Mixed Strategies (Chpt 7)

Evert (E) \ Navratilova (N)	DL	CC
DL	50, 50	80, 20
CC	90, 10	20, 80

No pure-strategy NE.

- Suppose E plays DL
  - N's best response: DL
  - E's payoff: 50
- Suppose E plays CC
  - N's best response: CC
  - E's payoff: 20

- Suppose E mixes her strategies
  - choose CC with some probability p
  - chooses DL with probab. 1 - p

$p = 0.75$

- What is N's best response?
  - We can calculate N's expected payoff to E's strategy
  - N's expected payoff if N plays DL:

$50(0.25) + 10(0.75) = 20$

- N's payoff if she plays CC:

  $20(0.25) + 80(0.75) = 65$

- N's Expected payoff is higher if she plays CC
- N's best response to E playing $p = 0.75$ is CC
- Everet's expected payoff:

  $(0.25)80 + (0.75)20 = 35$

- Suppose instead E chooses CC with prob. $p = 0.25$
- N's best response:
  - Payoff to DL:

    $50(0.75) + 10(0.25) = 40$

  - Payoff to CC:

    $20(0.75) + 80(0.25) = 35$

  - BR: DL
- E's expected payoff:

  $50(0.75) + 90(0.25) = 60$

  - Mixed strategy of CC with probability $p = 0.25$ gives E a higher payoff than playing either pure strat.

Exploiting the opponent's strategy

- Zero sum (fixed-sum), E doing better means N must be doing worse
- N can exploit E's pure strategy and do better
- N also exploits E's mixed strategy of CC with $p = 0.25$
- Question: Can E choose a strategy that can't be exploited?
- In other words, is there a strategy that E can play that makes N indifferent among strategies?
- E plays CC with prob. P
  - What is N's payoff to DL?
    $10p + 50(1-p)$
  - What is N's payoff to CC?
    $80p + 20(1-p)$

$$10p + 50(1-p) = 80p + 20(1-p)$$
$$10p + 50 - 50p = 80p + 20 - 20p$$
$$10p - 50p - 80p + 20p = 20 - 50$$

$$-100p = -30$$
$$100p = 30$$
$$p = 3/10$$
$$p = 0.3$$

- N's payoff to DL:

$$50(0.7) + 10(0.3) = 38$$

- N's payoff to CC:

$$80(0.3) + 20(0.70) = 38$$

- E's payoff (if N plays DL) (q=0)

$$(0.7)50 + (0.3)90 = 62$$

- E's payoff (if N plays CC) (q=1)

$$(0.7)80 + (0.3)20 = 62$$

Best response

- Both players can choose mixed strategies

  E plays CC with prob p

  N play CC with prob q

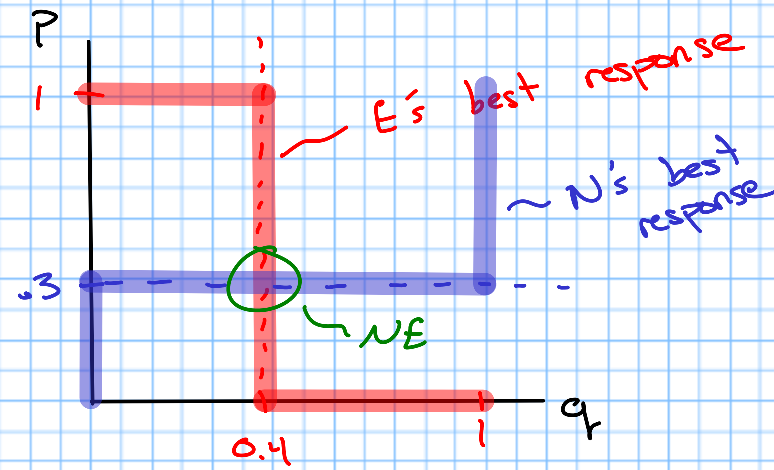
- E's payoff if N plays q

$$(50(1-p)+90p)(1-q) + (80(1-p)+20p)q$$
$$(50-50p+90p)(1-q) + (80-80p+20p)q$$
$$(50+40p)(1-q) + (80-60p)q$$
$$50+40p-(50q+40pq)+80q-60pq$$
$$50+40p-100pq+30q$$
$$50+30q+(40-100q)p$$

