Contents

1 First Order Differential Equations

1.1 Existence and Uniqueness of Solutions

3 cases for a solution at a point

1. No solution for Diff. Eq. at point
2. Infinitely many solutions at a point
3. One unique solution to Diff Eq. in neighborhood around point
   *needs to be continuous to be differentiable

For a Diff. Eq. $\frac{dy}{dx} = f(x, y)$
+ initial value (a, b), $y(a) = b$

1. If $f(x, y)$ is continuous around (a, b),
   Then atleast one solution exists on a open interval around x=a
   -if $f(x, y)$ is not continuous for open interval around x=a, a solution isn't guaranteed

2. If $\frac{\partial f}{\partial y}$ is also continuous around $(a.b)$, then the solution to the Diff. Eq. at (a, b) is unique on the open interval around x=a

1.2 Domain Restrictions

When the equations is divided by 0 for example, the of the solution is no longer true for $x = 0$. Sometimes if absolute values are present, x is assumed ≥ 0. The solution would only hold true for $x \geq 0$.

1.3 Separable Diff. Equations

If $\dfrac{dy}{dx}$ can be written as

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{1.3.1}$$

Then

$$G(y) = F(x) + C \tag{1.3.2}$$

Essentially if you can split the variable into two function multiplied together. You can integrate each function independently.

Proof:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{1.3.1}$$

$$g(y) \cdot \frac{dy}{dx} = f(x) \tag{1.3.3}$$

$$\int g(y) \, \frac{dy}{dx} \, dx = \int f(x) dx \tag{1.3.4}$$

$$G(y) = F(x) + C \tag{1.3.5}$$

1.4 Linear Diff. Equations

Linear Differential Equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.4.1}$$

Definition 1.1 (Integrating Factor)**.** *An integrating factor is expression an equation is multiplied by in order to make integration possible.*

The Linear Diff. Eq. looks very similar to the product rule:

$$\frac{d}{dy}[R(x)y = S(x)] \tag{1.4.2}$$

$$\frac{dy}{dx} \cdot R(x) + R'(x) \cdot y = S'(x) \tag{1.4.3}$$

$$\frac{dy}{dx} + \frac{R'(x)}{R(x)} \cdot y = \frac{S'(x)}{R'(x)} \tag{1.4.4}$$

Finding the Integrating Factor

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.4.1}$$

$$\rho(x) \cdot \frac{dy}{dx} + \rho(x) \cdot P(x)y = \rho(x) \cdot Q(x) \tag{1.4.5}$$

$$\Rightarrow \rho'(x) = \rho(x)P(x) \tag{1.4.6}$$

$$\rho(x) = e^{\int P(x)dx} \tag{1.4.7}$$

By how we defined $\rho(x)$

$$D_x[y \cdot \rho(x)] = \rho(x) \cdot Q(x) \tag{1.4.8}$$

$$\int D_x[y \cdot \rho(x)]\ dx = \int \rho(x) \cdot Q(x)\ dx \tag{1.4.9}$$

$$y \cdot \rho(x) = \int \rho(x) \cdot Q(x)\ dx \tag{1.4.10}$$

Example: $\frac{dy}{dx} + y\cot(x) = \cos(x)$

$$\begin{aligned}\rho(x) &= \exp\left(\int \cot(x)\ dx\right)\\ &= e^{ln\,|\sin(x)|}\\ &= \sin(x) \qquad \forall x \in \mathbb{R}, \sin(x) \neq 0\end{aligned}$$

$$\begin{aligned}\sin(x)\frac{dy}{dx} + y\cos(x) &= \sin(x)\cos(x)\\ y \cdot \sin(x) &= \int \sin(x)\cos(x)\ dx\\ y \cdot \sin(x) &= \frac{1}{2}\sin^2 x + C\\ y &= \frac{1}{2}\sin x + C\csc x\end{aligned}$$

1.4.1 Solving Linear Diff. Eq. with Complex Variables

Using Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.4.11}$$

The trigonometric function in the input (Q) are converted to a complex exponential, which is easier to integrate.

Convert y into the complex function $\widetilde{y}$
Solve for $\widetilde{y}$
$\widetilde{y} = y_1 + iy_2$
Solve for y_1 or y_2 depending on whether cos or sin was converted.

<u>Example</u>: $y' + ky = k\cos(\omega t)$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$
$$\widetilde{y}' + k\widetilde{y} = ke^{i\omega t}$$

$$\rho(x) = e^{kt}$$
$$\widetilde{y}\cdot e^{kt} = \int k\cdot e^{(k+i\omega)t}\,dt$$
$$\widetilde{y}\cdot e^{kt} = \tfrac{k}{k+i\omega}\cdot e^{(k+i\omega)t}$$

$$\widetilde{y} = \frac{1}{1+i\frac{\omega}{k}}\cdot e^{i\omega t}$$

Finding Real Part Using Polar Coordinates

$$\tfrac{1}{1+i\frac{\omega}{k}} = Ae^{-i\phi}$$
$$A = \left|\tfrac{1}{1+i\frac{\omega}{k}}\right| = \left(1+\left(\tfrac{\omega}{k}\right)^2\right)^{\frac{-1}{2}}$$
$$\phi = \arctan\left(\tfrac{\omega}{k}\right)$$
$$\widetilde{y} = \left(1+\left(\tfrac{\omega}{k}\right)^2\right)^{\frac{-1}{2}}\cdot e^{i(\omega t-\phi)}$$
$$y_1 = \left(1+\left(\tfrac{\omega}{k}\right)^2\right)^{\frac{-1}{2}}\cdot\cos\left(\omega t-\tan^{-1}\left(\tfrac{\omega}{k}\right)\right)$$

Using Cartesian Coordinates

$$\tfrac{1}{1+i\frac{\omega}{k}}[\cos\omega t + i\sin\omega t] \quad \cdot\tfrac{1-i\frac{\omega}{k}}{1-i\frac{\omega}{k}}$$
$$\tfrac{1}{1+\left(\frac{\omega}{k}\right)^2}\cdot\left(1-i\tfrac{\omega}{k}\right)\cdot[\cos\omega t + i\sin\omega t]$$

Re: $\tfrac{1}{1+\left(\frac{\omega}{k}\right)^2}\cdot\left[\cos\omega t + \tfrac{\omega}{k}\sin\omega t\right]$

With trig. identities: $y_1 = \left(1+\left(\tfrac{\omega}{k}\right)^2\right)^{\frac{-1}{2}}\cdot\cos\left(\omega t-\tan^{-1}\left(\tfrac{\omega}{k}\right)\right)$

1.5 Homogeneous Differential Equations

Definition 1.2 (Homogeneous)**.** *A function $f(x, y)$ is homogeneous of degree n if for any λ it satisfies*

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \tag{1.5.1}$$

A homogeneous diff. eg. can be written as

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} = F\left(\frac{y}{x}\right) \tag{1.5.2}$$

where A and B are both homogeneous functions of the same degree.

<u>How to Solve</u>

Make substitution

$$v = \frac{y}{x} \tag{1.5.3}$$

$$y' = x\frac{dv}{dx} + v \tag{1.5.4}$$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = F(v) \tag{1.5.5}$$

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x} \tag{1.5.6}$$

Now solve for V and substitute back in $\frac{y}{x}$ for V and solve for y.

1.6 Isobaric Differential Equations

Definition 1.3 (Isobaric)**.** *A function $f(x, y)$ is isobaric if for any t it satisfies*

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1.6.1}$$

An isobaric diff. eq. can be written as

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} \tag{1.6.2}$$

If A and B are both isobaric functions.

<u>How to Solve</u>

1. Give y and dy a weight m and x and dx a weight of 1.
2. Find the weight of each term in the diff. equation.
3. Find a value for m such that all the weights are equal.
4. Make substitution $y = vx^m$
5. Solve the now separable diff. eq.

Proof:

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} \tag{1.7.2}$$

$$y = vx^m \tag{1.6.3}$$

$$\frac{dy}{dx} = x^m \frac{dv}{dx} + mx^{m-1} \cdot V \tag{1.6.4}$$

$$x^m \frac{dv}{dx} + mx^{m-1} \cdot V = \frac{A(x, vx^m)}{B(x, vx^m)} \tag{1.6.5}$$

$$= x^{m-1} \frac{A(1,v)}{B(1,v)} \quad \leftarrow \text{ def. of isobaric} \tag{1.6.6}$$

$$= x^{m-1} C(v) \quad \leftarrow \text{ only dependent on v} \tag{1.6.7}$$

$$x^m \frac{dv}{dx} + mx^{m-1} \cdot V = x^{m-1} C(v) \tag{1.6.8}$$

$$x \frac{dv}{dx} + m \cdot V = C(v) \tag{1.6.9}$$

$$\frac{dv}{dx} = \frac{C(v) - m \cdot v}{x} \tag{1.6.10}$$

<u>Example:</u> $\frac{dy}{dx} = \frac{-1}{2yx}\left(y^2 + \frac{2}{x}\right)$

$\frac{dy}{dx} = \frac{-y}{2x} + \frac{-1}{x^2 y}$
$(m-1) = (m-1) = (-m-2)$
$m = \frac{-1}{2}$

$y = vx^{\frac{-1}{2}}$
$\frac{dy}{dx} = x^{\frac{-1}{2}} \frac{dv}{dx} - \frac{1}{2} v \cdot x^{\frac{-3}{2}}$

$x^{\frac{-1}{2}} \frac{dv}{dx} - \frac{1}{2} v \cdot x^{\frac{-3}{2}} = \frac{-vx^{\frac{-1}{2}}}{2x} - \frac{1}{vx^{\frac{3}{2}}}$
$\frac{dv}{dx} = \frac{-1}{xv}$
$\frac{1}{2} v^2 = -\ln|x| + C$

$\frac{1}{2} y^2 x = -\ln|x| + C$

1.7 Bernoulli Differential Equations

Bernoulli Diff. Eq.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1.7.1}$$

For $n \neq 0, 1$ use substitution

$$V = y^{1-n} \tag{1.7.2}$$

to reduce to a linear diff. equation.

Proof:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1.8.1}$$

$$v = y^{1-n} \tag{1.8.2}$$

$$y = v^{\frac{1}{1-n}} \tag{1.7.3}$$

$$\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{dv}{dx} \cdot v^{\frac{1}{1-n}-1} \tag{1.7.4}$$

Notice

$$v^{\frac{1}{1-n}-1} = v^{\frac{n}{1-n}} = \left(v^{\frac{1}{1-n}}\right)^n \tag{1.7.5}$$

Substituting

$$\frac{1}{1-n} \cdot \frac{dv}{dx} \cdot v^{\frac{1}{1-n}-1} + P(x) \cdot v^{\frac{1}{1-n}} = Q(x) \cdot v^{\frac{1}{1-n}-1} \tag{1.7.6}$$

$$\frac{1}{1-n} \cdot \frac{dv}{dx} + P(x) \cdot v^1 = Q(x) \tag{1.7.7}$$

1.8 Obvious and Embedded Derivative Substitutions

Embedded Derivative-

*Look for a piece that is a derivative of another piece

- Must include a y in order to have the $\frac{dy}{dx}$ by the chain rule

*Then replace not just $\frac{dy}{dx}$, but $f(x,y)\frac{dy}{dx}$ with $\frac{dx}{dx}$

Example: $3y^2\frac{dy}{dx} + y^3 = e^{-x}$

Obvious Substitution-

Look for any part that can be solved for y and make a substitution.

Not guaranteed to achieve any thing

Both of these techniques just simplify the diff. eq. or put it into a form which we know how to solve.

