Mathematics

Math Review Packet

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February 18, 2019

Contents

1 Trigonometry

1.1 Trig Identities and Formulas

Cofunction Identities

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) \tag{1.1.1}$$

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) \tag{1.1.2}$$

$$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right) \tag{1.1.3}$$

$$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right) \tag{1.1.4}$$

$$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right) \tag{1.1.5}$$

$$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right) \tag{1.1.6}$$

Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \tag{1.1.7}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \tag{1.1.8}$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \tag{1.1.9}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{1.1.10}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{1.1.11}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} \tag{1.1.12}$$

Quotient Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{1.1.13}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \tag{1.1.14}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.1.15}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{1.1.16}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{1.1.17}$$

Sum and Difference Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{1.1.18}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{1.1.19}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{1.1.20}$$

Odd/Even Formulas

$$\sin(-\theta) = -\sin(\theta) \tag{1.1.21}$$

$$\cos(-\theta) = \cos(\theta) \tag{1.1.22}$$

$$\tan(-\theta) = -\tan(\theta) \tag{1.1.23}$$

$$\csc(-\theta) = -\csc(\theta) \tag{1.1.24}$$

$$\sec(-\theta) = \sec(\theta) \tag{1.1.25}$$

$$\cot(-\theta) = -\cot(\theta) \tag{1.1.26}$$

Double Angle Formula

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{1.1.27}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{1.1.28}$$

$$= 2\cos^2\theta - 1 \tag{1.1.29}$$

$$= 1 - 2\sin^2\theta \tag{1.1.30}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{1.1.31}$$

Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}} \tag{1.1.32}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}} \tag{1.1.33}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta} \tag{1.1.34}$$

Squared Identities

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{1.1.35}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{1.1.36}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \tag{1.1.37}$$

Product to Sum Formulas

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \tag{1.1.38}$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)] \tag{1.1.39}$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \tag{1.1.40}$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \tag{1.1.41}$$

Sum to Product Identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{1.1.42}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.1.43}$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{1.1.44}$$

$$\cos A + \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.1.45}$$

1.2 Hyperbolic Trigonometry

Functions and Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{1.2.1}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{1.2.2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.2.3}$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} \tag{1.2.4}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} \tag{1.2.5}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{1.2.6}$$

Definitions of Inverse Functions

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{1.2.7}$$

$$\operatorname{arccosh} x = \ln\left(x \pm \sqrt{x^2 - 1}\right) \tag{1.2.8}$$

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \tag{1.2.9}$$

$$\operatorname{arccsch} x = \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right) \tag{1.2.10}$$

$$\operatorname{arcsech} x = \ln\left(\frac{1 \pm \sqrt{x^2 + 1}}{x}\right) \tag{1.2.11}$$

$$\operatorname{arccoth} x = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right) \tag{1.2.12}$$

Hyperbolic Identities

$$\sinh -x = -\sinh x \tag{1.2.13}$$

$$\cosh -x = \cosh x \tag{1.2.14}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.15}$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x \tag{1.2.16}$$

$$\sinh x + y = \sinh x \cosh y + \cosh x \sinh y \tag{1.2.17}$$

$$\cosh x + y = \cosh x \cosh y + \sinh x \sinh y \tag{1.2.18}$$

$$\sinh 2x = 2 \sinh x \cosh x \tag{1.2.19}$$

$$\cosh 2x = \sinh^2 x + \cosh^2 x \tag{1.2.20}$$

$$\sinh^2 x = \frac{1}{2}(-1 + \cosh 2x) \tag{1.2.21}$$

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x) \tag{1.2.22}$$

2 Matrices

2.1 Basic Operations

Adding and Subtracting Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \tag{2.1.1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix} \tag{2.1.2}$$

Multiplying Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \tag{2.1.3}$$

Properties of Matrix Operations

Associative Property of Addition:	$(A+B)+C = A+(B+C)$
Commutative Property of Addition:	$A+B = B+A$
Distributive Property:	$k(A \pm B) = kA \pm kB$
Associative Property of Matrix Multiplication:	$A(BC) = (AB)C$
Left Distribution Property:	$A(B+C) = AB+AC$
Right Distribution Property:	$(A+B)C = AC+BC$
Associative Property of Scalar Multiplication:	$k(AB) = (kA)B = A(kB)$

Determinants

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \color{blue}{a} & \color{red}{b} \\ \color{red}{c} & \color{blue}{d} \end{bmatrix} = \color{blue}{ad} - \color{red}{bc} \tag{2.1.4}$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \color{blue}{\ddots} & \color{blue}{\ddots} & \color{blue}{\ddots} \\ \color{blue}{\ddots} & \color{blue}{\ddots} & \color{blue}{\ddots} \\ \color{blue}{\ddots} & \color{blue}{\ddots} & \color{blue}{\ddots} \end{bmatrix} = \begin{bmatrix} \color{red}{\iddots} & \color{red}{\iddots} & \color{red}{\iddots} \\ \color{red}{\iddots} & \color{red}{\iddots} & \color{red}{\iddots} \\ \color{red}{\iddots} & \color{red}{\iddots} & \color{red}{\iddots} \end{bmatrix} = \color{blue}{(aei+bfg+cdh)} - \color{red}{(gec+dbi+ahf)} \tag{2.1.5}$$

