The Circular-Restricted Three-Body Problem

This document describes several MATLAB scripts that can be used to analyze and display orbits in the circular-restricted three-body problem (CRTBP). This is a special case of the general three-body problem where the primary and secondary bodies move in circular orbits about the common center of mass, and the effect of the gravitational attraction of the smallest body and any other perturbations such as solar radiation pressure are ignored.

Several of the applications described here are for orbital problems in the Earth-Moon-satellite system. However, the equations and algorithms are valid for any circular-restricted three-body system for which the mass ratio of the primary and secondary is between 0 and 1/2.

The classic textbook for this problem is *Theory of Orbits* by Victor Szebehely, Academic Press, New York, 1967. An excellent discussion about this problem can also be found in Chapter 8 of *An Introduction to Celestial Mechanics* by Forest Ray Moulton, Dover Publications, 1970.

Technical Discussion

The motion of the spacecraft or celestial body is modeled in a coordinate system which is rotating about the center-of-mass or barycenter of the Earth-Moon system. The motion of all three bodies is confined to the *x-y* plane. In the discussion that follows, subscript 1 is the Earth and subscript 2 is the Moon.

For convenience, the problem is formulated in *canonical* or non-dimensional units. The unit of length is taken to be the constant distance between the Earth and Moon, and the unit of time is chosen such that the Earth and Moon have an angular velocity ω about the barycenter equal to 1.

Kepler's third law for this formulation is given by

$$\omega^2 \left| m_1 m_2 \right|^3 = g\left(m_1 + m_2\right) = 1$$

The x and y coordinates of the Earth and Moon in this coordinate system are given by

$$x_1 = -\mu \qquad y_1 = 0$$

$$x_2 = 1 - \mu \qquad y_2 = 0$$

where $\mu =$ Earth-Moon mass ratio $= m_1 / m_2 \approx 1/81.27$.

The position and velocity vectors of the small body in this system are sometimes called "synodical" coordinates to distinguish them from "sidereal" coordinates defined in an inertial, non-rotating coordinate system.

The system of second-order *non-dimensional* differential equations of motion of a point-mass satellite or celestial body in this rotating coordinate system are given by

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - (1-\mu)\frac{x - x_1}{r_1^3} - \mu\frac{x - x_2}{r_2^3}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}$$

$$\frac{d^2z}{dt^2} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3}$$

where

$x = x$ component of position

$y = y$ component of position

$z = z$ component of position

$x_1 = -\mu$

$x_2 = 1 - \mu$

$\mu =$ Earth-Moon mass ratio $= m_1 / m_2 \approx 1/81.27$

$r_1^2 = (x - x_1)^2 + y^2 + z^2$

$r_2^2 = (x - x_2)^2 + y^2 + z^2$

The *Jacobi integral* for the three-body problem is given by

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C$$

In this equation V is the scalar speed of the small body and C is a constant of integration called the *Jacobi constant*.

crtbp1.m – coordinates and energy of the libration points

In his prize memoir of 1772, Joseph-Louis Lagrange proved that there are five *equilibrium points* in the circular-restricted three-body problem. If a satellite or celestial body is located at one of these points with zero initial velocity relative to the rotating coordinate system, it will stay there permanently provided there are no perturbations. These special positions are also called *libration* points or stationary Lagrange points. Three of these libration points are on the x-axis (collinear points) and the other two form equilateral triangles with the positions of the large primary mass and the small secondary mass.

This MATLAB script can be used to calculate the coordinates and energy of the five libration or equilibrium points of the CRTBP. For the collinear libration points, this script solves three nonlinear equations for the corresponding x coordinate using Brent's method.

The nonlinear equation for the L_1 libration point is given by

$$x - \frac{1-\mu}{(\mu + x)^2} + \frac{\mu}{(x - 1 + \mu)^2} = 0$$

The nonlinear equation for the L_2 libration point is

$$x - \frac{1-\mu}{(\mu + x)^2} - \frac{\mu}{(x - 1 + \mu)^2} = 0$$

and the nonlinear equation for the L_3 collinear libration point is

$$x + \frac{1-\mu}{(\mu + x)^2} + \frac{\mu}{(x - 1 + \mu)^2} = 0$$

The lower and upper bounds used during the root-finding search for all three equations are $x = -2$ and $x = +2$ respectively.