1.9 Exact Differential Equations

$$F(x,y) = C \tag{1.9.1}$$

$$\frac{d}{dx}[F(x,y) = C] \tag{1.9.2}$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0 \tag{1.9.3}$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \tag{1.9.4}$$

$$M = \frac{\partial F}{\partial x}, \quad N \frac{\partial F}{\partial y} \tag{1.9.5}$$

$$M dx + N dy = 0 \tag{1.9.6}$$

is an exact differential equation if M and N are partials of some some function F

Steps

1. Check for $F_{xy} = F_{yx}$, $M_y = N_x$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. $M = \dfrac{\partial F}{\partial x} \rightarrow F(x,y) = \int M dx + g(y)$
   $N = \dfrac{\partial F}{\partial y} \rightarrow F(x,y) = \int N dy + h(x)$
3. Find $\frac{\partial F}{\partial y}$. . . Must Equal N, you will have $g'(y)$
   or $\frac{\partial F}{\partial x}$. . . Must Equal M, you will have $h'(x)$
4. Solve for $g'(y)$ or $h'(x)$
5. Solve for $g(y)$ or $h(x)$: $g(y) = \int g'(y)dy$ or $h(x) = \int h'(x)dx$
6. Replace $g(y)$ or $h(x)$ in # 2
7. Now you have $F(x,y) = C$

1.9.1 Integrating Factor for Inexact Differential Equations

Given $M dx + N dy = 0$
What if $M_y \neq N_x$

Multiply by an integrating factor $U(x,y)$ to make it exact

$$\frac{\partial}{\partial y}[U(x,y) \cdot M] = \frac{\partial}{\partial x}[U(x,y) \cdot N] \tag{1.9.7}$$

$$\frac{\partial U}{\partial y}\cdot M + U\cdot\frac{\partial M}{\partial y} = \frac{\partial U}{\partial x}\cdot N + U\cdot\frac{\partial N}{\partial x} \tag{1.9.8}$$

$$U\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{\partial U}{\partial x}\cdot N - \frac{\partial U}{\partial y}\cdot M \tag{1.9.9}$$

This is too complicated so lets assume U contains only Xs

$$U\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{dU}{dx}\cdot N \tag{1.9.10}$$

$$\int\frac{dU}{U} = \int\frac{\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]}{N}dx \tag{1.9.11}$$

$$U = \exp\left(\int\frac{\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]}{N}dx\right) \tag{1.9.12}$$

U contains only Xs

$$U = \exp\left(\int\frac{M_y - N_x}{N}dx\right) \tag{1.9.13}$$

U contains only Ys

$$U = \exp\left(\int\frac{N_x - M_y}{N}dx\right) \tag{1.9.14}$$

If these contain the other variable then an integrating factor is impossible with only that variable.

1.10 Reducible Second Order Equations

Must be missing all Ys or all Xs

3 Cases

1. Y is gone: $P = \frac{dy}{dx} \to y'' = \frac{dP}{dx}$
2. X is gone: $P = \frac{dy}{dx} \to y'' = \frac{dP}{dx} = \frac{dP}{dy}\cdot\frac{dy}{dx} \to y'' = P\cdot\frac{dP}{dy}$
3. Both are gone: either substitution works

1.11 Miscellaneous Forms

For

$$\frac{dy}{dx} = F(ax + by + c) \tag{1.11.1}$$

Make Substitution

$$V = ax + by + c \tag{1.11.2}$$

$$\frac{dV}{dx} = a + b\cdot\frac{dy}{dx} \tag{1.11.3}$$

$$\frac{dV}{dx} = a + b\cdot F(V) \tag{1.11.4}$$

For

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g} \tag{1.11.5}$$

Substitute

$$x = X + \alpha$$

$$y = Y + \beta$$

Where α and β satisfy

$$a\alpha + b\beta + c = 0 \tag{1.11.6}$$

$$e\alpha + f\beta + g = 0 \tag{1.11.7}$$

Giving

$$\frac{dY}{dX} = \frac{aX + bY}{eX + fY} \tag{1.11.8}$$

Which is homogeneous and therefore solvable.

1.12 Higher Degree First Order Differential Equations

The term $\frac{dy}{dx}$ is raised to an power

Typically written in two forms

$$F(x, y, \frac{dy}{dx}) = 0 \tag{1.12.1}$$

$$\left(\frac{dy}{dx}\right)^n + a_{n-1}(x, y)\left(\frac{dy}{dx}\right)^{n-1} + \ldots + a_1(x, y)\frac{dy}{dx} + a_0(x, y) = 0 \tag{1.12.2}$$

3 Way to solve

1.) Soluble for $\frac{dy}{dx}$

2.)Soluble for x

3.) Soluble for y

Clairaut's Equation (Special Case)

1.12.1 Soluble for $\frac{dy}{dx}$

Must be forced into $(P - F_1)(P - F_2)\ldots(P - F_n) = 0$

Where $P = \frac{dy}{dx}$

Now solve each first degree equation for a solution $G_i(x, y) = 0$

The general solution is of the form

$$\prod_{i=1}^{n} G_i(x, y) = 0 \tag{1.12.3}$$

Example: $[(x+1)P-y][(x^2+1)P-2xy]=0$

$(x+1)P-y=0$

$\int \frac{dy}{y} = \int \frac{dx}{x+1}$

$\ln|y| = \ln|x+1| + C_1$

$0 = C(x+1)-y$

$(x^2+1)P = 2xy$

$\int \frac{dy}{y} = \int \frac{2x}{x^2+1}dx$

$\ln|y| = \ln|x^2+1| + C_1$

$0 = C(x^2+1)-y$

Solution: $[C(x+1)-y]\cdot[C(x^2+1)-y]=0$

1.12.2 Soluble for X

If

$$x = F(y,P) \tag{1.12.4}$$

Differentiating both sides by y yields

$$\frac{dx}{dy} = \frac{1}{P} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial P}\cdot\frac{dP}{dy} \tag{1.12.5}$$

Example: $3x = \frac{y}{P} - 6y^2P$

$\frac{3}{P} = \frac{1}{P} - 12yP + \left(\frac{-y}{p^2} - 6y^2\right)P'$

$0 = -(1+6yP^2)(2P+yP')$

$P' + \frac{2}{y}P = 0$

$y^2P' + 2yP = 0$

$y^2P = C$

$P = \frac{C}{y^2}$

Plugging back into problem and simplifying: $6C^2 + 3xC = y^3$

This is the general solution.

$1 + 6yP^2 = 0$

$P = \sqrt{\frac{-1}{6y}}$

Plugging back into problem and simplifying: $-3x^2 = 8y^3$

This is the singular solution.

1.12.3 Soluble for Y

If

$$y = F(x, P) \tag{1.12.6}$$

Differentiating both sides by x yields

$$\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial P} \cdot \frac{dP}{dx} = P \tag{1.12.7}$$

Example: $y = xp^2 + 2xp$

$P = P^2 + 2P + (2xp + 2x)P'$
$0 = (P + 1)(P + 2xP')$
$0 = \frac{P}{2x} + P'$
$0 = \frac{P}{2\sqrt{x}} + \sqrt{x}P'$
$P = \frac{C}{\sqrt{x}}$

Plugging back into problem and simplifying: $(y - c)^2 = 4cx$
This is the general solution.

$p = -1$

Plugging back into problem and simplifying: $y = -x$
This is the singular solution.

1.12.4 Clairaut's Equation

$$y = px + F(p) \tag{1.12.8}$$

Differentiating by x yields

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx} \tag{1.12.9}$$

$$0 = \frac{dp}{dx}\left(\frac{dF}{dp} + x\right) \tag{1.12.10}$$

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \tag{1.12.11}$$

$$p = c_1 \qquad y = c_1 x + c_2 \tag{1.12.12}$$

General Solution:

$$y = c_1 x + F(c_1) \tag{1.12.13}$$

Singular Solution:

$$\frac{dF}{dp} + x = 0 \tag{1.12.14}$$

2 Second Order Differential Equations

2.1 Second Order Linear Homogeneous ODEs with Constant Coefficients

Standard Form:

$$y'' + Ay' + By = 0 \tag{2.1.1}$$

Solution:

$$y = c_1 y_1 + c_2 y_2 \tag{2.1.2}$$

Where y_1 and y_2 are solutions.

Make substitution $y = e^{rx}$

$$r^2 e^{rx} + Are^{rx} + Be^{rx} = 0 \tag{2.1.3}$$

$$r^2 + Ar + B = 0 \tag{2.1.4}$$

This is called the characteristic equation of the system

Case 1: If the characteristic equation produces two distinct real roots r_1, r_2

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{2.1.5}$$

Case 2: If the characteristic equation produces two complex roots

$$r_1 = a + bi \qquad r_2 = a - bi$$

Complex Solution

$$y = e^{(a+bi)x} \tag{2.1.6}$$

Solutions:

$$e^{ax} \cos(bx), \qquad e^{ax} \sin(bx) \tag{2.1.7}$$

$$y = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)] \tag{2.1.8}$$

Theorem 2.1. *If $u(x) + iv(x)$ is a complex solution to $y'' + Ay' + By = 0$ then $u(x)$ and $v(x)$ are both real solutions.*

Proof.

$$(u + iv)'' + A(u + iv)' + B(u + iv) = 0 \tag{2.1.9}$$

$$[u'' + Au' + Bu] + i[v'' + Av' + Bv] = 0 \tag{2.1.10}$$

$$\therefore \quad \text{Real Part} = 0 = \text{Complex Part}$$

□

Case 3: If the characteristic equation produces two equal roots r

$$r = -a \tag{2.1.11}$$

$$(r + a)^2 = 0 \tag{2.1.12}$$

$$r^2 + 2ar + a^2 = 0 \tag{2.1.13}$$

$$y'' + 2ay' + a^2 y = 0 \tag{2.1.14}$$

Theorem 2.2. *If y_1 is one solution to $y'' + py' + qy = 0$ then there is another of the form $y_2 = y_1 \cdot u$*

Proof. Starting with one solution e^{-ax}

$$\begin{array}{ll} & a^2 \cdot [y = e^{-ax} \cdot u] \\ & 2a \cdot [y' = -ae^{-ax} \cdot u + e^{-ax} \cdot u'] \\ + & 1 \cdot \ [y'' = a^2 e^{-ax} \cdot u - 2ae^{-ax} \cdot u' + e^{ax} \cdot u''] \\ \hline & 0 = \quad 0 \quad + \quad 0 \quad + \quad u'' e^{ax} \end{array}$$

$$u'' = 0 \tag{2.1.15}$$

$$u = c_a x + c_b \tag{2.1.16}$$

$$y_2 = x \cdot e^{-ax} \tag{2.1.17}$$

Any choice of constants work, so the simplest is picked □

$$y = (c_1 + c_2 x)e^{rx} \tag{2.1.18}$$

Example: $y'' + 4y' + 3y = 0 \qquad y(0) = 1,\ y'(0) = 0$

$$\begin{aligned} r^2 + 4r + 3 &= 0 \\ r &= -1, -3 \\ y &= c_1 e^{-t} + c_2 e^{-3t} \\ y &= -c_1 e^{-t} - 3c_2 e^{-3t} \\ 1 &= c_1 + c_2 \\ 0 &= -c_1 - 3c_2 \\ c_2 = \tfrac{-1}{2} &\qquad c_1 = \tfrac{3}{2} \end{aligned}$$

$$y = \tfrac{3}{2} e^{-3t} - \tfrac{1}{2} e^{-t}$$

Example: $y'' + 4y' + 5y = 0 \qquad y(0) = 1,\ y'(0) = 0$

$$\begin{aligned} r^2 + 4r + 5 &= 0 \\ r &= -2 \pm i \\ y &= e^{-2t}(c_1 \cos t + c_2 \sin t) \\ y' &= -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(c_2 \cos t - c_1 \sin t) \\ 1 &= 1 \cdot (c_1 + 0) \\ 0 &= -2(1 \cdot 1 + 0) + 1 \cdot (c_2 - 0) \\ c_2 = 2 &\qquad c_1 = 1 \end{aligned}$$

$$y = e^{-2t}(\cos t + 2 \sin t)$$

Most General Complex Solution

$$y = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}, \qquad C_i \in \mathbb{C} \tag{2.1.19}$$