2.2 Applications of Matrices

Area of Triangle with Vertices $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$

$$\text{Area} = \pm\frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \tag{2.2.1}$$

$\pm$ to get positive area

Cramer's Rule for 2x2 System

$$ax + by = e$$

$$cx + dy = f$$

$$\text{Coefficient Matrix: A}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solutions

$$x = \frac{\begin{bmatrix} e & b \\ f & d \end{bmatrix}}{\det A} \tag{2.2.2}$$

$$y = \frac{\begin{bmatrix} a & e \\ c & f \end{bmatrix}}{\det A} \tag{2.2.3}$$

Cramer's Rule for 3x3 System

$$ax + by + cz = j$$

$$dx + ey + fz = k$$

$$gx + hy + iz = l$$

$$\text{Coefficient Matrix: A} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Solutions

$$x = \frac{\begin{bmatrix} j & b & c \\ k & e & f \\ l & h & i \end{bmatrix}}{\det A} \tag{2.2.4}$$

$$y = \frac{\begin{bmatrix} a & j & c \\ d & k & f \\ g & l & i \end{bmatrix}}{\det A} \tag{2.2.5}$$

$$z = \frac{\begin{bmatrix} a & b & j \\ d & e & k \\ g & h & l \end{bmatrix}}{\det A} \tag{2.2.6}$$

2.3 Identity and Inverse Matrices

Identity Matrix: AI=A IA=A

$$\text{2x2 Identity} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{3x3 Identity} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrices: A and B are inverse matrices if AB=I and BA=I. A matrix only has an inverse is its dterminant $\neq 0$.

Inverse of a 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2.3.1}$$

Using Inverse Matrices to Solve Systems of Equations

$$ax + by = e$$

$$cx + dy = f$$

$$\text{Coefficient Matrix: } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Variable Matrix: } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Matrix of Constants: } B = \begin{bmatrix} e \\ f \end{bmatrix}$$

1. Setup Equation

$$AX = B \tag{2.3.2}$$

2. Find A^{-1}
3. Multiply Each Side by A^{-1}

$$A^{-1}AX = A^{-1}B \tag{2.3.3}$$

$$X = A^{-1}B \tag{2.3.4}$$

$$X = \begin{bmatrix} \frac{d}{\text{det A}} & \frac{-c}{\text{det A}} \\ \frac{-b}{\text{det A}} & \frac{a}{\text{det A}} \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{de-cf}{\text{detA}} \\ \frac{af-be}{\text{detA}} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.3.5}$$

$$x = \frac{de - cf}{\text{detA}} \tag{2.3.6}$$

$$y = \frac{af - be}{\text{detA}} \tag{2.3.7}$$

3 Derivatives

3.1 Basic Rules

Constant Term

$$\frac{d}{dx}[C] = 0 \tag{3.1.1}$$

Constant Multiple Rule

$$\frac{d}{dx}[Cf(x)] = C \cdot \frac{d}{dx}[f(x)] \tag{3.1.2}$$

Sum/Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] \tag{3.1.3}$$

Power Rule

$$\frac{d}{dx}[ax^n] = n \cdot ax^{n-1} \tag{3.1.4}$$

Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x) \tag{3.1.5}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{3.1.6}$$

Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \tag{3.1.7}$$

Exponentials

$$\frac{d}{dx}[a^x] = a^x \cdot \ln(a) \tag{3.1.8}$$

Logarithms

$$\frac{d}{dx}[\log_a x] = \frac{1}{x \cdot \ln(a)} \tag{3.1.9}$$

Generalized Power Rule

$$\frac{d}{dx}[f(x)^{g(x)}] = f(x)^{g(x)}\left[g'(x) \cdot \ln f(x) + \frac{g(x)f'(x)}{f(x)}\right] \tag{3.1.10}$$

Inverse Functions

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} \tag{3.1.11}$$

3.2 Trigonometric Functions

Normal Trig Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x) \tag{3.2.1}$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \tag{3.2.2}$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \tag{3.2.3}$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x) \tag{3.2.4}$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x) \tag{3.2.5}$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x) \tag{3.2.6}$$

Inverse Trig Functions

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \tag{3.2.7}$$

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}} \tag{3.2.8}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \tag{3.2.9}$$

$$\frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{|\,x\,|\sqrt{x^2-1}} \tag{3.2.10}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|\,x\,|\sqrt{x^2-1}} \tag{3.2.11}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2} \tag{3.2.12}$$

