Suppose we are in time t and we have some amount of capital (K_t) and income (output, Y_t)

How much capital will we have tomorrow?

- Some of today's capital will depreciate

  K_t today $\rightarrow K_t - \delta K_t = (1-\delta) K_t$

- We will create new capital

  Y_t today $\rightarrow sY_t$ capital tomorrow

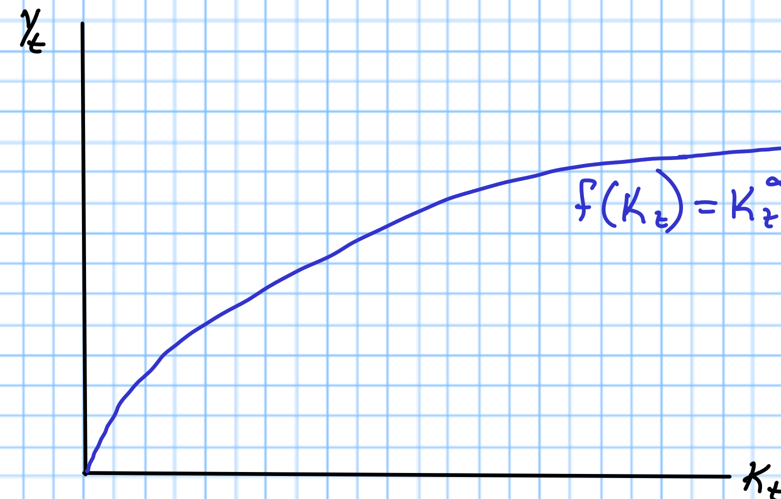
Putting it all together:

$$K_{t+1} = sY_t + (1-\delta) K_t$$

"Capital law of motion"

Production function

$$Y_t = f(K_t) \quad , \quad f''(K_t) < 0$$



For simplicity, assume
$f(K_t) = K_t^\alpha \quad 0 < \alpha < 1$

CLM: $\underline{K_{t+1}} = sY_t + \overline{(1-\delta)K_t}$

$K_{t+1} = sK_t^\alpha + (1-\delta)K_t$

$K_{t+1} = sK_t^\alpha + K_t - \delta K_t$

$K_{t+1} - K_t = sK_t^\alpha - \delta K_t$

$\Delta K_t = sK_t^\alpha - \delta K_t$

$\Delta K_t > 0 \iff sK_t^\alpha > \delta K_t$

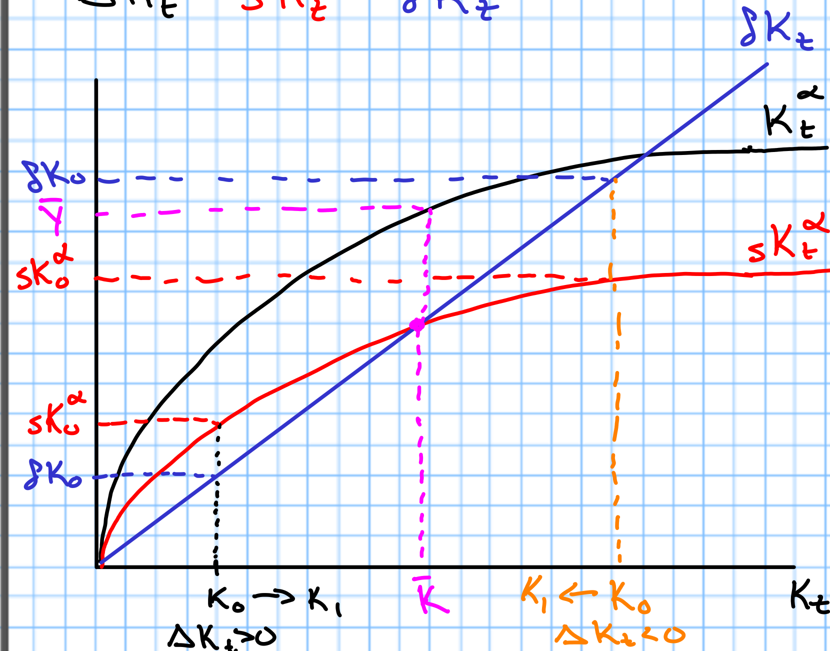
$\Delta K_t < 0 \iff sK_t^\alpha < \delta K_t$

$\Delta K_t = 0 \iff sK_t^\alpha = \delta K_t$

When $\Delta K_t = 0$, we say that we are at "steady state"

