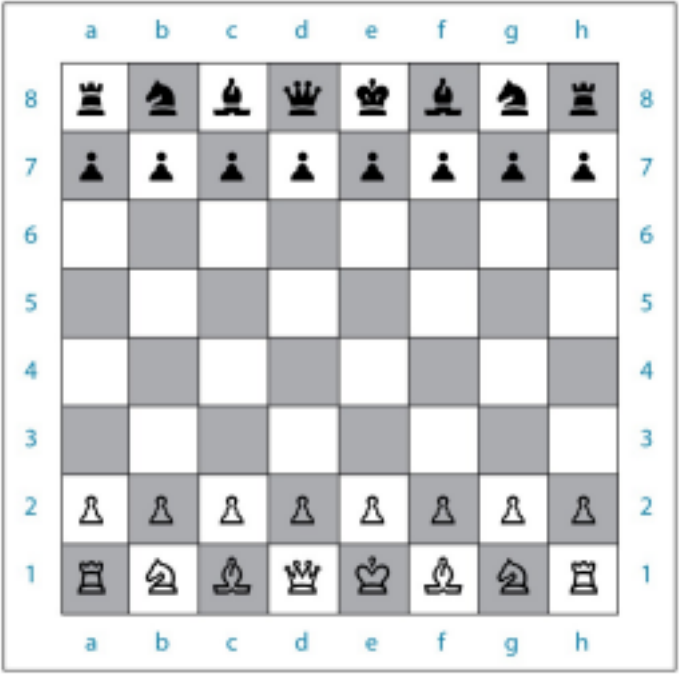
(a,a,a,a,a,a,a,a,r)

Limitations of rollback analysis

- Simple games can become difficult to express in extensive form
  - How many moves does the first player have in tic-tac-toe?
  - How many moves does the second player have?
- Some sequential games are *impossible* to express in extensive form!



X wins
X wins
Bottom left
Bottom right
X wins
X wins
Top right
Bottom right
X wins
X wins
Top right
Bottom left
Player X
Player X
Player X
Top right
X wins
Player X
Top right
Bottom left
Bottom right
Bottom left
X wins
Player X
X wins
Bottom right
Top left
Player O
Top left
X wins
Bottom right
Top left
X wins
Top left
Top left
X wins
Player X
Top right
Player O
Bottom left
Player X
Top right
Player O
Bottom left
Player X
X wins
Bottom right
Bottom right
X wins
Bottom right
Bottom right
Top left
X wins
Top left
Player O
Bottom right
X wins
Player X
Top left
Top right
Bottom left
Player X
X wins
Top right
X wins
Player X
Player X
Player X
Bottom left
Top right
Bottom left
Top left
Bottom left
Top left
Top right
X wins
X wins
X wins
X wins
X wins
X wins


a b c d e f g h
8 7 6 5 4 3 2 1

Chess

- 400 possible positions (nodes) after each player moves once
- 9 million after the third move
- 288 billion after the forth move
- 40 move game: More possible positions than fundamental particles in the universe