3.3 Hyperbolic Functions

Normal Hyperbolic Functions

$$\frac{d}{dx}[\sinh x] = \cosh x \tag{3.3.1}$$

$$\frac{d}{dx}[\cosh x] = \sinh x \tag{3.3.2}$$

$$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x \tag{3.3.3}$$

$$\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x \tag{3.3.4}$$

$$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x \tag{3.3.5}$$

$$\frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x \tag{3.3.6}$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}[\operatorname{arcsinh} x] = \frac{1}{\sqrt{1+x^2}} \tag{3.3.7}$$

$$\frac{d}{dx}[\operatorname{arccosh} x] = \frac{1}{\sqrt{1-x^2}} \tag{3.3.8}$$

$$\frac{d}{dx}[\operatorname{arctanh} x] = \frac{1}{1-x^2} \tag{3.3.9}$$

$$\frac{d}{dx}[\operatorname{arccsch} x] = \frac{-1}{x\sqrt{1-x^2}} \tag{3.3.10}$$

$$\frac{d}{dx}[\operatorname{arcsech} x] = \frac{-1}{x\sqrt{1-x^2}} \tag{3.3.11}$$

$$\frac{d}{dx}[\operatorname{arccoth} x] = \frac{1}{1-x^2} \tag{3.3.12}$$

4 Integrals

4.1 Techniques of Integration

U-Substitution

$$\int f(g(x))g'(x)dx = F(g(x)) + C \tag{4.1.1}$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du \tag{4.1.2}$$

Trig. Integrals

$$\int \sin^m x\ \cos^n x\,dx \tag{4.1.3}$$

Case 1: If power of sin is odd, keep one factor of sin and use the Pythagorean Identity to change $\sin^2 x \to 1 - \cos^2 x$. If power of cos is odd, keep one factor of cos and use the Pythagorean Identity to change $\cos^2 x \to 1 - \sin^2 x$. If both are odd, pick one that will get you to a power of 2 (if possible).

Case 2: If both powers are even, use half angle formulas: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

$$\int \tan^m x\ \sec^n x\,dx \tag{4.1.4}$$

Case 1: If power of tan is odd, keep one factor of $\sec x \tan x$ and use the Pythagorean Identity to change $\tan^2 x \to \sec^2 x - 1$.

Case 2: If power of sec is even, keep one factor of $\sec^2$ and use $\sec^2 x = \tan^2 x + 1$.

$$\int \cot^m x\ \csc^n x\,dx \tag{4.1.5}$$

Case 1: If power of cot is odd, keep one factor of $\csc x \cot x$ and use the Pythagorean Identity to change $\cot^2 x \to \csc^2 x - 1$.

Case 2: If power of csc is even, keep one factor of $\csc^2$ and use $\csc^2 x = \cot^2 x + 1$.

Trig. Substitutions

$$\begin{array}{llll}
\sqrt{a^2 - x^2} & x = a\sin\theta & dx = a\cos\theta d\theta & \sqrt{a^2 - x^2} = a\cos\theta \\
\sqrt{a^2 + x^2} & x = a\tan\theta & dx = a\sec^2\theta d\theta & \sqrt{a^2 + x^2} = a\sec\theta \\
\sqrt{x^2 - a^2} & x = a\sec\theta & dx = a\sec\theta\tan\theta d\theta & \sqrt{x^2 - a^2} = a\tan\theta
\end{array}$$

Integrating Inverse Functions

$$\int f^{-1}(x)dx = xf^{-1}(x) - (F \circ f^{-1})(x) + C \tag{4.1.6}$$

Partial Fractions

5 Series

5.1 Types of Series

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ or } \sum_{n=0}^{\infty} ar^{n} \tag{5.1.1}$$

Converges when $-1 < r < 1$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \tag{5.1.2}$$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots \tag{5.1.3}$$

Diverges

P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \ldots \tag{5.1.4}$$

Converges when $p > 1$
Diverges when $p \leq 1$

Telescoping Series

$$\sum_{n=1}^{\infty} (a_n - a_{n-1}) = a_\infty - a_0 \tag{5.1.5}$$

5.2 Testing for Convergence

Limit Test

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{then} \sum a_n \text{ diverges} \tag{5.2.1}$$

Type of Series

- Geometric: converges if $|\, r \,| < 1$
- Telescoping: find lim of $a_\infty - a_0$
- P-Series: converges when $p > 1$

Integral Test

$$\text{If} \int_1^\infty a(n)\, dn \text{ converges, then} \sum a_n \text{converges} \tag{5.2.2}$$

$$\text{If} \int_1^\infty a(n)\, dn \text{ diverges, then} \sum a_n \text{diverges} \tag{5.2.3}$$

Comparison Test

If all terms of a_n are positive and it acts like another series b_n.

$$\text{If } a_n \leq b_n \text{ and} \sum b_n \text{ converges, then} \sum a_n \text{converges} \tag{5.2.4}$$

$$\text{If } a_n \geq b_n \text{ and} \sum b_n \text{ diverges, then} \sum a_n \text{diverges} \tag{5.2.5}$$

$$\text{If } \lim_{n\to\infty} \frac{a_n}{b_n} \text{ exists, then both series converge or diverge} \tag{5.2.6}$$