- What p maximizes this payoff?
  - $p=0$ maximizes when $q=1$
  - $p=1$ maximizes when $q=0$
  - $p=0$ maximizes when $q=0.5$
  - calculate q such that $40-100q=0$

    $\longrightarrow q=0.4$

Best response: $p^* = \begin{cases} 0 & \text{if } q > .4 \\ \text{anything} & \text{if } q = .4 \\ 1 & \text{if } q < .4 \end{cases}$



- N's payoff:

$$[50(1-q)+20(q)](1-p)+[10(1-q)+80q]p$$
$$(50-50q+20q)(1-p)+(10-10q+80q)p$$
$$50-30q-50p+30pq+10p+70pq$$

$50 - 30q + 100pq - 40p$

$50 - 40p + (100p - 30)q$

N's best response:

$$q = \begin{cases} 0 & \text{if } p < .3 \\ \text{anything} & \text{if } p = .3 \\ 1 & \text{if } p > .3 \end{cases}$$

Nash equilibrium:

E: $p = 0.3$

N: $q = 0.4$

NE as beliefs

- At the NE, each player forms a belief about the other player's strategies, and chooses their best response to that belief, and the beliefs are accurate

Result

- If there is a mixed strategy equilibrium, all players will indifferent among their various strategies

To find E's NE mixed strategy, choose p (prob of cc) such that N has the same expected payoff regardless of their choice

- N's expected payoff when playing DL:

$$10p + 50(1-p)$$

- N's expected payoff when playing CC:

$$80p + 20(1-p)$$

$$10p + 50(1-p) = 80p + 20(1-p)$$

$$\cancel{10}p + 50 - \cancel{50}p = \cancel{80}p + 20 - \cancel{20}p$$

$$-100p = -30$$

$$p = \frac{30}{100} = 0.3$$

N's NE strategy:

Make E indifferent

$$\underbrace{80q + 50(1-q)}_{\text{payoff when playing DL}} = \underbrace{20q + 90(1-q)}_{\text{payoff when playing CC}}$$

$$\cancel{80}q + 50 - \cancel{50}q = \cancel{20}q + 90 - \cancel{90}q$$

$$100q = 40$$

$$q = \frac{40}{100} = 0.4$$

Prisoners' Dilemma

		Wife: confess	Wife: Deny
Husband	C	$-\underline{10}, -\underline{10}$ *	$-\underline{1}, -25$
	D	$-25, -\underline{1}$	$-3, -3$

Mixed-strategy

H chooses C with prob p

U chooses C with prob q

(Pure strategy NE: $p=1$, $q=1$)

$$-10p + (-1)(1-p) = -25p + (-3)(1-p)$$

$$-\cancel{10p} - 1 + \cancel{p} = -\cancel{25}p - 3 + \cancel{3p}$$

$$13p = -2$$

$$p = -2/13$$

but p must be between 0 and 1, Therefore no mixed strategy NE exists

Coordination game

		Sally: Starbucks	Sally: Local
H	S	1, 1 *	0, 0
	L	0, 0	2, 2 *

2 pure-strategy NE

p: prob H chooses Starbucks

q: prob Sally chooses Starbucks

H: $1p + 0(1-p) = 0p + 2(1-p)$

$$p = 2 - 2p$$

$$3p = 2$$

$$p = 2/3$$

Sally: $1q + 0(1-q) = 0q + 2(1-q)$

$q = 2/3$

NE: $p = 2/3,\ q = 2/3$

$p = 1,\ q = 1$

$p = 0,\ q = 0$

Payoffs when $p = q = 0$

H: 2

Sally: 2

Payoffs when $p = q = 1$

H: 1

Sally: 1

Payoffs when $p = q = 2/3$

H: $\left(\frac{2}{3}\cdot 1 + \frac{1}{3}0\right)\frac{2}{3} + \left(0\cdot\frac{2}{3} + 2\cdot\frac{1}{3}\right)\frac{1}{3}$

