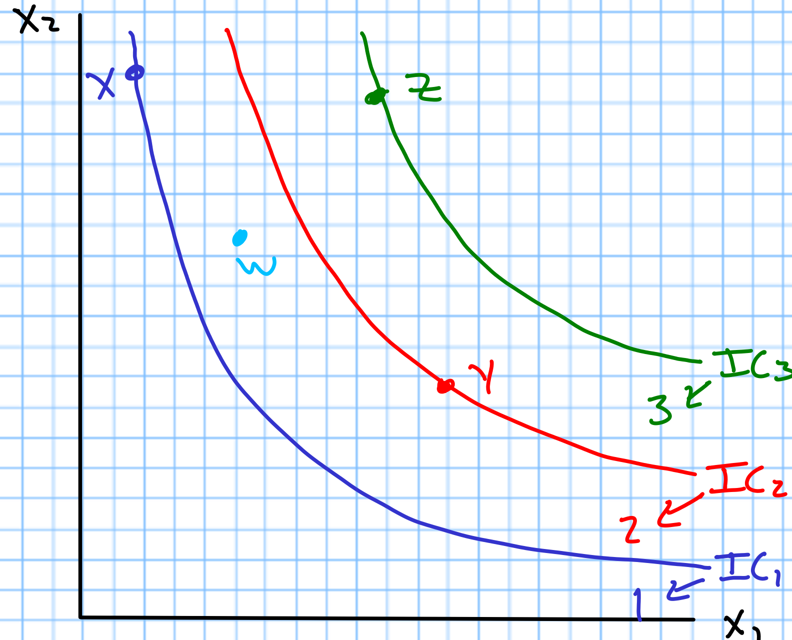
Utility



$Z \succ Y$, $Y \succ X$, $Z \succ X$

- Let's assign a <u>number</u> to each indifference curve
- If every point on IC_A is preferred to every point on IC_B, Then IC_A gets a larger number
- Now define a <u>function</u>
  - input: bundle
  - output: The number associated with the IC that passes through that bundle

- This allows us to compare bundles by comparing the value of the function evaluated at each bundle
- Call this function a <u>utility</u> function (output: utility)

$X = (x_1, x_2)$

$Y = (y_1, y_2)$

$Z = (z_1, z_2)$

$u(x_1, x_2) = 1$

$u(y_1, y_2) = 2$

$u(z_1, z_2) = 3$

$u(w_1, w_2) = 1.5$

Theorem: If preferences are rational, then there always exists a function $u(\cdot)$ such that: if $(x_1, x_2) \succ (y_1, y_2)$, then $u(x, x_2) > u(y_1, y_2)$ and if $(x_1, x_2) \sim (y_1, y_2)$ then $u(x_1, x_2) = u(y_1, y_2)$

<u>Example</u>

$u(x_1, x_2) = 1$

$u(y_1, y_2) = 2$

$u(z_1, z_2) = 3$

Suppose we have another function $v(\cdot) = 10u(\cdot)$

$v(x_1, x_2) = 10$

$v(y_1, y_2) = 20$

$v(z_1, z_2) = 30$

- suppose there are two bundles (x_1, x_2), (x_1', x_2')

and $u(x_1, x_2) > u(x_1', x_2')$

$10u(x_1, x_2) > 10u(x_1', x_2')$

$v(x_1, x_2) > v(x_1', x_2')$

Result: $v(\cdot)$ rank bundles the same as $u(\cdot)$

$\longrightarrow$ utility functions are not unique

- For a given set of preferences, there are infinitely many utility functions that we can choose!
- This includes any "monotonic transformation"

Examples:

$v(x_1, x_2) = 10u(x_1, x_2)$

$v(x_1, x_2) = u(x_1, x_2) + 5$

$v(x_1, x_2) = \ln(u(x_1, x_2))$

→ implications

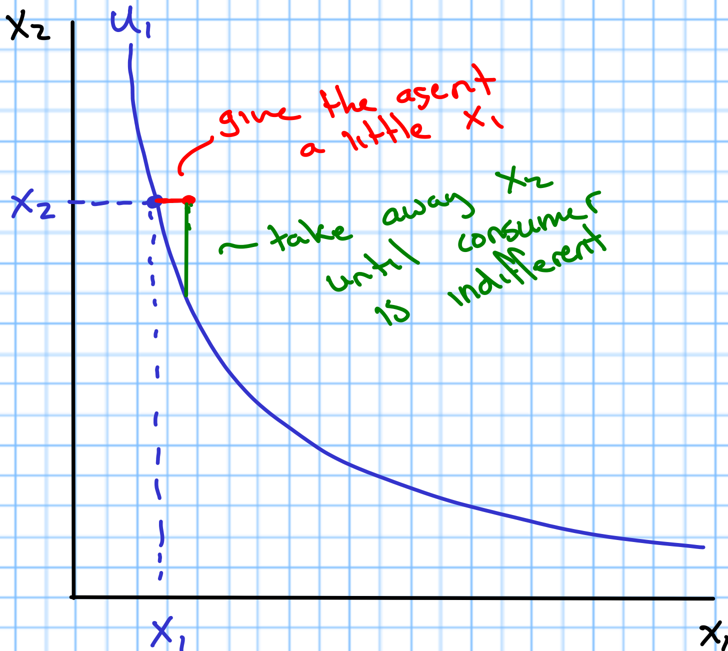
- Utility is an <u>ordinal</u> number, not a <u>cardinal</u> number
- Utility can only be used to rank alternatives it can't be used to tell us "how much" we like a certain bundle
- We can't compare utility across consumers!
- We can't say if one consumer "likes" a bundle more than another consumer

Example of ordinal numbers: Placing in a race:

Run a 5K

runners	time	Rank
John	45:00	3
Emily	13:00	1
Greg	22:00	2
	↑ cardinal number	↑ ordinal number

Slope of an IC (MRS)



$u(x_1, x_2) = u_1$

input: x_1, x_2

output: u

small change in x_1

how does this affect u?

→ "slope" of the utility function

→ marginal utility of good 1

- Extra utility you get from a small change in good 1

Agent gets MU_1 from the increase in good 1 (x_1)

Taking away a small amount of good 2 cause us to lose MU_2

How much x_2 do we have to lose?