Absolute Convergence

For $\sum(-1)^n a_n$ or $\sum(-1)^{n-1} a_n$

$$\text{If } \sum |a_n| \text{ converges, then} \sum a_n \text{is absolutely convergent.} \tag{5.2.7}$$

$$\text{If} \lim_{n\to\infty} a_n = 0, \text{ then} \sum (-1)^n a_n \text{ converges} \tag{5.2.8}$$

$$\text{If } a_{n+1} \leq a_n \text{ or } f'(x) < 0 \text{ then} \sum (-1)^n a_n \text{converges} \tag{5.2.9}$$

Ratio Test

use for factorials or "n"th powers

$$\text{If} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1, \text{then} \sum a_n \text{ is absolutely convergent.} \tag{5.2.10}$$

$$\text{If} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1, \text{then} \sum a_n \text{ is divergent.} \tag{5.2.11}$$

$$\text{If} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1, \text{then} \sum a_n \text{ is inconclusive.} \tag{5.2.12}$$

Root Test

use for "n"th powers

$$\text{If} \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1, \text{then} \sum a_n \text{ is absolutely convergent.} \tag{5.2.13}$$

$$\text{If} \lim_{n\to\infty} \sqrt[n]{|a_n|} > 1, \text{then} \sum a_n \text{ is divergent.} \tag{5.2.14}$$

$$\text{If} \lim_{n\to\infty} \sqrt[n]{|\ a_n\ |} = 1, \text{then} \sum a_n \text{ is inconclusive.} \tag{5.2.15}$$

5.3 Power Series

Power Series- a series with some variable "x" raised to some power

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 \tag{5.3.1}$$

P-series centered at c
Creates a function of x

Convergence of a Power Series

- at x=c, Domain: [c,c]
- for all x, Domain: $(-\infty, \infty)$
- for some $\mid x-c \mid < R$, Domain (c-R, c+R)

need to check endpoints

5.4 Taylor Series

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n \tag{5.4.1}$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 \tag{5.4.2}$$

Maclaurin Series- Taylor Series at c=0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n \tag{5.4.3}$$

5.5 List of Common Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \qquad (1,1) \tag{5.5.1}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} n x^{n-1} \qquad (1,1) \tag{5.5.2}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \qquad (1,1) \tag{5.5.3}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad (1,1) \tag{5.5.4}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad (1,1) \tag{5.5.5}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty, \infty) \tag{5.5.6}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1!} \qquad (-\infty, \infty) \tag{5.5.7}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} \qquad (-\infty, \infty) \tag{5.5.8}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1!} x^{2n+1} \qquad (-\infty, \infty) \tag{5.5.9}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n} \qquad (-\infty, \infty) \tag{5.5.10}$$

$$W(x) = x - x^2 + \frac{3x^3}{2} - \frac{8x^4}{3} \ldots = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \qquad \left(-\frac{1}{e}, \frac{1}{e}\right) \tag{5.5.11}$$

6 Vectors

6.1 Basic Properties

A vector $\vec{v}$ with an initial point at the origin and an terminal point P (v_1, v_2) is called an position vector and is denoted $\langle v_1, v_2 \rangle$.

Magnitude

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots} \tag{6.1.1}$$

Basic Properties

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, \dots \rangle \tag{6.1.2}$$

$$C \cdot \vec{v} = \langle C \cdot v_1, C \cdot v_{2,\dots} \rangle \tag{6.1.3}$$

Unit Vector

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} \tag{6.1.4}$$

Standard Basis Vectors

$$\hat{i} = \langle 1, 0, 0 \rangle \tag{6.1.5}$$

$$\hat{j} = \langle 0, 1, 0 \rangle \tag{6.1.6}$$

$$\hat{k} = \langle 0, 0, 1 \rangle \tag{6.1.7}$$

6.2 Dot Product

Adds the products of corresponding components of two vectors. Gives a scalar.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots = C \tag{6.2.1}$$

Properties

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \tag{6.2.2}$$

$$\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w} \tag{6.2.3}$$

$$(C\vec{v}) \cdot \vec{w} = C(\vec{w} \cdot \vec{v}) = \vec{v} \cdot (C\vec{w}) \tag{6.2.4}$$

$$\vec{0} \cdot \vec{v} = 0 \tag{6.2.5}$$

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \tag{6.2.6}$$

Another Definition (θ is the angle between the vectors)

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta \tag{6.2.7}$$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \tag{6.2.8}$$

$$\theta = \arccos \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \tag{6.2.9}$$

If $\theta = 0$ or π, the vectors are parallel.
If $\theta = \frac{\pi}{2}$, the vectors are orthogonal.