Making it Real

1. Multiply everything out and set imaginary part to zero
2. or Use the fact that if real: $(u + vi = u - iv)$

$$= C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} \tag{2.1.19}$$

$$= \overline{C}_1 e^{(a-bi)t} + \overline{C}_2 e^{(a+bi)t} \tag{2.1.20}$$

$$\therefore C_1 = \overline{C}_2, \quad C_2 = \overline{C}_1 \tag{2.1.21}$$

So in order to be real

$$= (c + di) e^{(a+bi)t} + (c - di) e^{(a+bi)t} \tag{2.1.22}$$

$$= e^{at} \left[c \left(e^{ibt} + e^{-ibt} \right) + id \left(e^{ibt} - e^{-ibt} \right) \right] \tag{2.1.23}$$

$$= e^{at} \left[2c \cos bt + 2d \sin bt \right] \tag{2.1.24}$$

$$y = e^{at} \left[c_1 \cos bt + c_2 \sin bt \right] \tag{2.1.25}$$

2.2 Theory of Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{2.2.1}$$

Solution $y = c_1y_1 + c_2y_2$ is called a linear combination.

Question 1: Why are $c_1y_1 + c_2y_2$ solutions?

Theorem 2.3 (Superposition Principle)**.** *If y_1 and y_2 are solutions to a linear homo. ODE. then $c_1y_1 + c_2y_2$ is also a solution.*

Proof.

$$y'' + py' + qy = 0 \tag{2.2.2}$$

$$D^2y + pDy + qy = 0 \tag{2.2.3}$$

$$(D^2 + pD + q)y = 0 \tag{2.2.4}$$

$\nearrow$

Definition 2.1. *Linear Operator -Turns a function of x into another function of x*

$$Ly = 0, \qquad L = D^2 + pD + q \tag{2.2.5}$$

Because L is linear

$$L(u_1 + u_2) = L(u_1) + L(u_2)$$
$$L(k \cdot u)) = k \cdot L(u)$$

$$Ly = 0 \tag{2.2.6}$$

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2) \tag{2.2.7}$$

$$L(c_1y_1 + c_2y_2) = c_1 \cdot 0 + c_2 \cdot 0 \tag{2.2.8}$$

$$L(c_1y_1 + c_2y_2) = 0 \tag{2.2.9}$$

□

Question 2: Why are $c_1y_1 + c_2y_2$ all of the solutions?

Theorem 2.4. *$\{c_1y_1 + c_2y_2\}$ is enough to satisfy any initial conditions.*

Proof. Initial Values: $y(x_0) = a, \quad y'(x_0) = b$

$$c_1y_1(x_0) + c_2y_2(x_0) = a \tag{2.2.10}$$

$$c_1y_1'(x_0) + c_2y_2'(x_0) = b \tag{2.2.11}$$

This system is solvable if

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \tag{2.2.12}$$

This is true if y_1 and y_2 are linearly independent.

□

This is called the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{2.2.13}$$

Theorem 2.5. *If y_1 and y_2 are solutions to $y'' + py' + qy = 0$, then*

$$W(y_1, y_2) \equiv 0 \quad \textit{for all } x \tag{2.2.14}$$

$$W(y_1, y_2) \quad \textit{is never 0 for all } x \tag{2.2.15}$$

<u>Finding Normalized Solutions:</u>
$\{c_1y_1 + c_2y_2\} = \{c_1'u_1 + c_2'u_2\}$- u_1 and u_2 are any other pair of ind. solutions.

$$u_1 = \bar{c}_1y_1 + \bar{c}_2y_2$$
$$u_2 = \bar{\bar{c}}_1y_1 + \bar{\bar{c}}_2y_2$$

$Y_1: \ y_1(0) = 1$, $y_1'(0) = 0$ $\qquad$ $Y_2: \ y_2(0) = 0$, $y_2'(0) = 1$

<u>Example</u>: $\qquad y'' - y = 0$

$$y_1 = e^x$$
$$y_2 = e^{-x}$$

$$y = c_1e^x + c_2e^{-x}$$
$$y' = c_1e^x - c_2e^{-x}$$

$Y_1: \ c_1 + c_2 = 1$
$c_1 - c_2 = 0$
$c_1 = c_2 = \frac{1}{2}$

$Y_2: \ c_1 + c_2 = 0$
$c_1 - c_2 = 1$
$c_1 = \frac{1}{2} c_2 = \frac{-1}{2}$

$Y_1 = \frac{e^x + e^{-x}}{2}$
$Y_1 = \sinh x$

$Y_2 = \frac{e^x - e^{-x}}{2}$
$Y_2 = \cosh x$

Why?

Y_1 and Y_2 normalized at 0 and IVP: $y(0) = y_0, \quad y'(0) = y_0'$
The solution is

$$y = y_0Y_1 + y_0'Y_2 \tag{2.2.16}$$

$$y(0) = a \cdot 1 + b \cdot 0$$
$$y'(0) = a \cdot 0 + b \cdot 1$$

Theorem 2.6 (Existence and Uniqueness Theorem)**.** *Given $y''+py'+qy = 0$ and p and q being continuous for all x, there is one and only one solution to the initial value: $y(0) = a, \quad y'(0) = b$*

Proof. Cont. of Question 2

Given a solution $u(x)$ w/ $u(0) = u_0, \quad u'(0) = u_0'$

Then $u_0 Y_1 + u_0' Y_2$ satisfies the same initial conditions

$\therefore \quad u = u_0 Y_1 + u_0' Y_2 \quad$ by Exi. & Uni. Theorem $\square$

2.3 Theory of Inhomogeneous Second Order ODES

Inhomogeneous:

$$y'' + p(x)y' + q(x)y = f(x) \tag{2.3.1}$$

Associated Homo. Equation: $y'' + p(x)y' + q(x)y = 0$
Complementary Solution: $y_h = c_1y_1 + c_2y_2$

Theorem 2.7. *Given $L(y) = f(x)$ (L being a linear operator)*
The solution is $y_p + y_h$

y_p is a particular solution to the equation.

Proof. All $y_p + c_1y_1 + c_2y_2$ are solutions.

$$L(y_p + c_1y_1 + c_2y_2) = L(y_p) + L(c_1y_1 + c_2y_2) \tag{2.3.2}$$

$$= f(x) + 0 = f(x) \tag{2.3.3}$$

□

Proof. There are no other solutions.
$u(x)$ is a solution.

$$L(u) = f(x) = L(y_p) \tag{2.3.4}$$

$$L(u - y_p) = 0 \tag{2.3.5}$$

$$u - y_p = c_1y_1 + c_2y_2 \tag{2.3.6}$$

$$\therefore \quad u = y_p + c_1y_1 + c_2y_2 \tag{2.3.7}$$

□

$$y'' + Ay' + By = f(t) \quad A, B \in \mathbb{R} \tag{2.3.8}$$

$$y = y_p + y_h \tag{2.3.9}$$

Definition 2.2 (Stable)**.** *An ODE is stable if y_h goes to 0 as $t \to \infty$*

y_h is called the trangent
y_p is called the steady-state solution

Roots	Solution	Stability Condition
$r_1 \neq r_2$	$c_1e^{r_1t} + c_2e^{r_2t}$	$r_1, r_2 < 0$
$r_1 = r_2$	$(c_1 + c_2x)e^{rt}$	$r_1 < 0$
$r = a \pm bi$	$e^{at}[c_1 \cos bt + c_2 \sin bt]$	$a < 0$

Some important $f(x)$s

$$e^{ax}, \qquad \sin \omega t,\ \cos \omega y, \qquad e^{ax} \sin \omega t,\ e^{ax} \cos \omega t, \qquad e^{(a \pm bi)t} \ \textit{or} \ e^{\alpha x}$$

$$y'' + Ay' + By = f(x) \tag{2.3.8}$$
$$(D^2 + AD + B)y = f(x) \tag{2.3.10}$$
$$p(D)y = f(x) \tag{2.3.11}$$

$$p(D)e^{\alpha x} = p(\alpha)e^{\alpha x} \tag{2.3.12}$$

Proof.
$$(D^2 + AD + B)e^{\alpha x} \tag{2.3.13}$$
$$D^2 e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x} \tag{2.3.14}$$
$$e^{\alpha x} \cdot \alpha^2 + Ae^{\alpha x} \cdot \alpha + Be^{\alpha x} \tag{2.3.15}$$
$$p(\alpha)e^{\alpha x} \tag{2.3.16}$$

□

Theorem 2.8 (Exponential Input Theorem)**.** *For* $(D^2 + AD + B)y = e^{\alpha x}$

$$y_p = \frac{e^{\alpha x}}{p(\alpha)} \tag{2.3.17}$$

Proof.
$$p(D)y_p = e^{\alpha x} \tag{2.3.18}$$
$$p(D)\frac{e^{\alpha x}}{p(\alpha)} = e^{\alpha x} \tag{2.3.19}$$
$$p(\alpha)\frac{e^{\alpha x}}{p(\alpha)} = e^{\alpha x} \tag{2.3.20}$$
$$e^{\alpha x} = e^{\alpha x} \tag{2.3.21}$$

□

Example: $y'' - y' + 2y = 10e^{-x}\sin x$

$$(D^2 - D + 2)\tilde{y}_p = 10e^{(-1+i)x}$$
$$\tilde{y}_p = \tfrac{10e^{(-1+i)x}}{(-1+i)^2-(-1+i)+2}$$
$$\tilde{y}_p = \tfrac{10}{3} \cdot \tfrac{(1+i)}{2} \cdot e^{-x}(\cos x + i \sin x)$$
$$y_p = \Im(\tilde{y}_p) = \tfrac{5}{3}e^{-x}(\cos x + \sin x)$$

$$y = e^{\frac{1}{2}x}\left(c_1 \cos\left(\tfrac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\tfrac{\sqrt{7}}{2}x\right)\right) + \tfrac{5}{3} \cdot e^{-x} \cdot \sqrt{2}\cos\left(x - \tfrac{\pi}{4}\right)$$