$-\frac{MU_1}{MU_2}$ = Slope of IC = $-MRS$

Marginal utility

"Slope" of the utility function

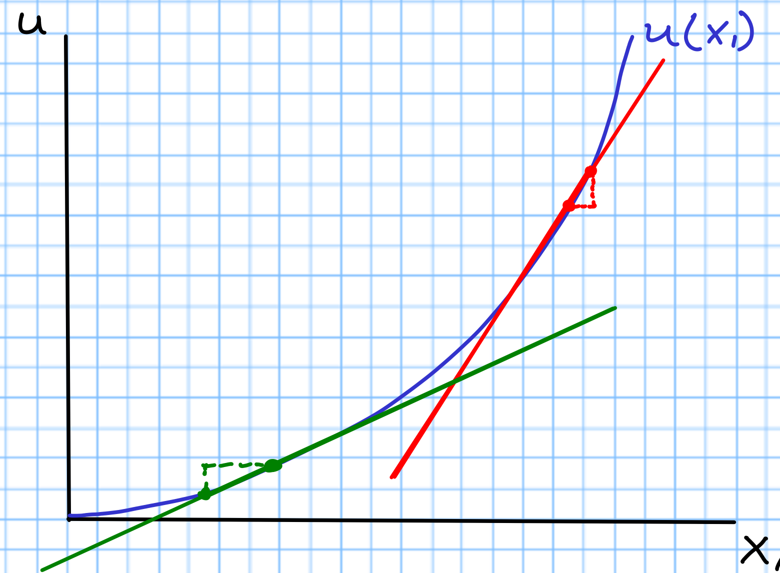
Example

$u(x_1) = 5x_1 + 3$

$MU_1 = 5$

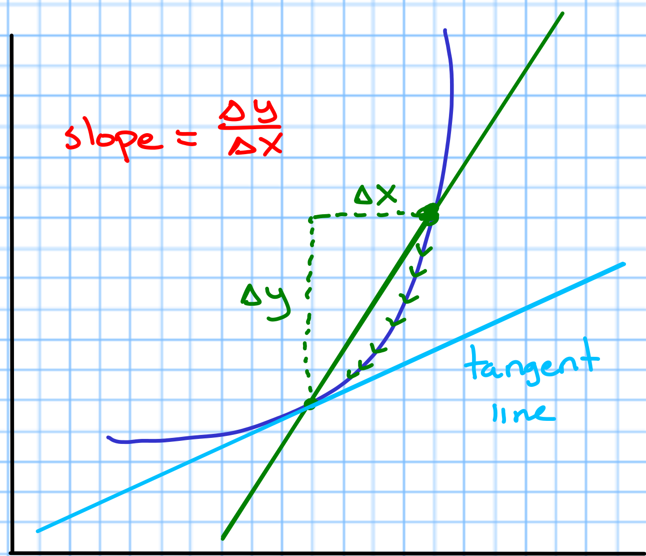
Example

$u(x_1) = x_1^2$



Problem: slope is not the same!

Calculating the slope of nonlinear functions (Calculus!)



- Tangent line touches a curve only once
- The slope of the tangent line is called a "derivative"
- "Power rule"

Suppose $f(x) = a x^b$

(a and b are constants)

Then the "derivative of f with respect to x"

$$\frac{df}{dx} = a \cdot b x^{b-1}$$

"derivative"