6.3 Cross Product

$$\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \hat{i} - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \hat{j} + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \hat{k} \tag{6.3.1}$$

Properties

1. $\vec{a} \times \vec{b}$ is a vector.
2. $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
3. $\vec{b} \times \vec{a}$ is orthogonal to both $\vec{a}$ and $\vec{b}$ but other direction.
4. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Another Definition

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta \tag{6.3.2}$$

Area of a Triangle with Sides $\vec{u}$ and $\vec{v}$.

$$A_t = \frac{1}{2} \|\vec{u} \times \vec{v}\| \tag{6.3.3}$$

Area of a Parallelogram with Sides $\vec{u}$ and $\vec{v}$.

$$A_p = \|\vec{u} \times \vec{v}\| \tag{6.3.4}$$

Volume of a Parallelepypite with Sides $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$V_p = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| \tag{6.3.5}$$

7 Vector Functions

7.1 Basics and Properties

Sketches a line through space with vectors starting at the origin.

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \tag{7.1.1}$$

Properties

$$\lim_{t\to a}\vec{r}(t) = \lim_{t\to a} f(t)\hat{i} + \lim_{t\to a} g(t)\hat{j} + \lim_{t\to a} h(t)\hat{k} \tag{7.1.2}$$

$$\frac{d}{dt}\vec{r}(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k} \tag{7.1.3}$$

$$\int \vec{r}(t) = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k} + C \tag{7.1.4}$$

Projectile Motion

$$\vec{r}(t) = (\vec{v}_0 \cos\alpha)t\hat{i} + [h + (\vec{v}_0 \sin\alpha)t - \frac{1}{2}gt^2]\hat{j} \tag{7.1.5}$$

$$\vec{v}_0 = \text{initial velocity}$$

$$\alpha = \text{angle of inclination}$$

$$h = \text{height above plane}$$

$$g = \text{acceleration of gravity}$$

7.2 Arc Length

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \tag{7.2.1}$$

$$L = \int_a^b \|\vec{r}'(t)\|dt \tag{7.2.2}$$

Reparameterizing by Arc Length

$$\text{Find} \quad S(t) = \int \|\vec{r}'(t)\|dt \tag{7.2.3}$$

$$\text{Solve for t in terms of S} \quad t = u(S) \tag{7.2.4}$$

$$\text{Substitute back into vector function}$$

$$\vec{r}(t) = f(u(S))\hat{i} + g(u(S))\hat{j} + h(t)u(\hat{S}) \tag{7.2.5}$$

7.3 TNB Frames

Frenet-Serret Frames

$\vec{T}$angent: Direction particle is heading.
$\vec{N}$ormal: Direction particle is turning.
$\vec{B}$inormal: Direction particle is twisting.

How to find TNB

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \tag{7.3.1}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \tag{7.3.2}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \tag{7.3.3}$$

7.4 Curvature, Torsion, and Osculating Circles

Curvature: "failure to be a line"

$$\kappa = \frac{\text{change in tangent vector}}{\text{arc length}} = \|\vec{T}'(S)\| \tag{7.4.1}$$

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \tag{7.4.2}$$

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \quad \text{"only for polynomials"} \tag{7.4.3}$$

Torsion: "failure to be contained in a plane"

$$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2} \tag{7.4.4}$$

$$\tau = \frac{-d\vec{B}}{ds} \cdot \vec{N} \tag{7.4.5}$$

At one point on a curve, there will be a circle that fits the curve "best" called a osculating circle with a radius of curvature

$$\rho = \frac{1}{\kappa} \tag{7.4.6}$$

The plane that contains $\vec{N}$ and $\vec{B}$ at the point is called the normal plane and contains all vectors orthogonal to $\vec{T}$.

8 Multivariable Functions

8.1 Lines and Planes

Lines need a point $(x_0,\ y_0,\ z_0)$ and direction vector $\langle a,\ b,\ c\rangle$.

Parametric Equation for a Line

$$x = at + x_0, \quad y = bt + y_0, \quad z = ct + z_0 \tag{8.1.1}$$

Symmetric Equation of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{8.1.2}$$

Planes need a point $P_0(x_0,\ y_0,\ z_0)$ and normal vector $\vec{n} = \langle a,\ b,\ c\rangle$.

Standard Form of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{8.1.3}$$

General Form

$$ax + by + cz = d \tag{8.1.4}$$

Angle Between Line and Plane

$$90° - \arccos \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\| \|\vec{v}\|} \tag{8.1.5}$$

8.2 Distances

Point $P_T(x_T,\ y_T,\ z_T)$ and a Plane $P_0(x_0,\ y_0,\ z_0)$ $\ \vec{n} = \langle a,\ b,\ c\rangle$.

$$D = \left| \frac{ax_T + by_T + cz_T + d}{\sqrt{a^2 + b^2 + c^2}} \right| \tag{8.2.1}$$

Parallel Planes

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} \tag{8.2.2}$$

Point P and Line Q and $\vec{v}$

$$D = \frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|} \tag{8.2.3}$$

Between Skewed Lines

Find direction vectors $\vec{v}_1$ and $\vec{v}_2$. Find common normal $\vec{n} = \vec{v}_1 \times \vec{v}_2$. Pick a point on each line P_1 and P_2. Take P_1 and $\vec{n}$ and P_2 and $\vec{n}$ to make equations of planes and solve for the distance between parallel planes.