The L_4 libration point is located at $x = 1/2 - \mu,\ y = \sqrt{3}/2$ and the L_5 libration point is at $x = 1/2 - \mu,\ y = -\sqrt{3}/2$. For these two libration points, $r_1 = r_2 = 1$.

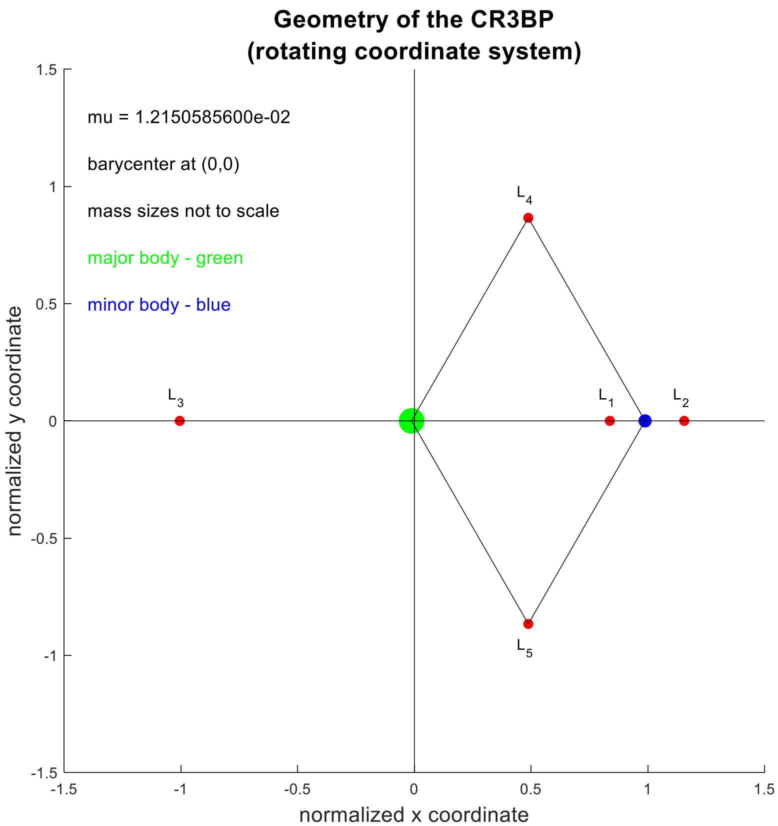
The following is a typical user interaction with this script.

```
              program crtbp1

 < equilibrium coordinates and energy >

 please input the value for the mass ratio
 ? 1.2150585600e-02
```

The following is the program output for this example.

```
                    program crtbp1

         < equilibrium coordinates and energy >


 mass ratio = 1.2150585600e-02

 location      x-coordinate       y-coordinate          energy

   L1          0.8369151258       0.0000000000      -1.5941705588e+00

   L2          1.1556821654       0.0000000000      -1.5860802304e+00

   L3         -1.0050626458       0.0000000000      -1.5060735753e+00

   L4          0.4878494144       0.8660254038      -1.4939985256e+00

   L5          0.4878494144      -0.8660254038      -1.4939985256e+00
```

This script will also create a graphics image of the configuration. The following illustrates the geometry of the CR3BP problem for this value of normalized gravity.



g3body.m – graphics display of three-body motion

This MATLAB script graphically displays motion of a spacecraft or small celestial body in the circular-restricted three-body problem. This algorithm and program examples are based on the methods described in "Periodic Orbits in the Restricted Three-Body Problem with Earth-Moon Masses", by R. A. Broucke, JPL TR 32-1168, 1968.