What if $p(\alpha) = 0$

<u>Exponential Shift Rule</u>

$$p(D)(e^{ax} \cdot u(x)) = e^{ax} p(D+a) u(x) \tag{2.3.22}$$

Check: $p(D) = D$ (simple polynomial)

$$De^{ax}u \tag{2.3.23}$$

$$e^{ax} \cdot Du + ae^{ax} \cdot u \tag{2.3.24}$$

$$e^{ax}(Du + au) \tag{2.3.25}$$

$$e^{ax}(D+a)u \tag{2.3.26}$$

$$p(D) = D^2$$

$$D^2 e^{ax} u \tag{2.3.27}$$

$$D(De^{ax}u) \tag{2.3.28}$$

$$D(e^{ax}(D+a)u) \tag{2.3.29}$$

$$e^{ax}(D+a)[(D+a)u] \tag{2.3.30}$$

$$e^{ax}(D+a)^2 u \tag{2.3.31}$$

$$(D^2 + AD + B)y = e^{ax}, \qquad p(a) = 0 \tag{2.3.32}$$

Then

$$y_p = \frac{xe^{ax}}{p'(a)} \tag{2.3.33}$$

If a is a double root:

$$y_p = \frac{x^2 e^{ax}}{p''(a)} \tag{2.3.34}$$

Proof. Simple Root Case

$$p(D) = (D-b)(D-a), \qquad a \neq b \tag{2.3.35}$$

$$p'(D) = (D-a) + (D-b) \tag{2.3.36}$$

$$p'(a) = a - b \tag{2.3.37}$$

$$p(D)\frac{e^{ax} \cdot x}{p'(a)} = e^{ax} \tag{2.3.38}$$

$$e^{ax} \cdot (D + a - b) D \frac{x}{p'(a)} \tag{2.3.39}$$

$$e^{ax}(D+a-b)\frac{1}{p'(a)} \tag{2.3.40}$$

$$e^{ax}\frac{a-b}{a-b} \tag{2.3.41}$$

$$e^{ax} = e^{ax} \tag{2.3.42}$$

□

Example: $y'' - 3y' + 2y = e^x$

$$p(D) = D^2 - 3D + 2$$
$$p'(D) = 2D - 3D$$
$$p'(1) = -1$$

$$y_p = -xe^x$$

2.4 Fourier Series

$$f(t) = c_0 + \sum_{n+1}^{\infty} [a_n \cos nt + b_n \sin nt] \tag{2.4.1}$$

Where $f(t)$ is a periodic function

$$\begin{array}{cccc} & \underline{\text{Input}} & & \underline{\text{Response}} \\ & b_n \sin nt & \rightsquigarrow & b_n y_n^s(t) \\ + & a_n \cos nt & \rightsquigarrow & a_n y_n^c(t) \\ \hline \end{array}$$

$$f(t) = \sum_{n+1}^{\infty} [a_n y_n^c(t) + b_n y_n^s(t)] + c_1$$

Because the ODE is linear

Definition 2.3. *Given two functions $u(t)$ and $v(t)$ which are continuous on $\mathbb{R}$ and have a period of 2π. They are orthogonal on $[-\pi, \pi]$ if*

$$\int_{-\pi}^{\pi} u(t)v(t)dt = 0 \tag{2.4.2}$$

Theorem 2.9. *From the set of functions*

$$\begin{array}{ll} \sin nt & n = 1, 2, 3, ... \\ \cos mt & m = 0, 1, 2, ... \end{array}$$

Any two distinct (can be same function) members are orthogonal on $[-\pi, \pi]$

Proof. $m \neq n$

They $(\sin nt, \cos nt)$satisfy the ODE

$$u_n'' + n_n^2 u = 0 \tag{2.4.3}$$

Let u_n, v_m be any distinct of the two functions

$$\int_{-\pi}^{\pi} u_n'' v_m dt = u_n' v_m|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u_n' v_m' dt \tag{2.4.4}$$

$$\int_{-\pi}^{\pi} u_n'' v_m dt = -\int_{-\pi}^{\pi} u_n' v_m' dt = \int_{-\pi}^{\pi} v_m'' u_n dt \tag{2.4.5}$$

$$\int_{-\pi}^{\pi} u_n'' v_m dt = -n^2 \int_{-\pi}^{\pi} u_n v_m dt \tag{2.4.6}$$

$$\int_{-\pi}^{\pi} v_m'' u_n dt = -m^2 \int_{-\pi}^{\pi} u_n v_m dt \tag{2.4.7}$$

$$-n^2 \int_{-\pi}^{\pi} u_n v_m dt = -m^2 \int_{-\pi}^{\pi} u_n v_m dt \tag{2.4.8}$$

$$\rightarrow \int_{-\pi}^{\pi} u_n v_m dt = 0 \tag{2.4.9}$$

□

Given $f(t)$ with period 2π find a_n and b_n

$$f(t) = ... + a_k \sin kt + ... + a_n \cos nt + ... \tag{2.4.10}$$

$$f(t) \cdot \cos nt = ... + a_k \cos nt \sin kt + ... + a_n \cos nt \cos nt + ... \tag{2.4.11}$$

$$\int_{-\pi}^{\pi} f(t) \cdot \cos nt \; dt = ... + \int_{-\pi}^{\pi} a_k \cos nt \sin kt \; dt + ... + \int_{-\pi}^{\pi} a_n \cos nt \cos nt \; dt + ... \tag{2.4.12}$$

$$\int_{-\pi}^{\pi} f(t) \cdot \cos nt \; dt = ... + 0 + 0 + 0 + ... + 0 + a_n \pi + 0 + ... \tag{2.4.13}$$

Formula for a_n and b_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \cos nt \; dt \tag{2.4.14}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \sin nt \; dt \tag{2.4.15}$$

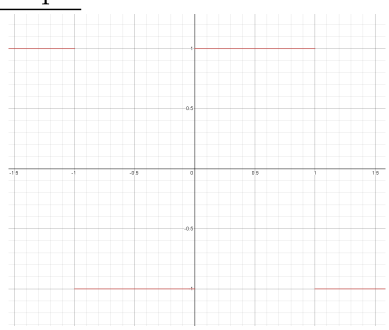
The constant term:

$$f(t) = c_0 + ... + a_n \cos nt + ... \tag{2.4.16}$$

$$\int_{-\pi}^{\pi} f(t) dt = 2\pi + 0 + 0 + 0 + ... \tag{2.4.17}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{a_0}{2} \tag{2.4.18}$$

Example: Find the fourier series of



$$a_n = \frac{1}{\pi} \left(\int_0^{\pi} \cos nt \; dt - \int_{-\pi}^{0} \cos nt \; dt \right) = 0$$

$$b_n = \frac{1}{\pi} \left(\int_0^{\pi} \sin nt \; dt - \int_{-\pi}^{0} \sin nt \; dt \right)$$

$$\frac{1}{\pi} \left(\frac{1 - \cos n\pi}{n} + \frac{1 - \cos n\pi}{n} \right) = \frac{2}{\pi n} (1 - \cos n\pi)$$

$$= \frac{2}{\pi n} (1 - (-1)^n) = b_n$$

$$f(t) = \frac{a_n}{2} + \sum_{n+1}^{\infty} [a_n \cos nt + b_n \sin nt]$$

$$f(t) = \sum_{n+1}^{\infty} \frac{2}{\pi n} (1 - (-1)^n) = b_n$$

$$= \tfrac{4}{\pi} \sin t + \tfrac{4}{3\pi} \sin 3t + \tfrac{4}{5\pi} \sin 5t + ...$$

How to shorten Process:

1. If $f(t)$ is an even function

   $\rightarrow f(t) = \frac{a_n}{2} + \sum_{n+1}^{\infty} a_n \cos nt$

   all b_n are zero

2. If $f(t)$ is an odd function

   $\rightarrow f(t) = \sum_{n+1}^{\infty} abn \sin nt$

   all a_n are zero

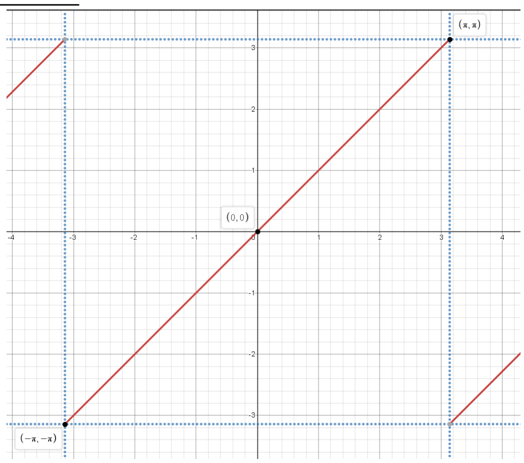
3. If $f(t)$ is an even function

   $\rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cdot \cos nt \ dt$

4. If $f(t)$ is an even function

   $\rightarrow b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cdot \cos nt \ dt$

Example: Find the Fourier series



$$b_n = \frac{2}{\pi} \int_0^{\pi} t \sin t \ dt$$
$$b_n = \frac{2}{\pi} \left[-t \frac{\cos nt}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{-\cos nt}{n} \ dt \right]$$
$$b_n = \frac{2}{\pi} \left[\frac{-\pi}{n} (-1)^n + \frac{\sin nt}{n^2} \Big|_0^{\pi} \right]$$
$$b_n = \frac{2}{n} (-1)^{n+1}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nt$$

Theorem 2.10. *If f is continuous at t_0, then $f(t_0)$ = sum of F.S. If f has a jump discontinuity, then F.S.= midpoint of the jump.*

Extension 1: period is 2L

$$\cos\frac{n\pi}{L}t \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cdot\cos\frac{n\pi}{L}t\;dt \tag{2.4.19}$$

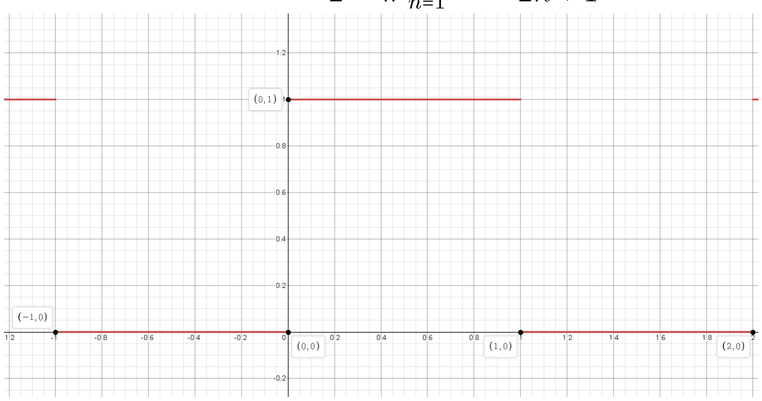
$$\sin\frac{n\pi}{L}t \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\cdot\sin\frac{n\pi}{L}t\;dt \tag{2.4.20}$$

Extension 2: if f is defined on $(0, L]$

Periodic Extension:
Either odd or even extensions making it periodic on $[-L, L]$, now do a F.S.

2.4.1 Using Fourier Series to Solve Differential Equations

Example: Use $f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n+1)\pi t)}{2n+1}$ to solve $x'' + \omega_0 x = f(t)$



Knowing that if $f(t) = \sin\omega t$, then $x_p = \frac{\sin\omega t}{\omega_0^2 - \omega^2}$

$f(t) = \cos\omega t \qquad x_p = \frac{\cos\omega t}{\omega_0^2 - \omega^2}$

More General: If $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\omega_n t + b_n\sin\omega_n t \qquad \omega_n = \frac{n\pi}{L}$

$$x_p = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty}\frac{a_n\cos\omega_n t}{\omega_0^2 - \omega_n^2} + \frac{b_n\sin\omega_n t}{\omega_0^2 - \omega_n^2}$$

If f is the sum of a bunch of functions, the particular solution will be the sum of the individual solutions because the ODE is linear. The constant term in front is treated as a cosine when $\omega_n = 0$

Another Method if $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n\sin nt$

Assume a solution in the form $x_p = c_0 + \sum_1^{\infty} c_n\sin nt$

$x_p = \qquad \sum_1^{\infty} -c_n n^2\sin nt$

$$c_0 + \sum_1^{\infty} c_n[\sin nt - n^2\sin nt] = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n\sin nt$$

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{b_n}{1-n^2} \tag{2.4.21}$$

2.5 Laplace Transform

Start with a power series

$$\sum_{n=0}^{\infty} a(n)x^n = A(x) \tag{2.5.1}$$

The continuous analogue where $n = 0, 1, 2, ... \to 0 \le t \le \infty$

$$\int_0^{\infty} a(t)x^t \; dt = A(x) \tag{2.5.2}$$

$$\begin{aligned} x &= e^{\ln x} \\ x^t &= \left(e^{\ln x}\right)^t \\ 0 \le x \le 1 \quad & \text{we want } \textstyle\int \text{ to converge} \\ \ln x &< 0 \\ -s &= \ln x \end{aligned}$$

Laplace Transform:

$$\int_0^{\infty} f(t)e^{-st} \; dt = F(s) \tag{2.5.3}$$

$$\mathscr{L}\{f(t)\} = F(s) \tag{2.5.4}$$

Linear Transform

$$\mathscr{L}(f+g) = \mathscr{L}(f) + \mathscr{L}(g) \tag{2.5.5}$$

$$\mathscr{L}(c \cdot f) = c \cdot \mathscr{L}(f) \tag{2.5.6}$$

Because the integral is linear operator.