Example

$u(x_1) = x_1^2$

$$\frac{du}{dx_1} = 2x^{2-1} = 2x_1^1 = 2x_1$$

Example

$u(x_1) = 3x_1^{-4}$

$$\frac{du}{dx_1} = 3\cdot(-4)x_1^{-4-1} = -12x_1^{-5}$$

Example

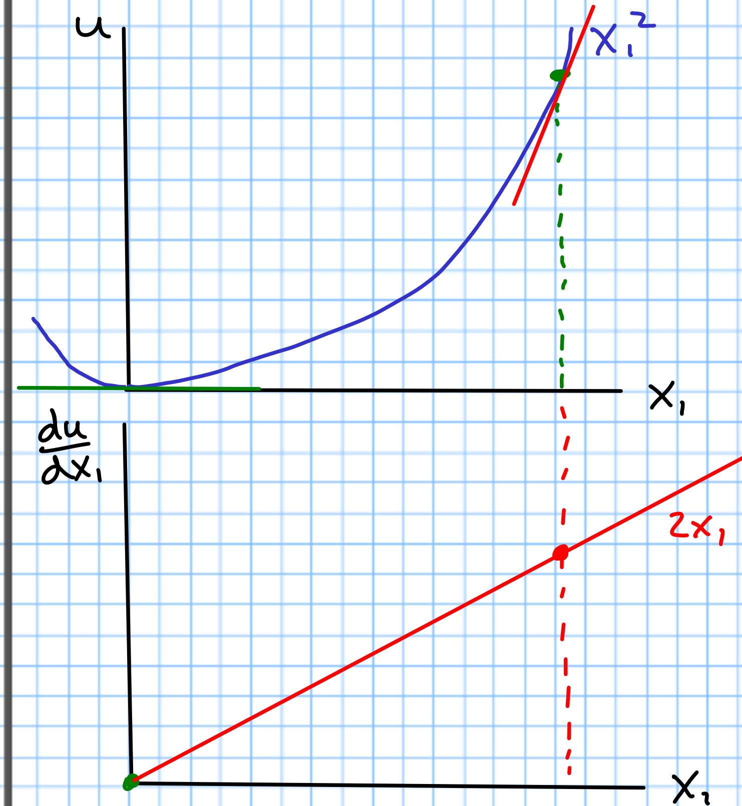
$u(x_1) = x_1^{1/3}$

$$\frac{du}{dx_1} = \frac{1}{3}x_1^{1/3-1}$$

$$= \frac{1}{3}x_1^{-2/3}$$

$\frac{1}{3} - \frac{3}{3}$

$-\frac{2}{3}$



Multivariable calculus

Example:

$$u(x_1, x_2) = x_1^2 x_2^3$$

- "Partial derivative of u with respect to x_1"

$$\frac{\partial u}{\partial x_1} = 2x_1 x_2^3$$

- "Partial derivative of u with respect to x_2"

$$\frac{\partial u}{\partial x_2} = 3x_1^2 x_2^2$$

Examples (from Quiz 3)

$$u(x_1, x_2) = x_1^3 x_2^2$$

Marginal utility of good 1

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

$$MU_1 = 3x_1^2 x_2^2$$

$$MU_2 = \frac{\partial u}{\partial x_2}$$

$$MU_2 = 2x_1^3 x_2$$

Example 2

$$u(x_1, x_2) = x_1 x_2$$

$$MU_1 = 1 x_1^{1-1} x_2$$

$$= 1 \cdot 1 \cdot x_2$$

$$= x_2$$

$$MU_2 = x_1$$

Example 3

$$u(x_1, x_2) = x_1^2 x_2^2$$

$$MRS = \frac{MU_1}{MU_2} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}$$

$$MU_1 = 2x_1 x_2^2$$

$$MU_2 = 2x_1^2 x_2$$

$$\frac{MU_1}{MU_2} = \frac{\cancel{2}\cancel{x_1} x_2^{\cancel{2}}}{\cancel{2} x_1^{\cancel{2}} \cancel{x_2}} = \frac{x_2}{x_1}$$

$x_1 = 4, \quad x_2 = 12$

$$MRS = \frac{12}{4} = 3$$

Example Cobb-Douglas utility

- A function is C-D if it can be written as $u(x_1, x_2) = x_1^a x_2^b$, where a and b are constants

$$MU_1 = a x_1^{a-1} x_2^b$$

$$MU_2 = b x_1^a x_2^{b-1}$$

$$MRS = \frac{MU_1}{MU_2} = \frac{\cancel{a} \cancel{x_1^{a}}{}^{-1} x_2^b}{b \cancel{x_1^a} x_2^{b-1}}$$

$$= \frac{a x_1^{-a} x_1^{a-1}}{b x_2^{-b} x_2^{b-1}}$$

$$= \frac{a x_1^{-1}}{b x_2^{-1}}$$

$$= \frac{a x_2}{b x_1}$$

memorize this if you want!

Example: $u(x_1, x_2) = 3x_1^{1/3} x_2^{2/3}$

$$MU_1 = \frac{1}{3} \cdot 3\, x_1^{-2/3}\, x_2^{2/3}$$

$$MU_2 = \frac{2}{3} \cdot 3\, x_1^{1/3}\, x_2^{-1/3}$$

$$MRS = \frac{\frac{1}{3}\cancel{3}\, x_1^{-2/3}\, x_2^{2/3}}{\frac{2}{3}\cancel{3}\, x_1^{1/3}\, x_2^{-1/3}}$$

$$= \frac{\frac{1}{3}\, x_2^{1/3}\, x_2^{2/3}}{\frac{2}{3}\, x_1^{2/3}\, x_1^{1/3}}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{x_2}{x_1}$$

$$= \frac{\frac{\cancel{3}}{2} \cdot \frac{1}{\cancel{3}}\, x_2}{x_1}$$

$$= \frac{x_2}{2x_1}$$