With the following substitutions, $y_1 = x,\ y_2 = \dot{x},\ y_3 = y$ and $y_4 = \dot{y},$ this script implements the following first-order equations of motion:

$$\dot{y}_1 = \frac{dy_1}{dt} = y_2 = \dot{x} \qquad \dot{y}_2 = \frac{dy_2}{dt} = 2y_4 + y_1 - (1-\mu)\frac{y_1+\mu}{r_1^3} - \mu\frac{y_1 - 1 + \mu}{r_2^3}$$

$$\dot{y}_3 = \frac{dy_3}{dt} = y_4 = \dot{y} \qquad \dot{y}_4 = \frac{dy_4}{dt} = -2y_2 + y_3 - (1-\mu)\frac{y_3}{r_1^3} - \mu\frac{y_3}{r_2^3}$$

where

$$r_1 = \sqrt{(y_1+\mu)^2 + y_3^2} \qquad r_2 = \sqrt{(y_1 - 1 + \mu)^2 + y_3^2}$$

This script begins by displaying the following main menu.

```
              program g3body

   < graphics display of three-body motion >


         main menu

    <1> periodic orbit about L1

    <2> periodic orbit about L2

    <3> periodic orbit about L3

    <4> user input of initial conditions

    selection (1, 2, 3 or 4)
```

The first three menu options will display typical *periodic* orbits about the respective libration point. The initial conditions for each of these example orbits are as follows:

(1) periodic orbit about the L_1 libration point

$$x_0 = 0.300000161 \qquad \dot{y}_0 = -2.536145497$$
$$\mu = 0.012155092 \qquad t_f = 5.349501906$$

(2) periodic orbit about the L_2 libration point

$$x_0 = 2.840829343 \qquad \dot{y}_0 = -2.747640074$$
$$\mu = 0.012155085 \qquad t_f = 11.933318588$$

(3) periodic orbit about the L_3 libration point

$$x_0 = -1/600000312 \qquad \dot{y}_0 = 2.066174572$$
$$\mu = 0.012155092 \qquad t_f = 6.303856312$$

Notice that each orbit is defined by an initial *x*-component of position, x_0, an initial *y*-component of velocity, $\dot{y}_0$, a value of Earth-Moon mass ratio μ, and an orbital period t_f. The initial *y*-component of position and *x*-component of velocity for each of these orbits is equal to zero. The software will draw the location of the Earth, Moon and libration point on the graphics screen.

If you would like to experiment with your own initial conditions, select option `<4>` from the main menu. The program will then prompt you for the initial *x* and *y* components of the orbit's position and velocity vector, and the values of μ to use in the simulation. The software will also ask for an initial and final time to run the simulation, and an integration step size. The value `0.01` is a good number for step size.

Here is a short list of initial conditions for several other periodic orbits that you may want to display with this script.

(1) Retrograde periodic orbit about m_1

$$x_0 = -2.499999883 \qquad \dot{y}_0 = 2.100046263$$

$$\mu = 0.012155092 \qquad t_f = 11.99941766$$

(2) Direct periodic orbit about m_1

$$x_0 = 0.952281734 \qquad \dot{y}_0 = -0.957747254$$

$$\mu = 0.012155092 \qquad t_f = 6.450768946$$

(3) Direct periodic orbit about m_1 and m_2

$$x_0 = 3.147603117 \qquad \dot{y}_0 = -3.07676285$$

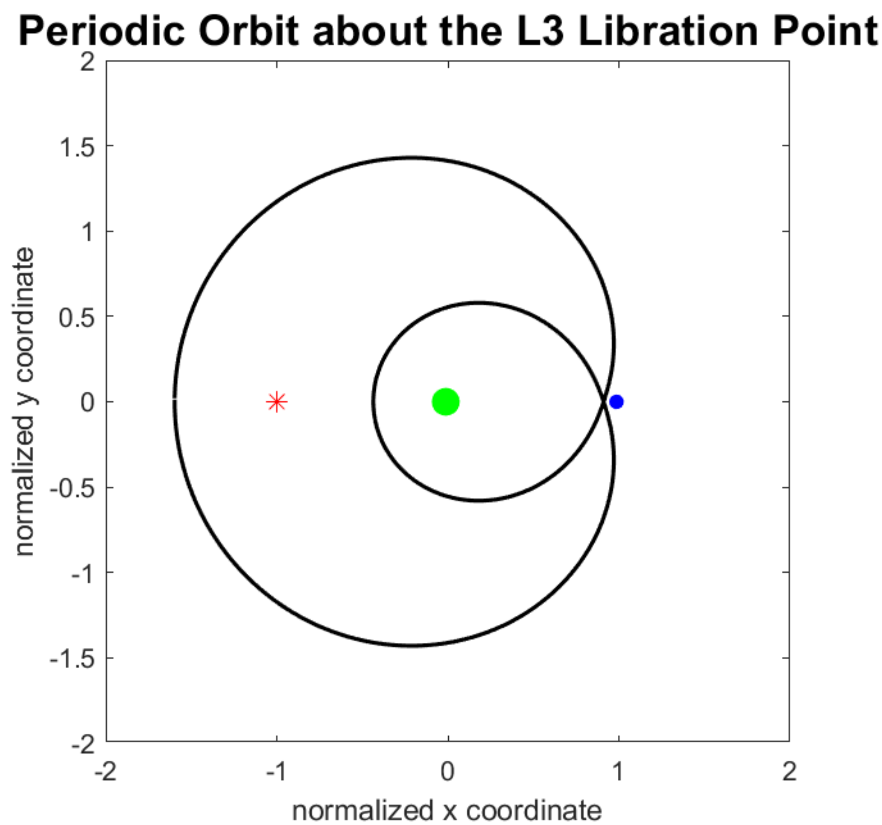
$$\mu = 0.012155092 \qquad t_f = 12.567475674$$