8.3 Cylinder and Surfaces

Cylinder

1. Equations only have two variables.
2. Directed along axis of missing variable.
3. Does not change along direction axis.

Surfaces

1. Has three variables.
2. Traces occur on coordinate planes or parallel to.
3. Shape changes along the axis.

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{8.3.1}$$

One-Sheet Hyperbolas

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8.3.2}$$

Two Sheet Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8.3.3}$$

Cones

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{8.3.4}$$

Paraboloids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz \tag{8.3.5}$$

Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz \tag{8.3.6}$$

8.4 Limits of Multivariable Functions

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{8.4.1}$$

Problem: infinite ways to approach (a,b).
To prove that L doesn't exist, its sufficient to show different values of L from different lines through (a,b).

9 Partial Derivatives

9.1 Basics

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x \tag{9.1.1}$$

Holds any other variables constant.
Slope of tangent line in x-direction.

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = z_y \tag{9.1.2}$$

Holds any other variables constant.
Slope of tangent line in y-direction.

Higher Derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{9.1.3}$$

h(x,y) is called a Harmonic Function if it satisfies the Laplace Equation

$$h_{xx} + h_{yy} = 0 \tag{9.1.4}$$

Differential

$$dz = f_x dx + f_y dy \tag{9.1.5}$$

Directional Derivative $\hat{u} = \langle u_1,\ u_2 \rangle$

$$D_{\hat{u}} f(x,y) = f_x u_1 + f_y u_2 \tag{9.1.6}$$

9.2 Multivariable Chain Rule

Suppose that "f" is a function of "x" and "y" where "x" and "y" are functions of some variable "t."

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{9.2.1}$$

"w" is a function of "x," "y," and "z." "x," "y," and "z" are based of variables "u" and "v."

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \tag{9.2.2}$$

9.3 Gradient

$$\nabla f(x,y) = f_x \hat{i} + f_y \hat{j} \tag{9.3.1}$$

$$D_{\hat{u}} f(x,y) = \nabla f(x,y) \cdot \hat{u} \tag{9.3.2}$$

Properties of the $\bigtriangledown$

1. If $\bigtriangledown f = \vec{0}$ then $D_{\hat{u}} f = 0$ of any $\hat{u}$.
2. $D_{\hat{u}} f(x, y)$ has its max value of $\| \bigtriangledown f(x, y)\|$ only when $\hat{u} = C \cdot \bigtriangledown f$.
3. $D_{\hat{u}} f(x, y)$ has its minimum value of $-\| \bigtriangledown f(x, y)\|$ at $\theta = \pi$ from max value.

9.4 Local Extrema of Functions

Second Derivative Test

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 \tag{9.4.1}$$

1. $D(a, b) > 0 \quad f_{xx}(a, b) > 0$ Concave up; relative min
2. $D(a, b) > 0 \quad f_{xx}(a, b) < 0$ Concave down; relative max
3. $D(a, b) < 0$ saddle/inflection point
4. $D(a, b) = 0$ no conclusions can be drawn

9.5 LaGrange Multipliers

Idea: to find the local min/max of a surface $f(x, y) = z$ along a constraint $g(x, y) = c$.

$$\bigtriangledown f(x, y) = \lambda \bigtriangledown g(x, y) \tag{9.5.1}$$

Plug λ back into equations used to find lambda to a coordinate.
Plug back into f(x,y) and do second derivative test to find out if its a max or min.

10 Multiple Integrals

10.1 Double Integrals

Definition:

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\ \Delta x \Delta y = \iint_R f(x,y)\ dxdy \tag{10.1.1}$$

Fubini's Theorem for General Regions

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\ dxdy \tag{10.1.2}$$

Double Integrals in Polar Coordinates

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) \cdot r\ drd\theta \tag{10.1.3}$$

10.2 Triple Integrals

$$\iiint_T f(x,y,z)\ dV \tag{10.2.1}$$

1. For $\iiint_T$ define T between two surfaces. This takes care of the first $\int$ and leaves $\iint_R$ just like before. But, R can be on XY, YZ, or XZ planes.
   <u>Two Tasks and Two Regions</u>
   (a) A area in $\mathbb{R}-3$
   (b) Then a very specific $\mathbb{R}-2$ region on the coordinate plane.
2. $\iiint_T f(x,y,z)dV$ can be evaluated in any order.
3. x/y/z Simple
   - <u>z-simple</u>: Define $\mathbb{R}-3$ region between two $z = f(x,y)$ functions, R will be in the <u>xy-plane</u>.
   - <u>y-simple</u>: Define $\mathbb{R}-3$ region between two $z = f(x,z)$ functions, R will be in the <u>xz-plane</u>.
   - <u>x-simple</u>: Define $\mathbb{R}-3$ region between two $z = f(y,z)$ functions, R will be in the <u>yz-plane</u>.

