Productivity in the Solow model

Production function:

$$Y_t = A_t K_t^{\alpha} (h_t L_t)^{1-\alpha}$$

A_t: Productivity

Solow model: Assume that

A_t grows exogenously

$$g = \frac{A_{t+1} - A_t}{A_t} \text{ is constant}$$

for all t

"Exogenous growth model"

Simplifying assumptions:

$$h_t = 1 \quad \forall t$$

$$L_t = \bar{L} \quad \forall t$$

$$Y_t = A_t K_t^{\alpha} (\bar{L})^{1-\alpha}$$

- Before: Divide both sides by $\bar{L}$ to get per-worker variables
- Now: Define a new variable

$$z_t = A_t^{1/1-\alpha}$$

$$\rightarrow A_t = z_t^{1-\alpha}$$

$$Y_t = z_t^{1-\alpha} K_t^{\alpha} (\bar{L})^{1-\alpha}$$

$$= K_t^{\alpha} (z_t \bar{L})^{1-\alpha}$$

Note: Z_t and A_t are capturing the same process

$A_t \uparrow$, $Z_t \uparrow$

$A_t \downarrow$, $Z_t \downarrow$

- Z_t is labor-augmenting technology
- Define $Z_t L_t$ as "effective workers"
  → equivalent number of constant productivity workers
- Let's express our production function in per effective worker terms
- before: $y_t = Y_t / L_t$
- now: $y_t = Y_t / Z_t L_t$
  now little variable will denote "per-effective worker"

$$y_t = Y_t / Z_t \bar{L} = \frac{K_t^\alpha (Z_t \bar{L})^{1-\alpha}}{Z_t \bar{L}}$$

$$= K_t^\alpha (Z_t \bar{L})^{1-\alpha} (Z_t \bar{L})^{-1}$$

$$= K_t^\alpha (Z_t \bar{L})^{-\alpha}$$

$$= \frac{K_t^\alpha}{(Z_t \bar{L})^\alpha}$$

$$= \left(\frac{K_t}{Z_t \bar{L}}\right)^{\alpha}$$

$$y_t = k_t^{\alpha}$$

CLM: $K_{t+1} = K_t + I_t - D_t$

$$\frac{K_{t+1}}{Z_t \bar{L}} = \frac{K_t}{Z_t \bar{L}} + \frac{\gamma Y_t}{Z_t \bar{L}} - \frac{\delta K_t}{Z_t \bar{L}}$$

$$\frac{K_{t+1}}{Z_t \bar{L}} = k_t + \gamma y_t - \delta k_t$$

$$Z_{t+1} = (1+\hat{z}) Z_t$$

$$Z_t = \frac{Z_{t+1}}{1+\hat{z}}$$

$$\frac{K_{t+1}}{\frac{Z_{t+1}\bar{L}}{1+\hat{z}}} = k_t + \gamma k_t^{\alpha} - \delta k_t$$

$$(1+\hat{z}) \frac{K_{t+1}}{Z_{t+1}\bar{L}} = k_t + \gamma k_t^{\alpha} - \delta k_t$$

$$(1+\hat{z}) k_{t+1} = k_t + \gamma k_t^{\alpha} - \delta k_t$$

$$k_{t+1} + \hat{z} k_{t+1} - k_t = \gamma k_t^{\alpha} - \delta k_t$$

$$k_{t+1} - k_t + \hat{z} k_{t+1} - \hat{z} k_t = \gamma k_t^{\alpha} - \delta k_t - \hat{z} k_t$$

$$\Delta k_t + \hat{z} \Delta k_t = \gamma k_t^{\alpha} - (\delta + \hat{z}) k_t$$

$$(1+\hat{z}) \Delta k_t = \gamma k_t^{\alpha} - (\delta + \hat{z}) k_t$$

Review: with pop. growth:

$$(1+n)\Delta k_t = \gamma k_t^\alpha - (\delta+n)k_t$$

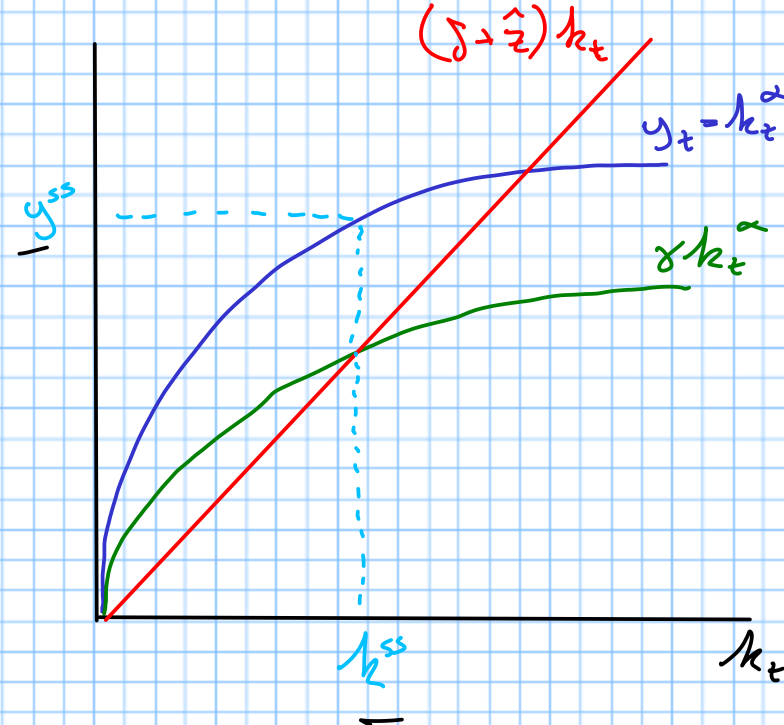
capital per worker

Suppose we had population growth and productivity growth:

$$(1+n)(1+\hat{z})\Delta k_t = \gamma k_t^\alpha - (\delta+\hat{z}+n)k_t$$

Let's consider the $n=0$:

$$(1+\hat{z})\Delta k_t = \gamma k_t^\alpha - (\delta+\hat{z})k_t$$



ss k: $\gamma k_t^{\alpha} = (\hat{z}+\delta) k_t$

$$k_t k_t^{-\alpha} = \frac{\gamma}{\hat{z}+\delta}$$

$$k_t^{1-\alpha} = \frac{\gamma}{\hat{z}+\delta}$$

$$k^{ss} = \left(\frac{\gamma}{\hat{z}+\delta}\right)^{1/1-\alpha}$$

$$y^{ss} = \left(\frac{\gamma}{\hat{z}+\delta}\right)^{\alpha/1-\alpha}$$

Suppose $n > 0$:

$$k^{ss} = \left(\frac{\gamma}{n+\hat{z}+\delta}\right)^{1/1-\alpha}$$

At steady-state, k^{ss} and y^{ss} are <u>constant</u>

$$k = \frac{K}{zL}$$

z increases at rate $\hat{z}$

So at k^{ss}, K_t is increasing at rate $\hat{z}$

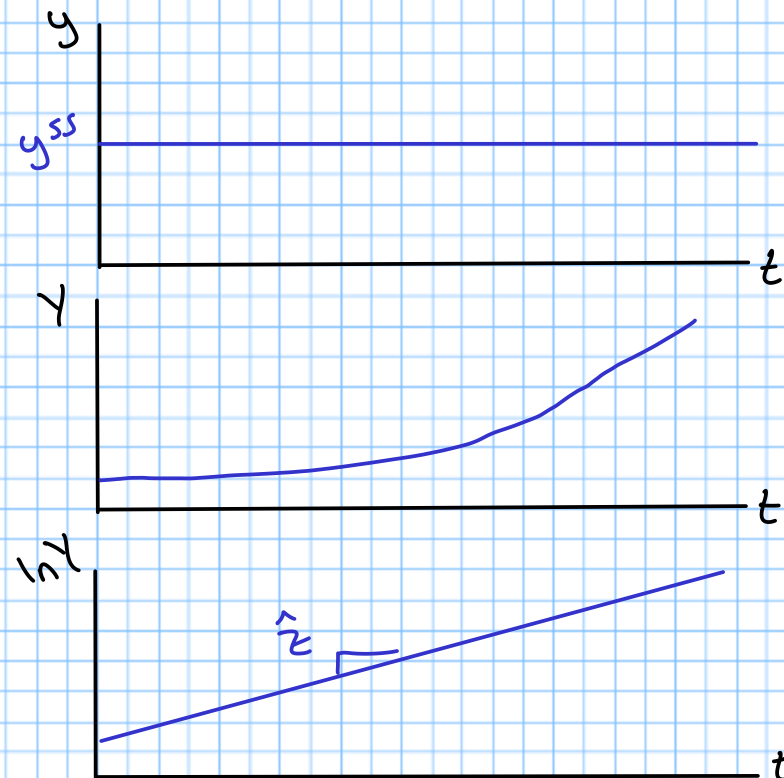
$$y_t = \frac{Y_t}{z_t L_t}$$

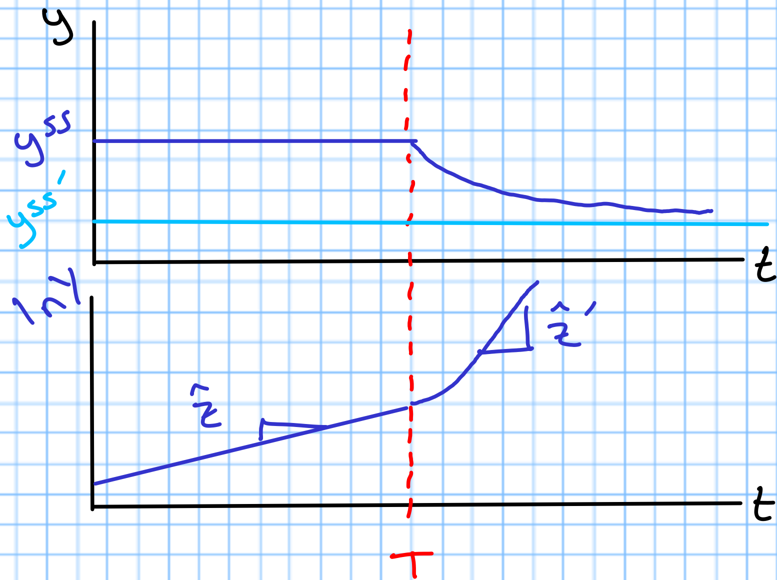
- At ss, Y_t increases at rate $\hat{z}$
- There is growth at the

Steady-state

Y and K both increasing at rate $\hat{z}$

Long-run growth rate of the economy is determined by the growth rate of productivity





At T, $\hat{z}$ increase to $\hat{z}'$