(4) Direct periodic orbit about m_2

$$x_0 = 1.399999991 \qquad \dot{y}_0 = -0.9298385561$$

$$\mu = 0.012155092 \qquad t_f = 13.775148738$$

The following is the graphic screen display for main menu option `<3>` and an integration step size of `0.01`. The locations of the Earth (green) and Moon (blue) are labeled with small, filled circles and the L_3 libration point is the small red asterisk.



Another excellent source of circular-restricted three-body problems is described in the report, "A Collection of Restricted Three-body Test Problems" by Philip W. Sharp, Report Series 460, Department of Mathematics, University of Auckland, March 2001. A copy of this report in PDF format can be found at the following web site:

https://www.math.auckland.ac.nz/~sharp/papers/preprints.html

The following is a summary of these test problems. Column 1 is the test problem number, column 2 is the initial *x* coordinate x_0, column 3 is the initial *y* speed $\dot{y}_0$ and column 4 is the orbital period t_f. For problems 1 to 15 $\mu = 0.012277471$ and for problems 16 to 20 $\mu = 0.000953875$.

1	0.994000E+00	-0.21138987966945026683E+01	0.54367954392601899690E+01
2	0.994000E+00	-0.20317326295573368357E+01	0.11124340337266085135E+02
3	0.994000E+00	-0.20015851063790825224E+01	0.17065216560157962559E+02
4	0.997000E+00	-0.16251217072210773125E+01	0.22929723423442969481E+02
5	0.879962E+00	-0.66647197988564140807E+00	0.63006757422352314657E+01
6	0.879962E+00	-0.43965281709207999128E+00	0.12729711861022426544E+02
7	0.879962E+00	-0.38089067106386964470E+00	0.19138746281183026809E+02

```
  8   0.997000E+00   -0.18445010489730401177E+01   0.12353901248612092736E+02

  9   0.100000E+01   -0.16018768253456252603E+01   0.12294387796695023304E+02

 10   0.100300E+01   -0.14465123738451062297E+01   0.12267904265603897140E+02

 11   0.120000E+01   -0.71407169828407848921E+00   0.18337451820715063383E+02

 12   0.120000E+01   -0.67985320356540547720E+00   0.30753758552146029263E+02

 13   0.120000E+01   -0.67153130632829144331E+00   0.43214375227857454128E+02

 14   0.120000E+01   -0.66998291305226832207E+00   0.55672334134347612727E+02

 15   0.120000E+01   -0.66975741517271092087E+00   0.68127906604713772763E+02

 16  -0.102745E+01    0.40334488290490413053E-01   0.18371316400018903965E+03

 17  -0.976680E+00   -0.61191623926410837000E-01   0.17733241131524483004E+03

 18  -0.766650E+00   -0.51230158665978820282E+00   0.17660722897242937108E+03

 19  -0.109137E+01    0.14301959822238380020E+00   0.82949461922342093092E+02

 20  -0.110137E+01    0.15354250908611454510E+00   0.60952121909407746612E+02
```

zvcurve1.m – graphics display of zero velocity curves through equilibrium points

This MATLAB script can be used to display zero relative velocity curves that pass through the five equilibrium or libration points of the Earth-Moon circular-restricted three-body problem (CRTBP). These two-dimensional drawings are also called *equipotential* curves. The zero velocity curves define regions of the *x-y* plane that a satellite or celestial body with a given energy level can or cannot attain.