Example: $\mathscr{L}(1)$

$$\begin{aligned} \int_0^{\infty} e^{-st} dt &= \lim_{r\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^r \\ &= \lim_{r\to\infty} \frac{e^{-sr}}{-s} - \frac{e^0}{-s} \\ &= \tfrac{1}{s}, \quad s > 0 \end{aligned}$$

Exponential Shift Formula

$$e^{at}f(t) \rightsquigarrow F(s-a) \tag{2.5.7}$$

Proof.

$$\mathscr{L}(e^{at}f(t)) = \int_0^{\infty} e^{at}f(t)e^{-st} \; dt \tag{2.5.8}$$

$$\int_0^{\infty} e^{-(s-a)t}f(t) \; dt \tag{2.5.9}$$

This is just a Laplace Transform in terms of $s - a$. □

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}, \quad s > a$$

Example: $\mathscr{L}(\sin at \text{ and } \cos at)$

$\cos at = \frac{e^{iat}+e^{-iat}}{2}$
$\mathscr{L}(\cos at) = \frac{1}{2}\mathscr{L}(e^{iat}) + \frac{1}{2}\mathscr{L}(e^{-iat})$
$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$

$$\mathscr{L}(\cos at) = \frac{s}{s^2+a^2}$$
$$\mathscr{L}(sinat) = \frac{a}{s^2+a^2}$$

Example: Find $\mathscr{L}^{-1}$ of $\frac{1}{s(s-3)}$

By partial fraction decomp.

$$\frac{1}{s(s-3)} = \frac{\frac{1}{3}}{s} + \frac{\frac{-1}{3}}{s+3}$$
$$\rightsquigarrow \mathscr{L}^{-1} \rightsquigarrow \frac{1}{3} - \frac{1}{3}e^{-3t}$$

Example: $\mathscr{L}(t^n)$

$$\int_0^\infty t^n e^{-st}dt = t^n \frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty nt^{n-1}\frac{e^{-st}}{-s}dt$$
$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$
$$= \frac{n}{s}\mathscr{L}(t^{n-1})$$

$\mathscr{L}(t^n) = \frac{n}{s}\mathscr{L}(t^{n-1}) = \ldots = \frac{n!}{s^n}\mathscr{L}(t^0)$

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

2.5.1 Using Laplace Transforms to Solve Linear ODEs

We need a condition that guarantees that $\mathscr{L}$

Growth Condition:

$$|f(t)| \leqslant Ce^{kt} \quad C, k > 0 \tag{2.5.10}$$

Must be true all $t > 0$

$f(t)$ must be of exponential type

Example: t^n

$\lim_{t\to\infty} \frac{t^n}{e^{kt}} = 0$
$\mathscr{L}$ does exist

Example: $\frac{1}{t}$

$$\int_0^\infty \frac{e^{-st}}{t} dt$$

Doesn't converge at $t = 0$
$\frac{1}{t}$ is not of exponential type

e^{t^2} is also not of exponetial type

Solving ODEs: $[\mathscr{L}(y(t)) = Y(s)]$

$$y'' + Ay' + By = h(t) \tag{2.5.11}$$

$$\downarrow \mathscr{L}$$

$$Y = \frac{p(s)}{q(s)} \tag{2.5.12}$$

$$\downarrow \mathscr{L}^{-1}$$

$$y(t) = y \tag{2.5.13}$$

Laplace Transform of a Derivative

$$\mathscr{L}(f'(t)) \tag{2.5.14}$$

$$= \int_0^\infty f'(t)e^{-st}dt \tag{2.5.15}$$

$$= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \tag{2.5.16}$$

$$= \lim_{t\to\infty} \frac{f(t)}{e^{st}} - \frac{f(0)}{1} + sF(s) \tag{2.5.17}$$

$$\mathscr{L}(f'(t)) = sF(s) - f(0) \tag{2.5.18}$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0) \tag{2.5.19}$$

Example: $y'' - y = e^{-y}, \quad y(0) = 1 \; y'(0) = 0$

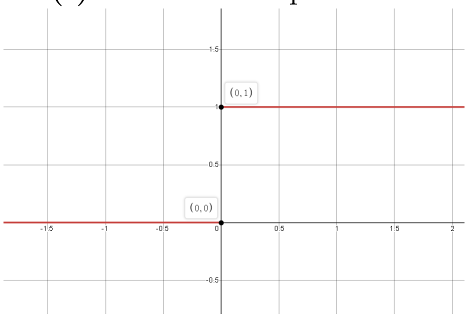
$\mathscr{L}(y'' - y) = \mathscr{L}(e^{-t})$
$s^2Y - s \cdot 1 - 0 - Y = \frac{1}{s+1}$ Plugged in initial conditions.
$Y \cdot (s^2 - 1) = \frac{1}{s+1} + s$
$Y \cdot (s^2 - 1) = \frac{s^2+s+1}{s+1}$

$$Y = \frac{s^2+s+1}{(s+1)^2(s-1)}$$
$$Y = \frac{-.5}{(s+1)^2} + \frac{.25}{s+1} + \frac{.75}{s-1}$$
$$y(t) = \mathscr{L}^{-1}\left(\frac{-.5}{(s+1)^2} + \frac{.25}{s+1} + \frac{.75}{s-1}\right)$$

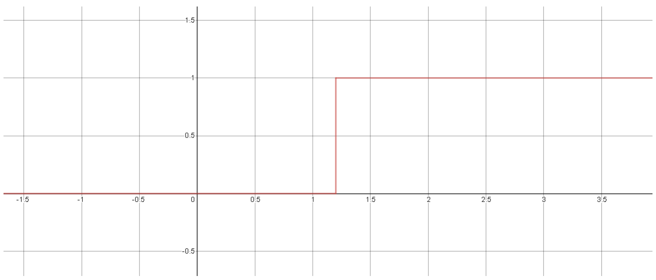
$$y = \frac{1}{4}e^{-t} + \frac{3}{4}e^{t} - \frac{1}{2}te^{-t}$$

2.5.2 Laplace Transforms for ODEs with Jump Discontinuities

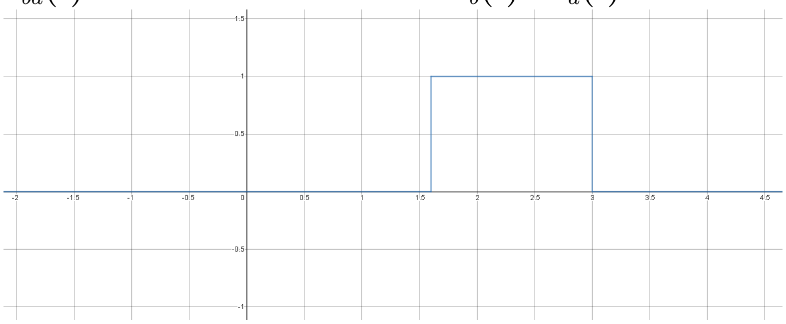
$u(t)$ is the unit step function



$u(t-a) = u_a(t)$

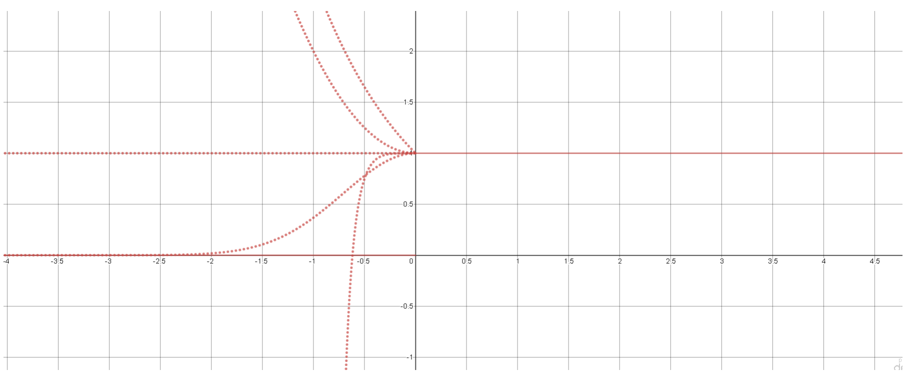


$u_{ba}(t)$ is the unit box function $= u_b(t) - u_a(t)$



$$\mathscr{L}(u(t)) = \int_0^\infty e^{-st}u(t)\ dt = \frac{1}{s}, \quad s > 0 \tag{2.5.20}$$

$$\mathscr{L}(1) = \frac{1}{s} = \mathscr{L}(u(t)) \tag{2.5.21}$$

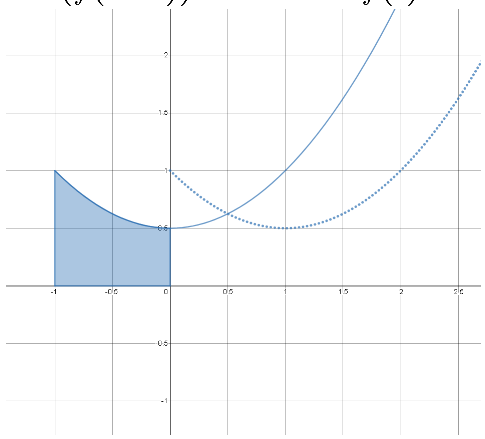


All these functions have the same $\mathscr{L}$, $\mathscr{L}$ doesn't care about $x < 0$.