$\frac{2}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} = \frac{4}{9} + \frac{2}{9}$

$= \frac{6}{9} = \frac{2}{3}$

Sally: $\frac{2}{3}$

Chicken

D

J	Swerve	Straight
Swerve	0,0	$-\underline{1}, \underline{1}$
Straight	$\underline{1}, -\underline{1}$	$-2,-2$

p: prob J plays Swerve

q: Prob D plays Swerve

J: $0p + (-1)(1-p) = 1p + (-2)(1-p)$

$-1 + \cancel{p} = \cancel{p} - 2 + 2p$

$1 = 2p$

$p = 1/2$

D: $q = 1/2$

Payoffs: $p = 1, q = 0$

J : -1

D : 1

$p = 0, q = 1$

J : 1

D : -1

$p = q = 1/2$

J: $\left(0 \cdot \frac{1}{2} + (-1)\frac{1}{2}\right)\frac{1}{2} + \left(1 \cdot \frac{1}{2} + (-2)\frac{1}{2}\right)\frac{1}{2}$

$-1/4 + -1/4 = -1/2$

D: $-1/2$

Navratilova (N)

Evert (E)

	DL	CC
DL	50 $\underline{50}$	$\underline{80}$ 20
CC	$\underline{90}$ 10	20 $\underline{80}$

N gets better at DL

N

E

	DL	CC
DL	30, 70	80, 20
CC	90, 10	20, 80

As before: p = prob that E chooses CC

$\begin{cases} q = \text{prob N chooses} \\ \text{CC} \end{cases}$

- Original NE: $p = 0.3$
  $q = 0.4$

- New NE:
  - E chooses p to make N indifferent:

$$10p + 70(1-p) = 80p + 20(1-p)$$

$$\cancel{10p} + 70 - \cancel{70p} = \cancel{80p} + 20 - \cancel{20p}$$

$$-120p = -50$$

$$p = 5/12 \approx 0.4167$$

  - N chooses q to make E indifferent:

$$80q + 30(1-q) = 20q + 90(1-q)$$

$$60q = 60(1-q)$$

$$120q = 60$$

$$q = 0.5$$

$\longrightarrow$ N is playing DL less frequently than before

Payoffs:

- Original game:
  - E: $(80(.4) + 50(0.6))0.7 + (20(.4) + 90(0.6))0.3$

$(32+30)(0.7)+(8+54)(0.3)$

$(62)(0.7)+(62)(0.3)=62$

- N payoff:

$10(0.3)+(50)(0.7)$

$3+35=38$

- New game:
  - E: $(30)(0.5)+(80)(0.5)=55$
  - N: $(70)(7/12)+10(5/12)$

    $245/6+25/6=270/6=45$

Risky and safe choices

- Football game
- 3rd down with 1 yard to gain

Probability of success:

Def.

Offense	run	pass
run	0.6	0.7
pass	0.8	0.3

Payoffs:

- Zero sum
- Payoffs depend on the situation

- If the play succeeds off. earns payoff V
  - V is large in "game-winning" situations
- Game table

		D	
		run	pass
O	run	0.6V, −0.6V	.7V, −.7V
	pass	.8V, −.8V	.3V, −.3V

NE: O choose r with prob p such that D is indifferent

$-0.6V(p) + (-0.8V)(1-p) =$
$-0.7V(p) + (-0.3)(1-p)$

$(0.1V)p = (0.5V)(1-p)$
$(0.1V)p = (0.5V) - (0.5V)p$
$(0.6V)p = 0.5V$

$$p = \frac{0.5V}{0.6V}$$

$p = 5/6$

- D: $(0.6V)q + (0.7V)(1-q) =$
  $(0.8V)q + (0.3V)(1-q)$
  $(0.4V)(1-q) = 0.2Vq$
  $0.4V = 0.6Vq$
  $q = 2/3$
- NE strategies do not depend on V