If we set the velocity components to zero in the equation for the Jacobi integral, we find that the Jacobi constant can be determined from the following expression:

$$C = -2E_0 = (1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right) - \mu(1-\mu)$$

where E_0 is the zero relative velocity *specific orbital energy* which is given by.

$$E_0 = -\frac{1}{2}\left(x^2 + y^2\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

From the Jacobi integral we can see that imaginary velocities occur whenever the following expression is smaller than the Jacobi constant C.

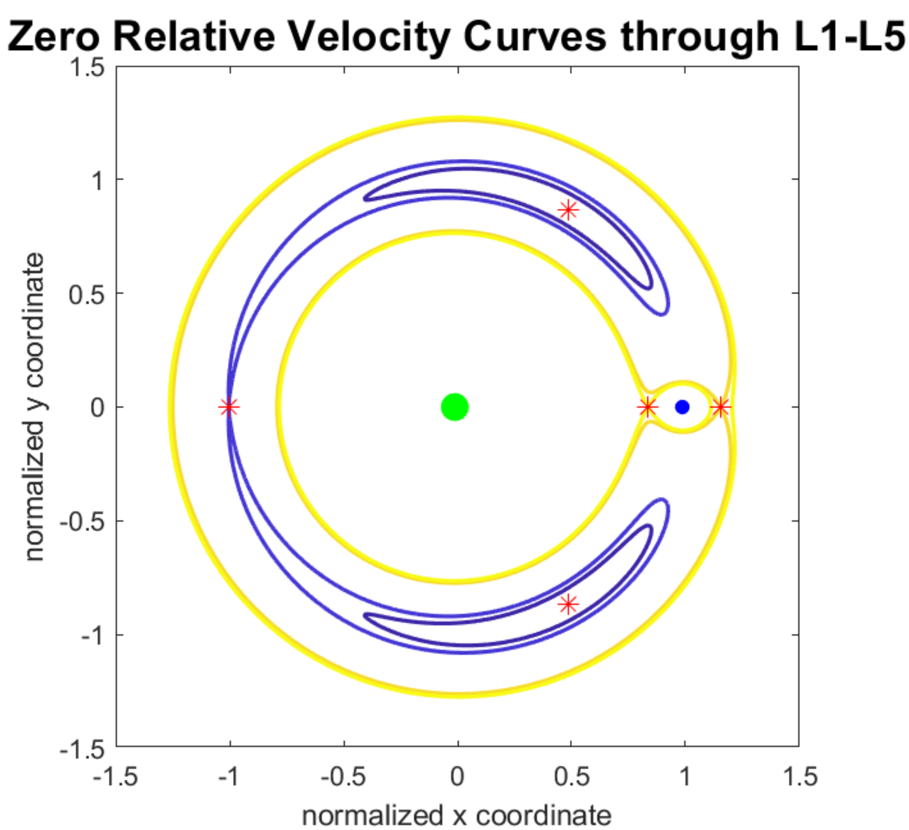
$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}$$

This MATLAB script computes and displays zero relative velocity contour plots that pass through the five libration points.

The following is a typical user interaction with this script.

```
                program zvcurve1

graphics display of zero relative velocity
curves through the crtbp equilibrium points

<1> L1 contour

<2> L2 contour

<3> L3 contour

<4> L4 and L5 contour

<5> all points contour

selection (1, 2, 3, 4 or 5)

?5
```

The following is a graphics display of the equipotential curves through each libration point. The libration points are labeled with red asterisks and the Earth (green) and Moon (blue) are shown as small, filled circles.



zvcurve2.m – graphics display of user-defined zero relative velocity curves

This MATLAB script can be used to display zero relative velocity contour curves of the circular-restricted three-body problem (CRTBP) for user-defined values of the Jacobi constant *C* and mass ratio μ as defined by the following expression:

$$C = (1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right) - \mu(1-\mu)$$