$K_{t+1} = K_t = \bar{K}$ is the steady-state level of capital

$\Delta K_t = sK_t^\alpha - \delta K_t$



$$\Delta K_t = sK_t^{\alpha} - \delta K_t$$

$$\Delta K_t = 0 \Rightarrow 0 = s\bar{K}^{\alpha} - \delta \bar{K}$$

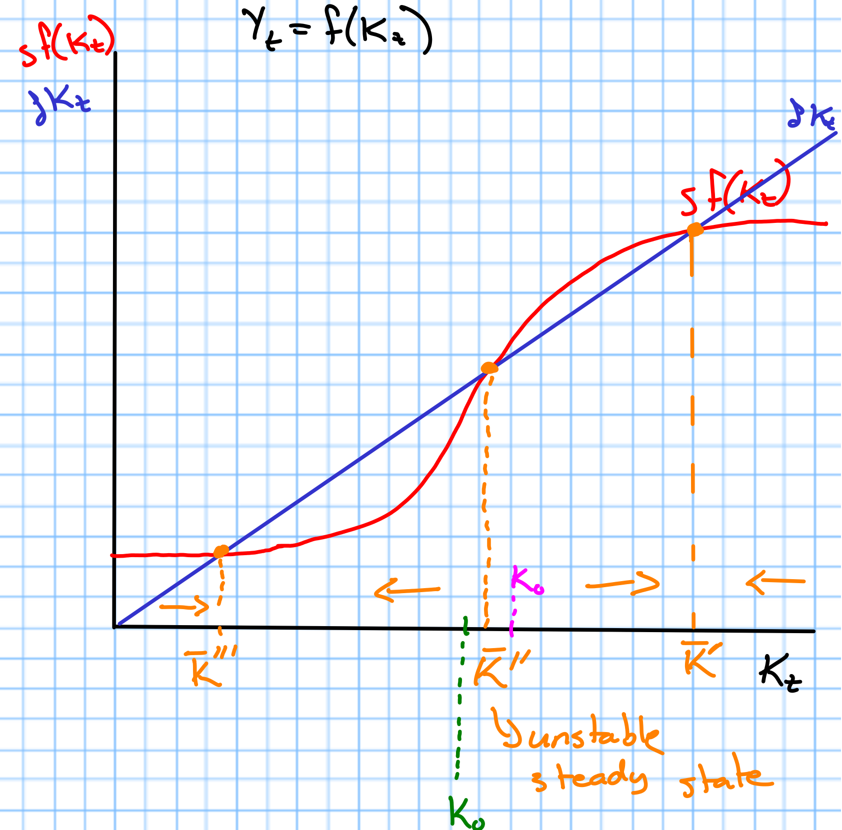
$$\frac{\delta \bar{K}}{\bar{K}^{\alpha}} = \frac{s\cancel{\bar{K}^{\alpha}}}{\cancel{\bar{K}^{\alpha}}}$$

$$\delta \bar{K}^{1-\alpha} = s$$

$$\bar{K}^{1-\alpha} = \frac{s}{\delta}$$

$$\bar{K} = \left(\frac{s}{\delta}\right)^{1/1-\alpha}$$

$$\bar{Y} = \bar{K}^{\alpha} = \left(\frac{s}{\delta}\right)^{\alpha/1-\alpha}$$



CLM: $K_{t+1} = (1-\delta)K_t + I_t$

$I_t = sY_t \rightarrow$ "Gross investment"

$s \rightarrow$ "Gross" savings rate

$Y_t \rightarrow$ Gross income

Instead consider "net" variables

$\tilde{Y}_t = Y_t - \delta K_t$ ("income net of depreciation")

CLM: $K_{t+1} = K_t - \delta K_t + I_t$

$K_{t+1} = K_t + (I_t - \delta K_t)$

$K_{t+1} = K_t + \tilde{I}_t$

$\uparrow$ net investment

$K_{t+1} = K_t + \tilde{s}\tilde{Y}_t$

On the "balanced growth path" (steady-state growth)

$\tilde{Y}_t$ grow at rate g

K_t grow at rate g

$\frac{K_t}{\tilde{Y}_t}$ is constant on BGP

$$\frac{K_{t+1}}{\tilde{Y}_t} = \frac{K_t}{\tilde{Y}_t} + \tilde{s}$$

Since $K_t/\tilde{Y}_t$ is constant,

$$\frac{K_t}{\tilde{Y}_t} = \frac{K_{t+1}}{\tilde{Y}_{t+1}} = \cdots$$

Growth rate of g means

$$\frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} = (1+g) \qquad \left(g = \frac{\tilde{Y}_{t+1} - \tilde{Y}_t}{\tilde{Y}_t} \right)$$

$$\frac{K_{t+1}}{\tilde{Y}_{t+1}} \cdot \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} = \frac{K_t}{\tilde{Y}_t} + \tilde{S}$$

$$\frac{K_{t+1}}{\tilde{Y}_{t+1}} (1+g) = \frac{K_t}{\tilde{Y}_t} + \tilde{S}$$

$$\frac{K_t}{\tilde{Y}_t} (1+g) = \frac{K_t}{\tilde{Y}_t} + \tilde{S}$$

$$\frac{K_t}{\tilde{Y}_t} g = \tilde{S}$$

$$\frac{K_t}{\tilde{Y}_t} = \frac{\tilde{S}}{g}$$

Piketty's "Second fundamental Law of Capitalism"

Wealth to (net) income ratio is equal to the (net) saving to growth ratio

Gross terms:

$$\frac{K_t}{Y_t} = \frac{s}{g+\delta}$$