Now if $\mathscr{L}(f(t)) = F(s)$

$$\mathscr{L}^{-1}(F(s)) = u(t) \cdot f(t) \tag{2.5.22}$$

Formula for $\mathscr{L}(f(t-a))$ in terms of $f(t)$ doesn't exist



The shaded blue piece is not used for $f(t)$
however it will be needed for $f(t-a)$

We can find a formulas for $u(t-a) \cdot f(t-a)$

$$\mathscr{L}(u(t-a) \cdot f(t-a)) = e^{-as}F(s) \tag{2.5.23}$$

$$\mathscr{L}(u(t-a) \cdot f(t)) = e^{-as}\mathscr{L}(f(t+a)) \tag{2.5.24}$$

Proof.

$$\int_0^\infty e^{-st}u(t-a)f(t-a)dt \qquad t_1 = t-a \tag{2.5.25}$$

$$\int_{-a}^\infty e^{-s(t_1+a)}u(t_1)f(t_1)dt_1 = e^{-as}\int_a^\infty e^{-st_1}u(t_1)f(t_1)dt_1 \tag{2.5.26}$$

$u(t)$ is zero when negative,bound simplified

$$e^{-as}\int_a^\infty e^{-st_1}f(t_1)dt_1 \tag{2.5.27}$$

$$e^{-as}F(s) \tag{2.5.28}$$

$$u(t-a) \cdot f(t-a) \rightsquigarrow e^{-as}\mathscr{L}(f(t)) \qquad \text{replace t by t+a in f} \tag{2.5.29}$$

$$u(t-a) \cdot f(t) \rightsquigarrow e^{-as}\mathscr{L}(f(t+a)) \tag{2.5.30}$$

□

Example: $\mathscr{L}(u_{ab}(t))$

$\mathscr{L}(u(t-a)) - \mathscr{L}(u(t-b))$

$\dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s}$

Example: $\mathscr{L} u(t-1) \cdot t^2$

$= e^{-s} \mathscr{L}(t+1)^2$
$= e^{-s} \mathscr{L}(t^2 + 2t + 1)$
$= e^{-s}(\frac{2}{s^2} + \frac{2}{s^2} + \frac{1}{s})$

Example: $\mathscr{L}^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right)$

$\mathscr{L}^{-1} \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$
$u(t) \sin t + u(t-\pi) \sin(t-\pi)$
$u(t)$ because we need a unique solution of this in order for formulas to work

$$f(t) = \begin{matrix} \sin t & 0 \le t \le \pi \\ 0 & t > \pi \end{matrix}$$

2.6 Convolution

Convolution

$$f(t) * g(t) \tag{2.6.1}$$

Given

$F(s) = \int_0^\infty e^{-st} f(t) dt$

$G(s) = \int_0^\infty e^{-st} g(t) dt$

$$F(s)G(s) = \int_0^\infty e^{-st} (f * g) dt \tag{2.6.2}$$

Why should a formula for $f * g$ exist

$F(x) = \sum_0^\infty a_n x^n$

$G(x) = \sum_0^\infty b_n x^n$

$$F(x)G(x) = \sum_0^\infty c_n x^n \tag{2.6.3}$$

$$\begin{array}{ccc} \multicolumn{3}{c}{(a_0 + a_1 x + a_2 x^2 ...)(b_0 + b_1 x + b_2 x^2 ...)} \\ x^0 & x^1 & x^2 ... \\ a_0 b_0 & (a_0 b_1 + b_1 a_0) & (a_0 b_2 + a_1 b_1 + a_2 b_0)... \end{array}$$

$$c_n = \sum_{t=0}^{n} a_t b_{n-t} \tag{2.6.4}$$

Back to $\mathscr{L}$

$$f * g = \int_0^t f(u) g(t-u)\ du \tag{2.6.5}$$

Remember $f * g = g * f$

Example: $t^2 * t$

$\int_0^t u^2 (t-u)\ du$

$\frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$

$t^2 * t$

$\vee \mathscr{L}$

$\frac{2}{s^3} \cdot \frac{1}{s^2} = \frac{2}{s^5}$

$\mathscr{L}^{-1}(\frac{2}{s^5}) = \frac{t^4}{12}$

Example: $f(t) * 1$

$= \int_0^t f(u) 1 du$

$= \int_0^t f(u)\ du$

Proof.

$$F(s)G(s) = \int_0^\infty e^{-st}(f * g)\ dt \tag{2.6.6}$$

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)\ dt \cdot \int_0^\infty e^{-sv} g(v)\ dt \tag{2.6.7}$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v)\ du\ dv \tag{2.6.8}$$

$$t = u + v$$
$$u = u \qquad v = t - u$$

$$\int_0^\infty \int_0^t e^{-st} f(u) g(t-u)\ du\ dt \tag{2.6.9}$$

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$= \int_0^\infty e^{-st} \int_0^t f(u) g(t-u)\ du\ dt \tag{2.6.10}$$

The middle is the formula for convolution.

□

Example: Radioactive waste is dumped. $f(t)$ is the dump rate where t = years

Starting at $t = 0$
Remembering radioactive decay, at time t, how much waste is in the dump?

$A_0 e^{-kt}$ is the amount left at time t

Amount dumped on $[u_i, u_{i+1}]$
$\approx f(u_i) \cdot \Delta u$

By time t, it has decayed to: $f(u_i)\Delta u \cdot e^{-k(t-u_i)}$
Total

$$= \lim_{\Delta u \to 0} \sum_{i=1}^{n} f(u_i) e^{-k(t-u_i)} \Delta u$$

$$= \int_0^t f(u) e^{-k(t-u)} du = f(t) * e^{-kt}$$

What if it didn't decay

$f(t) * 1 \qquad = \int_0^t f(u)\ du$ which is intuitive

$f(t) \qquad * \qquad g(t) \quad$ = amount of x at time t

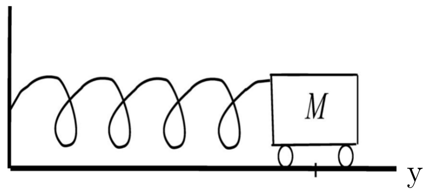
prod. of x how x changes

2.7 Dirac-Delta Function

Force = force × time $[a, b]$

Constant force

Impulse = $F \cdot (b - a)$



Lets say required constants are 1 and a unit impulse is applied.

This is modeled by

$$y'' + y = \frac{1}{h} u_{0h}(t) \tag{2.7.1}$$

$$\mathscr{L}\left(\frac{1}{h} u_{0h}(t)\right) \tag{2.7.2}$$

$$\frac{1}{h}\mathscr{L}\left(u(t) - u(t-h)\right) \tag{2.7.3}$$

$$\frac{1}{h}\left(\frac{1}{s} - \frac{e^{-hs}}{s}\right) \tag{2.7.4}$$

What if $h \to 0$

$$\lim_{h \to 0} \frac{1 - e^{-hs}}{hs} \tag{2.7.5}$$

$$\lim_{u \to 0} \frac{1 - e^{-u}}{u} = \lim_{u \to 0} \frac{e^{-u}}{1} = 1 \tag{2.7.6}$$

This is the Dirac Delta Function

$$\delta(t) \tag{2.7.7}$$

$$\mathscr{L}[\delta(t)] = 1 \tag{2.7.8}$$

$$\int_{-\infty}^{\infty} \delta(t)\ dt = 1 \tag{2.7.9}$$

$$\mathscr{L}[u(t)f(t) * \delta(t)] = F(s) \cdot 1$$

$$\mathscr{L}[u(t)f(t)] = F(s)$$

$$u(t)f(t) * \delta(t) = u(t)f(t) \tag{2.7.10}$$

$\delta(t)$ is the identity element of convolution

Example: $y'' + y = A\delta(t - \frac{\pi}{2}) \quad y(0) = 1, \quad y'(0) = 0$
"kicked with impulse A at time $\frac{\pi}{2}$"

$$
\begin{aligned}
&s^2Y - s + Y = Ae^{-\frac{\pi}{2}s} \\
&Y = \tfrac{s}{s^2+1}\tfrac{Ae^{-\frac{\pi}{2}s}}{s^2+1} \\
&y = \cos t + u(t - \tfrac{\pi}{2})A\sin\left(t - \tfrac{\pi}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
y = \cos t & \qquad 0 \le t \le \tfrac{\pi}{2} \\
(1 - A)\cos t & \qquad t \ge \tfrac{\pi}{2}
\end{aligned}
$$

$$y'' + ay' + by = f(t) \quad y(0) = 0, \quad y'(0) = 0 \tag{2.7.11}$$

$$s^2Y + asY + bY = F(s) \tag{2.7.12}$$

$$Y = F(s) \cdot \frac{1}{s^2 + as + b} \tag{2.7.13}$$

$\frac{1}{s^2+as+b} = W(s)$ Transfer function

$$y(t) = f(t) * \mathscr{L}^{-1}\left(\frac{1}{s^2 + as + b}\right) \tag{2.7.14}$$

$\mathscr{L}^{-1}\left(\frac{1}{s^2+as+b}\right) = w(t)$ Weight function

$$y(t) = \int_0^t f(u) \cdot w(t - u)\ du \tag{2.7.15}$$

But what is $w(t)$, well imagine the input being the Dirac Delta Function

$$y'' + ay' + by = \delta(t) \quad y(0) = 0, \quad y'(0) = 0 \tag{2.7.16}$$

$$s^2Y + asY + bY = 1 \tag{2.7.17}$$

$$Y = \frac{1}{s^2 + as + b} \tag{2.7.18}$$

$$y(t) = w(t) \tag{2.7.19}$$

3 Systems of Ordinary Differentials Equations

3.1 First Order Systems of ODEs

$$x' = f(x, y, t) = \frac{dx}{dt} \tag{3.1.1}$$

$$y' = g(x, y, t) = \frac{dy}{dt} \tag{3.1.2}$$

x and y are independent variables on t.

Linear Systems:

$$x' = ax + by + r_1(t) \tag{3.1.3}$$

$$y' = cx + dy + r_2(t) \tag{3.1.4}$$

$$x(t_0) = x_0 \quad y(t_0) = y_0 \tag{3.1.5}$$

$a,\ b,\ c,\ d$ can be functions of t. If $a,\ b,\ c,\ d$ are constants, the system is called a constant-coefficient system.

Definition 3.1. *A system is Homogeneous if it has no $r_1(t)$ or $r_2(t)$.*

A system has as many initial conditions as the total order in the system.

Example: An egg in water with yolk and white stuff.
$T_e(t)$ = temp. of water (constant for simplicity)
$T_2(t)$ = temp. of white stuff
$T_1(t)$ = temp. of yolk

$$\frac{dT_1}{dt} = a(T_2 - T_1)$$

$$\frac{dT_2}{dt} = a(T_1 - T_2) + b(T_e - T_2)$$

$$T_1' = -aT_1 + aT_2$$
$$T_2' = aT_1 - (a+b)T_2 + bT_e$$

Lets say $T_e = 0$
also a=2, b=3

$$T_1' = -2T_1 + 2T_2$$
$$T_2' = 2T_1 - 5T_2$$

To eliminate T_2 $\qquad T_2 = \dfrac{T_1' + 2T_1}{2}$

$$\left(\frac{T_1' + 2T_1}{2}\right)' = 2T_1 - 5\left(\frac{T_1' + 2T_1}{2}\right)$$
$$T_1'' + 7T_1' + 6T = 0$$

$$T_1 = c_1 e^{-6t} + c_2 e^{-t}$$
$$T_2 = \tfrac{1}{2} c_2 e^{-t} - 2c_1 e^{-6t}$$

$$T_1(0) = 40$$
$$T_2(0) = 45$$

$$40 = c_1 + c_2$$
$$45 = \tfrac{1}{2}c_2 - 2c_1$$

$$c_1 = -10 \qquad c_2 = 50$$

$$T_1 = 50e^{-t} - 10e^{-6t}$$
$$T_2 = 25e^{-t} + 20e^{-6t}$$

Definition 3.2. *An autonomous system has no "t"s.*

$$x' = f(x, y)$$
$$y' = g(x, y)$$

The solution is a parametric equations of the form
$x = x(t)$
$y = y(t)$

x' and y' give a vector field of the velocities of the solutions.

3.1.1 Eigenvalue Method of Solving First Order Systems of ODEs

A different method to the first example.

