Cobb-Douglas production functions

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

$$0 < \alpha < 1$$

Code: K[t]^α L[t]^(1-α)

Decreasing MPK:

$$MPK = \frac{\partial F(K_t, L_t)}{\partial K_t}$$

$$= \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

$$= \alpha \frac{L_t^{1-\alpha}}{K_t^{1-\alpha}}$$

$$= \alpha \left(\frac{L_t}{K_t}\right)^{1-\alpha}$$

Question: What happens to MPK as K_t increases?

MPK goes down.

Returns to scale:

$$F(zK_t, zL_t) = (zK_t)^{\alpha}(zL_t)^{1-\alpha}$$

$$= z^{\alpha} K_t^{\alpha} z^{1-\alpha} L_t^{1-\alpha}$$

$$= \underbrace{z^{\alpha} z^{1-\alpha}}_{z} \cdot \underbrace{K_t^{\alpha} L_t^{1-\alpha}}_{F(K_t, L_t)}$$

$\longrightarrow$ Constant RTS

Note: Book says $F(K, L) = A K^{\alpha} L^{1-\alpha}$

We'll ignore A for now $(A = 1)$

Let's set $z = 1/L_t$

$$F(zK_t, zL_t) = F(K_t/L_t, 1)$$

$$\frac{1}{L_t} F(K_t, L_t) = \left(\frac{K_t}{L_t}\right)^\alpha$$

$$Y_t = F(K_t, L_t)$$

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^\alpha$$

Define: $k_t = \dfrac{K_t}{L_t}$

$$y_t = \frac{Y_t}{L_t}$$

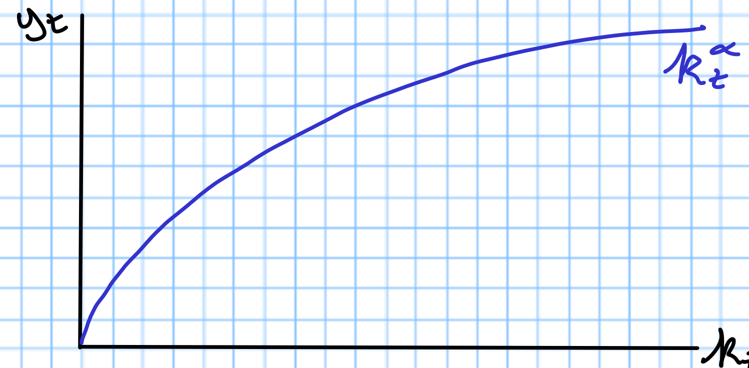
Then $y_t = k_t^\alpha$

→ Lower-case variable will mean "per worker"

$k_t = \dfrac{K_t}{L_t}$ capital per worker

$y_t = \dfrac{Y_t}{L_t}$ income per worker

$y_t = k_t^\alpha$ is the per-worker production function



Solow model

(AKA: Solow-Swan, neoclassical growth, exogenous growth)

Capital law of Motion

$$K_{t+1} = K_t + I_t - D_t$$

capital next year = capital this year + new capital (investments) - depreciation

Assume output is either consumed or saved

- Constant savings rate γ

$$I_t = \gamma Y_t$$

(consumption $(1-\gamma)Y_t$)

MPC

- Constant depreciation rate δ

$$D_t = \delta K_t$$

- Constant population

$$L_t = \bar{L}$$

CLM:

$$K_{t+1} = K_t + \gamma Y_t - \delta K_t$$

Now divide both sides by $\bar{L}$

$$\frac{K_{t+1}}{\bar{L}} = \frac{K_t}{\bar{L}} + \gamma \frac{Y_t}{\bar{L}} - \delta \frac{K_t}{\bar{L}}$$

$$k_{t+1} = k_t + \gamma y_t - \delta k_t$$

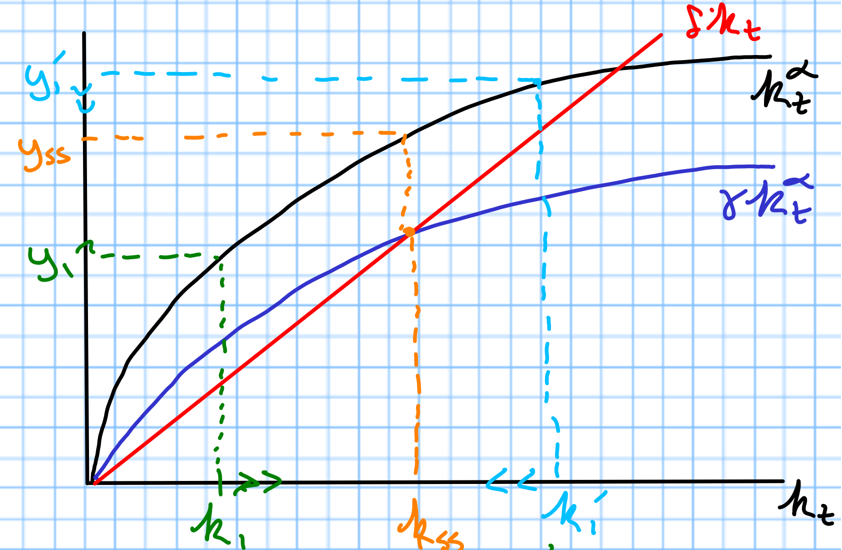
$$k_{t+1} - k_t = \gamma y_t - \delta k_t$$

$$\Delta k_t = \gamma y_t - \delta k_t$$

change in k from t to $t+1$

Recall: $y_t = k_t^{\alpha}$

$$\Delta k_t = \gamma k_t^{\alpha} - \delta k_t$$



when $k_t = k_1$ $\gamma k_1^{\alpha} > \delta k_1$

$\Delta k_1 > 0$

when $k_t = k_1'$ $\gamma k_1^{\alpha} < \delta k_1$

$\Delta k_1' < 0$

When $k_t = k_{ss}$ $\gamma k_t^\alpha = \delta k_t$

$\Delta k_t = 0$

k_{ss} is the steady state level of capital per worker

When $k_t = k_{ss}$, $\Delta k_t = 0$

$$0 = \gamma k_t^\alpha - \delta k_t$$

$$\frac{\gamma k_t^\alpha}{k_t^\alpha} = \frac{\delta k_t}{k_t^\alpha}$$

$$\gamma = \delta k_t^{1-\alpha}$$

$$k_t^{1-\alpha} = \frac{\gamma}{\delta}$$

$$k_{ss} = \left(\frac{\gamma}{\delta}\right)^{1/1-\alpha}$$

$$y_{ss} = k_{ss}^\alpha$$

$$= \left[\left(\frac{\gamma}{\delta}\right)^{1/1-\alpha}\right]^\alpha$$

$$y_{ss} = \left(\frac{\gamma}{\delta}\right)^{\alpha/1-\alpha}$$