10.3 C.O.M and Moments of Inertia

If an object has a density of $\rho(x, y, z)$ at any point (x, y, z) then the mass of the plate is:

$$m = \iiint_T \rho(x, y, z)\ dV \tag{10.3.1}$$

First Moments of Mass:

$$M_{yz} = \iiint_T x\rho(x, y, z)\ dV, \qquad \bar{x} = \frac{M_{yz}}{m} \tag{10.3.2}$$

$$M_{xz} = \iiint_T y\rho(x, y, z)\ dV, \qquad \bar{y} = \frac{M_{xz}}{m} \tag{10.3.3}$$

$$M_{xy} = \iiint_T z\rho(x, y, z)\ dV, \qquad \bar{z} = \frac{M_{xy}}{m} \tag{10.3.4}$$

$$\text{C.O.M}\ \ (\bar{x},\ \bar{y},\ \bar{z}) \tag{10.3.5}$$

Second Moments of Inertia:

$$I_x = \iiint_T (y^2 + z^2)\rho(x, y, z)\ dV \tag{10.3.6}$$

$$I_y = \iiint_T (x^2 + z^2)\rho(x, y, z)\ dV \tag{10.3.7}$$

$$I_z = \iiint_T (x^2 + y^2)\rho(x, y, z)\ dV \tag{10.3.8}$$

Radius of Gyration:

$$k_x = \sqrt{\frac{I_x}{m}} \tag{10.3.9}$$

$$k_y = \sqrt{\frac{I_y}{m}} \tag{10.3.10}$$

$$k_z = \sqrt{\frac{I_z}{m}} \tag{10.3.11}$$

10.4 Cylindrical and Spherical Coordinates

Cylindrical: $x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad z = z$

$$\iiint\limits_T f(x,y,z)\ dV = \int\limits_{\theta_1}^{\theta_2}\int\limits_{r_1}^{r_2}\int\limits_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\ dzrdrd\theta \tag{10.4.1}$$

Spherical:

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

$$r = \rho\sin\phi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint\limits_T f(x,y,z)\ dV = \int\limits_{\theta_1}^{\theta_2}\int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho 1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho d\phi d\theta \tag{10.4.2}$$

10.5 The Jacobian

For

$$\iint\limits_R f(x,y)\ dA \Rightarrow \iint f(g(u,v), h(u,v)) \cdot J \cdot dudv \tag{10.5.1}$$

J the Jacobian:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{10.5.2}$$

Generalizes to

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} & \cdots \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} & \cdots \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \tag{10.5.3}$$

11 Vector Calculus

11.1 Vector Fields

Vector Field: an object which gives a vector for each coordinate point

$$F(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k} \tag{11.1.1}$$

For which P, Q, and R are defined functions.

Conservative Vector Field:

$F(x,y,z)$ is conservative V.F. if $F(x,y,z) = \triangledown f(x,y,z)$

For some $f(x,y,z)$ called a potential function.

11.2 Divergence and Curl

Divergence and curl are two characteristics of how flow is behaving on a vector field in a small neighborhood aroud a given point "P."

Divergence- a measurement of how much fluid enters the neighborhood around "P" compared to how much leaves.

- If more fluid enters region than leaves, then the divergence will be negative. Think of this as fluid gathering at a point.
- If the same amount of fluid/flow enters as leave, divergence is "0" at "P" called incompressible.
- If more fluid/flow leaves than enters, divergence is positive at "P." Think of fluid leaving the point called divergent.

For $F(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$, $\quad \triangledown = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\operatorname{div} F = \triangledown \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{11.2.1}$$

Curl- measure of rotation of the vector field in the neighborhood around "P"

- If curl is positive at a point, the fluid is spinning counter-clockwise.
- If curl is negative at a point, the fluid is spinning clockwise.
- If curl is 0, there is no rotation a "P," called irrotational.

$$\operatorname{curl} F = \triangledown \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \tag{11.2.2}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \tag{11.2.3}$$

11.3 Line Integrals

Integrating over any curve instead of just the x axes.

$$m = \int_c f(x, y)\ ds \tag{11.3.1}$$

To define "c" use parametric equations
"ds" means with respect to arc length, little length of the curve
$f(x, y)$ height above each point on "c"
$f(x, y) \cdot ds$ = Area

$$m = \int_c 2 \text{ var.} \to \int_c 1 \text{ var.}$$

$$f(x, y) = f(x(t), y(t))$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

$$c = \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

$$\vec{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath}$$

$$\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

$$ds = \|\vec{r}'(t)\|\ dt \tag{11.3.2}$$

$$\int_c f(x, y)\ ds = \int_{t=a}^{t=b} f(x(t), y(t))\ \|\vec{r}'(t)\|\ dt \tag{11.3.3}$$

Other possibilities

1. Instead of $\int ds$ you get

$$\int dx + dy + dz \tag{11.3.4}$$

   Substitute $\frac{dx}{dt}$ from c into the integral as $dx = x'(t)\ dt$.