	Ans.	Sol.
$x = T_1$	$x' = -2x + 2y$	$x = c_1 e^{-t} + c_2 e^{-6t}$
$y = T_2$	$y' = 2x - 5y$	$y = \frac{c_1}{2} e^{-t} - 2c_2 e^{-6t}$

Que.
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Sol.
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

Process:

Substitute

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

This is now a system of equations:

$$\lambda a_1 = -2a_1 + 2a_2$$
$$\lambda a_2 = 2a_1 - 5a_2$$

This is 3 unkowns in 2 equations so lets treat λ as a parameter

$$(-2 - \lambda)a_1 + 2a_2 = 0$$
$$2a_1 + (-5 - \lambda)a_2 = 0$$

There exists a non-trivial solution if and only if:

$$\begin{vmatrix} -2 - \lambda & 2 \\ 2 & -5 - \lambda \end{vmatrix} = 0$$

This is the condition λ must satisfy

$$(\lambda + 2)(\lambda + 5) - 4 = 0$$
$$\lambda^2 + 7\lambda + 6 = 0$$
$$\lambda = -1, \ -6$$

Notice how the quadratic equation is the same as the characteristic equation gotten from the first method. These values of λ are called the eigenvalues of the coefficient matrix.

$\lambda = -1$ Find a_1 and a_2 $\lambda = -6$

$$-a_1 + 2a_2 = 0 \qquad 4a_1 + 2a_2 = 0$$
$$2a_1 - 4a_2 = 0 \qquad 2a_1 + a_2 = 0$$

Notice how one equation is a constant multiple of the other, this is an easy way to check if everything is going correctly. As these are the same equations, find any solution that works. Usually substitute 1 in and see what value the other unknown is. Then put the two values into what is known as a eigen vector. Then multiply by e to the eigenvalue times t.

$$a_2 = 1 \ \rightarrow \ a_1 = 2 \qquad a_2 = -2 \ \rightarrow \ a_1 = 1$$
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

By superposition:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

If a different eigen vector was chosen, it would have to be a constant multiple of the previous, which can be factored out into the unknown constants in front.

In General:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.1.6}$$

Trial

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \tag{3.1.7}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{3.1.8}$$

Solvable if:

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \tag{3.1.9}$$

$$\lambda^2 - (a + b)\lambda + ad - bc = 0 \tag{3.1.10}$$

This the characteristic equation which can also be written as:

$$\lambda^2 + \lambda \operatorname{trace} A + \det A = 0 \tag{3.1.11}$$

Where det is the determinant and trace is the sum of the main diagonal.

The two roots λ_1 and λ_2 (for now assume real and distinct) are the eigenvalues for A.

For each λ_i, find the associated:

$$\vec{\alpha}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} \tag{3.1.12}$$

By solving the system:

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \tag{3.1.13}$$

The general solution is now in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} e^{\lambda_2 t} \tag{3.1.14}$$

More in general:

$$\begin{pmatrix} x \\ y \\ \vdots \end{pmatrix} = \vec{x} \qquad \vec{x}' = A\vec{x} \tag{3.1.15}$$

$$\begin{pmatrix} a & b & \dots \\ c & d & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = A \qquad \text{Trial}: \quad \vec{x} = \vec{\alpha} e^{\lambda t} \tag{3.1.16}$$

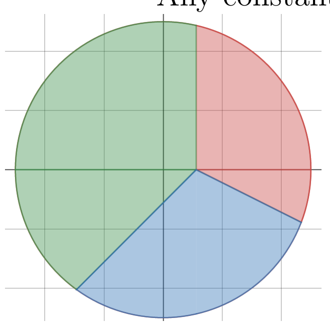
$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} = \vec{\alpha} \qquad \lambda\vec{\alpha} = A\vec{\alpha} \tag{3.1.17}$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = I \qquad (A - \lambda \cdot I)\vec{\alpha} = 0 \tag{3.1.18}$$

$$\text{Chara. Eq.} \quad |A - \lambda I| = 0 \tag{3.1.19}$$

now continue with the eigenvalues as before

<u>Example</u>: Image a tank divide into 3 sections where
X_i gives the temperature in tank i.
Any constants are 1.



$$x_1' = a(x_3 - x_1) + a(x_2 - x_1)$$
$$x_1' = -2ax_1 + ax_2 + ax_3$$

$$x_1' = -2x_1 + x_2 + x_3$$
$$x_2' = x_1 - 2x_2 + x_3$$
$$x_3' = x_1 + x_2 - 2x_3$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3 + 2 - 3(-2-\lambda) = 0$$

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 - 2 - 6 - 3\lambda = 0$$
$$\lambda^3 + 6\lambda^2 + 9\lambda = 0$$
$$\lambda(\lambda^2 + 6\lambda + 9) = 0$$
$$\lambda(\lambda + 3)^2 = 0$$

The eigenvalues are 0 and a repeated -3

$$\lambda = 0$$

$$(A - \lambda I)\vec{\alpha} = 0$$
$$A\vec{\alpha} = 0$$

$$-2a_1 + a_2 + a_3 = 0$$
$$a_1 - 2a_2 + a_3 = 0$$
$$a_1 + a_2 - 2a_3 = 0$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = -3$$

$$a_1 + a_2 + a_3 = 0$$
$$a_1 + a_2 + a_3 = 0$$
$$a_1 + a_2 + a_3 = 0$$

In this case, there are two possible solutions such that one is not a constant multiple of the other:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Notice how $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is not a new solution. It is not a constant multiple of the other but it is a linear combination, disqualifying it. The solution now is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Definition 3.3 (Complete Eigenvalue)**.** *If an eigenvalue λ is repeated, but you can find enough needed eigenvectors, then it is complete.*

Definition 3.4 (Defective Eigenvalue)**.** *If an eigenvalue λ is repeated, but you can not find enough needed eigenvectors, then it is defective.*

Theorem 3.1. *If a real $n \times n$ matrix which is symmetric (about diagonal) then all of its eigenvalues are complete.*

Complex Eigenvalues:

1. Calculate complex eigenvalues
2. Formulate solution: $\vec{\alpha}e^{(a+bi)t}$
3. Take the real and imaginary parts to get two solutions.

Example:
$$x' = x + 2y$$
$$y' = -x - y$$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\lambda^2 + 0\lambda + (-1 + 2) = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

$$\lambda = i$$

$$(1 - i)a_1 + 2a_2 = 0$$
$$-1a_1 + (-1 - i)a_2 = 0$$
$$a_1 = 1 \rightarrow a_2 = \tfrac{-1+i}{2}$$

$$\begin{pmatrix} 1 \\ \frac{-1+i}{2} \end{pmatrix} e^{it} = \left[\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + i\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right](\cos t + i \sin t)$$

Solutions:

$$\text{Real} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\sin t$$

$$\text{Imaginary} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}\sin t$$

3.2 How to Sketch Solutions to Systems of First Order ODEs

We will work with the cases of:

$$x' = -x + by \tag{3.2.1}$$

$$y' = cx - 3y \tag{3.2.2}$$

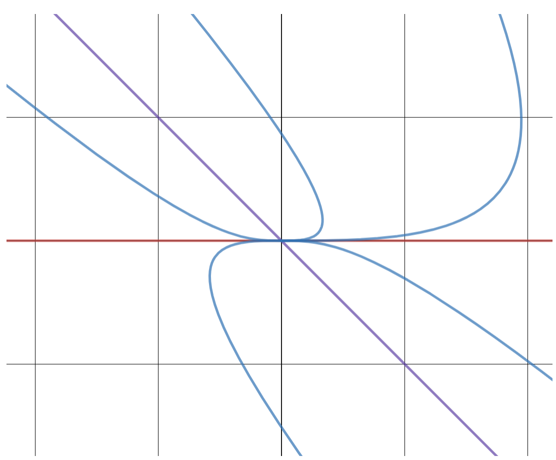
<u>First Case</u>: $x' = -x + 2y$
$y' = -3y$

Solving the system yields:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

1.) There are 4 easy solutions
$c_1 = \pm 1, \quad c_2 = 0$
$c_1 = 0, \quad c_2 = \pm 1$
These just straight lines, what about in between

2.) $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$
$\downarrow$ $\searrow$ dom. as $t \to \infty$
dom. as $tv \to -\infty$



In this case, all the lines go towards the origin as $t \to \infty$, making the origin a sync node. If the signs where flipped as in

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\underline{3t}} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\underline{t}} \tag{3.2.3}$$

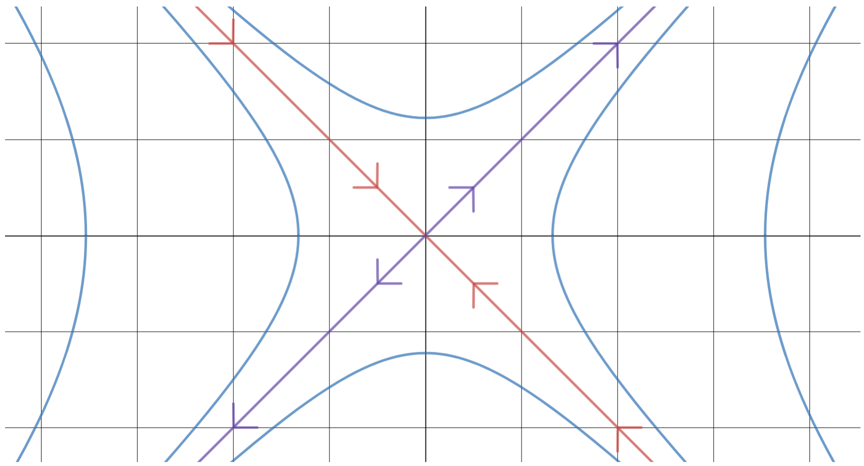
then the curves would go all outwards from the origin, making the origin a source node.

Example: $\qquad x' = -x + 3y$
$\qquad\qquad\qquad y' = 5x - 3y$

Solving the system yields:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

The four trivial solutions give line where one goes toward the origin and another away. This makes the origin a saddle point. Everything in between acts like a hyperbola.



Example: $\qquad$ Complex eigenvalues:

Solutions looks something like

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin t \right] e^{(\lambda t)} + c_2[\text{something similar}]$$

The $\cos t$ and $\sin t$ make circles which grow and decay because of the exponential. So essentially you get spirals. To check which direction, you plug a point into the initial problem to get a velocity vector.

3.3 Inhomogeneous Systems

Lets begin with the theory of Homogeneous Systems (constant coefficient matrix):

$$\vec{x}' = A\vec{x} \tag{3.3.1}$$

Theorem 3.2. *The general solution to $\vec{x}' = A\vec{x}$ is:*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 \tag{3.3.2}$$

Where $\vec{x}_1$ and $\vec{x}_2$ are two solutions which linearly independent of each other and not constant multiples of each other.

Theorem 3.3. *The Wronskian of two solutions*

$$W(\vec{x}_1,\ \vec{x}_2) = |\vec{x}_1\ \vec{x}_2| \tag{3.3.3}$$

Either: $W(t) \equiv 0$ (if $\vec{x}_1$, $\vec{x}_2$ are dependent)
or $W(t)$ is never 0 for any t.