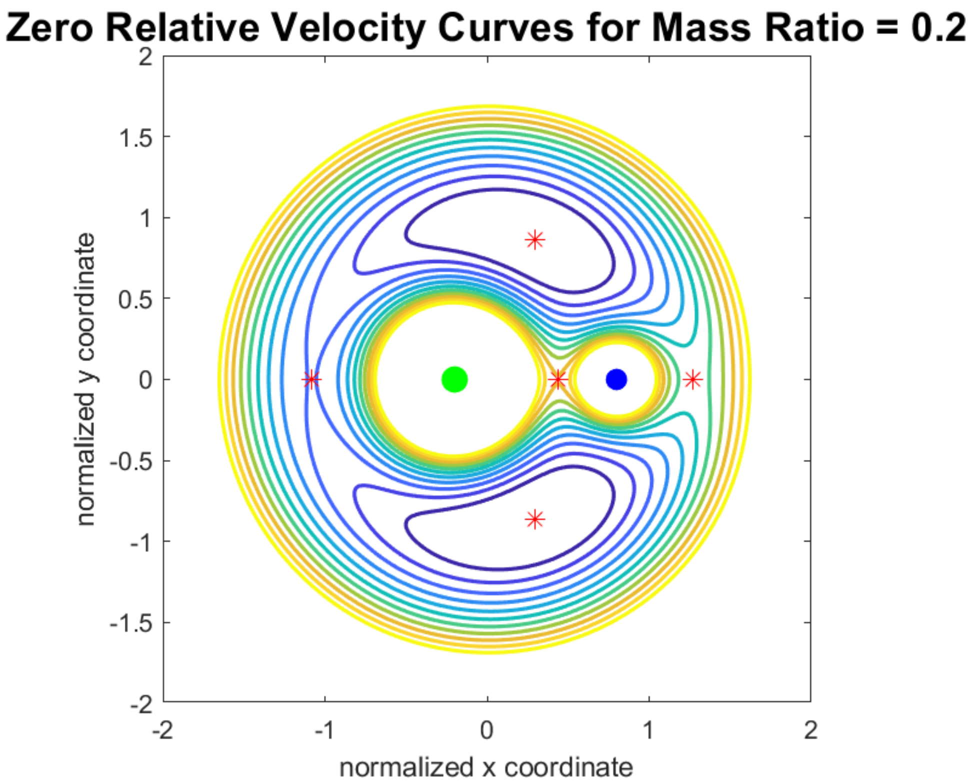
This application will prompt the user for the mass ratio and the contour levels to plot either as discrete values or as a range of values with a user-defined increment. The software will also ask you for the minimum and maximum values of the *x* and *y* coordinates and an increment for creating the mesh grid used by MATLAB to plot the contours. Please note that if the mesh grid increment is too large the script may not plot "complete" contour levels.

The following is a typical user interaction with this MATLAB script.

```
      program zvcurve2

graphics display of user-defined
 zero relative velocity curves

please input the value for the mass ratio
? 0.2

      contour level input menu

 <1> lower value, increment, upper value

 <2> discrete contour values

 selection (1 or 2)

? 1

please input the lower contour value
? 3.0

please input the contour increment
? 0.1

please input the upper contour value
? 4.0

would you like to label the contours (y = yes, n = no)
? n

please input the minimum value for the x and y coordinates
? -2

please input the maximum value for the x and y coordinates
? 2
```

```
please input the mesh grid value for x and y
(a value between 0.01 and 0.005 is recommended)
? 0.01
```

The following is the graphics display for this example. The libration points for this three-body system are labeled with red asterisks and the primary (green) and secondary (blue) bodies are shown as small, filled circles.



The software also allows the user to control what part of the contour plot is displayed and label the zero relative velocity contours. The following is a typical user interaction illustrating these script options.

```
      program zvcurve2

graphics display of user-defined
zero relative velocity curves

please input the value for the mass ratio
? 0.2

      contour level input menu

<1> lower value, increment, upper value

<2> discrete contour values

selection (1 or 2)

? 1
```

```
please input the lower contour value
? 3.0

please input the contour increment
? 0.1

please input the upper contour value
? 3.5

would you like to label the contours (y = yes, n = no)
? y

please input the minimum value for the x and y coordinates
? 0

please input the maximum value for the x and y coordinates
? 1.5

please input the mesh grid value for x and y
(a value between 0.01 and 0.005 is recommended)
? 0.01
```

The following is the graphics display for this example. The libration points for this three-body system are labeled with red asterisks and the secondary body is the small, filled blue circle.