2. If "c" isn't smooth split it into sections

$$m = \int_{c_1} + \int_{c_2} \tag{11.3.5}$$

Line Integral through a Vector Field:

For $F(x, y, z) = F(x(t), y(t), z(t))$ & $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k},\ a \le t \le b$

$$\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\ dt \tag{11.3.6}$$

$$w = \int_c F \cdot dr \tag{11.3.7}$$

11.4 Line Integral on a Conservative Vector Field

"F" is conservative if F = ∇f for some function.

Conservative Vector Field: no matter how you set from point a to point b, it is the same amount of work. The line integral is independent of path. We don't need to define a curve.

Fundamental Theorem of Line Integrals

$$\int_c F \cdot dr = \int_c \nabla f \cdot dr = f(b) - f(a) \tag{11.4.1}$$

#1: Show "F" is conservative

$F(x,y) = P\hat{\imath} + Q\hat{\jmath}$
If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then F is conservative.

$(x,y,z)P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$
Prove $\operatorname{curl} F = 0$

#2: $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$

Integrate a term
Differentiate with respect to another variable
Solve for a the extra function.

11.5 Green's Theorem

What if $\int Pdx + Qdy$ has a curve "C" that encloses a region on a plane and "C" is a simple, closed, curve that is traveled in the counter-clockwise, then the $\int$ becomes $\oint$

Green's Theorem:

$$\oint_c Pdx + Qdy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \tag{11.5.1}$$

Area Formula:

$$A = \frac{1}{2}\oint_c -ydx + xdy \tag{11.5.2}$$

11.6 Surface Integrals

$$m = \iint_S f(x,y,z)\ dS \tag{11.6.1}$$

Defining one variable as the surface:

$$\iint_S f(x,y,z)\ dS = \iint_R f(x,y,g(x,y))\ \sqrt{g_x^2 + g_y^2 + 1}\ dA \tag{11.6.2}$$

$$\iint_S f(x,y,z)\ dS = \iint_R f(x,g(x,z),z)\ \sqrt{g_x^2 + g_z^2 + 1}\ dA \tag{11.6.3}$$

$$\iint_S f(x,y,z)\ dS = \iint_R f(g(y,z),y,z)\ \sqrt{g_y^2 + g_z^2 + 1}\ dA \tag{11.6.4}$$

<u>Parametric Surfaces</u>:

For f(x,y,z) &

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$
$$f(x,y,z) = \vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k} \tag{11.6.5}$$

<u>Surface Area</u>:

$$\iint_D \|\vec{r}_u \times \vec{r}_v\|\ dA \tag{11.6.6}$$

"D" is the domain of the parameters of "u" and "v"

<u>Parametric Surface Integral</u>:

$$\iint_D f(\vec{r}(u,v))\ \|\vec{r}_u \times \vec{r}_v\|\ dA \tag{11.6.7}$$

$$\iint_S P\ dydx + Q\ dzdy + R\ dxdy = \iint_D \left(P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)} \right)\ du\ dv \tag{11.6.8}$$

<u>Flux</u>: If "F," a vector field contains a surface "S," then F describes the velocity of the flow/field at any point across a surface. The rate of flow across the surface is called flux.

$$F \cdot \vec{n}\ A(S) \tag{11.6.9}$$

For a small area of S. Across an entire surface it becomes:

$$\iint_S F \cdot d\mathbf{S} = \iint_S F \cdot \vec{n} dS \tag{11.6.10}$$

$$d\mathbf{S} = \vec{n} dS \tag{11.6.11}$$

For $z = g(x, y)$ and a vector field $F = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\iint\limits_S F \cdot dS = \iint\limits_D \left(-Pg_x - Qg_y + R\right) dA \tag{11.6.12}$$

$$\iint\limits_S F \cdot dS = \iint\limits_D F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v \; dA \tag{11.6.13}$$

<u>Divergence Theorem</u>: For when "S" is a simple closed surface

$$\text{Flux} = \iint\limits_S F \cdot dS = \iiint\limits_T \operatorname{div} F \; dV \tag{11.6.14}$$

11.7 Stoke's Theorem

If a curve "c" is not contained in a plane. (Green's Theorem except the region becomes a surface in 3-D)

$$\oint\limits_c F \cdot dr = \iint\limits_S \operatorname{curl} F \cdot \vec{n} \; dS \tag{11.7.1}$$

When evaluating, thinks of $\operatorname{curl} F$ as a vector field and use surface integrals.

$$\iint\limits_S \operatorname{curl} F \cdot \vec{n} \; dS = \iiint\limits_T \operatorname{div}\left(\operatorname{curl} F\right) \; dV \tag{11.7.2}$$