Fundamental Matrix for the System: $\mathbb{X} = \begin{bmatrix}\vec{x}_1 & \vec{x}_2\end{bmatrix}$ where $\vec{x}_1$ and $\vec{x}_2$ are two ind. solutions

Properties:

1.) $|\mathbb{X}| \neq 0$ for any t.
2.) $\mathbb{X}' = A\mathbb{X}$

Proof.

$$\begin{bmatrix}\vec{x}_1' & \vec{x}_2'\end{bmatrix} = A\begin{bmatrix}\vec{x}_1 & \vec{x}_2\end{bmatrix} = \begin{bmatrix}A\vec{x}_1 & A\vec{x}_2\end{bmatrix} \tag{3.3.4}$$

$$\Leftrightarrow \vec{x}_1' = A\vec{x}_1\ ,\ \vec{x}_2' = A\vec{x}_2 \tag{3.3.5}$$

□

Inhomogeneous Systems:

$x' = ax + by + r_1(t)$
$y' = cx + dy + r_2(t)$

$$\vec{x}' = A\vec{x} + \vec{r}(t) \tag{3.3.6}$$

Theorem 3.4. *The general solution is the sum of the solution to the homogeneous system and the particular solution.*

$$\vec{x}_g = \vec{x}_c + \vec{x}_p \tag{3.3.7}$$

We need a method to solve $\vec{x}' = A\vec{x} + \vec{r}$ for $\vec{x}_p$

Variation of Parameters:

$$\vec{x}_p = v_1(t)\vec{x}_1 + v_2(t)\vec{x}_2 \tag{3.3.8}$$

Where $\vec{x}_1$ and $\vec{x}_2$ are solutions to the homo. system.

$$\vec{x}_p = \mathbb{X}\vec{v} \tag{3.3.9}$$

$$\vec{x}_p = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1v_1 + x_2v_2 \\ y_1v_1 + y_2v_2 \end{pmatrix} \tag{3.3.10}$$

Substitute into system and see what $\vec{v}$ is

$$\vec{x}' = A\vec{x} + \vec{r} \tag{3.3.11}$$

$$\mathbb{X}'\vec{v} + \mathbb{X}\vec{v}' = A\mathbb{X}\vec{v} + \vec{r} \tag{3.3.12}$$

$$A\mathbb{X}\vec{v} + \mathbb{X}\vec{v}' = A\mathbb{X}\vec{v} + \vec{r} \tag{3.3.13}$$

$$\mathbb{X}\vec{v}' = \vec{r} \tag{3.3.14}$$

$$\vec{v}' = \mathbb{X}^{-1}\vec{r} \tag{3.3.15}$$

$$\vec{v} = \int \mathbb{X}^{-1}\vec{r}\ dt \tag{3.3.16}$$

$$\vec{x}_p = \mathbb{X}\int \mathbb{X}^{-1}\vec{r}\ dt \tag{3.3.17}$$

There isn't a Fundamental Matrix $\mathbb{X}$, any two solutions to the homo. system will work.

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 \tag{3.3.18}$$

$$\vec{x} = \mathbb{X}\vec{c} \tag{3.3.19}$$

Using that new definition of a solution, the most general FM looks like:

$$\begin{bmatrix} \mathbb{X}\vec{c}_1 & \mathbb{X}\vec{c}_2 \end{bmatrix} = \mathbb{X}\begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix} \tag{3.3.20}$$

$$= \mathbb{X}C \tag{3.3.21}$$

Where C is any 2x2 constant coefficient matrix.

3.4 Matrix Exponentials

For

$$\vec{x}' = A\vec{x} \tag{3.4.1}$$

We want a formula and not a algorithm.

For a 1×1 case: $x' = ax \rightsquigarrow x = ce^{at}$

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \ldots$$

$$\frac{de^{at}}{dt} = a + a^2t + \frac{a^3t^2}{2!} + \frac{a^3t^3}{3!} + \ldots$$

$$\frac{de^{at}}{dt} = ae^{at}$$

So e^{at} solves the 1×1 differential equation.

A fundamental matrix for $\vec{x}' = A\vec{x}$

$$\text{is} e^{At} := I_n + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} \tag{3.4.2}$$

This is the FM because it satisfies $\mathbb{X}' = A\mathbb{X}$ and $\det \mathbb{X} \neq 0$

<u>Example</u>: $x' = y$
$y' = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$
$$A^3 = I \cdot A = A$$
$$A^4 = A^3 \cdot A = A \cdot A = I_2$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{3!} + \ldots\ldots$$

$$\begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & t + \frac{t^3}{3!} + \ldots \\ t + \frac{t^3}{3!} + \ldots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

What about an initial value $\vec{x}(0) = \vec{x}_0$

Solution

$$\vec{x} = e^{At} \cdot \vec{c} \tag{3.4.3}$$

$$\vec{x}(0) = e^{A0} \cdot \vec{c} \tag{3.4.4}$$

$$\vec{x}_0 = I_2 \cdot \vec{c} \tag{3.4.5}$$

$$\vec{x}_0 = \vec{c} \tag{3.4.6}$$

$$\vec{x} = e^{At} \cdot \vec{x}_0 \tag{3.4.7}$$

Matrix Exponentials don't follow standard laws of exponents:

$$e^{A+B} \neq e^A \cdot e^B \tag{3.4.8}$$

This is only true in the special case of $AB = BA$

Three important cases of $AB = BA$

1.) if $A = cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

2.) $B = -A$

3.) $B = A^{-1}$

The second law leads to:

$$I = e^{A-A} = e^A \cdot e^{-A} \tag{3.4.9}$$

$$(e^A)^{-1} = e^{-A} \tag{3.4.10}$$

-1 is a inverse, not power

A third way of solving is:

$$\mathbb{X} \cdot \mathbb{X}^{-1}(0) \tag{3.4.11}$$

Because $\mathbb{X}^{-1}(0)$ is constant coefficient matrix, above is still a FM
Its value at 0 is $\mathbb{X}(0) \cdot \mathbb{X}^{-1}(0) = I_2$

These are the same two properties of e^{At}

$$\therefore e^{At} = \mathbb{X} \cdot \mathbb{X}^{-1}(0) \tag{3.4.12}$$

3.5 Decoupling Linear Systems

Find u and v

$$u = ax + by \tag{3.5.1}$$

$$v = cx + dy \tag{3.5.2}$$

such that in uv-coordinates, the two problems aren't related

$$u' = k_1 u \tag{3.5.3}$$

$$v' = k_2 v \tag{3.5.4}$$

Example: Imagine two tanks with heights y and x next to each other with the base of x being 1 and the base of y being 2 (meaning it the water level rises half as fast). Also remember flow rate $\propto$ area of hole $\cdot$ height difference.

$$x' = c(y - x) \qquad c = 2$$
$$y' = \tfrac{c}{2}(x - y)$$

$$x' = -2x + 2y$$
$$y' = x - y$$

Lets somethings better than height:

$u = x + 2y$ (total water)

$v = x - y$ (height difference)

$$u' = x' + 2y' = 0$$
$$v' = x' - y' = -3x + 3y = -3v$$

$$u = c_1$$
$$v = c_2 e^{-3t}$$

$$x = \tfrac{1}{3}(u + 2v) = \tfrac{1}{3}(c_1 + 2c_2 e^{-3t})$$
$$y = \tfrac{1}{3}(u - v) = \tfrac{1}{3}(c_1 - c_2 e^{-3t})$$

$$\vec{x} = \frac{1}{3}c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3}c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-3t}$$

To Decouple: all e-values must be real and complete

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_1' & a_2' \\ b_1' & b_2' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.5.5}$$

D - the decoupling matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \tag{3.5.6}$$

A D^{-1} works when = E the matrix whose columns are the e-vectors.

$$E = (\vec{\alpha}_1 \ \vec{\alpha}_2) \tag{3.5.7}$$

$$\vec{\alpha}_1 \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.8}$$

$$\vec{\alpha}_2 \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.9}$$

This is saying that what would be the e-vectors in xy-coordinates become the $\hat{\imath}$ and $\hat{\jmath}$ in uv-coordinates.

Now substitute these new variable into $\vec{x}' = A\vec{c}$ to see if the system was decoupled.

This gives a different definition of e-vectors:

$$A\vec{\alpha}_1 = \lambda_1\vec{\alpha}_1 \tag{3.5.10}$$

Where $\vec{\alpha}$ is an e-vector and λ is an e-value.

$$AE = A\begin{bmatrix}\vec{\alpha}_1 & \vec{\alpha}_2\end{bmatrix} = \begin{bmatrix}A\vec{\alpha}_1 & A\vec{\alpha}_2\end{bmatrix} \tag{3.5.11}$$

$$= \begin{bmatrix}\lambda_1\vec{\alpha}_1 & \lambda_2\vec{\alpha}_2\end{bmatrix} \tag{3.5.12}$$

$$= \begin{bmatrix}\vec{\alpha}_1 & \vec{\alpha}_2\end{bmatrix} \cdot \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \tag{3.5.13}$$

$$= E \cdot \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \tag{3.5.14}$$

Given as system $\vec{x}' = A\vec{x}$ and the new variables $\vec{x} = E\vec{u}$

$$E\vec{u}' = AE\vec{u} = E\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\vec{u} \tag{3.5.15}$$

$$\vec{u}' = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\vec{u} \tag{3.5.16}$$

$$u' = \lambda_1 u \qquad u = c_1e^{\lambda_1 t} \tag{3.5.17}$$

$$v' = \lambda_2 v \qquad v = c_2e^{\lambda_2 t} \tag{3.5.18}$$

Example: Decouple $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda^2 + 3\lambda = 0$
$\lambda = 0 \quad \lambda = -3$

$\lambda = 0$
$-2a_1 + 2b_1 = 0$
$a_1 = 1 = b_1 \quad \vec{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3$
$1a_1 + 2b_1 = 0$
$\vec{\alpha}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$E = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$D = E^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \tfrac{1}{3}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$u' = \tfrac{1}{3}(x' + 2y')$$
$$v' = \tfrac{1}{3}(-x' + y')$$

$$u' = \tfrac{1}{3}((-2x + 2y) + 2(x - y)) = 0$$
$$v' = \tfrac{1}{3}(-(-2x + 2y) + (x - y)) = \tfrac{1}{3}(3x - 3y) = -\tfrac{1}{3}v$$

$$u' = 0$$
$$v' = -\frac{1}{3}v$$

3.6 Closed Trajectories

Represents periodic behavior of a system. (Example: circles)

Definition 3.5 (Limit cycle)**.** *A closed trajectory which is stable and isolated (no others nearby).*

An closed trajectory is stable when nearby curves asymptote towards the closed trajectory. An unstable closed trajectory is the opposite.

Existence: not much is known except Poincaie-Bendixson Theorem. Today mostly computers are used.

For not non-existence, Bendixson's Theorem and Crit points are used.

Theorem 3.5 (Bendixson's Theorem)**.** *Given D a region of the plane, if $\operatorname{div}\vec{F} \neq 0$ in D, then there are no closed trajectories in D.*

Proof. Suppose a closed trajectory c exists in D which defines a region R.

$$\oint_c \vec{F}\cdot\hat{n}\ ds = \iint_R \operatorname{div}\vec{F}\ dA \tag{3.6.1}$$

$\vec{F}\cdot\hat{n}$ since $\vec{F}$ must perpendicular to the curve and therefore orthogonal to $\hat{n}$. The hypothesis is that $\operatorname{div}\vec{F} > 0$ everywhere or $\operatorname{div}\vec{F} < 0$ everywhere. It can't be both positive and negative because if you connected the two points with a line, because $\vec{F}$ is continuous, it would have to cross 0. The LHS = 0 and because $\operatorname{div}\vec{F} \neq 0$, the RHS is never zero which is a contradiction. Therefore there cannot be a closed trajectory. □

For example, if a $\operatorname{div}\vec{F} = 0$ along a curve c if a closed trajectory exists, it must intersect c.

Critical Points

Given a region D with a closed trajectory c inside. There must be a critical point inside c.

Theorem 3.6. *If D has no critical points, it has no closed trajectories.*